

**OPTIMAL REINSURANCE UNDER COMPETING
BENEFIT CRITERIA**

**RAKİP FAYDA ÖLÇÜTLERİNE BAĞLI OPTİMAL
REASÜRANS**

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To my family...

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- I did not do any distortion in the data set
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22/10/2015

BAŞAK BULUT KARAGEYİK

ABSTRACT

OPTIMAL REINSURANCE UNDER COMPETING BENEFIT CRITERIA

Başak BULUT KARAGEYİK

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In this thesis, we present a new approach to decide the optimal reinsurance under the minimum ruin probability and maximum profit of the insurance portfolio. Based on the fundamental works in the context of finite time ruin probability, we are concerned with the reinsurance level of an insurance policy.

In this context, we develop optimal reinsurance criteria which consider three quantities that affect the optimal reinsurance level: released capital, expected profit, and expected utility of resulting wealth. We aim to find the pair of initial surplus and reinsurance level which minimises the finite time ruin probability and maximises the output of these three quantities as well.

Firstly, we impose the classical risk model and ruin probability on problem of selecting an optimal reinsurance level in finite time horizons.

We consider the classical risk model under a reinsurance arrangement either excess of loss or proportional. We use the translated gamma process to approximate the compound Poisson process.

Having the results for the finite time ruin probability on the translated gamma process approximation, we then step further into the lower barrier model which is directly related to the insurer's surplus process. This modified surplus process does not depend on the traditional reinsurance arrangements which are either excess of loss and proportional reinsurance. Moreover, this surplus process is neither based on the individual nor the aggregate claim process, but based only on the insurer's surplus. The lower barrier model is interpreted as when the insurer's surplus falls below a certain lower barrier level, the reinsurance company makes a capital injection to close the gap and raises the surplus back to this level. The reinsurance premium is calculated according to this capital injection amount. The distribution function of the time to ruin is used to obtain the finite time ruin probability on lower barrier model.

We apply methods from multiple attribute decision making (MADM) in order to decide an optimal reinsurance or optimal barrier level. In particular, we apply the "The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS)" method with Mahalanobis distance. We consider the classical risk model under a reinsurance arrangement-either excess of loss or proportional-and we consider scenarios that have the same finite time ruin probabilities. We compare and contrast our results for the exponential and Pareto distribution which have different tail behaviors.

We build numerical analysis to show how the optimal reinsurance or lower barrier level are calculated under the minimum ruin probability constraint according to the maximum value of the optimal reinsurance criteria quantities. We compare and contrast our findings with those when decisions are based on different assumptions on sensitivity analysis.

Keywords: Risk Theory, finite and infinite time ruin probability, lower barrier model, optimal reinsurance, released capital, expected profit, expected utility, TOPSIS.

ÖZET

RAKİP FAYDA ÖLÇÜTLERİNE BAĞLI OPTİMAL REASÜRANS

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İflas olasılığı, bireysel hasar tutarı ve hasar sayısı dağılımı, prim hesaplama ilkeleri, sigorta şirketi ve reasürans şirketinin güvenlik yükleme faktörü gibi birçok varsayım üzerine kuruludur. Klasik risk modeli, toplam hasarın bileşik Poisson sürecine uyduğunu varsaymaktadır. Klasik risk modeli altında farklı bireysel hasar tutarı dağılımlarına göre sonlu ve sonlu olmayan zamanlı iflas olasılığı analizi aktüerya bilimlerinde önemli çalışma alanlarından biridir. Son yıllardaki çalışmalar sigorta şirketinin iflas olasılıklarının nasıl asgari düzeye düşürülebileceği üzerine yapılmaktadır. Geçmiş çalışmalar iflas olasılığının minimum düzeye indirilebilmesinin reasürans anlaşmaları, temettü ödemeleri ve yatırım stratejileri ile mümkün olduğunu göstermektedir.

İflas olasılığının minimum seviyeye düşürülmesi ile ilgili yapılan çalışmaların birçoğu genellikle reasürans anlaşmalarına dayanmaktadır.

Geleneksel reasürans anlaşmaları olan hasar fazlası ve orantılı reasürans anlaşmalarında reasürans primi reasürans seviyesine ve reasürans yükleme faktörüne göre belirlenmektedir. Hasar fazlası reasürans anlaşmalarında hasar tutarının belirlenen bir saklama payı seviyesinin üzerinde kalması durumunda reasürans şirketi devreye girer ve bu tutarı geçen kısım reasürans şirketi tarafından karşılanır. Orantılı reasürans anlaşmalarında ise tüm hasarın belirlenen bir oranı reasürans şirketine devredilir. Sigorta ve reasürans şirketi arasındaki prim ve hasar paylaşımı bu oran üzerinden yapılır. Bununla birlikte geleneksel olmayan reasürans anlaşmalarında, reasürans seviyelerinin yanında sigorta şirketinin başlangıç sermayesi de göz önünde bulundurulur. İflas olasılığını minimum seviyeye indirecek şekilde optimal reasürans seviyelerinin belirlenmesi üzerine literatürde birçok çalışma bulunmaktadır.

Sigorta şirketi iflas olasılığını minimum seviyeye indirmek istemesinin yanında portföy karını ve gelirini de maksimum seviyede tutmak istemektedir. Sigorta şirketinin net karı; prim gelirinden, beklenen hasar tutarı ve reasürans priminin çıkarılmasıyla elde edilmektedir.

Bu çalışmada, minimum iflas olasılığı kısıtı altında sigorta şirketinin karını maksimize edecek optimal reasürans seviyesinin belirlenmesini sağlayan yeni bir yöntem sunulmaktadır. Sonlu zamanlı iflas olasılığı kısıtı altında optimal reasürans seviyesinin belirlenmesi amaçlanmıştır.

Sigorta şirketinin optimal reasürans seviyesinin belirlenmesi için optimal reasürans kriterleri belirlenmiştir. Bu kriterler sırasıyla sigorta şirketinin reasürans anlaşması nedeniyle elde ettiği sermaye kazancı, beklenen karı ve sigorta şirketinin varlığının beklenen faydası olarak belirlenmiştir. Çalışmanın amacı, iflas olasılığının minimum seviyede olduğu durumda bu üç kriter değerini maksimize edecek optimal başlangıç sermayesi ve optimal reasürans seviyelerinin belirlenmesidir.

Tez iki temel bölümden oluşmaktadır. İlk olarak, optimal reasürans seviyesi sonlu zamanlı iflas olasılığına bağlı olarak klasik risk süreci mode-li altında incelenmiştir. Klasik risk modeli varsayımları altında hasar fazlası reasürans anlaşması için optimal saklama payı seviyeleri ve orantılı reasürans anlaşması için optimal reasürans oranları araştırılmıştır.

Bileşik Poisson sürecine sahip toplam hasar sürecinin elde edilmesinde, dönüştürülmüş gama süreci yaklaşımından yararlanılmıştır. Dönüştürülmüş gama yaklaşımı toplam hasar dağılımının sağ kuyruk bölgesine diğer yaklaşımlardan daha iyi uyum sağlaması ve uygulama kolaylığı nedeniyle tercih edilmektedir.

Sonlu zamanlı iflas olasılığı ile sigorta şirketinin sermaye kazancı, beklenen karı ve sigorta şirketinin varlığının beklenen faydası dönüştürülmüş gama süreci yaklaşımı kullanılarak elde edilmiştir. Optimal reasürans seviyeleri, iflas olasılığının minimum olduğu durumda bu üç kriterin değerini maksimum yapacak şekilde belirlenmiştir.

Klasik risk modeli, hasar fazlası ve orantılı reasürans anlaşmalarına bağlı olarak sonlu zamanlı iflas olasılığının 0.01 gibi minimum bir seviyede olduğu durum için incelenmiştir. Bireysel hasar tutarı dağılımı için farklı kuyruk yapısına sahip olan üstel ve Pareto dağılımları varsayılmıştır. Her iki dağılıma göre elde edilen optimal saklama payları ve optimal reasürans seviyeleri karşılaştırılmıştır.

İkinci olarak, alt bariyer model (lower barrier model) kapsamında optimal başlangıç sermayesi ve bariyer seviyelerinin belirlenmesi amaçlanmıştır. Alt bariyer model, değiştirilmiş risk süreci (modified surplus process) altında incelenmektedir. Bu süreç geleneksel reasürans anlaşmaları olan hasar fazlası ve orantılı reasürans anlaşmalarına dayanmamaktadır. Bununla birlikte bu reasürans anlaşması bireysel ve toplam hasar dağılımına da bağlı olmayıp doğrudan sigortacının risk sürecine göre şekillenmektedir.

Alt bariyer modelde, başlangıç sermayesinden küçük olacak şekilde ($k < u$) bir k bariyer seviyesi belirlenir. Reasürans şirketi, risk sürecinin bu seviyenin altına düşmesi durumunda devreye girer ve risk sürecini tekrar bariyer seviyesi olan k 'ye yükseltecek şekilde ek bir ödeme yapar. Reasürans şirketinin karşılayamayacağı miktarda bir hasar gelmesi durumunda risk süreci sıfırın altına düşecek ve iflas gerçekleşmiş olacaktır. Reasürans şirketi yaptığı bu ödemeye bağlı olarak reasürans primi talep eder. Alt bariyer modelde sonlu zamanlı iflas olasılığı, iflas zamanının dağılım fonksiyonuna bağlı olarak elde edilmektedir.

Optimal reasürans ve optimal bariyer seviyelerinin belirlenmesinde, Çok Nitelikli Karar Analizi'nden yararlanılmıştır. Bu kapsamda “Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS)” yöntemi, Mahalanobis uzaklık ölçümü kullanılarak uygulanmıştır.

Bu çalışmada ayrıca, sonlu zamanlı minimum iflas olasılığı kısıtı altında optimal reasürans kriterlerini maksimum yapacak şekilde optimal bariyer ve optimal reasürans seviyeleri sayısal olarak incelenmiştir. Ayrıca, duyarlılık analizi ile farklı varsayımlara göre sonuçlar karşılaştırılarak optimal reasürans ve optimal bariyer seviyelerindeki değişimler gözlemlenmiştir.

Tezin birinci bölümünde; çalışmanın amacı, aşamaları ve her bir bölümün kısa özeti verilmiştir. İkinci bölümde; klasik risk modeline giriş yapılmış ve bu modelin temel bileşenleri ve varsayımları açıklanmıştır. Bu kapsamda ilk olarak hasar tutarı dağılımının belirlenmesinde kullanılan yaklaşımlar karşılaştırılmış ve sonuç olarak dönüştürülmüş gama yaklaşımının diğer yaklaşımlara göre toplam hasar tutarının özellikle sağ kuyruk bölgesine daha iyi uyum sağlaması nedeniyle tercih edildiği belirlenmiştir. Ayrıca çalışmanın bu bölümünde sonlu ve sonlu olmayan zamanlı iflas olasılığının hesaplanmasına ilişkin formül ve yaklaşımlar sunulmuştur.

Üçüncü bölümde, iflas olasılığı kısıtı altında optimal reasürans seviyesinin belirlenmesi ile ilgili literatürdeki çalışmalar özetlenmiş ve temel alınan kriterlere bağlı olarak sınıflandırılmıştır. Çalışmalar baz alınarak üç önemli kriter tarafından oluşan optimal reasürans kriterleri belirlenmiştir. Bu kriterler sigorta şirketinin reasürans anlaşması nedeniyle elde ettiği sermaye kazancı, beklenen kar ve beklenen fayda olarak belirlenmiştir. Bu üç kriterin reasürans seviyelerine bağlı olarak nasıl hesaplandığı gösterilmiştir.

Dördüncü bölümde, iflas olasılığı kısıtı altında optimal reasürans modellemesi yapılmıştır. Reasürans anlaşmasının geçerli olması durumunda sigorta şirketinin sonlu zamanlı iflas olasılığı, dönüştürülmüş gama süreci yaklaşımı altında incelenmiştir.

Minimum iflas olasılığı kısıtı altında sigorta şirketinin sermaye kazancının, beklenen karının ve beklenen faydasının reasürans seviyesine bağlı olarak dönüştürülmüş gama süreci yaklaşımı altında nasıl hesaplanacağı araştırılmıştır.

Optimal başlangıç sermayesi ve reasürans seviyesinin belirlenmesi aşamasında çok nitelikli karar analizinden yararlanılmıştır. Bu kapsamda TOPSIS yöntemi kullanılmıştır. Belirlenen üç kriter sigorta şirketinin başlangıç sermayesi ve reasürans seviyelerine bağlı olarak elde edildiğinden kriterler arasında güçlü bir bağımlılığın olduğu görülmüştür. Bu nedenle öklid uzaklık ölçüsü yerine değişkenler arasındaki bağımlılığı da göz önünde bulunduran Mahalanobis uzaklık ölçüsü kullanılmıştır.

Beşinci bölümde, dönüştürülmüş gama yaklaşımı altında optimal reasürans analizi yapılmıştır. Öncelikle hasar fazlası reasürans anlaşmasına göre; optimal saklama payı seviyeleri ve optimal başlangıç sermayeleri farklı zaman tercihi ve yükleme faktörlerine göre hesaplanmış ve değerler karşılaştırılmıştır. Orantılı reasürans anlaşması için optimal reasürans oranı ve optimal başlangıç sermayeleri farklı zaman süreleri ve yükleme faktörleri için hesaplanmıştır. Her iki reasürans anlaşması için de bireysel hasar tutarının üstel ve Pareto dağılımlarına uyması durumları incelenmiş ve sonuçlar karşılaştırılmıştır. Ayrıca çalışmanın bu bölümünde kriter ağırlıklarının ve beklenen fayda fonksiyonu parametresinin değişiminin optimal reasürans seviyeleri üzerindeki etkileri duyarlılık analizi ile incelenmiştir.

Altıncı bölümde, değiştirilmiş risk sürecine bağlı olarak alt bariyer model araştırılmıştır. Nie vd. [1] tarafından önerilen k bariyer seviyesi ve u başlangıç sermayesi olmak üzere $T_{u,k}$ iflas zamanının dağılım fonksiyonundan yararlanılarak sonlu zamanlı iflas olasılığı incelenmiştir. Bireysel hasar tutarının üstel dağılması varsayımı altında, alt bariyer model için iflas zamanının dağılım fonksiyonu elde edilmiştir. Bu kapsamda reasürans primi, net yükleme faktörü kullanılarak modele dahil edilmiş ve böylece reasürans anlaşmasının etkisi gözlemlenmiştir.

Yedinci bölümde, iflas olasılığı kısıtı altında optimal bariyer seviyesi alt bariyer model için belirlenmiştir. Öncelikle alt bariyer model kapsamında sonlu zamanlı iflas olasılığının hesaplanması için gerekli formül ve yaklaşımlar açıklanmıştır.

Alt bariyer modele göre sırasıyla sigorta şirketinin sermaye kazancının, beklenen karının ve beklenen faydasının nasıl hesaplanacağı gösterilmiştir. Optimal başlangıç sermayesi ve optimal bariyer seviyesinin belirlenmesi aşamasında TOPSIS yöntemi Mahalanobis uzaklık ölçüsü ile kullanılmıştır.

Bireysel hasar tutarının üstel dağılması varsayımına göre optimal bariyer seviyeleri farklı zaman ve güvenlik yükleme faktörlerine göre hesaplanmıştır. Ayrıca çalışmanın bu bölümünde kriter ağırlıkları ve beklenen fayda fonksiyonu parametresi değişimlerinin optimal bariyer seviyeleri üzerindeki etkileri duyarlılık analizi ile incelenmiş ve sonuçlar değerlendirilmiştir.

Çalışmanın son bölümünde minimum iflas olasılığı kısıtı altında sigorta şirketinin sermaye kazancını, beklenen karını ve beklenen faydasını maksimum yapacak optimal reasürans seviyesinin belirlenmesi amacıyla geliştirilen model üzerine genel bir değerlendirme yapılmıştır. Önerilen yaklaşım ve model ile ilgili çeşitli yorum ve açıklamalar verilmiştir. Çalışmanın sonuçları hakkında genel bir değerlendirme ile birlikte gelecek araştırmalar için önerilerde bulunulmuştur.

Anahtar Kelimeler: Risk kuramı, sonlu ve sonlu olmayan zamanlı iflas olasılığı, alt bariyer model, optimal reasürans, sermaye kazancı, beklenen kar, beklenen fayda, TOPSIS.

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1. INTRODUCTION

Risk theory plays a vital role in actuarial work. It provides frameworks of a mathematical basis of the insurance risk. There are two points of view from which risk theory is considered: the individual and the collective risk theory. While the investigation of the claim on each individual claim policy is in the scope of the individual risk theory, total claim amount or total gain from all policies are examined within the context of collective risk theory. Collective risk theory considers two essential problems: finding the distribution function of total claim amount and finding the probability of ruin. Ruin theory has been one of the most important working area in actuarial science. The main aim of the research on the risk process is to indicate whether an insurance company will remain solvent or not. In addition, ruin probability is a measure that allows the insurance company to investigate whether the collected premiums are sufficient to pay the claims, starting with a certain initial surplus, on a finite time horizon.

A wide variety of models and assumptions are investigated under the consideration of the ruin theory. The classical risk models assume that the premium income has a constant rate and the aggregate claim amount has a compound Poisson process. Although the literature on the ruin probability on the classical risk model shows a variety of approaches, in the last few years there has been a growing interest on minimizing the ruin probability or maximizing the survival probability of the insurance company. An insurance company has to control its ruin probability and keep it at a minimum level to be able to sustain its existence. Previous studies indicate that by making a reinsurance arrangement, dividend payments or an investment, the ruin probability can be controlled and can be minimised. Most of the literature on minimizing the ruin probability is based on reinsurance arrangements. In traditional reinsurance arrangements; excess of loss reinsurance and proportional reinsurance, the reinsurance premium is calculated according to the reinsurance level and the reinsurance loading factor. However, in a new kind of reinsurance arrangement, the initial surplus also has an essential role in determining the reinsurance premium. Several publications on deciding the optimal reinsurance level which minimise the ruin probability have appeared in recent years.

In addition to minimizing the ruin probability, the gain, earnings or profit are also other important measures for the insurance company. An insurance company covers the benefit payment and reinsurance premium with their premium income. The balance will be the profit of the insurance company. The main purpose of the insurance company is to maximise their profit, gains or capital. Furthermore, insurance company decides the reinsurance level by using different essential measures except the insurance profit. Within this framework, we consider some important criteria such as expected utility and released capital for reinsurance decision making. We propose optimal reinsurance criteria which consist of three measures: released capital, expected profit and expected utility. We demonstrate determination of the optimal reinsurance level by using these measures under the fixed minimum ruin probability constraint.

The main purpose of this thesis is to develop a method to decide the optimal reinsurance considering three important criteria and the ruin probability. We investigate the optimal reinsurance level that maximises not only the insurance company's profit but also the released capital and expected utility, while the ruin probability is fixed at a minimum level. In this context, we present a new method to determine the optimal reinsurance level, barrier level and the corresponding initial surplus.

The goal of the thesis is twofold. First, we examine the optimal reinsurance methodology according to three criteria: released capital, expected profit and expected utility under the finite time ruin probability constraint. In the classical surplus process, we aim to determine the optimal reinsurance level and initial surplus based on the translated gamma process approximation. The translated gamma approximation is preferred to the other approximations because of its simplicity and having a better fitting on the right tails of the aggregate claim distribution. We consider two kinds of reinsurance arrangement: excess of loss and proportional. Optimal retention level for the excess of loss reinsurance and optimal proportion for the proportional reinsurance are calculated using the translated gamma process approximation. Under the classical risk model, the exponential and Pareto distribution are chosen as the individual claim amount distribution because of their different tail behaviors. Second, we focus on a different reinsurance arrangement in which the reinsurance level is defined as a barrier level. Having considered the modified surplus process, we aim to determine the optimal barrier level and initial surplus on lower barrier model.

We provide the finite time ruin probability for the lower barrier model. Then under the fixed ruin probability constraint, we apply optimal reinsurance criteria to determine the optimal barrier level.

This work presents a different view of optimal reinsurance decision. The originality of the method lies in the fact that consideration both of the minimum ruin probability and the maximum of the released capital, expected profit and expected utility. To our knowledge, this is the first study to deal with three essential criteria and the ruin probability together.

The thesis is organized as follows. In Chapter 2 we introduce the classical risk model and give definitions for components and assumptions of this model. In the first step, we summarise the approximation methods to estimate the aggregate claims process and we discuss their properties. Next we define the infinite and finite time ruin probabilities. We review exact formulas and approximations to obtain the ruin probability for finite and infinite time horizons. We conclude the chapter by discussing the comparison of methods for obtaining the aggregate claims and ruin probability.

In Chapter 3 we present a brief literature review on optimal reinsurance under the ruin probability constraint. We categorize methods which are used in the literature to determine which measures are essential. We define a new method called optimal reinsurance criteria which consist of three criteria: released capital, expected profit and expected utility. We show how these measures can be calculated and applied under the reinsurance arrangement.

In Chapter 4 we introduce the optimal reinsurance modelling under the ruin probability constraint. We examine the finite time ruin probability by using the translated gamma process approximation when a reinsurance arrangement is involved. We introduce how released capital, expected profit and expected utility can be modelled under the translated gamma process approximation. For the purpose of determining the optimal reinsurance level, we introduce the decision theory and multi attribute decision making. In this context, we use the “Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS)” method. We employ the Euclidean distance as an element of Minkowski’s distance measure and Mahalanobis distance.

The main problem is that released capital, expected profit and expected utility are dependent in respect to initial surplus and reinsurance level. Thus, we choose using Mahalanobis distance to apply optimal reinsurance modelling due to the relationship between the measures.

In Chapter 5 we present an analysis on optimal reinsurance which considers the finite time ruin probability under the translated gamma process approximation in detail. First, we examine the optimal retention level with respect to three optimal reinsurance criteria for the excess of loss reinsurance. We calculate the optimal retention level and initial surplus for the combinations of different time horizons and loading pairs. Second, we investigate the optimal proportion on the proportional reinsurance arrangement. Based on two assumptions about the individual claim amounts: exponential and Pareto distribution for each reinsurance arrangement, we show the differences on the optimal reinsurance level. Furthermore, we examine the sensitivity analysis of the model due to the choice of weight of the criteria and the parameter of the utility function. We also compare the optimal levels according to the maximum possible values of the attributes. We conclude the chapter by discussing the optimal reinsurance level under the translated gamma process approximation.

In Chapter 6 we introduce the lower barrier model on modified surplus process. We investigate the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u which is suggested by Nie et. al [2]. The distribution function of time to ruin, i.e finite time ruin probability on lower barrier model is showed for the exponential case. Reinsurance premium is calculated as “the aggregate amount needed to restore the modified surplus process to level k up to time t with initial surplus u ”. We incorporate the reinsurance premium into the calculation of the finite time ruin probability on lower barrier model. Thus, we obtain the specific formula for the net of reinsurance loading factor and hence, the effect of the reinsurance is evaluated on lower barrier level.

In Chapter 7 we present a detailed analysis on optimal barrier level which considers the finite time ruin probability. First, we introduce the necessary formulas and approximations about the ruin probabilities to obtain the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u under the exponential individual claim amount assumption.

Second, we investigate the calculation of three criteria on lower barrier model. In this context, we use the expected total claim amount paid by the reinsurer up to the time of ruin to calculate the reinsurance premium. Then, we obtain the optimal initial surplus and corresponding barrier level according to released capital, expected profit and expected utility under the fixed ruin probability constraint. In reinsurance decision process, we use the TOPSIS method with Mahalanobis distance. Moreover, we mainly investigate the sensitivity of the results to different weights of criteria, and different values for the parameter of the utility function. We also compare the optimal pair of initial surplus and barrier level according to the maximum possible value of each attribute. We conclude the chapter by discussing the optimal barrier levels.

Finally, in Chapter 8 we present our conclusions and give ideas for further research.

2. A REVIEW OF RISK THEORY

2.1. Introduction

In this chapter, we review the classical risk model in which most of the literature is based on ruin theory. The main concern of the chapter is to introduce the classical risk model and to discuss some corresponding factors with this model. This chapter is mainly based on three subsections: approximations to the aggregate claims, infinite time and finite time ruin probability.

In Section 2.2. we introduce the classical risk model and describe the assumptions of this model. Estimation of the aggregate claims process plays a vital role in the risk theory. However, in some cases, it is not possible to estimate the aggregate claims process. For this reason, some approximations are developed to estimate the aggregate claims. In Section 2.3. we give the approximations to the aggregate claims process by describing often used ones in the literature. In Section 2.4 and Section 2.5 we consider the infinite time ruin probability. In these sections, we address not only to the exact solution but also to the approximation techniques to obtain the infinite time ruin probability. We also discuss the consequences of the finite time ruin probability in Section 2.6 and Section 2.7. In Section 2.8 we conclude the chapter.

2.2. The Classical Risk Model

The classical risk process is based on initial surplus, premiums and claims. It is assumed that the surplus process starts with an initial level u and continues according to two opposing cash flows: the premium income per unit of time and the aggregate claim amount up to time t , denoted by $S(t)$. The insurer's surplus (or risk) process, $\{U(t)\}_{t \geq 0}$, is defined by

$$U(t) = u + ct - S(t).$$

The aggregate claim amount up to time t , $S(t)$, is

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where $N(t)$ denotes the number of claims that occur in the fixed time interval $[0, t]$. The aggregate claim amount consists of two main components. The preliminary component is the individual claim amounts, modelled as independent and identically distributed (*i.i.d.*) random variables $\{X_i\}_{i=1}^{\infty}$ with distribution function $F(x) = \Pr(X_1 \leq x)$ such that $F(0) = 0$ and X_i is the amount of the i^{th} claim. We use the notation f and m_k to represent the density function and k^{th} moment of X_1 , respectively, and it is assumed that $c > E[N] m_1$.

T_i is the time when the i^{th} claim occurs and $T_1, T_2, \dots$ are random variables such that $T_1 < T_2 < \dots$. A typical realization of the surplus process is shown in Figure 2.1

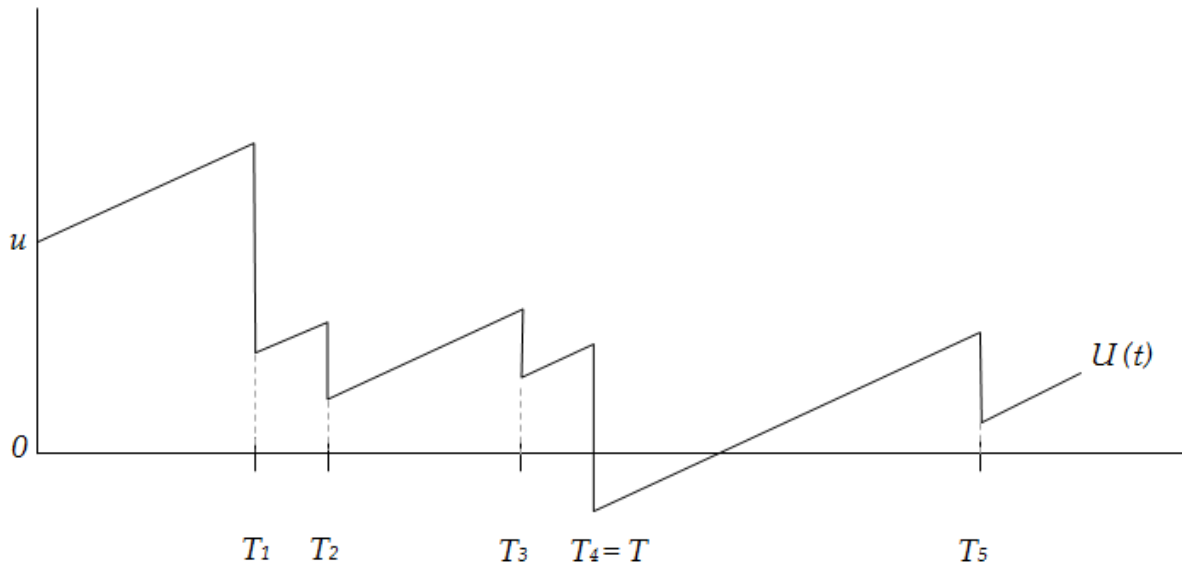


Figure 2.1: A realization of the surplus process

The other important component of the aggregate claims process is the counting process for the number of claims, $\{N(t)\}_{t \geq 0}$. In the classical risk process, it is assumed that $\{N(t)\}_{t \geq 0}$ is a Poisson process with parameter λ and hence the aggregate claims process $\{S(t)\}_{t \geq 0}$ is a compound Poisson process with parameters λ and F .

In Poisson process, the increments of the process have the following property:

$$N(t + h) - N(t) \sim \text{Poisson}(\lambda h)$$

for all $t > 0, h > 0$. As a result of the characteristic feature of the increments in Poisson process, this process ensures the following independency and stationary properties.

Independency: $N(t_i + h_i) - N(t_i)$ is independent for the intervals $(t_i, t_i + h_i), i = 1, 2, \dots$ and

Stationary: $N(t + h) - N(t)$ has Poisson (λh) distribution for every value of t [3].

The classical risk model ignores the expenses of the insurer and the interest for the initial surplus. However, this model is still preferred not only because it is simple but also it gives some insight about the characteristics of an insurance process. The focus on recent research has been on the finite and infinite time ruin probability.

2.3. Approximation to Claim Distribution

The recent research on actuarial studies pays attention to calculate the aggregate claims distribution especially in ratemaking and reserving. The literature on the calculation of aggregate claims distribution presents various approaches such as Heckman-Meyers method, Panjer method, Fast Fourier transform, and stochastic simulations. However, an important limitation of these calculations is the assumption of the availability of the claim frequency and the claim severity distribution. This assumption may not always be valid. The frequency and the severity distributions may not be obtained separately and only some information on aggregate claims can be available. In this situation, some approaches are used to estimate the aggregate claims distribution.

In the literature, several methods are proposed to estimate the aggregate claim distribution. These methods are generally built on two main methods. The first one depends on matching moments of the aggregate claims by a given distribution and obtains probabilities from the approximated distribution. The second one is based on applying some transformations to the aggregate claims random variable.

Besides these two methods, the aggregate claim distribution can be modelled by estimating the moment generating function. The compatibility of the approximation is based on the tail behavior.

Hardy [4] investigates most of the approximations for the aggregate claims distribution in detail and compares them according to the true and estimated right tail (4-standard deviation) probabilities. Hardy categorizes the approximations in terms of matching moments and transformation techniques.

There are various papers which describe and compare the approximations to the aggregate claims process. For several years, great effort has been devoted to estimate the aggregate claims process. Hardy [4] gives a comprehensive study on the estimation of aggregate claims process. Approximations to the aggregate claim process are also discussed in Pentikainen [5], [6]; Hipp [7],[8], and Papush [9].

We summarise the recently used approximations in the actuarial literature as follows.

2.3.1. The Normal Approximation

In the normal approximation, the aggregate distribution is estimated by assuming the same mean and variance as in normal distribution. This approximation depends on the central limit theorem. Hardy [4] emphasizes this situation as “so that sum of the independent random variables are in tendency to a normal random variable, as the number in the sum increases”.

This approximation can be preferred when the expected number of claims is large enough. However, the aggregate claims distribution which is estimated from the normal approximation cannot fit well in the right tail. Normal distribution has thinner tail than that of aggregate claims distribution and it is the biggest disadvantage of this approximation. Pentikainen [5] shows that the normal approximation ensures the consistency when the capacity of the risk business is rather large and the individual claims distribution is not heterogeneous. To improve the normal approximation, normal power approximation is proposed.

2.3.2. Normal Power Approximation

Generally, the normal approximation cannot be fitted well and the results are not realistic. Normal power approximation is formed by adding a transformation to the normal distribution to change the skewness.

Let μ_S , σ_S^2 and γ_S denote the mean, variance and coefficient of skewness of the aggregate claim distribution, respectively. Let $\Phi()$ denotes the standard normal distribution function. Hence, the normal power approximation to the distribution function is estimated as follows

$$F_S(x) = \Phi \left[\sqrt{\frac{9}{\gamma_S^2} + 1} + \frac{6(x - \mu_S)}{\gamma_S \sigma_S} - \frac{3}{\gamma_S} \right].$$

This approximation does not give results when $\gamma_S > 1$ [10].

2.3.3. The Translated Gamma Approximation

The gamma distribution has two parameters and its density function can be defined as

$$f(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)} \quad x, \alpha, \beta > 0.$$

The translated gamma distribution has a similar structure to the gamma distribution but it is assumed to be shifted by an amount k . The probability density function of the translated gamma distribution is

$$f(x) = \frac{(x - k)^{\alpha-1} e^{-\frac{(x-k)}{\theta}}}{\theta^\alpha \Gamma(\alpha)} \quad x, \alpha, \beta > 0, \quad -\infty < k < \infty.$$

According to the translated gamma process approximation, mean, variance and the coefficient of skewness are obtained as follows

$$\text{Mean} = \alpha \theta + k,$$

$$\text{Variance} = \alpha \theta^2,$$

$$\text{Coefficient of Skewness} = \frac{2}{\sqrt{\alpha}}.$$

An aggregate claims distribution can be fitted by a translated gamma distribution by matching the first three moments. Translated gamma distribution fits the aggregate claims better than the normal approximation particularly when the aggregate claims have a right-skewed distribution [11]. On the other hand, a translated gamma distribution fits poorly to the left tail of the distribution and this approximation might also give the probability for negative claim amounts [4].

In actuarial literature, normal, normal-power and the translated gamma process approximations are generally used to obtain the aggregate claim amount distribution. However, Bowers' Gamma approximation, Haldane's method, Wilson-Hilferty method and Esscher approximations are also used to obtain the aggregate claim amount distribution. Bowers [12] suggests a method to estimate the aggregate claims distribution function by using orthogonal polynomials. In this approximation, a functional form is employed to estimate the distribution function. Bowers' formula have some similarities with the method based on fitting a gamma distribution except for the translation. Major drawback of this approximation is that the first five moments of the aggregate claims distribution are needed. Haldane's method [13] is similar to the normal power approximation considering the application of a transformation to the original random variable. Wilson-Hilferty method is suggested by Wilson [14] as a simplified version of Haldane's method where h is a constant equal to $1/3$. Esscher approximation is described by Beard [10] and Gerber [15]. While in the other approximations, only the information of the moment of the underlying distribution is required, full knowledge of the underlying distribution is needed in Esscher approximation.

The literature on the aggregate claims distribution includes various approaches. Each of the approximation method has some limitations, advantages and disadvantages in comparison with the others. The required information may vary with respect to the relevant approximation. Individual claim amounts generally have right-skewed and heavy-tailed distributions, i.e. Pareto, lognormal.

The tail behavior of aggregate claim amounts gives information about the structure of the future claims. For this reason, actuarial studies generally concentrate on finding the best approximation which fits the right-tail correctly. The first step in modelling the claim amount distribution is to decide the correct approach.

In this study, the translated gamma approximation is preferred since it is easily applicable and it fits to the aggregate claims distribution better than the other approximations.

2.4. The Infinite Time Ruin Probability

The probability of ruin in continuous time for infinite time (ultimate ruin probability) is defined as

$$\psi(u) = Pr(U(t) < 0 \text{ for some } t > 0).$$

$\psi(u)$ is the probability that the insurer's surplus falls below zero at some time in the future, that is claims outgo exceed the initial surplus plus premium income.

In order to verify the validity of the classical risk model, some assumptions are made. It is usually assumed that the premium income is greater than the expected aggregate claim amount per unit of time ($c > \lambda\mu_1$). Otherwise, $\psi(u) = 1$ for all $u > 0$.

Some premium principles are also used with respect to the positive insurance loading factor $\theta > 0$. In general, expected value premium principle is applied such as $c = (1 + \theta)\lambda\mu_1$.

We show some formulas based upon the calculation of the infinite time ruin probability in the following subsections.

2.4.1. Pollaczek-Khinchine Formula

Let L is the sum of so-called ladder heights as

$$L = \sup_{0 \leq t < \infty} \left(u - U(t)_{t \geq 0} \right) = \sup_{0 \leq t < \infty} \left(\sum_{i=1}^{N(t)} X_i - ct \right),$$

and the ruin probability can be defined by using ladder heights L as

$$\psi(u) = P(L > u).$$

In order to present a ruin probability formula in terms of ladder heights as the sum of a sequence random variable $\{L_k\}_{k=1}^{\infty}$, the density of the sequence of independent and identically random variables must be defined.

Since the process of $(u - U(t)) = (S_t - ct)$ has stationary and independent increments, $\{L_k\}_{k=1}^{\infty}$ is a sequence of *i.i.d* variables with density function as

$$f_{L_1}(x) = \frac{\bar{F}_x(x)}{\mu},$$

where $\bar{F}_x(x) = 1 - F_x(x)$ and μ denotes the mean. Random variable L is expressed as with regard to K which is the number of ladder heights and is distributed as geometric distribution with parameter $q = \theta/(1 + \theta)$.

$$L = \sum_{i=1}^K L_i,$$

and L has compound geometric distribution.

Ladder heights distribution is also known as *integrated tail distribution* or *first order equilibrium distribution* and calculated as given below

$$F_{L_I}(x) = \frac{1}{\mu} \int_0^x \bar{F}(y) dy \quad x \geq 0.$$

Hence, Pollaczek-Khinchine formula is given in the following theorem.

Theorem 2.4.1. Pollaczek-Khinchine Formula

Pollaczek-Khinchine formula for the infinite time ruin probability is obtained as

$$\psi(u) = 1 - P(L \leq u) = 1 - \frac{\theta}{(1 + \theta)} \sum_{n=0}^{\infty} \left(\frac{1}{1 + \theta} \right)^n F_{L_1}^{n*}(u), \quad u \geq 0 \quad (2.1)$$

where $F_{L_1}^{n*}(x)$ denotes the n^{th} convolution of the distribution function F_{L_1} .

The Pollaczek-Khinchine formula is based on queuing theory and is suggested by Pollaczek [16] and reformed by Khinchine [17] with respect to probabilistic terms. In risk theory, the Pollaczek-Khinchine formula is referred to Beekman's convolution formula [18], [19].

Theorem 2.4.2. Beekman's Convolution Formula

The infinite time ruin probability is calculated by the tail of a compound geometric distribution as follows

$$\psi(u) = \phi(0) \sum_{n=1}^{\infty} (\psi(0))^n \bar{F}_{L_1}^{n*}(u), \quad u \geq 0 \quad (2.2)$$

where $\bar{F}_{L_1}^{n*}(x)$ denotes the n^{th} convolution of the distribution function $\bar{F}_{L_1}$.

The Pollaczec-Khinchine formula has two special forms. The first form is proposed for the zero initial surplus whereas the second one is proposed for the exponential individual claims.

2.4.2. The Infinite Time Ruin Probability with Zero Initial Surplus

The infinite time ruin probability with zero initial surplus formula depends only on the relative insurance loading factor θ , regardless of individual claims and number of claims distribution.

$$\psi(0) = \frac{1}{1 + \theta}.$$

2.4.3. The Infinite Time Ruin Probability with Exponential Claim Amounts

The Pollaczec-Khinchine formula turns into an explicit formula when the individual claim amounts are distributed exponentially with the parameter α as

$$\psi(u) = \frac{1}{1 + \theta} \exp\left(-\frac{\theta\alpha u}{1 + \theta}\right).$$

See Asmussen [20] for the proof.

This formula can be expressed in terms of $\psi(0)$ and adjustment coefficient, R ,

$$\begin{aligned}\psi(u) &= \psi(0) \exp\left(-\frac{\theta\alpha u}{1 + \theta}\right), \\ &= \psi(0) \exp(-Ru).\end{aligned}$$

where $R = \alpha - \lambda/c$ with $c = (1 + \theta)\lambda\alpha$.

2.5. Approximations to the Infinite Time Ruin Probability

When the individual claim amounts are distributed exponentially, simple analytic solution for the infinite time ruin probability can be obtained. Even though it is possible to estimate the infinite time ruin probability for the exponential case, for other claim amount distributions the Laplace transform does not work and some approximations are needed.

2.5.1. Cramer-Lundberg Approximation

In the classical risk process, $N(t)$ is assumed as a homogeneous Poisson process with intensity λ , $N(t) \sim \text{Poisson}(\lambda t)$, $U(t)$ is called a classical risk model or Cramer-Lundberg model.

Cramer condition is found by using Essecher transform of F ,

$$\frac{\rho}{\mu} \int_0^\infty e^{rx} (1 - F(x)) dx = 1,$$

where $\rho = 1/(1 + \theta)$.

Definition 2.5.1. *Adjustment coefficient*

From the Cramer condition it follows that

$$\frac{M_x(R) - 1}{r} = \mu(1 + \theta),$$

where $M_x(R)$ is the moment generating function of X .

In a risk process with claims distributed as $X \geq 0$ having $E[X] = \mu > 0$. The equation has a solution equal to zero and the adjustment coefficient R is the positive solution of the following equations.

$$1 + (1 + \theta)\mu_1(r) = \mu_X(r). \quad (2.3)$$

It is assumed that $X \geq 0$ and the first moment of the individual claims is $E[X] = \mu_1 > 0$ in the risk process.

The adjustment coefficient can also be obtained from $\lambda + cR = \lambda M_X(R)$. When S is the total claims in an interval of length 1; consequently $(c - S)$ is the profit in that interval. It is assumed that S is a compound Poisson with parameter λ and the moment generating function of S is $M_S(R) = e^{\lambda(M_X(R)-1)}$. Then R can be found as the positive root to any of the following equations [3].

$$e^{Rc} = E[e^{RS}] \quad \Leftrightarrow \quad M_{c-S}(-R) = 1 \quad \Leftrightarrow \quad c = \frac{1}{R} \log(M_S(R))$$

It should be noted that the adjustment coefficient is used as a risk aversion coefficient in the case of exponential utility function. When the individual claim is distributed exponentially with parameter $\beta = 1/\mu_1$, the corresponding adjustment coefficient is obtained as a positive root of the following equation:

$$1 + (1 + \theta)\mu_1(r) = M_X(r) = \frac{\beta}{\beta - r}.$$

where $R < \beta$. Hence, the explicit solution for the adjustment coefficient is

$$r = R = \frac{\theta\beta}{1 + \theta}.$$

For most of the other distributions, the explicit expression for the adjustment coefficient cannot be obtained.

Theorem 2.5.1. *Lundberg's inequality*

Lundberg's inequality for the ruin probability is

$$\psi(u) \leq e^{-Ru}.$$

In this inequality, for a compound Poisson risk process, u represents the initial capital and R represents the adjustment coefficient. The proof of this inequality is given in Kaas [3] by using the ruin probability that ruin occurs at or before the k^{th} claim.

When the surplus process $U(t)$ is based upon a compound Poisson aggregate claim process, $S(t)$, with $c > \lambda\mu$, the infinite time ruin probability for initial surplus $u \geq 0$ is obtained with positive relative security loading ($\theta > 0$) as [21]

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)} | T < \infty]}.$$

which R is the smallest positive root of

$$1 + (1 + \theta)\mu r = M_x(r).$$

A closed form of this equation cannot be obtained. However, it can be used to derive an inequality. Since $U(T)$ is necessarily negative when $T < \infty$ and so that the denominator of the equation exceeds 1. Thus, a simplified upper bound for the infinite time ruin probability yields that

$$\psi(u) \leq e^{-Ru}.$$

Cramer-Lundberg's asymptotic infinite time ruin probability formula for large value of initial surplus u is given as

$$\psi_{CL}(u) = Ce^{-Ru},$$

where

$$C = \frac{\theta\mu}{M'_X(R) - \mu(1 + \theta)}.$$

The main limitation of the Cramer-Lundberg approximation is that it requires the existence of the adjustment coefficient and takes light-tailed distributions into consideration. When the individual claim amounts have exponential distribution, this formula yields the exact infinite time ruin probability.

Theorem 2.5.2. *Integro-Differential Equation*

Let $\phi(u) = 1 - \psi(u)$ is the probability of the ruin never occurs. This probability is also known as non-ruin or survival probability. When $u > 0$, the infinite time ruin probability $\psi(u)$ satisfies the following equation, called Integro-Differential equation, [15]

$$\frac{d}{du}\phi(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c} \int_0^u f(x)\phi(u-x)dx,$$

and it follows that

$$\frac{d}{du}\psi(u) = \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c} \int_0^u f(x)\psi(u-x)dx - \frac{\lambda}{c}(1 - F(u)).$$

Infinite time ruin probability can also be expressed by using the defective renewal equation.

Theorem 2.5.3. Defective Renewal Equation

When $u > 0$, the infinite time ruin probability $\psi(u)$ satisfies the following equation, called defective renewal equation,

$$\psi(u) = \psi(0) \int_0^u \psi(u-s) dF_{L_1}(s) + \psi(0) \bar{F}_{L_1}(u), \quad u \geq 0.$$

where F_{L_1} is the ladder height distribution [15].

2.5.2. Approximation to the Infinite Time Ruin Probability with Exponential Claims Amounts

This approximation is derived by De Vylder [22]. For the sake of applicability, the first three moments must be finite. The ruin probability can be calculated under the exponential approximation as

$$\psi_E(u) = \exp \left\{ -1 - \frac{2\mu\theta u - M''}{\sqrt{(M'')^2 + (4/3)\theta\mu M'''}} \right\}.$$

In the literature, there are also other approximation methods to obtain the infinite time ruin probability such as Lundberg, Beekman-Bowers, Renyi, De Vylder, heavy-light traffic and heavy-tailed claims approximations. Lundberg approximation is originated from Grandell [23] and this approximation requires that the first three moments are finite. Beekman-Bowers approximation is suggested by Beekman [24] and then it is developed by Bowers [12]. This approximation needs the first three moments exist. Beekman-Bowers approximation gives the exact infinite time ruin probability when the individual claim amounts are assumed to be distributed exponentially. In Renyi approximation, gamma distribution is replaced with the exponential distribution for matching only the first moment. In this way, Renyi approximation is explained by using the simplified version of Beekman-Bowers approximation [23]. De Vylder approximation is suggested by De Vylder [25] to approximate the ruin probability by using different claims surplus process. This approximation gives the exact infinite time ruin probability when the individual claims are assumed to be distributed exponentially. “Heavy traffic and light traffic” is related with the queuing theory. Heavy traffic is explained as the average of the premiums slightly exceeds the expected claims while light traffic is described as the average of the premiums largely exceeds the expected claims.

Furthermore, while safety loading factor is small and positive in heavy traffic, in light traffic it is positive but large. Asmussen [20] suggests the heavy and light traffic approximations for the infinite time ruin probability. Heavy-light traffic approximation is proposed as a combination of heavy traffic and light traffic approximations [20]. Embrechts et al. [26] summarises the heavy-tailed distribution and its subclasses briefly and also show how asymptotic formula for the infinite time ruin probability can be obtained under these subclasses.

In this section, approximations to the infinite time ruin probability are presented. Asmussen [20] describes the exact formula and approximations to the infinite time ruin probability in detail. Burnecki et. al. [27] summarise the exact and approximation techniques for the infinite time ruin probability. In addition, Burnecki et. al. [27] demonstrate the effectiveness of the distribution on the approximations to the infinite time ruin probability as shown in Table 2.1

Table 2.1: The effectiveness of the distributions to the infinite time ruin probability

Method	Distribution						
	Exponential	Gamma	Weibull	Mix-Exponential	Log-normal	Pareto	Burr
Cramer- Lundberg	+	+	-	+	-	-	-
Exponential	+	+	+	+	+	$\alpha > 3$	$\alpha\tau > 3$
Lundberg	+	+	+	+	+	$\alpha > 3$	$\alpha\tau > 3$
Beekman-Bowers	+	+	+	+	+	$\alpha > 3$	$\alpha\tau > 3$
Renyi	+	+	+	+	+	$\alpha > 2$	$\alpha\tau > 2$
De Vylder	+	+	+	+	+	$\alpha > 3$	$\alpha\tau > 3$
Heavy Traffic	+	+	+	+	+	$\alpha > 2$	$\alpha\tau > 2$
Light Traffic	+	+	+	+	+	+	+
Heavy-Light Traffic	+	+	+	+	+	$\alpha > 2$	$\alpha\tau > 2$
Subexponential	-	-	$0 < \tau < 1$	-	+	+	+
Pollaczek-Khinchine	+	+	+	+	+	+	+

Table 2.1 shows that light traffic approximation and Pollaczek-Khinchine formula are applicable to any distribution without any assumption.

2.6. The Finite Time Ruin Probability

In this section, the finite time ruin probability is presented and the exact formula and approximations to the finite time ruin probability which are used in the literature are summarised.

As in infinite time case, we assume that the number of claims has Poisson process with rate λ and hence, the aggregate claims are assumed as the compound Poisson process. Premiums are also assumed as payable with a rate c per unit time.

The probability of ruin in continuous and the finite time is defined as

$$\psi(u, t) = \Pr(U(s) < 0 \quad \text{for some } s, \quad 0 < s \leq t),$$

where $\psi(u, t)$ is the probability that the insurer's surplus falls below zero in the finite time interval $(0, t]$. Contrary to infinite time case, there is no general finite time ruin probability formula such as Pollaczek-Khinchine formula.

