

INSTANTON METHODS AND THE DYON ATOM

by

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ABSTRACT

INSTANTON METHODS AND THE DYON ATOM

In quantum mechanics, perturbation theories have crucial importance for almost any practical calculation. However, perturbative expansions are generally asymptotic series due to the singularities on the complex coupling constant plane. In these cases, there is a non-zero error for the perturbative series. The divergent behaviour of an perturbative expansion can also be examined by using the non-perturbative techniques, such as WKB approximation and instanton methods. Furthermore, the non-perturbative methods can also be used to improve the error. In this thesis, instanton methods are applied to the perturbed Dyon atom. After a detailed discussion about the divergences of perturbation theory, the divergent behaviour of the energy coefficients of the one dimensional anharmonic oscillator is estimated by instanton methods. Then, the divergent behaviour of the perturbation series of the Dyon atom is calculated by a map from the estimation of 4 dimensional anharmonic oscillator, which is again done by instanton methods.

ÖZET

INSTANTON METODLARI VE DYON ATOMU

Pertürbasyon teorileri kuantum mekaniğindeki hemen her pratik hesaplamada çok önemli bir yere sahiptir. Ancak, pertürbatif açılımlar karmaşık çiftlenim sabiti uzayında bulunan tekilliklerden dolayı genellikle asimptotik seridirler. Bu durumlarda pertürbatif serilerin sıfırdan farklı hata payları vardır. Bir pertürbatif açılımın ıraksak davranışı WKB yaklaşımı ve instanton metodları gibi pertürbatif olmayan teknikler kullanılarak incellenilebilir. Ayrıca, pertürbatif olmayan metodlar hata payını geliştirmek için de kullanılabilirler. Bu tezde instanton metodları pertürbe edilmiş Dyon atomunda uygulanacaktır. Pertürbasyon teorilerinde ıraksamalarla ilgili detaylı bir çalışmadan sonra bir boyutlu harmonik olmayan salıngaçın enerji katsayılarının ıraksama davranışı hesaplanacaktır. Daha sonra da Dyon atomunun pertürbasyon serisinin ıraksak davranışı, yine instanton metodlarıyla yapılan 4 boyutlu harmonik olmayan salıngaç hesabından eşleştirme ile bulunacaktır.

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LIST OF SYMBOLS

A	Full vector electromagnetic potential
B	Magnetic Fields
D	Vector part of the vector electromagnetic potential
e	Electric charge
E	Energy of the anharmonic oscillators
E	Electric Fields
E_r	Real part of the energy of anharmonic oscillators
$F^{\alpha\beta}$	Electromagnetic Field Tensor
g	Magnetic charge
H	Hamiltonian
$\mathcal{H}$	Hamiltonian of the electromagnetic fields
H	Hamiltonian of the Dyon atom
J	Angular momentum operator of the dyon atom
k	Order of perturbative expansion
L	Angular momentum operator of the Coulomb system
$\mathcal{L}$	Lagrangian
N	Number of particles
$\mathcal{N}$	The normalization constant of path integrals
N_c	Critical umber of particles
p	Momentum operator
q	Coordinates of the anharmonic oscillator in one dimension
$q^{(a)}$	Total charge of a Dyon atom
$q_c(\tau)$	Classical path of the one dimensional anharmonic oscillator
r	Coordinates of the Dyon atom
$\hat{r}$	Radial unit vector
S_c	Classical action
S_E	Euclidean action
t	Time parameter

T	Temperature
$\mathbf{u}_c$	Classical path of the four dimensional anharmonic oscillator
$\mathbf{u}$	Coordinates of the anharmonic oscillator in four dimensions
x_i	Components of the coordinates of the Dyon atom in three dimensions
Z	Partition function of anharmonic oscillators
Z_0	Partition function of harmonic oscillators
β	Euclidean time parameter
Γ	Decay rate
θ_q	Argument of the coordinates of the one dimensional anharmonic oscillator
ε	Energy of the Dyon Atom
ϕ	Scalar electromagnetic potential
λ	Coupling constant of anharmonic oscillator
$\hbar$	Planck constant
τ	Imaginary time parameter
ψ	Wave function of the Dyon atom
Ψ	Wave function of the 4 dimensional anharmonic oscillator

LIST OF ACRONYMS/ABBREVIATIONS

3D	Three Dimensional
4D	Four Dimensional
QM	Quantum Mechanics
QFT	Quantum Field Theory
WKB	Wentzel–Kramers–Brillouin

1. INTRODUCTION

Perturbation theory is one of the most important tools of quantum mechanics (QM). This is because the Schrödinger equation, the fundamental equation of quantum mechanics, is not exactly solvable in most physical problems. One of the simplest examples for illustrating this is the Hydrogen atom. When the spin of both electron and proton is taken into account, the full Schrödinger equation can no longer be solved exactly. Instead, the additional effect is considered as a small perturbation and using standard Rayleigh-Schrödinger theory, the wave functions and energy eigenvalues are found as a series. A second standard application of perturbation theory is the system of the Hydrogen atom in an electromagnetic field. In this case, when the effects of the external fields are relatively small, perturbation theory is applicable. However, when the series that gives the energy eigenvalues is found, a difference with the first example is observed. While in the first case the series converges to a certain finite value, in the second case it has a divergent behaviour! In quantum field theory (QFT), perturbation theory is even more crucial because typically only non-interacting cases are exactly solvable¹. However, in all these QFT's the perturbation series diverges.

Before discussing how to properly interpret a divergent series, it is important to understand the physical origin of these divergences. This was explained by Freeman Dyson [1], as due to the appearance of a non-linear instability for the ground state at some (possibly complex) value of the coupling constant. Furthermore, the non-perturbative decay of the (unstable) ground state contains a lot of information about high order terms in the perturbation series. Therefore, it is possible to examine the divergent behaviour of the perturbation series by looking at the decay of ground states and vice-versa.

Both in QM and QFT, computing the high order corrections can be very difficult. Often it is more efficient to look at tunnelling of the ground state to examine high

¹Actually in some interacting cases, a precise definition of the theory that does not rely on perturbation theory is still missing.

order behaviour. In QM, the WKB approach can be used as in the famous example of the anharmonic oscillator [3]. Equivalent to WKB one can use the instanton method, via the Euclidean path integral [4, 5]. This second approach has the advantage that it generalizes to QFT.

On the other hand, in cases where a complete non-perturbative definition of the theory is lacking the study of high order perturbation theory gives insight about a non-perturbative completion. This has been of interest in gauge theory, string theory and holography, see e.g. [9–11].

Finally a comment on the proper interpretation of divergent series². Although at large orders the series diverges, it starts with a number of decreasing terms. One can find a best estimate by truncating the series at order $n_* \sim \lambda^{-1}$. This limits the precision and the best possible error is $\varepsilon_* \sim e^{-1/\lambda}$, which coincides with the size of non-perturbative effects and the error can only be improved by including their contribution.

Apart from the anharmonic oscillator, non-perturbative methods were also used to estimate the higher order corrections of the Hydrogen atom in electromagnetic fields [6, 7]. In these papers, the authors used the transformation rules that maps the coordinates of the three dimensional Coulomb systems to the simple harmonic oscillator. These rules are based on the equivalence of the symmetry group of these two systems [28, 29].

A dyon is a hypothetical particle that carries both electric and magnetic charges. Even though they have not been observed in nature, it is interesting to discuss problems involving dyons since they correspond to a more general form of electromagnetism. For example a Dyon atom, made of two interacting dyons, is a direct generalization of the Hydrogen atom, which reduces to it when the magnetic charges are set to zero [8]. The Dyon atom generalize the four dimensional anharmonic oscillator to a more general angular momentum eigenstates. See [12] for a review about the duality between the Dyon atom and the harmonic oscillator.

Because of the similarity between the Hydrogen atom and the Dyon atom, a simi-

²The discussion is limited to series with factorially growing coefficients.

lar divergent behaviour in the high orders of the perturbation series is expected. In this thesis, the application of the instanton method to the Dyon atom in the context quantum mechanics will be discussed and an estimate of the high order perturbative corrections will be presented.

The thesis will start with a general discussion of the divergent behaviour of perturbation theory in Chapter 2. After reviewing Dyson's arguments in Section 2.1, in Sections 2.2 and 2.3 the mathematical relations between divergence and tunnelling will be established. In these Sections, the relation between a branch cut in the complex coupling constant plane and the large order behaviour is discussed. This will be illustrated in the example of a simple integral which calculates the partition function of the zero dimensional anharmonic oscillator. Finally at the end of the chapter, the coefficients of the perturbative expansion of the one dimensional anharmonic oscillator will be estimated by the instanton method and a comparison with the results coming from the Rayleigh-Schrodinger theory will be given.

In Chapter 3 the interaction between two point Dyons will be discussed in the context of duality invariant electromagnetism. First the duality invariant formalism of classical electromagnetism will be explained (Sections 3.1, 3.2). Then the duality invariant form of the Schrödinger equation for two interacting Dyons will be derived in detail (Sections 3.3 and 3.4). The duality between the Dyon atom and the four dimensional harmonic oscillator is discussed in Section 3.6 and is based on the transformation rules derived in Section 3.5.

Finally in the last chapter the high order coefficients of the perturbative expansion for the energy of the Dyon atom will be estimated by the instanton method. This calculation is done for the four dimensional quartic anharmonic oscillator, which is equivalent to the perturbed Dyon atom, through the transformation of Chapter 3. The energy eigenvalues of the Dyon atom can be found by transforming back. They are presented in Equation 4.27.

2. DIVERGENCES OF PERTURBATION THEORY

In this chapter, the physical reason of the divergent behaviour of the large orders of perturbative expansions and its relation with non-perturbative physics will be discussed. The method based on the dispersion relation between the imaginary part of the ground state energy when the coupling constant is negative and high order perturbative expansion of the energy will be studied. It is illustrated in two examples of zero and one dimensional anharmonic oscillators. This chapter provides the technical basis for the study of the large order perturbations of the Dyon atom.

2.1. Large Order Behaviour of Perturbative Expansions

In this section, the underlying physical reasons of the divergent behaviour of the perturbative expansion at high order will be discussed. First, using a simple example, Dyson's arguments about a decaying unstable ground state will be recovered and then factorial divergence of the expansion coefficients

$$f_k \sim k! \tag{2.1}$$

will be found. In order to understand the general behaviour of the perturbation series, coupling constants will be considered as complex variables and the importance of the non-physical values of coupling constants will be understood.

Consider a system contains N particles [2]. Each particle has a kinetic energy T and each pair interacts with a potential V . The total energy is given as

$$E \sim NT + \frac{1}{2}e^2VN^2. \tag{2.2}$$

which has only one critical point(classical equilibrium) at

$$N_c = \frac{-T}{e^2 V} \quad (2.3)$$

In addition to that the second derivative of the energy function with respect to the number of particles can be found as

$$\frac{d^2 E}{dN^2} = e^2 V \quad (2.4)$$

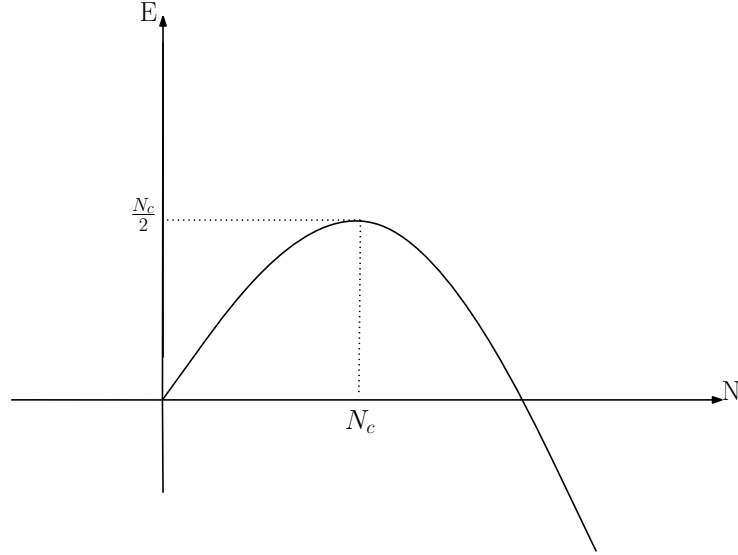


Figure 2.1. Energy graph with negative coupling.

For a classical system, there are two possibilities about the equilibrium of the system, which depends on the sign of e^2 . If the sign is positive, which corresponds to the physical world, the second derivative will be positive and the classical equilibrium would always be stable. However, when $e^2 < 0$, corresponding to the non-physical world, the equilibrium will be unstable under small perturbations around N_c . In Figure 2.1, it is easily seen even the negatively coupled system has a stable equilibrium at $N = 0$, when it is started from small values of N . On the other hand, if the number of particles, N , of the system exceeds the critical value, $N > N_c$, it starts to increase drastically without any extra work on the system. In fact, as the number of particles increases, the total energy of the system decreases and eventually it goes to $-\infty$, which is not

possible for a physical system.

The physical interpretation of a system with $e^2 < 0$ is given by Dyson [1]. Define a physical quantity such as $F(\lambda)$, where $\lambda = e^2$. Its perturbative expansion is given by the power expansion of λ

$$F(\lambda) = f_0 + f_1\lambda + f_2\lambda^2 + \dots = \sum_{k=0}^{\infty} f_k\lambda^k. \quad (2.5)$$

Consider a system has only electron and positron pairs. In the presence of negative coupling, like charges attract each other. Then, it is possible to bring electrons together in a region and bring positrons together in another one. Thus, electron-positron pairs would not occur, but, because of the spontaneous particle creation, the particle number does never stop to increase and the system would eventually be in an unstable state. Moreover, this instability would be seen as the divergence of the expansion of physical quantity $F(\lambda)$. However, as Dyson states, these “pathological states” would occur when the magnitude of potential energy, $\frac{1}{2}\lambda VN^2$, is much greater than the kinetic energy, NT , of particles. That means, for small number of particles, the repulsion of opposite(or attraction of like) charges is too low so that e^+ and e^- would annihilate each other quickly, before these forces become effective. Therefore, the series should not diverge, if the system is classical and the number of particles is small.

On the other hand, if the quantum effects are considered, by the spontaneous creation of pair particles, there is always a finite possibility for the system to be in a “pathological state”. Therefore, even the system starts with a very low kinetic energy and in a state with $N \ll N_c$, it would be in a “pathological state” with $N > N_c$ in finite time, which has a same energy with the state just before the tunnelling. After, the system in that “pathological state”, the number of the particles starts to increase very rapidly and tunnelling back to the “secure” states would not be possible.

The spontaneous particle creation can be shown by Feynmann diagrams as in Figure 2.2. Each node in the diagrams is related to the corresponding order of the expansion.

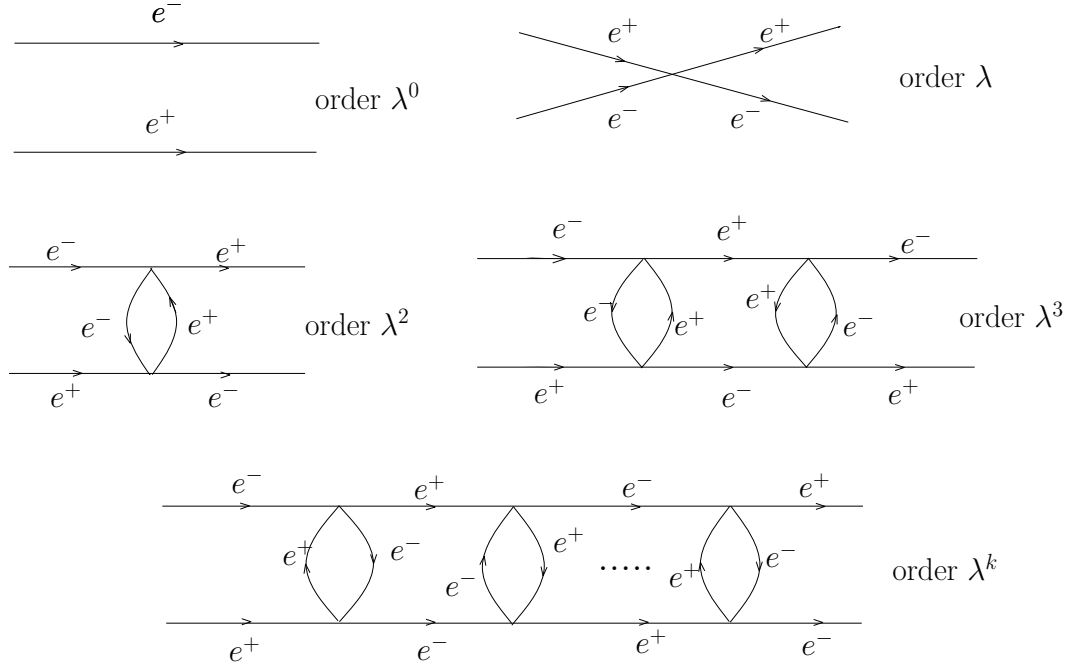


Figure 2.2. Feynmann Diagrams.

sion. When the diagrams are examined, it is obvious that the order k of the expansion corresponds to the interaction of k number of particles. Then, if Equation 2.5 is considered, one can easily deduced that the “pathological states”, $N > N_c$, are related to the orders $k > N_c$, which corresponds to the high orders of the expansion since N_c is typically a large number. Therefore, the divergence of the expansion in the equation is related to high orders. Moreover, since the system is stable for small N , the expansion should show a convergent behaviour at small orders. Then, the terms in Equation 2.5 will decrease up to order $k \simeq N_c$, where the instability, so that divergence, starts to be seen and they increase without any boundary at high orders. Therefore, a minima for the terms in the expansion is expected to be around order $k \simeq N_c$.

The divergence of the expansion can be seen in the following mathematical argument [2];

$$f_k(e^2)^k \simeq f_{k+1}(e^2)^{k+1} \quad \text{for } k \sim N_c$$

$$\frac{f_k}{f_{k+1}} \sim e^2$$

Recall the Equation 2.3

$$N_c = \frac{-T}{Ve^2}$$

Then, $N_c \sim \frac{1}{e^2}$ and

$$\frac{f_{k+1}}{f_k} \sim \frac{1}{e^2} \sim N_c \sim k$$

which leads to the Equation 2.1

$$f_k \sim k!$$

This relation shows the divergent behaviour of the expansion of $F(\lambda)$ for large k with non-physical coupling constant.

In addition to these arguments, following from the Dyson's argument [1], one can claim that the physical value $F(e^2)$ has a singularity at $e^2 = 0$ and is not convergent for values of $e^2 > 0$ neither. Convergence of $F(e^2)$ for $e^2 > 0$ leads the function $F(e^2)$ to be analytical at $e^2 = 0$ and its convergence at small negative values. Then, the function with negative e^2 would be well-behaved, which contradicts with behaviour of the system in the “fictitious” world explained above.

2.2. Large Order Behaviour and a Dispersion Relation

After the intuitive introduction in the previous section, now it is time to start to construct a mathematical relation between large order behaviour of the perturbative coefficients and decay of unstable ground state. In order to do that first the energy of the system, which is a complex function, will be related to the decay rate. Then, the dispersion relation, which relates the imaginary part any function with a branch cut

to its imaginary part, will be derived in detail.

$$f(z) = \frac{1}{\pi} \int_{-\infty}^0 \frac{\text{Im}f(x')}{x' - z} dx' \quad (2.6)$$

Furthermore, using the perturbative expansion

$$f(z) = \sum f_k k^\lambda$$

the dispersion relation for the expansion coefficient

$$f_k = \frac{1}{\pi} \int_{-\infty}^0 dx' \frac{\text{Im}f(x')}{x'^{k+1}} \quad (2.7)$$

will be stated. Since the imaginary part is directly related to the decay of the false vacuum, this equation will be crucial in the calculations of large order behaviour of perturbation series.

2.2.1. The Decay Rate as Imaginary Energy

Consider a potential $V(x)$, which has a false vacuum (metastable ground state) at $x = x_0$ as in Figure 2.3. Due to quantum tunnelling the classical equilibrium point, located at $x = x_0$, is unstable since it is not a global minimum. A quantum particle, which is localized at $x = x_0$ initially, tunnels through the barrier and since the potential is unbounded on the right side of the barrier, it will never return to its initial position again. Therefore, the magnitude of the wave function at the local minimum $x = x_0$, can be approximately represented as $|\psi| \sim e^{-|\Gamma| \frac{t}{\hbar}}$.

From the Schrödinger equation, the time dependent form of the wave function for this metastable ground state can be given as $\psi \sim e^{-\frac{i}{\hbar} E_0 t}$, where E_0 is the energy of the state. From the previous relation,

$$|\psi|^2 \sim e^{-\frac{i}{\hbar} E_0 t} e^{\frac{i}{\hbar} \bar{E}_0 t} \sim e^{-2|\Gamma| \frac{t}{\hbar}}$$

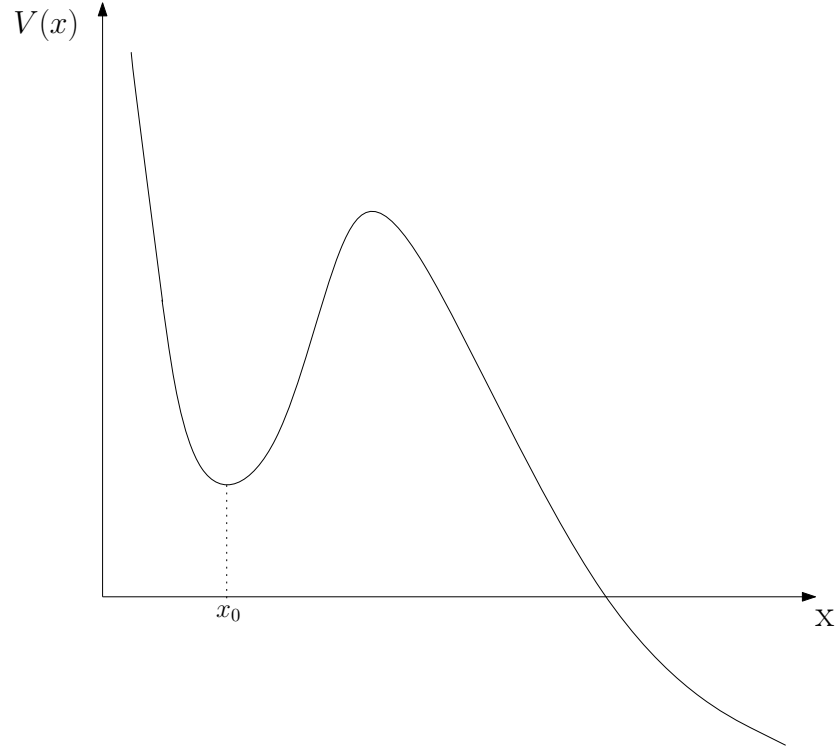


Figure 2.3. False Vacuum.

can be found. This relation shows the necessity of non-zero imaginary part of the ground state energy, which means $E_0 \neq \bar{E}_0$. Therefore, for $E_0 = E_r + iE_i$, where E_r and E_i represent the real and complex parts respectively,

$$|\Gamma| = -E_i$$

Thus the lifetime of the metastable state is given by $\tau = \frac{\hbar}{|\Gamma|}$, and so one can claim that the decay of the metastable state is directly related to the imaginary part of the complex energy of the corresponding state. As discussed in the Section 2.1, it is known that the decay corresponds to the high orders of the perturbative expansion. Therefore, it can be concluded that the imaginary part of the metastable ground state energy should be mathematically related to the high orders of the perturbative expansion. The precise formula is 2.7 which will be derived in the following subsection.

2.2.2. Derivation of The Dispersion Relation

A perturbative expansion can be represented as

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \quad (2.8)$$

In order to derive the dispersion relation 2.7 assume that the function $f(z)$ has the following properties

- $f(z)$ is analytic in the complex plane, except in the region $(-\infty, 0]$ along the real line.
- $|f(z)| \rightarrow 0$ faster than $\frac{1}{z} \rightarrow 0$ as $|z| \rightarrow \infty$.
- $f(z) \in \mathbb{R}$ for $z = \text{Re}(z)$

By a well-known expression for analytic functions, called the “Cauchy Integral Representation” [14, 15], the function $f(z)$ is given as

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z')}{z' - z} \quad (2.9)$$

where C is a closed contour such that $f(z)$ is analytic in its interior . Since $f(z)$ has a cut along $(-\infty, 0]$, a contour for the integral that represents $f(z)$ is given in Figure 2.4. Along this contour, the integral in Equation 2.9 is a summation of individual integrals along the paths Γ_1 , Γ_2 , γ_1 , γ_2 . However, from well-known theorems [14, 15], the individual integrals along the curves Γ_1 and Γ_2 are 0. Therefore, Equation 2.9 becomes,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} dz' \frac{f(z')}{z' - z} + \int_{\gamma_2} dz' \frac{f(z')}{z' - z}$$

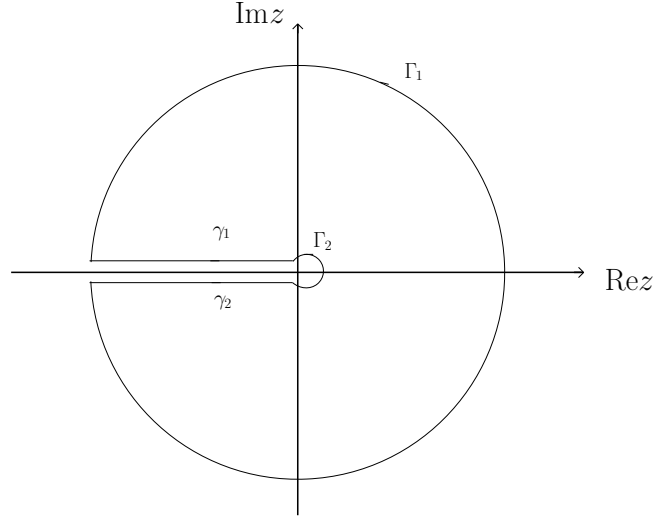


Figure 2.4. Contour for a function with a cut along the negative real axis.

which equals to

$$f(z) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \left\{ \int_{-\infty+i\epsilon}^{0+i\epsilon} \frac{f(z')}{z' - z} dz' + \int_{0-i\epsilon}^{-\infty-i\epsilon} \frac{f(z')}{z' - z} dz' \right\}$$

By the following change of variables,

$$z' = x' \pm i\epsilon dz' = dx'$$

and the integral becomes

$$f(z) = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \left\{ \int_{-\infty}^0 \frac{f(x' + i\epsilon)}{x' - z + i\epsilon} dx' - \int_{-\infty}^0 \frac{f(x' - i\epsilon)}{x' - z - i\epsilon} dx' \right\}.$$

Via the Schwarz reflection principle, that states $\bar{f}(z) = f(\bar{z})$, it then follows that

$$\begin{aligned} f(z) &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \left\{ \int_{-\infty}^0 \frac{f(x' + i\epsilon)}{x' - z + i\epsilon} dx' - \int_{-\infty}^0 \frac{\bar{f}(x' + i\epsilon)}{x' - z + i\epsilon} dx' \right\}. \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^0 \frac{(x' - z + i\epsilon)f(x' + i\epsilon) - (x' - z + i\epsilon)\bar{f}(x' + i\epsilon)}{(x' - z)^2 - \epsilon^2} dx' \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{-\infty}^0 \frac{(x' - z) [f(x' + i\epsilon) - \bar{f}(x' + i\epsilon)] - i\epsilon [f(x' + i\epsilon) + \bar{f}(x' + i\epsilon)]}{(x' - z)^2 - \epsilon^2} dx' \end{aligned}$$

When the limit is taken, the second term in the numerator vanishes and the equation becomes

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^0 \frac{f(x') - \bar{f}(x')}{x' - z} dx'$$

Using that $f(x') - \bar{f}(x') = 2i\text{Im}f(x')$, one finally achieves the dispersion relation 2.6

$$f(z) = \frac{1}{\pi} \int_{-\infty}^0 \frac{\text{Im}f(x')}{x' - z} dx'$$

Now, the relation between the expansion coefficients and the imaginary part of the function can be stated. The integral in Equation 2.6 can be expanded in a Taylor series around $z = 0$,

$$\begin{aligned} f(z) &= \frac{1}{\pi} \int_{-\infty}^0 \frac{\text{Im}f(x')}{x' - z} dx' = \frac{1}{\pi} \int_{-\infty}^0 \frac{\text{Im}f(x')}{x'(1 - \frac{z}{x'})} dx' \\ &= \frac{1}{\pi} \int_{-\infty}^0 \frac{\text{Im}f(x')}{x'} \left(1 + \frac{z}{x'} + \frac{z^2}{x'^2} + \dots \right) \\ &= \frac{1}{\pi} \sum_{k=0}^{\infty} \int_{-\infty}^0 \frac{\text{Im}f(x')}{x'} \left(\frac{z}{x'} \right)^k \end{aligned}$$

Comparing with the perturbative expansion in Equation 2.8 directly gives the Equation 2.7,

$$f_k = \frac{1}{\pi} \int_{-\infty}^0 dx' \frac{\text{Im}f(x')}{x'^{k+1}}$$

Note that since in the derivation of the dispersion relation a Taylor expansion around $z = 0$ is used, it requires very small z , which corresponds to the coupling constant in the physical systems.

2.3. Large Order Behaviour and Tunneling

The dispersion relation in Equation 2.7 is a very important tool in calculations of large order behaviour via non-perturbative physics. Before discussing a dynamical problem, it is beneficial to examine a zero dimensional system as a simple example. In this way, comparisons between the direct calculations and the estimation using the dispersion relation can be done. Moreover, estimating the large orders via steepest descent method, it is possible to verify the validity of the proposed method.