2.6.1. Exact Finite Time Ruin Probability with Exponential Claim Amounts

Asmussen [28] presents a finite time ruin probability formula when the individual claim amounts are distributed exponentially with $\beta = 1$ and the premium rate per unit time is equal to 1 ($c = 1$). Hence, the ruin probability for finite time is calculated as

$$\psi(u, T) = \lambda \exp\{-(1 - \lambda)u\} - \frac{1}{\pi} \int_0^\pi \frac{f_1(x)f_2(x)}{f_3(x)}, \quad (2.4)$$

where

$$f_1(x) = \lambda \exp\left\{2\sqrt{\lambda T} \cos(x) - (1 + \lambda)T + u(\sqrt{\lambda} \cos(x) - 1)\right\},$$

$$f_2(x) = \cos\left(u\sqrt{\lambda} \sin(x)\right) - \cos\left(u\sqrt{\lambda} \sin(x) + 2x\right),$$

and

$$f_3(x) = 1 + \lambda - 2\sqrt{\lambda} \cos(x).$$

An important implication of this method is to remove the restriction on the parameter of individual claims distribution and premium rate. When $\beta \neq 1$, the following equation is applied [20].

$$\psi_{\lambda,\beta}(u, T) = \psi_{\frac{\lambda}{\beta},1}(\beta u, \beta T), \quad (2.5)$$

and the following equation is valid when $c \neq 1$, [27]

$$\psi_{\lambda,c}(u, T) = \psi_{\frac{\lambda}{c},1}(\beta u, cT). \quad (2.6)$$

In the literature, there are also two finite time ruin probability formulas which are suggested by Seal [29] and Takacs [30]. Both of these methods depend on different numerical integrations. However, these methods may not be practical in all situations because of the unstability for the large value of t [20].

2.7. Approximation to the Finite Time Ruin Probability

In the literature, there are many approximations for the calculation of the finite time ruin probability. A few of these approximations are obtained by making some adjustments to the infinite time methods. Some of frequently used approximations for the finite time ruin probabilities are given below.

Method 2.7.1. *Prabhu's Formula* Prabhu [31] gives a formula for the finite time ruin probability in the classical risk process for $u \geq 0$. In this formula, $\psi(0, T)$ is expressed in terms of the distribution function.

For $T > 0$, Prabhu's formula is

$$\psi(u, T) = F(u + ct, t) - c \int_0^T (1 - \psi(0, T - t)) f(u + ct, t) dt,$$

where $f(x, t) = \frac{d}{dx} F(x, t)$ and $1 - \psi(0, T) = \frac{1}{cT} \int_0^{cT} F(x, T) dx$.

Seal [29] considers the exponential claims and shows how to apply Prabhu's formula in this case using the fact that a closed form exists for $F^{n*}(x)$ and its derivative.

In addition, Segerdahl [32] suggests a formula to obtain the finite time ruin probability. This method depends on adding a time factor to the Cramer-Lundberg approximation. This approximation requires the existence of the adjustment coefficient R and the moment generating function M_X . Therefore, this approximation can only be used for the light-tailed distributions.

In the calculation of the finite time ruin probability, some diffusion approximations are also developed. In these approximations, it is assumed that the claim amount distributions belong to the domain of attraction of the normal law or have a light tail [33], see also Grandell [34] and Asmussen [20].

2.7.1. Gamma and the Translated Gamma Process Approximation

2.7.1.1. The Gamma Process Approximation

The gamma process is first introduced into the risk theory by Dufresne et al. [35]. They identify the gamma process by using a non-increasing and non-negative function of x , denoted by $Q(x)$, with the properties $Q(x) \rightarrow 0$ as $x \rightarrow \infty$ and

$$\int_0^{\infty} Q(x) dx < \infty.$$

They assume that this function is differentiable and $-Q'(x)$ can be obtained as

$$-Q(x) = q(x) = \frac{a}{x} \exp\{-b x\}, \quad x > 0,$$

where a and b are positive constants. A process $\{S(t)\}_{t>0}$ can be defined by using the function $Q(x)$. Under the assumption that $\{S(t)\}_{t>0}$ is related to the aggregate claims, the expectation of the number of claims for an interval with length t is defined by

$$t Q(x) = at \int_x^{\infty} \frac{e^{-by}}{y} dy.$$

Furthermore, the expected aggregate claim amounts in an interval with length t is defined by

$$E[S(t)] = t \int_0^{\infty} y q(y) dy = at \int_0^{\infty} e^{-by} dy = \frac{at}{b}.$$

To obtain the distribution of $S(t)$, the equation given below is obtained by using Laplace transforms:

$$\begin{aligned} E[e^{-zS(t)}] &= \exp \left\{ t \int_0^{\infty} [e^{-zy} - 1] q(y) dy \right\}, \\ &= \exp \left\{ at \int_0^{\infty} \frac{e^{-(z+b)y} - e^{-by}}{y} dy \right\}, \\ &= \left(\frac{b}{z+b} \right)^{at}. \end{aligned}$$

This formula shows that the distribution of $S(t)$ is gamma with shape parameter $\alpha_t = at$ and scale parameter $\beta_t = b$, and the process is called *gamma process* [35].

We use the notation $\{S_G(t)\}_{t>0}$ for a Gamma (α, β) process. When $\alpha = \beta = 1$, the gamma process is called *standardized gamma process* and denoted by $\{S_{SG}(t)\}_{t>0}$. The relationship between these two processes is

$$S_G(t) = \beta^{-1} S_{SG}(\alpha t). \quad (2.7)$$

Dickson and Waters [36] discuss how to approximate ruin probabilities in the classical risk model by using gamma process. In particular, they approximate the compound Poisson process $\{S(t)\}_{t>0}$ by a Gamma process with Poisson parameter λ . The approximation is based on matching the moments of the compound Poisson process with the Gamma (α, β) process $\{S_G(t)\}_{t>0}$.

For each value of $t \geq 0$, the mean and variance can be matched as

$$E[S(t)] = \lambda t m_1 = \alpha t / \beta = E[S_G(t)],$$

$$V[S(t)] = \lambda t m_2 = \alpha t / \beta^2 = V[S_G(t)],$$

which gives $\beta = m_1/m_2$ and $\alpha = \lambda m_1^2/m_2$.

Dickson and Waters [36] also assume that the premium income is collected continuously at constant rate c per unit time such as

$$c > E[S_{SG}(1)] (= 1).$$

The finite time survival probability, denoted by $\delta(U, t) = 1 - \psi(u, t)$, is defined as in the classical risk process as

$$\delta(U, t) = \Pr(S(\tau) \leq U + c\tau \quad \text{for all } \tau, \quad 0 < \tau \leq t).$$

This probability can be calculated by using a gamma process approximation,

$$\delta_G(U, t) = \Pr(S_G(\tau) \leq U + c\tau \quad \text{for all } \tau, \quad 0 < \tau \leq t).$$

Dickson and Waters [36] suggest that $\delta_G(U, t)$ is calculated by standardized Gamma distribution as

$$\delta_G(U, t) = \delta_{SG}(\beta U, \alpha t). \tag{2.8}$$

As a result, $\delta_G(U, t)$ is calculated by using the standardized Gamma process by using βU as the parameter instead of U , and αt as the parameter instead of t in the standardized gamma process.

Dickson and Waters [36] also show that

$$\delta_{SG}(0, t) = F_{SG}(ct, t) - \frac{1}{c} F_{SG}(ct, t + 1) \quad (2.9)$$

and

$$\begin{aligned} \delta_{SG}(U, t) = & F_{SG}(U + ct, t) - c \int_0^t f_{SG}(U + cs, s) F_{SG}(c(t - s), t - s) ds \\ & + \int_0^t f_{SG}(U + cs, s) F_{SG}(c(t - s), t - s + 1) ds. \end{aligned} \quad (2.10)$$

where $f_{SG}(x, t)$ and $F_{SG}(x, t)$ are the density and distribution functions of a $Gamma(t, 1)$ random variable, respectively [36].

2.7.1.2. The Translated Gamma Process Approximation

The aggregate claims process $\{S(t)\}_{t>0}$ can be approximated by using a translated gamma process $\{S_{TG}(t)\}_{t>0}$. The structure is similar to a gamma process except for the parameter k .

For all $t > 0$, the translated gamma process can be defined by using the gamma process as follows:

$$S_{TG}(t) = S_G(t) + kt,$$

where $\{S_G(t)\}_{t>0}$ is a Gamma (α, β) process and k is a constant.

For all $t > 0$, the mean, variance and coefficient of skewness of $S(t)$ and $S_{TG}(t)$ are matched. According to the matching method, the parameters of the translated gamma process α , β and k are found as follows:

$$\begin{aligned} E[S(t)] &= \lambda t m_1 = \alpha t / \beta + kt = E[S_{TG}(t)], \\ V[S(t)] &= \lambda t m_2 = \alpha t / \beta^2 = V[S_{TG}(t)], \\ Sk[S(t)] &= \lambda t m_3 / (\lambda t m_2)^{3/2} = 2 / (\alpha t)^{1/2} = Sk[S_{TG}(t)]. \end{aligned}$$

The mean, variance and coefficient of skewness of $S(t)$ and $S_{TG}(t)$ for all t are matched.

These identities give the parameter values as

$$\alpha = 4\lambda m_2^3/m_3^2, \quad \beta = 2m_2/m_3, \quad \text{and} \quad k = \lambda(m_1 - 2m_2^2/m_3). \quad (2.11)$$

The finite time ruin probability for a translated gamma process is calculated by using a gamma process. Dickson and Waters [36] show that the non-ruin probability can be found as

$$\delta_{TG}(U, t) = \Pr(S_G(\tau) \leq U + (c - k)\tau \text{ for all } \tau, 0 < \tau \leq t).$$

Note that under this moment matching,

$$c - k = \theta \lambda p_1 + \alpha/\beta, \quad (2.12)$$

where θ is the premium loading factor. Dickson and Waters [36] calculate the loading factor for the translated gamma process. This loading factor can be written in terms of the loading factor of the compound Poisson process. The relationship between the loading factors can be explained as

$$(1 + \hat{\theta}) \frac{\alpha}{\beta} = c - k,$$

where $E[S(1)] = \lambda p_1 = \frac{\alpha}{\beta} + k$. Then the loading factor for the translated gamma process approximation can be obtained as

$$\hat{\theta} = \theta \left(1 + \frac{k\beta}{\alpha}\right). \quad (2.13)$$

According to the translated gamma process approximation, the finite time survival probability is obtained by using the premium loading factor as $\hat{\theta} = \theta (1 + k\beta/\alpha)$ when the aggregate claims process is the translated $Gamma(\alpha, \beta)$ process [36].

2.8. Interim Conclusion: A Review of Risk Theory

In this chapter, we have presented the risk theory literature. Great effort has been devoted to estimate the aggregate claims process correctly for several years. In the first step, we have summarised the approximation methods to estimate the aggregate claims process. In this context, the approximation methods are explained with their advantages and disadvantages. It can be concluded that the translated gamma process approximation fits better to the right tail of aggregate claims than the other approximations.

Furthermore, research's attention has been concentrated not only on estimation of the aggregate claims but also on calculation of the ruin probability. In recent years, ruin theory has become very popular subject among the researchers. Ruin probability can be obtained by exact formula according to the individual claim distribution. However, it is not always possible to find the exact formula. Therefore, the approximation methods are applied to obtain the ruin probability. In this chapter, we have described the exact formulas and approximation methods which are frequently used in actuarial studies.

Approximations are analyzed and compared with various aspects of closeness to the exact value. Thus, approximations are obtained as comparable with the exact value of ruin probability. The studies show that obtaining the finite time ruin probability is more complicated than the infinite time ruin probability. It should also be noted that the distribution of the individual claim amount plays a crucial role in choosing the best approximation for both finite and infinite time cases.

The literature on risk theory indicates that it is possible to obtain the exact formula for both finite and infinite time ruin probability when individual claims are distributed exponentially. On the other hand, if the individual claims are not exponentially distributed, some approximations are needed to calculate the ruin probabilities.

3. OPTIMAL REINSURANCE

3.1. Introduction

The main purpose of this chapter is to present the commonly used and the most preferred methods on optimal reinsurance under the ruin probability constraint. In this context, we review the literature to decide which one of these methods are substantial. In Section 3.2, we discuss the literature review on optimal reinsurance related with the ruin probability. The literature review on optimal reinsurance under the ruin probability constraint is mainly focus on four major aspects: reinsurance arrangement, adjustment coefficient, investment and dynamic optimization techniques, and dividend payments. In Section 3.3, we discuss the consequences of optimal reinsurance criteria. In this context, we propose a new method for the optimal reinsurance which contains three main measures: released capital, expected profit and expected utility. We also explain how these measures are calculated. Finally we conclude the chapter in Section 3.4.

3.2. Optimal Reinsurance Under the Ruin Probability Constraint

In a reinsurance arrangement, the optimal level and the type of reinsurance can be determined under a constraint such as ruin probability or risk measure. Most of the actuarial literature focus on the determination of an optimal level for a reinsurance arrangement. Many studies have been carried out on the calculation of optimal reinsurance under the ruin probability constraint. The studies that consider the optimal reinsurance level under the ruin probability constraint can be categorized according to the reinsurance principles, the adjustment coefficient, investment and dynamic optimization techniques and dividend payments. We mainly point out recent studies which discuss the optimal reinsurance under the ruin probability constraint.

Previous studies indicate that the optimal reinsurance on ultimate ruin probability is mostly related with the adjustment coefficient. Thus, such as in Cramer-Lundberg approximation, the ruin probability decreases fastly when the adjustment coefficient increases. The effect of the adjustment coefficient on the ultimate ruin probability under the Cramer-Lundberg approximation is examined in various papers.

One of the first examples of minimizing the ruin probability by using the adjustment coefficient is presented by Gerber [15]. Gerber shows the optimal retention level for the excess of loss reinsurance. Gerber [15] assumes that the expected claim amount for the reinsurer is a function of the individual claim amount in order to maximise the adjustment coefficient R .

The same results are obtained by Bowers [21] who suggests to consider the reinsurance to maximise the expected utility.

Waters [37] introduces the optimal levels of the excess of loss reinsurance for a portfolio of risks in order to minimise the ruin probability. The aggregate claims are distributed as a compound Poisson and the expected value premium principle is used for the calculation of the reinsurance premiums. Waters [37] aims to find an optimal level for the excess of loss reinsurance under the minimum ultimate ruin probability. The ruin probability is minimised by using the insolvency constant (as adjustment coefficient) of the insurer and this constant is defined as a function of the excess of loss reinsurance retention level. He also investigates how this insolvency constant can be maximised.

Waters [38] extends his study according to the different individual claims and premium principles. The results suggest that the optimal retention level is obtained to satisfy the equation of $M = R^{-1} \ln(1 + \theta)$ under the assumption of the aggregate claims are compound Poisson and premiums are calculated according to the expected value premium principle. It is also featured that the adjustment coefficient is a function of the retention level in a proportional case without any restrictive assumptions on the distribution of the annual claims. Moreover he finds that in a non-proportional case, it cannot be possible to generalize the distribution of annual claims or the way in which the reinsurance premium is calculated. This approach is proposed for the proportional case.

Centeno [39] assumes that the annual claims are distributed as compound Poisson and the premiums are calculated according to the expected value principle. She implements the combinations of quota-share with the excess of loss reinsurance which is suggested by Centeno [40] under the assumption of the claims described as an ordinary renewal process.

Centeno [41] also points out an approach for the insurers' adjustment coefficient as a function of the retention levels. Centeno [41] relates the retention level to a combination of quota-share with the excess of loss reinsurance in the Sparre Anderson model.

Hesselager [42] aims to find optimal reinsurance treaties in order to minimise the ruin probability. Hesselager [42] uses three different functions to identify the reinsurance arrangements and also uses Vajda's compensation functions to compare the reinsurer's adjustment coefficient of these functions.

Centeno [43] considers the optimal retention level M under the excess of loss reinsurance so that the upper bound of the probability of ruin is minimised. Centeno [43] improves Lundberg's inequality for the finite time to redefine the optimal retention limit for an excess of loss arrangement. Moreover Centeno [43] defines the optimal retention M , in the manner that minimises the upper bound of the ultimate ruin probability.

Dickson and Waters [44] indicate obtaining the optimal retention level when the reinsurance loading factor depends on the retention level. Dickson and Waters [44] discuss the effect of the reinsurance both on finite and infinite time ruin probabilities. Additionally, this paper demonstrates the feasibility of the translated gamma process on the optimal retention levels.

Liang and Guo [45] mention that the insurer's surplus process is defined by Brownian motion and Liang and Guo [45] propose an explicit expression in the diffusion approximation and jump-diffusion case to find an optimal strategy to minimise the ruin probability by maximizing the adjustment coefficient.

Guerra and Centeno [46] study on the optimal reinsurance policy considering both the adjustment coefficient and the expected utility. Guerra and Centeno [46] deal with the relationship between maximizing the adjustment coefficient and maximizing the expected utility of wealth.

Kolkovska [47] analyses the reinsurance treaties and the existence of the insurer's adjustment coefficient as a function of the retention levels. Kolkovska [47] proposes two different types of reinsurance treaties as a combination of excess of loss and quota-share.

Moreover, Kolkovska [47] shows some results obtained from these arrangements.

Centeno and Simoes [48] present a summary of the previous studies on optimal reinsurance.

The focus of recent studies is to construct the optimal reinsurance by reinsurance principles.

One of the first examples of optimal reinsurance by reinsurance principle is presented by De Finetti [49]. This paper clearly shows that the insurer's risk such as the variance of the profit and ruin probability can be reduced by using any reinsurance arrangement. He determines the optimal retention level for a non-life insurance portfolio under a constraint of a minimum variance of the insurer's profit for a fixed expected profit. De Finetti [49] shows how optimal levels should be calculated for both excess of loss and the proportional reinsurance under the minimum variance criterion for the insurer's expected profit. De Finetti [49] points that the retention level is proportional to the insurance loading factor and inversely proportional to the variance of the risk. Further details and proofs can be found in Bühlmann [50].

Dickson and Waters [51] consider the optimal reinsurance levels under the criterion of minimizing the finite time ruin probability for both discrete and continuous time in a non-life insurance portfolio. Dickson and Waters [51] focus on minimizing the ruin probability instead of the variance criterion in De Finetti's approach [49] and Dickson and Waters [51] describe a constraint as the expected profit per unit time is greater than zero or equal to it. In the paper, it is assumed that the aggregate claims process can be approximated by a translated gamma process.

Ignatov et al. [52] derive a formula for the expected profit under the probability of survival of the insurer. Ignatov et al. [52] consider estimating the optimal levels by using an optimality criterion for the expected profit formula from both the insurer's and the reinsurer's points of view. The optimality is explained as the levels that maximise the joint survival probability up to a finite time horizon of the cedent and the reinsurer.

Kaluszka [53] proposes the optimal reinsurance to minimise the ruin probability for the truncated stop loss reinsurance. Kaluszka [53] applies the truncated stop loss contracts for different pricing rules such as economic principle, generalized zero-utility principle, Esscher principle and mean-variance principle.

Kaluszka [53] indicates that the result is also valid for models not involving the ruin probability such as the expected utility, the Kahneman-Tversky value function of the cedent, the variance of the cedents cover and the dividend payment.

Dickson and Waters [54] minimise the ruin probability by using the retention levels in accordance with a dynamic reinsurance strategy. They derive a formula for the finite time ruin probability for discrete and continuous time by using the Bellman optimality principle. In addition, they show how this formula can be used both by approximating the compound Poisson aggregate claims distribution with a translated gamma distribution, and by approximating a compound Poisson process with a translated gamma process, respectively.

Kaishev and Dimitrova [55] generalize a joint survival optimal reinsurance model for the excess of loss reinsurance. Kaishev and Dimitrova [55] assume that the individual claim amounts are modelled by continuous dependent random variables with a joint distribution. Based on maximizing the probability of joint survival and level of premium income, the optimal levels of the excess of loss reinsurance are obtained.

Nie et al. [2] propose a different kind of reinsurance arrangement in which the reinsurer's payments are bounded above by a fixed level. According to Nie et al. [2], whenever the surplus falls between 0 and this fixed level, the reinsurance company makes an additional payment in such a way that the surplus process returns to the fixed level. The reinsurance premium is also calculated according to these capital injections. The optimal initial surplus and the fixed reinsurance level are calculated so that the ultimate ruin probability is minimised.

Several publications have appeared in recent years documenting on constructing optimal reinsurance by using investment and optimization techniques.

Hipp and Plum [56] present that the ruin probability can be minimised by investment in a risky asset in the classical Cramer-Lundberg model. In this model, the interest rate of the bond is assumed to be zero. They use the Bellman equation to determine the optimal strategy and give explicit solution for the exponential and Pareto claim size. They also investigate the effect of the optimal investment strategy and ruin probability according to the changes in some factors such as stock volatility.

Liu and Yang [57] develop the model which is proposed by Hipp and Plum [56] with respect to a non-zero interest rate. Liu and Yang [57] assume that the premiums are calculated by a constant rate and investment in a risky asset. Liu and Yang [57] use Hamilton-Jacobi-Bellman equation to determine the optimal investment strategy and compare these strategies under the various claim size distributions. Liu and Yang [57] further assume that the fraction of an amount available for investment in a risky asset is proportional to insurer's surplus.

Schmidli [58] examines the dynamic proportional reinsurance to minimise the ruin probability. Schmidli [58] develops an optimal unlimited proportional reinsurance strategy. He also compares the results of both the diffusion and Cramer-Lundberg cases. He uses the Hamilton-Jacobi-Bellman equation and presents a proof of the existency of the solution.

Schmidli [59] suggests the optimal investment and reinsurance strategies to minimise the ruin probability. A key limitation of this research is the assumption of investing in only a risky assets which are modelled by Black-Scholes formula. The optimal levels of investment and reinsurance are obtained by using Hamilton-Jacobi-Bellman approach. Schmidli [59] concludes that the investment and reinsurance decrease the ruin probability for larger initial surplus under the Pareto claim sizes. Taksar and Markussen [60] develop this approach in regards to the diffusion model.

Hipp and Vogt [61] examine the dynamic excess of loss reinsurance. They suggest a dynamic unlimited excess of loss reinsurance strategy to minimise the infinite time ruin probability. Hipp and Vogt use the classical Hamilton-Jacobi-Bellman equation to maximise the survival probability for an optimal strategy and give numerical examples for three different claim size distributions namely exponential, shifted exponential, and Pareto.

Castillo and Parrocha [62] describe the stochastic control methods to determine the optimal investment strategy which minimises the ruin probability. In this context, the existence of a solution of Hamilton-Jacobi-Bellman equation is proved and numerical algorithm is given. They consider an insurance business with a fixed amount of investment in a portfolio which consists of one risky and one risk free asset.

Gaier and Grandits [63] develop an optimal investment with positive interest force to minimise the ruin probability. They analyze the ruin probability for the proportional investment for the claims distributed by regularly varying tails which is one of subclass of heavy tailed.

Hipp [64] briefly reviews the methods for the optimal investment problem by using stochastic control theory on minimizing the ruin.

Schmidli [65] uses both investment and proportional reinsurance in a classical risk process and defines the Cramer-Lundberg approximation for the ruin probability. It is assumed that investment into a risky asset is modelled by a Brownian motion. Under the assumption of non existence of the exponential moments, the Lundberg exponent is redefined and an upper bound for the ruin probability is calculated by using Hamilton-Jacobi-Bellman equation.

Irgens and Paulsen [66] discuss the maximization of the expected utility of the asset of an insurance company under the reinsurance and investment constraints in a diffusion-perturbated classical risk model. The risky or non-risky assets are used for the investment and some of the company's risk is covered by the proportional reinsurance and the remaining part is covered by the excess of loss reinsurance. They use three different utility functions for the expected utility: exponential, logarithmic and power utility with termination at ruin. A key limitation of this paper is that the insurance company cannot invest in a risk-free and a risky asset at the same time.

Ma et. al [67] concentrate on the possibility of investing in a risk-free asset to find the optimal proportional reinsurance strategy in order to minimise the probability of ruin in the classical risk model. They conclude that optimal proportional is valid to minimise the ruin probability or to maximise the survival probability.

Kasumo [68] and Joseph [69] determine the role of investment on minimizing the ruin probability. While Kasumo [68] studies the proportional reinsurance case, Joseph [69] analyses the excess of loss reinsurance. The risk process is assumed as a perturbed-diffusion risk model and investment is modelled as a Black-Scholes model in both studies.

In addition, the optimal investment is examined with Hamilton-Jacobi-Bellman approach and Volterra Integro Differential equation which is transformed into a linear Volterra Integral Equation of second order is proved.

The previous studies show that Hamilton-Jacobi-Bellman equation is necessary in determination of the optimal investment and reinsurance.

In the literature there are also some papers on constructing optimal reinsurance using dividend payments. The optimal dividend problem is suggested by De Finetti [70]. He proposes a strategy that "maximises the expected value of discounted dividend payments until the time of ruin". This problem is firstly solved by Gerber [71] for the classical Cramer-Lundberg model.

Dickson and Waters [72] extend De Finetti [70]'s approach by allowing the process to continue after the ruin. They examine the optimal dividend strategy which maximises the expected discounted future dividends by using barrier level in modified surplus process. They assume that when the surplus reaches the level b , the premium income cannot be used in the surplus process and the premium income directly goes into paying dividends to shareholders. Hence, the premium income is only enough to pay dividend and ruin occurs when the modified surplus process falls below 0.

Bäurle [73] concentrates on the reinsurance policy in the presence of dividend strategies for piecewise-deterministic Markov processes.

Kaluszka [53] examines the optimal reinsurance level on a truncated stop loss reinsurance and provides the optimality according to several models with regard to maximizing the expected dividend payment. He assumes that the insurer pay dividends if the insurer's gain exceeds a level such as expenses plus profit after the reinsurance.

Azcue and Muler [74] find a dynamic strategy which contains both the reinsurance policy and the dividend distribution strategy that maximises the cumulative expected discounted dividend payouts for Cramer-Lundberg model.

Schmidli [75] focuses on two main topics to minimise the ruin probability by using the proportional reinsurance and to maximise the expected discounted dividend payout. On minimizing the ruin probability part, he proves the existency and uniqueness of the Hamilton-Jacobi-Bellman equation and verification theorem. The Hamilton-Jacobi-Bellman equation is used on maximization of the expected discounted dividend payout and obtaining the optimal strategy.

Thonhauser and Albrecher [76] suggest that a value function which considers both expected dividends and the time value of ruin to find the optimal dividend payment strategy for maximizing the expected value of the discounted dividend payments until ruin. The optimality of barrier strategy is obtained for both the diffusion model and the Cramer-Lundberg model with exponential claim sizes.

Beveridge et al. [77] examine to find out whether or not the expected present value of net income to shareholders can be increased by reinsurance when there is a constant dividend barrier. In addition, they use both De Vylder's approximation and Dickson and Waters's discrete time model [72] to identify an optimal reinsurance strategy.

Liang and Yao [78] present a nonlinear-singular stochastic optimal control on reinsurance rate and dividend payout process under the ruin probability constraints. They aim to maximise the expected present value of the dividend payments until the time of ruin. They also discuss solving the optimality problems such as optimal retention ratio, dividend payout level, optimal return function and optimal control strategy of the insurance company.

Meng and Siu [79] describe the optimal reinsurance and dividend strategy under the consideration of reinvestment or retained earnings. Meng and Siu aim to maximise the difference between the expected discounted dividends minus the expected discounted reinvestment until the time of ruin. In this context, the optimal reinsurance strategy is obtained under the excess of loss reinsurance.

Zhou and Yuen [80] concentrate on the optimal dividend strategy with the proportional reinsurance and capital injection in a large portfolio.

Zhou and Yuen [80] study on maximizing differences between the expected value of the discounted dividend payments and the discounted costs of capital injection. Optimal strategy is obtained in consideration of the proportional reinsurance, variance premium principle, and capital injection at ruin time.

Wu [81] focuses on maximizing the difference between the cumulative expected discounted dividend and the penalized discounted capital injections until the time of ruin. It is assumed that the company controls the dividend payments to the shareholders as well as capital injection. He assumes that optimal combination reinsurance strategy consists of only the excess of loss reinsurance.

The literature review shows that most of the studies focus on optimal reinsurance based only on a single constraint and the optimal reinsurance strategy change under different constraints.

3.3. Optimal Reinsurance Criteria

We aim to determine the essential factors for the insurer's income and profit after a reinsurance arrangement. For this reason, firstly, we specify the main factors for the insurance company. We develop a new approach which involves the released capital, the insurer's expected profit and expected utility. The difference between two initial surplus levels according to existence of a reinsurance arrangement is identified as the released capital. Expected profit is explained as the difference between net premium income and the expected claims, whereas the insurer's expected utility is calculated as the utility of the insurer's surplus by using an utility function.

3.3.1. Released Capital

The insurer's surplus must start with an adequate initial surplus. Without reinsurance, this initial surplus will usually be much greater than in the reinsurance case. While the ruin probability is fixed at a minimum level, the required initial surplus will be smaller than the reinsurance case.

The differences between these two amounts are called as savings amounts or gains by depending on a reinsurance arrangement. We refer to these as the released capital.

In this study, in the case of no-reinsurance, the required initial surplus subject to a fixed ruin probability is denoted by u_L and it is called as *the largest initial surplus*. Similarly, in a reinsurance case, the required initial surplus subject to the same ruin probability is denoted by u_S , and it is called as *the smallest initial surplus*.

3.3.2. Expected Profit

The profit of an insurance company consists of many items. According to the investment source and the expenses of the insurance company, the form of the insurance profit can be changed. Besides, the general form of the insurance profit is defined as income minus expenses. An insurance company obtains the benefit payments with their premium income. After the benefit payments, the net assets will be the profit of the insurance company. Insurance profit is formulated in many articles in the literature.

Schnieper [82] classifies the insurance company's profit and loss account in terms of the details of income and expenditure. He proposes an equation which identifies the increase amount in capital which consists of pure risk premium, profit loading for underwriting risk, the total claims amount, the investment income and capital gains. Forbes [83] examines the growth of an insurance company as a combination of the pairs of the underwriting and investment income, dividend policies and external financial income. Leen and Forbes [84] explain the changes in a non-life insurer's capital and surplus as a point of net income or loss after taxes, unrealized capital gains, the dividends payments, and external financing. Oyatoye and Arogundade [85] assume that the profit of an insurance company can be explained as a composite function consisting of pure risk premium, loading factor expenses and loading for profit.

In the theory and practice, the profit which is obtained only from insurance or reinsurance coverage is identified as *underwriting profit*.

We assume that the premiums and claims are the main items in insurance profit so we ignore other factors such as investment income, dividends payments and taxes.

Insurance and reinsurance premiums can be calculated by using different premium calculation principles. In this study, insurance and reinsurance premiums are calculated according to the expected value premium principle with insurance loading factors θ and reinsurance loading factor ξ , respectively, where $\xi > \theta$. The remaining premium income after payment of a reinsurance premium is called as the net premium.

For each value of $t > 0$, $E[X]$ denotes the expected individual claim amount and $E[Z]$ denotes the expected individual claim amounts paid by the reinsurer and $E[N]$ denotes the expected number of claims. Hence, net premium up to time t is denoted by $\Pi(t)_{net}$ and calculated as

$$\Pi_{net}(t) = (1 + \theta) E[N] E[X] - (1 + \xi) E[N] E[Z]. \quad (3.1)$$

$E[Z]$ changes depending on the retention level M under the excess of loss reinsurance or proportion p under the proportional reinsurance. The insurance and reinsurance loading factors are indicated as θ and ξ , respectively.

Expected claims paid by the insurer up to time t are denoted as $E[S_I(t)]$ and calculated by using the expected individual claim amount paid by the insurer and the number of claims that occurs in this time interval $[0, t]$.

Thus, insurance profit which is calculated as the premium minus the loss at time t , denoted as $P(t)$, defined by

$$P(t) = \Pi_{net}(t) - S_I(t). \quad (3.2)$$

3.3.3. Expected Utility

Expected utility theory depends on an analysis that measures the decision maker's choice between risky or uncertain statement by comparing their expected utility values. In this selection process, the utility values of outcomes are multiplied by their respective probabilities. Expected utility hypothesis is initiated by St. Petersburg paradox. Nicolas Bernoulli describes the St.Petersburg paradox (involving infinite expected values) in 1713. Daniel Bernoulli [86] proposes that a mathematical function should be used to measure the expected value depending on probability. It is assumed that expected utility increases with a decreasing rate when the corresponding wealth increases.

Von Neumann and Morgenstern [87] present four properties of the expected utility theory. They define a rational decision maker and clarify an individual's choice as if and only if there is a utility function such that the expected utility of one exceeds that of the other. Von Neumann and Morgenstern develop the expected utility from the point of view of an investor. According to this theory, an investor chooses the outcome that has the highest expected benefits among the opportunities under risky situations.

A utility function measures the utility as a function of wealth. It should be a twice-differentiable function and for all $x > 0$, the inequality $U'(x) > 0$ must hold.

The form of the utility function can change depending on the individual's preferences. If the utility function is concave, the person is a risk averse person; if the utility function is convex, the person is a risk taker, and finally if the utility function is linear, then the person is a risk neutral agent and in this situation, expected utility of wealth is equal to a linear function of the expected wealth.

Four common utility functions are used in the literature: exponential, logarithmic, quadratic and fractional power. However, most studies focus on the exponential utility case. Hence, we assume that the insurer's expected utility is calculated by using an exponential utility function. We prefer this utility function since the outcomes are independent of the insurer's initial wealth. Dickson [11] shows that the initial wealth of the investor has an independent role on the decision making process and the decision only depends on the investment's properties for the exponential utility function.

The exponential utility function, denoted by u , is defined as

$$u(x) = 1 - \exp(-Bx), \quad (3.3)$$

where B is the parameter of the utility function, $B > 0$.

From the insurer's point of view, the insurer's wealth at the end of t , $U(t)$, is defined as

$$U(t) = u + \Pi_{net} - S_I(t).$$

To avoid the confusion, initial surplus is represented by u_0 instead of u . Then

$$\begin{aligned}
E[\mathbf{u}(U(t))] &= E[\mathbf{u}(u_0 + \Pi_{net} - S_I(t))], \\
&= E[1 - \exp(-B(u_0 + \Pi_{net} - S_I(t)))], \\
&= 1 - [E[\exp(-B u_0)] \times E[\exp(-B \Pi_{net})] \times E[\exp(B S_I(t))]] \quad (3.4)
\end{aligned}$$

Under the excess of loss reinsurance, the insurer's responsibility is limited to the retention level M yielding that the premiums are calculated according to the expected value premium principle with the insurance loading factor θ and the reinsurance loading factor ξ . Hence, the expected utility of the insurer's wealth is found by using the Equation (3.4). The second term on the right side of Equation (3.4) can be obtained as follows

$$E[\exp(-B \Pi_{net})] = \exp[-B(1 + \theta)E[N]E[X] + B(1 + \xi)E[N]E[X - S_I]] \quad (3.5)$$

where $E[S_I]$ denotes the expected claim amounts paid by the insurer according to the retention level M .

Under the proportional reinsurance, the insurer's responsibility is limited to the proportion p . Premiums are calculated according to the expected value premium principle with the insurance loading factor θ and the reinsurance loading factor ξ .

Similar to the excess of loss case, the second term on the right side of Equation (3.4) is found as

$$E[\exp(-B \Pi_{net})] = \exp[-B(1 + \theta)E[N]E[X] + B(1 + \xi)E[N](1 - p)E[X]]. \quad (3.6)$$

Additionally, $(1 - p)E[X]$ denotes the expected claim amounts paid by the reinsurer.

In this context, we focus on the released capital, the insurer's expected profit and the expected utility of insurer's surplus. We will consider a set of outcomes which the ruin probability is the same for each outcome.

We assume that each criterion is weighted by the value w_i . The weight to each criterion depend on its importance in the decision making. Moreover, the sum of the weights of criteria must be equal to 1.

These criteria indicate the relationship between the profit, the expected utility of insurer's risk process and the released capital gains in the case of reinsurance arrangement.

Criteria essentially depend on two main components: the reinsurance level and the initial surplus. The optimal initial surplus and reinsurance level can be obtained with respect to these three criteria. Selection of the best alternative of the initial surplus and reinsurance level depends on the decision theory. We will introduce the decision theory and its subclass, the multi criteria decision analysis in the following chapter.

3.4. Interim Conclusion: Optimal Reinsurance

In this chapter, we have focused on reviewing the recent studies which discuss the optimal reinsurance under the ruin probability constraint. From the research that has been conducted, it is possible to conclude that the optimal reinsurance under the ruin probability constraint can be categorised into four topics: the adjustment coefficient, the reinsurance principles, investment with dynamic optimization techniques and dividend payments. Moreover, the authors' attention is concentrated not only on finding the optimal reinsurance in order to minimise the ruin probability but also on maximizing some criteria such as investment earnings, dividend payments, expected utility or expected profit. However, to the authors' best knowledge, very few publications exist in the literature that discuss more than one measure. In addition, most of the previous studies on optimal reinsurance under the ruin probability constraints do not take multiple criteria into account.

We have started by investigating the former studies to determine which one is substantial and we have designed a new method that consists of three main criteria to determine the optimal reinsurance under the ruin probability constraints. With this approach, we have explored the possibility of obtaining the optimal reinsurance according to investigation on three main criteria: expected profit, released capital and expected utility. We have also given information about these criteria for the two types of reinsurance which are excess of loss and proportional reinsurance.

Most of the previous studies on optimal reinsurance do not take into account more than one criterion together. In this context, we have addressed not only to minimum ruin probability but also to maximum expected profit, maximum expected utility and maximum released capital of insurance company under the excess of loss and the proportional reinsurance.

Finally, in order to obtain the optimal reinsurance, we need to take advantage of decision theory. The existence of relationship among these three criteria implies that a decision making method which involves the correlation between criteria is required.

4. OPTIMAL REINSURANCE ON THE TRANSLATED GAMMA PROCESS APPROXIMATION

4.1. Introduction

In this chapter, we investigate how three measures: released capital, expected profit and expected utility are calculated under the translated gamma process approximation. We also take advantage of the decision theory to obtain the optimal pair of initial surplus and reinsurance level. We review the advantages and disadvantages of the multi attribute decision making methods to decide which one is the best in order to obtain the optimal pairs. The key limitation for the decision is that the method should involve the dependency between the criteria.

The layout of the chapter is as follows.

In Section 4.2, we examine the translated gamma process approximation under the reinsurance arrangement. We derive the formula for the finite time ruin probability under the excess of loss and proportional reinsurance. We assume that the aggregate claims process is a compound Poisson process with the exponential and Pareto individual claim amounts.

In Section 4.3, we consider the gain of the insurer under the assumption of the ruin probability is fixed at a minimum level. From the insurers' point of view, we use the optimal reinsurance criteria which consist of the released capital, expected profit and expected utility. These three criteria are calculated according to two types of reinsurance arrangements and two types of individual claim assumptions under the translated gamma process approximation.

In Section 4.4 we examine the decision theory methods in order to find the optimal reinsurance. We discuss the multi-criteria decision making methods according to its characteristics and applicability on decision problem. The classification of multi-criteria decision making is investigated. We particularly focus on multi attribute decision making. We consider methods for Cardinal Preference of Attribute Given according to their convenience on obtaining the optimal reinsurance.

Methods for Cardinal Preference of Attribute Given are explained due to the their applicability, advantages and disadvantages. In this context, Linear Assignment Method (LAM), Simple Additive Weighting Method (SAW), Hierarchical Additive Weighting Method (HAWM), The Analytical Hierarchy Process (AHP), Elimination et Choice Translating Reality (ELECTRE), The VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR), The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), and TOPSIS Method with Mahalanobis Distance are examined.

In Section 4.5 we conclude the chapter.

4.2. The Translated Gamma Process Approximation with Reinsurance

We introduce the results from Dickson and Waters [36] in Section 2.7 in which they give a finite time ruin probability formula for the translated gamma process. Dickson and Waters [36] suggest that $\delta_G(U, t)$ (survival or non-ruin probability) is calculated by using the standardized gamma process as

$$\delta_G(U, t) = \delta_{SG}(\beta U, \alpha t).$$

In the standardized gamma process, $\delta_G(U, t)$ is calculated by using βU parameter instead of U , and αt parameter instead of t .

Dickson and Waters [36] show that how the survival probability is calculated for zero initial surplus and positive initial surplus as

$$\delta_{SG}(0, t) = F_{SG}(ct, t) - \frac{1}{c} F_{SG}(ct, t + 1),$$

and then

$$\begin{aligned} \delta_{SG}(U, t) = & F_{SG}(U + ct, t) - c \int_0^t f_{SG}(U + cs, s) F_{SG}(c(t - s), t - s) ds \\ & + \int_0^t f_{SG}(U + cs, s) F_{SG}(c(t - s), t - s + 1) ds. \end{aligned}$$

where $f_{SG}(x, t)$ and $F_{SG}(x, t)$ are the density and the distribution function of a $Gamma(t, 1)$ random variable, respectively [36].

This approximation can be generalized when a reinsurance arrangement is included. In order to verify the validity of the reinsurance, we carry out two main reinsurance arrangements: excess of loss and proportional reinsurance. Following sections define the translated gamma process approximation under the excess of loss and proportional reinsurance, respectively.

4.2.1. Excess of Loss Reinsurance on the Translated Gamma Process Approximation

Under an excess of loss reinsurance arrangement, the insurer and the reinsurer's expected individual claim amounts are calculated according to a constant retention level M .

When a claim X occurs, the insurer pays $Y = \min(X, M)$ and the reinsurer pays $Z = \max(0, X - M)$ with $X = Y + Z$. Hence, the distribution function of Y , $F_Y(x)$, is

$$F_Y(x) = \begin{cases} F_X(x) & \text{for } x < M, \\ 1 & \text{for } x \geq M, \end{cases}$$

and the moments of Y is

$$E[Y^n] = \int_0^M x^n f(x) dx + M^n (1 - F(M)). \quad (4.1)$$

Similarly, the moments of Z are

$$E[Z^n] = \int_M^\infty (x - M)^n f(x) dx. \quad (4.2)$$

In the classical risk model, it is assumed that the number of claims have a Poisson distribution with parameter Λ . According to the expected value premium principle with the insurance loading factor θ and the reinsurance loading factor ξ , the insurer's premium income per unit time after the reinsurance premium (i.e. net of reinsurance) is defined as

$$c^* = (1 + \theta) \Lambda E[X] - (1 + \xi) \Lambda E[Z], \quad (4.3)$$

where we assume that $\xi \geq \theta > 0$ and that $c^* > \Lambda E[Y]$.

The net of reinsurance loading factor, θ_{net} , is calculated by using the net of reinsurance premium income as

$$c^* = (1 + \theta_{net}) \Lambda E[Y], \quad (4.4)$$

and hence a formula for θ_{net} can be derived as

$$\theta_{net} = \frac{c^*}{\Lambda E[Y]} - 1. \quad (4.5)$$

Under the translated gamma process approximation, the insurance loading factor is reformed by using the parameters of the translated gamma process. Since $\hat{\theta} = \theta(1 + k\beta/\alpha)$, we have the following result for the net of reinsurance loading factor under the translated gamma process approximation:

$$\hat{\theta} = \theta_{net} \left(1 + \frac{k\beta}{\alpha}\right). \quad (4.6)$$

Therefore, $\delta_{SG}(\beta U, \alpha t)$ approximates $\delta(U, t)$ by using this premium loading factor $\hat{\theta}$, under the translated gamma process approximation.

Dickson [11] describes a constraint for the retention level M under the excess of loss reinsurance. When the individual claim amount has an exponential distribution with the parameter 1, we can obtain the net premium income as

$$\begin{aligned} c^* &= (1 + \theta) \Lambda - (1 + \xi) \Lambda \int_M^\infty (x - M) e^{-x} dx, \\ &= \Lambda \left(1 + \theta - (1 + \xi) e^{-M}\right), \end{aligned}$$

and by using the condition $c^* > \Lambda E[Y]$, the minimum level for the retention level M is found as

$$\Lambda \left(1 + \theta - (1 + \xi) e^{-M}\right) > \Lambda \left(\int_0^M x e^{-x} dx + M e^{-M}\right) = \Lambda (1 - e^{-M}), \quad (4.7)$$

and so it can be concluded that $M > \log(\xi/\theta)$ [11].

When the individual claim amount is distributed exponentially with parameter λ , the moments of the insurer's individual claim amount can be obtained in terms of the incomplete gamma function as

$$m_k = E[Y^k] = \frac{k}{\lambda^k} \gamma(k, \alpha M) \quad \text{for } k = 1, 2, \dots, \quad (4.8)$$

where $\gamma(k, M)$ is the incomplete gamma function and defined as

$$\gamma(k, M) = \int_0^M t^{k-1} e^{-t} dt.$$

When the individual claim amount has Pareto distribution with parameters γ and δ , the moments of the insurer's individual claim amount cannot be expressed in a simple closed form as in the exponential case. The first three moments of the insurer's individual claim amount are obtained as

$$m_1 = \left(1 + \frac{M}{\delta}\right)^{-\gamma} \left(\frac{-\delta - \gamma M + \delta \left[\frac{\delta+M}{\delta}\right]^\gamma}{\gamma - 1} + M \right), \quad (4.9)$$

$$m_2 = \left(1 + \frac{M}{\delta}\right)^{-\gamma} \left(\frac{-2\gamma\delta M - (\gamma - 1)\gamma M^2 + 2\delta^2 \left(-1 + \left(\frac{\delta+M}{\delta}\right)^{-\gamma}\right)}{(\gamma - 2)(\gamma - 1)} + M^2 \right), \quad (4.10)$$

and

$$m_3 = \left(1 + \frac{M}{\delta}\right)^{-\gamma} \left(\frac{-6\gamma\delta^2 M - 3(-1 + \gamma)\gamma\delta M^2 - (\gamma - 2)(\gamma - 1)\gamma M^3 + 6\delta^3 \left(-1 + \left(\frac{\delta+M}{\delta}\right)^{-\gamma}\right)}{(\gamma - 3)(\gamma - 2)(\gamma - 1)} + M^3 \right). \quad (4.11)$$

4.2.2. Proportional Reinsurance on the Translated Gamma Process Approximation

Under the proportional reinsurance, a fixed proportion is used instead of the retention level M in the excess of loss reinsurance. The insurer's liability is limited to the proportion p of each claim; conversely, the reinsurer pays proportion $1 - p$ of each claim. Therefore, the reinsurance premium is calculated depending on this same proportion.

Let Y and Z represent the part of the claim paid by the insurer and the reinsurer, respectively. Under the proportional reinsurance, $Y = pX$ and $Z = (1 - p)X$ with $X = Y + Z$. The distribution function of Y , $F_Y(x)$ and the density function $f_Y(x)$ are obtained as $F_X(x/p)$ and $\frac{1}{p} f_X(x/p)$, respectively [11].

According to the expected value premium principle with insurance loading factor θ and reinsurance loading factor ξ , the insurer's premium income per unit time after paying the reinsurance premium (i.e. net of reinsurance) is defined as

$$\begin{aligned} c^* &= (1 + \theta) \Lambda E[X] - (1 + \xi) \Lambda (1 - p)E[X], \\ &= \left(1 + \theta - (1 + \xi)(1 - p)\right) \Lambda E[X], \end{aligned}$$

where we assume that $\xi \geq \theta > 0$ and that $c^* > \Lambda E[Y]$.

The net of reinsurance loading factor is calculated by using the net of reinsurance premium income as

$$c^* = (1 + \theta_{net}) \Lambda pE[X], \quad (4.12)$$

and hence a formula for θ_{net} can be derived as

$$\theta_{net} = \frac{c^*}{\Lambda p E[X]} - 1. \quad (4.13)$$

Dickson [11] illustrates a constraint for the level of the proportion on the reinsurance arrangement under the condition of $c^* > \Lambda E[Y]$. It is assumed that the individual claim amount has an exponential distribution with the parameter 1. Hence, the constraint is estimated as

$$c^* = (1 + \theta - (1 + \xi)(1 - p)) > p, \quad (4.14)$$

and it gives the minimum proportion as

$$p > 1 - \frac{\theta}{\xi}. \quad (4.15)$$

From a practical point of view, θ can be equal to ξ under the proportional reinsurance. On the other hand since our main aim is to investigate the minimum ruin probability, we assume that $\xi > \theta$; see [44].

When the individual claim amount has an exponential distribution with parameter λ , the moments of Y are

$$m_k = E[Y^k] = \frac{k! p^k}{\lambda^k} \quad \text{for } k = 1, 2, \dots \quad (4.16)$$

When the individual claim amount has a Pareto distribution with parameters γ and δ , the moments of Y are

$$m_k = E[Y^k] = \frac{k! p^k \delta^k \Gamma(\gamma - k)}{\Gamma(k)} \quad \text{for } k = 1, 2, \dots \quad (4.17)$$

provided that $\gamma > k$.

4.3. Optimal Reinsurance Criteria on the Translated Gamma Process Approximation

In Chapter 3, substantial and commonly used criteria to obtain the optimal reinsurance under the ruin probability constraint are discussed. From the outcome of our investigation; the released capital, the expected profit and the expected utility are determined as the essential optimal reinsurance criteria. Under the translated gamma process approximation, calculation of these criteria is explained according to the excess of loss and the proportional reinsurance arrangement. Differences on pairs of insurance and reinsurance loading factors or time horizons cause significant changes in values of these criteria.

4.3.1. Released Capital on the Translated Gamma Process Approximation

In Section 3.3.1, we define the released capital as the differences between the largest and the required initial surplus. We draw attention to the differences between the insurer's initial surplus and the largest initial surplus according to a reinsurance arrangement. We suppose that the insurer's ruin probability is fixed at a minimum level such as 0.01.

In the case of no-reinsurance, the required initial surplus which makes the insurer's ruin probability equal to 0.01 is defined as *Largest Initial Surplus* and denoted by u_L . The smallest initial surplus is calculated in a similar way. It is not possible to get a ruin probability such as 0.01 if the initial surplus is above a certain level. The insurer's surplus must start with this adequate initial surplus called as *Smallest Initial Surplus* and denoted by u_S .

When the reinsurance arrangement is included to the insurer's surplus process, initial surplus and the reinsurance level which make the ruin probability equals to 0.01 are calculated. For each reinsurance level, there is a unique initial surplus which satisfies 0.01 ruin probability. The difference between the largest initial surplus and this value is called as savings amounts or gains by choosing a reinsurance arrangement. We refer to this value as the released capital. The representation of the initial surplus and the reinsurance level is given in Figure 4.1.

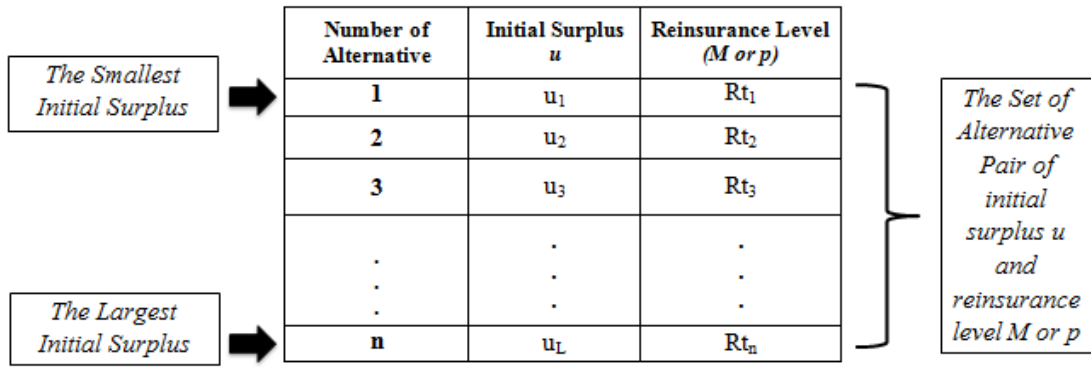


Figure 4.1: The representation of the initial surplus and the reinsurance level

In this study, the largest initial surplus, the smallest initial surplus, pair of the initial surplus and the reinsurance level are calculated according to the translated gamma process approximation. Each pair of the initial surplus and the reinsurance level are obtained which make the ruin probability equal to 0.01 under the translated gamma process approximation. The pair of initial surplus and reinsurance level is presented as follows.

$$\begin{aligned}
 \psi(u_1, Rt_1, t) &= 0.01, \\
 &\vdots \\
 \psi(u_n, Rt_n, t) &= 0.01.
 \end{aligned}$$

The following conclusions about the released capital can be drawn. Due to the relationship between the initial surplus and the reinsurance level, it can be observed that when the initial surplus increases, the corresponding reinsurance level increases as well. A set of initial surplus starts with the smallest initial surplus and ends with the largest initial surplus. While the reinsurance level is low at the smallest initial surplus, the reinsurance level is high at the largest initial surplus. The released capital is maximum at the smallest initial surplus level whereas it is minimum in fact close to zero, at the largest initial surplus.

4.3.2. Expected Profit on the Translated Gamma Process Approximation

Insurance profit which is calculated as the difference between the net insurance premium income and the insurer's expected total claim at time t , $P(t)$, is defined as

$$P(t) = \Pi_{net}(t) - S_I(t). \quad (4.18)$$

According to the translated gamma process approximation, the insurer's profit is changed by the type of the reinsurance arrangement.

Let us assume that $S(t)$ has a compound Poisson distribution with parameter Λt . Under an excess of loss reinsurance arrangement, the insurer's expected profit up to time t is

$$P(t) = \left((1 + \theta) \Lambda t E[X] - (1 + \xi) \Lambda t [E[X] - E[Y]] \right) - \left(\frac{\alpha t}{\beta} + kt \right), \quad (4.19)$$

where $E[X]$ denotes the expected individual claim amount and $E[Y]$ is the expected individual claim amount paid by the insurer. The parameters of α , β and k are calculated due to the retention level M by using Equation (2.11).

Under a proportional reinsurance arrangement, the insurer's expected profit up to time t is

$$P(t) = \left((1 + \theta) \Lambda t E[X] - (1 + \xi) \Lambda t (1 - p) E[X] \right) - \left(\frac{\alpha t}{\beta} + kt \right) \quad (4.20)$$

where α , β and k are calculated due to proportion p by using Equation (2.11).

The expected profit on the translated gamma process depends on the reinsurance level (retention level M or proportion p). An increase in the reinsurance level causes a decrease in the transferred risk to the reinsurance company. Hence, the reinsurance premium decreases, the net insurance premium income increases and eventually the expected profit of insurance company increases. The results show that under the fixed minimum ruin probability constraint, there is a positive correlation between the reinsurance level and the expected profit of the insurance company up to time t .

4.3.3. Expected Utility on the Translated Gamma Process Approximation

The exponential utility function, u is defined as

$$u(x) = 1 - \exp(-Bx), \quad B > 0$$

where B is the parameter of the utility function. From the insurer's point of view, the insurer's wealth at the end of t , $U(t)$, is defined as

$$U(t) = u + \Pi_{net} - S_I(t).$$

To avoid the confusion of the notations, initial surplus is represented by u_0 instead of u . The expected utility of insurer's wealth at time t is denoted as

$$\begin{aligned} E[u(U(t))] &= E[u(u_0 + \Pi_{net} - S_I(t))] , \\ &= E[1 - \exp(-B(u_0 + \Pi_{net} - S_I(t)))], \\ &= 1 - [E[\exp(-B u_0)] \times E[\exp(-B \Pi_{net})] \times E[\exp(B S_I(t))]]. \end{aligned} \quad (4.21)$$

We assume that the aggregate claims paid by the insurer is approximated by using the translated gamma process with parameters α , β , and k . In calculation of $E[\exp(-B S_I(t))]$, we take advantage of the moment generating function.

The moment generating function of the translated gamma distribution is obtained by applying a transformation to the gamma distribution. If X has a gamma distribution with parameters α and β , then the translated gamma distribution is obtained as $Y = k + X$ where k is a real constant. Thus, the moment generating function of Y is

$$\begin{aligned} M_Y(s) &= E[e^{sY}], \\ &= E[e^{s(X+k)}], \\ &= E[e^{sk}]E[e^{sX}], \\ &= e^{sk}M_X(s), \end{aligned} \tag{4.22}$$

where $M_X(s)$ denotes the moment generating function of X .

The moment generating function of a gamma distribution with parameters (α, β) is defined as

$$M_X(s) = \left(\frac{\beta}{\beta - s} \right)^{\alpha t}.$$

The relationship between the gamma and the translated gamma process can be defined as

$$M_{S_{TG}(t)}(s) = e^{kts} M_{S_G(t)}(s).$$

We need to find the moment generating function in terms of the parameter of the exponential utility function, B . Hence $M_{S_I}(B)$ is obtained as follows

$$M_{S_I(t)}(B) = e^{ktB} \left(\frac{\beta}{\beta - B} \right)^{\alpha t} \quad \text{for } B < \beta. \tag{4.23}$$

On the other hand, the main equation for the expected utility of the insurer's wealth at the end of time t is obtained as

$$E[\mathbf{u}(U(t))] = 1 - \left[\exp(-B u_0) \right] E \left[\exp(-B \Pi_{net}) \right] \left[e^{ktB} \left(\frac{\beta}{\beta - B} \right)^{\alpha t} \right]. \tag{4.24}$$

Moreover, the expected utility of insurance's wealth at the end of time t can be categorized into two types of reinsurance: the excess of loss and the proportional reinsurance. The expected utility of the insurer's wealth is found by using Equation (4.24).

Under the excess of loss reinsurance, $E\left[\exp\left(-B \Pi_{net}\right)\right]$ is obtained as follows

$$E\left[\exp\left(-B \Pi_{net}\right)\right] = \exp\left[-B(1+\theta)\Lambda t E[X] + B(1+\xi)\Lambda t E[X - S_I]\right] \quad (4.25)$$

where $E[S_I]$ denotes the expected claim amount paid by the insurer based on the retention level M under the translated gamma process. The parameters of the translated gamma process α , β and k are calculated based on the retention level M in Equation (2.11).

Under the proportional reinsurance, $E\left[\exp\left(-B \Pi_{net}\right)\right]$ is

$$E\left[\exp(-B \Pi_{net})\right] = \exp\left[-B(1+\theta)\Lambda t E[X] + B(1+\xi)\Lambda t(1-p)E[X]\right]. \quad (4.26)$$

where $(1-p)E[X]$ denotes the expected claim amounts paid by the reinsurer. Hence the expected claim amounts paid by the insurer can be found under the translated gamma process according to the values of p . The parameters of the translated gamma process α , β and k are calculated based on the proportion p as in Equation (2.11).

The expected profit and the released capital are monetary variables. However, the expected utility is in a different scale. Therefore, we use a normalization technique to compare these three criteria.

The calculations show that the expected utility increases exponentially. The slope gradient of the expected utility mainly depends on the risk averse coefficient i.e. parameter of the exponential utility function. In this study, we firstly assume that the parameter of the exponential utility function is equal to 0.02. In sensitivity analysis, we use different values and we examine the effect of the risk averse coefficient on the optimal reinsurance level. Three criteria for the pair of initial surplus and the reinsurance level is presented in Figure 4.2.

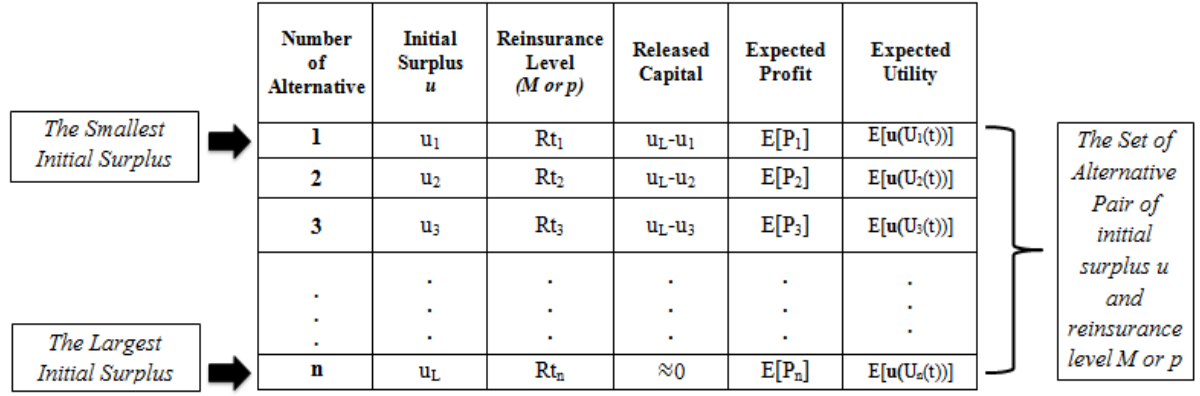


Figure 4.2: The representation of the released capital, expected profit and expected utility for the pair of initial surplus and reinsurance level

In order to obtain the optimal initial surplus and the reinsurance level, we use a multi-criteria decision analysis under the decision theory.

4.4. Multi Criteria Decision Theory

The decision problems that are encountered in real life often have a very complex structure. The underlying reason is the evaluation of more than one factors or objectives together. Decision making investigates identifying and choosing the best alternatives based on the values and preferences of the decision maker.

Multiple criteria decision making (MCDM) is used when making decisions considering the multiple, inconsistent criteria. Hwang and Yoon [88] describe the common characteristics of the MCDM problems with the following statements: each problem has multiple objectives/attributes; multiple criteria conflict with each other; each objective /attribute has a different unit of measurement and solutions are explained as the best alternative among all possible alternatives.

MCDM is a sub-discipline of the operations research which considers multiple, usually inter-relating criteria in a decision-making analysis. Hwang and Yoon [88] categorize the solution of a MCDM problem into five categories: an optimal solution, an ideal solution, a non-dominated solution, a satisfying solution and a preferred solution. Hwang and Yoon [88] also identify the optimal solutions as follows.

Definition 4.4.1. Optimal Solution

An optimal solution is one which gives the maximum value of each of the objective functions simultaneously. $\underline{x}^*$ is an optimal solution to a problem if $\underline{x}^* \in X$ and $f(\underline{x}^*) > f(x)$ for all $\underline{x} \in X$. An optimal solution is also known as a maximum solution or a superior solution. In general, optimal solution does not exist in a MCDM problem because of having conflict in objectives or attributes .

MCDM is classified into two categories depending on whether the problem is a selection problem or a design problem. Multiple-Attribute Decision Making (MADM) is developed to select the best alternative in a set whereas Multiple-Objective Decision Making (MODM) is developed to design the best alternative. Table 4.1 shows the comparison of the MADM and MODM methods made by Hwang and Yoon [88].

Table 4.1: Comparison of MADM and MODM methods

	MADM	MODM
Criteria	Attributes	Objective
Objective	Implicit	Explicit
Attribute	Explicit	Implicit
Constraint	Inactive	Active
Alternative	Finite Number Discrete	Infinite Number Continuous
Interaction with Decision Maker	Not Much	Mostly
Usage	Selection/Evaluation	Design

MODM is related to design a problem and it depends on making the final decision rather than selecting the best alternative. Therefore, we focus on MADM methods.

4.4.1. Multiple-Attribute Decision Making

Rao [89] defines MADM as “an approach which applies on solving problem by selection among a finite number of alternatives. MADM method indicates that how attribute information is to be processed in order to arrive at a choice”.

For a MADM problem, a decision matrix is used to express a multiple criteria decision problem as a matrix format. A decision matrix D is a $m \times n$ matrix. There are m alternative options which need to be assessed on n attributes (criteria).

The elements of X_{ij} specifies the value of alternative i , in regard to attribute (criteria) j . Each element, X_{ij} , is either a single numerical value or a single grade, represents the performance of alternative i on criterion j . Therefore, the components of the decision making analysis can be shown in a decision matrix as in Table 4.2.

Table 4.2: Decision matrix for MADM methods

Alternatives (A_i)	Attributes (Criteria) (C_j)				
	C_1	C_2	C_3	$\cdots$	C_n
A_1	X_{11}	X_{12}	X_{13}	$\cdots$	X_{1n}
A_2	X_{21}	X_{22}	X_{23}	$\cdots$	X_{2n}
A_3	X_{31}	X_{32}	X_{33}	$\cdots$	X_{3n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_m	X_{m1}	X_{m2}	X_{m3}	$\cdots$	X_{mn}

In MADM analysis, five basic components are used: attributes, objectives, goals, criteria and alternatives. The *Alternative* component is defined as monetary values for all the major costs and benefits when this is impractical [90]. The *Attribute* component is used to refer to a measurable criterion and it has the same meaning as performance parameter, component, factor, characteristic or property [88]. The *Objective* component is identified as why the decision maker is interested in a decision. The objectives must be well defined and should also be specific, measurable, agreed upon, and realistic. The *Goal* component means a target towards a desired result. In general, some constraints must be used to limit the goal in the decision making. Finally, the *criterion* component is defined as a measure of the effectiveness. The criterion appears as a form of attributes or objectives. Moreover, the criterion and the attribute are often interchangeable in the literature on multiple criteria analysis.

In MADM, the importance of the attributes can be described by the weights of the attributes. Furthermore, in some of MCDM problems, information about the relative importance of each attribute is required. A set of weights for n attributes are shown as

$$\mathbf{w}^T = (w_1, w_2, \cdots, w_n),$$

where $\sum_{j=1}^n w_j = 1$.

The weights should be determined to reflect the importance of each attribute. Hwang and Yoon [88] suggest four recently used techniques in MCDM to calculate the weights of the criteria namely the eigenvector method, the weighted least squares method, the entropy method, and the linear programming technique for multidimensional analysis of preference (LINMAP). In the eigenvector method and the weighted least squares method, a decision matrix is not required and the weights are calculated according to a positive pairwise comparison matrix. On the other hand, the other cases require the decision matrix.

Method 4.4.1. The Eigenvector Method

In the Eigenvector Method, let us define A as a $n \times n$ pairwise comparison matrix, I as the $n \times n$ identity matrix, λ as the vector of eigenvalues, and $\mathbf{w}$ as the set of weights. The set of eigenvalues (λ) are calculated that make the determinant of $(A - \lambda I)$ equal to 0.

The largest eigenvalue of λ is denoted by λ_{max} . Hence, the weights according to the eigenvector method are calculated as a solution of the following equation

$$(A - \lambda_{max}I)\mathbf{w} = 0.$$

Saaty [91] introduces a method which depends on the scaling of importance ratios by using the principle eigenvector of a positive pairwise comparison matrix. Moreover, Saaty proposes to use an intensity scale of importance for activities in the eigenvector method instead of the pairwise comparison matrix.

The scale and its description of Saaty's pairwise comparison matrix is given in Table 4.3. This pairwise comparison matrix is used to compare the criteria according to their importance [92].

Table 4.3: The scale and its description of Saaty's pairwise comparison matrix

Intensity of Importance	Definition	Explanation
1	Equal importance	Two criteria contribute equally to the objective.
3	Weak importance of one over another	Experience and judgment slightly favor one criterion over another.
5	Essential or strong importance	Experience and judgment strongly favor one criterion over another.
7	Demonstrated importance	A criterion is strongly favored and its dominance is demonstrated in practice.
9	Absolute importance	The evidence favoring one criterion over another is of the highest possible order of affirmation.
2, 4, 6, 8	Intermediate values between the two adjacent judgments	When compromise is needed.

Method 4.4.2. The Weighted Least Squares Method

The weighted least squares method is suggested by Chu et al. [93]. This method depends on the solution of a set of simultaneous linear algebraic equations as is more understandable than the Saaty's eigenvector method. Let us define $a_{ij} = w_i/w_j$ which consider the elements a_{ij} of Saaty's pairwise comparison matrix.

The weights are determined under the constrained optimization problem as

$$\min \sum_{i=1}^n \sum_{j=1}^n (a_{ij} w_j - w_i)^2,$$

with $\sum_{j=1}^n w_j = 1$. By using the Lagrangian function, the weights can be obtained by solving this equation.

$$\sum_{i=1}^n (a_{i\ell} w_\ell - w_i) a_{i\ell} - \sum_{j=1}^n (a_{\ell j} w_j - w_\ell) + \lambda = 0, \quad \ell = 1, 2, \dots, n.$$

Method 4.4.3. The Entropy Method

In the Entropy method, it is assumed that a criterion for the amount of uncertainty is represented by a discrete probability distribution, P_i . Let the decision matrix D of m alternatives and n attributes (criteria) be,

$$D = \begin{matrix} & X_1 & X_2 & \cdots & X_n \\ \begin{matrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{matrix} & \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \end{matrix}$$

The project outcomes of attribute j , P_{ij} can be defined as

$$P_{ij} = \frac{X_{ij}}{\sum_{j=1}^m X_{ij}}, \quad \forall i, j.$$

The entropy E_j of the set of project outcomes of attribute j is

$$E_j = -k \sum_{i=1}^m P_{ij} \ln P_{ij}, \quad \forall j,$$

where $k = 1/\ln(m)$ and $0 \leq E_j \leq 1$.

The degree of diversification d_j is calculated as $d_j = 1 - E_j$ for all j and then the weights are calculated as

$$w_j = \frac{d_j}{\sum_{j=1}^n d_j}, \quad \forall j.$$

Method 4.4.4. *The Linear Programming Technique for Multidimensional Analysis of Preference Model (LINMAP)*

In the LINMAP model, it is assumed that the decision maker has his ideal point which is denoted by his most preferred alternative location. The decision makers prefer the weights which are closer to decision maker's ideal point (in terms of a weighted Euclidean distance measure).

The weights also should provide the consistency principle. Consistency can be measured in two ways: by transitivity or numerically.

Consistency according to the transitivity can be explained by a simple example: If A is better than B and B is better than C then A must be better than C .

Consistency according to the numerically can be explained by using Saaty's pairwise comparison matrix (Table 4.3): If A compared to B is 2 and B compared to C is 3 then A compared to C should be $2 \times 3 = 6$. Unless these conditions are satisfied, some inconsistency would occur.

Inconsistent matrices have some specific properties. The first property is explained according to the Eigenvalue/Eigenvector method. The number of non-zero eigenvalues of the matrix is equal to its rank. In a consistent matrix, it should be equal to 1. The second property is related with the summation of the eigen values. While the summation of the eigen values should be equal to sum of the diagonal element of the consistent matrix, each element should be equal to 1 in an inconsistent matrix. Moreover, while a $n \times n$ consistent matrix has eigenvalue with value n , inconsistent matrices have more than one eigen value.

Another important issue in MCDA is the normalization. Normalization of the attribute values is necessary when criteria having different scales in the decision matrix are compared. Especially for some of MADM methods, normalization of the attributes is always necessary. By the normalization technique, the attribute values are converted to a comparable scale. There are different ways of the normalization techniques for the attribute values; however, vector normalization and the linear scale transformation are preferred in general.

Method 4.4.5. Vector Normalization

Vector normalization depends on the technique which each row vector of the decision matrix is divided by its norm [94]. The normalized decision matrix has normalized value r_{ij} which is calculated as

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^m x_{ij}^2}}. \quad (4.27)$$

Hwang and Yoon [88] state the advantage of this normalization technique as all criteria are measured in dimensionless units. Thus, attributes can be compared easily. On the other hand, they also mention that the major drawback of this normalization technique as it does not lead to measurement scales with an equal length.

Method 4.4.6. Linear Scale Transformation

Linear scale transformation changes according to a benefit or a cost criteria. In the case of a benefit criteria, the outcome of a certain criterion is divided by its maximum value whereas in the case of cost criteria, the outcome is calculated as a benefit criteria and the value is subtracted from 1. In the case of benefit criteria, r_{ij} is calculated as

$$r_{ij} = \frac{x_{ij}}{x_j^*}, \quad (4.28)$$

where $x_j^* = \max_i x_{ij}$. In the case of cost criteria, r_{ij} is calculated as

$$r_{ij} = 1 - \frac{x_{ij}}{x_j^*}. \quad (4.29)$$

Hwang and Yoon [88] emphasize the advantage of this normalization technique such that “all outcomes are transformed in a linear way, thus the relative order of magnitude of the outcomes remains equal”.

In a decision matrix, benefit and cost criteria can both exist. In this case, the linear scale transformation is changed because of the different bases, 0 for benefit criteria and 1 for cost criteria. The cost criteria can be expressed in terms of the benefit criteria as

$$r_{ij} = \frac{x_j^{min}}{x_{ij}}. \quad (4.30)$$

When the number of benefit criteria is greater than the number of cost criteria, all the attributes are assumed as the benefit criteria and Equation (4.28)-(4.30) are used.

A more complicated form of r_{ij} also can be given for the benefit and the cost criteria, respectively. In the case of benefit criteria, r_{ij} is obtained as

$$r_{ij} = \frac{x_{ij} - x_j^{\min}}{x_j^* - x_j^{\min}}, \quad (4.31)$$

In the case of cost criteria, r_{ij} is obtained as

$$r_{ij} = \frac{x_j^* - x_{ij}}{x_j^* - x_j^{\min}}. \quad (4.32)$$

MADM is a widely used tool and it is investigated in different types of working areas such as business, academic, and public [95].

Hwang and Yoon classify the methods for MADM according to different forms of preference information of the decision maker. The classification system has three categories: no information, information on attribute, and information on alternative. The classification also depends on three subclasses: type of needed information from the decision maker; typical characteristics of the information and the major methods which are formed in the first and the second subclasses. The structure of MADM methods is shown in Figure 4.3 [88]

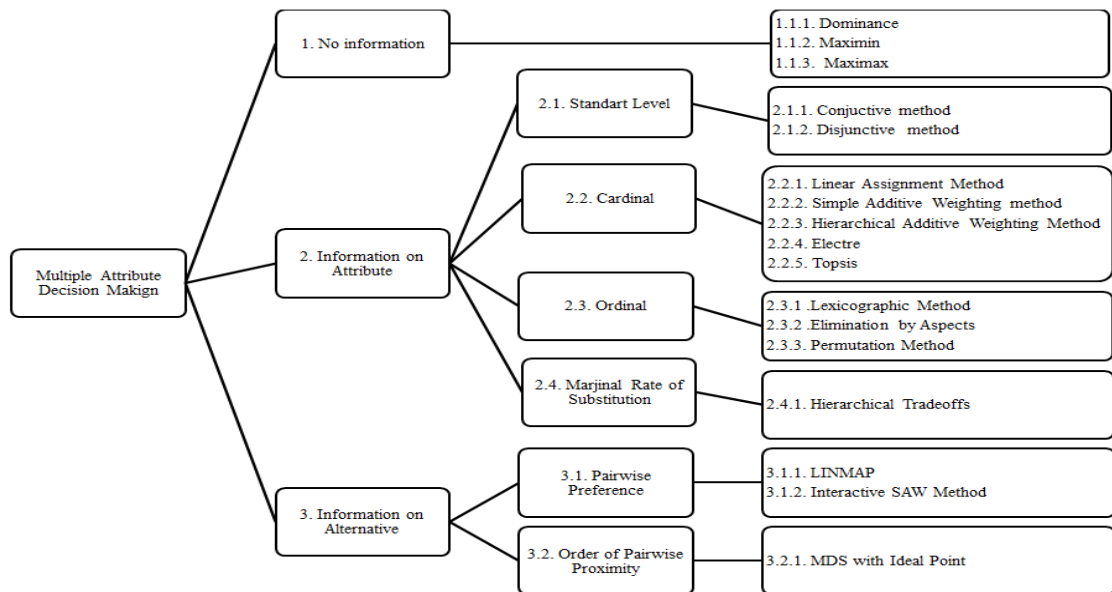


Figure 4.3: The structure of MADM methods

In *methods for no-preference information given*, the decision maker's preference information about the criteria is unnecessary and the best alternative is selected by using one of the dominance, maximin or maximax methods. In the dominance method, the alternatives are reduced by eliminating the dominated ones. In this method, no assumption is required. In the maximin method, the selection depends on the maximum of the smallest values of each alternative whereas in the maximax method, the best alternative is selected as the maximum of the highest values of each alternative.

Methods for information on alternative require that the decision maker is able to specify his preference about the alternatives. Therefore, in these methods an ideal point which explains the preferred alternative should be given.

In *methods for information on attribute*, the preference information of the decision maker either on attributes or alternatives can be used. The required information is defined in different ways such as standard level of each attribute, relative importance of each attribute by ordinal preference, relative importance of each attribute by cardinal preference, and marginal rate of substitution between attributes.

In *methods for standard level of attribute*, the decision maker assigns a minimum standard level for the alternatives. Any alternative which has an attribute value less than the standard level is rejected. Conjunctive and disjunctive method can be given as examples of this class. For *methods for ordinal preference of attribute*, the ordinal inter-attribute preference information is needed. Lexicographic method, elimination by aspects and permutation method can be given as examples of this class. In methods for *marginal rate of substitution between attributes*, the tradeoff information is explicitly utilized. The detailed information about all these methods can be found in Hwang and Yoon [88].

The methods for cardinal preference are widely used in decision theory. Detailed information about these methods is given briefly in the following section.

4.4.2. Methods for Cardinal Preference of Attribute Given

In these methods, the decision maker's cardinal preferences of attributes are required. In the expression of the inter-attribute preference information, the methods for cardinal preference of attribute given are commonly preferred. The methods for cardinal preference of attribute given can be categorized into five main methods: the linear assignment method (LAM), the simple additive weighting (SAW) method, the hierarchical additive weighting method (HAWM), the analytical hierarchy process (AHP), the elimination et and choice translating reality (ELECTRE), and the technique for order preference by similarity to the ideal solution (TOPSIS).

Method 4.4.7. Linear Assignment Method (LAM)

The linear assignment method is suggested by Bernardo and Blin [96] and it depends on a set of attribute rankings and a set of attribute weights. The linear assignment method uses a linear relationship between attribute interaction and their combinations. In this method, the ordinal data is preferred instead of the cardinal data. The first step of this method is to rank alternatives for each attribute and then to rank the summation of each alternative from the lowest to the highest sum.

In spite of this method's simplicity, it does not cover the basic linear requirement such as finding an overall ranking which simultaneously uses all the information, not just the sum of the ranks. Furthermore, this method may not be conventional in all applications because of the requirement of the attribute wise rankings.

Method 4.4.8. Simple Additive Weighting Method (SAW)

Churchman [97] first utilize the Simple Additive Weighting Method (SAW) to cope with a portfolio selection problem. The SAW method is probably the best known and widely used method for MADM analysis. In this method, the decision maker assigns the importance of weights to each of the attributes.

The decision maker assigns a set of importance weights to the attributes, $\underline{w} = w_1, w_2, \dots, w_n$ and then the most preferred alternative A^* is selected such that

$$A^* = \left\{ A_i \mid \max_i \frac{\sum_{j=1}^n w_j x_{ij}}{\sum_{j=1}^n w_j} \right\},$$

where x_{ij} is the outcome of the i^{th} alternative about the j^{th} attribute with a numerically comparable scale.

The SAW method requires that the value of attributes must be in a comparable numerical form. In order to obtain a comparable scale, firstly, a normalization technique is applied to the attributes. The detailed information about this method can be found in MacCrimon [98] and Hwang and Yoon [88].

Method 4.4.9. Hierarchical Additive Weighting Method (HAWM)

As in SAW method, the weighted averages for alternative A_i are given as

$$A^* = \frac{\sum_{j=1}^n w_j x_{ij}}{\sum_{j=1}^n w_j},$$

where $\sum_{j=1}^n w_j = 1$ and x_{ij} is a ratio scale.

Klee [99] indicates that the ratio scale X_{ij} is the subscore of the i^{th} alternative in association with j^{th} criterion. Then the vector $\underline{x}_j = (x_{1j}, x_{2j}, \dots, x_{ij}, \dots, x_{mj})$ represents the importance of A_i 's for each j^{th} criteria such that $\sum_{i=1}^m x_{ij} = 1$ for $j = 1, 2, \dots, n$. In this model, it is assumed that the combination of the (w_{y_j}, x_i) are converted into a matrix B . Let B_k be the priority matrix of k^{th} level for $k = 1, 2, \dots, h$. In addition, let W be the priority vector of the p^{th} level with respect to some element z in the $(p-1)^{st}$ level, and the priority vector W of the q^{th} level ($p < q$) is given as

$$W = B_q B_{q-1} \dots W'.$$

Let L_k represents the k^{th} level of a hierarchy of h levels and $W' = 1$.

Method 4.4.10. The Analytical Hierarchy Process (AHP)

The Analytical Hierarchy Process (AHP) first decomposes the decision problem into a system of hierarchies of objectives, attributes (criteria), and alternatives. The decision-maker evaluates the relative importance of the various elements by pairwise comparisons. Saaty [92] identifies the AHP as “a method that converts these evaluations to numerical values (weights or priorities), which are used to calculate a score for each alternative”.

The main procedure of AHP is given in the following steps:

1. Make pairwise comparisons:

A pairwise comparison matrix is obtained by using Saaty’s matrix [91] in Table 4.3 to use a scale of the relative importance.

By using the Saaty’s pairwise comparison matrix, an initial matrix is obtained. In this matrix, the principal diagonal contains entries of 1 as each factor is as important as itself and the rest of the entries has comparison importance. Making the pairwise comparison are applied to each attribute.

2. Synthesize judgments:

Synthesization is the identification and the calculation process of the priority of each criterion according to its contribution to the aim of the decision.

The synthesize judgments have the following steps:

Step 1: The summation of the values in each column of the pairwise comparison matrix is found,

Step 2: Each element is divided by its column total value (the normalized pairwise comparison matrix),

Step 3: The average of elements in each row is calculated (relative priorities).

3. *Check for consistency*

In the consistency step, several pairwise are compared by considering the consistency of the pairwise judgments. The degree of consistency is measured by the following steps.

Step 1: The pairwise comparison matrix and its relative priorities are multiplied,

Step 2: The weighted sum vector elements are divided by the associated priority value,

Step 3: A consistency index (CI) is calculated by the average of the values of Step 2 or that is to say the maximum eigen value λ_{max} such that

$$CI = \frac{\lambda_{max} - n}{n - 1}.$$

Step 4: A consistency ratio (CR) is calculated as

$$CR = \frac{CI}{RI}.$$

where RI denotes the random index and it is calculated according to the value of n as given below

n	3	4	5	6	7	8
RI	0.58	0.90	1.12	1.24	1.32	1.41

In general it is assumed that if $CR \leq 0.1$, consistency is acceptable. But if the consistency condition is not satisfied, pairwise comparisons should be revised.

In the AHP method, the geometric mean method is also commonly used to determine the relatively normalized weights of the attributes instead of the arithmetic mean. The detailed information about this method can be found in Saaty [92], [100], [101] and [102].

The elimination et choice translating reality (ELECTRE) method depends on the ‘outranking relationship’. Various ELECTRE models are developed according to the pattern of the problem statements. The ELECTRE I model is introduced by Benayoun et al. [103] to find the Kernel solution in a situation depending on true criteria and then restricted outranking relations are assigned [104].

Hwang and Yoon [88] clarify ELECTRE method as “Alternatives can be compared by using the pairwise comparison of alternatives based on the degree which evaluations of the alternatives. The preference weights are clarified confirm or contradict the pairwise dominance relationship between alternatives.”

In addition, The VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method is developed for multiple-criteria optimization of complex systems. This method focuses on ranking and selecting from a set of alternatives in the presence of conflicting criteria. This method provides a multiple-criteria ranking index based on the particular measure of “closeness” to the ideal solution [105].

4.4.3. The Technique for Order of Preference by Similarity to ideal Solution (TOPSIS)

TOPSIS is the acronym for “Technique for Order of Preference by Similarity to ideal Solution”. TOPSIS is suggested by Hwang and Yoon [88] to determine the best alternative based on the concepts of a compromised solution. The compromised solution can be regarded as choosing the solution with the shortest Euclidean distance from the ideal solution and the farthest Euclidean distance from the negative ideal solution. The ideal solution is defined as the one which maximises the benefit criteria and minimises the cost criteria. In contrast, the negative ideal solution is defined as the one which maximises the cost criteria and minimises the benefit criteria. The ranking of the alternatives is calculated according to the relative proximity to the ideal solution.

TOPSIS method is the most preferred decision technique, and according to most studies this method is also stated as the best alternative method among the MADM methods.

References to the TOPSIS method for real data can be found in a number of studies, including Wang and Hsu [106], Wu and Olson [107], Shih et al. [108], Jahanshahloo et al. [109], Bulgurcu [110], Zhu et al. [111], and Hosseini et al. [112].

TOPSIS methodology is applied to a wide range of areas: Behzadian et. al. [113] conduct a state of the art literature survey to taxonomize the research on TOPSIS application and methodologies according to nine application areas. The findings of this study suggests that application areas and global interest in TOPSIS method gradually increase.

In recent papers on group decision-making methods, TOPSIS is preferred as one of the major MADM methods due to some advantages over the other MADM methods.

Kim et al. [114] point out the advantages of TOPSIS method over other MADM methods as: this method has (i) sound logic that embodies the rationale of human choice, (ii) scalar values that considers the best and worst alternatives simultaneously, and (iii) a simple computation procedure that can easily be programmed into a spreadsheet.

Parkan and Wu [115] state that TOPSIS method is the best alternative among the other methods and its computational procedure is straightforward and fast.

Zanakis et al. [116] investigate the performance of SAW, multiplicative exponential weighting (MEW), TOPSIS, ELECTRE and AHP. The rank-reversal experiment shows that SAW and MEW methods performed the best, TOPSIS and AHP followed them. In addition, it is suggested that TOPSIS has the fewest rank reversals among these methods.

Yeah [117] explains the reasons why TOPSIS method has widely been used in MADM models as (i) its simplicity and comprehensibility in concept, (ii) its computational efficiency, and (iii) its ability to measure the relative performance of the decision alternatives in a simple mathematical form.

In addition, according to the selection of compensatory MADM methods for a specific ranking problem via sensitivity analysis of attribute weights, it is obtained that the ranking outcome produced by TOPSIS can best match the decision information embedded in the decision making. Moreover in this study, it is found that TOPSIS is the most sensitive method between Simple Summation Method (SSM), SAW, and Weighted Product Method (WPM).

Shih et. al. [108] state one more advantage of TOPSIS method as the performance measures of all alternatives on attributes can be visualized on a polyhedron, at least for two dimensions.

Also, Shih et. al. [108] suggest that this advantage makes TOPSIS method a major MADM techniques over the other MADM techniques such as AHP and ELECTRE. Hence, by using TOPSIS method, requirement of paired comparisons are reduced and the capacity limitation might not significantly dominate the process and it would be suitable for the cases with a large number of attributes or alternatives.

Velasquez and Hester [118] point out the advantage of TOPSIS method as it is a simple process, easy to use and program, and the number of steps remains the same regardless of the number of attributes. However, Velasquez and Hester [118] interpret the disadvantages of TOPSIS method as the use of Euclidean distance does not consider the correlation of attributes, difficult to weight, and keep consistency of judgment.

The procedures of the TOPSIS are described as follows

Suppose that D be a decision matrix and there are m alternatives $A_1, A_2, \dots, A_m$ and n decision attributes (criteria) $C_1, C_2, \dots, C_n$. Let x_{ij} denote the attribute (criteria) value of A_i on C_j for $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ in D matrix.

Step 1:

The decision matrix is normalized by using vector-normalization technique (*Method 4.4.5*).

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^m (x_{ij})^2}},$$

where r_{ij} is the normalized value, $i = 1, 2, \dots, m, j = 1, 2, \dots, n$.

Step 2:

Weighted-normalized values are calculated by using the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_n)$.

$$V_{ij}(x) = w_j r_{ij}, \quad i = 1, \dots, n; \quad j = 1, \dots, m.$$

Step 3:

The positive ideal points and negative ideal points (anti-ideal) be S^+ and S^- are determined, respectively. The positive ideal points are equivalent to maximum value under each criterion.

$$\begin{aligned} S^+ &= \{S_1^+, S_2^+, \dots, S_j^+, \dots, S_n^+\} \\ &= \{(\max_i V_{ij} \mid j \in J), (\min_i V_{ij} \mid j \in J') \mid i = 1, 2, \dots, m)\}. \end{aligned}$$

and the negative ideal points are equivalent to minimum value under each criterion.

$$\begin{aligned} S^- &= \{S_1^-, S_2^-, \dots, S_j^-, \dots, S_n^-\} \\ &= \{(\min_i V_{ij} \mid j \in J), (\max_i V_{ij} \mid j \in J') \mid i = 1, 2, \dots, m)\}. \end{aligned}$$

where $J = \{j = 1, 2, \dots, n \mid j \text{ associated with benefit criteria}\}$ and $J' = \{j = 1, 2, \dots, n \mid j \text{ associated with cost criteria}\}$.

Step 4:

The distance between each alternative is calculated by using n -dimensional Euclidean distance. The distance between alternative A_i and the ideal solution is

$$d_i^+ = \sqrt{\sum_{j=1}^n (V_{ij} - S_j^+)^2}, \quad i = 1, 2, \dots, m,$$

and the distance between alternative A_i and the negative-ideal solution is

$$d_i^- = \sqrt{\sum_{j=1}^n (V_{ij} - S_j^-)^2}, \quad i = 1, 2, \dots, m.$$

Step 5:

The relative closeness of each alternative to the ideal solution is calculated as

$$C_i = \frac{d_i^-}{d_i^+ + d_i^-}, \quad i = 1, 2, \dots, m,$$

where $C_i \in [0, 1]$ for $i = 1, \dots, n$. The results are sorted according to the value of C_i . A higher C_i means that A_i is a better solution.

A number of distance measure can be applied for TOPSIS method. The most commonly used distance measure depends on Minkowski's L_p metrics.

Definition 4.4.2. *Minkowski's L_p matrices*

Minkowski distance is a metric in a normed vector space. The L_p spaces are function spaces identified by using a natural generalization of the p -norm for finite-dimensional vector spaces.

For a real number $p \geq 1$, the p – norm or L_p -norm of x is defined as

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}. \quad (4.33)$$

These function spaces are called as Lebesgue spaces whose names come from Henri Lebesgue in Dunford et al. [119]. However it is also indicated that these function spaces are introduced by Frigyes Riesz [120]. Minkowski's L_p metrics are defined as

$$L_p(x, y) = \left\{ \sum_{j=1}^n |x_j - y_j|^p \right\}^{1/p},$$

where $p \geq 1$ with n dimensions.

Definition 4.4.3. *Euclidean Distance*

In Minkowski's L_p metrics with $p = 2$ is called as Euclidean distance calculated as

$$L_2(x, y) = \left\{ \sum_{j=1}^n |x_j - y_j|^2 \right\}^{1/2}.$$

Traditional TOPSIS method uses the Euclidean distance.

4.4.4. TOPSIS Method with Mahalanobis Distance

The traditional TOPSIS method is based on Euclidean distance measure and it is assumed that there is no relationship between the attributes. This approach suffers from information overlap and either overestimates or underestimates the attributes which take slack information [121].

When attributes are dependent and influence each other, application on TOPSIS based on Euclidean distances can lead to inaccurate estimation of relative significances of alternatives and cause the improper ranking results [122]. For this reason, another distance measure technique is suggested instead of Euclidean distance in the TOPSIS method.

The measurement of the distance between two objects is called Multivariate Chemo Metrical Technique and the most commonly used distance measures are Euclidean distance and Mahalanobis distance. De Maesschalck [123] compare the Euclidean and Mahalanobis distances in the original variable space and the principal component space is used to clarify the relationship between these distance measures [123].

Definition 4.4.4. *Mahalanobis Distance*

The Mahalanobis distance is also called quadratic distance and introduced by Mahalanobis [124]. For a multivariate vector $\mathbf{x} = (x_1, x_2, x_3, \dots, x_N)^T$ from a group of observations with mean $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \dots, \mu_N)^T$ and the covariance matrix Σ , the Mahalanobis distance is defined as follows

$$D_M(x) = \sqrt{(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})}.$$

The Mahalanobis distance standardizes via the factor of inverse of the covariance matrix Σ^{-1} . This distance measure depends on the covariance between variables and it can give some information about the similarities of an unknown sample to a known one. For this reason, Mahalanobis distance can be calculated by using the correlation matrix of a variable in the case of different patterns are analyzed. Mahalanobis distance depends on the correlation of the variables so that Mahalanobis distance measure depends on the scale-invariant case [121].

When the attributes are not related to one another, the weighted Mahalanobis distance and the weighted Euclidean distance will be equivalent [121].

References to the TOPSIS method with Mahalanobis distance can be found in a number of studies such as Wang and Wang [121], Garca and Ibarra [125], Chang et al. [126], and Lahby et al. [127].

The TOPSIS method with Mahalanobis distance has the following steps:

We suppose that there are m alternatives $A_1, A_2, \dots, A_m$ and n decision attributes (criteria) $C_1, C_2, \dots, C_n$. Let x_{ij} denote the attribute (criteria) value of A_i on C_j for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

The decision matrix consists of the element of X_{ij} . The decision matrix can be standardized as follow

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^m x_{ij}^2}}. \quad (4.34)$$

Let, the ideal solution and negative ideal solution (anti-ideal solution) be S^+ and S^- , respectively as in the case of TOPSIS and A_i denote the i^{th} alternative. Hence, the Mahalanobis distance from A_i to the ideal solution point is calculated as

$$d(r_i, S^+) = \sqrt{\{S_j^+ - r_{ij}\}^T \Omega^T \Sigma^{-1} \Omega \{S_j^+ - r_{ij}\}} \quad i = 1, 2, \dots, m. \quad (4.35)$$

Similarly, the Mahalanobis distance from A_i to the negative ideal solution point is calculated as

$$d(r_i, S^-) = \sqrt{\{S_j^- - r_{ij}\}^T \Omega^T \Sigma^{-1} \Omega \{S_j^- - r_{ij}\}} \quad i = 1, 2, \dots, m.$$

where ω is the weight vector such as $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ and Ω is defined as $\Omega = \text{diag}(\sqrt{\omega_1}, \sqrt{\omega_2}, \dots, \sqrt{\omega_n})$.

The closeness of each alternative is given as

$$c_i = \frac{d(r_i, S^-)}{d(r_i, S^-) + d(r_i, S^+)} \quad i = 1, 2, \dots, m. \quad (4.36)$$

The results for each alternative are sorted according to the value of c_i . A higher c_i suggests that A_i is a better solution.

In the TOPSIS method, the covariance matrix can be calculated directly from the original data or the normalized data. Antucheviciene et al. [122] propose that the original covariance matrix or the normalized covariance matrix can be used if there are no fewer alternatives than the criteria. Since, the inverse covariance matrix cannot be obtained due to the determinant rules. Antucheviciene et al. [122] also suggest to use the correlation matrix. Wang and Wang [121] use the covariance matrix calculated from the normalized data. Vega et al. [128] use the covariance matrix obtained from the normalized data and calculate Mahalanobis distance using a variety of normalization modes such as the distributive mode, the Euclidean mode, the ideal mode, the utility mode, the statistical standardization mode and non-normalized data.

4.5. Interim Conclusion: Optimal Reinsurance on the Translated Gamma Process Approximation

In this chapter, we have focused on the translated gamma process approximation where we reformulate the finite time ruin probability under the reinsurance arrangement. For two types of reinsurance arrangements: excess of loss and proportional reinsurance arrangements, the retention level M or proportion p have been added to the calculation as a new parameter and a new loading factor. We have obtained specific formulas by using the translated gamma process approximation for the compound Poisson process with the exponential and Pareto individual claim amounts.

We have considered the gain of the insurer under the fixed minimum ruin probability. From the insurers' point of view, we have generated a new method which consists of the main factors of the insurance's profit by making reinsurance. We have determined factors as released capital, expected profit per unit and expected utility of insurer's surplus process. We have given general information and formulas for the calculation of these three criteria under the translated gamma process approximation. We have calculated the released capital as the differences between the initial surplus whether in the case of reinsurance or not, whereas we have calculated the expected profit as the differences between premiums and paid claims by the insurer per unit time.

We have identified the third criteria as the expected utility of the insurer's surplus. In this context, we have assumed that the insurer's utility function is exponential function with parameter B .

We have also investigated the multi-criteria decision making under the consideration of the decision theory. It has been the main purpose of this chapter to draw attention to the multi-criteria decision making method in order to obtain the optimal reinsurance under the optimal reinsurance criteria. These criteria are investigated in Chapter 3: Optimal Reinsurance.

It is possible to conclude that the TOPSIS is the most suitable technique for optimal reinsurance criteria. This function depends on three main criteria: released capital, expected profit and expected utility. These criteria change according to the initial surplus and reinsurance level. While, the released capital and expected utility depend on both the initial surplus and the reinsurance level, the expected profit is only depend on the reinsurance level and it is independent from the initial surplus. The existence of these effects implies that these three criteria depend on the same components and they have different relationship levels.

It has been demonstrated that the key limitation of the optimal reinsurance criteria is dependency between criteria. However, TOPSIS method based on the Euclidean distance measure which does not consider the dependency among criteria. One possible solution to this problem is to use a different measure technique which involves the dependency. Mahalanobis distance is proposed as a distance measure by assuming the dependency among criteria. This distance measure requires the covariance or the correlation matrix of the criteria. Using the covariance or the correlation matrix, the relationship between the criteria can be expressed and the optimal levels can be obtained in respect of this statements.

Moreover, released capital, expected profit and expected utility are not on the same scale. Released capital and expected profit are based on monetary variable whereas the expected utility is based on scale variable between 0 and 1. Hence, the method to be used should involve the normalization technique. From the outcome of the investigation on Multi-attribute decision making, it is possible to conclude that the TOPSIS with Mahalanobis distance is a more convenient and appropriate method than the other multiple criteria decision making methods to determine the optimal reinsurance.

5. ANALYSIS OF THE OPTIMAL REINSURANCE ON THE TRANSLATED GAMMA PROCESS

5.1. Introduction

In this chapter, optimal pair of initial surplus and reinsurance level are obtained under a fixed ruin probability constraint for both cases of the excess of loss and the proportional reinsurance. The layout of the chapter is as follows.

In Section 5.2, we give the main assumptions of the calculations.