Consider the potential of the quartic anharmonic oscillator [2, 9],

$$V(x) = \frac{x^2}{2} + \lambda \frac{x^4}{4} \quad (2.10)$$

When $\lambda < 0$ the ground state at $x = 0$ becomes metastable when $\lambda < 0$. As a simple example, look at the zero dimensional analogue of the partition function

$$I(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} dx \quad (2.11)$$

which is well-defined only for $\text{Re}(\lambda) > 0$. From the discussion in Section 2.1, it thus follows that the perturbative expansion in λ

$$I(\lambda) = \sum_{k=0}^{\infty} I_k \lambda^k \quad (2.12)$$

is divergent.

First, assume that $\lambda \in \mathbb{R}$, and small i.e. $|\lambda| \ll 1$. The term $e^{\lambda \frac{x^4}{4}}$ can be expanded around $\lambda = 0$.

$$e^{-\lambda \frac{x^4}{4}} = 1 - \lambda \frac{x^4}{4} + \frac{\lambda}{2!} \left(\frac{x^4}{4}\right)^2 = \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa}{\kappa!} \frac{x^{4\kappa}}{4^\kappa} \lambda^\kappa$$

By inserting this expansion in Equation 2.11

$$I(\lambda) = 2 \frac{1}{\sqrt{2\pi}} \int_0^\infty dx e^{-\frac{x^2}{2}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{x^{4k}}{4^k} \lambda^k$$

If $\lambda < 0$, the divergence occurs when $|x| \rightarrow \infty$ but if the upper bound of the integral is taken as a finite number $r \in \mathbb{R}$ it would be convergent. Then, separate the integral into two parts,

$$I(\lambda) = 2 \frac{1}{\sqrt{2\pi}} \left\{ \int_0^r dx e^{-\frac{x^2}{2}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{x^{4k}}{4^k} g^k + \int_r^\infty dx e^{-\frac{x^2}{2}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{x^{4k}}{4^k} g^k \right\}$$

If r is chosen large enough, the second integral would be negligible with respect to the first one. Thus, $I(\lambda)$ becomes

$$I(\lambda) \simeq 2 \frac{1}{\sqrt{2\pi}} \int_0^r dx e^{-\frac{x^2}{2}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{x^{4k}}{4^k} \lambda^k$$

Now the summation is uniformly convergent on the domain of the integral. Therefore, the order of the integration and the summation can be changed. Then, the boundary of the integration can be extended to infinity again with the cost of losing convergence of the series. It has become an asymptotic series.

$$I(\lambda) = 2 \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \frac{(-1)^k}{4^k k!} \int_0^\infty dx x^{4k} e^{-\frac{x^2}{2}} \lambda^k$$

If $I(\lambda)$ is expressed as in Equation 2.12, the k 'th term of the summation becomes

$$I_k = 2 \frac{1}{\sqrt{2\pi}} \frac{(-1)^k}{4^k k!} \int_0^\infty dx x^{4k} e^{-\frac{x^2}{2}} \quad (2.13)$$

2.3.1. Exact Calculation of Series Coefficients

Now, the integral

$$J(k) = \int_0^\infty dx x^{4k} e^{-\frac{x^2}{2}} \quad (2.14)$$

can be evaluated exactly. The result that will be found by this calculation is valid for all k -values.

$$\begin{aligned} J(k) &= \int_0^\infty dx x^{4k} e^{-\frac{x^2}{2}} = \int_0^\infty dx x^{4k-1} x e^{-\frac{x^2}{2}} \\ &= \int_0^\infty dx x^{4k-1} \left(-\frac{d}{dx} e^{-\frac{x^2}{2}} \right) \\ &= -x^{4k-1} e^{-\frac{x^2}{2}} \Big|_0^\infty + \int_0^\infty dx (4k-1) x^{4k-2} e^{-\frac{x^2}{2}} \\ &= (4k-1) \int_0^\infty dx x^{4k-3} x e^{-\frac{x^2}{2}} \\ &= (4k-1)(4k-3) \int_0^\infty dx x^{4k-4} e^{-\frac{x^2}{2}} \end{aligned}$$

By continuing this process,

$$\begin{aligned} J(k) &= (4k-1)(4k-3)(4k-5)\dots 7*5*3 \int_0^\infty dx x^2 e^{-\frac{x^2}{2}} \\ &= (4k-1)!! \left\{ -x e^{\frac{x^2}{2}} \Big|_0^\infty + \int_0^\infty dx e^{-\frac{x^2}{2}} \right\} \end{aligned}$$

which yields by usual Gaussian integration

$$J(k) = (4k-1)!! \sqrt{\frac{\pi}{2}} \quad (2.15)$$

Therefore, k 'th term of summation is

$$I_k = 2 \frac{1}{\sqrt{2\pi}} \frac{(-1)^k}{4^k k!} (4k-1)!! \sqrt{\frac{\pi}{2}} = (-4)^{-k} \frac{(4k-1)!!}{k!} \quad (2.16)$$

which is valid for all k -values. Since for large k -values $(4k-1)!!$ dominates $k!$, I_k grows without bound in k .

2.3.2. Calculation by the Steepest Descent Method

The integral in Equation 2.14 can also be evaluated by the Steepest descent method, which is a good approximation for large k -values. First prepare the integral for this method.

$$J(k) = \int_0^\infty dx x^{4k} e^{-\frac{x^2}{2}} = \int_0^\infty dx (x^2)^{2k} e^{-\frac{x^2}{2}}$$

Make a change of variables

$$\frac{x^2}{2} = y$$

$$x dx = dy$$

Then,

$$\begin{aligned} J(k) &= \int_0^\infty dx (x^2)^{2k-\frac{1}{2}} x e^{-\frac{x^2}{2}} \\ &= \int_0^\infty dy (2y)^{2k-\frac{1}{2}} e^{-y} \\ &= 2^{2k-\frac{1}{2}} \int_0^\infty dy y^{2k-\frac{1}{2}} e^{-y} \end{aligned}$$

Define a new variable: $s = 2k - \frac{1}{2}$,

$$\begin{aligned} J(k) &= 2^s \int_0^\infty dy y^s e^{-y} \\ &= 2^s \int_0^\infty dy e^{s(\ln y - \frac{y}{s})} \end{aligned} \quad (2.17)$$

Extend the definition of y into the complex plane. For large s -values, the steepest descent method, Equation B.14 in Appendix B, gives

$$J(k) = 2^s e^{sw(y_0)} e^{i\alpha_0} \sqrt{\frac{2\pi}{sw_0''}} \quad (2.18)$$

where

$$w(y) = \ln y - \frac{y}{s}$$

and $y_0 = s$ is the saddle point of $w(y)$. Furthermore, α_0 and w'' are easily found using the definitions in Appendix B.

The second derivative of $w(y)$ is

$$w''(y) = -\frac{1}{y^2}.$$

At the saddle point

$$w''(y_0) = -\frac{1}{s^2} = \frac{1}{s^2} e^{i\pi}$$

Then,

$$\theta = \pi$$

and

$$w_0'' = \frac{1}{s^2}$$

so α_0 in Equation 2.18 is given as

$$\alpha_0 = \frac{\pm\pi + \theta}{2} = 0, \pi$$

Since the original path of the integral in Equation 2.17 is from 0 to ∞ , $\alpha_0 = 0$ should be chosen. Then, Equation 2.18 becomes

$$J(k) = 2^s e^{s \ln s - s} e^{i0} \sqrt{\frac{2\pi}{s \frac{1}{s^2}}} = 2^s e^{-s} s^s \sqrt{2\pi s}$$

By recalling $s = 2k - \frac{1}{2}$,

$$J(k) = 2^{2k - \frac{1}{2}} \left(2k - \frac{1}{2}\right)^{2k - \frac{1}{2}} \sqrt{2\pi \left(2k - \frac{1}{2}\right)} e^{-2k + \frac{1}{2}} \quad (2.19)$$

Then, Equation 2.13 becomes

$$I_k \simeq 2 \frac{1}{\sqrt{2\pi}} \frac{(-1)^k}{4^k k!} 2^{2k - \frac{1}{2}} \left(2k - \frac{1}{2}\right)^{2k - \frac{1}{2}} \sqrt{2\pi \left(2k - \frac{1}{2}\right)} e^{-2k + \frac{1}{2}}$$

By some simple calculations

$$\begin{aligned} I_k &\simeq \frac{(-1)^k}{k!} \sqrt{2} \left(2k - \frac{1}{2}\right)^{2k} e^{-2k + \frac{1}{2}} \\ &\simeq \frac{(-1)^k \sqrt{2} \left(2k - \frac{1}{2}\right)^{2k} e^{-2k + \frac{1}{2}}}{k^k e^{-k} \sqrt{2\pi k}} \\ &\simeq \frac{(-1)^k \sqrt{2} (4k - 1)^{2k} e^{-k + \frac{1}{2}}}{\sqrt{\pi} 4^k k^{k + \frac{1}{2}}} \end{aligned}$$

In the 2nd line Stirling formula

$$k! = \Gamma(k+1) \simeq k^k e^k \sqrt{2\pi k}$$

is used.

Moreover, since k is taken large, $(4k-1)^{2k} \simeq (4k)^{2k}$. Then, multiply and divide I_k by $\sqrt{2\pi k}$.

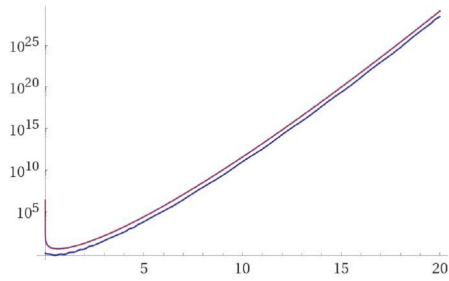
$$\begin{aligned} I_k &\sim \frac{(-1)^k 4^k k^k e^{-k+\frac{1}{2}}}{\sqrt{\pi} k^{\frac{1}{2}}} \times \frac{\sqrt{2\pi k}}{\sqrt{2\pi k}} \\ &\sim \frac{(-1)^k 4^k k! e^{\frac{1}{2}}}{\sqrt{2\pi k}} \\ &\sim \frac{(-1)^k e^{\frac{1}{2}}}{\sqrt{2\pi}} 4^k (k-1)! \end{aligned}$$

Ignoring the k dependent pre-factors, it is finally found that

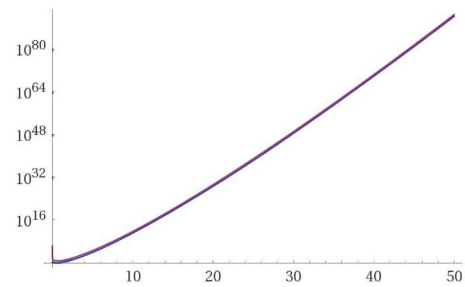
$$I_k \sim (-4)^k (k-1)! \quad (2.20)$$

As expected from the previous Section 2.1, at large orders, $k \gg 1$, I_k is dominated by the factorial term.

As it can be seen in Figures (2.5) and (2.6), the exact result 2.15 and the result for the large orders 2.20, which is achieved by using the steepest descent method, matches perfectly at large orders.



(a) Plot for $k = 0$ to $k = 20$. There is an obvious difference between exact result and the estimation for large orders.



(b) Plot for $k = 0$ to $k = 50$. The difference almost vanishes in this plot.

Figure 2.5. Comparison between exact result and the steepest descent methods.

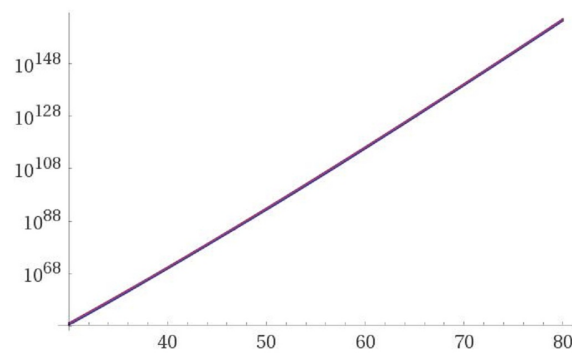


Figure 2.6. The behaviours of two solutions are exactly the same for $k > 30$.

2.3.3. Relation between large order and tunnelling

Now the large order behaviour will be derived by using the imaginary part of $I(\lambda)$. From previous discussions, it is known that the large order behaviour is also related to decay of the metastable state, located near a minimum of the potential, $V'(x) = 0$, which corresponds to $x = 0$ in this case. Tunnelling appears when $\lambda < 0$ and by the dispersion relation 2.7, the coefficients of the expansion are related to the tunneling amplitude as

$$I_k = \frac{1}{\pi} \int_{-\infty}^0 d\lambda \frac{\text{Im} I(\lambda)}{\lambda^{k+1}} \quad (2.21)$$

The calculation of $\text{Im} I(\lambda)$ is handled by an analytic continuation from the region $\text{Re}(\lambda) > 0$, where the integration is well-defined, to the region $\lambda < 0$ and this will introduce a branch cut. For some additional details to the discussion here see Appendix A.2.

While going from the $\text{Re}(\lambda) > 0$ region to the $\text{Re}(\lambda) < 0$ region, both x and λ are extended to the complex plane. The main objective of this extension is keeping the term $\text{Re}(\lambda x^4)$ positive so that no divergences occurs when $\text{Re}(\lambda) < 0$. This goal is achieved by rotating the x -plane as $\arg \lambda$ changes accordingly. After the extension of the domain of the variable x to the complex plane, the integral in Equation 2.11 becomes

$$I(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\Gamma} e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} dx \quad (2.22)$$

where the contour Γ is arranged such that

$$\Gamma : \quad \arg x = -\frac{1}{4} \arg \lambda \quad (2.23)$$

Remember that the main goal was reaching the value $-|\lambda|$. Then, $|\arg \lambda|$ changes from 0 to π . However, rotation in λ -plane can be performed in two directions, i.e. clockwise

or counter-clockwise and these two different rotations give different answers due to the branch cut along the negative axis in λ -plane.

Then, in x -plane, the contour Γ becomes

$$\begin{aligned}\Gamma_+ : \quad \arg x &= -\frac{\pi}{4} \\ \Gamma_- : \quad \arg x &= \frac{\pi}{4}\end{aligned}$$

where Γ_+ and Γ_- correspond to the rotations in counter-clockwise and clockwise directions respectively. See Figure 2.7.

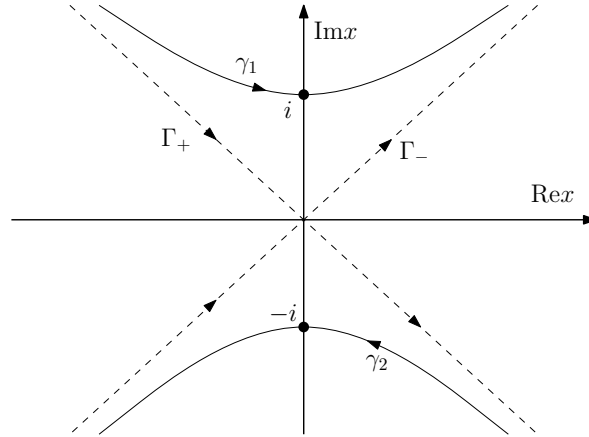


Figure 2.7. The original contours, i.e. Γ_+ , Γ_- , and the deformed contours, i.e. γ_1 , γ_2 on x -plane.

Define, $\lambda \equiv -|\lambda| \pm i\varepsilon$, for the contours $\Gamma_{\pm}$

$$\lim_{\varepsilon \rightarrow 0} I(-|\lambda| \pm i\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{\Gamma_{\pm}} e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} dx \quad (2.24)$$

$$(2.25)$$

Then, using

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} \{I(-|\lambda| + i\varepsilon) - I(-|\lambda| - i\varepsilon)\} &= \lim_{\varepsilon \rightarrow 0} \{I(-|\lambda| + i\varepsilon) - \bar{I}(-|\lambda| + i\varepsilon)\} \\ &= \lim_{\varepsilon \rightarrow 0} 2i \operatorname{Im} I(-|\lambda| + i\varepsilon)\end{aligned}$$

the imaginary part of $I(\lambda)$ becomes

$$\text{Im}I(\lambda) = \frac{1}{2i} \frac{1}{\sqrt{2\pi}} \left(\int_{\Gamma_+} \frac{dx}{\sqrt{\lambda}} e^{-\frac{1}{\lambda} \left(\frac{x^2}{2} + \frac{x^4}{4} \right)} - \int_{\Gamma_-} \frac{dx}{\sqrt{\lambda}} e^{-\frac{1}{\lambda} \left(\frac{x^2}{2} + \frac{x^4}{4} \right)} \right) \quad (2.26)$$

In both integrals, the variable x is rescaled as $x \rightarrow \frac{x}{\sqrt{\lambda}}$. Since there is no singularity in the x -plane, the contours Γ_1 and Γ_2 shown in Figure 2.7 can be deformed to make them pass the saddle points of rescaled potential $V(x) = \frac{1}{\lambda} \left(\frac{x^2}{2} + \frac{x^4}{4} \right)$. The saddle points of the potential are found as $x = 0$ and $x = \pm i$. The contribution of the saddle point $x = 0$ is the same for both integral in Equation 2.26. Therefore, the contribution of this saddle point to $\text{Im}I(\lambda)$ is 0, even though, when individual integrals are considered, the largest order contributions come from this saddle point. Then, the other saddle points, i.e. $x = \pm i$, should be considered. The deformed contours, passing through $x = \pm i$, are shown as γ_1 and γ_2 in Figure 2.7 and by using the steepest descent method both integrals can be evaluated.

The second derivative of $V(x)$,

$$V''(x) = 1 + 3x^2 \Big|_{x=\pm i} = -2 = 2e^{\pm i\pi}.$$

Then,

$$\theta = \pm\pi$$

$$v_0'' = 2$$

and

$$\alpha = \frac{-\pi \pm \pi}{2} = 0, -\pi.$$

Use $\alpha \equiv \alpha_1 = 0$ for the saddle point $x = i$ and use $\alpha \equiv \alpha_2 = -\pi$ for the saddle point $x = -i$. Since $V(x_c) = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$, for both saddle points, $x = \pm i$, Equation 2.26

becomes

$$\begin{aligned} \text{Im}I(\lambda) &\simeq \frac{1}{2i} \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{i\alpha_1}}{\sqrt{\lambda}} \sqrt{-\frac{2\pi\lambda}{2}} - \frac{e^{i\alpha_2}}{\sqrt{\lambda}} \sqrt{-\frac{2\pi\lambda}{2}} \right\} e^{-\frac{V(x_c)}{\lambda}} \\ &= \frac{1}{2i} \frac{1}{\sqrt{2\pi}} \{e^0 i\sqrt{\pi} - e^{i\pi} i\sqrt{\pi}\} e^{\frac{1}{4\lambda}} \end{aligned}$$

and

$$\text{Im}I(\lambda) \simeq 2^{-1/2} e^{\frac{1}{4g}} \quad (2.27)$$

Now, by using Equations 2.27 and 2.21 the coefficients of the expansion

$$I(\lambda) = \sum I_k \lambda^k$$

can be evaluated as

$$I_k \sim \left(-\frac{1}{4}\right)^k \Gamma(k) \sim (-4)^k (k-1)! \quad (2.28)$$

which exactly matches with the Equation 2.20. Therefore, this simple example verifies the direct connection between the large order behaviour and the imaginary part of the physical value when the coupling constant, i.e. λ , small and negative.

2.4. Instantons in the One Dimensional Anharmonic Oscillator

After the discussion on the zero dimensional toy model of anharmonic oscillator in the last section, in this section the large order behaviour of the perturbative energy coefficients of the one dimensional anharmonic oscillator in quantum mechanics will be estimated by using the non-perturbative physics. This estimation, initially, was done by WKB approximation in [3]. Later using the Euclidean path integral formalism same

results for imaginary part of the false vacuum energy

$$\text{Im}E = \frac{4}{\sqrt{2\pi}} \frac{1}{\sqrt{-\lambda}} e^{4/3\lambda} \quad (2.29)$$

and for the high order energy correction

$$E_k = \left(\frac{6}{\pi^3}\right)^{1/2} (-1)^{k+1} \left(\frac{3}{4}\right)^k \Gamma(k + \frac{1}{2}) \quad (2.30)$$

were found [4, 5]. Both approaches utilizes the branch cut along the negative real axis on the λ plane. For a rigorous proof of the cut see Appendix A. In this section, the estimation using the path integral formalism will be done in a more pedagogical way based on the discussions in [9, 13, 17].

In the path integral formalism, bound states energies are found using the partition function originally invented in the statistical physics. The relation between the partition function and the energy states is given as

$$Z(\beta) = \sum_n e^{-\beta E_n} \quad (2.31)$$

Since the imaginary part of the energy eigenvalues in the presence of negative coupling is taken into consideration, its relation with the imaginary part of the partition function is important.

$$\begin{aligned} \text{Im}Z &\sim \text{Im}e^{-\beta E} = \text{Im}e^{-\beta(E_r + i\Gamma)} \\ &= e^{-\beta E_r} \text{Im} \left(1 - i\beta\Gamma + \frac{1}{2} (i\beta\Gamma)^2 - \dots \right) \end{aligned}$$

Then, at the leading order, the partition function is

$$\text{Im}Z \sim -\beta e^{-\beta E_r} \Gamma \quad (2.32)$$

or the ground state energy is

$$\text{Im}E_0 = \lim_{\beta \rightarrow \infty} -\frac{1}{\beta} e^{\beta E_r} \text{Im}Z \quad (2.33)$$

In the path integral formalism the partition function for the one dimensional quartic anharmonic oscillator is written as

$$Z(\beta) = \mathcal{N} \int_{q(-\beta/2)=q(\beta/2)} \mathcal{D}q e^{-S_E(q)} \quad (2.34)$$

S_E is called the Euclidean action, in which time is taken imaginary. The Euclidean action is found from the usual one by the transformation $t \rightarrow i\tau$.

$$S_E(q) = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{q}^2 + \frac{1}{2} q^2 + \frac{1}{4} \lambda q^4 \right) \quad (2.35)$$

and $\mathcal{N}$ is the normalization given as

$$\mathcal{N} = \frac{1}{Z_0}$$

where Z_0 is the partition function of the simple harmonic oscillator.

Note that even though the relation between the partition function and the ground state energy originally comes from statistical mechanics and β is defined as $\beta = \frac{1}{k_b T}$, in the Euclidean path integral formalism, β is a imaginary time parameter [13].

As discussed in previous sections, high order corrections are related to the imaginary part of the energy eigenvalues at negative coupling. However, when $\lambda < 0$, the Euclidean action S_E blows up as $\beta \rightarrow \infty$. Therefore, instead of taking the coupling constant negative directly, analytical continuation methods should be used as in the simple integral example Section 2.3.3. Starting from positive valued λ , negative valued λ can be achieved by clockwise or counter-clockwise rotations. The important point is rotating the coordinates q as the coupling constant rotates in such a way that the

conditions

$$\operatorname{Re}(\lambda q^4) > 0 \quad , \quad \operatorname{Re}(q^2) > 0$$

and

$$\operatorname{Re}(\dot{q}^2) > 0$$

are satisfied. Then, setting

$$q \rightarrow qe^{-i\theta_q}$$

and

$$\lambda \rightarrow \lambda e^{-i\theta_\lambda}$$

yields the conditions

$$-\frac{\pi}{2} \leq 4\theta_q + \theta_\lambda \leq \frac{\pi}{2} \tag{2.36}$$

$$-\frac{\pi}{2} \leq 2\theta_q \leq \frac{\pi}{2} \tag{2.37}$$

If the rotation of λ is in clockwise direction, $\theta_q = -\pi$, the interval for q becomes

$$\frac{\pi}{8} \leq \theta_q \leq \frac{3\pi}{8}$$

and if the rotation is in counter-clockwise direction, $\theta_q = \pi$, the interval is

$$-\frac{\pi}{4} \leq \theta_q \leq -\frac{\pi}{8}$$

Furthermore, Equation 2.37 simply leads to

$$-\frac{\pi}{4} \leq \theta_q \leq \frac{\pi}{4}$$

Then, the intersections of these intervals are for the clockwise rotation

$$\frac{\pi}{8} \leq \theta_q \leq \frac{\pi}{4}$$

and for counter-clockwise rotation

$$-\frac{\pi}{4} \leq \theta_q \leq -\frac{\pi}{8}$$

Because of the branch cut along the negative axis on the complex λ plane, these rotations yield complex conjugate partition functions. Therefore, the imaginary part can be written as

$$2i \operatorname{Im} Z(\beta) = \int_{q_+(-\frac{\beta}{2})=q_+(\frac{\beta}{2})} \mathcal{D}q e^{-S(q)} - \int_{q_-(-\frac{\beta}{2})=q_-(\frac{\beta}{2})} \mathcal{D}q e^{-S(q)} \quad (2.38)$$

where q_+ is the paths for counter-clockwise rotations and q_- is for the clockwise ones.

For small values of coupling constant, the path integral is dominated by the paths satisfying the condition $\delta S = 0$. Therefore, as in the example of the simple integral in Section 2.3.3, the steepest descent method yields a good approximation. Let $q(\tau) = q_c(\tau) + \mu(\tau)$, where $q_c(\tau)$ corresponds to the paths satisfying $\delta S = 0$ and $V(q) = \frac{1}{2}q^2 + \frac{1}{4}q^4$. Expanding $S(q)$ around q_c gives

$$Z(\beta) = \mathcal{N} e^{-S_c} \int \mathcal{D}\mu e^{-\Sigma(\mu)} \quad (2.39)$$

where

$$S_c = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{q}_c^2 + V(q_c) \right) \quad (2.40)$$

and

$$\Sigma(\mu) = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{\mu}^2 + \frac{1}{2} V''(q_c) \mu^2 + \dot{q}_c \dot{\mu} + V'(q_c) \mu \right) \quad (2.41)$$

The simplest case is considering the constant solutions $\delta S_c = 0$, which means setting $\dot{q}_c(\tau) = 0$. In this case, the solutions are the saddle points of the potential $V(q_c)$

$$q_c(\tau) = 0$$

and

$$q_c(\tau) = \pm \sqrt{-\frac{1}{\lambda}}$$

However, both solutions yield unimportant contributions to the imaginary part of $Z(\beta)$. First consider the $q_c(\tau) = 0$ solution. Its contributions to the partition function after both rotations are the same. Therefore, they cancel each other in the Equation 2.38. Then, the solutions

$$q_c(\tau) = \pm \sqrt{-\frac{1}{\lambda}}$$

yield

$$S_c = \int_{-\beta/2}^{\beta/2} d\tau \left(-\frac{1}{2\lambda} + \frac{1}{4\lambda} \right) = -\frac{\beta}{4\lambda}$$

This result contributes to the partition function exponentially

$$Z(\beta) \sim e^{\beta/4\lambda}$$

which is negligible as $\beta \rightarrow \infty$ since λ is negative. Therefore, the contributions of $q_c(\tau) = \pm \sqrt{-\frac{1}{\lambda}}$ are also zero. Since no important contribution comes from the constant

solutions, non-constant ones should be considered. These solutions are also called instanton solutions.

2.4.1. Instanton Solutions

Non-constant solutions of $\delta S_c = 0$ are simply non-trivial solutions of the Euler-Lagrange equations.

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{q}_c} - \frac{\partial \mathcal{L}}{\partial q_c} = 0$$

Then the non-constant solutions of the Equation 2.40 is

$$\ddot{q}_c - V'(q_c) = 0 \tag{2.42}$$

Note that the sign difference with the usual classical equation of motion is because the system is considered in imaginary time. This is equivalent to the particle moving under the influence of an inverted potential with the boundary conditions of the partition function $q_c(-\beta/2) = q_c(\beta/2)$ See Figure 2.8.

Furthermore, the particular values of $q(\pm\beta/2)$ can be established by multiplying the Equation 2.42 and re-arranging the action accordingly.

$$\dot{q}_c \ddot{q}_c - \dot{q}_c V'(q_c) = \frac{d}{d\tau} \left(\frac{\dot{q}_c^2}{2} - V(q_c) \right) = 0 \tag{2.43}$$

or

$$\frac{\dot{q}_c^2}{2} - V(q_c) = E \tag{2.44}$$

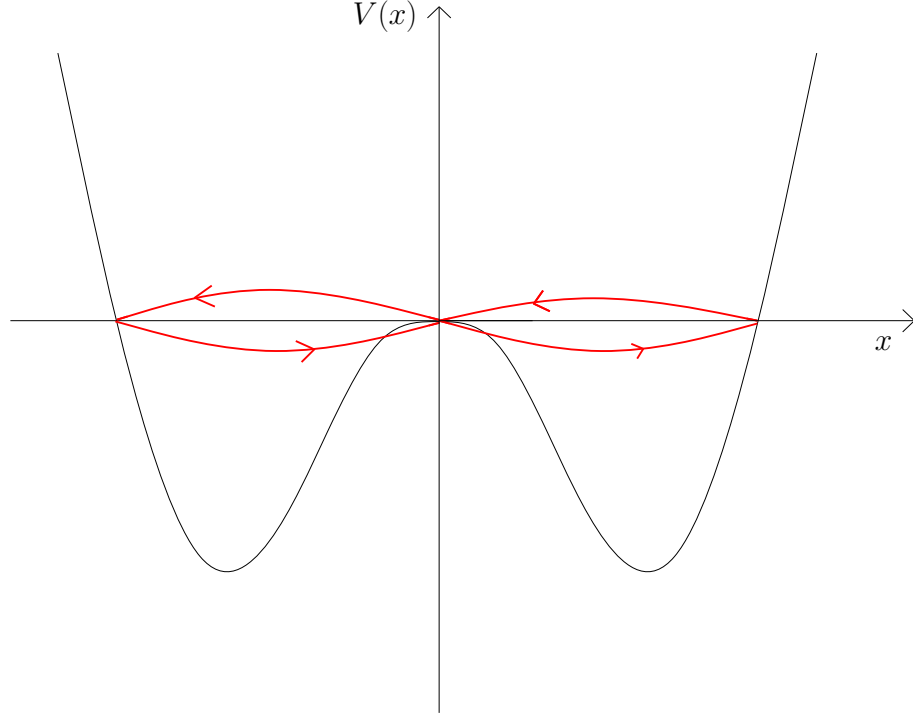


Figure 2.8. The periodic motions of the instantons.

where E is a constant and corresponds to the total energy of the system with the inverted potential. Then, the classical part of the action can be written as

$$S_c = \int_{-\beta/2}^{\beta/2} d\tau (2V(q_c) + E) \quad (2.45)$$

Since E is a constant, in the limit of $\beta \rightarrow \infty$, S_c is finite if and only if $E = 0$. Therefore, only the solutions with zero total energy are taken into account. Combining this with the periodic boundary conditions coming from the definition of the partition function shows why the motion of the particle should be as in the Figure 2.8.