In Section 5.3, we discuss numerical examples for the compound Poisson process with individual claim amounts which have exponential and Pareto distributions under the excess of loss reinsurance. We calculate the optimal initial surplus and retention level employing TOPSIS method. In the context of the TOPSIS method, we use the Mahalanobis distance instead of Euclidean distance. We compare the optimal pair of initial surplus and the retention level considering different time horizons and different loading factors.

In Section 5.4 we discuss numerical examples for the compound Poisson process with individual claim amounts which have exponential and Pareto distributions under the proportional reinsurance. As in the excess of loss reinsurance, we use TOPSIS method with Mahalanobis distance instead of Euclidean distance. We compare the optimal pair of initial surplus and the proportion considering different time horizons and different loading factors.

In Section 5.5, we present the sensitivity analysis on the calculations according to the weights of the criteria, the parameter of the exponential utility function and the maximum possible value of each attribute. We measure the sensitivity on the vector-normalized covariance matrix, the original outcomes covariance matrix, and the correlation matrix by comparing the results in the weight set which one of the criteria is dominant.

In Section 5.6 we discuss the results and in Section 5.7 we conclude the chapter.

5.2. Analysis of the Optimal Reinsurance

In this section, we present some numerical examples under the optimal reinsurance criteria. We start with making the main assumptions for the calculations. The assumptions are as follows:

- The individual claim amount has exponential distribution with a probability density function $p(x) = e^{-x}$ and Pareto distribution with a probability density function $p(x) = \frac{\alpha}{\beta} [1 + \frac{x}{\beta}]^{-(\alpha+1)}$ with $\alpha = 3$ and $\beta = 4$,
- For the number of claims per unit time, we assume a Poisson distribution with the parameter $\lambda = 500$.
- Four different premium loading factors combinations (θ, ξ) , namely $(0.1, 0.15)$, $(0.1, 0.2)$, $(0.1, 0.3)$ and $(0.2, 0.3)$ are assumed.
- For time horizons, $t = 0.1, t = 0.5, t = 1, t = 5, t = 10$ and $t = 20$ are used.
- Four different sets of weights; equal weights and three set of weights in which one criterion is more important than the others are assumed. The sets of weights are given in Table 5.1.

Table 5.1: The weights in the TOPSIS method

Set of Weights	Released Capital	Expected Profit	Expected Utility
Scenario 1	0.3	0.3	0.3
Scenario 2	0.5	0.25	0.25
Scenario 3	0.25	0.5	0.25
Scenario 4	0.25	0.25	0.5

Individual claim amounts are assumed as the exponential and Pareto distributions which have different tail structures to investigate the differences between the optimal values. We assume that although the expectations of the aggregate claims are the same, the variances are different. Optimal pairs of initial surplus and reinsurance level are discussed both for the excess of loss and proportional reinsurance.

A set of alternatives which consists of pairs of initial surplus and the reinsurance level are calculated under a ruin probability constraint. By using these pairs, the released capital, expected profit and expected utility are calculated. In decision analysis part, we use the TOPSIS method with Mahalanobis distance to select the optimal initial surplus and reinsurance level.

5.3. Excess of Loss Reinsurance

The excess of loss reinsurance is analyzed with respect to two types of individual claims: exponential and Pareto distributions.

5.3.1. Exponential Claims on Excess of Loss Reinsurance

The algorithm for the exponential claims with the excess of loss reinsurance can be summarised as follows.

Step 1: *Calculation of the finite time ruin probability under the translated gamma process approximation.*

In calculation of the finite time ruin probability, we obtain the parameters of the translated gamma process and the loading factor, $\hat{\theta}$, by using Equation (2.11) and Equation (2.13), respectively. Then we integrate this loading factor into Equation (2.10).

Step 2: *Calculation of the largest initial surplus under the ruin probability which is calculated according to the translated gamma process is fixed at a minimum level such as 0.01.*

The largest initial surplus is obtained from Equation (2.10) as 0.01. The largest initial surplus depends on only θ and t since the reinsurance is not involved in this case. When the individual claim has an exponential distribution, the largest initial surplus which is calculated according to two different insurance loading factors $\theta = 0.1$ and $\theta = 0.2$ for various values of t is given as in Table 5.2.

Table 5.2: The largest initial surplus, u_L , for the exponential claims on the translated gamma process

t	$\theta = 0.1$	$\theta = 0.2$
0.1	23.943	20.246
0.5	39.970	26.171
1	45.855	26.569
5	49.630	26.591
10	49.638	26.591
20	49.638	26.591

Step 3: *Calculation of the smallest initial surplus under the ruin probability calculated according to the translated gamma process is fixed at a minimum level such as 0.01*

When the reinsurance arrangement is involved, parameters of the translated gamma process are calculated according to Equation (2.11) and Equation (4.8). Hence, by using these parameters, the loading factor, $\hat{\theta}$, is calculated by Equation (4.6) and then we integrate this loading factor into Equation (2.10). The smallest initial surplus which makes the ruin probability equal to 0.01 is calculated by using the one dimensional optimization technique which searches the interval from lower to upper for a minimum or maximum value of the Equation (2.10) in the **R** Programming. The corresponding retention level M of the smallest initial surplus is calculated.

In calculation of the smallest initial surplus u_S , and the corresponding retention level M , four different premium loading factors (θ, ξ) as (0.1, 0.15), (0.1, 0.2), (0.1, 0.3) and (0.2, 0.3) for various values of t are used. These loading factors are the same as in Dickson and Waters [44]. Under the excess of loss reinsurance for the exponential claims amount, the smallest initial surplus for various values of t and the loading factors are given in Table 5.3. These smallest initial surplus are used as a starting point of the alternative set.

Table 5.3: The smallest initial surplus, u_S , for the exponential claims under the excess of loss reinsurance

t	$\theta = 0.1 \ \& \ \xi = 0.15$	$\theta = 0.1 \ \& \ \xi = 0.2$	$\theta = 0.1 \ \& \ \xi = 0.3$	$\theta = 0.2 \ \& \ \xi = 0.3$
0.1	6.766	10.588	14.877	6.766
0.5	14.761	23.062	32.137	13.330
1	20.749	31.516	40.272	14.250
5	27.730	38.263	45.736	14.367
10	27.797	38.307	45.758	14.367
20	27.798	38.307	45.758	14.367

Step 4: *Constitution of the alternative set which consists of the pair of initial surplus and retention level*

We design a set of alternatives which consist of the pair of initial surplus and retention level. We begin with the smallest initial values u_S and increase this value by 0.1 or 0.05 to the largest initial surplus u_L . We calculate the corresponding retention level for each initial surplus. Hence we obtain an outcome set which represents all the possible combinations of the retention level and the corresponding initial surplus. Each pair of outcome set is denoted as (u_M, M) . The set of u_M starts with the smallest initial surplus u_S and it ends with the largest initial surplus u_L .

Step 5: *Calculation of the released capital, the expected profit and the expected utility according to the initial surplus and the retention level*

The set of (u_M, M) is used in the optimal reinsurance criteria. In this equation, released capital is calculated as $u_L - u_M$, expected profit is calculated by Equation (4.19) and expected utility is calculated by Equation (4.24) and (4.25). For the calculation of expected profit, parameters of the translated gamma process α , β and k are needed. These parameters can be calculated according to the retention level, M . However, in calculation of the expected utility, not only the retention level but also the initial surplus u_M are needed. In Table 5.4, we give the results for the three criteria when $\theta = 0.1$ and $\xi = 0.3$ for different time horizons.

Table 5.4: The released capital, expected profit and expected utility when $\theta = 0.1$ and $\xi = 0.3$ for the exponential claims under the excess of loss reinsurance

t	Number of Alternative	Initial Surplus u_M	Retention level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.1	1	14.877	1.099	9.066	0.000	0.253
	2	14.977	1.119	8.966	0.102	0.256
	3	15.077	1.140	8.866	0.202	0.259
	4	15.177	1.161	8.766	0.300	0.261
	5	15.277	1.181	8.666	0.397	0.264
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	89	23.677	6.170	0.266	4.969	0.425
	90	23.777	6.764	0.166	4.983	0.426
	91	23.877	7.900	0.066	4.994	0.427
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.5	1	32.137	1.298	7.833	4.522	0.501
	2	32.237	1.462	7.733	7.609	0.529
	3	32.337	1.536	7.633	8.853	0.541
	4	32.437	1.596	7.533	9.793	0.549
	5	32.537	1.648	7.433	10.574	0.556
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	77	39.737	6.907	0.233	24.925	0.696
	78	39.837	7.599	0.133	24.962	0.697
	79	39.937	9.269	0.033	24.993	0.698
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
1	1	40.272	1.977	5.583	29.237	0.720
	2	40.372	2.172	5.483	32.907	0.737
	3	40.472	2.262	5.383	34.382	0.744
	4	40.572	2.336	5.283	35.493	0.749
	5	40.672	2.402	5.183	36.415	0.754
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	54	45.572	6.865	0.283	49.843	0.818
	55	45.672	7.411	0.183	49.909	0.819
	56	45.772	8.379	0.083	49.966	0.819
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
5	1	45.736	2.669	3.894	197.989	0.984
	2	45.836	2.897	3.794	208.629	0.986
	3	45.936	3.007	3.694	212.925	0.987
	4	46.036	3.098	3.594	216.162	0.988
	5	46.136	3.181	3.494	218.844	0.988
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	37	49.336	6.984	0.294	249.305	0.993
	38	49.436	7.511	0.194	249.590	0.993
	39	49.536	8.404	0.094	249.832	0.993
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
10	1	45.758	2.676	3.880	396.786	0.999
	2	45.858	2.905	3.780	417.890	1.000
	3	45.958	3.015	3.680	426.420	1.000
	4	46.058	3.106	3.580	432.849	1.000
	5	46.158	3.189	3.480	438.177	1.000
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	37	49.358	7.047	0.280	498.695	1.000
	38	49.458	7.604	0.180	499.253	1.000
	39	49.558	8.598	0.080	499.723	1.000
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
20	1	45.758	2.676	3.880	793.573	1.000
	2	45.858	2.905	3.780	835.785	1.000
	3	45.958	3.015	3.680	852.843	1.000
	4	46.058	3.106	3.580	865.701	1.000
	5	46.158	3.189	3.480	876.358	1.000
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	37	49.358	7.047	0.280	997.390	1.000
	38	49.458	7.605	0.180	998.506	1.000
	39	49.558	8.599	0.080	999.447	1.000

We extend this calculation for various time horizons and different loading factors. Finally, we have a large set of outcome under different assumptions. We can see that each criterion has a different pattern. In the excess of loss reinsurance, when the initial surplus increases, corresponding retention level also increases in order to get a fixed ruin probability such as 0.01. When the pair (u_M, M) increases, the released capital decreases. However, both of the expected profit and expected utility increase with different slopes.

Figure 5.1 shows the graph of the three criteria when $\theta = 0.1$ & $\xi = 0.3$ for the exponential claims under the excess of loss reinsurance. The x axis of this graph shows the number of the alternative for the pair (u_M, M) while the results of the three criteria appear on the y axis. We prefer to use a plot with three y -axes and one shared x -axis because of the different scale. While the main y -axis shows the released capital, the second and third y axes show the expected profit and the expected utility, respectively.

It may be clearly seen that the released capital declines steadily to the level zero when the initial surplus closes to u_L . The expected profit increases with a decreasing slope. The first alternative begins with the maximum released capital and the minimum expected profit and the minimum expected utility. Conversely, the last alternative has the maximum expected profit and expected utility and the minimum released capital.

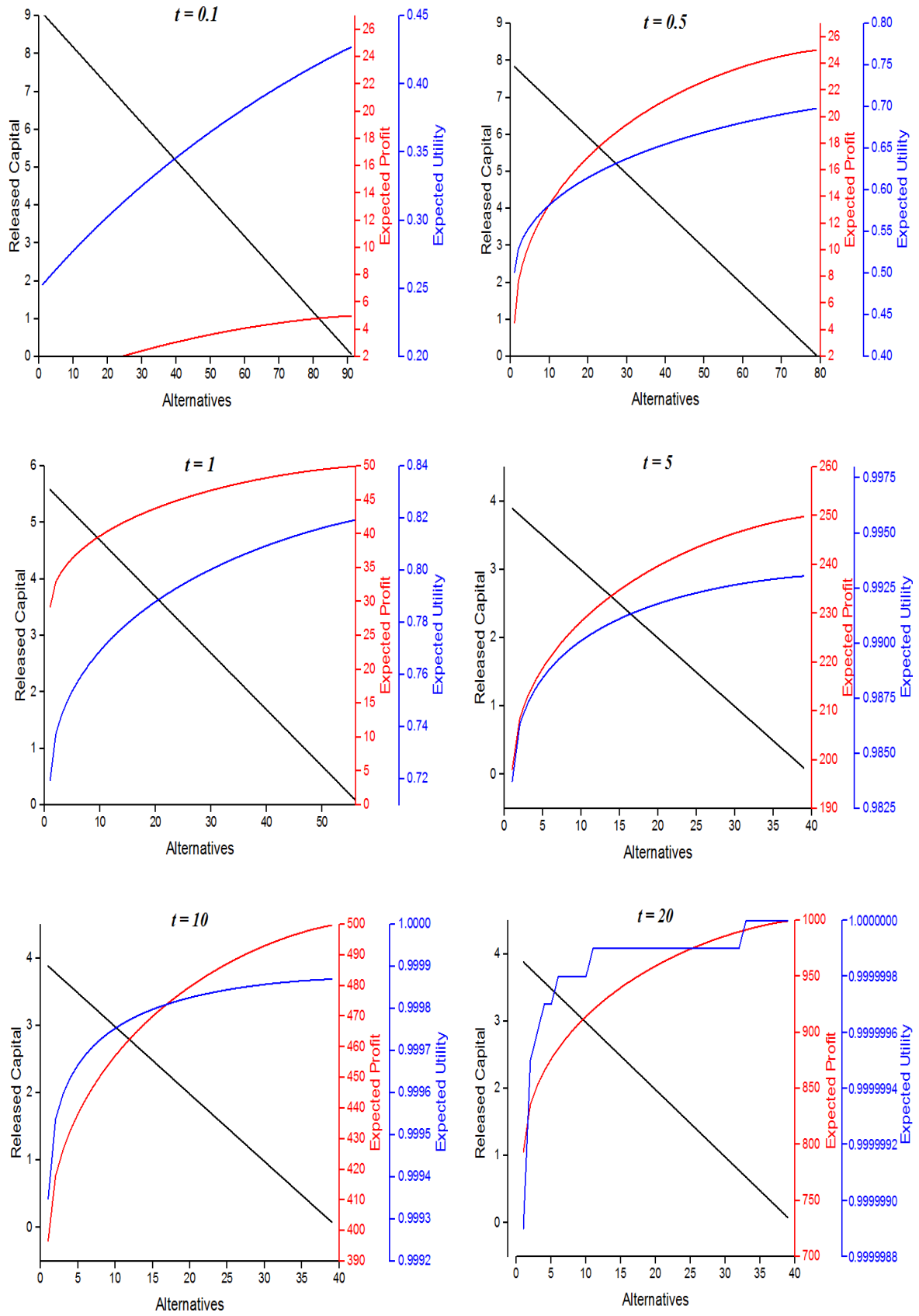


Figure 5.1: The graph of released capital, expected profit and expected utility when $\theta = 0.1$ and $\xi = 0.3$ under the excess of loss reinsurance for the exponential claims

Step 6: *Decision of the optimal pair of initial surplus and retention level under the TOPSIS method with Mahalanobis distance*

In order to decide the optimal pair of initial surplus and retention level (u_M, M) , we use the TOPSIS method with Mahalanobis distance. This method is explained in Section 4.4.4. In the TOPSIS method with Mahalanobis distance, we investigate the relationship between the criteria by using the vector-normalized covariance matrix, the original covariance matrix and the correlation matrix. For these three approaches, the optimal pairs (u_M, M) are obtained. Relative proximity to the ideal solution for each alternative is used in deciding the optimal pairs (u_M, M) .

In optimal reinsurance criteria, alternatives are not on the same scale as can be seen in Figure 5.1. In addition, the variance of the outcomes have an important role. The expected profit criteria has the highest variance of outcomes. For the case of $t = 1$, $\theta = 0.1$ and $\xi = 0.3$, the variance of the released capital, the expected profit and the expected utility are 2.6600, 24.8127 and 0.0006, respectively.

Hence, we use the vector-normalization technique to reduce the variance and to convert criteria to a comparable form. In the covariance matrix of the vector-normalized outcomes, the values are smaller than the original outcomes. To compare the three criteria on the same scale, we use the correlation matrix. In the correlation matrix, values are formed as scale-variance.

Let X is a random vector with X_i is random scalar defined as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}.$$

Each random variables has a finite variance.

Therefore, the covariance matrix Σ is identified as a matrix whose (i, j) entry represents the covariance as

$$\Sigma_{ij} = \text{cov}(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)]$$

where $\mu_i = E(X_i)$ and $\mu_j = E(X_j)$.

Hence, the covariance matrix is shown as

$$\Sigma = \begin{bmatrix} E[(X_1 - \mu_1)(X_1 - \mu_1)] & E[(X_1 - \mu_1)(X_2 - \mu_2)] & \cdots & E[(X_1 - \mu_1)(X_n - \mu_n)] \\ E[(X_2 - \mu_2)(X_1 - \mu_1)] & E[(X_2 - \mu_2)(X_2 - \mu_2)] & \cdots & E[(X_2 - \mu_2)(X_n - \mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ E[(X_n - \mu_n)(X_1 - \mu_1)] & E[(X_n - \mu_n)(X_2 - \mu_2)] & \cdots & E[(X_n - \mu_n)(X_n - \mu_n)] \end{bmatrix}$$

The inverse of the covariance matrix, Σ^{-1} is known as the concentration matrix or the precision matrix [129], [130].

The correlation coefficient $\rho_{X,Y}$ between two random variables X and Y with expected values μ_X and μ_Y and standard deviations σ_X and σ_Y is defined as:

$$\rho_{X,Y} = \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}.$$

It should be noted that the normalized version of the covariance is defined as the correlation coefficient because it shows the magnitude of the linear relation. Hence, the covariance of the normalized data should be equal to the correlation coefficient. However, in this study, normalization represents the vector-normalization technique which is the first step of the TOPSIS method. Therefore the covariance matrix of the normalized data is not equal to the correlation matrix.

The covariance and the correlation matrices of the vector-normalized and the original criteria outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ are given below.

The covariance matrix of the original outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ is obtained as

	Released Capital	Expected Profit	Expected Utility
Released Capital	2.6600	-7.5900	-0.0363
Expected Profit	-7.5900	24.8127	0.1179
Expected Utility	-0.0363	0.1179	0.0006

The covariance matrix of the vector-normalized outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ is obtained as

	Released Capital	Expected Profit	Expected Utility
Released Capital	0.0045	-0.0009	-0.0003
Expected Profit	-0.0009	0.0002	0.0001
Expected Utility	-0.0003	0.0001	0.0000

The correlation matrix of the vector-normalized or original outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ is defined as

	Released Capital	Expected Profit	Expected Utility
Released Capital	1	-0.9343	-0.9405
Expected Profit	-0.9343	1	0.9997
Expected Utility	-0.9405	0.9997	1

We can see that the covariance matrix of the released capital, expected profit and expected utility outcomes in the original case has much greater variation from the others. It can be clearly seen from the covariance matrix that, the relationship between the released capital and the pair of expected profit and expected utility is negative while the relationship between the expected profit and expected utility is positive.

Additionally, the variance of a set of outcomes for the released capital is higher than the variance of the corresponding outcomes for the expected utility. Moreover, the biggest variance is observed in the expected profit. The variance of the expected profit is very high and hence it has a dominant impact in the decision making. For this reason, we focus on both the covariance and the correlation cases to show the fluctuation between these situations.

In this study, we also consider four different sets of weights; equal weights and three set of weights under which one criterion is more important than the others given in Table 5.1. Furthermore, the optimal pair (u_M, M) is obtained according to six different time horizons: $t = 0.1, 0.5, 1, 5, 10, 20$. It should also be noted that when $t > 20$ the expected utility criteria will lose its efficiency and all outcomes of this criteria are equal to 1. Therefore, when $t > 20$, optimality only depends on two criteria: the released capital and the expected profit. For different sets of weights (Scenario 1-4 in Table 5.1), the optimal pair (u_M, M) by using the vector-normalized covariance matrix, the original covariance matrix and the correlation matrix are given in Table 5.5 through Table 5.8, respectively.

In Table 5.5, it is assumed that each criterion has the same importance so that the weights are $1/3$. The optimal levels are obtained according to the three different matrices: covariance matrix of vector-normalized data and original data and the correlation matrix.

Table 5.5: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 1 under the excess of loss reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	23.866	7.754	23.866	7.754	6.766	0.405
	0.5	29.361	1.951	39.761	7.166	36.861	3.727
	1	20.750	0.405	44.450	5.042	38.650	2.792
	5	27.730	0.845	40.530	2.635	31.130	1.381
	10	27.797	0.856	35.197	1.862	29.997	1.246
	20	37.398	2.151	31.298	1.400	29.398	1.171
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	23.888	8.140	23.888	8.140	11.888	0.867
	0.5	28.162	1.537	39.962	10.905	36.462	3.457
	1	39.416	2.760	45.316	6.188	43.116	4.037
	5	38.263	1.535	45.463	3.606	39.463	2.003
	10	38.307	1.545	42.907	2.830	38.907	1.848
	20	43.307	2.935	40.507	2.252	38.707	1.783
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	23.877	7.900	23.877	7.900	16.977	1.563
	0.5	39.937	9.269	39.937	9.269	36.037	2.995
	1	41.272	2.731	45.772	8.379	43.272	3.798
	5	45.736	2.669	48.236	4.852	46.036	3.098
	10	45.758	2.676	47.458	4.139	45.858	2.905
	20	47.558	4.218	46.758	3.626	45.858	2.905
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	20.166	7.635	20.166	7.635	7.566	0.538
	0.5	20.930	2.554	25.430	5.358	21.530	2.748
	1	14.250	0.812	23.850	3.613	18.050	1.760
	5	20.267	2.293	17.467	1.631	15.567	1.226
	10	19.667	2.137	15.767	1.270	15.267	1.158
	20	19.567	2.119	14.967	1.085	15.267	1.158

The results can be summarised into two categories: before and after $t = 1$. When $t = 0.1$, the optimal levels are smaller than the case of $t = 0.5$. The most likely explanation of this situation is that the time horizon is too small so the premium income is more sufficient to cover the expected loss than the case of $t = 0.5$. After that time, the optimal levels increase to the level of $t = 1$. We can see that time increases enough such as $t > 1$, the optimal pairs (u_M, M) start to decrease and change in the optimal levels gets smaller. After a sufficient time period, the optimal initial surplus and retention level do not vary and the results become stable for the infinite time. Nonetheless, some distortion can be observed since the expected utility loses its efficiency when t is sufficiently large. The highest optimal levels are obtained according to the normalized covariance matrix, whereas the lowest optimal levels are obtained according to the correlation matrix.

Table 5.6: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 2 under the excess of loss reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.1$ & $\xi = 0.15$	0.1	6.766	0.405	23.866	7.754	6.766	0.405
	0.5	14.761	0.405	39.661	6.690	35.061	3.091
	1	20.750	0.405	44.050	4.715	34.050	2.027
	5	27.730	0.845	39.330	2.437	29.130	1.137
	10	27.797	0.856	34.397	1.764	28.697	1.075
	20	33.398	1.644	30.898	1.353	28.498	1.044
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	10.588	0.693	23.888	8.140	11.588	0.826
	0.5	26.762	1.320	39.962	10.905	35.662	3.160
	1	31.516	0.961	45.216	5.973	41.316	3.304
	5	38.263	1.535	44.763	3.365	38.863	1.840
	10	38.307	1.545	42.307	2.679	38.607	1.747
	20	38.307	1.545	40.307	2.205	38.507	1.706
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	4.877	1.099	23.877	7.900	16.577	1.468
	0.5	32.837	1.785	39.937	9.269	35.537	2.789
	1	40.972	2.575	45.772	8.379	42.872	3.559
	5	45.736	2.669	48.036	4.647	45.936	3.007
	10	45.758	2.676	47.258	3.986	45.858	2.905
	20	45.758	2.676	46.658	3.555	45.858	2.905
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	6.766	0.405	20.166	7.635	7.566	0.538
	0.5	13.330	0.676	25.230	5.041	19.230	2.086
	1	14.250	0.812	23.250	3.316	15.750	1.270
	5	14.367	0.840	16.967	1.524	14.867	1.058
	10	17.467	1.631	15.669	1.248	14.767	1.031
	20	16.667	1.461	14.867	1.058	14.767	1.031

In Table 5.6, Scenario 2, it is assumed that more weight is given to the released capital criterion. The lowest and the highest optimal levels are obtained according to the normalized and the original covariance matrix, respectively, as in Table 5.5. The optimal levels according to all matrices give smaller results than Scenario 1, Table 5.5. Some decrease in the optimal levels can be observed according to Scenario 1 for all matrices. However, the biggest change in the optimal levels are observed in the vector-normalized covariance matrix case. Increasing the importance of the released capital causes the optimal levels move towards to the level that the released capital has the highest value. Any significant difference cannot be observed according to the original covariance matrix with Scenario 1.

Table 5.7: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 3 under the excess of loss reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.1$ & $\xi = 0.15$	0.1	6.766	0.405	23.866	7.754	10.166	0.766
	0.5	37.261	3.914	39.761	7.166	38.861	5.090
	1	42.850	4.033	44.450	5.042	42.650	3.945
	5	44.630	3.538	40.530	2.635	34.930	1.829
	10	42.997	3.121	35.197	1.862	32.497	1.539
	20	41.698	2.849	31.198	1.388	31.398	1.412
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.1$ & $\xi = 0.2$	0.1	10.588	0.693	23.888	8.140	14.288	1.231
	0.5	35.562	3.126	39.962	10.905	37.562	3.985
	1	44.416	4.917	45.316	6.188	44.716	5.227
	5	47.462	4.557	45.463	3.606	40.663	2.290
	10	46.507	4.033	42.907	2.830	39.607	2.037
	20	45.607	3.657	40.507	2.252	39.107	1.906
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.1$ & $\xi = 0.3$	0.1	14.877	1.099	23.877	7.900	18.077	1.848
	0.5	35.237	2.673	39.937	9.269	36.737	3.315
	1	43.672	4.065	45.772	8.379	43.772	4.137
	5	49.136	6.308	48.236	4.852	46.336	3.333
	10	48.858	5.689	47.458	4.139	45.958	3.015
	20	48.458	5.100	46.758	3.626	45.858	2.905
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.2$ & $\xi = 0.3$	0.1	6.766	0.405	20.166	7.635	11.366	1.198
	0.5	24.730	4.461	25.430	5.358	24.030	3.905
	1	24.650	4.116	23.850	3.613	21.150	2.547
	5	22.667	3.064	17.367	1.609	17.067	1.546
	10	21.967	2.807	15.767	1.270	16.467	1.419
	20	21.767	2.739	14.967	1.085	16.267	1.377

In Table 5.7, Scenario 3, it is assumed that more weight is given to the expected profit. The lowest and the highest optimal levels are obtained according to the normalized and the original covariance matrix, respectively, as in Table 5.5 and Table 5.6. The optimal levels according to all matrices give higher results than under Scenario 1 except $t = 0.1$. We can see that the changes in the optimal levels for the original covariance matrix and the correlation matrix are not so large, since the expected profit has a dominant impact on the other criteria. The outcomes of the expected profit have the largest variance. Any significant differences cannot be observed according to the original covariance matrix with Scenario 1 and Scenario 2.

Table 5.8: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 4 under the excess of loss reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	23.866	7.754	23.866	7.754	6.766	0.405
	0.5	33.761	2.754	39.761	7.166	34.061	2.826
	1	25.950	1.102	44.850	5.473	36.250	2.354
	5	27.730	0.845	41.730	2.856	30.930	1.358
	10	27.797	0.856	36.907	1.977	29.997	1.246
	20	37.298	2.137	31.598	1.435	29.398	1.171
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	23.888	8.140	23.888	8.140	10.588	0.693
	0.5	39.962	10.905	39.962	10.045	35.362	3.061
	1	40.116	2.943	45.316	6.188	44.716	5.227
	5	38.263	1.535	46.063	3.840	39.463	2.003
	10	38.307	1.545	43.407	2.962	38.907	1.848
	20	43.207	2.909	40.807	2.322	38.707	1.783
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	23.877	7.900	23.877	7.900	15.777	1.288
	0.5	39.937	9.269	39.937	9.269	35.637	2.829
	1	45.772	8.379	45.772	8.379	43.072	3.675
	5	46.036	3.098	48.436	5.093	46.036	3.098
	10	45.758	2.676	47.758	4.385	45.858	2.905
	20	45.758	2.676	46.958	3.767	45.858	2.905
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	20.166	7.635	20.166	7.635	6.766	0.405
	0.5	18.730	1.965	25.630	5.770	19.030	2.037
	1	14.350	0.901	24.450	3.975	16.750	1.481
	5	20.067	2.240	17.767	1.696	15.567	1.226
	10	19.667	2.137	15.967	1.313	15.267	1.158
	20	19.467	2.087	14.967	1.085	15.267	1.158

In Table 5.8, Scenario 4, it is assumed that more weight is given to the expected utility. The lowest and the highest optimal levels are obtained according to the vector-normalized and the original covariance matrix, respectively, as in Table 5.5, Table 5.6 and Table 5.7. We can see that the optimal levels are higher than Scenario 1, but not as high as Scenario 3.

5.3.2. Pareto Claims on Excess of Loss Reinsurance

In this section, we assume that the individual claim amounts have a Pareto distribution with a probability density function $p(x) = \frac{\alpha}{\beta} [1 + \frac{x}{\beta}]^{-(\alpha+1)}$. Parameters of the Pareto distribution are assumed as $\alpha = 3$ and $\beta = 4$ to have the same expectation as the exponential distribution given in Section 5.3.1. The algorithm for the Pareto claims with the excess of loss reinsurance have a similar structure to the exponential claims.

The largest initial surplus value, u_L which makes the ruin probability equal to 0.01 for a fixed t is calculated in case of no-reinsurance, given in Table 5.9.

Table 5.9: The largest initial surplus, u_L , for the Pareto claims on the translated gamma process

t	$\theta = 0.1$	$\theta = 0.2$
0.1	34.337	30.858
0.5	57.815	42.827
1	68.450	44.755
5	79.598	45.090
10	79.772	45.090
20	79.774	45.090

The results are comparable with the exponential case. It is shown that when the individual claim amount has a Pareto distribution, the required largest initial surplus is higher than the exponential case.

When reinsurance arrangement is involved, moments of the Pareto distribution are calculated according to Equation (4.9)-(4.10) and (4.11) and then the parameters of the translated gamma process are calculated according to Equation (2.11). The set of the feasible smallest initial surplus, u_S , for various values of t with four different loading combinations (θ, ξ) as $(0.1, 0.15)$, $(0.1, 0.2)$, $(0.1, 0.3)$ and $(0.2, 0.3)$ are calculated. The smallest initial surplus for various values of t and loading factors are given in Table 5.10.

Table 5.10: The smallest initial surplus, u_S , for the Pareto claims under the excess of loss reinsurance

t	$\theta = 0.1 \ \& \ \xi = 0.15$	$\theta = 0.1 \ \& \ \xi = 0.2$	$\theta = 0.1 \ \& \ \xi = 0.3$	$\theta = 0.2 \ \& \ \xi = 0.3$
0.1	21.173	33.409	45.711	15.455
0.5	30.222	44.316	57.415	15.692
1	21.173	33.409	45.711	15.455
5	30.222	44.316	57.415	15.692
10	30.379	44.506	57.613	15.692
20	30.382	44.510	57.616	15.692

As seen in Table 5.10, the required smallest initial surplus for the Pareto claims is higher than the exponential case. The same situation is observed for the largest initial surplus.

In Table 5.11, we show the calculation of the three criteria (the released capital, expected profit and expected utility) when $\theta = 0.1$ and $\xi = 0.3$. From this table, it can be seen that when the pair (u_M, M) increases, the released capital decreases as in the exponential case. Although similar comment can be made, some different results can also be observed. Even though it is seen that there is an increase in the expected profit and the expected utility as retention level increase, slope of this increase differs from the exponential case.

The analysis indicates that because of the nature of the high variance of the Pareto distribution, the differences between the starting point (smallest initial surplus) and the ending point (largest initial surplus) under each criterion are higher than those in the exponential claims case. In particular, we can see a huge range in the expected profit and released capital by comparing with the exponential claims.

Figure 5.2 shows the results under the three criteria when $\theta = 0.1$ and $\xi = 0.3$ for the Pareto claims under the excess of loss reinsurance.

Table 5.11: The released capital, expected profit and expected utility when $\theta = 0.1$ and $\xi = 0.3$ for the Pareto claims under the excess of loss reinsurance

t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.1	1	15.890	1.341	18.447	0.049	0.268
	2	15.990	1.364	18.347	0.127	0.270
	3	16.090	1.387	18.247	0.204	0.273
	4	16.190	1.410	18.147	0.279	0.275
	5	16.290	1.434	18.047	0.354	0.278
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	183	34.090	241.950	0.247	5.000	0.528
	184	34.190	407.740	0.147	5.000	0.529
	185	34.290	1285.000	0.047	5.000	0.529
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.5	1	34.477	1.488	23.338	2.601	0.505
	2	34.577	1.550	23.238	3.502	0.514
	3	34.677	1.604	23.138	4.250	0.521
	4	34.777	1.653	23.038	4.900	0.528
	5	34.877	1.699	22.938	5.482	0.533
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	232	57.577	306.884	0.238	25.000	0.775
	233	57.677	526.961	0.138	25.000	0.776
	234	57.777	1910.971	0.038	25.000	0.776
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
1	1	45.711	2.061	22.740	18.751	0.693
	2	45.811	2.264	22.640	22.241	0.711
	3	45.911	2.357	22.540	23.652	0.718
	4	46.011	2.431	22.440	24.720	0.724
	5	46.111	2.497	22.340	25.612	0.728
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	226	68.211	336.529	0.240	50.000	0.871
	227	68.311	573.121	0.140	50.000	0.871
	228	68.411	2004.297	0.040	50.000	0.871
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
5	1	57.415	3.318	22.183	169.691	0.976
	2	57.515	3.563	22.083	178.370	0.979
	3	57.615	3.676	21.983	181.930	0.980
	4	57.715	3.767	21.883	184.644	0.981
	5	57.815	3.847	21.783	186.916	0.981
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	220	79.315	341.032	0.283	250.000	0.993
	221	79.415	520.282	0.183	250.000	0.993
	222	79.515	1121.160	0.083	250.000	0.993
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
10	1	57.613	3.369	22.159	339.383	0.998
	2	57.713	3.614	22.059	356.740	0.999
	3	57.813	3.727	21.959	363.861	0.999
	4	57.913	3.818	21.859	369.287	0.999
	5	58.013	3.898	21.759	373.832	0.999
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	220	79.513	377.136	0.259	499.999	1.000
	221	79.613	608.146	0.159	500.000	1.000
	222	79.713	1621.064	0.059	500.000	1.000
t	Number of Alternative	Initial Surplus u_M	Retention Level M	Released Capital $u_L - u_M$	Expected Profit P	Expected Utility $E[u[U(t)]]$
20	1	57.616	3.371	22.158	686.713	1.000
	2	57.716	3.615	22.058	720.214	1.000
	3	57.816	3.728	21.958	734.005	1.000
	4	57.916	3.819	21.858	744.530	1.000
	5	58.016	3.899	21.758	753.356	1.000
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	220	79.516	378.826	0.258	999.999	1.000
	221	79.616	612.581	0.158	1000.000	1.000
	222	79.716	1653.219	0.058	1000.000	1.000

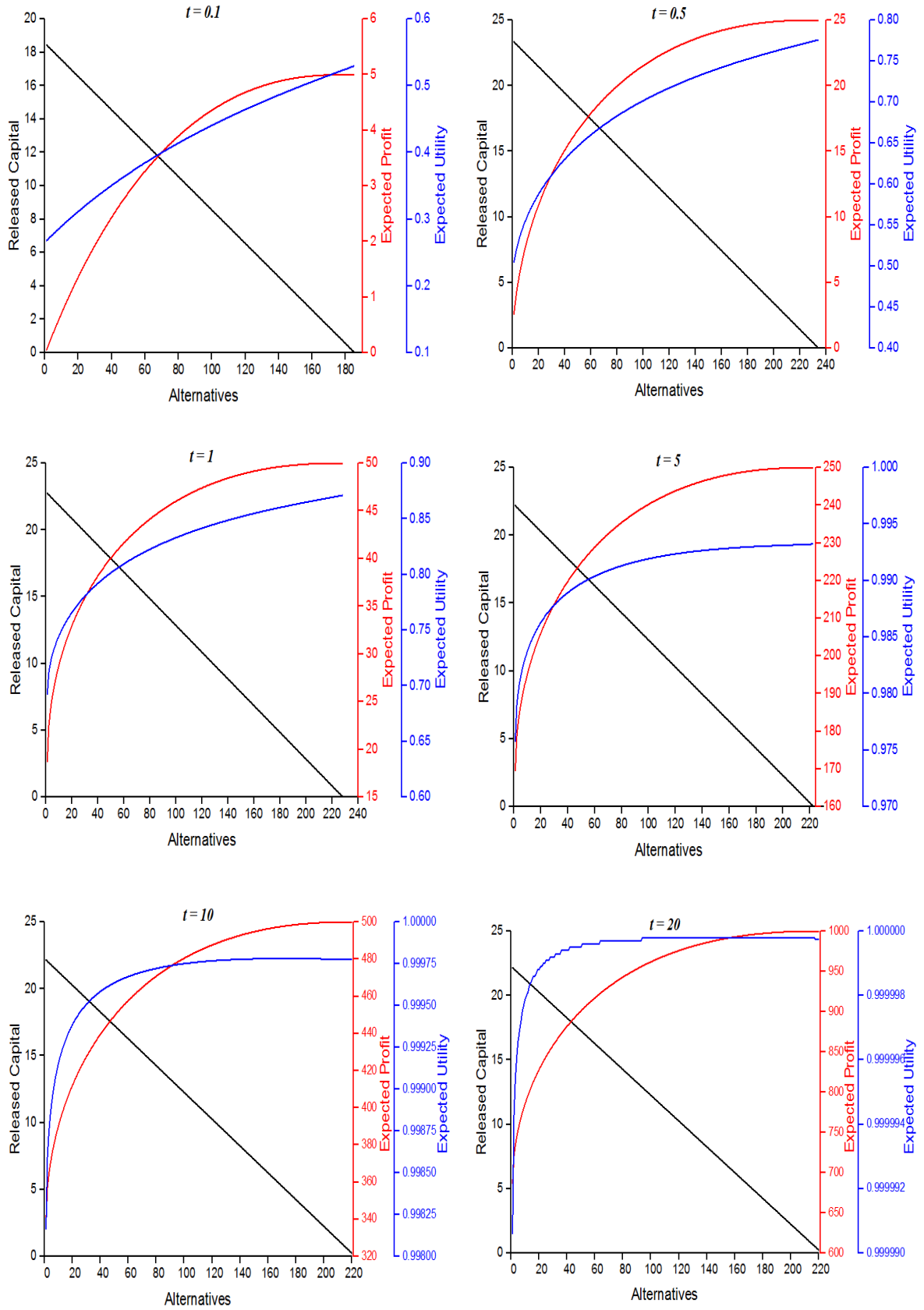


Figure 5.2: The graph of the released capital, expected profit and expected utility when $\theta = 0.1$ and $\xi = 0.3$ under the excess of loss reinsurance for the Pareto claims

The optimal pairs (u_M, M) are shown in Tables 5.12 through Table 5.15 for the vector-normalized covariance matrix, the original covariance matrix and the correlation matrix.

We can make similar comments as in the exponential claims. The highest optimal levels are obtained according to the normalized covariance matrix, whereas the lowest optimal levels changes between the original covariance matrix and the correlation matrix. We can observe that the optimal pair (u_M, M) decreases when t increases. At the end of a sufficient time period such as $t = 20$, Scenario 1 (Table 5.12) and Scenario 4 (Table 5.15) give very close optimal levels compared to most of four different loading situations as in the exponential claims. It is clear that the optimal levels for the Pareto claims are higher than the exponential claims in each scenario and different loading factors. The results are consistent in good agreement with the exponential case has been shown in the previous section.

Table 5.12: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 1 under the excess of loss reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	34.250	687.312	34.250	687.312	6.950	0.437
	0.5	25.321	1.469	57.721	769.050	25.921	1.540
	1	40.173	2.865	68.073	215.961	48.173	4.627
	5	53.522	4.605	62.422	7.593	41.822	2.507
	10	51.779	4.203	40.079	3.662	38.179	2.031
	20	50.082	3.854	38.982	2.121	36.282	1.800
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	34.298	1532.929	34.298	1532.929	11.098	0.791
	0.5	57.745	1029.200	57.745	1029.000	31.945	2.094
	1	40.609	2.527	68.409	1922.980	48.209	4.339
	5	61.116	6.650	70.116	13.425	51.216	3.570
	10	60.006	6.170	60.306	6.289	48.306	2.903
	20	58.710	5.686	53.010	3.995	47.010	2.612
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	34.290	1285.000	34.290	1285.000	15.890	1.341
	0.5	57.777	1910.971	3.777	1910.971	38.877	3.059
	1	67.411	2004.297	68.411	2004.297	50.411	4.333
	5	67.915	10.307	75.615	29.215	61.315	5.750
	10	67.613	9.936	68.613	10.924	59.313	4.675
	20	66.616	9.073	64.816	7.748	58.616	4.290
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	30.762	668.576	30.762	668.576	6.962	0.440
	0.5	20.084	2.138	42.784	1924.792	22.084	2.633
	1	28.855	4.753	43.055	53.988	26.855	3.949
	5	27.892	4.323	24.392	3.132	20.392	2.103
	10	27.192	4.054	19.292	1.857	19.392	1.879
	20	26.892	3.945	17.092	1.375	19.092	1.813

In Table 5.12, it is assumed that each criterion has the same importance level so that the weights are equal to $1/3$. The optimal levels are obtained according to the three different matrices. The results are in good agreement with the excess of loss case. Hence, the results can be categorized with regard to $t \leq 1$ and $t > 1$ as in the excess of loss reinsurance. When $t \leq 1$, the results are not conclusive. The optimal pair of initial surplus and the retention level in the case of $t = 0.1$ is smaller than the case of $t = 0.5$. We can see that as time increases sufficiently enough, the optimal pairs (u_M, M) decrease and the changes in the optimal levels get smaller. For all time horizons and loading pairs, it is observed that the optimal levels according to the correlation matrix are obtained between the optimal levels which are calculated according to the vector-normalized and the original covariance matrix.

Table 5.13: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 2 under the excess of loss reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	6.950	0.437	34.250	687.312	6.950	0.437
	0.5	22.521	1.159	57.721	769.050	23.621	1.276
	1	35.373	2.144	67.973	171.935	44.373	3.672
	5	30.222	0.914	60.022	6.567	36.822	1.867
	10	45.979	3.124	47.479	3.374	34.479	1.585
	20	44.582	2.904	38.282	2.044	33.382	1.452
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	11.098	0.791	34.298	1532.929	11.098	0.791
	0.5	28.345	1.492	57.745	1029.200	30.745	1.884
	1	39.409	2.298	68.409	1922.980	46.309	3.804
	5	44.316	1.761	68.416	11.437	48.416	2.938
	10	55.806	4.752	59.106	5.829	46.606	2.519
	20	54.910	4.497	52.210	3.797	45.810	2.324
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	15.890	1.341	34.290	1285.000	15.890	1.341
	0.5	36.277	2.211	57.777	1910.971	38.177	2.827
	1	48.011	3.361	68.411	2004.297	49.611	4.002
	5	65.015	7.938	74.515	23.391	74.515	23.391
	10	57.613	3.369	67.813	10.122	58.513	4.230
	20	64.316	7.421	64.216	7.358	58.116	3.973
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	6.962	0.440	30.762	668.576	6.962	0.440
	0.5	19.084	1.912	42.784	1924.792	20.284	2.185
	1	25.855	3.602	42.355	38.384	23.755	2.960
	5	24.892	3.281	23.592	2.904	18.291	1.639
	10	24.192	3.074	19.092	1.813	17.592	1.487
	20	23.992	3.016	16.992	1.352	17.492	1.465

In Table 5.13, under Scenario 2, it is assumed that more weight is given to the released capital criterion. The lowest and the highest optimal levels are obtained according to the vector-normalized and the original covariance matrix, respectively. The optimal levels according to all matrices give smaller or equal values than the values under Scenario 1. However, the biggest change in optimal levels is in the vector-normalized covariance matrix case. Increasing the importance of the released capital causes the optimal levels move towards to the level that the released capital has the highest value.

Table 5.14: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 3 under the excess of loss reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	6.950	0.437	34.250	687.312	6.950	0.437
	0.5	26.821	1.651	57.721	769.050	32.621	2.684
	1	51.673	5.818	68.073	215.961	53.073	6.421
	5	60.422	6.723	62.322	7.545	47.522	3.385
	10	57.679	5.739	48.979	3.643	43.179	2.696
	20	55.482	5.097	38.882	2.119	40.682	2.351
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	11.098	0.791	34.298	1532.929	11.098	0.791
	0.5	26.745	1.242	57.745	1029.200	36.445	3.020
	1	46.809	3.938	68.409	1922.980	51.109	5.331
	5	66.616	9.836	70.116	13.425	54.816	4.483
	10	64.606	8.404	60.306	6.289	51.006	3.511
	20	62.510	7.264	53.010	3.995	49.110	3.081
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	15.890	1.341	34.290	1285.000	16.490	1.481
	0.5	34.477	1.488	57.777	1910.971	40.977	3.811
	1	47.411	3.116	68.411	2004.297	52.311	5.194
	5	72.015	16.061	75.615	29.215	63.215	6.797
	10	71.013	14.049	68.613	10.924	60.413	5.242
	20	69.216	11.594	64.816	7.748	59.316	4.676
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
	0.1	6.962	0.441	30.762	668.576	6.962	0.441
	0.5	24.484	3.332	42.784	1924.792	26.084	3.886
	1	33.755	7.759	43.055	53.988	30.355	5.480
	5	30.992	5.785	24.392	3.132	23.092	2.768
	10	29.992	5.257	19.292	1.857	21.892	2.460
	20	29.592	5.062	17.092	1.375	21.592	2.386

In Table 5.14, under Scenario 3, it is assumed that more weight is given to the expected profit. The lowest and the highest optimal levels are obtained according to the normalized and the original covariance matrix, respectively, as in Table 5.12 and Table 5.13. We can see that the changes in the optimal levels for the original covariance matrix and the correlation matrix are not so big. We can see that as time increases sufficiently enough such as $t > 1$, the optimal pairs (u_M, M) start to decrease and the changes in the optimal levels get smaller.

Table 5.15: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 4 under the excess of loss reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.1$ & $\xi = 0.15$	0.1	34.250	687.312	34.250	687.312	10.750	0.882
	0.5	57.721	769.050	57.721	769.050	17.821	0.696
	1	54.773	7.293	68.173	291.675	43.973	3.586
	5	51.622	4.175	64.722	8.832	41.522	2.466
	10	51.479	4.139	50.779	3.994	38.179	2.031
	20	50.082	3.854	39.782	2.233	36.282	1.800
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.1$ & $\xi = 0.2$	0.1	34.298	1532.929	34.298	1532.929	11.098	0.791
	0.5	57.745	1029.200	57.745	1029.200	27.245	1.320
	1	68.409	1922.980	68.409	1922.980	45.709	3.649
	5	58.416	5.997	71.816	16.164	50.916	3.500
	10	59.406	5.940	61.606	6.841	48.306	2.903
	20	58.510	5.616	53.910	4.227	47.010	2.612
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.1$ & $\xi = 0.3$	0.1	34.290	1285.000	34.290	1285.000	15.890	1.341
	0.5	57.777	1910.971	57.777	1910.971	37.177	2.504
	1	68.411	2004.297	68.411	2004.297	49.411	3.921
	5	63.015	6.680	76.615	37.931	61.215	5.698
	10	66.513	8.990	69.313	11.709	59.313	4.675
	20	66.416	8.912	65.516	8.233	58.616	4.290
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
$\theta = 0.2$ & $\xi = 0.3$	0.1	30.762	668.576	30.762	668.576	11.562	1.279
	0.5	42.784	1924.792	42.784	1924.792	16.884	1.445
	1	26.455	3.807	3.807	76.191	25.355	3.439
	5	27.792	4.284	4.284	3.405	20.392	2.103
	10	27.192	4.054	4.054	1.945	19.392	1.879
	20	26.692	3.873	17.192	1.398	19.092	1.813
	t	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M

In Table 5.15, under Scenario 4, it is assumed that more weight is given to the expected utility. The highest optimal pairs are obtained according to the original covariance matrix whereas the lowest optimal pairs are obtained according to the correlation matrix. We can see that the optimal levels are higher than Scenario 1, however difference is not as high as Scenario 3.