Taking $E = 0$, the Equation 2.44 becomes

$$\frac{\dot{q}_c^2}{2} = V(q_c) \quad (2.46)$$

and the particular value of $q_c(\theta)$ for the potential $V(q) = \frac{1}{2}q^2 + \frac{1}{4}\lambda q^4$ can be found as

$$\begin{aligned} d\tau &= dq_c \frac{1}{2V(q_c)} \\ (\tau - \tau_0) &= \int_0^{q_c(\tau)} \frac{dq}{\sqrt{q^2 + \frac{\lambda}{2}q^4}} \end{aligned}$$

Set $q = e^y$

$$\begin{aligned} \tau - \tau_0 &= \int_{-\infty}^{y^*} \frac{dy}{\sqrt{1 + \frac{\lambda}{2}e^{2y}}} = -\tanh^{-1} \left(\sqrt{1 + \frac{\lambda}{2}e^{2y}} \right) \Big|_{-\infty}^{y^*} \\ &= -\tanh^{-1} \left(\sqrt{1 + \frac{\lambda}{2}e^{2y^*}} \right) \end{aligned}$$

Then,

$$\begin{aligned} -\tanh(\tau - \tau_0) &= \sqrt{1 + \frac{\lambda}{2}e^{2y^*}} \\ -\frac{2}{\lambda}(1 - \tanh^2(\tau - \tau_0)) &= e^{2y^*} \end{aligned}$$

Converting the variable back to q_c , the classical solution of $q_c(\tau)$ is found as

$$q_c(\tau) = \pm \left(-\frac{2}{\lambda} \right)^{1/2} \frac{1}{\cosh(\tau - \tau_0)} \quad (2.47)$$

Putting this solution into the action

$$\begin{aligned} S_c &= \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{\dot{q}_c^2}{2} + V(q_c) \right) = \int_{-\beta/2}^{\beta/2} d\tau \dot{q}_c^2 \\ &= \left(-\frac{2}{\lambda} \right) \int_{-\beta/2}^{\beta/2} d\tau \frac{\tanh^2(\tau - \tau_0)}{\cosh^2(\tau - \tau_0)} \\ &= \left(-\frac{2}{\lambda} \right) \int_{-\beta/2}^{\beta/2} d\tau \tanh^2(\tau - \tau_0) \frac{d}{d\tau} [\tanh(\tau - \tau_0)] \\ &= \left(-\frac{2}{3\lambda} \right) \int_{-\beta/2}^{\beta/2} d\tau \frac{d}{d\tau} [\tanh^3(\tau - \tau_0)] \\ &= \left(-\frac{2}{3\lambda} \right) \tanh^3(\tau - \tau_0) \Big|_{-\beta/2}^{\beta/2} \end{aligned}$$

Since $\lim_{\beta \rightarrow \pm\infty} \tanh(\tau - \tau_0) = \pm 1$

$$S_c = -\frac{4}{3\lambda} \quad (2.48)$$

This is the value of the classical action as $\beta \rightarrow \infty$. Then, these particular solutions in the Equation 2.47, contribute to the partition function proportional to $e^{4/3\lambda}$. Note that this contribution corresponds to the trajectory oscillating once between $q_c = 0$ and $q_c = \pm \frac{2}{\lambda}$. For n oscillations, the contribution to the partition function will be $e^{n4/3\lambda}$. Since λ is negative and small, $n = 1$ becomes the leading order contribution and the other will be neglected.

Now contributions from the fluctuations around the classical paths can be calculated in the following way. Recall the Equation 2.41.

$$\Sigma(\mu) = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{\mu}^2 + \frac{1}{2} V''(q_c) \mu^2 + \dot{q}_c \dot{\mu} + V'(q_c) \mu \right)$$

and first consider the integration of the last two terms

$$\int_{-\beta/2}^{\beta/2} d\tau (\dot{q}_c \dot{\mu} + V'(q_c) \mu) = \int_{-\beta/2}^{\beta/2} d\tau \frac{d}{d\tau} (\mu \dot{q}_c) - \int_{-\beta/2}^{\beta/2} d\tau \mu (\ddot{q}_c - V'(q_c))$$

where the first integral on the right hand side is zero since $\mu(\pm\beta/2) = 0$. The second one is also zero since the term in the parenthesis corresponds to the equation of motion of the instanton in the Equation 2.46. Then, the Equation 2.41 becomes

$$\begin{aligned} \Sigma(\mu) &= \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau \left(\dot{\mu}^2 + V''(q_c) \mu^2 \right) \\ &= \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{d}{d\tau} (\mu \dot{\mu}) - \mu \ddot{\mu} + V''(q_c) \mu^2 \right) \\ &= \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau \mu \left(-\frac{\partial^2}{\partial \tau^2} + V''(q_c) \right) \mu \end{aligned}$$

Using this result and the classical action in Equation 2.48, the partition function simply becomes

$$Z(\beta) = \mathcal{N} e^{4/3\lambda} (\det M)^{-1/2} \quad (2.49)$$

where

$$M = -\frac{\partial^2}{\partial \tau^2} + V''(q_c)$$

However, one of the eigenvalues of M is zero and this is called the “zero mode”. The zero mode causes $(\det M)^{-1/2}$ to blow up. The existence of this zero eigenvalue can be shown as follows

$$\left(\frac{\partial^2}{\partial \tau^2} - V''(q_c) \right) \dot{q}_c = \frac{d}{d\tau} (\ddot{q}_c - V'(q_c)) = 0$$

The reason of the zero mode is the time translation invariance in the classical solution $q_c(\tau)$ which means any $q_c(\tau + \tau_0) = q$ is also a solution of the classical equation of motion. This implies that the action is also invariant under time translation

$$S(q_c) = S(q)$$

Therefore, the second derivative $\frac{\partial^2 S}{\partial \tau^2}$, which corresponds to the operator M , causes no change in this direction.

The divergence due to the zero mode is easily curable by removing the time translation symmetry in the action. This is done by introducing the Faddeev-Popov technique.

2.4.2. Zero-modes and the Fadeev-Popov Technique

As it is discussed the zero-mode problem arises from the time translation invariance which is due to the arbitrary parameter τ_0 , which is called the collective

coordinate, of the classical paths in the Equation 2.47. In order to eliminate it first introduce the following integral

$$\int_{-\infty}^{\infty} d\sigma e^{-\sigma^2/2\gamma} = \sqrt{2\pi\gamma} \quad (2.50)$$

where γ is an arbitrary constant, which will vanish at the end of the calculations. Define the integration variable σ using the collective time coordinate

$$\sigma = \int d\tau \dot{q}_c(\tau) [q(\tau + \tau_0) - q_c(\tau)] \quad (2.51)$$

where $q_c(\tau)$ is the classical path in the Equation 2.47. Then,

$$\begin{aligned} \frac{d\sigma}{d\tau_0} &= \frac{d}{d\tau_0} \int d\tau \dot{q}_c(\tau) [q(\tau + \tau_0) - q_c(\tau)] \\ &= \int d\tau \dot{q}_c(\tau) \frac{d}{d\tau_0} (q(\tau + \tau_0)) \int d\tau \dot{q}_c(\tau) \frac{d}{d\tau} (q(\tau + \tau_0)) \\ &= \int d\tau \dot{q}_c(\tau) \dot{q}(\tau + \tau_0) \end{aligned}$$

and Equation 2.50 can be written as

$$1 = \frac{1}{\sqrt{2\pi\gamma}} \int d\tau_0 \int d\tau \dot{q}_c(\tau) \dot{q}(\tau + \tau_0) e^{-\frac{1}{2\gamma} [\int d\tau \dot{q}_c(\tau) (q(\tau + \tau_0) - q_c(\tau))]^2} \quad (2.52)$$

Then, putting this into Equation 2.34 yields

$$Z(\beta) = \frac{1}{\sqrt{2\pi\gamma}} \int \mathcal{D}q \int_{-\beta/2}^{\beta/2} d\tau_0 \int_{-\beta/2}^{\beta/2} d\tau \dot{q}_c(\tau) \dot{q}(\tau + \tau_0) e^{-S_\gamma(q)} \quad (2.53)$$

where

$$S_\gamma = \int d\tau \left(\frac{1}{2} \dot{q}^2(\tau) + \frac{1}{2} q^2(\tau) + \frac{\lambda}{4} q^4(\tau) \right) + \frac{1}{2\gamma} \int d\tau \dot{q}_c(\tau) [q(\tau + \tau_0) - q_c(\tau)]^2$$

The action S_γ is not invariant under time translation since $q_c(\tau)$ appears in the action explicitly.

Variation of S_γ ,

$$\frac{\delta S_\gamma}{\delta q(\tau)} = \frac{\delta S(q)}{\delta q} + \frac{1}{\gamma} \frac{\delta}{\delta q(\tau)} \left[\int d\tau \dot{q}_c(\tau) q(\tau + \tau_0) \right] \int d\tau \dot{q}_c(\tau) [q(\tau + \tau_0) - q_c(\tau)]$$

where $S(q)$ is the action of quartic anharmonic oscillator. Apply the following change of variables

$$\tau \rightarrow \tau' - \tau_0$$

and

$$\tau \rightarrow \tau'' - \tau_0$$

in the integrals. Then,

$$\begin{aligned} \frac{\delta S_\gamma}{\delta q(\tau)} &= \frac{\delta S(q)}{\delta q} + \frac{1}{\gamma} \frac{\delta}{\delta q(\tau)} \left[\int d\tau'' \dot{q}_c(\tau'' - \tau_0) q(\tau'') \right] \int d\tau' \dot{q}_c(\tau' - \tau_0) [q(\tau') - q_c(\tau' - \tau_0)] \\ &= \frac{\delta S(q)}{\delta q} + \frac{1}{\gamma} \int d\tau'' \dot{q}_c(\tau'' - \tau_0) \delta(\tau'' - \tau) \int d\tau' \dot{q}_c(\tau' - \tau_0) [q(\tau') - q_c(\tau' - \tau_0)] \\ &= \frac{\delta S(q)}{\delta q} + \frac{1}{\gamma} \dot{q}_c(\tau - \tau_0) \int d\tau' \dot{q}_c(\tau' - \tau_0) [q(\tau') - q_c(\tau' - \tau_0)] \end{aligned}$$

Recalling

$$\left. \frac{\delta S(q)}{\delta q} \right|_{q(\tau)=q_c(\tau-\tau_0)} = 0$$

the equation of motion for the action S_γ is

$$q(\tau) = q_c(\tau - \tau_0) \tag{2.54}$$

Putting this equation into the Equation 2.53, the integration over τ_0 turn into a straightforward calculation and the partition function becomes

$$Z(\beta) = \mathcal{N} \frac{\beta}{\sqrt{2\pi\gamma}} \int \mathcal{D}q \int d\tau \dot{q}_c(\tau) \dot{q}_c(\tau) e^{-S_\gamma(q)} \quad (2.55)$$

Then, the result of the path integral becomes

$$Z(\beta) = \mathcal{N} \frac{\|\dot{q}_c\|^2 \beta}{\sqrt{2\pi\gamma}} e^{-S_c} (\det M_\gamma)^{-1/2} \quad (2.56)$$

where $S_c = -\frac{4}{3\lambda}$ as it was calculated before. Since the time translation invariance is eliminated $\det M_\gamma$ would not be zero; therefore, $Z(\beta)$ wouldn't have a divergence.

To calculate the determinant, first find the operator $\frac{\delta^2 S_\gamma}{\delta q(\tau_1) \delta q(\tau_2)}$, which corresponds to M_γ

$$\begin{aligned} \frac{\delta^2 S_\gamma}{\delta q(\tau_1) \delta q(\tau_2)} &= \frac{\delta^2 S}{\delta q(\tau_1) \delta q(\tau_2)} + \frac{1}{\gamma} \frac{\delta}{\delta q(\tau_2)} \left[\dot{q}_c(\tau_1 - \tau_0) \int d\tau' \dot{q}_c(\tau' - \tau_0) [q(\tau') - q_c(\tau' - \tau_0)] \right] \\ &= M + \frac{1}{\gamma} \dot{q}_c(\tau_1 - \tau_0) \dot{q}_c(\tau_2 - \tau_0) \end{aligned}$$

and the eigenvalue of M_γ corresponding to the eigenvector $\dot{q}_c$ is found as

$$\begin{aligned} M_\gamma \dot{q}_c(\tau) &= M \dot{q}_c(\tau) + \frac{1}{\gamma} \dot{q}_c(\tau_1 - \tau_0) \dot{q}_c(\tau_2 - \tau_0) \dot{q}_c(\tau) \\ &= \frac{1}{\gamma} \int d\tau_1 \delta(\tau_1 - \tau) \int d\tau_2 \delta(\tau_2 - \tau) \dot{q}_c(\tau - \tau_0) \dot{q}_c(\tau - \tau_0) \dot{q}_c(\tau) \\ &= \frac{1}{\gamma} \|\dot{q}_c\|^2 \dot{q}_c(\tau) \end{aligned}$$

Now the partition function can be calculated without encountering any divergences.

Recall the partition function of simple harmonic oscillator

$$Z_0(\beta) = \int \mathcal{D}q e^{-S^{(0)}}$$

where

$$S_0 = \int d\tau \left(\frac{1}{2} \dot{q}^2 + \frac{1}{2} q^2 \right)$$

In the first order approximation the path integral is given as

$$Z_0(\beta) = e^{-S_c^{(0)}} (\det M_0)^{-1/2}$$

where $M_0 = -\frac{\partial^2}{\partial \tau^2} + 1$. Then, the partition function of anharmonic oscillator in Equation 2.56 becomes

$$Z(\beta) = \frac{\beta \|\dot{q}_c\|^2}{\sqrt{2\pi\gamma}} e^{4/3\lambda} \left(\frac{\det M_\gamma}{\det M_0} \right)^{-1/2} e^{S_c^{(0)}} \quad (2.57)$$

It is more suitable to calculate $\frac{\det M_\gamma}{\det M_0}$ at once. Consider,

$$\begin{aligned} \frac{\det M_\gamma}{\det M_0} &= \det [(M + \eta |0\rangle \langle 0|) M_0^{-1}] \\ &= \lim_{\epsilon \rightarrow 0} \det [(M - \epsilon + \eta |0\rangle \langle 0|) (M_0 - \epsilon)^{-1}] \\ &= \lim_{\epsilon \rightarrow 0} \det [(M - \epsilon)(\eta |0\rangle \langle 0| (M - \epsilon)^{-1})(M_0 - \epsilon)^{-1}] \\ &= \lim_{\epsilon \rightarrow 0} \left\{ \det [(M - \epsilon) (M_0 - \epsilon)^{-1}] \det [1 + \eta |0\rangle \langle 0| (M - \epsilon)^{-1}] \right\} \end{aligned}$$

where $\eta |0\rangle \langle 0|$ is the part of the normalized zero-mode and

$$\eta = \frac{\|\dot{q}_c\|^2}{\gamma}$$

The part

$$\eta |0\rangle \langle 0| (M - \epsilon)^{-1} = -\frac{\eta}{\epsilon}$$

since the $|0\rangle \langle 0|$ eigenvector corresponds to the zero-mode of M . Then,

$$\frac{\det M_\gamma}{\det M} = \lim_{\epsilon \rightarrow 0} \left[\left(1 - \frac{\eta}{\epsilon} \right) \det (M - \epsilon) (M_0 - \epsilon)^{-1} \right] \quad (2.58)$$

Defining,

$$\det' MM_0^{-1} \equiv \lim_{\epsilon \rightarrow 0} \left[-\frac{1}{\epsilon} \frac{\det(M - \epsilon)}{\det(M_0 - \epsilon)} \right] \quad (2.59)$$

Equation 2.58 becomes

$$\frac{\det M_\gamma}{\det M} = \det' MM_0^{-1} \eta = \det' MM_0^{-1} \frac{\|\dot{q}_c\|^2}{\gamma} \quad (2.60)$$

and the Equation 2.56 is written as

$$Z(\beta) = \frac{\beta \|\dot{q}_c\|^2}{\sqrt{2\pi\gamma}} e^{4/3\lambda} e^{S_c^{(0)}} \left(\frac{\|\dot{q}_c\|^2}{\gamma} \det' MM_0^{-1} \right)^{-1/2} \quad (2.61)$$

In general, the ratio of the determinants in the Equation 2.59 is given as [4, 13]

$$\frac{\det(M - \epsilon)}{\det(M_0 - \epsilon)} = \frac{\Gamma(1 - z)\Gamma(z)}{\Gamma(1 + \kappa + z)\Gamma(z - \kappa)} \quad (2.62)$$

where

$$z = \sqrt{1 - \epsilon}$$

and

$$\kappa = \frac{N}{N - 1}$$

In the last equation N is associated to the power of the anharmonic part which is 2 for the quartic anharmonic oscillator. Then in the leading order

$$\det' MM_0^{-1} = \lim_{\epsilon \rightarrow 0} \left[-\frac{1}{\epsilon} \frac{\det(M - \epsilon)}{\det(M_0 - \epsilon)} \right] = -\frac{1}{12} \quad (2.63)$$

Furthermore, the value of the classical action of the simple harmonic oscillator is

$$S_c^{(0)} = \frac{\beta}{2}$$

Then,

$$Z(\beta) = \frac{\beta}{\sqrt{2\pi}} \parallel \dot{q}_c \parallel e^{4/3\lambda} e^{-\beta/2} \left(-\frac{1}{12} \right)^{-1/2} \quad (2.64)$$

Furthermore, recalling

$$q_c(\tau) = \pm \left(-\frac{2}{\lambda} \right)^{1/2} \frac{1}{\cosh(\tau)}$$

$$\parallel \dot{q}_c \parallel = \left(\int d\tau \dot{q}_c^2 \right)^{1/2} = \pm \frac{2}{\sqrt{-3\lambda}}$$

Each sign corresponds to one of the saddle points around which the classical paths were found. Then, putting all the results into the Equation 2.38 gives

$$\begin{aligned} 2i\text{Im}Z(\beta) &= 2 \frac{\beta}{\sqrt{2\pi}} \frac{2}{\sqrt{-3\lambda}} e^{-\beta/2} e^{4/3\lambda} \frac{2\sqrt{3}}{i} \\ \text{Im}Z(\beta) &= -\frac{4\beta}{\sqrt{2\pi}} \frac{1}{\sqrt{-\lambda}} e^{-\beta/2} e^{4/3\lambda} \end{aligned}$$

and recalling the Equation 2.33

$$\text{Im}E = \lim_{\beta \rightarrow \infty} \frac{1}{\beta} e^{\beta E_r} \left(-\frac{4\beta}{\sqrt{2\pi}} \frac{1}{\sqrt{-\lambda}} e^{-\beta/2} e^{4/3\lambda} \right) \quad (2.65)$$

since E_r is just the ground state energy of the harmonic oscillator,

$$\text{Im}E = \frac{4}{\sqrt{2\pi}} \frac{1}{\sqrt{-\lambda}} e^{4/3\lambda}$$

Recalling, the dispersion relation integral

$$\begin{aligned} E_k &= \frac{1}{\pi} \int_{-\infty}^0 d\lambda \frac{\text{Im} E}{\lambda^{k+1}} \\ &= -\frac{4}{\sqrt{2\pi^3}} \int_{-\infty}^0 \frac{e^{4/3\lambda}}{\lambda^{k+3/2}} \end{aligned}$$

and finally the result in Equation 2.30 is found

$$E_k = -\left(\frac{6}{\pi^3}\right)^{1/2} (-1)^{k+1} \left(\frac{3}{4}\right)^k \Gamma\left(k + \frac{1}{2}\right)$$

As expected in the higher order, E_k is dominated by the gamma function, thus it has a factorially divergent behaviour.

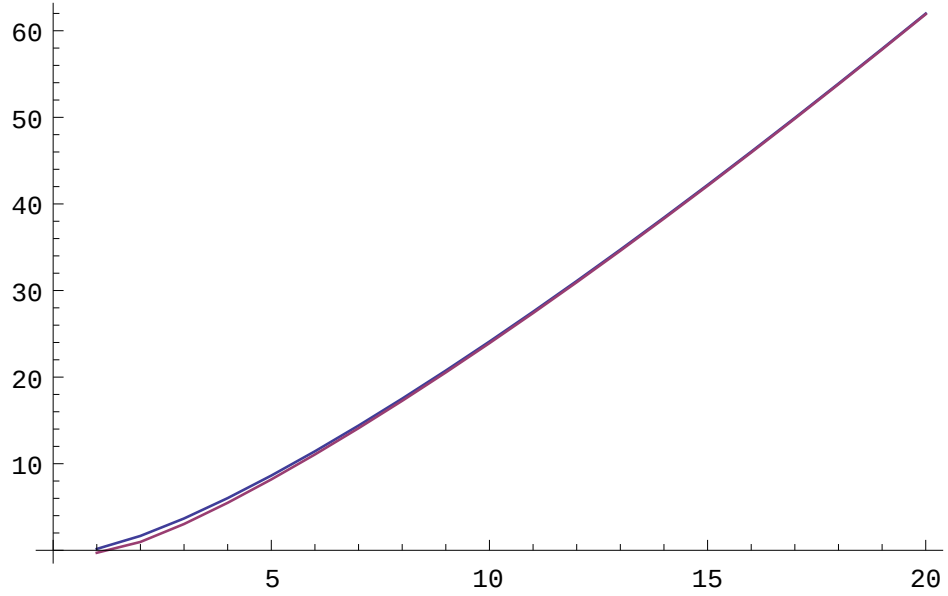


Figure 2.9. Comparison between the the perturbative corrections of Rayleigh-Schrödinger theory (Red line) and by using instantons (Blue line). As expected, even though in the low order they have a difference, they starts to match as the order of the perturbation expansion grows.

Validity of Equation 2.30 can be shown by comparing it with the results of Rayleigh-Schrödinger perturbation theory. The comparison is given in Figures 2.9 and 2.10

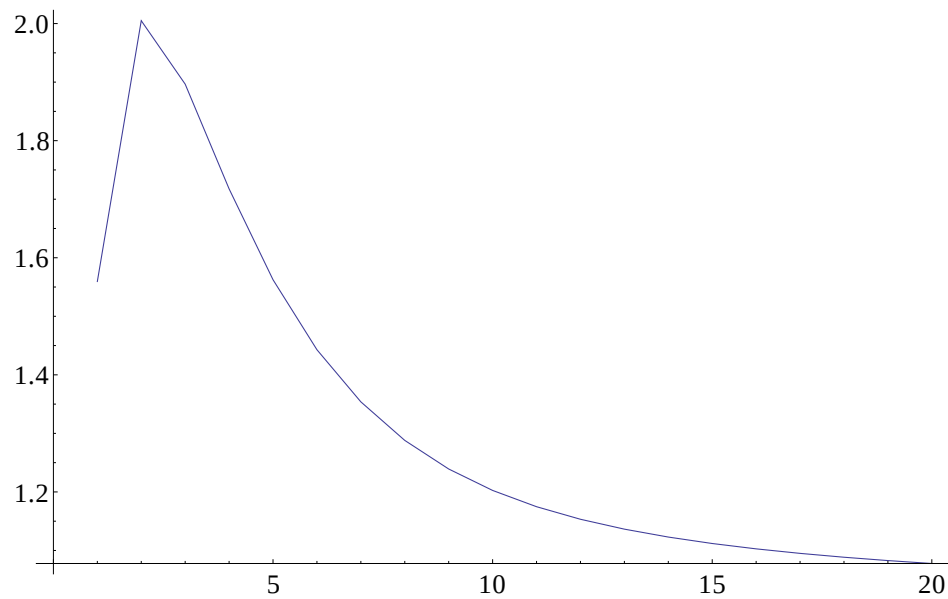


Figure 2.10. Comparison between the the perturbative corrections of Rayleigh-Schrödinger theory (Red line) and by using instantons (Blue line). As expected, even though in the low order they have a difference, they starts to match as the order of the perturbation expansion grows.

as expected the results of these two estimation matches more and more in the large orders.

3. THE DYON ATOM

The main motivation for this thesis is finding the large order behaviour of the expansion of perturbed Dyon atom. As discussed in the introduction, the Dyon atom can be considered as a generalization of the Hydrogen atom and both have $SO(3) \times SO(3) \sim SO(4)$ symmetry group. This hints at a possible relation to the four dimensional harmonic oscillator, which has also $SO(4)$ symmetry. Indeed in this chapter this equivalence will be made explicit. Since the instanton method for the anharmonic oscillator is well understood, this duality is the crucial tool for the estimation of the high orders.

Before constructing the transformation rules in Section 3.5 and applying them in Section 3.6, from Section 3.1 to 3.3 the Schrödinger equation of the Dyon atom

$$\left\{ \frac{1}{2m} \left[(\mathbf{p} - \mu \mathbf{D})^2 + \frac{\mu^2}{2mr^2} \right] - \frac{\alpha}{r} \right\} |\pi\rangle = E |\pi\rangle$$

will be constructed starting from classical electromagnetism. The Schrödinger equation describes the interaction of two Dyons in the center of mass frame. To properly derive the motion of a dyon, electromagnetism should be written in the duality invariant form. The duality invariant action in vacuum

$$S = \frac{1}{2} \int d^4x \left(\mathbf{B}^{(a)} \cdot \mathbf{E}^{(b)} \varepsilon_{ab} - \mathbf{B}^{(a)} \cdot \mathbf{B}^{(b)} \delta_{ab} \right)$$

which intrinsically have the duality transformations

$$\begin{aligned} \mathbf{E} &\rightarrow \mathbf{B} \\ \mathbf{B} &\rightarrow -\mathbf{E} \end{aligned}$$

will be introduced. The generalization to the cases with electromagnetic sources will be easy.

3.1. Duality Invariance in Vacuum

Historically, classical electromagnetism is constructed empirically by many scientists. In the presence of no electric or magnetic source, these results can be written in an elegant mathematical formulation as

$$\partial_\alpha F^{\alpha\beta} = 0 \quad (3.1)$$

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0 \quad (3.2)$$

where $F^{\alpha\beta}$ and $\tilde{F}^{\alpha\beta}$ are the electromagnetic field tensor and its dual tensor respectively. They are defined as

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \quad (3.3)$$

and

$$\tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad (3.4)$$

where four-potential A_α consists of scalar (ϕ) and vector ($\vec{A}$) potentials.

$$A^\alpha = \left(\frac{\phi}{c}, \vec{A} \right) \quad (3.5)$$

The Equation 3.2 can be considered as Bianchi identity of $F^{\alpha\beta}$.

$$\partial_\alpha \tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} \partial_a F_{\gamma\delta} = 0 \quad (3.6)$$

It is easier to get a deeper understanding by looking at this feature by considering electromagnetic potentials and tensors as differential forms [24]. In this case Equation 3.2 can be written as

$$dF = 0 \tag{3.7}$$

where d is the exterior derivative and $F = F_{\mu\nu}dx^\mu dx^\nu$ is a 2-form defined as

$$F = dA$$

where A is a 1-form corresponding to the four potential and defined as

$$dA = A_\mu dx^\mu$$

The reason for introducing the electromagnetism by using differential forms is the following identity

$$ddw = 0$$

for any 1-form w . Using this identity, the Bianchi identity in Equation 3.6 can be written as

$$dF = ddA = 0 \tag{3.8}$$

This last equation reveals more information than the one in tensor formalism in Equation 3.6. Equation 3.8 is equivalent with

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

which comes from the vector product properties. This property will be examined again later in the discussion of magnetic monopoles.

In electromagnetism the objects which can be measured are electric and magnetic fields. They are defined as

$$\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}} \quad (3.9)$$

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (3.10)$$

Since the electromagnetic theory includes special relativity, the physical quantities should transform according to Lorentz transformations. Furthermore, the equations of motion, so that action, consist of these physical quantities should be invariant under Lorentz transformations.