It is shown that the same pattern can be observed for both the exponential and Pareto case under the excess of loss reinsurance. The results indicate that the changes between the optimal pairs which are obtained from all matrices are higher than the exponential case.

5.4. Proportional Reinsurance

We now consider the proportional reinsurance to obtain the optimal initial surplus and the proportion. The approach is the same as the one adopted for the excess of loss reinsurance.

5.4.1. Exponential Claims on Proportional Reinsurance

Similar to the study of excess of loss reinsurance, it is assumed that the individual claim amount has an exponential distribution with a probability density function $p(x) = e^{-x}$. The largest initial surplus for the exponential case is the same as the excess of loss or the proportional case since the largest initial surplus does not depend on a reinsurance arrangement. The largest initial surplus value for various values of t , $\theta = 0.1$ and $\theta = 0.2$ are given in Table 5.2 in Section 5.3.1. When the proportional reinsurance arrangement is involved, parameters of the translated gamma process are calculated according to equations (2.11) and (4.16). By using these parameters, the loading factor $\hat{\theta}$ is calculated by using Equation (4.6) and then we integrate this loading factor into Equation (2.10) as in the case of no-reinsurance. The smallest values of u and proportion p are found by using the one dimensional optimization technique in order to make the ruin probability equal to 0.01. The smallest initial surplus u_S for various values of t with four different loading combinations $\theta = 0.1/\xi = 0.15$, $\theta = 0.1/\xi = 0.2$, $\theta = 0.1/\xi = 0.3$, $\theta = 0.2/\xi = 0.3$ are calculated and given in Table 5.16.

Table 5.16: The smallest Initial Surplus, u_S , for the exponential distribution under the proportional reinsurance

t	$\theta = 0.1 \ \& \ \xi = 0.15$	$\theta = 0.1 \ \& \ \xi = 0.2$	$\theta = 0.1 \ \& \ \xi = 0.3$	$\theta = 0.2 \ \& \ \xi = 0.3$
0.1	27.920	41.742	45.855	22.387
0.5	43.063	49.560	49.630	22.790
1	27.920	41.742	45.855	22.387
5	43.063	49.560	49.630	22.790
10	43.292	49.574	49.638	22.790
20	43.296	49.574	49.638	22.790

As seen in Table 5.16, the required smallest initial surplus in the proportional reinsurance is higher than the excess of loss reinsurance even for the same individual claim assumption. For the exponential claim amount distribution, the required smallest initial surplus for the proportional and the excess of loss reinsurance can be compared in Table 5.16 and Table 5.3.

In the excess of loss reinsurance, we use the loading factors $\theta = 0.1$ and $\xi = 0.3$ to show the pattern. In the proportional reinsurance, for this pair of loading factor, the obtained set of alternative is so small, hence the smallest initial surplus is almost equal to the largest initial surplus. Thus, the optimal initial surplus is obtained as the smallest initial values. For this reason, to show the results of the three criteria, we prefer to present the case of $\theta = 0.2$ and $\xi = 0.3$.

The results of the three criteria for the pair (u_p, p) when $\theta = 0.2$ and $\xi = 0.3$ are given in Table 5.17.

Figure 5.3 shows the graph of three criteria for $\theta = 0.2$ and $\xi = 0.3$ under the proportional reinsurance with individual claim amount has an exponential distribution. It can be clearly seen that the released capital declines steadily and reaches zero at the point where the last alternative stands. On the other hand, the expected profit increases with a decreasing slope. The first alternative has the maximum released capital but minimum expected profit and expected utility. Conversely, the last alternative has the maximum expected profit and expected utility but the minimum released capital. This situation is same as in the excess of loss reinsurance case. In comparison with the excess of loss reinsurance, it is seen that the difference between the largest and the smallest observations in the expected profit and the expected utility are larger than that in the excess of loss case. The pattern for the released capital is nearly same as in the excess of loss reinsurance.

The optimal pairs (u_p, p) are shown in Tables 5.18 through 5.21 using the vector-normalized covariance matrix, the original covariance matrix and the correlation matrix according to four different scenarios.

Table 5.17: The released capital, expected profit and expected utility when $\theta = 0.2$ and $\xi = 0.3$ for the exponential claims under the proportional reinsurance

t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.1	1	9.344	0.333	10.901	0.000	0.169
	2	9.444	0.340	10.801	0.098	0.172
	3	9.544	0.346	10.701	0.195	0.175
	4	9.644	0.353	10.601	0.293	0.178
	5	9.744	0.359	10.501	0.390	0.181
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	108	20.044	0.988	0.201	9.820	0.439
	109	20.144	0.994	0.101	9.910	0.441
	110	20.244	1.000	0.001	9.999	0.443
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.5	1	19.751	0.413	6.420	5.944	0.391
	2	19.851	0.469	6.320	10.172	0.439
	3	19.951	0.493	6.220	11.986	0.459
	4	20.051	0.512	6.120	13.408	0.474
	5	20.151	0.528	6.020	14.629	0.487
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	63	25.951	0.986	0.220	48.969	0.753
	64	26.051	0.993	0.120	49.438	0.756
	65	26.151	0.999	0.020	49.905	0.758
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
1	1	22.387	0.594	4.182	39.032	0.686
	2	22.487	0.640	4.082	46.047	0.724
	3	22.587	0.661	3.982	49.164	0.740
	4	22.687	0.678	3.882	51.647	0.752
	5	22.787	0.692	3.782	53.822	0.762
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	40	26.287	0.981	0.282	97.194	0.897
	41	26.387	0.988	0.182	98.194	0.899
	42	26.487	0.995	0.082	99.190	0.901
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
5	1	22.790	0.634	3.801	0.000	0.325
	2	22.890	0.676	3.701	30.118	0.624
	3	22.990	0.696	3.601	44.274	0.715
	4	23.090	0.711	3.501	55.191	0.770
	5	23.190	0.725	3.401	64.428	0.808
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	37	26.390	0.987	0.201	206.631	0.988
	38	26.490	0.993	0.101	209.648	0.989
	39	26.590	1.000	0.001	212.626	0.989
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
10	1	22.790	0.634	3.801	450.662	1.000
	2	22.890	0.676	3.701	514.578	1.000
	3	22.990	0.696	3.601	543.563	1.000
	4	23.090	0.711	3.501	566.854	1.000
	5	23.190	0.725	3.401	587.218	1.000
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	37	26.390	0.987	0.201	979.860	1.000
	38	26.490	0.993	0.101	989.902	1.000
	39	26.590	1.000	0.001	999.891	1.000
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
20	1	22.790	0.634	3.801	901.324	1.000
	2	22.890	0.676	3.701	1029.154	1.000
	3	22.990	0.696	3.601	1087.126	1.000
	4	23.090	0.711	3.501	1133.708	1.000
	5	23.190	0.725	3.401	1174.435	1.000
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	37	26.390	0.987	0.201	1959.720	1.000
	38	26.490	0.993	0.101	1979.804	1.000
	39	26.590	1.000	0.001	1999.783	1.000

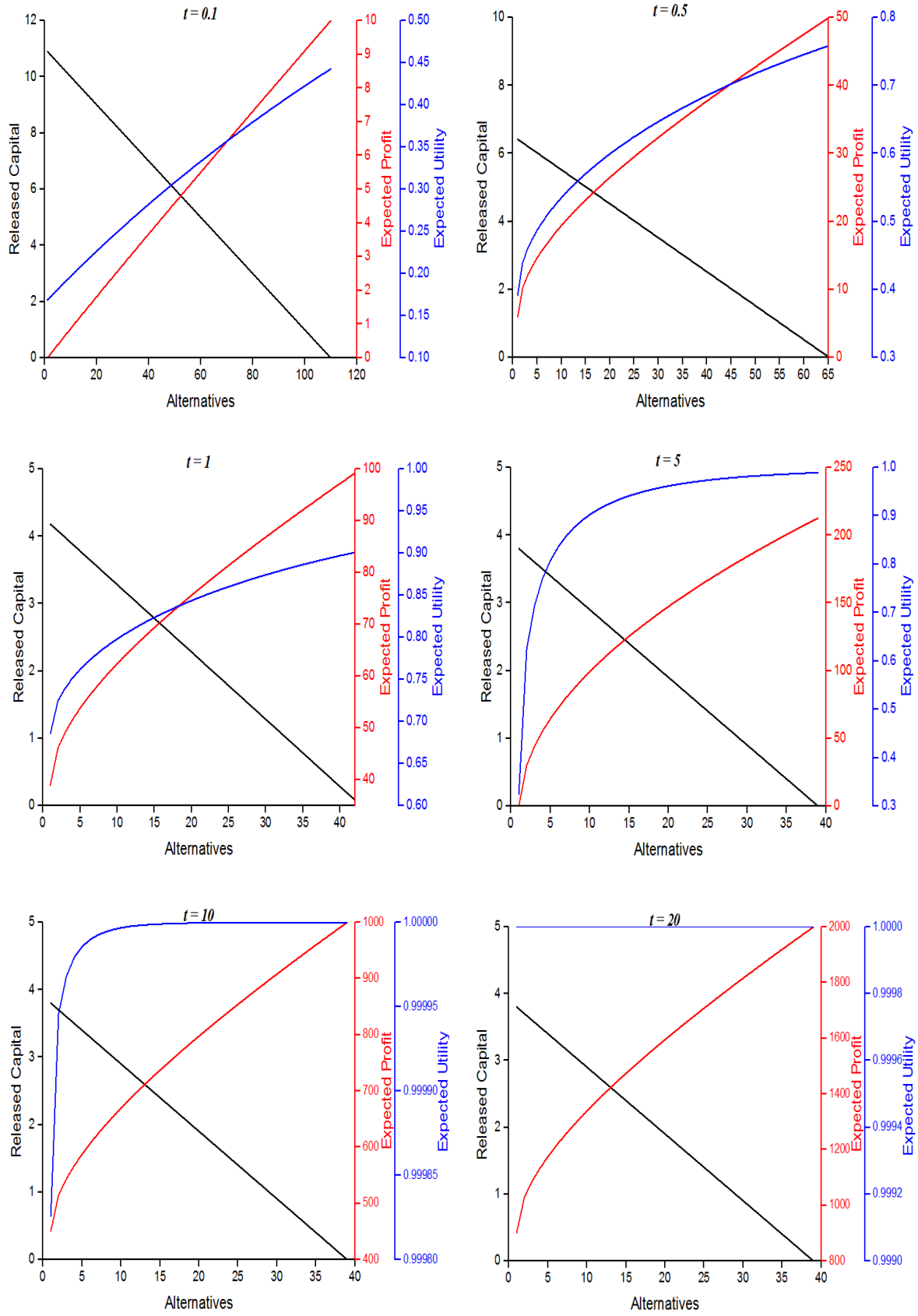


Figure 5.3: The graph of the released capital, expected profit and expected utility when $\theta = 0.2$ and $\xi = 0.3$ under the proportional reinsurance for the exponential claims

Under the proportional reinsurance, differences between the smallest feasible initial surplus and the largest initial surplus are not as much as in the excess of loss reinsurance case. For this reason, particularly for certain cases, the number of alternatives is so small. In cases of $\theta = 0.1$ & $\xi = 0.2$ and $\theta = 0.1$ & $\xi = 0.3$, after time horizons greater than 5, the smallest feasible initial surplus almost equal to the largest initial surplus. In these cases, a set of alternatives cannot be obtained and there is only one possible solution that can be reached for the optimal levels.

We can see that, in the exponential claims under the proportional reinsurance, the optimal levels do not vary for the original covariance matrix, the vector-normalized covariance matrix and the correlation matrix. Hence, for the three different matrices, one for each pair of optimal level is obtained in each loading factor and time case.

For all the scenarios based on the different weights in Table 5.1, the optimal levels are very close to each other.

Table 5.18: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 1 under the proportional reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	23.844	0.996	23.844	0.996	23.844	0.996
	0.5	39.966	0.999	31.266	0.719	39.966	0.999
	1	45.820	0.999	40.820	0.836	45.820	0.999
	5	43.163	0.667	45.863	0.846	43.363	0.693
	10	43.292	0.648	44.792	0.790	43.492	0.694
	20	43.296	0.649	43.296	0.649	43.296	0.649
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	23.917	0.999	14.017	0.500	23.917	0.999
	0.5	39.948	0.999	37.448	0.886	39.948	0.999
	1	45.842	0.999	44.842	0.936	45.842	0.999
	5	49.560	0.964	49.560	0.964	49.560	0.964
	10	49.574	0.965	49.574	0.965	49.574	0.965
	20	49.574	0.965	49.574	0.965	49.574	0.965
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	23.889	0.997	20.789	0.802	23.889	0.997
	0.5	39.602	0.904	39.802	0.964	39.902	0.986
	1	45.855	0.999	45.855	0.999	45.855	0.999
	5	49.630	0.999	49.630	0.999	49.630	0.999
	10	49.638	0.999	49.638	0.999	49.638	0.999
	20	49.638	0.999	49.638	0.999	49.638	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	20.244	0.999	16.544	0.779	20.244	0.999
	0.5	26.151	0.999	24.851	0.917	23.351	0.815
	1	26.487	0.995	24.887	0.883	22.787	0.692
	5	22.790	0.634	23.590	0.767	23.790	0.789
	10	24.490	0.849	23.090	0.711	22.890	0.676
	20	22.290	0.833	22.790	0.634	22.790	0.634

In Table 5.18, it is assumed that each criterion has the same importance so the weights are $1/3$. The optimal levels are obtained according to the three different matrices. We can see that the optimal pair of initial surplus and proportion is obtained as no reinsurance case for small values of t such as $t \leq 1$.

Table 5.19: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 2 under the proportional reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
$\theta = 0.1$ & $\xi = 0.15$	0.1	9.344	0.333	23.844	0.996	23.844	0.996
	0.5	19.966	0.333	29.866	0.673	19.966	0.333
	1	27.920	0.333	39.720	0.799	29.220	0.400
	5	43.063	0.633	45.463	0.827	43.263	0.682
	10	43.292	0.648	44.592	0.779	43.392	0.680
	20	43.296	0.649	43.296	0.649	43.296	0.649
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	14.017	0.500	14.017	0.500	14.017	0.500
	0.5	29.948	0.500	36.748	0.854	31.848	0.609
	1	41.841	0.630	44.542	0.916	43.242	0.816
	5	49.560	0.964	49.560	0.964	49.560	0.964
	10	49.574	0.965	49.574	0.965	49.574	0.965
	20	49.574	0.965	49.574	0.965	49.574	0.965
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	18.689	0.667	20.489	0.783	18.689	0.667
	0.5	39.502	0.825	39.802	0.964	39.902	0.986
	1	45.855	0.999	24.587	0.861	22.587	0.661
	5	49.630	0.999	49.630	0.999	49.630	0.999
	10	49.638	0.999	49.638	0.999	49.638	0.999
	20	49.638	0.999	49.638	0.999	49.638	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	9.344	0.333	15.844	0.736	9.344	0.333
	0.5	19.951	0.493	24.551	0.897	20.651	0.592
	1	22.487	0.640	24.587	0.861	22.587	0.661
	5	22.790	0.634	23.490	0.759	23.190	0.725
	10	22.790	0.634	23.090	0.711	22.890	0.676
	20	22.790	0.634	23.090	0.711	22.890	0.676

In Table 5.19, under Scenario 2, it is assumed that more weight is given to the released capital criterion. The highest optimal levels are obtained according to the original covariance matrix whereas the lowest optimal pairs are obtained according to the vector-normalised covariance matrix. We can observe the same effect under the excess of loss reinsurance. The optimal levels according to all matrices give smaller or equal results than under Scenario 1.

Table 5.20: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 3 under the proportional reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
$\theta = 0.1$ & $\xi = 0.15$	0.1	23.844	0.996	23.844	0.996	23.844	0.996
	0.5	39.966	0.999	30.066	0.680	39.966	0.999
	1	45.820	0.989	40.620	0.829	45.820	0.999
	5	49.563	0.997	45.863	0.846	43.863	0.736
	10	49.592	0.998	44.792	0.790	43.792	0.724
	20	43.296	0.649	43.296	0.649	43.296	0.649
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	23.917	0.999	23.917	0.999	23.165	0.999
	0.5	39.948	0.999	37.248	0.877	39.948	0.999
	1	45.842	0.999	44.842	0.936	45.842	0.999
	5	49.560	0.964	49.560	0.964	49.560	0.964
	10	49.574	0.965	49.574	0.965	49.574	0.965
	20	49.574	0.965	49.574	0.965	49.574	0.965
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	23.889	0.997	19.689	0.732	23.889	0.997
	0.5	39.902	0.986	39.802	0.964	39.902	0.986
	1	45.855	0.999	45.855	0.999	45.855	0.999
	5	49.630	0.999	49.630	0.999	49.630	0.999
	10	49.638	0.999	49.638	0.999	49.638	0.999
	20	49.638	0.999	49.638	0.999	49.638	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	20.244	0.999	16.044	0.748	20.244	0.999
	0.5	26.151	0.999	24.851	0.916	26.151	0.999
	1	24.487	0.995	24.787	0.876	23.887	0.805
	5	26.590	0.999	23.590	0.770	25.090	0.895
	10	26.590	0.999	23.090	0.711	22.990	0.696
	20	22.790	0.634	22.790	0.634	22.790	0.634

In Table 5.20, Scenario 3, it is assumed that more weight is given to the expected profit. The optimal levels which are obtained according to all matrices give larger or equal values than Scenario 1. Nonetheless, the optimal pairs do not overchange as in the excess of loss reinsurance.

Table 5.21: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 4 under the proportional reinsurance for the exponential claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	9.344	0.333	23.844	0.996	23.844	0.996
	0.5	19.966	0.333	33.266	0.784	39.966	0.999
	1	27.920	0.333	42.020	0.876	45.820	0.999
	5	43.063	0.633	46.263	0.864	43.363	0.693
	10	43.292	0.648	45.092	0.806	43.492	0.694
	20	43.296	0.649	43.296	0.649	43.296	0.649
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	14.017	0.500	14.017	0.500	23.917	0.999
	0.5	29.948	0.500	38.048	0.914	39.948	0.999
	1	41.742	0.570	45.042	0.949	45.842	0.999
	5	49.560	0.964	49.560	0.964	49.560	0.964
	10	49.574	0.965	49.574	0.965	49.574	0.965
	20	49.574	0.965	49.574	0.965	49.574	0.965
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	18.689	0.667	21.889	0.872	23.889	0.997
	0.5	39.702	0.938	39.802	0.964	39.902	0.986
	1	45.855	0.999	45.855	0.999	45.855	0.999
	5	49.630	0.999	49.630	0.999	49.630	0.999
	10	49.638	0.999	49.638	0.999	49.638	0.999
	20	49.638	0.999	49.638	0.999	49.638	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	9.344	0.333	17.644	0.845	20.244	0.999
	0.5	19.751	0.413	25.151	0.936	21.351	0.660
	1	22.387	0.594	25.187	0.905	22.687	0.678
	5	22.790	0.634	23.790	0.789	23.690	0.780
	10	22.790	0.634	23.190	0.725	22.890	0.676
	20	22.890	0.676	22.890	0.676	22.890	0.676

In Table 5.21, Scenario 4, it is assumed that more weight is given to the expected utility. The optimal levels which are obtained according to all matrices give greater than or equal values than Scenario 1 however not as high as Scenario 3.

5.4.2. Pareto Claims on Proportional Reinsurance

In the part of the study, it is assumed that the individual claim amounts have a Pareto distribution with probability density function $p(x) = \frac{\alpha}{\beta} \left[1 + \frac{x}{\beta}\right]^{-(\alpha+1)}$ with $\alpha = 3$ and $\beta = 4$.

In the case of no-reinsurance, the largest initial surplus value is same as in the excess of loss case. The largest initial surplus value for various values of t , $\theta = 0.1$ and $\theta = 0.2$ are given in Table 5.9 in Section 5.3.2.

When the reinsurance arrangement is involved, the parameters of the translated gamma process are calculated according to Equations (2.11) and (4.17). After that, by using these parameters, the loading factor $\hat{\theta}$ is calculated by Equation (4.6) and then we integrate this loading factor into Equation (2.10) as in the case of no-reinsurance.

The smallest values of u and proportion p are found by calculating the ruin probability function (Equation 2.10) as 0.01 by using one dimensional optimization technique. The smallest initial surplus value u_s and proportion p which satisfy $\psi(u, t) = 0.01$, are found.

The set of smallest initial surplus (u_s) and proportion p for various values of t with four different loading combinations $\theta = 0.1/\xi = 0.15$, $\theta = 0.1/\xi = 0.2$, $\theta = 0.1/\xi = 0.3$, $\theta = 0.2/\xi = 0.3$ are calculated and given in Table 5.22.

Table 5.22: The smallest initial surplus, u_s , for the Pareto claims under the proportional reinsurance

t	$\theta = 0.1 \ \& \ \xi = 0.15$	$\theta = 0.1 \ \& \ \xi = 0.2$	$\theta = 0.1 \ \& \ \xi = 0.2$	$\theta = 0.2 \ \& \ \xi = 0.3$
0.1	35.539	50.617	68.277	34.139
0.5	66.121	79.074	79.598	37.461
1	68.235	79.417	79.772	37.463
5	68.370	79.422	79.774	37.463
10	68.371	79.422	79.774	37.463
20	68.371	79.422	79.774	37.463

The results under three criteria when $\theta = 0.2$ and $\xi = 0.3$ according to the feasible smallest initial surplus and proportion p are given in Table 5.23.

Figure 5.4 shows the results under three criteria when $\theta = 0.2$ and $\xi = 0.3$ for the Pareto claims under the proportional reinsurance.

Table 5.23: The Released Capital, Expected Profit and Expected Utility when $\theta = 0.2$ and $\xi = 0.3$ for the Pareto claims under the proportional reinsurance

t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.1	1	12.714	0.333	18.144	0.000	0.222
	2	12.814	0.337	18.044	0.057	0.224
	3	12.914	0.341	17.944	0.114	0.227
	4	13.014	0.345	17.844	0.171	0.229
	5	13.114	0.349	17.744	0.228	0.231
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	180	30.614	0.991	0.244	9.867	0.541
	181	30.714	0.995	0.144	9.922	0.542
	182	30.814	0.998	0.044	9.976	0.543
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
0.5	1	25.784	0.333	17.043	0.000	0.393
	2	25.884	0.340	16.943	0.526	0.400
	3	25.984	0.347	16.843	1.036	0.407
	4	26.084	0.354	16.743	1.531	0.413
	5	26.184	0.360	16.643	2.014	0.420
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	169	42.584	0.992	0.243	49.404	0.814
	170	42.684	0.995	0.143	49.649	0.815
	171	42.784	0.999	0.043	49.894	0.816
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
1	1	34.139	0.465	10.616	19.697	0.636
	2	34.239	0.503	10.516	25.410	0.672
	3	34.339	0.519	10.416	27.861	0.687
	4	34.439	0.532	10.316	29.781	0.698
	5	34.539	0.543	10.216	31.426	0.707
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	105	44.539	0.992	0.216	98.864	0.922
	106	44.639	0.996	0.116	99.390	0.923
	107	44.739	1.000	0.016	99.953	0.924
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
5	1	37.461	0.610	7.629	207.252	0.987
	2	37.561	0.640	7.529	230.027	0.991
	3	37.661	0.654	7.429	240.154	0.992
	4	37.761	0.664	7.329	248.212	0.993
	5	37.861	0.674	7.229	255.203	0.994
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	75	44.861	0.992	0.229	493.766	1.000
	76	44.961	0.995	0.129	496.488	1.000
	77	45.061	0.999	0.029	499.204	1.000
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
10	1	37.463	0.610	7.627	414.897	1.000
	2	37.563	0.640	7.527	460.350	1.000
	3	37.663	0.654	7.427	480.573	1.000
	4	37.763	0.664	7.327	496.670	1.000
	5	37.863	0.674	7.227	510.639	1.000
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	75	44.863	0.992	0.227	987.647	1.000
	76	44.963	0.995	0.127	993.091	1.000
	77	45.063	0.999	0.027	998.523	1.000
t	Number of Alternative	Initial Surplus u_p	Proportion p	Released Capital $u_L - u_p$	Expected Profit P	Expected Utility $E[u[U(t)]]$
20	1	37.463	0.610	7.627	829.794	1.000
	2	37.563	0.640	7.527	920.699	1.000
	3	37.663	0.654	7.427	961.147	1.000
	4	37.763	0.664	7.327	993.340	1.000
	5	37.863	0.674	7.227	1021.279	1.000
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	75	44.863	0.992	0.227	1975.294	1.000
	76	44.963	0.995	0.127	1986.182	1.000
	77	45.063	1.000	0.027	1999.100	1.000

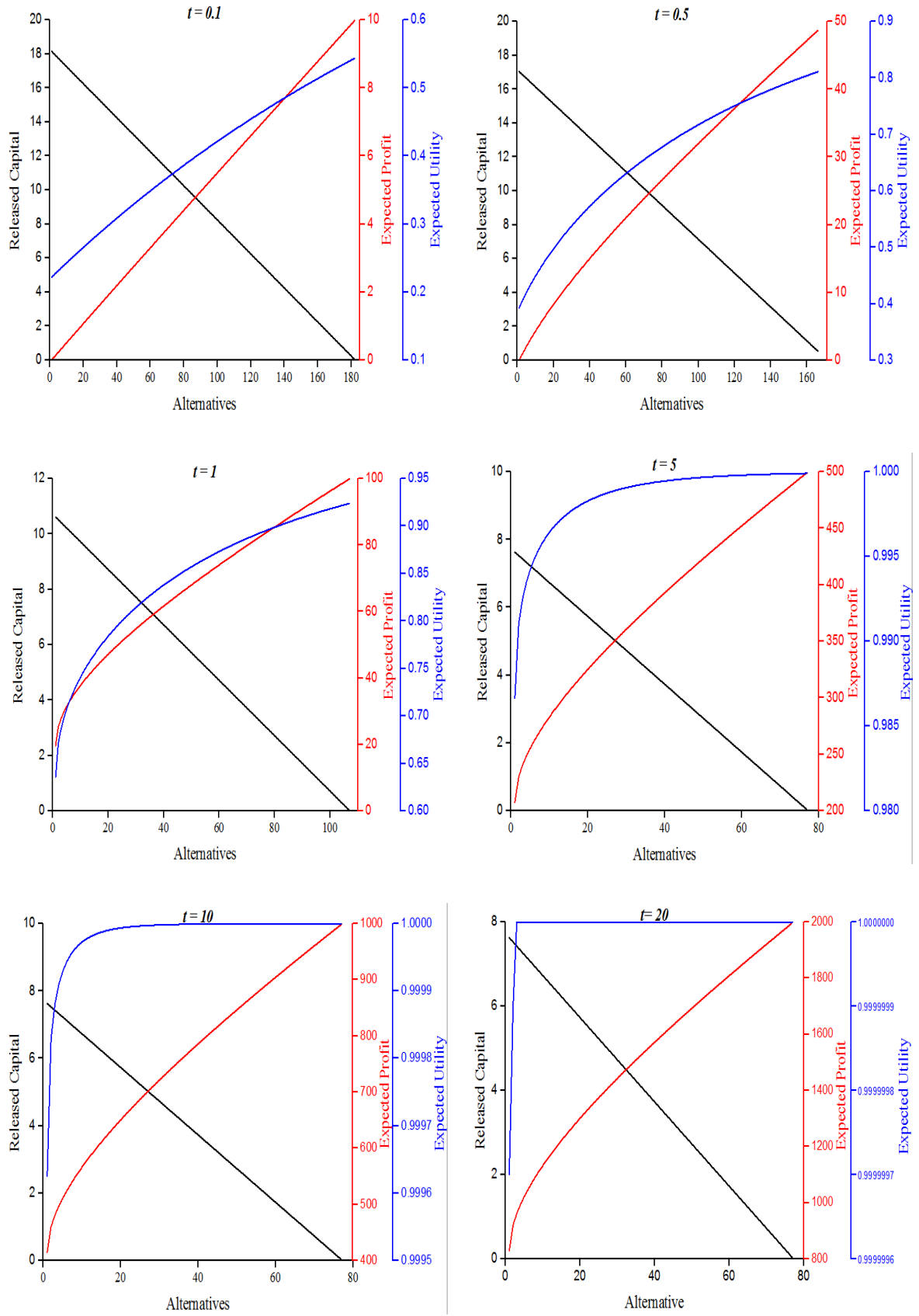


Figure 5.4: The graph of released capital, expected profit and expected utility when $\theta = 0.2$ and $\xi = 0.3$ under the proportional reinsurance for the Pareto claims

It can clearly be seen that the released capital decline steadily and become almost equal to zero at the last alternative. The range of the released capital is higher than in the excess of loss case. The expected profit increases with a decreasing slope. The difference between the largest and the smallest values of expected profit is larger than the excess of loss case. The pattern for the expected utility is the same as under the excess of loss reinsurance.

The optimal pairs (u_p, p) are shown in Tables 5.24 through Table 5.27 using the vector-normalized covariance matrix, the original covariance matrix and the correlation matrix according to the four different scenarios respectively.

As in the case of exponential claims, the differences between the smallest feasible initial surplus and the largest initial surplus are not as large as in the excess of loss reinsurance case. For this reason, for some particular cases, the number of alternatives is small. For example, in the case $\theta = 0.1$ and $\xi = 0.3$, after $t = 5$, the smallest feasible initial surplus is almost equal to the largest initial surplus. In these cases, a set of alternatives cannot be obtained and there is only one possible solution that can be calculated for the optimal levels.

We can see that, the optimal levels have greater variability than the exponential claims with the proportional reinsurance in the original covariance matrix, the normalized covariance matrix and the correlation matrix.

Table 5.24: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 1 under the proportional reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	34.314	0.999	34.314	0.999	34.314	0.999
	0.5	57.784	0.999	28.484	0.392	57.784	0.999
	1	68.439	0.999	53.239	0.706	68.439	0.999
	5	66.121	0.566	73.121	0.849	67.121	0.658
	10	74.535	0.877	72.135	0.812	68.635	0.678
	20	68.370	0.634	70.370	0.756	68.670	0.676
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	34.271	0.998	34.271	0.998	34.271	0.998
	0.5	57.776	0.999	48.776	0.772	57.776	0.999
	1	68.417	0.999	64.117	0.878	68.417	0.999
	5	79.374	0.979	79.274	0.968	79.074	0.919
	10	79.617	0.983	79.517	0.969	79.417	0.937
	20	79.422	0.938	79.422	0.938	79.422	0.938
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	34.228	0.996	25.428	0.666	34.228	0.996
	0.5	57.768	0.998	56.168	0.925	57.768	0.998
	1	68.277	0.929	68.277	0.929	68.277	0.929
	5	79.598	0.999	79.598	0.999	79.598	0.999
	10	79.772	0.999	79.772	0.999	79.772	0.999
	20	79.774	0.999	79.772	0.999	79.772	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	30.814	0.998	12.714	0.333	30.814	0.998
	0.5	42.784	0.999	38.784	0.866	42.784	0.999
	1	44.739	0.999	41.239	0.873	35.939	0.642
	5	37.461	0.610	39.161	0.757	37.761	0.664
	10	37.463	0.610	38.363	0.711	37.663	0.644
	20	40.463	0.819	37.863	0.674	37.663	0.654

In Scenario 1, Table 5.24, the optimal levels are calculated according to the assumption of the same importance for each criterion. It is found that the optimal initial surplus is larger than the exponential case. Optimal strategy is the case of no reinsurance when $t < 1$ according to vector-normalized covariance matrix and the correlation matrix.

Table 5.25: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 2 under the proportional reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	12.714	0.333	34.314	0.999	12.714	0.333
	0.5	25.784	0.333	28.184	0.385	25.784	0.333
	1	35.539	0.333	51.139	0.664	35.539	0.333
	5	66.121	0.566	72.321	0.829	66.621	0.629
	10	68.235	0.627	71.535	0.795	68.435	0.663
	20	68.370	0.634	70.170	0.749	68.570	0.668
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	19.071	0.500	34.271	0.998	19.071	0.500
	0.5	38.676	0.500	47.476	0.738	38.676	0.500
	1	53.317	0.500	63.217	0.852	56.717	0.641
	5	79.074	0.919	79.174	0.953	79.074	0.919
	10	79.417	0.937	79.417	0.937	79.417	0.937
	20	79.422	0.938	79.422	0.938	79.422	0.938
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	25.428	0.666	25.428	0.666	25.428	0.666
	0.5	51.568	0.667	55.768	0.906	57.768	0.998
	1	68.277	0.929	68.277	0.929	68.277	0.929
	5	79.598	0.999	79.598	0.999	79.598	0.999
	10	79.772	0.999	79.772	0.999	79.772	0.999
	20	79.774	0.999	79.772	0.999	79.772	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	12.714	0.333	12.714	0.333	12.714	0.333
	0.5	25.784	0.333	37.684	0.829	27.784	0.448
	1	34.339	0.519	40.539	0.846	34.839	0.570
	5	37.461	0.610	38.861	0.740	37.561	0.640
	10	37.463	0.610	38.163	0.697	37.563	0.640
	20	37.463	0.610	37.863	0.674	37.563	0.640

We can see that, in Scenario 2, Table 5.25, more weight is given to the released capital so that the optimal levels get closer to the levels where the released capital is at its maximum. Hence, the optimal levels are calculated as lower than those under Scenario 1.

Table 5.26: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 3 under the proportional reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	34.314	0.999	34.314	0.999	34.314	0.999
	0.5	57.784	0.999	25.784	0.333	57.784	0.999
	1	68.439	0.999	51.639	0.674	68.439	0.999
	5	79.521	0.998	73.121	0.849	68.821	0.726
	10	79.735	0.999	72.035	0.809	69.235	0.711
	20	79.770	0.999	70.370	0.756	69.070	0.701
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	34.271	0.998	34.271	0.998	34.271	0.998
	0.5	57.776	0.999	47.476	0.738	57.776	0.999
	1	68.417	0.999	64.017	0.875	68.417	0.999
	5	79.574	0.998	79.274	0.968	79.174	0.953
	10	79.717	0.995	79.517	0.969	79.417	0.937
	20	79.422	0.938	79.422	0.938	79.422	0.938
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	34.328	0.999	34.328	0.999	34.328	0.999
	0.5	57.768	0.998	56.068	0.920	57.768	0.998
	1	68.277	0.929	68.277	0.929	68.277	0.929
	5	79.598	0.999	79.598	0.999	79.598	0.999
	10	79.772	0.999	79.772	0.999	79.772	0.999
	20	79.772	0.999	79.772	0.999	79.772	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	30.814	0.998	30.814	0.998	30.814	0.998
	0.5	42.784	0.999	38.584	0.859	42.784	0.999
	1	44.739	0.999	41.239	0.873	44.739	0.999
	5	45.061	0.999	39.161	0.757	38.161	0.697
	10	45.063	0.999	38.363	0.711	38.063	0.690
	20	45.063	0.999	37.863	0.674	37.963	0.682

In Scenario 3, Table 5.26, the highest optimal levels are obtained. In this case, more weight is given to the expected profit. Therefore, the optimal levels get closer to the level which the expected profit is maximum.

Table 5.27: The optimal pairs (u_p, p) in the TOPSIS method with Mahalanobis distance for Scenario 4 under the proportional reinsurance for the Pareto claims

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
$\theta = 0.1$ & $\xi = 0.15$	0.1	34.314	0.999	34.314	0.999	34.314	0.999
	0.5	25.784	0.333	34.184	0.513	57.784	0.999
	1	35.539	0.333	56.339	0.767	68.439	0.999
	5	66.121	0.566	74.021	0.872	67.121	0.658
	10	68.235	0.627	72.735	0.829	68.635	0.678
	20	68.370	0.634	70.670	0.766	68.670	0.676
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	19.071	0.500	34.271	0.998	34.271	0.998
	0.5	38.676	0.500	50.976	0.828	57.776	0.999
	1	53.317	0.500	65.117	0.907	68.417	0.999
	5	79.074	0.919	79.274	0.968	79.074	0.919
	10	79.417	0.937	79.517	0.969	79.417	0.937
	20	79.422	0.938	79.422	0.938	79.422	0.938
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	25.428	0.667	25.428	0.667	34.228	0.996
	0.5	51.568	0.667	56.568	0.944	57.768	0.998
	1	68.277	0.929	68.277	0.929	68.277	0.929
	5	79.598	0.999	79.598	0.999	79.598	0.999
	10	79.772	0.999	79.772	0.999	79.772	0.999
	20	79.772	0.999	79.772	0.999	79.772	0.999
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_p	Optimal p	Optimal u_p	Optimal p	Optimal u_p	Optimal p
	0.1	12.714	0.333	39.684	0.439	30.814	0.998
	0.5	25.784	0.333	41.939	0.896	42.784	0.999
	1	34.139	0.465	39.461	0.899	35.539	0.619
	5	37.463	0.610	39.461	0.772	37.761	0.664
	10	37.463	0.610	38.463	0.717	37.663	0.654
	20	37.463	0.610	37.963	0.682	37.663	0.654

In Scenario 4, Table 5.27, the optimal levels are higher than Scenario 1. The expected utility criterion does not have the same influential impact as expected profit. Hence, the values are not as high as under Scenario 3.

Based on the results, it can be concluded that the vector-normalized covariance matrix reflect the effect of each criteria better than the other matrices.

5.5. Sensitivity Analysis on Optimal Reinsurance

Sensitivity analysis is used to determine how different values of an independent variable affect dependent variables under a given set of assumptions. This technique is applied in a way to depend on one or more input variables. By this way, the outcome of a decision is determined in the case of different variables.

Optimal reinsurance criteria has some key assumptions on the calculation of optimal levels such as weights of the criteria, and the parameter of the exponential utility function. Besides these assumptions, the relationship between optimal levels and maximum value of each criteria is also investigated in the sensitivity analysis.

5.5.1. Effect of the Changes in Weights of the Criteria

The sensitivity analysis on the changes in the weights of the criteria measure the performance of the weights. Optimal levels are calculated according to four different sets of weights scenario as given in Table 5.1. In this section, we observe the differences in the optimal pair of initial surplus and reinsurance level according to the changes in weights of released capital criteria. Since the excess of loss reinsurance has more importance and has more adequate alternatives in each time horizon and loading factor scenario, we conduct the sensitivity analysis only in the excess of loss reinsurance case.

The optimal pair of initial surplus and retention level is calculated when the weight of the released capital, the expected profit and the expected utility change between 0 and 1. In comparison with the optimal levels according to different criteria, we use the vector normalized covariance matrix.

The optimal pair which is obtained based on the changes in the weight of the released capital are given in Table 5.28 through 5.33 under the exponential individual claim assumption.

Table 5.28: Changes in weights of the released capital for the exponential claims under the excess of loss reinsurance when $t = 0.1$

t=0.1 Exponential			$\theta = 0.1 \text{ \& } \xi = 0.15$		$\theta = 0.1 \text{ \& } \xi = 0.2$		$\theta = 0.1 \text{ \& } \xi = 0.3$		$\theta = 0.2 \text{ \& } \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.5	0.5	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.0025	0.49875	0.49875	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.005	0.4975	0.4975	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.01	0.495	0.495	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.05	0.475	0.475	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.1	0.45	0.45	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.15	0.425	0.425	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.2	0.4	0.4	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.25	0.375	0.375	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.3	0.35	0.35	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.333	0.333	0.333	23.867	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.4	0.3	0.3	7.466	0.475	10.588	0.693	14.877	1.0987	8.266	0.653
0.41	0.295	0.295	6.966	0.425	10.588	0.693	14.877	1.0987	7.966	0.603
0.42	0.29	0.29	6.766	0.405	10.588	0.693	14.877	1.0987	7.766	0.571
0.43	0.285	0.285	6.766	0.405	10.588	0.693	14.877	1.0987	7.466	0.522
0.45	0.275	0.275	6.766	0.405	10.588	0.693	14.877	1.0987	6.866	0.422
0.5	0.25	0.25	6.766	0.405	10.588	0.693	14.877	1.0987	6.766	0.405
0.550	0.225	0.225	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.600	0.200	0.200	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.650	0.175	0.175	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.700	0.150	0.150	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.750	0.125	0.125	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.800	0.100	0.100	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.850	0.075	0.075	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.900	0.050	0.050	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
0.950	0.025	0.025	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405
1.000	0.000	0.000	6.766	0.405	10.588	0.693	14.877	1.099	6.766	0.405

Table 5.28 presents the optimal pair of initial surplus and retention level according to the changes in the weight of the released capital between 0 and 1 when $t = 0.1$. It can be seen that the optimal levels are stable until the weight of the released capital is equal to 1/3. At this point, a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes a decrease in optimal levels.

Table 5.29: Changes in weights of the released capital for the exponential claims under the excess of loss reinsurance when $t = 0.5$

t=0.5 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	39.961	10.787	39.962	10.905	39.937	9.269	26.130	8.959
0.0025	0.49875	0.49875	39.261	5.660	39.962	10.905	39.937	9.269	25.430	5.358
0.005	0.4975	0.4975	38.961	5.212	39.962	10.905	39.937	9.269	25.330	5.191
0.01	0.495	0.495	38.461	4.691	39.962	10.905	39.937	9.269	25.130	4.905
0.05	0.475	0.475	36.461	3.561	39.962	10.905	39.937	9.269	24.530	4.281
0.1	0.45	0.45	35.061	3.091	39.962	10.905	39.937	9.269	23.830	3.777
0.15	0.425	0.425	33.861	2.778	39.962	10.905	39.937	9.269	23.230	3.444
0.2	0.4	0.4	32.761	2.536	39.962	10.905	39.937	9.269	22.730	3.209
0.25	0.375	0.375	31.561	2.306	29.162	1.699	39.937	9.269	22.030	2.926
0.3	0.35	0.35	30.361	2.104	28.562	1.601	39.937	9.269	20.930	2.554
0.333	0.333	0.333	29.361	1.951	28.162	1.537	39.937	9.269	20.930	2.554
0.4	0.3	0.3	26.961	1.628	27.562	1.443	33.137	1.906	13.330	0.676
0.41	0.295	0.295	26.561	1.579	27.462	1.427	33.137	1.906	13.330	0.676
0.42	0.29	0.29	25.961	1.507	27.362	1.412	33.037	1.867	13.330	0.676
0.43	0.285	0.285	14.861	0.416	27.362	1.412	33.037	1.867	13.330	0.676
0.45	0.275	0.275	14.761	0.405	27.162	1.381	32.937	1.826	13.330	0.676
0.5	0.25	0.25	14.761	0.405	26.762	1.320	32.837	1.785	13.330	0.676
0.550	0.225	0.225	14.761	0.405	26.362	1.258	32.737	1.742	13.330	0.676
0.600	0.200	0.200	14.761	0.405	25.962	1.197	32.637	1.697	13.330	0.676
0.650	0.175	0.175	14.761	0.405	25.662	1.151	32.537	1.648	13.330	0.676
0.700	0.150	0.150	14.761	0.405	25.262	1.089	32.437	1.596	13.330	0.676
0.750	0.125	0.125	14.761	0.405	24.862	1.025	32.337	1.536	13.330	0.676
0.800	0.100	0.100	14.761	0.405	24.462	0.960	32.337	1.536	13.330	0.676
0.850	0.075	0.075	14.761	0.405	24.062	0.892	32.237	1.462	13.330	0.676
0.900	0.050	0.050	14.761	0.405	23.662	0.819	32.237	1.462	13.330	0.676
0.950	0.025	0.025	14.761	0.405	23.162	0.717	32.237	1.462	13.330	0.676
1.000	0.000	0.000	14.761	0.405	23.062	0.693	32.137	1.298	13.330	0.676

Table 5.29 presents the optimal pair of initial surplus and retention level according to the changes in the weight of the released capital between 0 and 1 when $t = 0.5$. From this table, it is observed that the increase in the weight of the released capital causes a decrease in optimal levels.

Table 5.30: Changes in weights of the released capital for the exponential claims under the excess of loss reinsurance when $t = 1$

t=1 Exponential			$\theta = 0.1 \text{ \& } \xi = 0.15$		$\theta = 0.1 \text{ \& } \xi = 0.2$		$\theta = 0.1 \text{ \& } \xi = 0.3$		$\theta = 0.2 \text{ \& } \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	45.850	11.778	45.816	9.348	45.772	8.379	26.550	9.927
0.0025	0.49875	0.49875	45.150	5.921	44.916	5.479	45.772	8.379	26.550	9.927
0.005	0.4975	0.4975	44.850	5.473	44.616	5.116	45.772	8.379	26.550	9.927
0.01	0.495	0.495	43.650	4.450	44.416	4.918	45.772	8.379	26.450	7.732
0.05	0.475	0.475	42.450	3.862	44.416	4.200	45.772	8.379	25.950	5.653
0.1	0.45	0.45	41.250	3.441	42.716	3.844	45.772	8.379	25.250	4.640
0.15	0.425	0.425	39.950	3.085	42.016	3.552	45.772	8.379	24.450	3.975
0.2	0.4	0.4	39.950	3.085	41.416	3.337	45.772	8.379	23.750	3.560
0.25	0.375	0.375	38.650	2.792	40.716	3.115	45.772	8.379	22.950	3.184
0.3	0.35	0.35	37.150	2.506	39.916	2.889	45.772	8.379	22.150	2.875
0.333	0.333	0.333	20.750	0.405	39.416	2.760	41.272	2.731	14.250	0.812
0.4	0.3	0.3	20.750	0.405	38.216	2.479	41.072	2.628	14.250	0.812
0.41	0.295	0.295	20.750	0.405	38.016	2.435	41.072	2.628	14.250	0.812
0.42	0.29	0.29	20.750	0.405	37.816	2.392	41.072	2.628	14.250	0.812
0.43	0.285	0.285	20.750	0.405	31.516	0.961	41.072	2.628	14.250	0.812
0.45	0.275	0.275	20.750	0.405	31.516	0.961	41.072	2.628	14.250	0.812
0.5	0.25	0.25	20.750	0.405	31.516	0.961	40.972	2.575	14.250	0.812
0.550	0.225	0.225	20.750	0.405	31.516	0.961	40.872	2.520	14.250	0.812
0.600	0.200	0.200	20.750	0.405	31.516	0.961	40.772	2.462	14.250	0.812
0.650	0.175	0.175	20.750	0.405	31.516	0.961	40.772	2.462	14.250	0.812
0.700	0.150	0.150	20.750	0.405	31.516	0.961	40.672	2.402	14.250	0.812
0.750	0.125	0.125	20.750	0.405	31.516	0.961	40.672	2.402	14.250	0.812
0.800	0.100	0.100	20.750	0.405	31.516	0.961	40.572	2.336	14.250	0.812
0.850	0.075	0.075	20.750	0.405	31.516	0.961	40.472	2.262	14.250	0.812
0.900	0.050	0.050	20.750	0.405	31.516	0.961	40.472	2.262	14.250	0.812
0.950	0.025	0.025	20.750	0.405	31.516	0.961	40.372	2.172	14.250	0.812
1.000	0.000	0.000	20.750	0.405	31.516	0.961	40.272	1.977	14.250	0.812

Table 5.30 presents the optimal levels according to the changes in the weight of the released capital between 0 and 1 when $t = 1$. It is shown that the optimal levels decrease until the weight of the released capital is equal to 1/3. At this point, a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes a decrease in optimal levels.

Table 5.31: Changes in weights of the released capital for the exponential claims under the excess of loss reinsurance when $t = 5$

t=5 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	49.630	15.360	49.563	8.929	49.536	8.404	26.567	9.636
0.0025	0.49875	0.49875	49.630	15.360	49.563	8.929	49.536	8.404	26.567	9.636
0.005	0.4975	0.4975	49.630	15.360	49.563	8.929	49.536	8.404	26.567	9.636
0.01	0.495	0.495	49.630	15.360	49.563	8.929	49.536	8.404	26.567	9.636
0.05	0.475	0.475	48.830	5.986	49.263	6.877	49.536	8.404	25.667	5.127
0.1	0.45	0.45	47.330	4.616	48.563	5.516	49.336	6.984	24.467	3.976
0.15	0.425	0.425	45.730	3.892	47.763	4.764	49.036	6.061	23.467	3.411
0.2	0.4	0.4	44.130	3.399	46.863	4.211	48.736	5.498	22.467	2.987
0.25	0.375	0.375	42.530	3.019	46.063	3.840	48.436	5.083	21.667	2.706
0.3	0.35	0.35	27.730	0.845	38.263	1.535	48.036	4.647	20.767	2.431
0.333	0.333	0.333	27.730	0.845	38.263	1.535	48.036	4.647	20.267	2.293
0.4	0.3	0.3	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.41	0.295	0.295	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.42	0.29	0.29	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.43	0.285	0.285	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.45	0.275	0.275	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.5	0.25	0.25	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.5	0.25	0.25	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.55	0.225	0.225	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.6	0.2	0.2	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.65	0.175	0.175	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.7	0.15	0.15	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.75	0.125	0.125	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.8	0.1	0.1	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.85	0.075	0.075	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.9	0.05	0.05	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.95	0.025	0.025	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
1	0	0	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840

Table 5.31 presents the optimal levels according to the changes in the weight of the released capital between 0 and 1 when $t = 5$. From this table, it can be seen that the optimal levels decrease until the weight of the released capital is equal to 0.3. At this point, a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes a decrease in optimal levels.