For free fields, by setting $c = 1$, the Lorentz invariant action is given as [21]

$$S = \frac{1}{2} \int d^4x F^{\alpha\beta} F_{\alpha\beta} \quad (3.11)$$

where

$$F^{\alpha\beta} F_{\alpha\beta} = 2 (B^2 - E^2) \quad (3.12)$$

is a Lorentz invariant term. Varying this action according to the least action principle, $\delta S = 0$, gives the equations of motion in 3.1 as expected and Equation 3.2 comes from the Bianchi identity. Using the definitions in Equations 3.3 and 3.10, their explicit forms, respectively, can be written as

$$\nabla \cdot \mathbf{E} = 0 \quad , \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0 \quad (3.13)$$

$$\nabla \cdot \mathbf{B} = 0 \quad , \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3.14)$$

An important aspect of the Equation 3.13 and 3.14 is the symmetry under the exchange of electric and magnetic fields.

$$\begin{aligned}\mathbf{E} &\rightarrow \mathbf{B} \\ \mathbf{B} &\rightarrow -\mathbf{E}\end{aligned}$$

However, the action in Equation 3.11 is not symmetric under these duality transformations since the quantity $F^{\alpha\beta}F_{\alpha\beta}$ is not symmetric. In order to construct an action, which is symmetric under duality transformations, auxiliary fields must be introduced. Then, the system will have 2 electrical fields, say $E^{(1)}$ $E^{(2)}$, and 2 magnetic fields, say $B^{(1)}$ $B^{(2)}$. The action can be written as [22, 23].

$$S = \frac{1}{2} \int d^4x \left(\mathbf{B}^{(a)} \cdot \mathbf{E}^{(b)} \varepsilon_{ab} - \mathbf{B}^{(a)} \cdot \mathbf{B}^{(b)} \delta_{ab} \right) \quad (3.15)$$

where $a, b = 1, 2$. Using the definitions in Equations 3.9 and 3.10, the action becomes

$$S = \frac{1}{2} \int d^4x \left[\varepsilon_{\alpha\beta} \left(-\nabla \times \mathbf{A}^{(a)} \cdot \dot{\mathbf{A}}^{(b)} - \nabla \times \mathbf{A}^{(a)} \cdot \nabla \phi^{(b)} \right) - \nabla \times \mathbf{A}^{(a)} \cdot \nabla \times \mathbf{A}^{(b)} \delta_{\alpha\beta} \right] \quad (3.16)$$

By varying the action, the corresponding equations of motion can be found. However, before starting the variation, it is better to discuss the gauge transformations, which leave the action, and hence the equations of motions, invariant. In this way, more information about the new structure will be obtained.

First consider the role of ϕ . It appears in the action, the invariant transformations can be guessed by considering the following term

$$\nabla \times \mathbf{A}^{(a)} \cdot \nabla \phi^{(b)}$$

This term can be written as

$$\nabla \times \mathbf{A}^{(a)} \cdot \nabla \phi^{(b)} = \nabla \cdot (\phi^{(b)} \nabla \times \mathbf{A}^{(a)}) - \nabla \cdot (\nabla \times \mathbf{A}^{(a)}) \phi^{(b)}$$

The second term is manifestly zero and the first term vanishes after by integration. Therefore, the term $\nabla \times \mathbf{A}^{(a)} \cdot \nabla \phi^{(b)}$ in the action 3.16 has no importance in determining the equations of motion. Then, since ϕ exists only in this part, any change by a function in the scalar potential does not lead to a difference in the action, so that in the equations of motion. Therefore, first gauge transformation is

$$\phi \rightarrow \phi + \Gamma \tag{3.17}$$

which means the scalar field can be set to any arbitrary value. For simplicity it is taken zero. Therefore, the electric fields can be re-defined as

$$\mathbf{E} = -\dot{\mathbf{A}} \tag{3.18}$$

In this way, both electric and magnetic fields are defined by only the vector potential.

Furthermore, the expression $\nabla \times \mathbf{A}$, which is common in every term, is not affected by a change in vector potential by the gradient of a function. Therefore, the other gauge transformation is

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda \tag{3.19}$$

which is the same with the one in the common formalism of classical electromagnetism.

Now, by setting $\phi = 0$, the variation of the simplified action can be written as

$$\begin{aligned} \delta S = & \frac{1}{2} \int d^4x \varepsilon_{ijk} \left[\varepsilon_{ab} \left(-\partial_j \delta A_k^{(a)} \partial_0 A_i^{(b)} - \partial_j A_k^{(a)} \partial_0 \delta A_i^{(b)} \right) \right. \\ & \left. - 2\delta_{ab} \partial_j A_k^{(a)} \varepsilon_{imn} \partial_m \delta A_n^{(b)} \right] \end{aligned} \quad (3.20)$$

Consider the variations term by term. The first term in the parenthesis,

$$\begin{aligned} \delta S_A^1 &= -\frac{1}{2} \int d^4x \varepsilon_{ab} \varepsilon_{ijk} \partial_j \delta A_k^{(a)} \partial_0 A_i^{(b)} \\ &= \frac{1}{2} \int d^4x \varepsilon_{ab} \varepsilon_{ijk} \partial_j \dot{A}_i^{(b)} \delta A_k^{(a)} \\ &= -\frac{1}{2} \int d^4x \varepsilon_{ab} \nabla \times \dot{\mathbf{A}}^{(b)} \cdot \delta \mathbf{A}^{(a)} \end{aligned} \quad (3.21)$$

The second term in the same parenthesis,

$$\begin{aligned} \delta S_A^2 &= -\frac{1}{2} \int d^4x \varepsilon_{ab} \varepsilon_{ijk} \partial_j A_k^{(a)} \partial_0 \delta A_i^{(b)} \\ &= \frac{1}{2} \int d^4x \varepsilon_{ab} \partial_0 \left(\varepsilon_{ijk} \partial_j A_k^{(a)} \right) \delta A_i^{(b)} \\ &= -\frac{1}{2} \int d^4x \varepsilon_{ab} \dot{\mathbf{B}}^{(b)} \cdot \delta \mathbf{A}^{(a)} \end{aligned} \quad (3.22)$$

And finally, the last term in Equation 3.20,

$$\begin{aligned} \delta S_A^3 &= -\frac{1}{2} \int d^4x 2 \delta_{ab} \varepsilon_{ijk} \partial_j A_k^{(a)} \varepsilon_{imn} \partial_m \delta A_n^{(b)} \\ &= \frac{1}{2} \int d^4x 2 \delta_{ab} \varepsilon_{ijk} \partial_m \left(\varepsilon_{ijk} \partial_j A_k^{(a)} \right) \delta A_n^{(b)} \\ &= \frac{1}{2} \int d^4x 2 \delta_{ab} \nabla \times \mathbf{B}^{(a)} \cdot \delta \mathbf{A}^{(b)} \end{aligned} \quad (3.23)$$

Thus, combining the Equations 3.21-3.23,

$$\delta S_A = \frac{1}{2} \int d^4x \left(-\varepsilon_{ab} \dot{\mathbf{B}}^{(b)} - \varepsilon_{ab} \nabla \times \dot{\mathbf{A}}^{(b)} + 2 \delta_{ab} \nabla \times \mathbf{B}^{(b)} \right) \cdot \delta \mathbf{A}^{(b)} \quad (3.24)$$

Recall, the re-defined electric field in Equation 3.18

$$\mathbf{E}^{(b)} = -\dot{\mathbf{A}}^{(b)}$$

Therefore, the equations of motion coming from Equation 3.24 is

$$-\varepsilon_{ab}\dot{\mathbf{B}}^{(b)} + \varepsilon_{ab}\nabla \times \mathbf{E}^{(b)} + 2\delta_{ab}\nabla \times \mathbf{B}^{(b)} = 0 \quad (3.25)$$

Note that in addition to 3.25, the Bianchi identity still holds:

$$\nabla \cdot \mathbf{B}^{(b)} = 0 \quad (3.26)$$

$$\nabla \times \mathbf{E}^{(b)} + \frac{\partial \mathbf{B}^{(b)}}{\partial t} = 0 \quad (3.27)$$

Putting 3.27 into Equation 3.25 gives

$$2\varepsilon_{ab}\nabla \times \mathbf{E}^{(b)} + 2\delta_{ab}\nabla \times \mathbf{B}^{(b)} = 2\nabla (\varepsilon_{ab}\mathbf{E}^{(b)} + \delta_{ab}\mathbf{B}^{(b)}) = 0 \quad (3.28)$$

Therefore,

$$\varepsilon_{ab}\mathbf{E}^{(b)} + \delta_{ab}\mathbf{B}^{(b)} = C \quad (3.29)$$

where C is a constant. Because of the gauge transformations, it can simply be chosen as zero. Then, the resulting equation is

$$\varepsilon_{ab}\mathbf{E}^{(b)} = -\mathbf{B}^{(a)} \quad (3.30)$$

First observation arises from this result shows that even though initially four fields are taken in hand, only two of them are really independent as expected. Any two of them can be chosen as basis fields and the other two are found by Equation 3.30. If $E^{(1)}$ and $B^{(1)}$ are chosen as the fundamental ones, the duality transformation in the original

theory is achieved.

$$\begin{aligned}\mathbf{E}^{(1)} &\rightarrow \mathbf{B}^{(2)} \\ \mathbf{B}^{(1)} &\rightarrow -\mathbf{E}^{(2)}\end{aligned}$$

Another observation is if the Equation 3.30 is put into the Equation 3.15 manifestly Lorentz invariant action in Equation 3.11 is recovered. Therefore, even though it is not manifest in the first place, the necessary condition of Lorentz invariance is also satisfied by imposing the equation of motion into the action.

3.2. Duality Invariance in the Presence of a Source

Now following the arguments in Section 3.1, duality invariant form of the electromagnetic fields with sources will be discussed in details and it will be shown that with an addition of proper Lorentz invariant source term

$$\frac{1}{2} \int d^4x \varepsilon_{\alpha\beta} J^{\mu(b)} A_{\mu}^{(a)} \quad (3.31)$$

to the action, the equation of motion in 3.30 is kept.

In the ordinary classical electromagnetism, when it is mentioned about a source term, it is an electric source because no magnetic charge has been observed in the nature yet. Therefore, setting $c = 1$ and $\varepsilon_0 = \mu_0 = 1$, the Maxwell equations can be written as

$$\partial_{\alpha} F^{\alpha\beta} = J_e^{\beta} \quad (3.32)$$

$$\partial_{\alpha} \tilde{F}^{\alpha\beta} = 0 \quad (3.33)$$

or explicitly

$$\nabla \cdot \mathbf{E} = e, \quad \nabla \cdot \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (3.34)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3.35)$$

J_e^β is the electric four-current is defined as

$$J_e^\beta = (e, \mathbf{J})$$

In this case, however, the symmetry under the duality transformations

$$\begin{aligned}\mathbf{E} &\rightarrow \mathbf{B} \\ \mathbf{B} &\rightarrow -\mathbf{E}\end{aligned}$$

does not hold anymore. On the other hand, theoretically there is no restriction for magnetic charge and current. Thus, they can be put into the Equations 3.34 by hand. Then, the Maxwell equations becomes

$$\nabla \cdot \mathbf{E} = e, \quad \nabla \cdot \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \quad (3.36)$$

$$\nabla \cdot \mathbf{B} = g, \quad \nabla \cdot \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{K} \quad (3.37)$$

where g and $\mathbf{K}$ are magnetic charge and current respectively. These equations are symmetric under the duality transformations with the additional transformations

$$\begin{aligned}e &\leftrightarrow g \\ \mathbf{J} &\leftrightarrow \mathbf{K}\end{aligned}$$

However, definition of $\mathbf{B}$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

forces the first equation in 3.37 to be zero. This feature has been discussed in the vacuum case in Section 3.1 and it is stated a deeper reason comes from the Bianchi identity (Equation 3.8). This obstacle can be overcome by defining an appropriate

vector potential. A general form of this appropriate vector potential is [8]

$$\mathbf{A}_+(\mathbf{r}) = \frac{g}{4\pi\mathbf{r}} \frac{-\hat{n} \times \mathbf{r}}{r + \mathbf{r} \cdot \hat{n}} \quad (3.38)$$

where $r = |\mathbf{r}|$ and $\hat{n}$ is an arbitrary unit vector. With this definition the vector potential is singular at the origin but this is not important since the magnetic charge is put at the origin and self-effects are not considered. The important singularity is the one along the straight line in the opposite direction to the unit vector $\hat{n}$. This singularity is called “Dirac” string. However, singularity along a straight line is not a desired situation. The solution is first restricting the definition of $\mathbf{A}$ into a region in which it is well-defined and defining another vector potential which is well-defined where the first “Dirac” string exist. Then the second vector potential can be defined by setting $\hat{n} \rightarrow -\hat{n}$.

$$\mathbf{A}_-(\mathbf{r}) = \frac{g}{4\pi\mathbf{r}} \frac{\hat{n} \times \mathbf{r}}{r - \mathbf{r} \cdot \hat{n}} \quad (3.39)$$

In this case the “Dirac” string is in the opposite direction of the first one, parallel to $\hat{n}$ (see Figure 3.1). Thus, by defining $\mathbf{A}_+$ and $\mathbf{A}_-$ only in the half space where they don’t have any singularity, vector potential can be assigned to whole space without any problem. In fact $\mathbf{A}_+$ and $\mathbf{A}_-$ are not independent from each other, they are connected with a gauge transformation. Therefore, examining physical situation in one region is enough.

A convenient choice for the direction of $\hat{n}$ is the z -axis. Then $\mathbf{A}_+$ and $\mathbf{A}_-$ becomes

$$\mathbf{A}_\pm(\mathbf{r}) = \frac{g}{4\pi\mathbf{r}} (\pm x_2, \mp x_1, 0) = \mp \frac{g}{4\pi\mathbf{r}} \frac{\sin \theta}{(1 \mp \cos \theta)} \hat{\phi} \quad (3.40)$$

The gauge transformation between $\mathbf{A}_+$ and $\mathbf{A}_-$ is now obvious. Let

$$\mathbf{A}_+(\mathbf{r}) = \mathbf{A}_-(\mathbf{r}) + \nabla \Lambda(\mathbf{r})$$

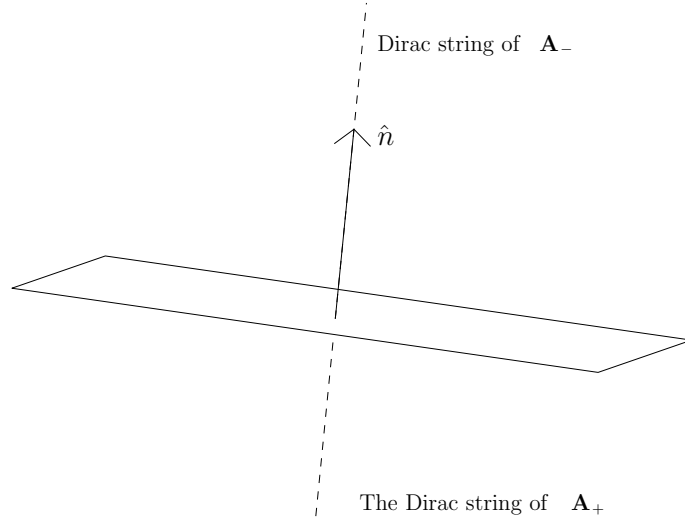


Figure 3.1. Dirac strings for an arbitrarily chosen $\hat{n}$. The string above the plane corresponds to the vector $\mathbf{A}_-$ and the one below the plane corresponds to $\mathbf{A}_+$.

Then,

$$\nabla\Lambda(\mathbf{r}) = \frac{g \sin \theta}{4\pi\mathbf{r}} \left(-\frac{1}{1 - \cos \theta} - \frac{1}{1 + \cos \theta} \right) \hat{\phi} = \frac{-2\rho_m}{r \sin \theta} \hat{\phi}$$

Both definitions in Equation 3.40 yields to the appropriate magnetic field in the presence of a magnetic charge.

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{g}{4\pi r^3} \mathbf{r}$$

After these appropriate definitions, which will be helpful in the discussion of Dyon-Dyon interaction in Quantum Mechanics, now duality symmetric action in the presence of both electric and magnetic charges will be examined.

In Section 3.1, the electromagnetic action in vacuum was defined in Equation 3.15 as

$$S_0 = \frac{1}{2} \int d^4x \left(\mathbf{B}^{(a)} \cdot \mathbf{E}^{(a)} \varepsilon_{ab} - \mathbf{B}^{(a)} \cdot \mathbf{B}^{(b)} \delta_{ab} \right)$$

Even though it is not manifest in the first place, at the end of the section it was found that the Lorentz invariance requirements is satisfied by this choice. Therefore, an additional term due to the electromagnetic sources should be Lorentz invariant as well. Inspiring from the usual Lorentz invariant action [21], an additional term can be chosen as in Equation 3.31. Then the full action becomes

$$S = \frac{1}{2} \int d^4x (\mathbf{B}^{(a)} \cdot \mathbf{E}^{(a)} \varepsilon_{ab} - \mathbf{B}^{(a)} \cdot \mathbf{B}^{(b)} \delta_{ab}) + \frac{1}{2} \int d^4x \varepsilon_{\alpha\beta} J^{\mu(b)} A_\mu^{(a)} \quad (3.41)$$

In the additional term

$$J^{\mu(b)} = \begin{pmatrix} j^\mu \\ k^\mu \end{pmatrix}$$

where $j^\mu = (e, \mathbf{J})$ is the 4-potential of the electric part and $k^\mu = (g, \mathbf{K})$ is the 4-potential of the magnetic part. Furthermore in this notation, sometimes charges and 3-currents are represented as

$$\rho^{(a)} = \begin{pmatrix} e \\ g \end{pmatrix}, \quad \mathbf{J}^{(a)} = \begin{pmatrix} \mathbf{J} \\ \mathbf{K} \end{pmatrix} \quad (3.42)$$

respectively. Furthermore, $A_\mu^{(a)}$ is the 4-potential for both electrical and magnetic charges. To preserve the duality between the electrical and magnetic charges the singularity initially associated with the magnetic charge can be assigned to the electrical one either. With this choice the general definition of the spatial part would change as

$$\mathbf{A}_\pm^{(a)} = \pm \frac{\rho^{(a)}}{\mathbf{r}} \frac{\hat{n} \times \mathbf{r}}{r \pm \mathbf{r} \cdot \hat{n}} \quad (3.43)$$

First re-check the gauge transformations starting from adding an arbitrary function to scalar potential, $\phi \rightarrow \phi + \Gamma$. Using the definitions of electric and magnetic fields, the

action can be written as

$$S' = S + \frac{1}{2} \int d^4x \varepsilon_{\alpha\beta} (\nabla \times \mathbf{A}^{(a)} \cdot \nabla \Gamma^{(b)} + \rho^{(a)} \Gamma^{(b)})$$

where S is same with in Equation 3.41. By applying integration by parts and eliminating the surface term,

$$S' = S + \frac{1}{2} \int d^4x \varepsilon_{\alpha\beta} (-\nabla \cdot \nabla \times \mathbf{A}^{(a)} + \rho^{(a)}) \Gamma^{(b)}$$

where the term in the parenthesis is zero due to the first equation in 3.37.

Similarly, by applying the transformation $\mathbf{A} \rightarrow \mathbf{A} + \nabla \Gamma$ the action can be written as

$$\begin{aligned} S' = S & - \frac{1}{2} \int d^4x \left\{ \varepsilon_{\alpha\beta} \left[\nabla \times \mathbf{A}^{(a)} \cdot \nabla \dot{\Gamma}^{(b)} + \nabla \times \nabla \Gamma^{(a)} \cdot \dot{\mathbf{A}}^{(b)} + \nabla \times \nabla \Gamma^{(a)} \cdot \nabla \phi^{(b)} \right] \right. \\ & \delta_{\alpha\beta} \left[(\nabla \times \mathbf{A}^{(a)}) \cdot (\nabla \times \nabla \Gamma^{(b)}) + (\nabla \times \nabla \Gamma^{(a)}) \cdot (\nabla \times \nabla \mathbf{A}^{(b)}) \right. \\ & \left. \left. + (\nabla \times \nabla \Gamma^{(a)}) \cdot (\nabla \times \nabla \Gamma^{(b)}) \right] - \varepsilon_{\alpha\beta} \mathbf{J}^{(a)} \cdot \nabla \Gamma^{(a)} \right\} \end{aligned}$$

Since

$$\nabla \times \nabla f = 0$$

for any function all the terms in the second bracket vanish. Furthermore,

$$\nabla \times \nabla \Gamma^{(a)} \cdot \mathbf{A}^{(b)} + \nabla \times \nabla \Gamma^{(a)} \cdot \nabla \phi^{(b)} = -\nabla \times \nabla \Gamma^{(a)} \cdot \mathbf{E}^{(b)}$$

Then applying integration by parts

$$\begin{aligned}
S' &= S - \frac{1}{2} \int d^4x \varepsilon_{\alpha\beta} \left[\frac{d}{dt} (-\nabla \times \mathbf{A}^{(a)}) \cdot \nabla \Gamma^{(b)} + \nabla \times \mathbf{E}^{(b)} \cdot \nabla \Gamma^{(a)} - \mathbf{J}^{(b)} \cdot \nabla \Gamma^{(a)} \right] \\
&= S - \frac{1}{2} \int d^4x \varepsilon_{\alpha\beta} \left[\dot{\mathbf{B}}^{(b)} + \nabla \times \mathbf{E}^{(b)} - \mathbf{J}^{(b)} \right] \cdot \nabla \Gamma^{(a)}
\end{aligned}$$

where the term in the parenthesis vanishes because of the second equation in 3.37, which originally comes from the Bianchi identity. This proves the invariance of the action under the transformation $\mathbf{A} \rightarrow \mathbf{A} + \nabla \Gamma$ where Γ is an arbitrary function.

Now as in the Section 3.1, setting $\phi = 0$ and varying the action in Equation 3.41 with respect to $\mathbf{A}$ gives

$$\delta S = \frac{1}{2} \int d^4x \left(\varepsilon_{\alpha\beta} \dot{\mathbf{B}}^{(b)} - \varepsilon_{\alpha\beta} \nabla \times \mathbf{E}^{(b)} - 2\delta_{\alpha\beta} \nabla \times \mathbf{B}^{(b)} - \varepsilon_{\alpha\beta} \mathbf{J}^{(b)} \right) \cdot \delta \mathbf{A}^{(a)} = 0 \quad (3.44)$$

which yields, by again using Equation 3.37,

$$-\varepsilon_{\alpha\beta} \nabla \times \mathbf{E}^{(b)} - \delta_{\alpha\beta} \nabla \times \mathbf{B}^{(b)} = 0$$

Therefore, as expected, the equations of motion, which correspond to the duality transformations, are the same with the vacuum case (Equation 3.30).

$$\varepsilon_{ab} \mathbf{E}^{(b)} = -\mathbf{B}^{(a)}$$

Since only two of four fields are independent, in the discussion of dyon-dyon interaction, the electric fields are suppressed and magnetic fields, $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$, are going to be considered as basis.

3.3. Particle Motion in Duality Invariant Electromagnetism

In this section, the motion of Dyons with charge $q^{(a)} = (e, g)$ will be discussed.

After deriving the classical equations of motions

$$m\ddot{x}^i + \varepsilon_{ab} q^{(b)} (\mathbf{B}^{(a)} \times \dot{\mathbf{x}})^i - q^{(b)} B^{i(b)} = 0$$

by using an appropriate Lorentz invariant action, the Hamiltonian of the system

$$\mathcal{H} = \frac{1}{2m} (\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}^{(b)})^2$$

will be extracted.

3.3.1. Classical Motion of a Dyon in Electromagnetic Fields

Even though the aim of this section is deriving equations of motion and Hamiltonian in classical dynamics, for completeness it is convenient to start the discussion with a Lorentz invariant action.

For a dyon particle moving in the presence of two magnetic fields $B^{(1)}$ and $B^{(2)}$, by using the interaction term in the previous discussion, the action can be written as

$$S = -m \int ds + \int d^4x \varepsilon_{ab} J_\mu^{(b)} A^{\mu(a)} \quad (3.45)$$

where $ds^2 = dx^\mu dx_\mu$. Note that the interaction between the fields are ignored.

In the rest frame,

$$ds^2 = d\tau^2, \quad c = 1$$

where τ is the proper time of the particle. Furthermore, again in the rest frame, second integral can be written as

$$\varepsilon_{ab} \int d^3x q^{(b)} \int d\tau U_\mu A^{\mu(a)} = \varepsilon_{ab} q^{(b)} \int d\tau U_\mu A^{\mu(a)}$$

where $U^\mu = \frac{dx^\mu}{d\tau}$ is 4-velocity. Then equation (3.45) becomes

$$S = \int d\tau \left(-m + \varepsilon_{ab} q^{(b)} \frac{dx_\mu}{d\tau} A^{\mu(a)} \right) \quad (3.46)$$

To find the equation of motion, this action should be varied. However, $d\tau$ varies as the trajectory varies. Therefore, a new variable, ξ , which doesn't vary with the trajectory of the particle, should be introduced [21].

$$\delta S = \int d\xi \left[-m \frac{d\delta\tau}{d\xi} + \varepsilon_{ab} q^{(b)} \left(\frac{dx_\mu}{d\xi} \delta A^{\mu(a)} + \frac{d\delta x_\mu}{d\xi} A^{\mu(a)} \right) \right] \quad (3.47)$$

Using the following trick

$$\frac{1}{2} \delta(d\tau)^2 = d\tau d\delta\tau = dx^\mu d\delta x_\mu$$

first term of the integrand can be written as

$$\frac{d\delta\tau}{d\xi} = \frac{d\tau}{d\tau} \frac{d\delta\tau}{d\xi} = \frac{dx^\mu}{d\tau} \frac{d\delta x_\mu}{d\xi}$$

Then, equation (3.47) becomes

$$\begin{aligned} \delta S &= \int d\xi \left[-m U^\mu \frac{d\delta x_\mu}{d\xi} + \varepsilon_{ab} q^{(b)} \left(\frac{dx_\mu}{d\xi} \delta A^{\mu(a)} - \frac{dA^{\mu(a)}}{d\xi} \delta x_\mu \right) \right] \\ &= \int d\xi \left[m \frac{dU^\mu}{d\xi} \delta x_\mu + \varepsilon_{ab} q^{(b)} \left(\frac{dx_\mu}{d\xi} \partial^\nu A^{\mu(a)} \delta x_\nu - \partial^\nu A^{\mu(a)} \frac{dx_\nu}{d\xi} \delta x_\mu \right) \right] \end{aligned}$$

Now ξ can be set to τ ,

$$\delta S = \int d\tau \left[m \ddot{x}^\nu + \varepsilon_{ab} q^{(b)} \left(\partial^\nu A^{\mu(a)} - \partial^\mu A^{\nu(a)} \right) \dot{x}_\mu \right] \delta x_\nu = 0 \quad (3.48)$$

Then, the term in the brackets gives the equations of motion of the particle.

$$m \ddot{x}^\nu + \varepsilon_{ab} q^{(b)} \left(\partial^\nu A^{\mu(a)} - \partial^\mu A^{\nu(a)} \right) \dot{x}_\mu = 0 \quad (3.49)$$

Recalling the chosen gauge $A^0 = \phi = 0$ and the equations

$$\begin{aligned}\mathbf{E}^{(a)} &= -\dot{\mathbf{A}}^{(a)} \\ \varepsilon_{\alpha\beta}\mathbf{E}^{(b)} &= -\mathbf{B}^{(a)}\end{aligned}$$

for $\nu = 0$, the equation of motion is,

$$\begin{aligned}m\ddot{x}^0 - \varepsilon_{ab}q^{(b)}\dot{A}^{i(a)}\dot{x}^i &= m\ddot{x}^0 - \varepsilon_{ab}q^{(b)}\dot{x}^i E^{i(a)} \\ &= m\ddot{x}^0 - q^{(b)}\dot{x}^i B^{i(b)} = 0\end{aligned}\tag{3.50}$$

and for $\nu = i$ where $i = 1, 2, 3$,

$$\begin{aligned}m\ddot{x}^i + \varepsilon_{ab}q^{(b)}(\partial^i A^{j(a)} - \partial^j A^{i(a)})\dot{x}^\mu &= m\ddot{x}^i + \varepsilon_{ab}q^{(b)}(\partial^i A^{j(a)} - \partial^j A^{i(a)})\dot{x}^j \\ &\quad + \varepsilon_{ab}q^{(b)}\partial^0 A^{i(a)}\dot{x}_0 \\ &= 0\end{aligned}$$

Using the vector relation

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = (A^i B^j - A^j B^i) C^i\tag{3.51}$$

in the second term and

$$\dot{x}_0 = -\dot{x}^0 = -1$$

in the third term, the 2nd equation of motion in equation is found as

$$m\ddot{x}^i + \varepsilon_{ab}q^{(b)}(\mathbf{B}^{(a)} \times \dot{\mathbf{x}})^i - q^{(b)}B^{i(b)} = 0\tag{3.52}$$

This is the equation of motion of a charged particle, under the influence of both electric and magnetic fields. In equation (3.52), the second term is equivalent to the interaction of an electric charge with a magnetic field and the last term equivalent to

the interaction of an electric charge with an electric field in the usual electromagnetism.

When the classical limit is taken, proper time derivatives become ordinary derivatives since as $v \rightarrow 0$, in the first order approximation $\gamma \rightarrow 1$, which is hidden in the proper time derivative. Therefore, even though (3.50) and (3.52) are the relativistic equations of motions, in the classical limit, the structures of the equations are the same. In this case, while (3.52) represent the classical motion of the particle and it is the only equation of motion, (3.50) gives the rate of change of the total energy of the particle, which is implicit in the equation of motion of time-like component in the relativistic case.