Table 5.32: Changes in weights of the released capital for the exponential claims under the excess of loss reinsurance when $t = 10$

t=10 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	49.597	9.550	49.607	9.830	49.558	8.598	26.567	9.636
0.0025	0.49875	0.49875	49.597	9.550	49.607	9.830	49.558	8.598	26.567	9.636
0.005	0.4975	0.4975	49.597	9.550	49.607	9.830	49.558	8.598	26.567	9.636
0.01	0.495	0.495	49.597	9.550	49.607	9.830	49.558	8.598	26.467	7.683
0.05	0.475	0.475	48.397	5.428	49.107	6.415	49.558	8.598	25.367	4.747
0.1	0.45	0.45	46.397	4.150	48.207	5.127	49.258	6.653	24.067	3.726
0.15	0.425	0.425	44.497	3.498	47.207	4.397	48.958	5.878	22.967	3.186
0.2	0.4	0.4	42.797	3.076	46.307	3.942	48.558	5.229	21.967	2.807
0.25	0.375	0.375	41.097	2.736	45.407	3.583	48.258	4.868	21.067	2.519
0.3	0.35	0.35	39.597	2.479	44.507	3.281	47.958	4.565	20.267	2.293
0.333	0.333	0.333	27.797	0.856	38.307	1.545	45.758	2.676	19.667	2.137
0.4	0.3	0.3	27.797	0.856	38.307	1.545	45.758	2.676	18.767	1.920
0.41	0.295	0.295	27.797	0.856	38.307	1.545	45.758	2.676	18.667	1.897
0.42	0.29	0.29	27.797	0.856	38.307	1.545	45.758	2.676	18.467	1.851
0.43	0.285	0.285	27.797	0.856	38.307	1.545	45.758	2.676	18.367	1.828
0.45	0.275	0.275	27.797	0.856	38.307	1.545	45.758	2.676	18.167	1.783
0.5	0.25	0.25	27.797	0.856	38.307	1.545	45.758	2.676	17.467	1.631
0.550	0.225	0.225	27.797	0.856	38.307	1.545	45.758	2.676	16.967	1.524
0.600	0.200	0.200	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
0.650	0.175	0.175	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
0.700	0.150	0.150	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
0.750	0.125	0.125	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
0.800	0.100	0.100	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
0.850	0.075	0.075	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
0.900	0.050	0.050	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
0.950	0.025	0.025	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840
1.000	0.000	0.000	27.797	0.856	38.307	1.545	45.758	2.676	14.367	0.840

Table 5.32 presents the optimal levels according to the changes in the weight of the released capital between 0 and 1 when $t = 10$. The optimal levels decrease until the weight of the released capital is equal to 1/3. At this point, a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes a decrease in optimal levels.

Table 5.33: Changes in weights of the released capital for the exponential claims under the excess of loss reinsurance when $t = 20$

t=20 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	49.598	9.571	49.607	9.840	49.558	8.599	26.567	9.636
0.0025	0.49875	0.49875	49.598	9.571	49.607	9.840	49.558	8.599	26.567	9.636
0.005	0.4975	0.4975	49.598	9.571	49.607	9.840	49.558	8.599	26.567	9.636
0.01	0.495	0.495	49.598	9.571	49.607	9.840	49.558	8.599	26.467	7.683
0.05	0.475	0.475	47.798	4.914	48.807	5.843	49.458	7.605	25.267	4.639
0.1	0.45	0.45	45.498	3.808	47.507	4.582	49.058	6.094	23.867	3.615
0.15	0.425	0.425	43.398	3.214	46.407	3.987	48.758	5.521	22.767	3.104
0.2	0.4	0.4	41.598	2.830	45.507	3.620	48.358	4.980	21.767	2.739
0.25	0.375	0.375	39.898	2.528	44.607	3.313	48.058	4.661	20.867	2.460
0.3	0.35	0.35	38.398	2.294	43.807	3.073	47.758	4.385	20.067	2.240
0.333	0.333	0.333	37.398	2.151	43.307	2.935	47.558	4.218	19.567	2.112
0.4	0.3	0.3	35.698	1.925	42.507	2.728	45.758	2.676	18.567	1.874
0.41	0.295	0.295	35.398	1.887	42.307	2.679	45.758	2.676	18.467	1.851
0.42	0.29	0.29	35.198	1.862	42.207	2.654	45.758	2.676	18.367	1.828
0.43	0.285	0.285	34.998	1.837	42.107	2.630	45.758	2.676	18.167	1.783
0.45	0.275	0.275	34.498	1.776	38.307	1.545	45.758	2.676	17.967	1.739
0.5	0.25	0.25	33.398	1.644	38.307	1.545	45.758	2.676	17.367	1.609
0.550	0.225	0.225	32.298	1.516	38.307	1.545	45.758	2.676	16.867	1.503
0.600	0.200	0.200	27.798	0.856	38.307	1.545	45.758	2.676	16.367	1.398
0.650	0.175	0.175	27.798	0.856	38.307	1.545	45.758	2.676	15.967	1.313
0.700	0.150	0.150	27.798	0.856	38.307	1.545	45.758	2.676	15.567	1.226
0.750	0.125	0.125	27.798	0.856	38.307	1.545	45.758	2.676	15.267	1.158
0.800	0.100	0.100	27.798	0.856	38.307	1.545	45.758	2.676	14.967	1.085
0.850	0.075	0.075	27.798	0.856	38.307	1.545	45.758	2.676	14.467	0.926
0.900	0.050	0.050	27.898	0.919	38.307	1.545	45.758	2.676	14.467	0.926
0.950	0.025	0.025	27.898	0.919	38.307	1.545	45.758	2.676	14.467	0.926
1.000	0.000	0.000	27.798	0.856	38.307	1.545	45.758	2.676	14.367	0.840

Table 5.33 presents the optimal levels according to the changes in the weight of the released capital between 0 and 1 when $t = 20$. From this table, it can be seen that the increase in the weight of the released capital causes a decrease in optimal levels.

More detailed examples for the changes in the weight of released capital for the exponential claims under the excess of loss reinsurance for $t = 1$ can be found in Appendix A. Table A.1 presents the optimal levels according to changes in the weight of the released capital while the weight of the expected profit is two times of the weight of the expected utility. Table A.2 presents the optimal level according to changes in the weight of the released capital while the weight of the expected utility is two times of the weight of the expected profit.

More detailed examples for the changes in the weight of expected profit for the exponential claims under the excess of loss reinsurance can be found in Appendix A. Table A.5 presents the optimal level according to changes in the weight of the expected profit while the weight of the released capital is equal to the weight of the expected utility. Table A.6 presents the optimal level according to changes in the weight of the expected profit while the weight of the released capital is two times of the weight of the expected utility. Table A.7 presents the optimal level according to changes in the weight of the expected profit while the weight of the expected utility is two times of the weight of the released capital.

More detailed examples for the changes in the weight of expected utility for the exponential claims under the excess of loss reinsurance can be found in Appendix A. Table A.11 presents the optimal level according to changes in the weight of the expected utility while the weight of the released capital is equal to the weight of the expected profit. Table A.12 presents the optimal level according to changes in the weight of the expected utility while the weight of the released capital is two times of the weight of the expected profit. Table A.13 presents the optimal level according to changes in the weight of the expected utility while the weight of the expected profit is two times of the weight of the released capital.

We investigate the sensitivity analysis in respect of not only on exponential claims but also Pareto claims for the excess of loss reinsurance. The optimal pair of initial surplus and retention level according to varying released capital for the Pareto claims are given in Table 5.34 through Table 5.39.

Table 5.34: Changes in weights of the released capital for Pareto claims under the excess of loss reinsurance when $t = 0.1$

t=0.1 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.0025	0.49875	0.49875	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.005	0.4975	0.4975	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.01	0.495	0.495	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.05	0.475	0.475	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.1	0.45	0.45	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.15	0.425	0.425	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.2	0.4	0.4	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.25	0.375	0.375	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.3	0.35	0.35	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.333	0.333	0.333	34.250	687.312	34.298	1532.929	34.290	1285.000	30.762	668.576
0.4	0.3	0.3	7.550	0.500	11.098	0.791	15.890	1.341	8.862	0.765
0.41	0.295	0.295	7.050	0.447	11.098	0.791	15.890	1.341	8.562	0.713
0.42	0.29	0.29	6.950	0.437	11.098	0.791	15.890	1.341	8.162	0.644
0.43	0.285	0.285	6.950	0.437	11.098	0.791	15.890	1.341	7.962	0.610
0.45	0.275	0.275	6.950	0.437	11.098	0.791	15.890	1.341	7.262	0.491
0.500	0.250	0.250	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.550	0.225	0.225	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.600	0.200	0.200	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.650	0.175	0.175	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.700	0.150	0.150	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.750	0.125	0.125	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.800	0.100	0.100	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.850	0.075	0.075	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.900	0.050	0.050	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440
0.950	0.025	0.025	6.950	0.437	11.098	0.791	15.890	1.341	6.962	0.440

Table 5.34 presents the optimal level according to the weight of the released capital changes between 0 and 1 when $t = 0.1$. From this table, it can be seen that the optimal levels are stable until the weight of the released capital is equal to $1/3$. At this point, a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes a decrease in optimal levels. Similar comments can be done as in the exponential case.

Table 5.35: Changes in weights of the released capital for Pareto claims under the excess of loss reinsurance when $t = 0.5$

t=0.5 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.0025	0.49875	0.49875	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.005	0.4975	0.4975	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.01	0.495	0.495	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.05	0.475	0.475	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.1	0.45	0.45	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.15	0.425	0.425	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.2	0.4	0.4	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.25	0.375	0.375	57.721	769.050	57.745	1029.200	57.777	1910.971	42.784	1924.792
0.3	0.35	0.35	57.721	769.050	57.745	1029.200	57.777	1910.971	20.384	2.208
0.333	0.333	0.333	25.321	1.469	57.745	1029.200	57.777	1910.971	20.084	2.138
0.4	0.3	0.3	24.121	1.332	29.845	1.733	37.277	2.536	19.684	2.046
0.41	0.295	0.295	23.921	1.309	29.645	1.700	37.177	2.504	19.584	2.023
0.42	0.29	0.29	23.721	1.287	29.545	1.684	37.077	2.471	19.584	2.023
0.43	0.285	0.285	23.621	1.276	29.345	1.652	36.977	2.439	19.484	2.001
0.45	0.275	0.275	23.321	1.244	29.045	1.603	36.777	2.375	19.384	1.978
0.5	0.25	0.25	22.521	1.159	28.345	1.492	36.277	2.211	19.084	1.912
0.550	0.225	0.225	21.821	1.086	27.745	1.398	35.877	2.077	18.784	1.846
0.600	0.200	0.200	21.021	1.005	27.145	1.305	35.577	1.972	18.484	1.781
0.650	0.175	0.175	20.321	0.936	26.645	1.227	35.377	1.900	18.184	1.717
0.700	0.150	0.150	19.621	0.868	26.145	1.149	35.177	1.823	17.884	1.653
0.750	0.125	0.125	18.721	0.781	25.645	1.070	34.977	1.742	17.484	1.569
0.800	0.100	0.100	17.921	0.705	25.045	0.972	34.777	1.653	17.084	1.486
0.850	0.075	0.075	16.921	0.611	24.545	0.887	34.677	1.604	16.684	1.403
0.900	0.050	0.050	15.721	0.496	24.045	0.796	34.477	1.488	16.084	1.278
0.950	0.025	0.025	15.121	0.436	24.045	0.796	34.477	1.488	15.384	1.126
1.000	0.000	0.000	15.121	0.436	24.045	0.796	34.477	1.488	14.084	0.684

Table 5.35 presents the optimal level according to the weight of the released capital between 0 and 1 when $t = 0.5$. The optimal levels are stable until the weight of the released capital is equal to 1/3. At this point a sudden decline is observed (it is given in bold) and after that the optimal levels decrease.

Table 5.36: Changes in weights of the released capital for Pareto claims under the excess of loss reinsurance when $t = 1$

t=1 Pareto			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	68.373	1030.142	68.409	1922.980	68.411	2004.297	44.655	905.073
0.0025	0.49875	0.49875	67.673	107.688	68.409	1922.980	68.411	2004.297	42.255	36.862
0.005	0.4975	0.4975	67.373	79.070	68.409	1922.980	68.411	2004.297	41.155	25.637
0.01	0.495	0.495	66.873	55.377	68.409	1922.980	68.411	2004.297	39.255	16.670
0.05	0.475	0.475	63.173	18.354	68.409	1922.980	68.411	2004.297	35.255	9.224
0.1	0.45	0.45	47.973	4.570	68.409	1922.980	68.411	2004.297	33.555	7.591
0.15	0.425	0.425	45.573	3.945	68.409	1922.980	68.411	2004.297	32.355	6.685
0.2	0.4	0.4	43.973	3.586	68.409	1922.980	68.411	2004.297	31.355	6.043
0.25	0.375	0.375	42.573	3.301	68.409	1922.980	68.411	2004.297	30.355	5.480
0.3	0.35	0.35	40.873	2.986	40.809	2.566	68.411	2004.297	29.455	5.029
0.333	0.333	0.333	40.173	2.865	40.609	2.527	68.411	2004.297	28.855	4.753
0.4	0.3	0.3	38.173	2.544	40.209	2.450	48.711	3.641	27.755	4.291
0.41	0.295	0.295	38.173	2.544	40.109	2.431	48.611	3.601	27.455	4.174
0.42	0.29	0.29	37.473	2.439	40.009	2.412	48.511	3.561	27.255	4.098
0.43	0.285	0.285	37.473	2.439	39.909	2.393	48.411	3.521	27.255	4.098
0.45	0.275	0.275	36.973	2.366	39.709	2.355	48.311	3.481	26.855	3.949
0.5	0.25	0.25	35.373	2.144	39.409	2.298	48.011	3.361	25.955	3.635
0.550	0.225	0.225	33.673	1.924	38.909	2.205	47.711	3.240	24.655	3.222
0.600	0.200	0.200	31.373	1.648	38.509	2.131	47.411	3.116	23.455	2.876
0.650	0.175	0.175	29.273	1.413	38.309	2.094	47.211	3.032	16.155	1.163
0.700	0.150	0.150	27.073	1.178	37.709	1.984	47.011	2.946	16.055	1.136
0.750	0.125	0.125	25.573	1.021	37.309	1.911	46.811	2.857	15.955	1.108
0.800	0.100	0.100	24.373	0.893	36.809	1.819	46.611	2.764	15.855	1.078
0.850	0.075	0.075	23.773	0.827	36.409	1.745	46.411	2.664	15.855	1.078
0.900	0.050	0.050	23.073	0.746	35.809	1.631	46.211	2.556	15.755	1.045
0.950	0.025	0.025	22.373	0.657	34.909	1.451	46.011	2.431	15.655	1.009
1.000	0.000	0.000	21.273	0.454	33.409	0.937	45.711	2.061	15.455	0.867

Table 5.36 presents the optimal level according to the weight of the released capital between 0 and 1 when $t = 1$.

Table 5.37: Changes in weights of the released capital for Pareto claims under the excess of loss reinsurance when $t = 5$

t=5 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	79.522	1227.698	79.516	1140.205	79.515	1121.160	44.992	938.281
0.0025	0.49875	0.49875	75.822	31.165	76.216	34.197	76.515	36.820	41.592	27.031
0.005	0.4975	0.4975	74.922	25.813	75.716	30.252	76.115	32.983	40.692	21.470
0.01	0.495	0.495	73.622	20.776	74.916	25.603	75.815	30.610	39.692	17.414
0.05	0.475	0.475	68.122	11.423	71.416	15.428	73.815	20.762	35.992	9.932
0.1	0.45	0.45	64.322	8.594	68.816	11.855	72.415	16.922	33.692	7.614
0.15	0.425	0.425	61.422	7.138	66.716	9.914	71.315	14.733	32.092	6.448
0.2	0.4	0.4	59.022	6.200	65.016	8.702	70.315	13.144	30.792	5.674
0.25	0.375	0.375	56.822	5.487	63.516	7.815	69.415	11.948	29.592	5.062
0.3	0.35	0.35	54.822	4.930	62.016	7.056	68.415	10.813	28.592	4.612
0.333	0.333	0.333	53.522	4.605	61.1644	6.650	67.915	10.307	27.892	4.323
0.4	0.3	0.3	51.022	4.048	59.216	5.888	66.815	9.309	26.692	3.873
0.41	0.295	0.295	50.622	3.966	58.916	5.778	66.615	9.142	26.492	3.803
0.42	0.29	0.29	50.322	3.906	58.716	5.706	66.315	8.899	26.292	3.733
0.43	0.285	0.285	49.922	3.827	58.416	5.600	66.315	8.899	26.192	3.699
0.45	0.275	0.275	30.222	0.914	44.316	1.761	65.915	8.589	25.792	3.566
0.5	0.25	0.25	30.222	0.914	44.316	1.761	65.015	7.938	24.892	3.281
0.550	0.225	0.225	30.222	0.914	44.316	1.761	57.415	3.318	23.992	3.016
0.600	0.200	0.200	30.222	0.914	44.316	1.761	57.415	3.318	23.092	2.768
0.650	0.175	0.175	30.222	0.914	44.316	1.761	57.415	3.318	22.092	2.510
0.700	0.150	0.150	30.222	0.914	44.316	1.761	57.415	3.318	20.992	2.243
0.750	0.125	0.125	30.222	0.914	44.316	1.761	57.415	3.318	15.992	1.090
0.800	0.100	0.100	30.322	0.983	44.316	1.761	57.515	3.563	15.892	1.054
0.850	0.075	0.075	30.322	0.983	44.316	1.761	57.515	3.563	15.892	1.054
0.900	0.050	0.050	30.322	0.983	44.416	1.884	57.515	3.563	15.892	1.054
0.950	0.025	0.025	30.322	0.983	44.416	1.884	57.515	3.563	15.892	1.054
1.000	0.000	0.000	30.222	0.914	44.316	1.761	57.415	3.318	15.692	0.917

Table 5.37 presents the optimal level according to the weight of the released capital when it is between 0 and 1 when $t = 5$.

Table 5.38: Changes in weights of the released capital for Pareto claims under the excess of loss reinsurance when $t = 10$

t=10 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	77.579	50.859	77.206	44.084	77.613	51.214	44.992	938.300
0.0025	0.49875	0.49875	76.079	32.038	77.406	47.403	78.713	98.428	41.292	24.892
0.005	0.4975	0.4975	74.879	24.998	76.606	36.554	78.013	61.690	40.292	19.651
0.01	0.495	0.495	73.279	19.448	75.406	27.467	77.413	47.269	39.192	15.885
0.05	0.475	0.475	66.979	10.357	71.006	14.576	74.713	23.683	35.292	9.121
0.1	0.45	0.45	62.679	7.690	67.906	10.863	72.913	17.827	32.992	7.070
0.15	0.425	0.425	59.679	6.422	65.806	9.183	71.413	14.717	31.292	5.956
0.2	0.4	0.4	57.179	5.584	64.006	8.052	70.213	12.858	29.992	5.257
0.25	0.375	0.375	54.979	4.963	62.406	7.214	69.113	11.476	28.792	4.698
0.3	0.35	0.35	52.979	4.471	60.906	6.536	68.213	10.512	27.792	4.284
0.333	0.333	0.333	51.779	4.203	60.006	6.170	67.713	10.028	27.192	4.054
0.4	0.3	0.3	49.379	3.718	58.306	5.545	66.513	8.990	25.892	3.599
0.41	0.295	0.295	49.079	3.662	58.106	5.477	66.313	8.831	25.792	3.566
0.42	0.29	0.29	48.679	3.588	57.806	5.376	66.213	8.753	25.592	3.501
0.43	0.285	0.285	48.379	3.533	57.606	5.310	65.913	8.525	25.492	3.469
0.45	0.275	0.275	47.679	3.409	57.106	5.148	65.613	8.303	25.092	3.343
0.5	0.25	0.25	45.979	3.124	55.706	4.723	57.613	3.369	24.192	3.074
0.550	0.225	0.225	44.279	2.858	54.406	4.359	57.613	3.369	23.392	2.849
0.600	0.200	0.200	42.479	2.596	44.606	1.916	57.613	3.369	22.292	2.560
0.650	0.175	0.175	30.679	1.058	44.606	1.916	57.613	3.369	21.692	2.411
0.700	0.150	0.150	30.679	1.058	44.606	1.916	57.613	3.369	20.892	2.219
0.750	0.125	0.125	30.679	1.058	44.606	1.916	57.613	3.369	19.892	1.990
0.800	0.100	0.100	30.579	1.033	44.606	1.916	57.613	3.369	18.992	1.791
0.850	0.075	0.075	30.579	1.033	44.606	1.916	57.613	3.369	17.792	1.530
0.900	0.050	0.050	30.579	1.033	44.606	1.916	57.613	3.369	15.992	1.090
0.950	0.025	0.025	30.579	1.033	44.606	1.916	57.613	3.369	15.892	1.054
1.000	0.000	0.000	30.379	0.935	44.506	1.794	57.613	3.369	15.692	0.917

Table 5.38 presents the optimal level according to the weight of the released capital between 0 and 1 when $t = 10$.

Table 5.39: Changes in weights of the released capital for Pareto claims under the excess of loss reinsurance when $t = 20$

t=20 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.5	0.5	79.182	170.140	79.710	1490.600	79.316	217.351	44.992	938.272
0.0025	0.49875	0.49875	75.382	27.503	76.510	35.575	79.316	217.351	41.092	23.640
0.005	0.4975	0.4975	74.082	21.870	75.510	28.051	79.316	217.351	40.192	19.242
0.01	0.495	0.495	72.182	16.901	74.210	22.114	76.316	33.476	38.992	15.340
0.05	0.475	0.475	65.482	9.272	69.510	12.536	73.516	19.444	35.092	8.908
0.1	0.45	0.45	61.182	7.014	66.510	9.694	71.716	15.261	32.692	6.853
0.15	0.425	0.425	58.082	5.868	64.310	8.227	70.416	13.143	30.992	5.785
0.2	0.4	0.4	55.582	5.124	62.510	7.264	69.316	11.711	29.692	5.110
0.25	0.375	0.375	53.382	4.565	61.010	6.580	66.816	9.237	28.592	4.612
0.3	0.35	0.35	51.482	4.140	59.610	6.017	66.816	9.237	27.492	4.168
0.333	0.333	0.333	50.082	3.854	58.810	5.722	66.816	9.237	26.892	3.945
0.4	0.3	0.3	47.982	3.462	56.810	5.055	66.116	8.678	25.692	3.533
0.41	0.295	0.295	47.682	3.410	56.810	5.055	66.016	8.602	25.492	3.469
0.42	0.29	0.29	47.382	3.358	56.710	5.024	65.816	8.452	25.292	3.405
0.43	0.285	0.285	46.882	3.272	56.510	4.962	65.716	8.378	25.192	3.374
0.45	0.275	0.275	46.382	3.189	56.110	4.842	65.416	8.162	24.792	3.251
0.5	0.25	0.25	44.782	2.935	55.010	4.525	63.816	7.109	23.99206	3.016236
0.55	0.225	0.225	43.282	2.711	53.910	4.227	63.816	7.109	23.192	2.795
0.6	0.2	0.2	41.682	2.486	52.510	3.870	62.516	6.353	22.292	2.560
0.65	0.175	0.175	40.082	2.272	51.610	3.653	62.516	6.353	21.492	2.362
0.7	0.15	0.15	38.482	2.069	50.610	3.419	61.616	5.865	20.692	2.172
0.75	0.125	0.125	36.882	1.872	49.310	3.125	57.716	3.615	19.792	1.968
0.8	0.1	0.1	34.982	1.645	44.910	2.056	57.716	3.615	18.892	1.769
0.85	0.075	0.075	30.982	1.119	44.810	2.017	57.716	3.615	17.992	1.574
0.9	0.05	0.05	30.882	1.101	44.810	2.017	57.716	3.615	17.092	1.375
0.95	0.025	0.025	30.782	1.081	44.710	1.972	57.716	3.615	15.992	1.090
1	0	0	30.382	0.935	44.510	1.795	57.616	3.371	15.692	0.917

Table 5.39 presents the optimal levels according to the weight of the released capital when it changes between 0 and 1 when $t = 20$. Similar comments can be done for Table 5.36 through Table 5.39. It is clear that the increase in the weight of the released capital causes a decrease in optimal levels.

More detailed examples for the changes in the weight of released capital for the Pareto claims under the excess of loss reinsurance for $t = 1$ can be found in Appendix A. Table A.3 presents the optimal level according to changes in the weight of the released capital while the weight of the expected profit is two times of the weight of the expected utility. Table A.4 presents the optimal level according to changes in the weight of the released capital while the weight of the expected utility is two times of the weight of the expected profit.

More detailed examples for the changes in the weight of expected profit for the Pareto claims under the excess of loss reinsurance can be found in Appendix A. Table A.8 presents the optimal level according to changes in the weight of the expected profit while the weight of the released capital is equal to the weight of the expected utility. Table A.9 presents the optimal level according to changes in the weight of the expected profit while the weight of the released capital is two times of the weight of the expected utility. Table A.10 presents the optimal level according to changes in the weight of the expected profit while the weight of the expected utility is two times of the weight of the released capital.

More detailed examples for the changes in the weight of expected utility for the Pareto claims under the excess of loss reinsurance can also be found in Appendix A. Table A.14 presents the optimal level according to changes in the weight of the expected utility while the weight of the released capital is equal to the weight of the expected profit. Table A.15 presents the optimal level according to changes in the weight of the expected utility while the weight of the released capital is two times of the weight of the expected profit. Table A.16 presents the optimal level according to changes in the weight of the expected utility while the weight of the expected profit is two times of the weight of the released capital.

Based on the results, it can be concluded that the increase in the weight of the released capital causes a decrease in the optimal levels. However, the increase in the weight of the expected profit or expected utility cause an increase in the optimal levels. Under the assumption of the Pareto claims, it can be observed that the optimal levels are higher than the exponential claims.

In this section, particular attention is paid to the measure of the importance of the weight of the criteria. An important implication of these findings is the change in the weight of each criteria which causes a difference on the optimal levels. When the weight of the expected profit and expected utility are changed, the similar pattern can be observed. However, it should be noted that an increase in the released capital causes a decrease in the optimal levels whereas the increase in the weight of the expected profit or expected utility causes an increase in the optimal level. The reason is that each criteria's maximum level are different.

The maximum level of the released capital is observed at the beginning of the alternative set such as minimum reinsurance level. At the same time, the maximum levels of expected profit and expected utility are observed at the end of the alternative set such as maximum reinsurance level.

Moreover, under the excess of loss reinsurance, for the exponential claims, when $t = 1$ and $t = 20$ for $\theta = 0.1$ and $\xi = 0.3$, we compare the optimal levels according to three different dominance set of weights. We aim to show the sensitivity of the matrices according to the importance of the criteria in Table 5.40.

The results according to the first set of weight (0.99, 0.005 and 0.005) are compared with Scenario 2 in Table 5.6. In this case, we expect that the optimal levels would be less than or equal to those under Scenario 2 because of the increase of the importance of the released capital.

The results according to the second set of weights (0.005, 0.99 and 0.005) are compared with Scenario 3 in Table 5.7 and we expect the levels obtained would be greater than or equal to those of Scenario 3 because of the increasing importance of the expected profit.

The results according to the third set of weights (0.005, 0.005 and 0.99) are compared with Scenario 4 in Table 5.8. We expect that the optimal levels would be greater than or equal to those of Scenario 4, because of the increasing importance of the expected utility.

Table 5.40: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for dominant weights under the excess of loss reinsurance for the exponential claims

Weights of Criteria 0.99 & 0.005 & 0.005	The Vector normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
t=1	40.272	1.977	45.772	8.379	40.872	2.520
t=20	45.758	2.676	45.858	2.905	45.758	2.676
Weights of Criteria 0.005 & 0.99 & 0.005	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
t=1	45.672	7.411	45.772	8.379	45.372	6.178
t=20	45.772	8.379	45.772	8.379	45.772	8.379
Weights of Criteria 0.005 & 0.005 & 0.99	Optimal u_M	Optimal M	Optimal u_M	Optimal M	Optimal u_M	Optimal M
t=1	45.772	8.379	45.772	8.379	45.772	8.379
t=20	46.258	3.267	46.958	3.767	45.858	2.905

We can observe that the vector normalized covariance matrix reflect the effect of the weights better than the other matrices.

5.5.2. Effect of the Changes in the Parameter of the Adjustment Coefficient

We also consider the consequences of the parameter of the exponential utility function. Under the excess of loss reinsurance, differences on the parameter of the exponential utility causes a change in optimal levels. Such cases are depicted in the following tables.

Table 5.41: Changes in the parameter of the exponential utility function for the exponential individual claims under the excess of loss reinsurance for $t = 0.1$

t=0.1	$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.35$	
B	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.015	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.016	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.017	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.018	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.019	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.02	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.021	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.022	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.023	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.024	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.025	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.026	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.027	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.028	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.029	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635
0.03	23.866	7.754	23.888	8.140	23.877	7.900	20.166	7.635

Table 5.41 presents the changes in the parameter of the exponential utility function for the exponential individual claims under the excess of loss reinsurance for $t = 0.1$. It can be seen that the retention level and corresponding initial surplus do not change according to the parameter of the exponential utility function.

Table 5.42: Changes in the parameter of the exponential utility function for the exponential individual claims under the excess of loss reinsurance for $t = 5$

t=5	$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.35$	
B	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.015	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.016	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.017	27.730	0.845	38.263	1.535	45.736	2.669	14.367	0.840
0.018	27.730	0.845	38.263	1.535	45.736	2.669	20.367	2.320
0.019	27.730	0.845	38.263	1.535	45.736	2.669	20.367	2.320
0.02	27.730	0.845	38.263	1.535	45.736	2.669	20.267	2.293
0.021	27.730	0.845	38.263	1.535	45.736	2.669	20.267	2.293
0.022	27.730	0.845	38.263	1.535	45.736	2.669	20.167	2.266
0.023	27.730	0.845	38.263	1.535	45.736	2.669	20.167	2.266
0.024	27.730	0.845	38.263	1.535	45.736	2.669	20.067	2.240
0.025	27.730	0.845	38.263	1.535	45.736	2.669	20.067	2.240
0.026	27.730	0.845	38.263	1.535	45.736	2.669	20.067	2.240
0.027	27.730	0.845	38.263	1.535	45.736	2.669	19.967	2.214
0.028	27.730	0.845	38.263	1.535	45.736	2.669	19.967	2.214
0.029	27.730	0.845	38.263	1.535	45.736	2.669	19.967	2.214
0.03	27.730	0.845	38.263	1.535	45.736	2.669	19.967	2.214

A similar comment to Table 5.41 can be made for Table 5.42.

Table 5.43: Changes in the parameter of the exponential utility function for the exponential individual claims under the excess of loss reinsurance for $t = 20$

t=20	$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.35$	
B	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.015	37.798	2.207	43.607	3.017	47.658	4.301	19.567	2.112
0.016	37.698	2.193	43.507	2.990	47.658	4.301	19.567	2.112
0.017	37.598	2.179	43.507	2.990	47.758	4.301	19.567	2.112
0.018	37.598	2.179	43.407	2.962	47.558	4.218	19.567	2.112
0.019	37.498	2.165	43.407	2.962	47.558	4.218	19.567	2.112
0.02	37.398	2.151	43.307	2.935	47.558	4.218	19.567	2.112
0.021	37.398	2.151	43.307	2.935	47.558	4.218	19.567	2.112
0.022	37.298	2.137	43.307	2.935	47.558	4.218	19.467	2.087
0.023	37.298	2.137	43.307	2.935	47.558	4.218	19.467	2.087
0.024	37.298	2.137	43.207	2.909	47.558	4.218	19.467	2.087
0.025	37.198	2.123	43.207	2.909	47.558	4.218	19.467	2.087
0.026	37.198	2.123	43.207	2.909	47.558	4.218	19.467	2.087
0.027	37.198	2.123	43.207	2.909	47.558	4.218	19.467	2.087
0.028	37.198	2.123	43.207	2.909	47.558	4.218	19.467	2.087
0.029	37.198	2.123	43.107	2.882	47.558	4.218	19.467	2.087
0.03	37.098	2.110	43.107	2.882	47.558	4.218	19.467	2.087

Table 5.43 represents the changes in the parameter of the exponential utility function for the exponential individual claims under the excess of loss reinsurance for $t = 20$.

It can be seen that the retention level and corresponding initial surplus decrease when the parameter of the exponential utility function increases. Based on the results it can be concluded that as the parameter of the utility function B increases, the insurer is more risk averse (as measured by the coefficient of risk aversion).so that the retention level M should decrease.

We can observe a small decrease in the optimal levels when $t = 10$ and $t = 20$, however the changes is so small. For many combinations of time and loading factors we find that the parameter of the utility function B has no effect on the optimal pairs.

This analysis is extended using the Pareto individual claim and the proportional reinsurance arrangement. It is obvious that the same pattern is observed.

5.5.3. Maximum Possible Value of Each Attribute

We give a different comparison technique with individual outcomes under each criterion. In this context, the question under discussion measures the sensibility of the optimal pairs. We choose the same pair of loading factors $\theta = 0.1$ and $\xi = 0.3$ and $t = 1$ under the excess of loss reinsurance for the exponential claim amount. We assume that each criterion has the same importance (Scenario 1 in Table 5.1). We calculate the optimal initial surplus and retention level for three different matrices. For the pairs of initial surplus and retention level, the released capital, expected profit and expected utility and percentage of the maximum possible values of these criteria are compared to measure the sensibility of the optimal pairs. The values are given in Table 5.44.

Table 5.44: Percentage of the maximum possible values of the released capital, expected profit and expected utility when $t = 1$ and $\theta = 0.1$ & $\xi = 0.3$

Weights of Criteria 1/3 & 1/3 & 1/3 (Scenario 1)	Optimal Initial Surplus u_k	Optimal Barrier Level k	Released Capital	Maximum Possible Released Capital	Percentage	Expected Profit	Maximum Possible Expected Profit	Percentage	Expected Utility	Maximum Possible Expected Utility	Percentage
The Vector normalized Covariance Matrix	41.272	2.731	4.583	5.583	82.088%	40.231	49.966	80.517 %	0.772	0.819	94.261 %
The Original Covariance Matrix	45.772	8.379	0.083	5.583	1.486%	49.966	49.966	100%	0.819	0.819	100%
The Correlation Matrix	43.272	3.798	2.583	5.583	46.265%	46.636	49.966	0.933 %	0.801	0.819	97.802 %

The discussion will point to the sensibility of the optimal pair for three different matrices. A closer look at the relationship between the outcomes indicate that the optimal levels obtained for the original covariance matrix mostly depend on the expected profit and the expected utility. We can see that the percentage of the released capital to maximum possible released capital is %1.5 whereas the percentage of the expected profit and expected utility to their maximum possible values are %100. The reason is that the variance of the expected profit is significantly higher than the other criteria. Higher variance causes to shift the optimal levels to the point where the expected profit has its maximum value. These results provide the confirmatory evidence that the optimal pairs obtained for the original covariance matrix do not look sensible because of the high variance impact. According to the optimal levels which are obtained by using the vector-normalized covariance matrix, the values are transformed into the same scale. In this case, the percentage of the released capital, expected profit and expected utility to their possible maximum values are close to each other in comparison with the original covariance matrix. In the correlation matrix, the percentages of the three criteria to their maximum values are observed between the original and the vector-normalized covariance matrix.

In order to obtain a more comparable form, the analysis is extended according to Scenario 1 and Scenario 2 for the exponential and Pareto claims under the excess of loss and proportional reinsurance by using vector-normalized covariance matrix. Tables 5.45, 5.46 and 5.47 show the percentage of the maximum possible value of each attribute.

Table 5.45: Maximum possible value of each attribute for the exponential claims under the excess of loss reinsurance

		Percentages of the maximum possible values under Scenario 1			Percentages of the maximum possible values under Scenario 2		
t	Premium Loading Factors	Released Capital	Expected Profit	Expected Utility	Released Capital	Expected Profit	Expected Utility
0.1	$\theta = 0.1, \xi = 0.15$	0%	100%	100%	100%	0%	29%
	$\theta = 0.1, \xi = 0.2$	0%	100%	100%	100%	0%	44%
	$\theta = 0.1, \xi = 0.3$	1%	100%	100%	100%	0%	59%
	$\theta = 0.2, \xi = 0.3$	1%	100%	100%	100%	0%	28%
0.5	$\theta = 0.1, \xi = 0.15$	42%	79%	86%	100%	0%	36%
	$\theta = 0.1, \xi = 0.2$	70%	57%	79%	78%	47%	74%
	$\theta = 0.1, \xi = 0.3$	0%	100%	100%	91%	50%	82%
	$\theta = 0.2, \xi = 0.3$	41%	88%	93%	100%	24%	51%
1	$\theta = 0.1, \xi = 0.15$	100%	0%	40%	100%	0%	40%
	$\theta = 0.1, \xi = 0.2$	45%	87%	95%	100%	24%	68%
	$\theta = 0.1, \xi = 0.3$	82%	81%	94%	87%	77%	93%
	$\theta = 0.2, \xi = 0.3$	100%	33%	66%	100%	33%	66%
5	$\theta = 0.1, \xi = 0.15$	100%	36%	89%	100%	36%	89%
	$\theta = 0.1, \xi = 0.2$	100%	57%	96%	100%	57%	96%
	$\theta = 0.1, \xi = 0.3$	100%	79%	99%	100%	79%	99%
	$\theta = 0.2, \xi = 0.3$	52%	85%	100%	100%	35%	97%
10	$\theta = 0.1, \xi = 0.15$	100%	36%	98%	100%	36%	98%
	$\theta = 0.1, \xi = 0.2$	100%	57%	100%	100%	57%	100%
	$\theta = 0.1, \xi = 0.3$	100%	79%	100%	100%	79%	100%
	$\theta = 0.2, \xi = 0.3$	57%	82%	100%	75%	71%	100%
20	$\theta = 0.1, \xi = 0.15$	56%	83%	100%	74%	71%	100%
	$\theta = 0.1, \xi = 0.2$	56%	89%	100%	100%	57%	100%
	$\theta = 0.1, \xi = 0.3$	54%	96%	100%	100%	79%	100%
	$\theta = 0.2, \xi = 0.3$	57%	82%	100%	81%	65%	100%

Table 5.45 presents the maximum possible value of each attribute for the exponential claims under the excess of loss reinsurance. From this table, it can be seen that the optimal levels move towards the maximum levels of the released capital under Scenario 2. In this case, the increase in the percentage of the released capital can be observed.

Table 5.46: Maximum possible value of each attribute for Pareto claims under the excess of loss reinsurance

		Percentages of the maximum possible values under Scenario 1			Percentages of the maximum possible values under Scenario 2		
t	Premium Loading Factors	Released Capital	Expected Profit	Expected Utility	Released Capital	Expected Profit	Expected Utility
0.1	$\theta = 0.1, \xi = 0.15$	0%	100%	100%	99%	0%	24%
	$\theta = 0.1, \xi = 0.2$	0%	100%	100%	97%	1%	37%
	$\theta = 0.1, \xi = 0.3$	0%	100%	100%	94%	1%	51%
	$\theta = 0.2, \xi = 0.3$	0%	100%	100%	100%	1%	24%
0.5	$\theta = 0.1, \xi = 0.15$	76%	55%	67%	83%	44%	61%
	$\theta = 0.1, \xi = 0.2$	0%	100%	100%	87%	40%	67%
	$\theta = 0.1, \xi = 0.3$	0%	100%	100%	92%	43%	75%
	$\theta = 0.2, \xi = 0.3$	79%	70%	79%	83%	66%	77%
1	$\theta = 0.1, \xi = 0.15$	60%	80%	88%	70%	70%	83%
	$\theta = 0.1, \xi = 0.2$	79%	68%	85%	83%	64%	84%
	$\theta = 0.1, \xi = 0.3$	0%	100%	100%	90%	69%	89%
	$\theta = 0.2, \xi = 0.3$	54%	91%	96%	65%	86%	94%
5	$\theta = 0.1, \xi = 0.15$	53%	91%	100%	100%	32%	87%
	$\theta = 0.1, \xi = 0.2$	52%	94%	100%	100%	50%	95%
	$\theta = 0.1, \xi = 0.3$	53%	97%	100%	66%	94%	100%
	$\theta = 0.2, \xi = 0.3$	59%	90%	100%	69%	84%	100%
10	$\theta = 0.1, \xi = 0.15$	57%	89%	100%	68%	82%	100%
	$\theta = 0.1, \xi = 0.2$	56%	93%	100%	68%	88%	100%
	$\theta = 0.1, \xi = 0.3$	55%	96%	100%	100%	68%	100%
	$\theta = 0.2, \xi = 0.3$	61%	88%	100%	71%	82%	100%
20	$\theta = 0.1, \xi = 0.15$	60%	88%	100%	71%	81%	100%
	$\theta = 0.1, \xi = 0.2$	60%	92%	100%	71%	87%	100%
	$\theta = 0.1, \xi = 0.3$	59%	95%	100%	70%	93%	100%
	$\theta = 0.2, \xi = 0.3$	62%	88%	100%	72%	81%	100%

Table 5.46 presents the maximum possible value of each attribute for the Pareto claims under the excess of loss reinsurance. Since the variance of the Pareto claims is larger than the exponential case, the dominance effect of the expected profit is greater than the exponential case. Because of that reason even for Scenario 2 in which more weight is given to the released capital, the maximum possible values of the released capital is smaller than the exponential case.

Table 5.47: Maximum possible value of each attribute for the exponential claims under the proportional reinsurance

		Percentages of the maximum possible values under Scenario 1			Percentages of the maximum possible values under Scenario 2		
t	Premium Loading Factors	Released Capital	Expected Profit	Expected Utility	Released Capital	Expected Profit	Expected Utility
0.1	$\theta = 0.1, \xi = 0.15$	1%	100%	100%	100%	0%	40%
	$\theta = 0.1, \xi = 0.2$	0%	100%	100%	100%	0%	56%
	$\theta = 0.1, \xi = 0.3$	1%	100%	100%	100%	0%	72%
	$\theta = 0.2, \xi = 0.3$	0%	100%	100%	100%	0%	38%
0.5	$\theta = 0.1, \xi = 0.15$	0%	100%	100%	100%	0%	46%
	$\theta = 0.1, \xi = 0.2$	0%	100%	100%	100%	0%	63%
	$\theta = 0.1, \xi = 0.3$	79%	74%	95%	100%	50%	89%
	$\theta = 0.2, \xi = 0.3$	0%	100%	100%	97%	24%	61%
1	$\theta = 0.1, \xi = 0.15$	0%	100%	100%	100%	0%	51%
	$\theta = 0.1, \xi = 0.2$	0%	100%	100%	98%	26%	78%
	$\theta = 0.1, \xi = 0.3$	100%	100%	100%	100%	100%	100%
	$\theta = 0.2, \xi = 0.3$	2%	100%	100%	98%	46%	80%
5	$\theta = 0.1, \xi = 0.15$	98%	50%	95%	100%	45%	94%
	$\theta = 0.1, \xi = 0.2$	100%	100%	100%	100%	100%	100%
	$\theta = 0.1, \xi = 0.3$	100%	100%	100%	100%	100%	100%
	$\theta = 0.2, \xi = 0.3$	100%	0%	33%	100%	0%	33%
10	$\theta = 0.1, \xi = 0.15$	100%	47%	99%	100%	47%	99%
	$\theta = 0.1, \xi = 0.2$	100%	100%	100%	100%	100%	100%
	$\theta = 0.1, \xi = 0.3$	100%	100%	100%	100%	100%	100%
	$\theta = 0.2, \xi = 0.3$	55%	77%	100%	100%	45%	100%
20	$\theta = 0.1, \xi = 0.15$	100%	47%	100%	100%	47%	100%
	$\theta = 0.1, \xi = 0.2$	100%	100%	100%	100%	100%	100%
	$\theta = 0.1, \xi = 0.3$	100%	100%	100%	100%	100%	100%
	$\theta = 0.2, \xi = 0.3$	61%	75%	100%	100%	45%	100%

Table 5.47 presents the maximum possible value of each attribute for the exponential claims under the proportional reinsurance. Since the optimal pair of initial surplus and reinsurance levels are calculated as no-reinsurance case ($p = 0.999$), the percentage of the expected profit and expected utility are larger than the released capital. In some cases, the main limitation of the experimental result is that there is only one possible pair of initial surplus and proportion level. This case can be observed when percentages of the maximum possible values of each of three criteria has a percentage of 100.

5.6. Discussion

In this chapter, optimal pair of initial surplus and reinsurance level are obtained according to four different sets of weights. The matrices which are used in the TOPSIS method with Mahalanobis distance are formed by employing the original outcomes covariance matrix, the vector-normalized outcomes covariance matrix and the correlation matrix.

We compare the optimal levels according to three different matrices, four different pairs of loading factors, and four different sets of weights of each criteria. We observe that in most cases, the correlation matrix gives results between those calculated by employing the vector-normalized covariance matrix and the original covariance matrix. In addition, in the correlation matrix, after a certain time period, stable optimal levels are observed, However, making general comments for both the vector-normalized and the original outcomes matrices are rather difficult. In these cases, even after a sufficient time horizons, stability on the optimal levels cannot be observed as in the correlation matrix.

The main objective of this section is to give a comprehensive perspective of our approach in respect of three matrices approach. In this context, we compare our results according to time horizons, individual claim distributions and the type of reinsurance arrangements.

We start by investigating the optimal levels under the excess of loss and proportional reinsurance by time horizon. In most cases, when $t = 0.1$, the optimal initial surplus and retention levels are calculated lower than the other time horizons. Since the expected loss up to time t can be paid by the collected premium up to time t , the lower initial surplus is enough for obtaining the fixed ruin probability. The smallest initial surplus for the short time horizon is less than the values in other time horizons. In order to get a constant ruin probability 0.01, small initial surplus levels and corresponding low retention levels are calculated.

After $t = 5$, as time increases, the optimal levels decrease and after a sufficient time level such as $t = 20$, optimal levels reach their fixed values.

The optimal initial surplus and reinsurance levels obtained under the excess of loss and proportional reinsurance can be investigated according to changes in the weight of the criteria.

We address to not only the same weights assumption but also the different sets of weights are used.

Under the excess of loss reinsurance, in Scenario 1, the importance of each criterion is assumed to be the same. The optimal levels that are obtained according to different importance levels are compared with the results of Scenario 1.

In Scenario 2, for both of the exponential and the Pareto claim amounts, the lowest optimal levels among the other scenarios are obtained. Since the importance level of the released capital is significantly higher than that of the other criteria. In Scenario 2, the optimal levels are shifted to the point where the released capital profit has the maximum value.

In Scenario 3, the optimal levels are obtained as slightly greater than the other scenarios. the importance level of the expected profit is significantly higher than that of the other criteria. Thus, optimal levels shifted towards the levels where the expected profit is maximum.

In Scenario 4, the importance of the expected utility criteria is higher than that of the other criteria. Optimal levels are observed as very close to the optimal values under Scenario 1, together with the values lying between Scenarios 2 and 3. The underlying reason is that the expected utility is also maximum at the point where the expected profit is maximum. However, the variance of the expected profit is greater than that of expected utility.

Under the proportional reinsurance, in cases of $\theta = 0.1$ & $\xi = 0.2$, and $\theta = 0.1$ & $\xi = 0.3$, after the first year, the number of alternatives decrease. For this reason, in most cases, the smallest feasible initial surplus is very close to the largest initial surplus so the optimal levels are calculated to be the same as the smallest feasible initial surplus. This situation is rather similar for both the exponential and the Pareto claim amounts. For the exponential claims, the changes in the optimal levels are not as large as in the excess of loss case. For all scenarios, after $t = 20$, we get the same optimal levels even for the four different scenarios and three matrix choices. Under the Pareto claims, similar to the excess of loss case, in Scenarios 2 and 3 the lowest and the highest optimal levels are obtained, respectively. The optimal values obtained in Scenario 4 are between the values in Scenarios 2 and 3, and these values are very close to the values in Scenario 1.

Furthermore, in this case, similar to the excess of loss case, the optimal levels for the Pareto claim amounts are calculated higher than for the exponential claims.

When the Pareto and the exponential claims are compared, it can be concluded that the optimal levels obtained for Pareto claims are higher than those obtained for the exponential claims. This case can be explained by the fact that the exponential distribution has a light tail whereas the Pareto distribution has a heavy tail.

We have also considered the consequence of dominance weights situations. The results are shown in Table 5.40. We aim to measure the sensitivity of our model under the covariance and the correlation matrices. An important practical issue is to evaluate the three matrices according to the results. Based on the results, it can be concluded that the vector normalized covariance matrix reflect the effect of the weights better than the other matrices.

Summing up the results, it can be concluded that our method into obtaining the optimal levels under the three criteria based on the translated gamma process is a new and different method. The originality of our approach lies in the fact that although several studies have indicated that the focus is only one criteria, we use a method depending on three main criteria under the minimum fixed finite time ruin probability.

5.7. Interim Conclusion: Analysis of the Optimal Reinsurance on the Translated Gamma Process Approximation

In this chapter, we have given numerical analysis under the assumption of the aggregate claim process is a compound Poisson with individual claim amounts have exponential and Pareto distributions. For the excess of loss and proportional reinsurance arrangements, we have calculated the optimal initial surplus and reinsurance level under the TOPSIS method with Mahalanobis distance. For different time horizons and different loading factors, we have calculated the optimal pairs. In addition, we have used the covariance matrix of the vector-normalized and the original outcomes and also the correlation matrix to explain the dependency between the criteria.

We have discussed our results and have made some general comments about our approach according to different scenarios and set of weights.

Moreover, we have measured the sensitivity of our approach on the vector-normalized covariance matrix, the original outcomes covariance matrix and the correlation matrix by comparing the results in the dominance weight set.

The proposed method can readily be used in practice. The technique is applicable to different type of reinsurance arrangement or to different claim assumptions. This research is concerned with the optimal reinsurance on the translated gamma process approximation; however, the optimal reinsurance criteria with TOPSIS method can also be applicable to other approximations. The findings suggest that this approach can be useful to determine the optimal reinsurance level under the ruin probability constraint according to not only the expected profit but also the released capital and the expected utility.

6. LOWER BARRIER MODEL ON MODIFIED SURPLUS PROCESS

6.1. Introduction

A type of barrier level is related to the ruin probability which depends on whether the underlying surplus process has reached or exceeded a predetermined barrier level. In absorbing barrier problem, it is aimed to find the probability which the surplus process attains the level b ($b > u$) prior to ruin. To the authors' best knowledge, very few publications are available in the literature that address the issue of absorbing barrier problem. Most of the studies focus on the other types of upper barrier problem such as dividend problems.

In the case of lower barrier model, a lower barrier k , where $0 \leq k \leq u$, is used instead of the upper barrier level b where $b > u$. In this modified surplus process, each time the surplus drops below this lower barrier level k but not below zero, an amount in the manner restore the insurer's surplus back to level k is paid by the reinsurer. When a claim which causes the insurer's surplus fall from a level above k to a level below zero occurs, the reinsurance company cannot make a payment and the ruin occurs. This reinsurance arrangement differs radically from the traditional type of reinsurance arrangements discussed in most actuarial papers. Furthermore, the excess of loss reinsurance and proportional reinsurance arrangements depend on the individual or aggregate claims whereas this reinsurance arrangement does not depend on neither the individual claims nor the aggregate claims. It is related to the surplus process falls below the lower barrier model where $0 \leq k \leq u$ [1].

The layout of this chapter for the next four sections is as follows: In Section 6.2, we introduce the modified surplus process on lower barrier model briefly by including the notation and assumptions. In Section 6.3, we concern the lower barrier model for infinite time and show formulas for the ruin probability. In Section 6.4, we focus on the lower barrier model for finite time and state formulas for the ultimate ruin probability. In Section 6.5, we show the calculation of reinsurance premium on lower barrier model. In Section 6.6, we conclude the chapter.

6.2. Modified Surplus Process

Modified surplus process is shaped with a lower barrier k , where $0 \leq k \leq u$. The main difference between the modified and the classical risk processes is that the former one includes a fixed barrier level. In the classical risk process, whenever the insurer's risk process falls below zero, ruin occurs. However, in the modified surplus process, each time the surplus drops below k but not below zero, a capital injection is made in order to fill the gap between the new level and k . Hence, the insurer's surplus process continues from level k . Ruin is observed when a claim occurs such that any capital injection cannot fill the gap [1]. A realization of the modified surplus process with lower barrier k is given in Figure 6.1 [1].

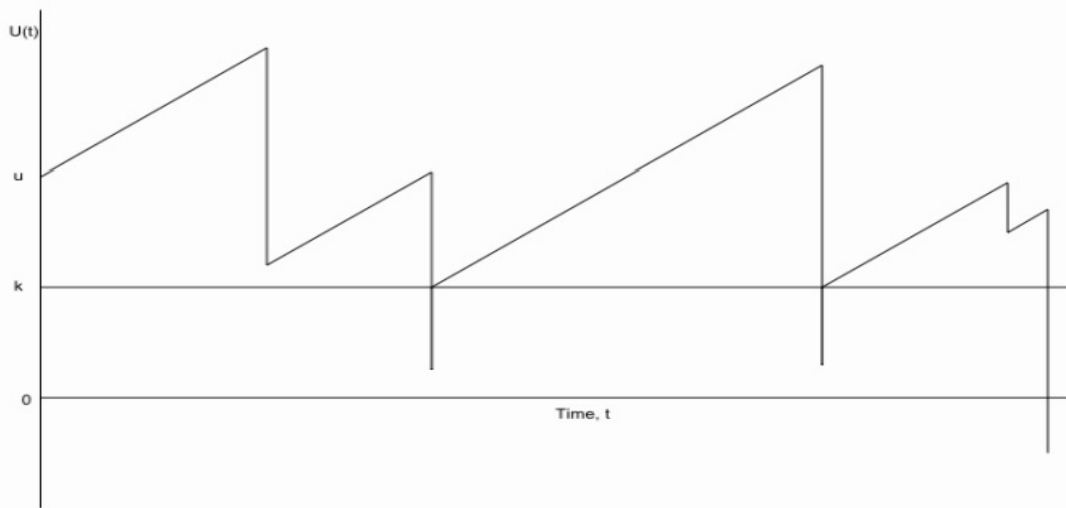


Figure 6.1: A realization of the modified surplus process with lower barrier k

Modified surplus process can be explained using two main components: severity of ruin and its density function.

6.2.1. The Severity of Ruin

Let the initial surplus is u , and the time of ruin is T_u which is denoted by

$$T_u = \inf\{t : U(t) < 0\}$$

where $U(t)$ is the surplus process at time t .

For all $t > 0$, $T_u = \infty$ if $U(t) \geq 0$ and hence $\psi(u) = \Pr(T_u < \infty)$ [11].

When ruin occurs, the insurer's deficit at ruin or severity of ruin is, at most y , denoted as

$$G(u, y) = \Pr(T_u < \infty \quad \text{and} \quad U(T_u) \geq -y),$$

and it yields that

$$\lim_{y \rightarrow \infty} G(u, y) = \psi(u).$$

Therefore, the defective distribution with (defective) density is

$$g(u, y) = \frac{\partial}{\partial y} G(u, y).$$

When the initial surplus equals to zero, the defective distribution with density is obtained by using the maximum of aggregate loss process [11].

Aggregate loss process, $\{L(t)\}_{t \geq 0}$, is defined by

$$L(t) = S(t) - ct, \quad \text{for all } t \geq 0$$

so that $U(t) = u - L(t)$.

Random variable L denotes the maximum of aggregate loss process and it is defined as

$$\begin{aligned} \phi(u) &= \Pr(U(t) \geq 0 \quad \text{for all } t > 0), \\ &= \Pr(L(t) \leq u \quad \text{for all } t > 0), \\ &= \Pr(L \leq u). \end{aligned}$$

where ϕ is the distribution function of L with $L(0) = 0$. Since $\phi(0) = \Pr(L = 0)$, L has a mixed distribution with a mass of probability at zero [11].

Let L^* be the Laplace transform of the random variable L .

Definition 6.2.1. *Laplace Transform*

Let $h(y)$ be a function defined for all $y \geq 0$. Then the Laplace transform of h is

$$h^*(s) = \int_0^{\infty} e^{-sy} h(y) dy. \quad (6.1)$$

Let X to be a random variable with $X \sim H$, where $H(0) = 0$. Then the Laplace transform of X random variable is

$$E[e^{-sX}] = \int_0^{\infty} e^{-sy} dH(y).$$

When X has a continuous distribution with density function h ,

$$E[e^{-sX}] = h^*(s).$$

Dickson [11] shows the Laplace transform of L as

$$\begin{aligned} L^*(s) &= E[e^{-sL}] = \int_0^{\infty} e^{-su} d\phi(u), \\ &= \phi(0) + \int_0^{\infty} e^{-su} \left(\frac{d}{du} \phi(u) \right) du, \\ &= \phi(0) + s\phi^*(s) - \phi(0), \\ &= s\phi^*(s). \end{aligned} \quad (6.2)$$

In Equation (6.2), $\phi(u)$ represents the survival probability (non-ruin probability) and defined by $1 - \psi(u)$. The survival probability can be established by adapting the reasoning used to prove Lundberg's inequality as

$$\phi(u) = \int_0^{\infty} \lambda e^{-\lambda t} \int_0^{u+ct} f(x) \phi(u + ct - x) dx dt.$$

Substituting $s = u + ct$, we have

$$\begin{aligned}\phi(u) &= \frac{1}{c} \int_u^\infty \lambda e^{-\lambda(s-u)/c} \int_0^s f(x) \phi(s-x) dx ds, \\ &= \frac{\lambda}{c} e^{\lambda u/c} \int_u^\infty e^{-\lambda s/c} \int_0^s f(x) \phi(s-z) dx ds,\end{aligned}$$

and the differentiation of the equation gives

$$\begin{aligned}\frac{d}{du} \phi(u) &= \frac{\lambda^2}{c^2} e^{\lambda u/c} \int_u^\infty e^{-\lambda s/c} \int_0^s f(x) \phi(s-x) dx ds - \frac{\lambda}{c} \int_0^u f(x) \phi(u-x) dx, \\ &= \frac{\lambda}{c} \phi(u) - \frac{\lambda}{c} \int_0^u f(x) \phi(u-x) dx.\end{aligned}$$

By using one of the properties of Laplace transform;

$$\int_0^\infty e^{-sy} \left(\frac{d}{dy} h(y) \right) dy = sh^*(s) - h(0),$$

it can be obtained as

$$s\phi^*(s) - \phi(0) = \frac{\lambda}{c} \phi(s)^* - \frac{\lambda}{c} f(s)^* \phi(s)^*,$$

$$\phi^*(s) = \frac{c\phi(0)}{cs - \lambda(1 - f^*(s))}. \quad (6.3)$$

We recall from Equation (6.2) that

$$L^*(s) = \frac{cs\phi(0)}{cs - \lambda(1 - f^*(s))}. \quad (6.4)$$

Let K be “the amount of the first record high of the aggregate loss process” and it is defined as $K(x) = \Pr(L_1 \leq x)$. Let k be the associated density function.

Then applying the Laplace transform,

$$E[e^{-sL}] = E[E(e^{-sL}|N)] = E[k^*(s)^N],$$

is attained where $k^*(s) = E[e^{-sL_1}]$.

Let N has a geometric distribution, so that L has a compound geometric distribution. The Laplace transform is given as

$$E[e^{-sL}] = \frac{\phi(0)}{1 - \psi(0)k^*(s)}. \quad (6.5)$$

where $E[e^{-sL}] = s\phi^*(s)$ in Equation (6.2) .

By using Equation (6.4) and Equation (6.5), $k^*(s)$ is obtained from the following equation

$$\frac{cs\phi(0)}{cs - \lambda(1 - f^*(s))} = \frac{\phi(0)}{1 - \psi(0)k^*(s)},$$

as

$$k^*(s) = \frac{\lambda(1 - f^*(s))}{cs\psi(0)}.$$

Hence, by using the properties of Laplace transform Equation (6.1), the density function of the amount of the first record high of the aggregate loss process is

$$k(x) = \frac{\lambda(1 - F(x))}{c\psi(0)}.$$

Let the probability of a first record high of the aggregate loss process is $\psi(0)$. Thus, $k(y)$ is obtained as

$$k(y) = \frac{g(0, y)}{\psi(0)}.$$

and hence the defective distribution with density when the initial surplus equal to zero is

$$\begin{aligned} g(0, y) &= k(y) \psi(0), \\ &= \frac{\lambda}{c\psi(0)} [1 - F(y)] \psi(0), \\ &= \frac{\lambda}{c} [1 - F(y)]. \end{aligned} \quad (6.6)$$

Dickson [11] explains the insurer's severity of ruin $G(u, y)$ such that if ruin occurs with a deficit of at most y , then on the first occasion on which the surplus falls below its initial level u either

- The surplus falls to $u - x (\geq 0)$, ruin subsequently occurs from this surplus level with a deficit of at most y , or
- Ruin occurs at this fall with a deficit of at most y .

Dickson [11] states $G(u, y)$ as

$$\begin{aligned} G(u, y) &= \int_0^u g(0, x)G(u - x, y)dx + \int_u^{u+y} g(0, x)dx, \\ &= \psi(0) \int_0^u k(x)G(u - x, y)dx + \psi(0) \eta(u, y), \end{aligned}$$

where $\eta(u, y)$ is obtained as

$$\begin{aligned} \eta(u, y) &= \int_u^{u+y} k(x) = K(u + y) - K(u), \\ &= \int_u^{u+y} \frac{\lambda}{c\psi(0)} e^{-\alpha x} dx, \\ &= \frac{\lambda}{c\psi(0)} \frac{e^{-\alpha x}}{-\alpha} \Big|_u^{u+y}, \\ &= \frac{\lambda}{c\psi(0)} \frac{e^{-\alpha(u+y)} - e^{-\alpha u}}{-\alpha}, \\ &= \frac{\lambda}{c\psi(0)} e^{-\alpha(u)} (1 - e^{-\alpha y}). \end{aligned}$$

By using the Laplace transform, which is given in Equation (6.1),

$$G^*(s, y) = \int_0^\infty e^{-su} G(u, y) du$$

and

$$\eta^*(s, y) = \int_0^\infty e^{-su} \eta(u, y) du$$

It is attained that

$$\begin{aligned}
\eta^*(s, y) &= \int_0^\infty e^{-su} \eta(u, y) du, \\
&= \int_0^\infty e^{-su} \frac{\lambda}{\alpha c \psi(0)} e^{-\alpha u} (1 - e^{-\alpha y}) du, \\
&= \frac{\lambda}{\alpha c \psi(0)} \int_0^\infty e^{-u(s+\alpha)} - e^{-su-\alpha u-\alpha y} du, \\
&= \frac{\lambda}{\alpha c \psi(0)} \int_0^\infty e^{-u(s+\alpha)} du - \int_0^\infty e^{-su+\alpha u+\alpha y} du, \\
&= \frac{\lambda}{\alpha c \psi(0)} \left(\frac{1}{s+\alpha} - \frac{e^{-\alpha y}}{s+\alpha} \right), \\
&= \frac{\lambda}{\alpha c \psi(0)} \frac{1 - e^{-\alpha y}}{s+\alpha}.
\end{aligned}$$

and

$$\begin{aligned}
G^*(s, y) &= \frac{\psi(0) \eta^*(s, y)}{1 - \psi(0) k^*(s)}, \\
&= \frac{\lambda(1 - e^{-\alpha y})}{c\alpha^2 + cs\alpha - \alpha\lambda}.
\end{aligned}$$

Hence,

$$\begin{aligned}
G(u, y) &= \frac{\lambda}{\alpha c} e^{-(\alpha-\lambda c)u} (1 - e^{-\alpha y}), \\
&= \psi(u) (1 - e^{-\alpha y}).
\end{aligned} \tag{6.7}$$

Note that $\frac{\lambda}{\alpha c} e^{-(\alpha-\lambda c)u}$ represents the exact ultimate ruin probability $\psi(u)$ [11].

6.2.2. Estimation of the Severity of Ruin on the Translated Gamma Process

Now we focus on estimating $G(u, y)$ on the translated gamma process. We examine the ultimate ruin probability on the translated gamma process in order to provide the Equation (6.7). In the literature, there are two methods which lead to a formula and an approximation in order to obtain the ultimate ruin probability under the translated gamma process approximation. We describe both methods respectively, then we compare the results.

The first one depends on the standardized Gamma process. The ultimate ruin probability for a standardized Gamma process is described by Dufresne et al. [35]. They show that ultimate ruin probability for a standardized Gamma process when the premium loading factor θ can be calculated using

$$\psi_{SG}(u) = 1 - \sum_{n=0}^{\infty} \frac{\theta}{(1+\theta)^{n+1}} G^{n*}(u), \quad (6.8)$$

where

$$G(x) = 1 - \exp(-x) + x \int_x^{\infty} \frac{e^{-y}}{y} dy. \quad x \geq 0 \quad (6.9)$$

Dickson and Waters [44] apply a formula for the expression of $\int_x^{\infty} \frac{e^{-y}}{y} dy$ in Equation (6.9).

$$E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt \quad (|\arg x| < \pi),$$

$E_1(x)$ can be defined as

$$E_1(x) = -\gamma - \ln(x) - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n n!} \quad (|\arg x| < \pi),$$

where $\gamma = 0.5772156649\dots$ is Euler's constant. This specific formula is suggested by Abramowitz and Stegun [131].

Dickson and Waters [44] suggest the discretizing of $G(x)$ function on $0, h, 2h, \dots$ with $h = 0.005$, by using crude rounding method. Then they apply the Panjer's recursive method [132], since $\psi_{SG}(u)$ is the tail probability of a compound geometric distribution.

The second one is related with the adjustment coefficient. Dickson and Waters [44] suggest an approximation for the ultimate ruin probability which depends on the adjustment coefficient. They develop the adjustment coefficient for a translated gamma process with parameter α, β and k .

They aim to find a retention level that maximises the adjustment coefficient and minimises the ultimate ruin probability with the translated Gamma process. They suggest that a unique positive adjustment coefficient R which satisfies the following equation

$$e^{(R(c-k))} = \left(1 - \frac{R}{\beta}\right)^{-\alpha}. \quad (6.10)$$

We compare the results of two approaches for the ultimate ruin probability under the translated Gamma process approximation with the exact ruin probability. We assume that the number of claims have Poisson distribution with $\lambda = 1$, the individual claim amount has an exponential distribution with parameter 1, and insurance loading factor is equal to 0.1. Table 6.1 shows the exact and approximated ruin probabilities which are obtained using two approaches. The second column of the table shows the exact ruin probabilities $\psi(u)$. The third column states the ultimate ruin probability under the translated gamma process $\psi_{TG}(u)$ which is given in Equation (6.8). The last column presents the ruin probabilities which are obtained by using the adjustment coefficient for the translated gamma process $\psi_{TG_R}(u)$ in Equation (6.10). Our results show that the ruin probabilities for all cases are very close to each other.

Table 6.1: The comparison of infinite time exact ruin probability and infinite time ruin probability with the translated gamma process

$\psi(u)$ when $\theta = 0.1, \lambda = 1, \alpha = 1$			
u	$\psi(u)$	$\psi_{TG}(u)$	$\psi_{TG_R}(u)$
0	0.909090900	0.9090910000	0.9090910000
1	0.830091600	0.831766100	0.8301760000
2	0.757957200	0.757591300	0.7581100000
3	0.692091300	0.690882100	0.6923010000
4	0.631949000	0.630833400	0.5322050000
5	0.577033100	0.576119500	0.5773250000
6	0.526889300	0.525835200	0.5272090000
7	0.481103000	0.480283600	0.4814440000
8	0.439295500	0.438684700	0.4396510000
9	0.401121100	0.400418900	0.4014860000
10	0.366263900	0.365741300	0.3666350000
15	0.232481100	0.232212100	0.2328340000
20	0.147564200	0.147534200	0.1478630000
25	0.093664370	0.093670980	0.0939016000
30	0.059452180	0.059472660	0.0596329000
35	0.037736470	0.037785520	0.0378703000

$\psi(u)$ when $\theta = 0.1, \lambda = 1, \alpha = 1$			
u	$\psi(u)$	$\psi_{TG}(u)$	$\psi_{TG_R}(u)$
40	0.023952710	0.023990420	0.0240498000
45	0.015203660	0.015231770	0.0152730000
50	0.009650315	0.009677404	0.0096992600
55	0.006125406	0.006144292	0.0061595900
60	0.003888018	0.003901084	0.0039116900
65	0.002467867	0.002478536	0.0024841500
70	0.001566445	0.001573659	0.0015758000
75	0.000994280	0.000999143	0.0010018500
80	0.000631106	0.000634810	0.0006362350
85	0.000400586	0.000403059	0.0004040460
90	0.000254266	0.000255917	0.0002565920
95	0.000161392	0.000162607	0.0001629510
100	0.000102441	0.000103252	0.0001034830
150	0.000001087	0.000001132	0.0000011041
200	0.0000000115	0.0000000000	0.0000000118
250	0.00000000012	0.000000000000	0.0000000001

Since, the ruin probabilities are remarkably close to each other, the possibility of using $\psi_{TG}(u)$ or $\psi_{TGR}(u)$ instead of exact ruin probability on the $G(u, y)$ function in Equation (6.7) is evaluated.

Under the classical risk model with exponential claims, $G(u, y)$ decomposes as $\psi(u)$ times the exponential cumulative distribution function, but this is due to the memoryless property of the exponential distribution. There is no decomposition in other cases such as Erlang (2) claims or mixed exponential claims. So, we realise that if we want to approximate $G(u, y)$, we cannot do it by approximating $\psi(u)$.

Moreover, it cannot be possible to approximate $G(u, y)$ based on the translated gamma process because of the substantial property of the translated gamma process approximation. The deficit at ruin will basically be 0 under the translated gamma process approximation. In this case, the deficit is 0 because the gamma process has an infinite number of small jumps (similar to the movement of Brownian motion). So ruin occurs when the surplus process crosses zero, but the amount which it crosses 0 is negligible.

We conclude our discussion of the severity of ruin under the translated gamma process approximation with one considerable warning on applicability. In this section we have explained the calculation of the ultimate ruin probability by using the translated gamma process approximation. On obtaining the severity of ruin, we cannot use any approximation even it gives very close results to the exact value. It simply means that the only solution to approximate $G(u, y)$ is to use the exact ruin probability.

6.3. Lower Barrier Model for Infinite Time

6.3.1. The Ultimate Ruin Probability

Nie et al. [2] suggest a formula for the calculation of the ultimate ruin probability under the lower barrier model. The ultimate ruin probability for the modified surplus process with the lower barrier at k is denoted by $\psi_k(u)$. This probability must be treated according to two cases $u = k$ and $u > k$.

(i) Consider the case $u = k$

Nie [1] obtains a formula for $\psi_k(u)$ as

$$\psi_k(k) = \int_0^k g(0, y) \psi_k(k) dy + \int_k^\infty g(0, y) dy,$$

where $g(u, y)$ and $G(u, y)$ are the density and distribution functions of the deficit at ruin in a classical risk model, respectively. This formula depends on the condition on the amount of the first drop below lower level k . Hence, it can be written as

$$\psi_k(k) = \frac{\psi(0) - G(0, k)}{1 - G(0, k)}. \quad (6.11)$$

(ii) Consider the case $u > k \geq 0$,

In this case, the ultimate ruin probability formula is generalized as given below. The survival probability of the classical risk process and modified risk process are defined as $\bar{\psi}(u) = 1 - \psi(u)$ and $\bar{\psi}_k(u) = 1 - \psi_k(u)$, respectively. The survival probability is obtained by conditioning on the amount of the first drop below level k as

$$\bar{\psi}_k(u) = \bar{\psi}(u - k) + G(u - k, k) \bar{\psi}_k(k),$$

hence

$$\begin{aligned} \psi_k(u) &= \psi(u - k) - G(u - k, k) (1 - \psi_k(k)), \\ &= \psi(u - k) - G(u - k, k) \frac{1 - \psi(0)}{1 - G(0, k)}. \end{aligned} \quad (6.12)$$

6.4. Lower Barrier Model for Finite Time

6.4.1. Finite Time Ruin Probability

Nie et. al [133] extend the analysis according to time horizons. They suggest an explicit formula for the finite time ruin probability on lower barrier model. Let $W_{u,k}(t)$ be the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u and defined as

$$W_{u,k}(t) = \Pr[T_{u,k} \leq t].$$

Hence, $w_{u,k}$ is the defective density function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u defined as $w_{u,k} = \frac{d}{dt}W_{u,k}$. The lower barrier model for the finite time is examined by considering two cases for u and k .

Nie et al. [1] start with the case when u is equal to k . Under the classical risk model, let $G(u, \cdot)$ be the defective distribution function of the deficit at ruin. The probability that ruin occurs on the n^{th} occasion, $n = 1, 2, 3, \dots$ that the surplus falls below k is

$$G(0, k)^{n-1}(\psi(0) - G(0, k)). \quad (6.13)$$

When ruin occurs on the n^{th} occasion that the surplus drops below k , the ruin time is defined as the sum of n random variables [1]. The distribution function of the first $n - 1$ independent and identically random variables, D , is defined as

$$D(t) = \Pr[\tau \leq t] = \Pr[T_0 \leq t | T_0 < \infty \quad \text{and} \quad Y \leq k],$$

where T_0 is the time to first drop such as Y below the initial surplus level under the classical risk model. Hence,

$$D(t) = \frac{\Pr[T_0 \leq t \quad \text{and} \quad Y \leq k]}{\Pr[T_0 < \infty \quad \text{and} \quad Y \leq k]}. \quad (6.14)$$

Let $W(u, y, t)$ be the joint distribution function of the deficit at ruin and the time to ruin given initial surplus u . The corresponding joint density function $w(u, y, t)$ is defined as $\frac{\partial^2}{\partial y \partial t}W(u, y, t)$.

Equation (6.14) yields that

$$D(t) = \frac{W(0, k, t)}{G(0, k)} \quad (6.15)$$

Further, τ_L denotes the waiting time between the $(n-1)^{th}$ and n^{th} drop, and has a distribution function as

$$\begin{aligned} D_L(t) &= \Pr[\tau_L \leq t], \\ &= \Pr[T_0 \leq t | T_0 < \infty \text{ and } Y > k], \\ &= \frac{\Pr[T_0 \leq t \text{ and } Y > k]}{\Pr[T_0 < \infty \text{ and } Y > k]}. \end{aligned}$$

and hence

$$D_L(t) = \frac{\psi(0, t) - W(0, k, t)}{\psi(0) - G(0, k)}. \quad (6.16)$$

Thus, the defective distribution function of $T_{k,k}$ i.e. finite time ruin probability for k initial surplus and k barrier level is obtained as

$$W_{k,k}(t) = \Pr(T_{k,k} \leq t) = \sum_{n=1}^{\infty} G(0, k)^{n-1} (\psi(0) - G(0, k)) D^{(n-1)*} * D_L(t), \quad (6.17)$$

where $D^{(n-1)*} * D_L(t)$ shows the convolution of the $(n-1)$ fold convolution of the function D with the function D_L .

According to Nie et. al [1], when $u > k$, there are two possible situations where ruin can happen:

- a)** Ruin happens at the first time when the surplus process drops below level k ,
- b)** Ruin does not happen at the first time the surplus process drops below level k , and the surplus restarts from k and ruin subsequently occurs.

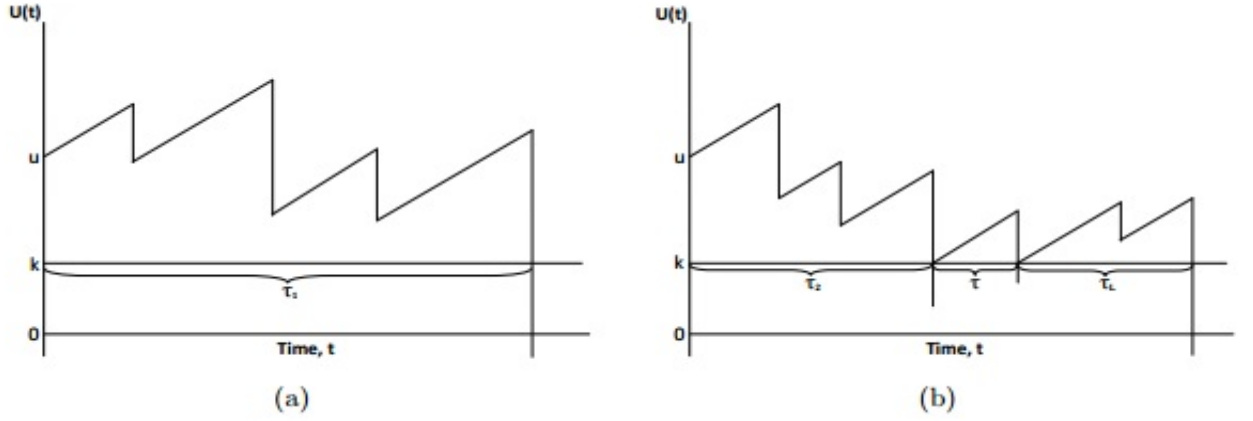


Figure 6.2: Ruin probability on lower barrier model

In Figure 6.2, (a) and (b) represent the first and second situations, respectively [1]. In the first situation, ruin happens at the first time the surplus process drops below level k . The probability of the ruin happens at the first time the surplus process drops below level k is $\psi(u - k) - G(u - k, k)$.

The waiting time for such an event is denoted by τ_1 and its distribution function, D_1 , is

$$\begin{aligned}
 D_1(t) &= \Pr(\tau_1 \leq t), \\
 &= \Pr[T_{u-k} \leq t | T_{u-k} < \infty \text{ and } Y_{u-k} > k], \\
 &= \frac{\Pr[T_{u-k} \leq t \text{ and } Y > k]}{\Pr[T_{u-k} < \infty \text{ and } Y > k]}, \\
 &= \frac{\psi(u - k, t) - W(u - k, k, t)}{\psi(u - k) - G(u - k, k)}.
 \end{aligned}$$

In the second situation, it is assumed that the ruin does not occur at the first time which the surplus drops below k . The probability is defined as $G(u - k, k)$. The waiting time until the surplus first drops below k without ruin occurring is presented by τ_2 .

The distribution function of τ_2 , $D_2(t)$, is stated as

$$\begin{aligned}
D_2(t) &= \Pr(\tau_2 \leq t), \\
&= \Pr[T_{u-k} \leq t | T_{u-k} < \infty \text{ and } Y_{u-k} \leq k], \\
&= \frac{\Pr[T_{u-k} \leq t \text{ and } Y \leq k]}{\Pr[T_{u-k} < \infty \text{ and } Y \leq k]}, \\
&= \frac{W(u-k, k, t)}{G(u-k, k)}.
\end{aligned}$$

Thus, the defective distribution function of $T_{u,k}$ is shown as

$$\begin{aligned}
W_{u,k}(t) &= \Pr[T_{u,k} \leq t], \\
&= [\psi(u-k) - G(u-k, k)]D_{u-k,1} + G(u-k, k)D_2 * W_{k,k}(t). \quad (6.18)
\end{aligned}$$

where $D_2 * W_{k,k}(t)$ denotes the convolution of the function D_2 with the function $W_{k,k}$ which is achieved from Equation (6.17).

Nie et. al [1] compare the ruin time distributions depending on two individual claim size distributions: exponential and Erlang (2). These distributions are preferred because of their simplicity.

6.4.2. Exponential Case on Lower Barrier Model

Nie et. al [1] show how the defective distribution function of $T_{u,k}$ is calculated under the assumption of exponential claim size.

The exponential distribution with parameter $\alpha > 0$ has a density function

$$f(x) = \alpha e^{-\alpha x}.$$

As a result of memorylessness property of the exponential distribution, under the classical risk model, $W(u, y, t)$ can be obtained as

$$W(u, y, t) = \psi(u, t)(1 - e^{-\alpha y})$$

which gives

$$W(0, k, t) = \psi(0, t)(1 - e^{-\alpha k})$$

Consider the first situation when $u = k$. $D(t)$ and $D_L(t)$ are respectively given by

$$\begin{aligned} D(t) &= \frac{W(0, k, t)}{G(0, k)}, \\ &= \frac{\psi(0, t)(1 - e^{-\alpha k})}{\psi(0)(1 - e^{-\alpha k})}, \\ &= \frac{\psi(0, t)}{\psi(0)}. \end{aligned}$$

and

$$\begin{aligned} D_L(t) &= \frac{\psi(0, t) - W(0, k, t)}{\psi(0) - G(0, k)}, \\ &= \frac{\psi(0, t) - \psi(0, t)(1 - e^{-\alpha k})}{\psi(0) - \psi(0)(1 - e^{-\alpha k})}, \\ &= \frac{\psi(0, t)(e^{-\alpha k})}{\psi(0)(e^{-\alpha k})}, \\ &= \frac{\psi(0, t)}{\psi(0)} = D(t). \end{aligned}$$

Equation (6.13) gives

$$\begin{aligned} G(0, k)^{n-1}(\psi(0) - G(0, k)) &= [\psi(0)(1 - e^{-\alpha k})]^{n-1}(\psi(0) - \psi(0)(1 - e^{-\alpha k})), \\ &= [\psi(0)^{n-1}(1 - e^{-\alpha k})]^{n-1}[\psi(0) - \psi(0)(1 - e^{-\alpha k})], \\ &= \psi(0)^n(1 - e^{-\alpha k})^{n-1} - \psi(0)^n(1 - e^{-\alpha k})^n, \\ &= \psi(0)^n(1 - e^{-\alpha k})^{n-1}(1 - (1 - e^{-\alpha k})), \\ &= \psi(0)^n(1 - e^{-\alpha k})^{n-1}e^{-\alpha k}. \end{aligned}$$

By using $\psi(0)^n * D^{n*}(t) = \psi^{n*}(0, t)$, the defective distribution function $W_{k,k}(t)$ is derived as

$$\begin{aligned}
W_{k,k}(t) &= \sum_{n=1}^{\infty} (\psi(0)(1 - e^{-\alpha k})^{n-1} (\psi(0) - \psi(0)(1 - e^{-\alpha k})\psi(0, t)\psi(0)^{n-1} \frac{\psi(0, t)}{\psi(0)}), \\
&= \sum_{n=1}^{\infty} \psi(0)^{n-1} (1 - e^{-\alpha k n-1}) (\psi(0)e^{-\alpha k}) \left(\frac{\psi(0, t)}{\psi(0)} \right)^{n-1} \frac{\psi(0, t)}{\psi(0)}, \\
&= \sum_{n=1}^{\infty} \psi(0)^n e^{-\alpha k} (1 - e^{-\alpha k})^{n-1} \left(\frac{\psi(0, t)}{\psi(0)} \right)^n, \\
&= \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^{n-1} e^{-\alpha k} \psi^{n*}(0, t). \tag{6.19}
\end{aligned}$$

where D^{n*} and ψ^{n*} represent the n-fold convolution of the functions D and ψ , respectively.

Next, consider the situation of $u > k$. It is straightforward to find $D_1(t)$ and $D_2(t)$ as

$$\begin{aligned}
D_1(t) &= \frac{W(u - k, k, t)}{G(u - k, k)}, \\
&= \frac{\psi(u - k, t)(1 - e^{-\alpha k})}{\psi(u - k)(1 - e^{-\alpha k})}, \\
&= \frac{\psi(u - k, t)}{\psi(u - k)}.
\end{aligned}$$

and

$$\begin{aligned}
D_2(t) &= \frac{\psi(u - k, t) - W(u - k, k, t)}{\psi(u - k) - G(u - k, k)}, \\
&= \frac{\psi(u - k, t) - \psi(u - k, t)(1 - e^{-\alpha k})}{\psi(u - k) - \psi(u - k)(1 - e^{-\alpha k})}, \\
&= \frac{\psi(u - k, t)(e^{-\alpha k})}{\psi(u - k)(e^{-\alpha k})}, \\
&= \frac{\psi(u - k, t)}{\psi(u - k)} = D_1(t).
\end{aligned}$$

The probability of ruin by time t with a deficit of at most y at ruin is obtained as

$$\begin{aligned}
W_{u,k}(t) &= [\psi(u-k) - G(u-k, k)]D_{u-k,1} + G(u-k, k)D_{u-k,2} * W_{k,k}(t) \\
&= \psi(u-k, t)e^{-\alpha k} + \psi(u-k, t)(1 - e^{-\alpha k}) * W_{k,k}(t) \\
&= e^{-\alpha k}\psi(u-k, t) + \psi(u-k, t)(1 - e^{-\alpha k}) \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^{n-1} e^{-\alpha k} \psi^{n*}(0, t) \\
&= e^{-\alpha k}\psi(u-k, t) + \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^n e^{-\alpha k} \psi^{n*}(0, t) \psi(u-k, t) \tag{6.20}
\end{aligned}$$

The defective probability density function of the time to ruin (the density of $T_{u,k}$) is obtained as

$$w_{k,k}(t) = \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^{n-1} e^{-\alpha k} w_0^{n*}(t),$$

where $w_u(t) = \frac{d}{dt}\psi(u, t)$.

Hence, the defective probability density function of the time to ruin is

$$w_{u,k}(t) = e^{-\alpha k}w(u-k, t) + \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^n e^{-\alpha k} w^{n*}(0) * w_{u-k}(t).$$

where $w^{n*}(0) * w_{u-k}(t)$ is the convolution of functions $w_{u,k}(t)$ and $w^{n*}(0)(t)$.

$$w_{u,k}(t) = e^{-\alpha k}w(u-k, t) + \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^n e^{-\alpha k} \int_0^t w^{n*}(0, t-s)w(u-k, s)ds.$$

6.5. Calculation of Reinsurance Premium

Nie et. al [1] define the reinsurance premium as “the expected total claim amount for the reinsurer up to the time of ruin”. Nie et. al [1] assume that the aggregate amount needed to restore the modified surplus process to k up to time t , given initial surplus u , be $S_{u,k}(t)$. Thus, $S_{u,k} = S_{u,k}(T_{u,k})$ is the expected total claim amount for the reinsurer up to the time of ruin. It is started with the case $u = k$, $E(S_{u,k})$ is calculated by using the idea of Pafumi [134] as given below.

When $u = k$, the expected total claim amounts by the reinsurer is obtained as

$$\begin{aligned}
 E(S_{k,k}) &= \int_0^k (y + E(S_{k,k}))g(0, y)dy, \\
 &= yg(0, y)dy + E(S_{k,k})G(0, k), \\
 &= \frac{\int_0^k yg(0, y)dy}{1 - G(0, k)}.
 \end{aligned} \tag{6.21}$$

Recall from Equation (6.6) and (6.7), $E(S_{k,k})$ can be calculated.

When $u > k$, $E(S_{u,k})$ is calculated by using the same way,

$$\begin{aligned}
 E(S_{u,k}) &= \int_0^k (y + E(S_{k,k}))g(0, y)dy \\
 &= \int_0^k y g(u - k, y)dy + E(S_{k,k})G(u - k, k).
 \end{aligned} \tag{6.22}$$

where $E[S_{k,k}]$ is obtained as in Equation (6.21).

6.5.1. Calculation of Insurer's Net Premium Income Per Unit Time

We assume that the insurer's net premium income per unit time is calculated by the difference between the premium that the insurer receives to cover the risk and reinsurance premium to cover its liability. In calculation of the net premium income, the expected value premium principle is employed.

Insurance company determines a fixed loading factor on the expected premium principle. Therefore, the insurance premium is calculated as the multiplied of expected payments to the insured by a loading factor. Reinsurance company also determines a fixed loading factor to afford its liability according to a premium assumption. We assume that reinsurance premium is calculated according to the same premium principle. Insurance loading factor is denoted by θ , and reinsurance loading factor is denoted by ξ . Thus, there is a decrease in the insurance premium due to the reinsurance arrangement.

The net of reinsurance premium income after paying the reinsurance premium is calculated as

$$\text{Net of Reinsurance Premium} = (1 + \theta) E[X] E[N] - (1 + \xi) E(S_{u,k}) \quad (6.23)$$

where $E[S_{u,k}]$ denotes the reinsurance premium, by initial surplus u and barrier level k . The net of reinsurance is denoted by c_{net} .

As a consequence of the reinsurance arrangement, net of reinsurance premium income equals to multiplying the net liability of insurance company by a net of loading factor, θ_{net} . Net of loading factor θ_{net} is clarified the insurance's loading factor after reinsurance arrangement.

$$\begin{aligned} c_{net} &= (1 + \theta_{net})(E[S] - E(S_{u,k})), \\ \frac{c_{net}}{(E[S] - E(S_{u,k}))} &= (1 + \theta_{net}), \\ \theta_{net} &= \frac{c_{net}}{(E[S] - E(S_{u,k}))} - 1. \end{aligned} \quad (6.24)$$

It should be noted that, in the rest of the calculation, θ_{net} is used instead of θ as an insurance loading factor. In this way, the effect of reinsurance can be incorporated into the finite time ruin probability on lower barrier model.

6.6. Interim Conclusion: Lower Barrier Model on Modified Surplus Process

In this chapter, we have focused on the modified surplus process on lower barrier model. Firstly, we have discussed the estimation of the severity of ruin which is the main component of the lower barrier model under the translated gamma process approximation. As a result of our investigation, we have realized that it is not possible to get the severity of ruin under the translated gamma process approximation. Hence, we have adopted the calculation on the exact ruin probabilities.

Then, we have showed the calculation of the ultimate ruin probability on lower barrier model for two cases: $u = k$ and $u > k$. Then, we have introduced how the finite time ruin probability is calculated on lower barrier model. The finite time ruin probability is related to the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u . The distribution function of time to ruin, i.e finite time ruin probability on lower barrier model is showed for the exponential case.

On lower barrier model, reinsurance premium is calculated as “the aggregate amount needed to restore the modified surplus process to level k up to time t with initial surplus u ”. We have added the reinsurance premium into the calculation of the finite time ruin probability on lower barrier model. We have developed an approach which depends on the net of reinsurance loading factor of the insurance company. Thus, we have obtained the specific formula for the net of reinsurance loading factor and we can evaluate the effect of the reinsurance on lower barrier level.

7. OPTIMAL BARRIER ON LOWER BARRIER MODEL

7.1. Introduction

In this chapter, optimal pair of initial surplus and barrier level are obtained under the constraint of a fixed ruin probability on lower barrier model. The layout of the chapter is as follows.

In Section 7.2, we examine the finite time ruin probability on lower barrier model. We explain the relevant ruin probability formulas in the calculation of the defective distribution function of $T_{u,k}$ (the time to ruin for the process with the initial surplus u and lower barrier k). We assume that the aggregate claims process is a compound Poisson process with the exponential individual claim amounts.

In Section 7.3, we consider the optimal reinsurance criteria under the assumption of the ruin probability to be fixed at a minimum level. From the insurers' point of view, we use the optimal reinsurance criteria: released capital, expected profit and expected utility. These three criteria are calculated according to the lower barrier model assumption.

In Section 7.4, we analyse the optimal barrier level. We give some numerical examples for the compound Poisson process with individual claim amounts which have exponential distribution. We calculate the optimal initial surplus and barrier level under TOPSIS method. In the context of TOPSIS method, we use the Mahalanobis distance instead of Euclidean distance. We compare the optimal pair of initial surplus and barrier level considering different time horizons and different loading factors.

In Section 7.5, we present the sensitivity analysis of the calculation on the optimal barrier level. In this section, particular attention is paid to the weights of the criteria, the parameter of the exponential utility function, and the maximum possible value of each attribute. We measure the sensitivity of the vector-normalized covariance matrix, the original outcomes covariance matrix, and the correlation matrix by comparing the results in the weight set in which one of the criteria is dominant. In Section 7.6 we discuss the results and in Section 7.7 we conclude the chapter.

7.2. Finite Time Ruin Probability on Lower Barrier Model

In this chapter, the algorithm of the finite time ruin probability on lower barrier model is showed. The convenient formulas and approaches for the calculation of the finite time ruin probability on lower barrier model are given in detail.

The defective distribution function of $T_{u,k}$, the time to ruin, for the process with the lower barrier, with initial surplus u and lower barrier k is obtained from Equation (6.20) in Chapter 6 as

$$\begin{aligned} W_{u,k}(t) &= [\psi(u-k) - G(u-k, k)]D_{u-k,1} + G(u-k, k)D_{u-k,2} * W_{k,k}(t), \\ &= (\psi(u-k) - \psi(u-k)(1 - e^{-\alpha k})) \frac{\psi(u-k, t)}{\psi(u-k)} + \\ &\quad \psi(u-k)(1 - e^{-\alpha k}) \frac{\psi(u-k, t)}{\psi(u-k)} * W_{k,k}(t). \end{aligned}$$

where $W_{k,k}(t)$ is attained from Equation (6.19) in Chapter 6 as

$$W_{k,k}(t) = \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^{n-1} e^{-\alpha k} \psi^{n*}(0, t).$$

It is obvious that $W_{u,k}(t)$ can be derived from the fundamental functions of $\psi(u)$, $\psi(u, t)$ and $\psi(0, t)$. The detailed information about the calculation of these functions is given as follows.

7.2.1. Calculation of $\psi(u)$

In calculation of $\psi(u)$, the exact ruin probability is used as

$$\psi(u) = \psi(0) \exp\{-Ru\},$$

with $R = \alpha - \frac{\lambda}{c_{net}}$.

The net premium income, c_{net} , is obtained from Equation (6.23) as

$$c_{net} = (1 + \theta_{net}) (E[S] - E[S_{u,k}]).$$

Hence, the ultimate ruin probability is

$$\psi(u) = \frac{1}{1 + \theta} \exp \left\{ - \left(\alpha - \frac{\lambda}{c_{net}} \right) u \right\}. \quad (7.1)$$

7.2.2. Calculation of $\psi(u, t)$

Some formulas and approximations are developed to calculate the ruin probability in finite time. The detailed information about these approaches is given in Chapter 2.

We aim to calculate the exact finite time ruin probability with initial surplus u at time t . Asmussen [20] suggests an explicit formula which is valid just under the assumption that claim amounts has an exponential distribution. For this reason, we use the finite time ruin probability formula for the exponential case which is given in Equation (2.4). The detailed information about this method can be found in Section 2.6.1.

7.2.3. Calculation of $\psi(0, t)$

When initial surplus equals to zero, $u = 0$, most of the approximations are not valid. The formula for the exact survival probability is

$$1 - \psi(0, t) = \int_0^{ct} F(x, t) dx \quad (7.2)$$

where $F(x, t) = Pr(S(t) < x)$, and $S(t)$ has a compound Poisson distribution. This formula is often called Seal's formula but originating from Prabhu [31]. See Asmussen [20] for the proof of this theorem.

7.2.4. Calculation of $W_{k,k}(t)$

According to Equation (6.19), we have that

$$W_{k,k}(t) = \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^{n-1} e^{-\alpha k} \psi^{n*}(0, t).$$

$W_{k,k}(t)$ can be calculated by using the Beekman's convolution formula [24] (*see Theorem 2.4.2 in Section 2.4*).

In other words, we have

$$\psi(u) = 1 - \sum_{n=0}^{\infty} H^{*n}(u)p(1-p)^n. \quad (7.3)$$

In most practical situations, this equation is obtained by using Panjer's recursive formula. This formula requires discretization of $H(\cdot)$. In such circumstances, Beekman's formula yields approximated ruin probabilities.

Obviously, there is an affiliation between $\psi(u)$ and $W_{k,k}(t)$. Therefore, we aim to find a relationship between $\psi^{n*}(0, t)$ and $H^{*n}(u)$. Thus, we assume that individual claims have an exponential distribution with $\alpha = 1$, and number of claims have a Poisson distribution with $\lambda = 500$. In calculation of $W_{k,k}(t)$, we need to change the starting point of the lower limit of summation and n^{th} power of the expression.

The algorithm is designed by using **R** and Wolfram Mathematica programming languages.

7.3. Optimal Reinsurance Criteria on Lower Barrier Model

In Chapter 3, the released capital, expected profit and expected utility are determined as the essential optimal reinsurance criteria. In Chapter 4 and Chapter 5, we examine these criteria under the translated gamma process approximation on the classical risk model. In this section, we calculate these three criteria on lower barrier model assumption.

As is the case with classical risk model under the translated gamma process approximation, differences on pairs of insurance and reinsurance loading factors or time horizons cause significant changes in values of these criteria. Instead of the traditional reinsurance arrangements, on modified surplus process, the ruin probability is directly related to the surplus process instead of the insurer's individual or aggregate claims.

7.3.1. Released Capital

In Section 3.3.1, we define the released capital as the difference between the largest and the required initial surplus of each alternative which makes the ruin probability equal to certain level.

We suppose that the insurer's ruin probability on lower barrier model is fixed at a minimum level 0.01.

In the case of no-reinsurance, the required initial surplus which makes the insurer's ruin probability equal to 0.01 is defined as *the Largest Initial Surplus* and denoted by u_L . In reinsurance case, it is not possible to get a ruin probability such as 0.01 if the initial surplus is below a certain level. The insurer's surplus on modified surplus process must start with this adequate initial surplus called as *Smallest Initial Surplus* and denoted by u_S .

According to the reinsurance arrangement on modified surplus process, not only initial surplus but also the lower barrier level which make the ruin probability equal to 0.01 are calculated. For each barrier level, there is a unique initial surplus which satisfies 0.01 ruin probability. The difference between the largest initial surplus and this value is called as savings amounts or gains by choosing this reinsurance arrangement. We refer to this value as the released capital. Figure 7.1 shows a set of 407 pairs of initial surplus and lower barrier level based on exponentially distributed individual claim amounts.