3.3.2. Hamiltonian for a Dyon in Electromagnetic Fields

In order to discuss interactions in quantum mechanics, which will be done in the next subsection, it is necessary to find the Hamiltonian of the system. Hamiltonian of the system is given by Legendre transformations

$$\mathcal{H} = \mathbf{p} \cdot \dot{\mathbf{x}} - \mathcal{L}$$

where $\mathcal{L}$ is the Lagrangian of the system and $\mathbf{P}$ is the canonical momentum. Since only the non-relativistic quantum mechanics will be considered, from now on Lagrangians and Hamiltonians are written in the domain of non-relativistic mechanics. Using the proper classical kinetic term and setting $\phi = 0$, the Lagrangian of the system is written as

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{x}}^2 + \varepsilon_{ab}q^{(a)}\dot{\mathbf{x}} \cdot \mathbf{A}^{(b)} \quad (3.53)$$

Also canonical momentum is

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = m\dot{\mathbf{x}} + \varepsilon_{ab}q^{(a)}\mathbf{A}^{(b)} \quad (3.54)$$

Then

$$\mathcal{H} = \left(m\dot{\mathbf{x}} + \varepsilon_{ab} q^{(a)} \mathbf{A}^{(b)} \right) \dot{\mathbf{x}} - \frac{1}{2} m \dot{\mathbf{x}}^2 - \varepsilon_{ab} q^{(a)} \mathbf{A}^{(a)} = \frac{1}{2} m \dot{\mathbf{x}}^2 \quad (3.55)$$

and since

$$\dot{\mathbf{x}} = \frac{1}{m} \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}^{(b)} \right)$$

the Hamiltonian of a dyon in an electromagnetic field is

$$\mathcal{H} = \frac{1}{2m} \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}^{(b)} \right)^2 \quad (3.56)$$

The structure of the Hamiltonian is different than the one in usual electromagnetism. Here at first sight it seems that the term corresponds to the interaction between electric charge and electric field in the usual case is missing. However, using the gauge transformation $\mathbf{A} \rightarrow \mathbf{A} + \nabla\Gamma$, it can be shown that it is just hidden in $\mathcal{H}$.

Before transforming the Hamiltonian, it is useful to relate the function Γ and vector potential $\mathbf{A}$. Recall the magnetic and electric field definitions in gauge $\phi = 0$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and

$$\mathbf{E} = -\dot{\mathbf{A}}$$

For a particle with both electric and magnetic charges (e, g) , associated fields are given as

$$\mathbf{E} = e \frac{\mathbf{r}}{r^3}$$

and

$$\mathbf{B} = g \frac{\mathbf{r}}{r^3}$$

Using gauge transformation, set $\mathbf{A} = \mathbf{A}_m + \nabla\Gamma$ gives

$$\nabla \times (\mathbf{A}_m + \nabla\Gamma) = \nabla \times \mathbf{A}_m = g \frac{\mathbf{r}}{r^3} \quad (3.57)$$

and similarly

$$\mathbf{E} = -\dot{\mathbf{A}}_m - \frac{\partial}{\partial t} \nabla\Gamma$$

where $\mathbf{A}_m$ can be chosen time independent or in other words $\nabla\Gamma$ is chosen such that it has the all time dependent parts of original vector potential $\mathbf{A}$. Therefore,

$$\mathbf{E} = -\frac{\partial}{\partial t} \nabla\Gamma = e \frac{\mathbf{r}}{r^3} \quad (3.58)$$

Finally, using the duality equation

$$\mathbf{E}^{(b)} = -\varepsilon_{ab} \mathbf{B}^{(a)}$$

the relation between Γ and $\mathbf{A}_m$ is written as

$$\frac{\partial}{\partial t} \nabla\Gamma^{(b)} = -\varepsilon_{ab} q^{(a)} \frac{\mathbf{r}}{r^3} = -\varepsilon_{ab} \nabla \times \mathbf{A}_m^{(a)}$$

which yields

$$\nabla\Gamma^{(b)} = -\varepsilon_{ab} (\nabla \times \mathbf{A}_m^{(a)}) t \quad (3.59)$$

Then, the Hamiltonian in equation (3.56) can be written as

$$\begin{aligned}\mathcal{H} &= \frac{1}{2m} \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} + \varepsilon_{ab} q^{(a)} \varepsilon_{cd} \nabla \times \mathbf{A}_m^{(c)} t \right)^2 \\ &= \frac{1}{2m} \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} + q^{(a)} \nabla \times \mathbf{A}_m^{(a)} t \right)^2\end{aligned}\quad (3.60)$$

For completeness the equation of motion of a dyon can be derived from the Hamiltonian in equation (3.60) using

$$\dot{p}^i = -\frac{\partial \mathcal{H}}{\partial x_i}$$

where

$$\frac{\partial \mathcal{H}}{\partial x_i} = \frac{1}{m} \left(p^j - \varepsilon_{ab} q^{(a)} A_+^{j(b)} q^{(a)} (\nabla \times A^{(a)})^j t \right) \left(-\varepsilon_{ab} q^{(a)} \partial^i \mathbf{A}_m^{j(b)} + \partial^i (\nabla \times A^{(a)})^j t \right)$$

and

$$p^j = m \dot{x}^j + \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{j(b)} - q^{(a)} (\nabla \times A^{(a)})^j t$$

Then,

$$-m \ddot{x}^i - \varepsilon_{ab} q^{(a)} \frac{d}{dt} A_m^{i(b)} - q^{(a)} \frac{d}{dt} \left[(\nabla \times A_m^{(a)})^i t \right] = \dot{x}^j (-\varepsilon_{ab} q^{(a)} \partial^i A_m^{j(b)})$$

$$m \ddot{x}^i + \varepsilon_{ab} q^{(a)} \dot{x}^j \partial^j A^{i(b)} - q^{(a)} (\nabla \times A_m^{(a)})^i = \varepsilon_{ab} q^{(a)} \dot{x}^j \partial^i A_m^{j(b)}$$

$$m \ddot{x}^i + \varepsilon_{ab} q^{(a)} (\dot{x}^j \partial^j A_m^{i(b)} - \dot{x}^j \partial^i A_m^{j(b)}) - q^{(a)} B^{i(a)} = 0$$

Using the identity in equation (3.51),

$$m\ddot{\mathbf{x}} - \varepsilon_{ab} q^{(a)} \dot{\mathbf{x}} \times (\nabla \times \mathbf{A}^{(b)}) - q^{(a)} B^{(a)} = 0$$

$$m\ddot{\mathbf{x}} - \varepsilon_{ab} q^{(a)} \dot{\mathbf{x}} \times \mathbf{B}^{(b)} - q^{(a)} B^{(a)} = 0 \quad (3.61)$$

This is same with the equation of motion in equation (3.52), which is derived by varying the action.

3.4. Quantum Mechanics in Duality Invariant Form

After the convincing results in classical electromagnetism, it is time to discuss the quantum mechanics of a dyon in the presence of electromagnetic fields and derive the Schrödinger equation for the Dyon atom.

Both Hamiltonians in equations (3.56) and (3.60) can be used to express the Schrödinger equation. The wave functions are also transformed by the gauge transformation as the corresponding Hamiltonian are transformed. These transformations is done in such a way that in the end the matrix elements of both Hamiltonians should be the same [26].

$$\langle \psi | \mathcal{H} | \psi \rangle = \langle \psi' | \mathcal{H}' | \psi' \rangle \quad (3.62)$$

where

$$\mathcal{H} = \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} - \varepsilon_{ab} q^{(a)} \nabla \Gamma^{(b)} \right)^2 \quad (3.63)$$

and

$$\mathcal{H}' = \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} \right)^2 \quad (3.64)$$

Furthermore, the norm of a state ket should not be affected by gauge transformations either.

$$\langle \psi | \psi \rangle = \langle \psi' | \psi' \rangle$$

This requires the relation between $|\psi\rangle$ and $|\psi'\rangle$ should be in the following form

$$|\psi'\rangle = e^{i\Lambda} |\psi\rangle \quad (3.65)$$

where Λ will be determined later in order to be consistent with the original transformation $\mathbf{A} \rightarrow \mathbf{A} + \Gamma$. Putting the equation (3.65) into the equation (3.62) yields

$$\mathcal{H} = e^{-i\Lambda} \mathcal{H}' e^{i\Lambda} \quad (3.66)$$

Then by choosing Λ as a function of $\mathbf{x}$ and t ,

$$\begin{aligned} \mathcal{H} &= e^{-i\Lambda} \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} \right)^2 e^{i\Lambda} \\ &= e^{-i\Lambda} \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} \right) e^{i\Lambda} e^{-i\Lambda} \left(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} \right) e^{i\Lambda} \end{aligned}$$

Since Λ is a function of $\mathbf{x}$,

$$\begin{aligned} e^{-i\Lambda} \mathbf{p} e^{i\Lambda} &= e^{-i\Lambda} [\mathbf{p}, e^{i\Lambda}] + \mathbf{p} \\ &= -ie^{-i\Lambda} \nabla e^{i\Lambda} + \mathbf{p} \\ &= \mathbf{p} + \nabla \Lambda \end{aligned}$$

where the commutation relation

$$[p_i, G(\mathbf{x})] = -i \frac{\partial G}{\partial x_i}$$

is used. Then, transformed Hamiltonian is achieved as

$$\mathcal{H} = (\mathbf{p} + \nabla\Lambda - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)}) \quad (3.67)$$

Comparing the equations (3.63) and (3.67) shows that

$$\Lambda = -\varepsilon_{ab} q^{(a)} \Gamma^{(b)} \quad (3.68)$$

Then, the Schrödinger equation for $\mathcal{H}$ is

$$\mathcal{H} |\psi\rangle = (\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)} - \varepsilon_{ab} q^{(a)} \nabla \Gamma^{(b)})^2 |\psi\rangle = i \frac{\partial}{\partial t} |\psi\rangle \quad (3.69)$$

In order to find a more usual form of the Schrödinger equation, apply the gauge transformations in reverse direction

$$\mathcal{H} \rightarrow e^{i\Lambda} \mathcal{H} e^{-i\Lambda}$$

and

$$|\psi\rangle \rightarrow e^{i\Lambda} |\psi\rangle$$

This yields

$$\begin{aligned} e^{i\Lambda} \mathcal{H} e^{-i\Lambda} e^{i\Lambda} |\psi\rangle &= i e^{i\Lambda} \frac{\partial}{\partial t} |\psi\rangle \\ &= i \left[\frac{\partial}{\partial t} (e^{i\Lambda} |\psi\rangle) - \left(\frac{\partial}{\partial t} e^{i\Lambda} \right) |\psi\rangle \right] \end{aligned}$$

Then, the transformed equation can be written in terms of $\mathcal{H}'$ and ψ'

$$\mathcal{H}' |\psi\rangle = i \frac{\partial}{\partial t} |\psi'\rangle + \frac{\partial \Lambda}{\partial t} |\psi'\rangle \quad (3.70)$$

Recall, the equation (3.59)

$$\nabla \Gamma^{(b)} = -\varepsilon_{cb} (\nabla \times \mathbf{A}_m^{(c)}) t$$

Using this relation, it can be deduced that

$$\varepsilon_{ab} \frac{\partial}{\partial t} \nabla \Gamma^{(b)} = -\mathbf{B}^{(a)} = -q^{(a)} \frac{\mathbf{r}}{r^3}$$

and

$$\varepsilon_{ab} \frac{\partial}{\partial t} \Gamma^{(b)} = q^{(a)} \frac{1}{|\mathbf{r}|}$$

Then, using the equation (3.68)

$$\frac{\partial \Lambda}{\partial t} = -q^{(a)} q^{(a)} \frac{1}{|\mathbf{r}|}$$

Therefore, time independent form of the Schrödinger equation in equation (3.69) can be written explicitly as

$$\left[(\mathbf{p} - \varepsilon_{ab} q^{(a)} \mathbf{A}_m^{(b)})^2 + \frac{q^{(a)} q^{(a)}}{|\mathbf{r}|} \right] |\psi'\rangle = E |\psi'\rangle \quad (3.71)$$

This is duality invariant form of the Schrödinger equation for a Dyon moving in the field of another Dyon. In equation (3.71), $\mathbf{A}_m^{(b)}$ corresponds to the vector potential of source Dyon. Recalling its general form

$$\mathbf{A}_m^{(b)} = \frac{q^{(b)}}{r} \frac{-\hat{n} \times \mathbf{r}}{r + \mathbf{r} \cdot \hat{n}} \equiv q^{(b)} \mathbf{D}_m(\mathbf{r})$$

Schrödinger equation can be written as

$$\left[(\mathbf{p} - \varepsilon_{ab} q^{(a)} q^{(b)} \mathbf{D}_m(\mathbf{r}))^2 + \frac{q^{(a)} q^{(a)}}{|\mathbf{r}|} \right] |\psi'\rangle = E |\psi'\rangle \quad (3.72)$$

This form can be easily generalized to the case of two interacting Dyons. When the Schrödinger equation for 2 interacting dyons is written in this form, it will be symmetric under the exchange of dyons as it should be. In the next subsection this equation will be written and transformed to the center of mass frame. The resulting equation will be closely related to the Schrödinger equation of Hydrogen atom.

3.4.1. Schrodinger Equation for 2 Interacting Dyons

The Schrödinger equation for two interacting Dyon can be written as [8]

$$\left\{ \sum_{i=1}^2 \sum_{j=1}^2 \left[\frac{1}{2m_i} \left(\mathbf{p}_i - \varepsilon_{ab} q_i^{(a)} q_j^{(b)} \mathbf{D}_m(\mathbf{r}_i - \mathbf{r}_j) \right)^2 + \delta_{ab} \frac{q_i^{(a)} q_j^{(b)}}{|\mathbf{r}_i - \mathbf{r}_j|} \right] \right\} |\psi\rangle = E |\psi\rangle \quad (3.73)$$

Defining new variables

$$\mu \equiv e_1 g_2 - e_2 g_1$$

$$-\alpha \equiv e_1 e_2 + g_1 g_2$$

and using the fact

$$\mathbf{D}(\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{D}(\mathbf{r}_2 - \mathbf{r}_1)$$

the kinetic part of the Hamiltonian becomes

$$\begin{aligned} \mathbf{K} &= \frac{1}{2m_1} (\mathbf{p}_1 - \mu \mathbf{D}(\mathbf{r}_1 - \mathbf{r}_2))^2 + \frac{1}{2m_2} (\mathbf{p}_2 + \mu \mathbf{D}(\mathbf{r}_2 - \mathbf{r}_1))^2 \\ &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - 2\mu \left(\frac{\mathbf{p}_1}{2m_1} - \frac{\mathbf{p}_2}{2m_2} \right) \mathbf{D}(\mathbf{r}_1 - \mathbf{r}_2) + \frac{\mu^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{D}^2(\mathbf{r}_1 - \mathbf{r}_2) \\ &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - 2\mu \frac{1}{2} \left(\frac{\mathbf{p}_1 m_2 - \mathbf{p}_2 m_1}{m_1 + m_2} \frac{m_1 + m_2}{m_1 m_2} \right) \mathbf{D}(\mathbf{r}_1 - \mathbf{r}_2) \\ &+ \frac{\mu^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{D}^2(\mathbf{r}_1 - \mathbf{r}_2) \end{aligned}$$

Introduce new variables for relative position and momentum

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\mathbf{p} = \frac{\mathbf{p}_1 m_2 - \mathbf{p}_2 m_1}{m_1 + m_2}$$

for center of mass momentum

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$

and for total and reduced masses

$$M = m_1 + m_2$$

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$$

Then,

$$\begin{aligned} K &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - 2\mu \mathbf{D}(\mathbf{r}) \frac{\mathbf{p}}{2m} + \frac{\mu^2}{2m} \mathbf{D}(\mathbf{r}) \\ &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{1}{2m} [-\mathbf{p}^2 + \mathbf{p}^2 - 2\mu \mathbf{D}(\mathbf{r}) \mathbf{p} + \mu^2 \mathbf{D}^2(\mathbf{r})] \\ &= \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{\mathbf{p}^2}{2m} + \frac{1}{2m} [\mathbf{p} - \mu \mathbf{D}(\mathbf{r})]^2 \end{aligned}$$

where first two terms can be written as

$$\begin{aligned} \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} &= \frac{1}{2} \frac{\mathbf{p}_1^2 m_2 + \mathbf{p}_2^2 m_1}{m_1 m_2} \frac{m_1 + m_2}{m_1 + m_2} \\ &= \frac{m_2^2 \mathbf{p}_1^2 + m_2 m_1 \mathbf{p}_1^2 + m_1^2 \mathbf{p}_2^2 + m_2 m_1 \mathbf{p}_2^2}{2(m_1 + m_2) m_1 m_2} \end{aligned}$$

and third term can be written as

$$\begin{aligned}\frac{\mathbf{p}^2}{2m} &= \frac{(\mathbf{p}_1 m_2 - \mathbf{p}_2 m_1)^2}{2(m_1 + m_2)^2} \frac{m_1 + m_2}{m_1 m_2} \\ &= \frac{\mathbf{p}_1^2 m_2^2 + \mathbf{p}_2^2 m_1^2 - 2m_1 m_2 \mathbf{p}_1 \mathbf{p}_2}{2(m_1 + m_2) m_1 m_2}\end{aligned}$$

Using these relations

$$\begin{aligned}\mathcal{K} &= \frac{\mathbf{p}_1^2 m_2^2 + 2\mathbf{p}_1 \mathbf{p}_2}{2(m_1 + m_2)} + \frac{1}{2m} [\mathbf{p} - \mu \mathbf{D}(\mathbf{r})]^2 \\ &= \frac{1}{2(m_1 + m_2)} (\mathbf{p}_1 + \mathbf{p}_2)^2 + \frac{1}{2m} [\mathbf{p} - \mu \mathbf{D}(\mathbf{r})]^2\end{aligned}$$

and finally using the center of mass momentum and total mass formulae the Hamiltonian is written as

$$\mathcal{H} = \frac{1}{2M} \mathbf{P}^2 + \frac{1}{2m} (\mathbf{p} - \mu \mathbf{D})^2 - \frac{\alpha}{r} \quad (3.74)$$

where $r = |\mathbf{r}|$. Note that in the center of mass frame $\mathbf{P} = 0$. Therefore, it is more convenient to examine the system in the center of mass frame with Schrödinger equation

$$\left(\frac{1}{2m} (\mathbf{p} - \mu \mathbf{D})^2 - \frac{\alpha}{r} \right) |\psi\rangle = E |\psi\rangle \quad (3.75)$$

This result is very similar to the Schrödinger equation of Hydrogen atom which basically describes the interaction of two electrically charged particles in the center of mass frame.³ However, the additional vector potential term, $\mu \mathbf{D}$, is a fundamental difference from the Hydrogen atom. This term exists because of the interaction between electric and magnetic charges when they are in motion. Initially, the existence of this term seems to spoil the symmetries of the system as an external magnetic field applied on the Hydrogen atom would do but the addition of an extra term would cure this.

Before adding an extra term, first examine the Schrödinger equation. The kinetic

³In the Hydrogen atom case, the center of mass is approximately the position of the proton since proton is much heavier than electron. Therefore, the proton is considered as stationary.

term in the equation (3.75) is

$$\boldsymbol{\pi} \equiv \mathbf{p} - \mu \mathbf{D}$$

It should correspond to the momentum operator of the system. However, the commutation relations between π_i and x_i are found as [8]

$$[x_i, x_j] = 0 \quad (3.76)$$

$$[\pi_i, x_j] = -i\delta_{ij} \quad (3.77)$$

$$[\pi_i, \pi_j] = i\mu\varepsilon_{ijk}\frac{x_k}{r^3} \quad (3.78)$$

Last relation is different than the usual momentum commutation relations. Therefore, the operator $\boldsymbol{\pi}$ is not a correct momentum operator. In fact it can be shown that this problems is due to the additional term which comes from the interactions of electric and magnetic charges. In order to define the correct momentum, first define the angular momentum. From classical electrodynamics, it can be written as [8]

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi} - \mu \hat{\mathbf{r}} \quad (3.79)$$

where the term $\mu \hat{\mathbf{r}}$ exists due to the interaction of the fields of the electric and magnetic charges [25]. It can be found by evaluating the integral

$$\int d\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$$

The angular momentum in equation (3.79) satisfies the following commutation relations

$$[J_i, H] = 0 \quad (3.80)$$

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \quad (3.81)$$

The operator $\boldsymbol{\pi}$ can be written in terms of $\mathbf{J}$ in the following way.

$$\begin{aligned}\mathbf{r} \times \mathbf{J} &= \mathbf{r} \times (\mathbf{r} \times \boldsymbol{\pi}) \\ &= (\mathbf{r} \cdot \boldsymbol{\pi})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\boldsymbol{\pi} \\ &= (\mathbf{r} \cdot \boldsymbol{\pi})\mathbf{r} - r^2\boldsymbol{\pi}\end{aligned}$$

Arranging the last equation, $\boldsymbol{\pi}$ is found as,

$$\boldsymbol{\pi} = \frac{\hat{r}}{2} (\hat{r} \cdot \boldsymbol{\pi} + \boldsymbol{\pi} \cdot \hat{r}) + \frac{1}{2} (\hat{r} \times \mathbf{J} - \mathbf{J} \times \hat{r}) \quad (3.82)$$

In this way, the radial and angular parts are written separately. Then, defining the radial part as $\boldsymbol{\pi}_r$

$$\boldsymbol{\pi}^2 = \boldsymbol{\pi}_r^2 + (\mathbf{J} \times \hat{r})^2 + 2\hat{r} \cdot (\mathbf{J} \times \hat{r})(\hat{r} \cdot \boldsymbol{\pi}) \quad (3.83)$$

Last term is vanishes and the second one can be written as

$$\begin{aligned}(\mathbf{J} \times \hat{r})^2 &= \frac{1}{r^2} \varepsilon_{ijk} J_j x_k \varepsilon_{lmn} J_l x_m x_n \\ &= \frac{1}{r^2} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{mk}) J_j \hat{x}_k J_m \hat{x}_n \\ &= \frac{1}{r^2} (J^2 - (\mathbf{J} \cdot \hat{r})(\mathbf{J} \cdot \hat{r}))\end{aligned}$$

From the definition of the angular momentum operator

$$\mathbf{J} \cdot \hat{r} = \mu$$

Then, equation (3.83) becomes

$$\boldsymbol{\pi}^2 = \boldsymbol{\pi}_r^2 + \frac{1}{r^2} (J^2 - \mu^2) \quad (3.84)$$

and the Hamiltonian is written as

$$H = \frac{\pi_r^2}{2m} + \frac{J^2}{2mr^2} - \frac{\mu^2}{2mr^2} - \frac{\alpha}{r} \quad (3.85)$$

Note that the difference between the symmetries of the Dyon atom and the Hydrogen atom is now more apparent. First two terms in the equation (3.85) gives the same kinetic term with the Hamiltonian of the Hydrogen atom. Thus, the third term is the additional factor that spoils the full $SO(4)$ symmetry of the system. In order to restore the symmetry, it is convenient to add an extra potential term to the Hamiltonian in the equation (3.75). Then, the appropriate Schrödinger equation is written as

$$\left\{ \frac{1}{2m} \left[(\mathbf{p} - \mu \mathbf{D})^2 + \frac{\mu^2}{r^2} \right] - \frac{\alpha}{r} \right\} |\psi\rangle = E |\psi\rangle \quad (3.86)$$

3.5. A 3D – 4D Duality

As it was stated that in order to estimate the large order behaviours of perturbative expansion by using the instanton methods in the Euclidean path integral, the action of the Dyon atom should be transformed to a form of anharmonic oscillator. The transformation rules between 3 dimensional Kepler type potentials and 4 dimensional harmonic oscillator are widely known and used in the transformation of the perturbative Coulomb systems [6, 7]. Furthermore, since with the addition of $\frac{\mu^2}{2mr^2}$, the potential of the Dyon atom becomes Kepler type, same transformation rules are also applied to map the charge-dyon system to the harmonic oscillators [12]. However, in those papers, the equivalence of the symmetry groups of Kepler type potentials and the harmonic oscillator left unexplained. In this section, first the $SO(3) \times SO(3)$ group algebra of the Kepler type potential will be derived and the equivalence $SO(3) \times SO(3) \sim SO(4)$ will be established. Then, using this equivalence as a motivation, the transformation

rules

$$\begin{aligned}
 x_1 &= 2[u_1 u_3 - u_2 u_4] \\
 x_2 &= 2[u_1 u_4 + u_2 u_3] \\
 x_3 &= (u_1^2 + u_2^2 - u_3^2 - u_4^2) \\
 r &= u^2
 \end{aligned}$$

will be derived.

Kepler-like potentials, $V(r) \sim \frac{1}{r}$, manifestly has a spherical symmetry and this fact leads to the conservation of angular momentum of the whole system. However, it can be found another vector, which is also invariant under time translation. In classical Kepler problem, this additional vector is called Laplace-Runge-Lenz vector and together with the angular momentum, it reveals a larger symmetry than the rotational one [27].

The Coulomb potential has the same structure with Kepler potential, so that it has the same symmetries. In this case, the rotational symmetry can be represented simply by a commutation relation

$$[H, L_i] = 0$$

where

$$H = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} - \frac{e^2}{r}$$

is the Hamiltonian and

$$L_i = \varepsilon_{ijk} x_j p_k$$

is the angular momentum operator. The existence of additional symmetry implies another operator, say V , which should also commute with the Hamiltonian.

$$[V, H] = 0$$

In addition to the similarity with the classical case, another motivation for larger symmetries can be found by looking the energy levels of the Hydrogen atom, which is governed by Coulomb potential [28]. It is well-known that the states of the Hydrogen atom is labelled by quantum numbers n and l which are principal quantum number and angular momentum quantum number respectively. The ground state is labelled by $l = 0$ and this shows the rotational symmetry of the ground state. Therefore, from group theory perspective it can be said that the ground state can be explained by $SO(3)$ group algebra. The first excited state, however, is labelled by $l = 0$ or $l = 1$. The states of $l = 0$ and $l = 1$ have their own rotational symmetries so that their own $SO(3)$ algebras. This implies the larger symmetry that the Hydrogen atom should have. Below the additional time translation invariant operator, V , will be derived starting from the classical Kepler problem and the larger symmetry of the Hydrogen atom will be established.

The Newton second law for a general central force is [27]

$$\mathbf{F} = \dot{\mathbf{p}} = f(r)\hat{r} = f(r)\frac{\mathbf{r}}{r}$$

Then,

$$\begin{aligned}\dot{\mathbf{p}} \times \mathbf{L} &= m \frac{f(r)}{r} [\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})] \\ &= m \frac{f(r)}{r} [\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - (\mathbf{r} \mathbf{r})\dot{\mathbf{r}}] \\ &= m \frac{f(r)}{r} \left[\frac{1}{2} \frac{d}{dt} (\dot{\mathbf{r}})^2 \mathbf{r} - r^2 \dot{\mathbf{r}} \right]\end{aligned}$$

which can be written as

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = mf(r)(\dot{\mathbf{r}}\mathbf{r} - r\dot{\mathbf{r}})$$

For Kepler type problems, define $mf(r) \equiv K/r^2$ and put it into the above equation

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = K \left(\frac{\dot{\mathbf{r}}\mathbf{r}}{r^2} - \frac{\dot{\mathbf{r}}}{r} \right) \quad (3.87)$$

$$= K \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right) \quad (3.88)$$

Then, new time translation invariant vector can be defined as

$$\mathbf{V} \equiv \mathbf{p} \times \mathbf{L} - K \frac{\mathbf{r}}{r} \quad (3.89)$$

where the relation

$$\frac{d\mathbf{V}}{dt} = 0$$

is obvious from the Equation 3.88.

In addition to the time translation invariance, the vectors $\mathbf{L}$ and $\mathbf{V}$ are also orthogonal to each other

$$\mathbf{V} \cdot \mathbf{L} = (\mathbf{p} \times \mathbf{L}) \cdot \mathbf{L} - \frac{K}{r} (\mathbf{r} \cdot (\mathbf{r} \times \dot{\mathbf{r}})) = 0$$

Therefore, the time translation invariance of $\mathbf{V}$, indeed, corresponds to an additional symmetry of the system.

Now, using the form in Equation 3.89, the corresponding operator in quantum mechanics can be defined as

$$V_i = \frac{1}{2} \varepsilon_{ijk} (p_j L_k - p_k L_j) - me^2 \frac{x_i}{r} \quad (3.90)$$

Time translation invariance requires the following commutation relation

$$[V_i, H] = 0$$

for any $i = 1, 2, 3$.

$$[V_i, H] = \frac{1}{2}\varepsilon_{ijk} \{[p_j, H]L_k - [p_k, H]L_j\} - me^2\left[\frac{x_i}{r}, H\right] \quad (3.91)$$

Here the following general commutation relations can be used [26]

$$[x_i, G(\mathbf{p})] = i\hbar \frac{\partial G}{\partial p_i} \quad (3.92)$$

and

$$[p_i, G(\mathbf{x})] = -i\hbar \frac{\partial G}{\partial x_i} \quad (3.93)$$

Put the explicit form of the Hamiltonian for Coulomb interaction, $H = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} - \frac{e^2}{r}$, into the Equation 3.91

$$[V_i, H] = -\frac{i\hbar}{2}\varepsilon_{ijk} \left(\frac{\partial H}{\partial x_j} L_k - \frac{\partial H}{\partial x_k} L_j \right) - \frac{e^2}{2} \left[\frac{x_i}{r}, p^2 \right] - me^4 \left[\frac{x_i}{r}, \frac{1}{r} \right] \quad (3.94)$$

The terms in the curly bracket comes from the following calculation

$$[p_j, p^2/2m - e^2/r] = -e^2[p_j, 1/r] = -i\hbar e^2 \frac{\partial}{\partial x_j} (1/r) = -i\hbar \frac{\partial H}{\partial x_j}$$

Using the Hamiltonian equation motion $\frac{\partial H}{\partial x} = -\frac{\partial p}{\partial t}$ and defining $g(\mathbf{x}) \equiv \frac{x_i}{r}$

$$\begin{aligned} [V_i, H] &= \frac{i\hbar}{2}\varepsilon_{ijk} \left(\frac{\partial p_j}{\partial t} L_k - \frac{\partial p_k}{\partial t} L_j \right) - \frac{e^2}{2} \left([g(\mathbf{x}), p_l] p_l + p_l [g(\mathbf{x}), p_l] \right) \\ &= i\hbar \frac{d}{dt} \left(\frac{1}{2}\varepsilon_{ijk} (p_j L_k - p_k L_j) \right) - i\hbar \frac{e^2}{2} \left(\frac{\partial g(\mathbf{x})}{\partial x_l} p_l + p_l \frac{\partial g(\mathbf{x})}{\partial x_l} \right) \end{aligned}$$

where

$$\frac{\partial g}{\partial x_l} p_l = m \frac{\partial g}{\partial x_l} \frac{\partial x_l}{\partial t} = m \frac{\partial g}{\partial t}$$

Then,

$$[V_i, H] = i\hbar \frac{d}{dt} \left\{ \frac{1}{2} \varepsilon_{ijk} (p_j L_k - p_k L_j) \right\} - i\hbar \frac{e^2}{2} 2m \frac{\partial g}{\partial t} \quad (3.95)$$

$$= i\hbar \frac{d}{dt} \left(\frac{1}{2} \varepsilon_{ijk} (p_j L_k - p_k L_j) - m e^2 \frac{x_i}{r} \right) \quad (3.96)$$

From the definition in Equation 3.90

$$[V_i, H] = i\hbar \frac{dV_i}{dt} = 0 \quad (3.97)$$

which proofs the time translation invariance of operator V .

Furthermore the commutation of operator V with itself is [29]

$$[V_i, V_j] = (-2mH) i\hbar \varepsilon_{ijk} L_k$$

Thus, it is convenient to define a new operator $\bar{V} \equiv \frac{V}{\sqrt{-2mH}}$ so that

$$[\bar{V}_i, \bar{V}_j] = i\hbar \varepsilon_{ijk} L_k$$

Then, the commutation relations that the operators L and $\bar{V}$ obeys are given as

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k \quad (3.98)$$

$$[L_i, \bar{V}_j] = i\hbar \varepsilon_{ijk} \bar{V}_k \quad (3.99)$$

$$[\bar{V}_i, \bar{V}_j] = i\hbar\varepsilon_{ijk}L_k \quad (3.100)$$

Thus the operators L and $\bar{V}$ form a closed algebra which is larger than $SO(3)$. The exact structure of this algebra can be found by defining new operators using L and $\bar{V}$ and this structure will reveal the full symmetry of the Coulomb potential. Thus define new operators

$$\chi_i^+ \equiv \frac{1}{2}(L_i + \bar{V}_i) \quad (3.101)$$

$$\chi_i^- \equiv \frac{1}{2}(L_i - \bar{V}_i) \quad (3.102)$$

The commutations of new operators are

$$\begin{aligned} [\chi_i^+, \chi_j^-] &= \frac{1}{4} \left(\{[L_i, L_j] - [L_i, \bar{V}_j] + [\bar{V}_i, L_j] - [\bar{V}_i, \bar{V}_j]\} \right. \\ &= \frac{1}{4} \{i\hbar\varepsilon_{ijk}L_k - i\hbar\varepsilon_{ijk}\bar{V}_k - i\hbar\varepsilon_{jik}\bar{V}_k - i\hbar\varepsilon_{ijk}L_k\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\chi_i^\pm, \chi_j^\pm] &= \frac{1}{4} \{i\hbar\varepsilon_{ijk}L_k \pm i\hbar\varepsilon_{ijk}\bar{V}_k \mp i\hbar\varepsilon_{jik}\bar{V}_k \pm i\hbar\varepsilon_{ijk}L_k\} \\ &= i\hbar\varepsilon \frac{1}{2}(L_k \pm \bar{V}_k) \\ &= i\hbar\chi_k^\pm \end{aligned}$$

Therefore, both χ^+ and χ^- are generators of $SU(2)$ algebra and since they commute with each other, the group algebra is $SU(2) \times SU(2)$. Since the algebras of the $SU(2)$

and $SO(3)$ are the same, it can be said that the symmetry group of the Coulomb system is $SO(3) \times SO(3)$.

Now the full symmetry group of Coulomb-like systems are understood as $SO(3) \times SO(3)$ (or $SU(2) \times SU(2)$). $SO(3) \times SO(3)$ group algebra, however, is also equivalent to $SO(4)$ group algebra and this will be explained next.

3.5.1. $SO(3) \times SO(3)$ and $SO(4)$ equivalence

In 4 dimension the generators of $SO(4)$ can be written as [29]

$$L_i = \varepsilon_{ijk} x_j p_k$$

and

$$K_i = x_i p_4 - x_4 p_i$$

where $i = 1, 2, 3$.

The operators L_i are the usual rotation generators in $3D$ and the operators K_i generate the rotations involving the 4th dimension. The commutation relations which are satisfied by these operators are

$$\begin{aligned}
 [L_i, K_j] &= \varepsilon_{ilk} [x_l p_k, x_j p_4 - x_4 p_j] \\
 &= \varepsilon_{ilk} \{x_l [p_k, x_j] p_4 + x_j [x_l, p_4] p_k - x_l [p_k, x_4] p_j - x_4 [x_l, p_j] p_k\} \\
 &= \varepsilon_{ilk} \{-i\hbar x_l p_4 \delta_{kj} - i\hbar x_4 p_k \delta_{lj}\} \\
 &= -i\hbar \{\varepsilon_{ilj} x_l p_4 + \varepsilon_{ijk} x_4 p_k\} \\
 &= i\hbar \varepsilon_{ijk} \{x_k p_4 - x_4 p_k\} \\
 &= i\hbar \varepsilon_{ijk} K_k
 \end{aligned} \tag{3.103}$$

The commutation of K with itself is

$$\begin{aligned}
 [K_i, K_j] &= [x_i p_4 - x_4 p_i, x_j p_4 - x_4 p_j] \\
 &= -x_4 [p_i, x_j] p_4 - x_j [x_4, p_4] p_i - x_i [p_4, x_4] p_j - x_4 [x_i, p_j] p_4 \\
 &= -i\hbar x_j p_i + i\hbar x_i p_j
 \end{aligned}$$

The last term is non-zero only if $i \neq j$, thus it can be written as

$$[K_i, K_j] = i\hbar \varepsilon_{ijk} L_k \quad (3.104)$$

and finally from 3 dimensional mechanics

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k \quad (3.105)$$

These commutation relations are the same with the ones of the operators of L and $\bar{V}$ in the Coulomb system. Thus new operators can be defined to reveal another group algebra that is generated by L_i and K_i as in the Equations 3.101 and 3.102. Then, define

$$J_i^+ = \frac{1}{2} (L_i + K_i) \quad (3.106)$$

$$J_i^- = \frac{1}{2} (L_i - K_i) \quad (3.107)$$

where the commutation relations are

$$[J_i^+, J_j^+] = i\hbar \varepsilon_{ijk} J_k^+ \quad (3.108)$$

$$[J_i^-, J_j^-] = i\hbar \varepsilon_{ijk} J_k^- \quad (3.109)$$

$$[J_i^+, J_j^-] = 0 \quad (3.110)$$

which reveal $SO(3) \times SO(3)$ symmetry as expected. This result show the equivalence of the $SO(3) \times SO(3)$ algebra and $SO(4)$ algebra. This is equivalent to saying that the full group algebra structure of a system with Coulomb interaction in 3 dimension should be same with the group structure of a system with $SO(4)$ symmetry.

A simple quantum mechanical system with $SO(4)$ is a harmonic oscillator in 4 dimension. Therefore, based on the similar symmetry structure of 3D Coulomb interaction and 4D harmonic oscillator, a transformation between the coordinates that these systems are described can be proposed. The transformation rules will be established in the next subsection.

3.5.2. From the 3D Coulomb to the 4D Harmonic Potential

It is a well-known fact that the rotations in 2 dimensions are represented either as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.111)$$

or on a complex plane by defining $z = x + iy$

$$z' = e^{i\phi} z \quad (3.112)$$

This relation shows the equivalence between groups $SO(2)$ and $U(1)$. Thus an object that has a fixed length in $\mathbb{R}^2$ can be represented in $\mathbb{C}$ either.

A similar connection can be constructed between objects in $\mathbb{R}^4$ and a corresponding complex plane [30]. Consider real numbers a, b, c, d constructing a vector, (a, b, c, d) , in

4D and let

$$M = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are called quaternions and corresponds to the following complex matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Then, the complex matrix M is

$$M = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}$$

or by defining usual complex numbers as $z_1 = a + ib$ and $z_2 = c + id$

$$M = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \tag{3.113}$$

Therefore, an object in $\mathbb{R}^4$ can be represented by two complex numbers in $\mathbb{C}^2$ space.

Now, finally transformation between coordinates in 3D and in 4D can be established. Due to the relation between $\mathbb{R}^2$ and $\mathbb{C}$, it is convenient to represent $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$. Therefore, define a complex number for two coordinates in $\mathbb{R}^3$

$$z = x_1 + ix_2 \tag{3.114}$$

and leave the other coordinate real

$$x = x_3 \quad (3.115)$$

where x_1, x_2, x_3 are the Cartesian coordinates in $\mathbb{R}^3$. For the coordinates in $\mathbb{R}^4$, say (u_1, u_2, u_3, u_4) , define 2 complex numbers

$$z_1 = u_1 + iu_2 \quad (3.116)$$

and

$$z_2 = u_3 + iu_4 \quad (3.117)$$

In Hopf coordinates [12], the coordinates of 4 dimensional space can be written as

$$u_1 = u \cos \xi_1 \cos \eta$$

$$u_2 = u \sin \xi_1 \cos \eta$$

$$u_3 = u \cos \xi_2 \cos \eta$$

$$u_4 = u \sin \xi_2 \cos \eta$$

where $u = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$. Then the Equations 3.116 and 3.117 become

$$z_1 = ue^{i\xi_1} \cos \eta \quad (3.118)$$

$$z_2 = ue^{i\xi_2} \sin \eta \quad (3.119)$$

On the other hand, if the coordinates of the 3 dimensional space is written in the usual

spherical coordinates, the Equations 3.114 and 3.115 become

$$z = r \sin \theta e^{i\phi} = 2r \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{i\phi} \quad (3.120)$$

$$x = r \cos \theta = r \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) \quad (3.121)$$

Compring the equations from 3.118 to 3.121, it is convenient to set

$$\eta = \frac{\theta}{2}$$

$$\xi_1 = \frac{\phi + \psi}{2}$$

and

$$\xi_2 = \frac{\phi - \psi}{2}$$

where ψ is the 4 coordinates on hypersphere. Then, the Hopf coordinates become

$$u_1 = u \cos \frac{\phi + \psi}{2} \cos \frac{\theta}{2} \quad (3.122)$$

$$u_2 = u \sin \frac{\phi + \psi}{2} \cos \frac{\theta}{2} \quad (3.123)$$

$$u_3 = u \cos \frac{\phi - \psi}{2} \sin \frac{\theta}{2} \quad (3.124)$$

$$u_4 = u \sin \frac{\phi - \psi}{2} \sin \frac{\theta}{2} \quad (3.125)$$

and

$$z_1 = u e^{i \frac{\phi + \psi}{2}} \cos \frac{\theta}{2} \quad (3.126)$$

$$z_2 = u e^{i \frac{\phi - \psi}{2}} \sin \frac{\theta}{2} \quad (3.127)$$

Recall, the Equation 3.120 which can be written as

$$z = 2\frac{r}{u^2}z_1z_2 = 2\frac{r}{u^2}(u_1 + iu_2)(u_3 + iu_4) \quad (3.128)$$

Therefore, from z 's original definition $z = x_1 + ix_2$

$$x_1 = 2\frac{r}{u^2}[u_1u_3 - u_2u_4] \quad (3.129)$$

and

$$x_2 = 2\frac{r}{u^2}[u_1u_4 + u_2u_3] \quad (3.130)$$

Similarly, from 3.121,

$$x = x_3 = r \left[\frac{u_1^2 + u_2^2}{u^2} - \frac{u_3^2 + u_4^2}{u^2} \right] = \frac{r}{u^2}(u_1^2 + u_2^2 - u_3^2 - u_4^2) \quad (3.131)$$

The Equations 3.129, 3.130 and 3.131 give the transformation rules from coordinates in 3D to coordinates in 4D. Finally, note that setting $r = u^2$ simplifies the transformation equation without any contradiction. Therefore, with this choice the transformation rules can also be written as followings.

$$x_1 = 2[u_1u_3 - u_2u_4] \quad (3.132)$$

$$x_2 = 2[u_1u_4 + u_2u_3] \quad (3.133)$$

$$x_3 = (u_1^2 + u_2^2 - u_3^2 - u_4^2) \quad (3.134)$$

$$r = u^2 \quad (3.135)$$

Now using the Equations 3.132 -3.135, 3 dimensional Coulomb-like potential can be mapped onto the 4 dimensional harmonic oscillator. Note that these rules also applies

on the perturbed Coulomb potential. In these case , while the unperturbed part is mapped to harmonic oscillator, which will be the unperturbed potential in 4 dimension, the perturbation yields to some anharmonicity, which will be treated as perturbations.

In the next section, the hamiltonian in the Equation 3.86 with an addition of perturbation potential will be transformed to the anharmonic oscillator in 4 dimensions.

3.6. Duality Between the Dyon Atom and the Harmonic Oscillator

In this section, the Schrödinger equation of Dyon-Dyon interaction, which also includes a general perturbation potential, will be transformed to the Schrödinger equation of anharmonic oscillator in 4D step by step. The motivation for the transformation and necessary relations between coordinates of Dyon-Dyon system and anharmonic oscillator was given in the last chapter. The discussion in this section is based on the paper in references [12]. In addition to those papers, here the transformation of perturbed Hamiltonian is considered but for simplicity only radial perturbations will be taken into account. Since during the transformations, the parameter, such as energy and coupling constant, in the Schrödinger equation of the Dyon atom is also transformed to some other parameters in the equation of anharmonic oscillator, the energy of the anharmonic oscillator will be given as a function of the parameters of the Dyon atom.

$$\varepsilon = \varepsilon(\sqrt{-8E}, \beta/16) \quad (3.136)$$

where E is the energy of the Dyon atom and β coupling constant in the additional radial potential. With this relation , re-mapping the energy corrections of anharmonic oscillator, which will be found in the next chapter, would be possible.

The Schrödinger equation with a radial perturbation term is

$$\left\{ \frac{1}{2m} \left[(\mathbf{p} - \mu \mathbf{D})^2 + \frac{\mu^2}{r^2} \right] - \frac{\alpha}{r} + V(r) \right\} \psi'(\mathbf{r}) = E \psi'(\mathbf{r}) \quad (3.137)$$

where

$$\mathbf{D} = \frac{-\hat{n} \times \mathbf{r}}{r(r + \mathbf{r} \cdot \hat{n})}$$

Also, recall that $\frac{\mu^2}{r^2}$ was added to the unperturbed Hamiltonian in order to preserve the $SO(4)$ symmetry of the potential.

Using the definition of momentum operator, $\mathbf{p} = -i\nabla$, kinetic term can be written as

$$(-i\nabla - \mu\mathbf{D})^2 = -(\nabla - i\mu\mathbf{D})^2 \quad (3.138)$$

Furthermore, choosing the Dirac singularity along the positive z -axis, vector $\mathbf{D}$ becomes

$$\mathbf{D} = \frac{1}{r(r + x_3)} (x_2, -x_1, 0)$$

Then, by this choice and ignoring the minus sign the Equation 3.138 can be expanded as

$$\begin{aligned} (\nabla - i\mu\mathbf{D})^2 &= (\partial_j - i\mu\mathbf{D}_j)(\partial_j - i\mu\mathbf{D}_j) \\ &= \left(\partial_1 - i\mu\frac{x_2}{r(r+x_3)}\right)^2 + \left(\partial_2 + i\mu\frac{x_1}{r(r+x_3)}\right)^2 + \partial_3^2 \\ &= \partial_1^2 + \partial_2^2 + \partial_3^2 - i\mu\frac{\partial}{\partial x_1}\frac{x_2}{r(r+x_3)} - i\mu\frac{x_2}{r(r+x_3)}\frac{\partial}{\partial x_1} \\ &\quad + i\mu\frac{\partial}{\partial x_2}\frac{x_1}{r(r+x_3)} + i\mu\frac{x_1}{r(r+x_3)}\frac{\partial}{\partial x_2} + \left(i\mu\frac{x_2}{r(r+x_3)}\right)^2 + \left(i\mu\frac{x_1}{r(r+x_3)}\right)^2 \end{aligned}$$

Combination of forth and sixth terms gives the divergence of $\mathbf{D}$, which is zero by its definition

$$\frac{\partial}{\partial x_1}\frac{x_2}{r(r+x_3)} - \frac{\partial}{\partial x_2}\frac{x_1}{r(r+x_3)} = \nabla \cdot \mathbf{D} = 0$$

Furthermore,

$$\nabla \cdot \mathbf{D}\psi = (\nabla \cdot \mathbf{D})\psi + \mathbf{D} \cdot \nabla\psi = \mathbf{D} \cdot \nabla\psi$$

Since third and fifth terms in the expansion correspond to $\mathbf{D} \cdot \nabla$, it is convenient to write

$$(\nabla - i\mu\mathbf{D})^2 = \nabla^2 - \frac{\mu^2}{r^2(r+x_3)^2}(x_1^2 + x_2^2) + \frac{2i\mu}{r(r+x_3)}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)$$

This equation can also be written in spherical coordinates,

$$\begin{aligned} (\nabla - i\mu\mathbf{D}) &= \nabla^2 - \frac{\mu^2(1 - \cos\theta)}{r^2(1 + \cos\theta)} + \frac{2i\mu}{r^2(1 + \cos\theta)}\frac{\partial}{\partial\phi} \\ &= \nabla^2 + \frac{\mu^2}{r^2} - \frac{2\mu^2}{r^2(1 + \cos\theta)} + \frac{2i\mu}{r^2(1 + \cos\theta)}\frac{\partial}{\partial\phi} \end{aligned}$$

Recalling the minus sign in the Equation 3.138, the term $\frac{\mu^2}{r^2}$ cancels with the same term in the Equation 3.137. Then, the Schrödinger equation becomes

$$\left\{ -\frac{1}{2m} \left[\nabla^2 - \frac{2\mu^2}{r^2(1 + \cos\theta)} + \frac{2i\mu}{r^2(1 + \cos\theta)}\frac{\partial}{\partial\phi} \right] - \frac{\alpha}{r} + V(r) \right\} \psi'(\mathbf{r}) = E\psi'(\mathbf{r}) \quad (3.139)$$

This equation represents the Dyon-Dyon interaction in 3 dimensions. However, μ , which is quantized, can be considered as the eigenvalue of some angular momentum operator. In this case, the Equation 3.139 becomes the result of a separation of variable. This process is very similar to the separation of variable process in the solution of Hydrogen atom. In that case, it yields the radial equation which includes angular momentum quantum numbers. The separation can be reversed and as a result, the full equation that explicitly includes the additional angular momentum operator can be found.

In order to this the eliminated part of the wave function, which is $e^{i\mu(\gamma+\phi)}$ (see [12])

for details), should be added. Multiply the Schrödinger equation by $e^{i\mu(\gamma+\phi)}$

$$e^{i\mu(\gamma+\phi)} \mathbf{H}' \psi'(r, \theta, \phi) = e^{i\mu(\gamma+\phi)} E \psi'(r, \theta, \phi)$$

where $\mathbf{H}'$ is the Hamiltonian in Equation 3.139. Setting, $e^{i\mu(\gamma+\phi)} \psi'(r, \theta, \phi) = \psi(r, \theta, \phi, \gamma)$ yields

$$\mathbf{H} \psi = E \psi$$

where $\mathbf{H}$ is the Hamiltonian which includes the additional angular momentum operators manifestly. Then, recall the explicit form of Laplacian in spherical coordinates,

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

and consider the terms of $\mathbf{H}'$ those involve ϕ and γ ,

$$\begin{aligned} & e^{i\mu(\phi+\gamma)} \left[\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{2i\mu}{r^2 (1 + \cos \theta)} \frac{\partial}{\partial \phi} - \frac{2\mu^2}{r^2 (1 + \cos \theta)} \right] \psi'(r, \theta, \phi) \\ = & e^{i\mu(\phi+\gamma)} \frac{1}{r^2 \sin^2 \theta} \left[\frac{\partial^2}{\partial \phi^2} + 2i\mu (1 - \cos \theta) \frac{\partial}{\partial \phi} - 2\mu^2 (1 - \cos \theta) \right] \psi'(r, \theta, \phi) \\ = & e^{i\mu(\phi+\gamma)} \frac{1}{r^2 \sin^2 \theta} \left[\left(\frac{\partial^2}{\partial \phi^2} + 2i\mu \frac{\partial}{\partial \phi} - \mu^2 \right) - 2 \cos \theta \left(i\mu \frac{\partial}{\partial \phi} - \mu^2 \right) - \mu^2 \right] \psi'(r, \theta, \phi) \\ = & \frac{1}{r^2 \sin^2 \theta} \left[\frac{\partial^2}{\partial \phi^2} - 2 \cos \theta \frac{\partial^2}{\partial \phi \partial \gamma} + \frac{\partial^2}{\partial \gamma^2} \right] e^{i\mu(\phi+\gamma)} \psi'(r, \theta, \phi) = \psi(r, \theta, \phi, \gamma) \end{aligned}$$

Since the other terms does not interfere with $e^{i\mu(\phi+\gamma)}$, the Schrödinger equation can be written as

$$\left[-\frac{1}{2m} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{J^2}{r^2} \right) - \frac{\alpha}{r} + V(r) \right] \psi(r, \theta, \phi, \gamma) = E \psi(r, \theta, \phi, \gamma) \quad (3.140)$$

where

$$J = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left(\frac{\partial^2}{\partial \phi^2} - 2 \cos \theta \frac{\partial^2}{\partial \phi \partial \gamma} + \frac{\partial^2}{\partial \gamma^2} \right)$$

which is the total angular momentum operator in 4 dimensional hyperspherical coordinates.

The rules stated in the last section, Equations 3.132-3.135, can be used to transform the Equation 3.140 to the Schrödinger equation of anharmonic oscillator in 4 dimension. Since the perturbation potential $V(r)$ is chosen to have only radial components, the transformation is done simply by using only $r = u^2$.

Then, the kinetic term in Equation 3.140 transforms as

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{J^2}{r^2} &= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{J^2}{r^2} \\ &= \frac{1}{4u^2} \frac{\partial^2}{\partial u^2} - \frac{1}{4u^3} \frac{\partial}{\partial u} + \frac{1}{u^3} \frac{\partial}{\partial u} - \frac{J^2}{u^4} \\ &= \frac{1}{4u^2} \left(\frac{\partial^2}{\partial u^2} + \frac{3}{u} \frac{\partial}{\partial u} - \frac{4J^2}{u^2} \right) \end{aligned}$$

where the term in the parenthesis is the Laplacian of hyperspherical coordinates in the 4 dimensions. Representing it with the symbol Δ_u , the Schrödinger equation becomes

$$\left[-\frac{1}{2m} \frac{1}{4u^2} \Delta_u - \frac{\alpha}{u^2} + V(u^2) \right] \Psi(u, \theta, \phi, \gamma) = E \Psi(u, \theta, \phi, \gamma) \quad (3.141)$$

and multiplying whole equation by $4u^2$ and arranging it, a more usual form is obtained

$$\left[-\frac{1}{2m} \Delta_u - 4Eu^2 + 4u^2 V(u^2) \right] \Psi(u, \theta, \phi, \gamma) = 4\alpha \Psi(u, \theta, \phi, \gamma) \quad (3.142)$$

Finally, defining new variables as

$$\frac{1}{2} m \omega^2 = -4E \quad (3.143)$$

and

$$\varepsilon = 4\alpha \quad (3.144)$$

The Schrödinger equation for 4 dimensional anharmonic oscillator is achieved

$$\left[-\frac{1}{2m} \Delta_u + \frac{1}{2} m \omega^2 u^2 + 4u^2 V(u^2) \right] \Psi(u, \theta, \phi, \gamma) = \varepsilon \Psi(u, \theta, \phi, \gamma) \quad (3.145)$$

Now, the high order terms of the perturbative expansion can be found by using the instanton methods. Also a mapping to the expansion coefficients of the Dyon system can be done. Because of the relation 3.143, the energy of the Dyon atom can be considered as a parameter of the energy of the anharmonic oscillator.

Furthermore, defining the perturbation potential as a polynomial and considering only the simplest case,

$$V(r) = \beta r$$

where β is a small constant, yields

$$\left[-\frac{1}{2m} \Delta_u + \frac{1}{2} m \omega^2 u^2 + \frac{\lambda}{4} u^4 \right] \Psi(u, \theta, \phi, \gamma) = \varepsilon \Psi(u, \theta, \phi, \gamma) \quad (3.146)$$

where

$$\lambda = 16\beta \quad (3.147)$$

Therefore, λ can be considered as another parameter of the energy ε . Then using the Equations 3.143 and 3.147, ε can be written as a function of E and β as in 3.136

$$\varepsilon(\omega, \lambda) = \varepsilon(\sqrt{-8E}, \beta/16)$$

This equation will be the main relation in mapping to the energy corrections of Dyon-Dyon system after finding them in the anharmonic oscillator case in the next chapter.

4. HIGH ORDERS AND INSTANTONS IN THE DYON ATOM

In this chapter, the large order behaviour of the Dyon atom will be found by a map from the large order terms of the four dimensional anharmonic oscillator. As in section 2.4, the large order terms of one dimensional case are found by the instanton method. In that section, the comparison between large order expansion terms found by instanton methods and terms found by Rayleigh-Schrödinger perturbation theory is also given. In the four dimensional case, there will be an additional zero mode of the system coming from the rotational invariance of the system. Throughout the chapter, only the additional calculations due to the extra dimensions will be discussed in detail, these are based on [4,6,7]. The details of remaining calculations can be found in Section 2.4.

4.1. Instanton Calculations for the 4D Anharmonic Oscillator

After the transformation in the last section, with the appropriate choices of the constants λ and ω , the potential of the 4 dimensional isotropic anharmonic was written as

$$V(u) = \frac{1}{2}m\omega^2 u^2 + \frac{\lambda}{4}u^4 \quad (4.1)$$

Recall the dispersion relation in Equation 2.7

$$\varepsilon_k = \frac{1}{\pi} \int_{-\infty}^0 d\lambda' \frac{\text{Im}\varepsilon(\lambda')}{\lambda'^{k+1}} \quad (4.2)$$

which is the tool of the estimation of the expansion terms using the imaginary part of the energy.

In order to calculate the energy, recall the partition function in Equation 2.34,

$$Z(\beta) = \mathcal{N} \int_{\mathbf{u}(-\beta/2)=\mathbf{u}(\beta/2)} \mathcal{D}\mathbf{u} e^{-S_E(\mathbf{u})} \quad (4.3)$$

where $\mathcal{N} = Z_0^{-1}$ and Z_0 is the partition function of the simple harmonic oscillator. Then, setting $m = 1$ and $\omega = 1$ for simplicity in the calculations, for the potential in the Equation 4.1, the Euclidean action is written as

$$S_E(\mathbf{u}) = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{u}^2 + \frac{1}{2} u^2 + \frac{\lambda}{4} u^4 \right) \quad (4.4)$$

Expanding the action $S_E(\mathbf{u})$ around the classical path, $\mathbf{u} = \mathbf{u}_c + \mathbf{y}$ gives

$$Z(\beta) = \mathcal{N} e^{-S_c(\mathbf{u}_c)} \int_{\mathbf{y}(-\beta/2)=\mathbf{y}(\beta/2)} \mathcal{D}\mathbf{y} e^{-\Sigma(\mathbf{y})} \quad (4.5)$$

where

$$S_c(\mathbf{u}_c) = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{u}_c^2 + \frac{1}{2} u_c^2 + \frac{\lambda}{4} u_c^4 \right)$$

Even though the classical action has the same form with the one dimensional case, here $S_c(\mathbf{u}_c)$ has an $O(4)$ symmetry. Any spontaneous choice of the classical path direction breaks this symmetry. The effects of this additional symmetry breaking will be discussed later. First define the classical path as

$$\mathbf{u}_c = l \hat{u}$$

where $\hat{u}$ is a unit vector in 4 dimensions which determines the direction. Since the value of classical action is not affected by the spontaneous choice of this direction, for now leaving the direction undetermined, $S_c(\mathbf{u}_c)$ is written as

$$S_c(\mathbf{u}_c) = \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{l}^2 + \frac{1}{2} l^2 + \frac{\lambda}{4} l^4 \right) \quad (4.6)$$

Now it is exactly same with the one dimensional action. Therefore, copying the results from the Appendix 2.4, the classical path is

$$l(\tau) = \pm \left(-\frac{2}{\lambda} \right)^{1/2} \frac{1}{\cosh(\tau - \tau_0)} \quad (4.7)$$

and the value of classical action is

$$S_c(\mathbf{u}_c) = -\frac{4}{3\lambda} \quad (4.8)$$

The important difference appears in the rest of the expanded action. An expansion up to second order yields

$$\Sigma(\mathbf{y}) = \int_{-\beta/2}^{\beta/2} d\tau \left(\dot{\mathbf{u}}_c \dot{\mathbf{y}} + y^i \frac{\partial V(\mathbf{u})}{\partial u^i} \Big|_{\mathbf{u}=\mathbf{u}_c} + \frac{1}{2} m \dot{\mathbf{y}}^2 + \frac{1}{2} y^i y^j \frac{\partial^2 V(\mathbf{u})}{\partial u^i \partial u^j} \Big|_{\mathbf{u}=\mathbf{u}_c} \right) \quad (4.9)$$

where

$$\frac{\partial^2 V(\mathbf{u})}{\partial u^i \partial u^j} = \delta_{ij} + \lambda u^i u^j + \frac{\lambda}{2} u^2 \delta_{ij}$$

After integrating by parts the first two terms vanish as in the one dimensional case. Then,

$$\begin{aligned} \Sigma(\mathbf{u}) &= \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2} \dot{\mathbf{y}}^2 + \frac{1}{2} \left[\delta_{ij} + \lambda u_c^i u_c^j + \frac{\lambda}{2} u_c^2 \delta_{ij} \right] y^i y^j \right) \\ &= \int_{-\beta/2}^{\beta/2} d\tau y^i \left(-\frac{1}{2} \frac{\partial^2}{\partial \tau^2} \delta_{ij} + \frac{1}{2} \delta_{ij} + \frac{\lambda}{2} u_c^2 \delta_{ij} + \lambda u_c^i u_c^j \right) y^j \end{aligned}$$

In this term, the direction of the classical path is important because of the terms involving u_c^i . For simplicity, the classical path is chosen as

$$\mathbf{u}_c = (l, 0, 0, 0)$$

This choice yields

$$\begin{aligned}\Sigma(\mathbf{y}) &= \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau y^1 \left(-\frac{\partial^2}{\partial \tau^2} + 1 + 3\lambda l^2 \right) y^1 \\ &+ \frac{1}{2} \int_{-\beta/2}^{\beta/2} d\tau \mathbf{z} \left(-\frac{\partial^2}{\partial \tau^2} + 1 + \lambda l^2 \right) \mathbf{z}\end{aligned}\quad (4.10)$$

where $\mathbf{z}$ is a 3 dimensional vector, which corresponds to the dimensions other than the direction of classical path. . Then, the partition function becomes

$$Z(\beta) = \mathcal{N} e^{-S_c} \int \mathcal{D}y^1 e^{-\int d\tau y^1 M y^1} \int \mathcal{D}\mathbf{z} e^{-\int d\tau \mathbf{z} T \mathbf{z}} \quad (4.11)$$

$$= \mathcal{N} e^{-S_c} (\det M)^{-1/2} (\det T)^{-3/2} \quad (4.12)$$

where

$$M = -\frac{\partial^2}{\partial \tau^2} + 1 + 3\lambda l^2$$

and

$$T = -\frac{\partial^2}{\partial \tau^2} + 1 + \lambda l^2$$

As in the one dimensional case, these determinants have zero-modes; therefore, without normalization they blow up.

Zero-mode of M is the same with the one dimensional case. It is due to the time translational invariance due to the arbitrary integration constant, τ_0 , of classical path u_c . On the other hand, the zero-mode of the operator T is an additional one. Since all the zero-modes occur as a consequence of a symmetry of the system, this second zero mode is due to the rotational symmetry of the system. Or in other words it is due to the freedom of choosing the direction of the classical path. The eigenvector associated with the zero eigenvalue for operator T is $\mathbf{u}_c(\tau)$. This can be simply proven

by checking the equation

$$\begin{aligned} \left(\frac{\partial^2}{\partial \tau^2} + 1 + \lambda u_c^2 \right) \mathbf{u}_c &= -\ddot{\mathbf{u}}_c + \mathbf{u}_c + \lambda \mathbf{u}_c^3 \\ &= -\ddot{\mathbf{u}}_c + \frac{\partial V(\mathbf{u})}{\partial u} \Big|_{\mathbf{u}=\mathbf{u}_c} = 0 \end{aligned}$$

since the last equation is the classical equation of motion in imaginary time.

As in the one dimensional case, applying Fadeev-Popov technique, the operators involving the appropriate collective coordinates M_γ and T_γ can be written and the divergences due to both of the zero-modes can be eliminated.

Recall, $\mathcal{N} = Z_0^{-1}$ where

$$\begin{aligned} Z_0 &= \int \mathcal{D}\mathbf{u} e^{-S^{(0)}} = e^{-S_c^{(0)}} \int \mathcal{D}y^1 e^{-\Sigma(y^1)} \int \mathcal{D}\mathbf{z} e^{-\Sigma(\mathbf{z})} \\ &= e^{-S_c^{(0)}} (\det M_0)^{-1/2} (\det T_0)^{-3/2} \end{aligned} \quad (4.13)$$

where

$$S_c^{(0)} = \frac{\beta}{2}$$

is the classical action of simple harmonic oscillator. Then, referring to the Equation 2.57, the partition function becomes

$$Z(\beta) = 2\pi^2 \frac{\beta}{\sqrt{2\pi\gamma}} e^{-S_c} e^{S_c^{(0)}} \|u_c\|^2 \left(\frac{\det M_\gamma}{\det M_0} \right)^{-1/2} \left[\|u_c\|^2 \left(\frac{\det T_\gamma}{\det T_0} \right)^{-1/2} \right]^3 \quad (4.14)$$

where $2\pi^2$ comes from the surface integral in 4 dimensional space and

$$\|u_c\|^2 = \int d\tau u_c^2$$

comes from the collective coordinates introduced to cure the divergences due to the rotational invariance.

Since the first determinant ratio corresponds to the 1 dimensional path integral, referring to the Section 2.4, the ratio of determinant can be written as

$$\frac{\det M_\gamma}{\det M_0} = \det' M M_0^{-1} \eta \quad (4.15)$$

where

$$\eta = \frac{\|\dot{u}_c\|^2}{\gamma}$$

is the eigenvalue corresponding to the zero mode and by definition

$$\det' M M_0^{-1} = \lim_{\epsilon \rightarrow 0} \left[-\frac{1}{\epsilon} \frac{\det(M - \epsilon)}{\det(M_0 - \epsilon)} \right]$$

Recalling the Equation 2.62, the ratio in the last equation is given in the most general form as

$$\frac{\det(M - \epsilon)}{\det(M_0 - \epsilon)} = \frac{\Gamma(1 - z)\Gamma(z)}{\Gamma(1 + \kappa + z)\Gamma(z - \kappa)} \quad (4.16)$$

where

$$z = \sqrt{1 - \epsilon}$$

and

$$\kappa = \frac{N}{N - 1}$$

For the quartic anharmonic oscillator $N = 2$. Then, in the leading order,

$$\det' M M_0^{-1} = -\frac{1}{12} \quad (4.17)$$

Similarly

$$\frac{\det T_\gamma}{\det T_0} = \det' T T_0^{-1} \nu$$

where

$$\nu = \| u_c \|$$

is the normalized zero-mode eigenvalue and

$$\det' T T_0^{-1} = \lim_{\epsilon \rightarrow 0} \left[-\frac{1}{\epsilon} \frac{\det(T - \epsilon)}{\det(T_0 - \epsilon)} \right] = \lim_{\epsilon \rightarrow 0} \left[-\frac{1}{\epsilon} \frac{\Gamma(1 - z)\Gamma(z)}{\Gamma(1 + \kappa + z)\Gamma(z - \kappa)} \right]$$

where

$$\kappa = \frac{1}{N - 1}$$

this time. In the leading order,

$$\det' T T_0^{-1} = \frac{1}{2} \quad (4.18)$$

Putting all these results, the partition function becomes

$$Z(\beta) = \sqrt{2\pi^3} \| \dot{u}_c \| \| u_c \|^3 e^{4/3\lambda} e^{-\beta/2} \left(-\frac{1}{12} \right)^{-1/2} \left(\frac{1}{2} \right)^{-3/2} \quad (4.19)$$

Recall

$$u_c = \pm \left(-\frac{2}{\lambda} \right)^{1/2} \frac{1}{\cosh(\tau - \tau_0)}$$

Then,

$$\| u_c \| = \left(\int d\tau u_c^2 \right)^{1/2} = \frac{2}{\sqrt{-\lambda}}$$

and

$$\| \dot{u}_c \| = \left(\int d\tau \dot{u}_c^2 \right)^{1/2} = \frac{2}{\sqrt{-3\lambda}}$$

Then,

$$\begin{aligned} Z(\beta) &= \sqrt{2\pi^3} \beta e^{-\beta/2} e^{4/3\lambda} \frac{16}{3\sqrt{3}} \frac{1}{\lambda^2} \frac{4\sqrt{2}\sqrt{3}}{i} \\ &= \pi^{3/2} \frac{256}{3i} \beta \frac{e^{-\beta/2} e^{4/3\lambda}}{\lambda^2} \end{aligned}$$

As it is explained in the Section 2.4, the imaginary part of the ground state energy eigenvalue is found as

$$\text{Im}\varepsilon_0 = \lim_{\beta \rightarrow \infty} -\frac{1}{\beta} e^{-\beta\varepsilon_r} \text{Im}Z(\beta)$$

where $\text{Im}Z$ is determined by the difference between the negative coupling results obtained by clockwise and counter-clockwise path rotations as stated in 2.38

$$2i \text{Im}Z(\beta) = \int_{q_+(-\frac{\beta}{2})=q_+(\frac{\beta}{2})} \mathcal{D}q e^{-S(q)} - \int_{q_-(-\frac{\beta}{2})=q_-(\frac{\beta}{2})} \mathcal{D}q e^{-S(q)}$$

where q_+ and q_- are the paths corresponding to counter-clockwise and clockwise rotations respectively. Since, as in the one dimensional case, the contribution coming from these rotations are complex conjugate,

$$\text{Im}Z(\beta) = -\frac{128}{3} \pi^{3/2} \beta \frac{e^{-\beta/2} e^{4/3\lambda}}{\lambda^2}$$

Then, since the real part of the ground state is the same with the ground state energy of the harmonic oscillator

$$\varepsilon_r = \frac{1}{2}$$

the imaginary part of the ground state energy is

$$\text{Im}\varepsilon_0 = \frac{128}{3}\pi^{3/2}\frac{e^{4/3\lambda}}{\lambda^2} \quad (4.20)$$

Recall the dispersion relation 4.2

$$\varepsilon_k = \frac{128}{3}\pi^{1/2} \int_{-\infty}^0 d\lambda \frac{e^{4/3\lambda}}{\lambda^{k+3}} \quad (4.21)$$

$$= -\frac{128}{3}\pi^{1/2}(-1)^{-2-k} \left(\frac{3}{4}\right)^{2+k} \Gamma(k+2) \quad (4.22)$$

$$= -96\pi^{1/2} \left(-\frac{3}{4}\right)^k \Gamma(k+2) \quad (4.23)$$

Recall the parameter ω , which is initially set to 1. In Equation 4.19, $\|u_c\|$ and $\|\dot{u}_c\|$ contain ω parameter. If it is added

$$\|u_c\| = \frac{2}{\sqrt{-\omega\lambda}}$$

and

$$\|\dot{u}_c\| = \frac{2\omega}{\sqrt{-3\lambda}}$$

then

$$\|u_c\| \|\dot{u}_c\|^3 \sim \omega^{5/2}$$

and Equation 4.23 becomes

$$\varepsilon_k = -96 \omega^{5/2} \pi^{-3/2} \left(-\frac{3}{4}\right)^k \Gamma(k+2) \quad (4.24)$$

4.2. The Results for the Dyon Atom

Now the energy corrections of anharmonic oscillator will be mapped to energy corrections of the Dyon atom [6, 7]. Recall Equation 3.136 and set it to a fix number,

$$\varepsilon(\omega, \lambda) = \varepsilon(\sqrt{-8E}, \beta/16) = 1 \quad (4.25)$$

and scale ε as

$$\varepsilon(\omega, \lambda) = \omega \tilde{\varepsilon}(\tilde{\lambda})$$

where

$$\tilde{\lambda} = \frac{\lambda}{\omega^4}$$

Then from Equation 4.25,

$$\omega = \frac{1}{\tilde{\varepsilon}(\tilde{\lambda})}$$

Writing $\varepsilon(\omega, \lambda)$ as a perturbation series

$$\varepsilon = \sum \varepsilon_k \lambda^k = \omega \sum \varepsilon_k \tilde{\lambda}^k = 1$$

yields

$$\omega = \frac{1}{\sum \varepsilon_k \tilde{\lambda}^k} = \frac{1}{\varepsilon_0} \left(1 + a_1 \tilde{\lambda} + a_2 \tilde{\lambda}^2 + \dots\right) \quad (4.26)$$

In the leading order

$$\omega = \frac{1}{\varepsilon_0}$$

In the quartic anharmonic oscillator case, $\varepsilon_0 = 2$ is the ground state energy of the four dimensional harmonic oscillator.

The transformation $\lambda = 16\beta$ changes the Equation 4.23 to

$$-96\pi^{1/2} \left(-\frac{3 \cdot 16}{4} \right)^k \Gamma(k+2) = -96\pi^{1/2} (-12)^k \Gamma(k+2)$$

Multiplying this with $\omega^{5/2} = \varepsilon_0^{-5/2} = 32^{-5/2}$ results in the perturbative expansion coefficients of the Dyon atom in the presence of a linear radial perturbation as

$$E_k = -4\sqrt{3}\pi^{-3/2} (-12)^{k+2} \Gamma(k+2) \quad (4.27)$$

Therefore, the large order terms of the perturbative expansion diverge factorially as expected, since E_k is dominated by the Γ function for $k \gg 1$.

APPENDIX A: ANALYTICITY OF THE ANHARMONIC OSCILLATOR

A.1. A Rigorous Proof

Consider Hamiltonian for a general anharmonic oscillator

$$H = p^2 + mx^2 + \lambda x^\alpha \tag{A.1}$$

where m is defined positive and λ is perturbation variable.

In the main text, it is discussed in the case of quartic oscillator, $\alpha = 4$, that the perturbation series is divergent because of the behaviour of the potential near infinities. Furthermore, it is explained, again for $\alpha = 4$ case, that the divergent behaviour is due to a branch cut along the negative real axis on the complex λ -plane. In this part a mathematical proof for the existence of this branch cut is given by following the arguments of Barry Simon in his paper on analyticity of anharmonic oscillator [19]. Here, α is taken general and first, it is shown that the singularity at the origin of λ -plane exists, whenever $\alpha > 2$. Then, the existence of the branch cut is shown the cases when $\alpha > 2$ and an even number.

A.1.1. Existence of Singularity at $\lambda = 0$

Consider a unitary operator $U(\beta)$ such that

$$U(\beta)f(x) = \beta^m f(\beta^n x)$$

and the action of inverse operator is

$$U^{-1}(\beta)f(x) = \beta^{-m}f(\beta^{-n}x)$$

Then,

$$\begin{aligned} U(\beta)x^kU^{-1}(\beta)f(x) &= U(\beta) \left(\beta^{-m}x^k f(\beta^{-n}x) \right) \\ &= \beta^{-m}\beta^m\beta^{n\cdot k}x^kf(x) \end{aligned}$$

and

$$U(\beta)x^kU^{-1}(\beta) = \beta^{n\cdot k}x^k \quad (\text{A.2})$$

In addition, using the relation $p \sim \frac{d}{dx}$

$$U(\beta)p^kU^{-1}(\beta) = \beta^{-n\cdot k}p^k \quad (\text{A.3})$$

Now consider unitary transformation of the Hamiltonian given in A.1

$$\begin{aligned} H' &= U(\beta)HU^{-1}(\beta) \\ &= U(\beta) \left(p^2 + mx^2 + \lambda x^\alpha \right) U^{-1}(\beta) \\ &= \beta^{-2n}p^2 + m\beta^{2n}x^2 + \lambda\beta^{\alpha\cdot n}x^\alpha \\ &= \beta^{-2n} \left(p^2 + m\beta^{4n}x^2 + \lambda\beta^{(\alpha+2)b}x^\alpha \right) \end{aligned}$$

The Schrödinger equations for H and H' can be written as

$$H |\Psi\rangle_i = E_i(m, \lambda, \alpha) |\Psi\rangle_i$$

and

$$H' |\Psi'\rangle_i = \beta^{-2n} E_i(m\beta^{4n}, \lambda\beta^{(\alpha+2)n}, \alpha) |\Psi'\rangle_i$$

Since H and H' are related with a unitary transformation, their corresponding energy eigenvalues should be the same.

$$\beta^{-2n} E_i(m\beta^{4n}, \lambda\beta^{(\alpha+2)n}) = E_i(m, \lambda)$$

Choosing $\beta = \lambda^{\frac{-1}{n(\alpha+2)}}$ simplifies it to

$$\lambda^{\frac{2}{\alpha+2}} E_i(m\lambda^{\frac{-1}{\alpha+2}}, 1) = E_i(m, \lambda) \quad (\text{A.4})$$

For simplicity set $m = 1$ and define a new variable $\gamma \equiv \lambda^{\frac{-1}{\alpha+2}}$. Then Equation A.4 becomes

$$\gamma^{-2} E_i(\gamma^4, 1) = E_i(1, \gamma^{-(\alpha+2)}) \quad (\text{A.5})$$

and multiplying both sides by γ^4 gives

$$\gamma^2 E_i(\gamma^4, 1) = \gamma^4 E_i(1, \gamma^{-(\alpha+2)}) \quad (\text{A.6})$$

Since E_i represents the i^{th} order corrections for the energy eigenvalues, it is given as a polynomial of γ . Therefore, by defining a new function

$$f_i(\gamma) \equiv \gamma^2 E_i(\gamma^4, 1)$$

it is straightforward to deduce that $f_i(\gamma)$ is analytic along the real γ axis, except infinities. The analyticity of $f_i(\gamma)$ as $\gamma \rightarrow \infty$ should be checked separately.

Suppose, $f_i(\gamma)$, so that $\gamma^4 E_i(1, \gamma^{-(\alpha+2)})$, is analytic near $\gamma = \infty$ as well. This means considering an integral on a closed contour should give 0.

$$\oint_C f_i(\gamma) d\gamma = 0$$

Note that the domain of the integral path on the complex plane is related to the value of α . If $\alpha \leq 2$, then the maximum angle between the path and positive real axis of γ is determined by the pre-factor γ^4 . It would be $\frac{\pi}{2}$ in this case and the corresponding contour is given in Figure A.1 where the path C_2 is a part of a circle at infinity.

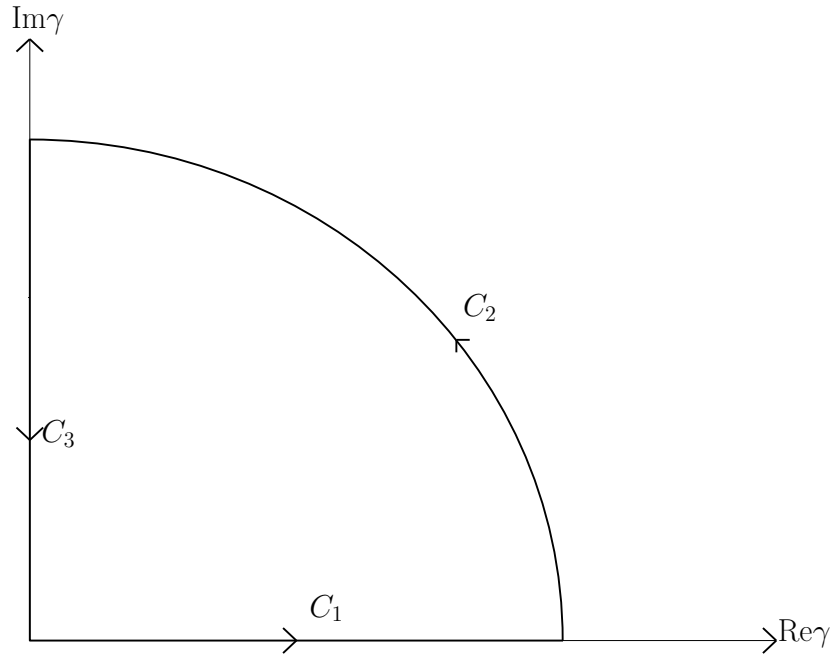


Figure A.1. Contour for $\alpha \leq 2$.

Then, the contour integral becomes

$$\begin{aligned} \oint_C f_i(\gamma) d\gamma &= \int_{C_1} f_i(\gamma) d\gamma + \int_{C_2} f_i(\gamma) d\gamma + \int_{C_3} f_i(\gamma) d\gamma \\ &= \int_0^\infty f_i(\gamma) d\gamma + \int_{-\infty}^0 f_i(\gamma e^{\pi i/2}) d\gamma \\ &= \int_0^\infty [f_i(\gamma) - f(\gamma e^{\pi i/2})] d\gamma = 0 \end{aligned}$$

where the integral on the path C_2 in the first line is zero since $f(\gamma)$ is assumed to be analytic near infinities. In the last line, the integrand should be zero. Therefore,

$$f(\gamma) = f(\gamma e^{\pi i/2}) \quad (\text{A.7})$$

Recalling the definition $f(\gamma) = \gamma^2 E_i(\gamma^4, 1)$ leads to

$$E_i(\gamma^4, 1) = e^{\pi i} E_i(\gamma^4 e^{2\pi i}, 1) = -E_i(\gamma^4, 1) \quad (\text{A.8})$$

On the other hand, if $\alpha > 2$, the maximal angle is determined by $\gamma^{-(\alpha-2)}$. In this case, it would be $\frac{2\pi}{\alpha+2}$, which is always less than $\frac{\pi}{2}$. The contour of the case for $\alpha > 2$ is given in the Figure A.2.

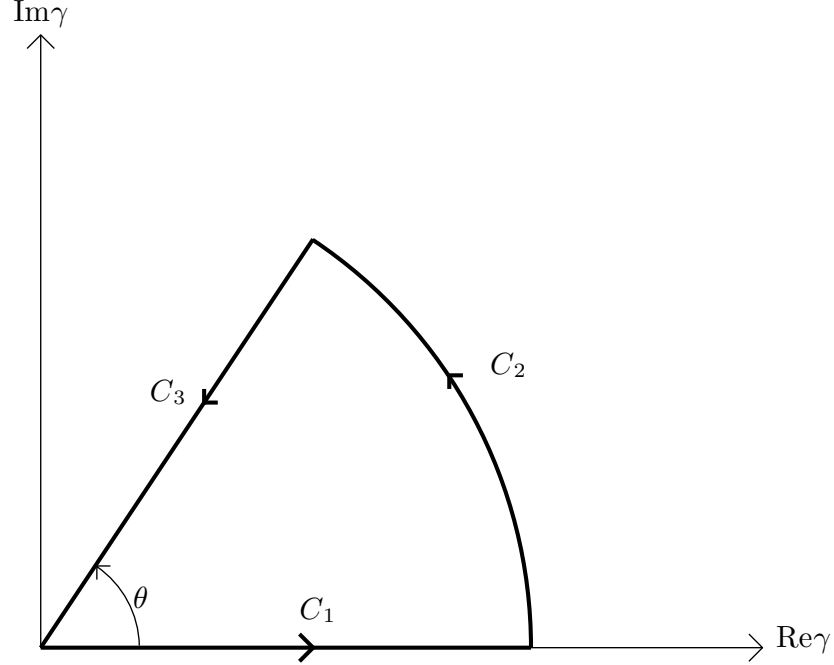


Figure A.2. Contour for $\alpha > 2$, where $\theta = \frac{2\pi}{\alpha+2}$.

Very similar to the previous one, the integral becomes

$$\begin{aligned}\oint_C f_i(\gamma) d\gamma &= \int_0^\infty f_i(\gamma) d\gamma + \int_{-\infty}^0 f_i(\gamma e^{2\pi i/(\alpha+2)}) d\gamma \\ &= \int_0^\infty [f_i(\gamma) - f(\gamma e^{2\pi i/(2+\alpha)})] d\gamma = 0\end{aligned}$$

Therefore,

$$E_i(\gamma^4, 1) = e^{4\pi i/(\alpha+2)} E_i(\gamma^4 e^{8\pi i/(\alpha+2)}, 1) \quad (\text{A.9})$$

Different than the Equation A.8, the right hand side of the Equation A.9 has always an imaginary part for any $\alpha > 2$. This situation leads a contradiction since energy eigenvalues of a hamiltonian, which is a self-adjoint operator, should be real. Therefore, this imaginary part should be eliminated in some way, which is only possible, if the integral along the C_2 line is not zero. Therefore, the function $f(\gamma)$, so that $E_i(1, \gamma^{-(\alpha+2)})$, can not be analytic at $\gamma = \infty$. Recalling the relation $\gamma = \lambda^{\frac{-1}{\alpha+2}}$, it is concluded that $E_i(1, \lambda)$ is not analytic at $\lambda = 0$ for $\alpha > 2$. Note that since the Equation A.8 does not lead to a contradiction, the initial assumption is still valid and $E_i(1, \lambda)$ is analytic at $\lambda = 0$ for $\alpha \leq 2$. These results match with the previous discussions on the asymptotic behaviour of the potential $V(x) = x^2 + \lambda x^\alpha$.

A.1.2. Branch Cut on Complex λ Plane

Now, the nature of the singularity at $\lambda = 0$ for $\alpha > 2$ cases can be examined. In order to do that first Herglotz functions will be introduced [20]. By definition, for a function $g(z)$ if one of the followings is true

$$\begin{cases} \text{Im}g(z) > 0 & \text{when } \text{Im}z > 0 \\ \text{Im}g(z) = 0 & \text{when } \text{Im}z = 0 \\ \text{Im}g(z) < 0 & \text{when } \text{Im}z < 0 \end{cases}$$

$g(z)$ is called Herglotz function. A simple deduction leads to if $g(z)$ is an entire and Herglotz function, it has on the highest order a linear term. The importance of the Herglotz functions in the search of the nature of the singularity comes from the following theorem: Consider, a hamiltonian $H(x) = H_0(x) + \gamma V'(x)$. If the perturbation potential $V'(x)$ is one-signed and perturbation around a bound state is regular, then the perturbed energy eigenvalue $E(\gamma)$ should be a Herglotz function [20].

Now suppose the singularity at $\gamma = \infty$, found previously, is an isolated one. Then, $f_i(\gamma)$, $E_i(\gamma^4, 1)$, $E_i(1, \gamma^{-(\alpha+2)})$ are all entire functions and the perturbation which gives energy eigenvalues $E_i(\gamma^4, 1)$ is a regular one. In addition, if the potential is an one-signed function, which indicates α is an even number, $E_i(\gamma^4, 1)$ should be a Herglotz linear function.

From now on, only the potentials which has even power in their anharmonic part will be taken into account. Therefore, to avoid confusion another variable m , defined as $\alpha = 2m$, where $m \in \mathbb{N}^+$, is introduced⁴.

The Laurent expansion of a complex function $E_i(\gamma, 1)$ near $\gamma = \infty$ is given as

$$E_i(\gamma, 1) = \sum_{l=-\infty}^{\infty} \varepsilon_l \gamma^l$$

Then Laurent expansion of Herglotz linear $E_i(\gamma^4, 1)$ becomes

$$E_i(\gamma, 1) = \sum_{l=-\infty}^1 \varepsilon_l \gamma^{4l}$$

Since $\gamma^2 E_i(\gamma^4, 1) = \gamma^4 E_i(1, \gamma^{-2(m+1)})$,

$$E_i(1, \gamma^{-2(m+1)}) = \sum_{l=-\infty}^1 \varepsilon_l \gamma^{4l-2}$$

⁴Note that in the cases of α is an odd number, potentials do not have lower bounds. As a result of this property, they do not have proper bound states. Therefore, they are not the particular interest of this thesis.

Defining, $\delta = 1/\gamma = \lambda^{\frac{1}{2m+1}}$,

$$E_i(1, \delta^{2(m+1)}) = \sum_{l=-\infty}^1 \varepsilon_l \delta^{2-4l} = \sum_{l=-1}^{\infty} \varepsilon_{-l} \delta^{2+4l} \quad (\text{A.10})$$

The last series is convergent for small enough δ but not for $\delta = 0$ since there is a singularity for the $l = -1$ term in this case.

In perturbation theory, an eigenvalue $E_i(1, \lambda)$ can be written as an asymptotic series

$$E_i(1, \lambda) = \sum_{k=0}^{\infty} E_k \lambda^k \quad (\text{A.11})$$

This series is convergent for small λ , only if $E_i(1, \lambda)$ is analytic around $\lambda = 0$. In Section A.1.1, however, it is shown that for the anharmonic oscillator $E_i(1, \lambda)$ is not analytic near $\lambda = 0$, thus the summation in Equation A.11 can not be convergent. As a result, since $\delta^{2(m+1)} = \lambda$, the expansion in Equation A.10 can not be convergent for small but not zero δ either. Because of this contradiction, $\gamma = \infty$ can not be an isolated singularity. Therefore, for $E_i(1, \lambda)$, $\lambda = 0$ is not an isolated singularity but a branch point.

A.2. Proof via Analytical Continuation

This appendix, which is based on [18], the existence of the branch cut on λ -plane will be shown by using analytical continuation. The methods used here are useful in the calculations in Sections 2.3.3 and 2.4

Consider the integral

$$I(\lambda) = \int_0^{\infty} e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} dx = \int_0^{\infty} e^{-\frac{x^2}{2} - \lambda \frac{x^4}{4}} dx \quad (\text{A.12})$$

Then, extend the domain of x to complex plane and define a new integral such as

$$I(\lambda) = \int_{(0)} e^{-\frac{x^2}{2} - \lambda \frac{x^4}{4}} dx \quad (\text{A.13})$$

where (0) means that the integral is taken along the line $\arg x \equiv \theta_x = 0$ from $x = 0$ to $x = \infty$. Since the integral should be finite on this line, $\arg \lambda \equiv \theta_\lambda$ is restricted in the region

$$|\theta_\lambda| \leq \frac{\pi}{2} \quad (\text{A.14})$$

In addition to that the rotation on the x -plane is restricted by the term $e^{-\frac{x^2}{2}}$. In order

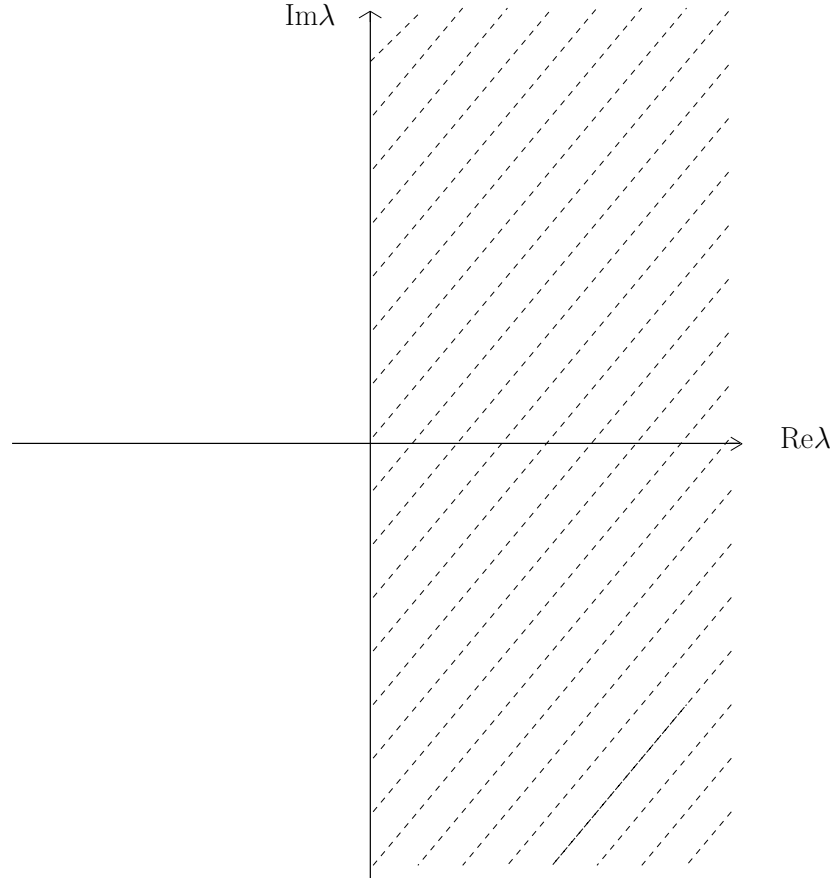


Figure A.3. Original domain.

to satisfy the condition

$$\operatorname{Re}(x^2) = |x| > 0$$

the restriction should be

$$|2\theta_x| \leq \frac{\pi}{2}$$

which results in

$$-\frac{\pi}{4} \leq \theta_x \leq \frac{\pi}{4} \quad (\text{A.15})$$

Also, the term $e^{-\lambda \frac{x^4}{4}}$ impose the condition

$$-\frac{\pi}{2} \leq 4\theta_x + \theta_\lambda \leq \frac{\pi}{2} \quad (\text{A.16})$$

The integration line, which is initially along the positive real axis on x -plane, can be rotated as long as it stays in the region stated in Equation A.15.

First consider the rotation from $\theta_x = 0$ to the angle $\theta'_x = \frac{\pi}{4} - \frac{\xi}{4}$, where $0 \leq \xi \leq \pi$. Then, Equation A.16 becomes

$$-\frac{3\pi}{2} + \xi \leq \theta_\lambda \leq -\frac{\pi}{2} + \xi \quad (\text{A.17})$$

The relevant regions for the Equations A.14 and A.17 are shown in the Figures A.3 and A.4. It is obvious that the intersection region is

$$-\frac{\pi}{2} \leq \theta_\lambda \leq -\frac{\pi}{2} + \xi \quad (\text{A.18})$$

Now, consider the path in the Figure A.5. Since inside the region A there is no

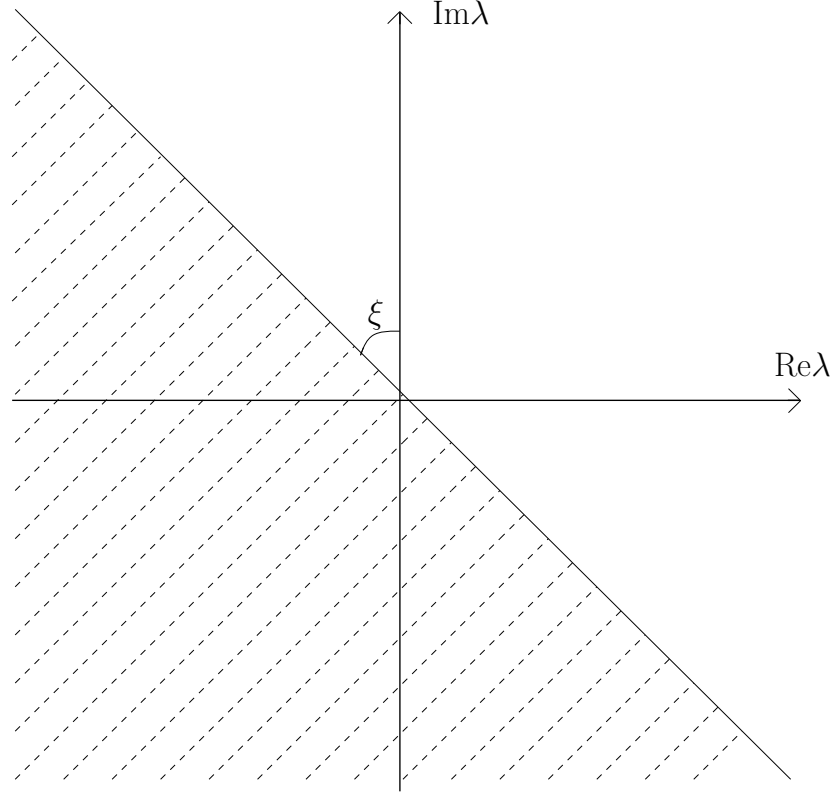


Figure A.4. Shaded region is corresponds to the domain after analytical continuation.

singularity, path integral on the contour C , which is the sum of the paths γ_1, Γ and (γ_2) , should be 0.

$$\int_C dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} = \left(\int_{\gamma_1} + \int_{\Gamma} + \int_{\gamma_2} \right) dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} \quad (\text{A.19})$$

where Γ is the arc at $|x| = \infty$. Its contribution to the path integral is 0 due to “Jordan Lemma”. Also, the path γ_1 corresponds to the line integral along the angles (0) and the path γ_2 corresponds to $(\frac{\pi}{2} - \frac{\xi}{4})$ in the reverse direction. Therefore, the Equation A.19 becomes

$$\int_{(0)} dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} - \int_{(\frac{\pi}{4} - \frac{\xi}{4})} dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} = 0$$

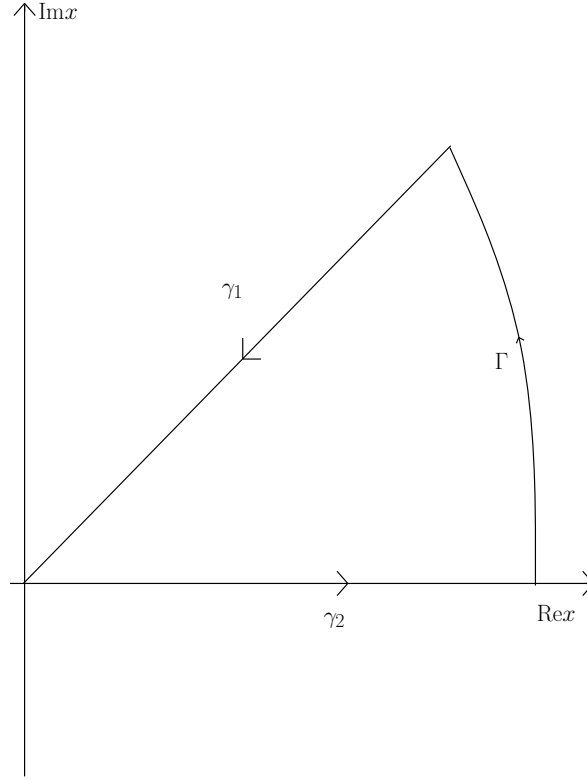


Figure A.5. Contour C.

Then,

$$\int_{(0)} dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} = \int_{\left(\frac{\pi}{4} - \frac{\xi}{4}\right)} dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} \quad (\text{A.20})$$

Equation A.13 is valid in the region given by Equation A.14. However, the equation A.20, which is valid in the intersection given by Equation A.18, allows the extension of the definition of the integral A.13 to the region in Equation A.17. Then, by analytic continuation the domain of A.13 can be redefined as

$$-\frac{3\pi}{2} + \xi \leq \theta_\lambda \leq \frac{\pi}{2} \quad (\text{A.21})$$

Same approach can be applied to a rotation from $\theta_x = 0$ to $\theta_x'' = -\frac{\pi}{4} + \frac{\xi}{4}$. Then, by similar calculations new domain for θ_λ is found such as

$$\frac{\pi}{2} - \xi \leq \theta_\lambda \leq \frac{3\pi}{2} - \xi \quad (\text{A.22})$$

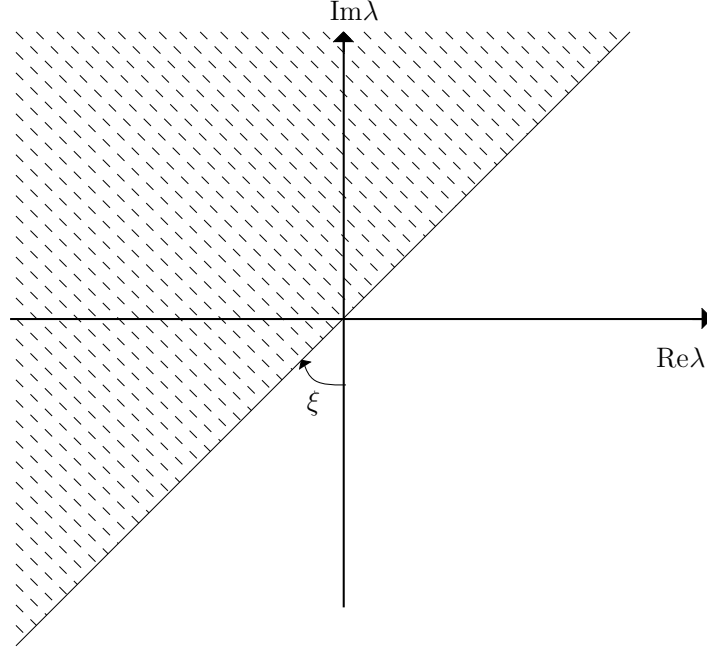


Figure A.6. Shaded region is corresponds to the domain after the second analytical continuation.

which has an intersection with the original domain in Equation A.14 such as

$$\frac{\pi}{2} - \xi \leq \theta_\lambda \leq \frac{\pi}{2} \quad (\text{A.23})$$

Again, similar to the above situation, by the help of Jordan Lemma, it is found that

$$\int_{(0)} dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} = \int_{(-\frac{\pi}{4} + \frac{\xi}{4})} dx e^{-\left(\frac{x^2}{2} + \lambda \frac{x^4}{4}\right)} \quad (\text{A.24})$$

and by using analytic continuation the domain of θ_λ can be extended to region

$$-\frac{\pi}{2} \leq \theta_\lambda \leq \frac{3\pi}{2} - \xi \quad (\text{A.25})$$

The combination of the regions in Equations A.14, A.21 and A.25 covers whole λ -plane. Thus, the integral

$$I(\lambda) = \int_{(0)} e^{-\frac{x^2}{2} - \lambda \frac{x^4}{4}} dx$$

is defined on the whole λ -plane and the only possible singularities would be $\lambda = 0$ or $\lambda = \infty$. If $I(\lambda)$ has these singularities, it has a cut on λ -plane and therefore, it is a multivalued function. Then, to check if the cut is exist or not compare $I(\lambda)$ and $I(ge^{2\pi i})$. A rotation from $I(\lambda)$ to $I(ge^{2\pi i})$ corresponds to the same point and they should result in the same value if the function is single-valued.

Consider, integral on angle $\theta_x = \frac{\pi}{4} - \delta$ where $0 \leq \delta \leq \frac{\pi}{4}$ and at point $\lambda = \lambda_0$ on the λ -plane.

$$I(\lambda_0) = \int_{(\frac{\pi}{4}-\delta)} dx e^{-\frac{x^2}{2}} e^{-\lambda_0 \frac{x^4}{4}} \quad (\text{A.26})$$

Then, let λ_0 to go $g_0 e^{i\frac{\pi}{2}}$ and the integration becomes

$$I(\lambda_0 e^{i\frac{\pi}{2}}) = \int_{(\frac{\pi}{4}-\delta)} dx e^{-\frac{x^2}{2}} e^{-\lambda_0 e^{i\frac{\pi}{2}} \frac{x^4}{4}} \quad (\text{A.27})$$

By using the fact that the integration along the path at $|x| = \infty$, change the contour to the line on the angle $\theta_x = -\frac{\pi}{4} + \delta$. The Equation A.27 becomes

$$I(\lambda_0 e^{i\frac{\pi}{2}}) = \int_{(-\frac{\pi}{4}+\delta)} dx e^{-\frac{x^2}{2}} e^{-\lambda_0 e^{i\frac{\pi}{2}} \frac{x^4}{4}} \quad (\text{A.28})$$

Then, again apply a rotation on λ -plane and let λ_0 be $\lambda_0 e^{\frac{3\pi}{2}i}$.

$$I(\lambda_0 e^{2\pi i}) = \int_{(-\frac{\pi}{4}+\delta)} dx e^{-\frac{x^2}{2}} e^{\left(-\lambda_0 e^{2\pi i} \frac{x^4}{4}\right)} \quad (\text{A.29})$$

Apply a change of variables, $x = x e^{-i\frac{\pi}{4}}$.

$$I(\lambda_0 e^{2\pi i}) = \int_{(\delta)} dx e^{\left(-\frac{x^2}{2} e^{-i\frac{\pi}{2}}\right)} e^{\left(-\lambda_0 e^{\pi i} \frac{x^4}{4}\right)} \quad (\text{A.30})$$

$$= \int_{(\delta)} dx e^{+i\frac{x^2}{2}} e^{\lambda_0 \frac{x^4}{4}} \quad (\text{A.31})$$

Recall, Equation A.26 and by using Jordan Lemma again it is equivalent to

$$I(\lambda_0) = \int_{(\delta)} dx e^{-\frac{x^2}{2}} e^{-\lambda_0 \frac{x^4}{4}} \quad (\text{A.32})$$

Then, subtract Equation A.32 from the Equation A.31

$$I(\lambda_0 e^{2\pi i}) - I(\lambda_0) \int_{(\delta)} dx \left\{ e^{+i\frac{x^2}{2}} e^{\lambda_0 \frac{x^4}{4}} - e^{-\frac{x^2}{2}} e^{-\lambda_0 \frac{x^4}{4}} \right\} \neq 0 \quad (\text{A.33})$$

This inequality implies that $I(\lambda)$ has a branch cut on the λ -plane.

APPENDIX B: THE STEEPEST DESCENT METHOD

The steepest descent method(or saddle point method) is a useful approximation in the study of integrals such that

$$I(s) = \int_C dz g(z) e^{sw(z)} \quad (\text{B.1})$$

where $s \in \mathbb{R}$ and it is very large and positive. Moreover, the functions $g(z)$ and $w(z)$ are assumed to be analytic complex functions on the z -plane.

For very large s , the integrand would be very large either. In fact, in the limit $s \rightarrow \infty$, it blows up. Therefore, the usual integration techniques are not valid for this type of problems.

The method of steepest descents based on contour deformation, which is applicable since $g(z)$ and $w(z)$ are assumed to be analytic in the whole plane. Another assumption should be made here that the function $g(z)$ varies slowly so that it doesn't make important impact on variation of the whole integrand. After this assumption, the work will be to find a small region that $|e^{sw(z)}|$ is much larger than it is any other small region and to deform the contour C to pass this region. Then evaluating the integral in this region gives a good approximation for $I(s)$.

Suppose, the dominant region exists around point

$$z_0 = x_0 + iy_0 \quad ; \quad x_0, y_0 \in \mathbb{R}$$

where z_0 is an maximum point, as in the Figure B.1b. Since the functions are analytic the contour can be deformed to make it pass through the point z_0 as in Figure B.1a

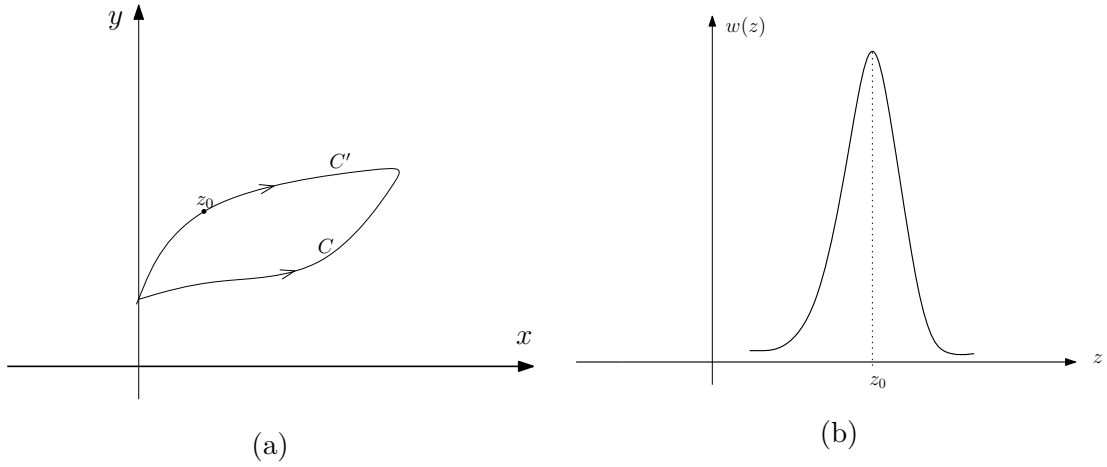


Figure B.1. Simple deformation to make the contour passed through maxima z_0 and the behaviour of the function $w(z)$ around its maxima z_0 .

From basic multivalued calculus, it is known that for an extremum point of a function $u(x, y)$, there are four possibility:

- (i) If $u_{xx} < 0$ and $u_{xx}u_{yy} - u_{xy}^2 > 0$; Then, z_0 is a *maximum*
- (ii) If $u_{xx} > 0$ and $u_{xx}u_{yy} - u_{xy}^2 > 0$; Then, z_0 is a *minimum*
- (iii) If $u_{xx}u_{yy} - u_{xy}^2 < 0$; Then, z_0 is a *saddle point*
- (iv) If $u_{xx}u_{yy} - u_{xy}^2 = 0$; Then, the property of z_0 is a *can not be determined*

In addition to that real and complex parts of a complex function can be represented as functions with 2 variables.

$$w(z) = u(x, y) + iv(x, y) \quad (\text{B.2})$$

For this function Cauchy-Riemann conditions states

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (\text{B.3})$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (\text{B.4})$$

From these equations for the function $u(x, y)$ it is achieved that ⁵

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (\text{B.5})$$

Therefore, in complex space, all the extremum points of the functions $u(x, y)$ and $v(x, y)$ fall into the third case. Therefore, all the extremum points of the complex function $w(z)$ should be saddle points.

Then, one must be careful when choosing the deformed path that passing through the saddle point. An easy observation shows that

$$|e^{sw(z)}| = e^{su(x,y)}$$

thus only $u(x, y)$ part is important while choosing the most appropriate path. Therefore, the most suitable path is the one makes $u(x, y)$ dominant in a small region around z_0 , as much as possible. This is possible by choosing the steepest descent path, which is the path that $u(x, y)$ decreases significantly just outside of the region around z_0 .

Say the steepest descent path is C' in Figure B.2. Then, the integral in Equation B.1 becomes

$$I(s) = \int_{C'} dz g(z) e^{sw(z)}$$

Since $e^{sw(z)}$ is dominated around the region $z_0 = (x_0, y_0)$, the integral can be approximated to

$$I(s) \simeq \int_{C''} dz g(z) e^{sw(z)} \quad (\text{B.6})$$

The new contour C'' , shown in the Figure B.3, lies in the region $(z_0 - \varepsilon, z_0 + \varepsilon)$ and $\varepsilon = \delta e^{i\alpha_0}$ where δ is the small displacement and α_0 is a constant.

⁵Same result is achieved for the function $v(x, y)$

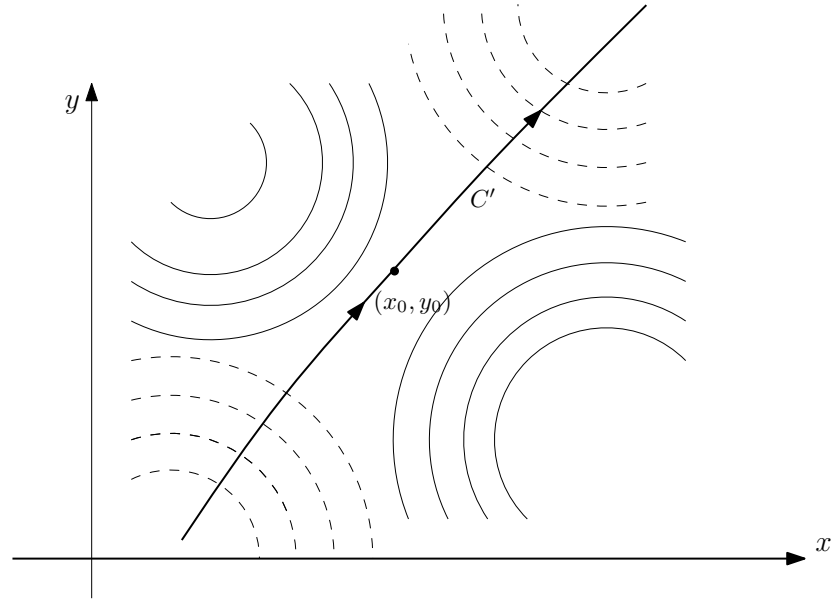


Figure B.2. The contour C' that passed through the saddle point (x_0, y_0) after the contour deformation.

A very important property of analytic functions

$$w(z) = u(x, y) + iv(x, y)$$

is “the lines of constant $v(x, y)$ are always perpendicular to the lines of constant $u(x, y)$ ” [16]. Therefore, the steepest descent path C'' , which is perpendicular to the constant lines of $u(x, y)$, is always along the constant lines of $v(x, y)$. This property can be shown by a simple deduction [15]:

Define;

$$\delta w = w(z) - w(z_0)$$

Then,

$$\delta w = u(z) + iv(z) - u(z_0) - iv(z_0)$$

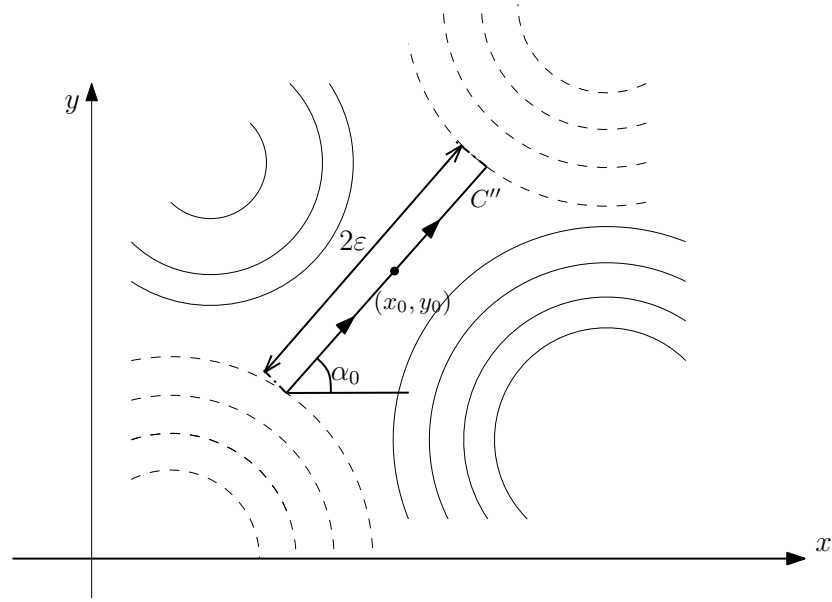


Figure B.3. The contour C'' that has total length 2ε , where ε represented as $\delta e^{i\alpha_0}$ in the complex plane.

and

$$\delta w = \delta u + i\delta v$$

Therefore, for a particular δw , the maximal change in $u(x, y)$ is achieved by choosing $\delta v = 0$. This result supports the idea of choosing the most appropriate path by just considering the function $u(x, y)$.

Along C''

$$z = z_0 + \delta e^{i\alpha_0} \tag{B.7}$$

and

$$dz = d\delta e^{i\alpha_0}$$

Then, the Equation B.6 becomes

$$I = e^{i\alpha_0} \int_{-\varepsilon}^{\varepsilon} d\delta g(z_0 + \delta e^{i\alpha_0}) e^{sw(z_0 + \delta e^{i\alpha_0})} \quad (\text{B.8})$$

Since $g(z)$ is assumed to be slowly varying

$$g(z_0 + \delta e^{i\alpha_0}) \simeq g(z_0).$$

Taylor expansion for $w(z)$ around z_0 is

$$w(z) = w(z_0) + \left. \frac{dw}{dz} \right|_{z=z_0} (z - z_0) + \left. \frac{d^2w}{dz^2} \right|_{z=z_0} \frac{(z - z_0)^2}{2} + \dots$$

Since $z = z_0$ is taken as a critical point, and by using B.7

$$w(z) \simeq w(z_0) + \frac{1}{2} \left. \frac{d^2w}{dz^2} \right|_{z=z_0} \delta^2 e^{2i\alpha_0} \quad (\text{B.9})$$

This approximation is done on the steepest descent path, thus $\delta v = 0$, i.e. $\text{Im}w(z)$ is constant, for this approximation. Then $\text{Im}w(z) = \text{Im}w(z_0)$ on the path which leads to the existence of pure real quantity. Then, in the first order approximation

$$\frac{\delta}{2} \left. \frac{d^2w}{dz^2} \right|_{z=z_0} e^{2i\alpha_0}$$

is a pure real quantity.

Define

$$\left. \frac{d^2w}{dz^2} \right|_{z=z_0} = w_0'' e^{i\theta_0}$$

where $w_0'' \in \mathbb{R}$ and $w_0'' > 0$. Then, the quantity

$$e^{i(\theta_0 + 2\alpha_0)}$$

must be a real quantity. Therefore, its argument should satisfy the following condition

$$\theta_0 + 2\alpha_0 = \pm n\pi$$

where $n = 0, 1, 2, \dots$

By using this condition and the Equation B.7 can be written as

$$w(z) \simeq w(z_0) + \frac{1}{2}\delta^2 w_0'' e^{\pm n\pi} = w(z_0) \pm \frac{1}{2}\delta^2 w_0'' \quad (\text{B.10})$$

where

+: for even integers

–: for odd integers

The the integral B.8 becomes

$$I \simeq e^{i\alpha_0} g(z_0) e^{sw(z_0)} \int_{-\varepsilon}^{\varepsilon} d\delta e^{\pm (s\delta^2 w_0'')/2} \quad (\text{B.11})$$

Remember, δ is the distance from z_0 so δ^2 will increase as the path moves away from z_0 . Since, the point z_0 is chosen as a maximum, the integrand should be the biggest at $z = z_0$ and decrease as it moves along C'' . Therefore, only the integrand with “minus” sign in the exponential factor contributes along this path.

$$I \simeq e^{i\alpha_0} g(z_0) e^{sw(z_0)} \int_{-\varepsilon}^{\varepsilon} d\delta e^{-\frac{s}{2}\delta^2 w_0''} \quad (\text{B.12})$$

and since the only contribution comes from the region $[-\varepsilon, \varepsilon]$, the boundaries of the integral can be extended to $(-\infty, \infty)$. Then,

$$I \simeq e^{i\alpha_0} g(z_0) e^{sw(z_0)} \int_{-\infty}^{\infty} d\delta e^{-\frac{s}{2}\delta^2 w_0''} \quad (\text{B.13})$$

which is a simple Gaussian integral. The result is

$$I \simeq e^{i\alpha_0} g(z_0) e^{sw(z_0)} \sqrt{\frac{2\pi}{sw_0''}} \quad (\text{B.14})$$

Notice that for larger s , the narrower and bigger Gaussian peak occurs in the Equation B.13. Therefore, as s gets bigger the better approximation is achieved.

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