As can be seen from Figure 7.1, initial surplus and lower barrier level inversely related. Hence, when the initial surplus increases, the corresponding barrier level which make the ruin probability equal to 0.01 decreases.

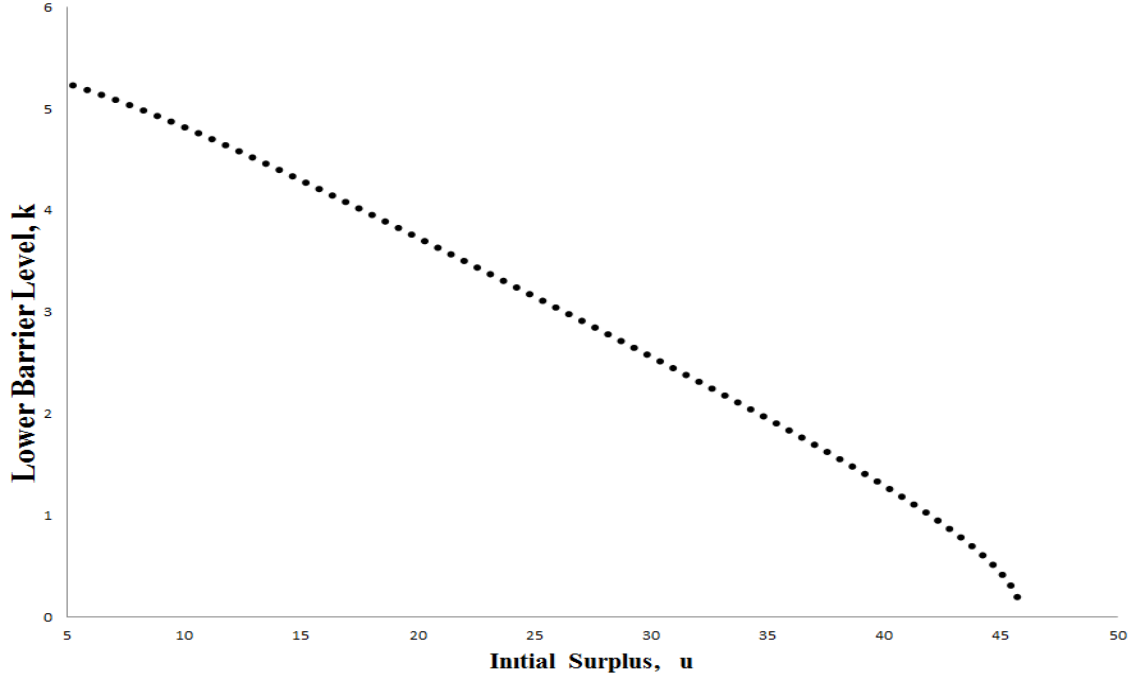


Figure 7.1: The relation between the initial surplus and lower barrier level

$\psi(u, k, t)$ represents the finite time ruin probability with initial surplus u and lower barrier k at time t . The pair of initial surplus and lower barrier level is represented as follows.

$$\begin{aligned} \psi(u_1, k_1, t) &= 0.01, \\ &\vdots \\ \psi(u_n, k_n, t) &= 0.01. \end{aligned}$$

where k_n denotes the lower barrier level and u_n denotes the initial surplus corresponding to each lower barrier level.

Due to the relationship between the initial surplus and lower barrier level, it can be observed that when the initial surplus increases, the corresponding barrier level decreases. A set of initial surplus starts with the smallest initial surplus and ends with the largest initial surplus. At the beginning of each set of alternatives, the barrier level is very close to the smallest initial surplus. However it should be noted that $u > k$ constraint must be hold. Released capital is maximum at the smallest initial surplus level whereas it is minimum in fact close to zero, at the largest initial surplus. The same pattern is observed under the translated gamma process on the classical risk model.

7.3.2. Expected Profit

Insurance profit which is calculated as the difference between the net insurance premium income and the insurer's expected total claim at time t , $P(t)$, is defined as

$$P(t) = \Pi_{net}(t) - S_I(t).$$

Under the lower barrier model on modified surplus process, according to the expected premium principle with insurance loading factor θ and reinsurance loading factor ξ ; insurer's expected profit up to time t is

$$P(t) = ((1 + \theta) E[S] - (1 + \xi) E[S_{u,k}]) - E[S_I], \quad (7.4)$$

where $E[S]$ denotes the aggregate claim amount. $E[S_{u,k}]$ is the expected total claim amount paid by the reinsurer up to the time of ruin and it is calculated using Equation (6.22) in Section 6. The net of expected claim amount paid by the insurer is represented by $E[S_I]$. Hence, it is assumed that

$$E[S] = E[S_{u,k}] + E[S_I]$$

Contrary to the classical risk model under the translated gamma process approximation, the expected profit depends on not only the lower barrier level but also its corresponding initial surplus.

A decrease in the lower barrier level causes a decrease in the transferred risk to the reinsurance company. Hence, the reinsurance premium decreases. Therefore, the net insurance premium income increases and eventually the expected profit of the insurance company increases.

The results show that under the fixed minimum ruin probability constraint on lower barrier model, there is a positive correlation between the lower barrier level and the expected profit of the insurance company up to time t .

7.3.3. Expected Utility

The exponential utility function, $\mathbf{u}$ is defined as

$$\mathbf{u}(x) = 1 - \exp(-Bx), \quad B > 0$$

where B is the parameter of the utility function. From the insurer's point of view, the insurer's wealth at the end of t , $U(t)$, is defined as

$$U(t) = u + \Pi_{net} - S_I(t).$$

where Π_{net} denotes the net of reinsurance, premium income and $S_I(t)$ denotes the net of insurance liability.

The net premium income, Π_{net} , is calculated as the difference between the whole premium income and reinsurance premium. According to the expected premium principle, the net premium income on lower barrier model is calculated as

$$\Pi_{net} = (1 + \theta)E[S] - (1 + \xi)E[S_{u,k}].$$

Therefore, the net of insurance liability on lower barrier model is calculated as the difference between the expected total aggregate claim and the expected total claim amounts for the reinsurer up to time of ruin.

To avoid the confusion of the notations, initial surplus is represented by u_0 instead of u . The expected utility of insurance's wealth at time t is denoted as

$$\begin{aligned} E[\mathbf{u}(U(t))] &= E[\mathbf{u}(u_0 + \Pi_{net} - S_I(t))] , \\ &= E[1 - \exp(-B(u_0 + \Pi_{net} - S_I(t)))] , \\ &= 1 - [E[\exp(-B u_0)] \times E[\exp(-B \Pi_{net})] \times E[\exp(B S_I(t))]] . \end{aligned} \quad (7.5)$$

In this way, we obtain the expected utility of the insurer's wealth on lower barrier model. As it is clear from Equation (7.5), the expected utility criterion depends on both the lower barrier and the corresponding initial surplus.

A decrease in the lower barrier level leads an increase in the expected utility of the insurer's wealth.

The calculation shows that the expected utility of the insurer's wealth increases exponentially as the lower barrier level decreases. The slope gradient of expected utility mainly depends on the risk averse coefficient, i.e. parameter of the exponential utility function. In this study, we firstly assume that the parameter of the exponential utility function is equal to 0.02 as in Section 5. In sensitivity analysis, we examine the effect of the optimal levels according to the different values of the exponential utility parameter. Figure 7.2 presents three criteria for the pair of initial surplus and lower barrier level

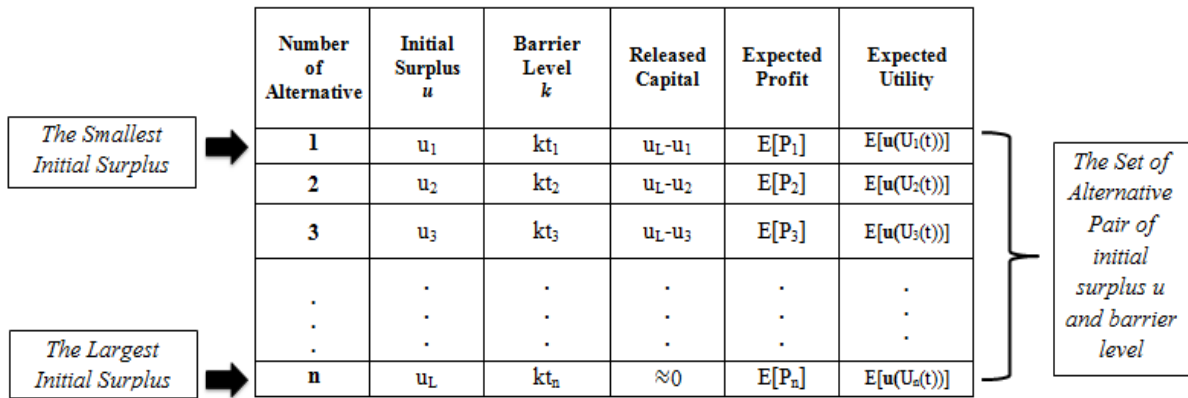


Figure 7.2: The representation of the released capital, expected profit and expected utility for the pair of initial surplus and lower barrier level

In order to obtain the optimal initial surplus and lower barrier level, we use the TOPSIS method with Mahalanobis distance. The detailed information about this method can be found in Chapter 4.

7.4. Analysis of Optimal Lower Barrier Level

In this section, we consider some numerical examples for the optimal barrier on lower barrier model. We start by investigating main assumptions of the calculation. Some of the main assumptions of this method are given as follows.

- The individual claim amount has an exponential distribution with probability density function $p(x) = e^{-x}$,
- For the number of claims per unit of time, we assume a Poisson distribution with parameter $\lambda = 500$.
- Four different loading combinations (θ, ξ) , namely $(0.1, 0.15)$, $(0.1, 0.2)$, $(0.1, 0.3)$ and $(0.2, 0.3)$ are employed.
- For time horizons, $t = 0.1, t = 0.5, t = 1, t = 5, t = 10$ and $t = 20$ are used.
- Four different sets of weights; equal weights and three set of weights under which one criterion is more important than the others are assumed. The sets of weights are given in Table 5.1.

We assume that the individual claim amount has the exponential distribution since the exponential claims provides practical results in the calculation of the finite time ruin probability on lower barrier model.

A set of alternatives which consists of pairs of initial surplus u , and the lower barrier level are calculated under the ruin probability constraint. By using these pairs, the optimal reinsurance criteria-the released capital, the expected profit and the expected utility-are calculated. The algorithm for the exponential claims on lower barrier model can be summarised as follows.

Step 1: *Calculation of the finite time ruin probability on lower barrier model*

Let $W_{u,k}(t)$ be the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u and it is defined in Equation (6.20) as

$$W_{u,k}(t) = e^{-\alpha k} \psi(u - k, t) + \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^n e^{-\alpha k} \psi^{n*}(0, t) \psi(u - k, t)$$

First, $\psi(u, t)$ is calculated by using the exact ruin probability formula which is given by Equation (2.4) [20]. Second, $\psi(0, t)$ is calculated by using Prabhu's formula [31] which is given in Equation (7.2). At last, $\psi^{n*}(0, t)$ is calculated by using Beekman's convolution formula [24] which is given by Equation (7.3).

Step 2: *Calculation of the largest initial surplus under the finite time ruin probability on lower barrier model which is fixed at a minimum level such as 0.01.*

Largest initial surplus is calculated by using Equation (6.20). The largest initial surplus which makes $W_{u,k}(t)$ formula equal to 0.01 without a lower barrier level is calculated. The pair of largest initial surplus and zero barrier level make the finite time ruin probability equal to 0.01. Hence, the largest initial surplus depends on θ and t . Assuming individual claim has an exponential distribution, the largest initial surplus according to two different insurance loading factors $\theta = 0.1$ and $\theta = 0.2$ for various values of t is given in Table 7.1.

Table 7.1: Largest initial surplus, u_L , for the exponential claims

t	$\theta = 0.1$	$\theta = 0.2$
0.1	23.895	20.188
0.5	39.939	26.118
1	45.826	26.515
5	49.599	26.537
10	49.608	26.537
20	49.608	26.537

Moreover, we can obtain the same largest initial surplus values for each loading pairs and time horizons according to the exact finite time ruin probability formula for the exponential case which is given in Section 2, Equation (2.4). It should also be noted that these largest initial surpluses are very close to the values which are obtained under the translated gamma process approximation in Table (5.2).

Step 3: *Calculation of the smallest initial surplus under the ruin probability which is fixed at a minimum level such as 0.01*

The smallest initial surplus which makes the ruin probability equal to 0.01 in Equation (6.20) is calculated by using the one dimensional optimization technique. Then, the corresponding lower barrier level k to the smallest initial surplus is calculated.

In calculation of the smallest initial surplus u_S , and the lower barrier level k , four different loading combinations $\theta = 0.1 \ \& \ \xi = 0.15$, $\theta = 0.1 \ \& \ \xi = 0.2$, $\theta = 0.1 \ \& \ \xi = 0.3$ and $\theta = 0.2 \ \& \ \xi = 0.3$ for various values of t are used. These loading factors are the same as in Dickson and Waters [44]. For the exponential claim amounts, the smallest initial surplus for various values of t and loading factors are given in Table 7.2. These smallest initial surplus are used as a starting point of the alternative set.

Table 7.2: The smallest initial surplus, u_S , for the exponential claims on lower barrier model

t	$\theta = 0.1 \ \& \ \xi = 0.15$	$\theta = 0.1 \ \& \ \xi = 0.2$	$\theta = 0.1 \ \& \ \xi = 0.3$	$\theta = 0.2 \ \& \ \xi = 0.3$
0.1	4.485	4.486	4.487	4.420
0.5	4.526	4.526	4.528	4.442
1	5.227	5.228	5.230	5.137
5	6.334	6.335	6.339	6.240
10	7.030	7.031	7.035	6.937
20	7.683	7.685	7.689	7.592

Step 4: *Constitute the alternative set which consists of the pair of initial surplus and lower barrier level.*

We design a set of alternatives which consists of the pair of initial surplus and lower barrier level. We begin with the smallest initial surplus u_S and increase this value by 0.1 or 0.05 to the largest initial surplus u_L . We calculate the corresponding lower barrier level for each initial surplus. Hence we obtain an outcome set which represents all possible combinations of the lower barrier level and corresponding initial surplus. Each pair of this outcome set is denoted by (u_k, k) . The set of u_k starts with the smallest initial surplus u_S and it ends with the largest initial surplus u_L .

Step 5: *Calculation of the released capital, expected profit and expected utility according to the initial surplus and lower barrier level*

The set of (u_k, k) is used in the optimal reinsurance criteria. In this equation, released capital is calculated as $u_L - u_k$, expected profit is calculated by Equation (7.4) and expected utility is calculated by Equation (7.5). Instead of optimal reinsurance level which depends on the classical risk model, both initial surplus and lower barrier level play vital roles in the calculation of each of three criteria under the modified surplus process. Therefore, the expected total claim amount paid by the reinsurance is calculated by using both the lower barrier level and the initial surplus.

In Table 7.3, we give the results for the three criteria when $\theta = 0.1$ and $\xi = 0.3$ on the lower barrier model

We extend this calculation for various time horizons and different loading factors. Finally, we have a large set of outcome under different assumptions. We can see that each criterion has a different pattern. When the initial surplus increases, its corresponding lower barrier level decreases in order to get a fixed ruin probability such as 0.01. When the initial surplus increases and the corresponding barrier level decreases, the released capital decreases. However, both of the expected profit and expected utility increase with different slopes. The reason is the difference between the variance of the expected profit and the expected utility.

Table 7.3: The released capital, expected profit and expected utility when $\theta = 0.1$ and $\xi = 0.3$ for the exponential claims on lower barrier model

t	Number of Alternative	Initial Surplus u_k	Lower Barrier Level k	Released Capital $u_L - u_k$	Expected Profit P	Expected Utility $E[\mathbf{u}[U(t)]]$
0.1	1	4.487	4.487	19.408	4.747	0.646
	2	4.587	4.473	19.308	4.750	0.647
	3	4.687	4.460	19.208	4.753	0.648
	4	4.787	4.446	19.108	4.756	0.649
	5	4.887	4.432	19.008	4.759	0.650
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	193	23.687	0.065	0.208	5.000	0.771
	194	23.787	0.034	0.108	5.000	0.771
	195	23.887	0.002	0.008	5.000	0.772
t	Number of Alternative	Initial Surplus u_k	Lower Barrier Level k	Released Capital $u_L - u_k$	Expected Profit P	Expected Utility $E[\mathbf{u}[U(t)]]$
0.5	1	4.528	4.528	35.411	23.727	0.646
	2	4.628	4.520	35.311	23.741	0.647
	3	4.728	4.511	35.211	23.755	0.648
	4	4.828	4.503	35.111	23.769	0.649
	5	4.928	4.494	35.011	23.783	0.650
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	353	39.728	0.032	0.211	25.000	0.834
	354	39.828	0.017	0.111	25.000	0.834
	355	39.928	0.002	0.011	25.000	0.834
t	Number of Alternative	Initial Surplus u_k	Lower Barrier Level k	Released Capital $u_L - u_k$	Expected Profit P	Expected Utility $E[\mathbf{u}[U(t)]]$
1	1	5.230	5.230	40.596	47.247	0.650
	2	5.330	5.223	40.496	47.276	0.651
	3	5.430	5.216	40.396	47.304	0.652
	4	5.530	5.208	40.296	47.331	0.653
	5	5.630	5.201	40.196	47.359	0.653
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	405	45.630	0.239	0.196	50.000	0.852
	406	45.730	0.190	0.096	50.000	0.853
	407	45.830	0.115	0.000	50.000	0.853
t	Number of Alternative	Initial Surplus u_k	Lower Barrier Level k	Released Capital $u_L - u_k$	Expected Profit P	Expected Utility $E[\mathbf{u}[U(t)]]$
5	1	6.339	6.339	43.260	235.452	0.657
	2	6.439	6.332	43.160	235.594	0.657
	3	6.539	6.326	43.060	235.735	0.658
	4	6.639	6.320	42.960	235.874	0.659
	5	6.739	6.314	42.860	236.013	0.660
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	431	49.339	1.240	0.260	249.983	0.863
	432	49.439	1.205	0.160	249.984	0.863
	433	49.539	1.167	0.060	249.985	0.863
t	Number of Alternative	Initial Surplus u_k	Lower Barrier Level k	Released Capital $u_L - u_k$	Expected Profit P	Expected Utility $E[\mathbf{u}[U(t)]]$
10	1	7.035	7.035	42.573	470.472	0.661
	2	7.135	7.029	42.473	470.757	0.662
	3	7.235	7.023	42.373	471.040	0.663
	4	7.335	7.018	42.273	471.320	0.664
	5	7.435	7.012	42.173	471.597	0.664
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	424	49.335	2.060	0.273	499.891	0.863
	425	49.435	2.032	0.173	499.895	0.863
	426	49.535	2.002	0.073	499.899	0.863
t	Number of Alternative	Initial Surplus u_k	Lower Barrier Level k	Released Capital $u_L - u_k$	Expected Profit P	Expected Utility $E[\mathbf{u}[U(t)]]$
20	1	7.689	7.689	41.920	940.511	0.665
	2	7.789	7.683	41.820	941.081	0.666
	3	7.889	7.678	41.720	941.646	0.667
	4	7.989	7.672	41.620	942.207	0.668
	5	8.089	7.666	41.520	942.762	0.669
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	418	49.389	2.838	0.220	999.574	0.863
	419	49.489	2.814	0.120	999.585	0.863
	420	49.589	2.789	0.020	999.595	0.863

Figure 7.3 shows the graph of the three criteria when $\theta = 0.1$ & $\xi = 0.3$ for the exponential claims on lower barrier model. The x axis of this graph shows the number of alternative pair (u_k, k) while the results of three criteria appear on the y axis. We prefer to use a plot with three y -axes and one shared x -axis because of the different scale. While the main y -axis shows the released capital, second and third y -axes show the expected profit and expected utility, respectively.

It is clearly seen that the released capital declines steadily to the level zero when the initial surplus closes to u_L . The expected profit increases with a decreasing slope. The first alternative begins with the maximum released capital but minimum expected profit and minimum expected utility. Conversely, the last alternative has the maximum expected profit and expected utility but minimum released capital.

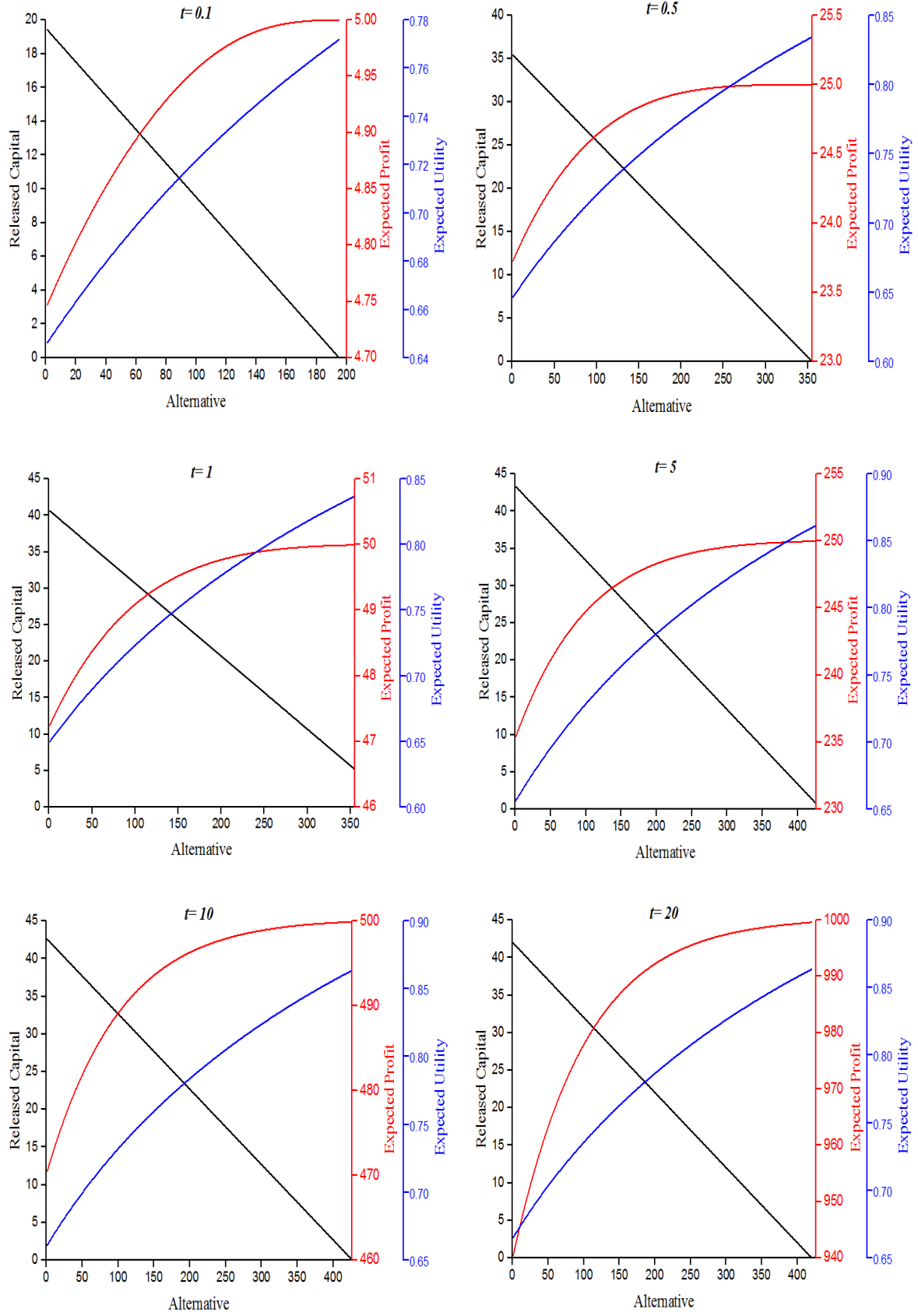


Figure 7.3: The graph of released capital, expected profit and expected utility when $\theta = 0.1$ and $\xi = 0.3$ on lower barrier model

Step 6: *Deciding the optimal pair of initial surplus and lower barrier level under the TOPSIS method with Mahalanobis distance*

In order to decide the optimal pair of initial surplus and lower barrier level (u_k, k) , we use the TOPSIS method with Mahalanobis distance. In the TOPSIS method with Mahalanobis distance, we investigate the relationship between the criteria by using the vector-normalized covariance matrix, the original covariance matrix and the correlation matrix. For these three approaches, the optimal pairs (u_k, k) are obtained. Relative proximity to the ideal solution for each alternative is used to decide the optimal pairs (u_k, k) .

Alternatives are not on the same scale as shown by Figure 7.3. In addition, the variance of the outcomes have an important role. The released capital criteria has the highest variance of outcomes. For the case of $t = 1$, $\theta = 0.1$ and $\xi = 0.3$, the variance of the released capital, expected profit and expected utility are 138.380, 0.526 and 0.003, respectively.

We use the vector-normalization technique to reduce the variance and convert criteria into a comparable form. In the covariance matrix of the vector-normalized outcomes, the values are smaller than the original outcomes. To compare the three criteria on the same scale, we use the correlation matrix. In the correlation matrix, values are formed as scale-variance.

The covariance and the correlation matrix of the vector-normalized and the original criteria outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ are given below respectively.

The covariance matrix of the original outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ is

	Released Capital	Expected Profit	Expected Utility
Released Capital	138.380	-7.457	-0.668
Expected Profit	-7.457	0.526	0.039
Expected Utility	-0.668	0.039	0.003

The highest variance is observed in the released capital. The variance of the expected profit is higher than the variance of the expected utility.

The covariance matrix of the vector-normalized outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ is

$$\begin{array}{c}
 \text{Released Capital} \quad \text{Expected Profit} \quad \text{Expected Utility} \\
 \begin{array}{c}
 \text{Released Capital} \\
 \text{Expected Profit} \\
 \text{Expected Utility}
 \end{array}
 \begin{bmatrix}
 0.000618212 & -0.000015804 & -0.000090619 \\
 -0.000015804 & 0.000000529 & 0.000002496 \\
 -0.000090619 & 0.000002496 & 0.000013547
 \end{bmatrix}
 \end{array}$$

The highest variance is still observed in the released capital. Contrary to the vector-normalized covariance, the variance of the expected utility is higher than the variance of the expected profit in the original covariance matrix.

The correlation matrix of the vector-normalized or original outcomes when $t = 1$, $\theta = 0.1$ and $\xi = 0.3$ is

$$\begin{array}{c}
 \text{Released Capital} \quad \text{Expected Profit} \quad \text{Expected Utility} \\
 \begin{array}{c}
 \text{Released Capital} \\
 \text{Expected Profit} \\
 \text{Expected Utility}
 \end{array}
 \begin{bmatrix}
 1 & -0.874 & -0.990 \\
 -0.874 & 1 & 0.933 \\
 -0.990 & 0.933 & 1
 \end{bmatrix}
 \end{array}$$

We can see that the covariance matrix of the released capital, expected profit and expected utility outcomes in the original case has much higher variation than the others. It can be clearly seen from the covariance matrix, the relationship between the released capital and the pair of expected profit and expected utility is negative while the relationship between the expected profit and expected utility is positive.

Moreover the variance of each criteria increases when the time horizon increases. The covariance and the correlation matrix of the vector-normalized and the original criteria outcomes when $t = 10$, $\theta = 0.1$ and $\xi = 0.3$ are given below.

The covariance matrix of the original outcomes when $t = 10$, $\theta = 0.1$ and $\xi = 0.3$ is

	Released Capital	Expected Profit	Expected Utility
Released Capital	151.585	-83.686	-0.695
Expected Profit	-83.686	58.637	0.412
Expected Utility	-0.695	0.412	0.003

The highest variance is observed in the released capital. The variance of the expected profit is higher than the variance of the expected utility. We can see that the variance of the expected profit increases when the time horizon increases. The change in the variance of the expected utility is very small when time horizon increases.

The covariance matrix of the vector-normalized outcomes when $t = 10$, $\theta = 0.1$ and $\xi = 0.3$ is

	Released Capital	Expected Profit	Expected Utility
Released Capital	0.0005873	-0.0000162	-0.0000847
Expected Profit	-0.0000162	0.0000006	0.0000025
Expected Utility	-0.0000847	0.0000025	0.0000125

The highest variance is still observed in the released capital. Contrary to the vector-normalized covariance, the variance of the expected utility is higher than the variance of the expected profit in original covariance matrix.

The correlation matrix of the vector-normalized or original outcomes when $t = 10$, $\theta = 0.1$ and $\xi = 0.3$ is

	Released Capital	Expected Profit	Expected Utility
Released Capital	1	-0.888	-0.990
Expected Profit	-0.888	1	0.944
Expected Utility	-0.990	0.944	1

The variance of the released capital is very high and hence it has a dominant impact in the optimal barrier decision making. For this reason, we focus on both the covariance and the correlation matrices to show the fluctuation between these criteria.

On the other hand, it should be noted that we have found that the expected profit has a dominant impact on optimal reinsurance analysis in Chapter 5..

The optimal pair (u_k, k) is obtained according to six different time horizons: $t=0.1, 0.5, 1, 5, 10, 20$. It should also be noted that when $t > 20$ the expected utility criteria will lose its efficiency and all outcomes of this criteria are equal to 1. Therefore, when $t > 20$, optimality only depends on two criteria: released capital and expected profit. For different sets of weights (Scenario 1-4 in Table 5.1), the optimal pair (u_k, k) which are calculated by using the vector-normalized covariance matrix, the original covariance matrix and the correlation matrix are given in Table 7.4 through Table 7.7, respectively.

Table 7.4: The optimal pairs (u_k, k) in the TOPSIS method with Mahalanobis distance for scenario 1 for the exponential claims on lower barrier model

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	9.485	3.660	23.885	0.003	4.485	4.485
	0.5	4.526	4.526	39.926	0.002	4.526	4.526
	1	5.227	5.227	45.827	0.119	5.227	5.227
	5	6.334	6.334	49.534	1.168	6.334	6.334
	10	7.030	7.030	49.530	2.002	7.030	7.030
	20	7.683	7.683	49.583	2.787	7.683	7.683
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	9.486	3.662	23.886	0.003	4.486	4.486
	0.5	4.526	4.526	39.926	0.002	4.526	4.526
	1	5.228	5.228	45.828	0.117	5.228	5.228
	5	6.335	6.335	49.335	1.168	6.335	6.335
	10	7.031	7.031	49.531	2.002	7.031	7.031
	20	7.685	7.685	49.685	2.788	7.685	7.685
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	9.387	3.685	23.887	0.002	4.487	4.487
	0.5	4.528	4.528	39.928	0.002	4.528	4.528
	1	5.230	5.230	45.830	0.115	5.230	5.230
	5	6.339	6.339	49.539	1.167	6.339	6.339
	10	7.035	7.035	49.535	2.002	7.035	7.035
	20	7.689	7.689	49.589	2.789	7.689	7.689
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.420	4.420	20.120	0.023	4.420	4.420
	0.5	4.442	4.442	26.042	0.016	4.442	4.442
	1	5.137	5.137	26.537	0.169	5.137	5.137
	5	6.240	6.240	25.640	1.885	6.240	6.240
	10	6.937	6.937	26.537	2.445	6.937	6.937
	20	7.592	7.592	26.492	3.334	7.592	7.592

In Table 7.4, it is assumed that each criterion has the same importance so that the weights are defined as $1/3$. The optimal initial surplus and lower barrier levels are obtained according to three different matrices. The results show that the optimal barrier level increases as time horizon increases. Possible explanation is when time horizons increases, the need for the reinsurance increases. Hence, the barrier level increases.

From the results that has been conducted, it is possible to conclude that the highest optimal levels are obtained according to the original covariance matrix, whereas the lowest optimal levels are obtained according to the correlation matrix and the vector-normalized covariance matrix. The reason is that the variance of three criteria is high and so the covariance matrix of the original data is higher than the other cases. The optimal pairs which are obtained by using the covariance matrix of the original data, are equal to levels in which the expected profit and expected utility are maximum.

The correlation matrix gives the optimal pairs as the starting point of the alternatives. As we can see, the optimal initial surplus equals to the smallest initial surplus when the correlation matrix is used. In this case, the optimal barrier level equals to the optimal initial surplus because each term in the subtraction has three decimal places. However, it should be noted that $u > k$.

Table 7.5: The optimal pairs (u_k, k) in the TOPSIS method with Mahalanobis distance for scenario 2 for the exponential claims on lower barrier model

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.485	4.485	23.885	0.003	4.485	4.485
	0.5	4.526	4.526	39.926	0.002	4.526	4.526
	1	5.227	5.227	45.827	0.419	5.227	5.227
	5	6.334	6.334	49.534	1.168	6.334	6.334
	10	7.030	7.030	49.330	2.002	7.03	7.03
	20	7.683	7.683	49.583	2.787	7.683	7.683
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.486	4.486	23.886	0.003	4.486	4.486
	0.5	4.526	4.526	39.926	0.002	4.526	4.526
	1	5.228	5.228	45.828	0.117	5.228	5.228
	5	6.135	6.135	49.535	1.168	6.335	6.335
	10	7.031	7.031	49.531	2.002	7.031	7.031
	20	7.685	7.685	49.585	2.788	7.685	7.685
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.487	4.487	23.887	0.002	4.487	4.487
	0.5	4.528	4.528	39.928	0.002	4.528	4.528
	1	5.230	5.230	45.830	0.115	5.230	5.230
	5	6.339	6.339	49.539	1.167	6.339	6.339
	10	7.035	7.035	49.535	2.002	7.035	7.035
	20	7.689	7.689	49.589	2.789	7.689	7.689
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.420	4.420	20.120	0.023	4.420	4.420
	0.5	4.442	4.442	26.042	0.016	4.442	4.442
	1	5.137	5.137	26.537	0.169	5.137	5.137
	5	6.240	6.240	25.640	1.885	6.240	6.240
	10	6.937	6.937	26.537	2.445	6.937	6.937
	20	7.592	7.592	26.492	3.334	7.592	7.592

In Table 7.5, Scenario 2, it is assumed that more weight is given to the released capital criterion. The optimal levels according to all matrices give equal or higher results than Scenario 1, Table 7.4. Increasing the importance of the released capital causes the optimal levels move towards to the level that the released capital has the highest value.

Table 7.6: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 3 for the exponential claims on lower barrier model

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
$\theta = 0.1$ & $\xi = 0.15$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.485	4.485	23.885	0.003	4.485	4.485
	0.5	4.526	4.526	39.926	0.002	4.526	4.526
	1	5.227	5.227	45.827	0.119	5.227	5.227
	5	6.334	6.334	49.534	1.168	6.334	6.334
	10	7.030	7.030	49.530	2.002	7.030	7.030
	20	7.683	7.683	49.583	2.787	7.683	7.683
$\theta = 0.1$ & $\xi = 0.2$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.486	4.486	23.886	0.003	4.486	4.486
	0.5	4.526	4.526	39.926	0.002	4.526	4.526
	1	5.228	5.228	45.828	0.117	5.228	5.228
	5	6.335	6.335	49.535	1.168	6.335	6.335
	10	7.031	7.031	49.521	2.002	7.031	7.031
	20	7.685	7.685	49.585	2.788	7.685	7.685
$\theta = 0.1$ & $\xi = 0.3$	t	Optimal u_M	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.487	4.487	23.887	0.002	4.487	4.487
	0.5	4.528	4.528	39.928	0.002	4.528	4.528
	1	5.230	5.230	45.830	0.115	5.230	5.230
	5	6.339	6.339	49.539	1.167	6.339	6.339
	10	7.035	7.035	49.535	2.002	7.035	7.035
	20	7.689	7.689	49.589	2.789	7.689	7.689
$\theta = 0.2$ & $\xi = 0.3$	t	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	0.1	4.420	4.420	20.120	0.023	4.420	4.420
	0.5	4.442	4.442	26.042	0.016	4.442	4.442
	1	5.137	5.137	26.537	0.169	5.137	5.137
	5	6.240	6.240	25.640	1.885	6.240	6.2040
	10	6.937	6.937	26.537	2.445	6.937	6.937
	20	7.592	7.592	26.492	3.334	7.592	7.592

In Table 7.6, Scenario 3, it is assumed that more weight is given to the expected profit. According to the original covariance matrix, the optimal levels are obtained as the largest initial surplus and the lowest barrier level. The variance of the expected profit has an important role in the original covariance matrix. In the vector-normalized covariance matrix, the optimal levels do not change according to Scenario 2, Table 7.5. The correlation matrix gives the same optimal level for Scenario 1, Scenario 2 and Scenario 3.

The highest and the lowest optimal levels are obtained according to the vector-normalized and the original covariance matrix, respectively, as in Table 7.4 and Table 7.5. The optimal barrier levels according to all matrices give higher or equal results than under Scenario 1.

We can see that the changes in the optimal levels for the original covariance matrix and the correlation matrix are not so large. One possible reason of it is the released capital has a dominant impact as these outcomes have the largest variance. Any significant differences cannot be observed according to the original covariance matrix with Scenario 1 and 2.

Table 7.7: The optimal pairs (u_M, M) in the TOPSIS method with Mahalanobis distance for Scenario 4 for the exponential claims on lower barrier model

Loading Factors	Time	The Vector-normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
		Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
$\theta = 0.1$ & $\xi = 0.15$	t						
	0.1	23.885	0.003	23.885	0.003	4.485	4.485
	0.5	39.926	0.002	39.926	0.002	4.526	4.526
	1	45.827	0.119	45.827	0.119	5.227	5.227
	5	49.534	1.168	49.534	1.168	6.334	6.334
	10	49.530	2.002	49.530	2.002	7.030	7.030
	20	49.583	2.787	49.583	2.787	49.583	2.787
$\theta = 0.1$ & $\xi = 0.2$	t						
	0.1	23.886	0.003	23.886	0.003	4.486	4.486
	0.5	39.926	0.002	39.926	0.002	4.526	4.526
	1	45.828	0.117	45.828	0.117	5.228	5.228
	5	49.535	1.168	49.535	1.168	6.335	6.335
	10	49.531	2.002	49.531	2.002	7.031	7.031
	20	49.535	2.788	49.535	2.788	7.685	7.685
$\theta = 0.1$ & $\xi = 0.3$	t						
	0.1	23.887	0.002	23.887	0.002	4.487	4.487
	0.5	39.928	0.002	39.928	0.002	4.528	4.528
	1	45.830	0.115	45.830	0.115	5.230	5.230
	5	49.539	1.167	49.539	1.167	6.339	6.339
	10	49.535	2.002	49.535	2.002	7.035	7.035
	20	49.589	2.789	49.589	2.789	7.689	7.689
$\theta = 0.2$ & $\xi = 0.3$	t						
	0.1	20.120	0.023	20.120	0.023	4.420	4.420
	0.5	26.042	0.016	26.042	0.016	4.442	4.442
	1	26.537	0.169	26.537	0.169	5.137	5.137
	5	25.640	1.885	25.640	1.885	6.240	6.240
	10	26.537	2.445	26.537	2.445	6.937	6.937
	20	26.492	3.334	26.492	3.334	7.592	7.592

In Table 7.7 under Scenario 4, it is assumed that more weight is given to the expected utility. According to the vector-normalized covariance matrix, the variance of the expected utility is higher than the expected profit, hence the effect of the expected utility on the optimal levels can be observed easily. The highest optimal initial surplus and the lowest initial barrier level are observed in the scenarios according to the vector-normalized covariance matrix. The optimal levels are the same as the other scenarios for the correlation matrix.

The lowest and the highest barrier levels are obtained according to the vector-normalized and the original covariance matrix, respectively, in Table 7.4, Table 7.5 and Table 7.6.

7.5. Sensitivity Analysis on Optimal Barrier Level

In this section, we investigate the results by changing one or more parameters. This analysis provides the “robustness” of the final decision of the optimal initial surplus and lower barrier level. We examine the differences on the optimal levels by changing the weights of each criteria and the parameter of the utility function. Beyond this analysis, the relationship between optimal levels and maximum possible value of each criterion are also investigated.

7.5.1. Effect of the Changes in the Weights of the Criteria

Criteria weights reflect the importance of the criteria. However, the weights of the criteria can cause some uncertainties. An effective method is comparison weights of criteria and observe the differences. The simplest way to observe the changes in the optimal levels is to allow only one single criterion vary while the others keep constant. We investigate two different approaches to measure the effect of the changes in the weights of the criteria. First, we assume that the weight of one criterion is increased from 0 to 1 step by step, while the other criteria have same importancy. The increase is continued until the all possible ranking are done. We observe the differences on the optimal pair of initial surplus and lower barrier level according to the changes in the weights of the criteria.

Second, we assume that the weight of one criterion is increased from 0 to 1 step by step while the other criteria have a relationship such as twofold of each other. Similarly, weights of other criteria are increased to find the optimal initial surplus and lower barrier level. Therefore, we can observe the general sensitivity trend under different assumptions.

The optimal pair of initial surplus and barrier level are calculated when the weight of the released capital, expected profit and expected utility changes between 0 and 1. In comparison of the optimal levels according to different criteria, we use the vector normalized covariance matrix.

The optimal pairs which are obtained based on the released capital are given in Table 7.8-Table 7.13 under the exponential individual claim on lower barrier model.

Table 7.8: Changes in weights of the released capital for the exponential claims on lower barrier model when $t = 0.1$

t=0.1			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*
0	0.5	0.5	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.0025	0.49875	0.49875	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.0050	0.4975	0.4975	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.01	0.495	0.495	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.05	0.475	0.475	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.1	0.45	0.45	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.15	0.425	0.425	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.2	0.4	0.4	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.25	0.375	0.375	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.3	0.3500	0.3500	23.885	0.003	23.886	0.003	23.887	0.002	20.120	0.023
0.333	0.333	0.333	21.585	0.702	9.586	3.643	9.487	3.666	4.420	4.420
0.4	0.3	0.3	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.41	0.295	0.295	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.42	0.29	0.29	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.43	0.285	0.285	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.45	0.275	0.275	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.5	0.2500	0.2500	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.55	0.225	0.225	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.6	0.2	0.2	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.65	0.175	0.175	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.7	0.15	0.15	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.75	0.125	0.125	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.8	0.1	0.1	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.85	0.075	0.075	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.9	0.1	0.1	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
0.95	0.025	0.025	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420
1	0	0	4.485	4.485	4.486	4.486	4.487	4.487	4.420	4.420

Table 7.8 presents the optimal pair of initial surplus and lower barrier level according to the weight of the released capital which changes between 0 and 1 when $t = 0.1$. It can be seen that the optimal levels are stable until the weight of the released capital is equal to 1/3. At this point a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes an increase in optimal barrier levels and a decrease in initial surplus.

Table 7.9: Changes in weights of the released capital for the exponential claims on lower barrier model when $t = 0.5$

t=0.5			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*
0	0.5	0.5	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.0025	0.49875	0.49875	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.0050	0.4975	0.4975	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.01	0.495	0.495	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.05	0.475	0.475	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.1	0.45	0.45	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.15	0.425	0.425	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.2	0.4	0.4	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.25	0.375	0.375	39.926	0.002	39.926	0.002	39.928	0.002	26.042	0.016
0.3	0.3500	0.3500	33.926	0.883	34.026	0.869	34.328	0.825	23.342	0.577
0.333	0.333	0.333	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.4	0.3	0.3	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.41	0.295	0.295	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.42	0.29	0.29	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.43	0.285	0.285	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.45	0.275	0.275	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.5	0.2500	0.2500	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.55	0.225	0.225	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.6	0.2	0.2	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.65	0.175	0.175	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.7	0.15	0.15	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.75	0.125	0.125	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.8	0.1	0.1	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.85	0.075	0.075	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.9	0.1	0.1	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
0.95	0.025	0.025	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442
1	0	0	4.526	4.526	4.526	4.526	4.528	4.528	4.442	4.442

Table 7.9 presents the optimal pair of initial surplus and retention level according to the weight of the released capital which changes between 0 and 1 when $t = 0.5$. Table 7.9 indicates that the increase in the weight of the released capital causes an increase in optimal barrier levels and a decrease in initial surplus. It can be seen that the optimal levels are stable until the weight of the released capital is equal to 1/3.

Table 7.10: Changes in weights of the released capital for exponential claims on lower barrier model when $t = 1$

t=1			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*
0	0.5	0.5	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0025	0.49875	0.49875	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0050	0.4975	0.4975	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.01	0.495	0.495	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.05	0.475	0.475	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1	0.45	0.45	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.15	0.425	0.425	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.2	0.4	0.4	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.25	0.375	0.375	44.927	0.456	45.028	0.431	45.430	0.314	26.537	0.169
0.3	0.3500	0.3500	37.627	1.617	37.728	1.604	38.030	1.565	24.337	1.034
0.333	0.333	0.333	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.4	0.3	0.3	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.41	0.295	0.295	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.42	0.29	0.29	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.43	0.285	0.285	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.45	0.275	0.275	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.5	0.2500	0.2500	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.55	0.225	0.225	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.6	0.2	0.2	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.65	0.175	0.175	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.7	0.15	0.15	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.75	0.125	0.125	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.8	0.1	0.1	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.85	0.075	0.075	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.9	0.1	0.1	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.95	0.025	0.025	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
1	0	0	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137

Table 7.10 presents the optimal level according to the weight of the released capital between 0 and 1 when $t = 1$. It is shown that the optimal levels are stable until the weight of the released capital is equal to $1/3$. At this point a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes an increase in optimal barrier levels and a decrease in initial surplus.

Table 7.11: Changes in weights of the released capital for exponential claims on lower barrier model when $t = 5$

t=5			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*	u_{k*}	k^*
0	0.5	0.5	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.0025	0.49875	0.49875	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.0050	0.4975	0.4975	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.01	0.495	0.495	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.05	0.475	0.475	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.1	0.45	0.45	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.15	0.425	0.425	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.2	0.4	0.4	49.534	1.168	49.535	1.168	49.539	1.167	25.640	1.885
0.25	0.375	0.375	47.534	1.682	47.735	1.642	48.039	1.579	25.640	1.885
0.3	0.3500	0.3500	39.834	2.798	39.935	2.787	40.239	2.751	25.640	1.885
0.333	0.333	0.333	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.4	0.3	0.3	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.41	0.295	0.295	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.42	0.29	0.29	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.43	0.285	0.285	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.45	0.275	0.275	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.5	0.2500	0.2500	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.55	0.225	0.225	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.6	0.2	0.2	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.65	0.175	0.175	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.7	0.15	0.15	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.75	0.125	0.125	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.8	0.1	0.1	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.85	0.075	0.075	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.9	0.1	0.1	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
0.95	0.025	0.025	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240
1	0	0	6.334	6.334	6.335	6.335	6.339	6.339	6.240	6.240

Table 7.11 presents the optimal level according to the changes in weight of the released capital between 0 and 1 when $t = 5$. It is shown that the optimal levels are stable until the weight of the released capital is equal to $1/3$. At this point a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes an increase in optimal barrier levels and a decrease in initial surplus.

Table 7.12: Changes in weights of the released capital for exponential claims on lower barrier model when $t = 10$

t=10			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.5	0.5	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.0025	0.49875	0.49875	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.0050	0.4975	0.4975	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.01	0.495	0.495	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.05	0.475	0.475	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.1	0.45	0.45	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.15	0.425	0.425	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.2	0.4	0.4	49.530	2.002	49.531	2.002	49.535	2.002	26.537	2.445
0.25	0.375	0.375	47.630	2.442	47.831	2.403	48.035	2.364	26.537	2.445
0.3	0.3500	0.3500	40.030	3.546	40.131	3.535	40.335	3.513	25.137	2.926
0.333	0.333	0.333	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.4	0.3	0.3	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.41	0.295	0.295	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.42	0.29	0.29	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.93
0.43	0.285	0.285	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.45	0.275	0.275	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.5	0.2500	0.2500	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.55	0.225	0.225	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.6	0.2	0.2	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.65	0.175	0.175	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.7	0.15	0.15	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.75	0.125	0.125	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.8	0.1	0.1	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.85	0.075	0.075	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.9	0.1	0.1	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.95	0.025	0.025	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
1	0	0	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937

Table 7.12 presents the optimal level according to the changes in weight of the released capital between 0 and 1 when $t = 10$. The optimal levels decrease until the weight of the released capital is equal to $1/3$. At this point a sudden decline is observed (it is given in bold) and after that the optimal levels are constant. It is clear that the increase in the weight of the released capital causes an increase in optimal barrier levels and a decrease in initial surplus.

Table 7.13: Changes in weights of the released capital for exponential claims on lower barrier model when $t = 20$

t=20			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.5	0.5	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.0025	0.49875	0.49875	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.0050	0.4975	0.4975	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.01	0.495	0.495	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.05	0.475	0.475	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.1	0.45	0.45	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.15	0.425	0.425	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.2	0.4	0.4	49.583	2.787	49.585	2.788	49.589	2.789	26.492	3.334
0.25	0.375	0.375	47.883	3.151	47.985	3.132	48.189	3.096	26.492	3.334
0.3	0.3500	0.3500	40.283	4.247	40.385	4.236	40.589	4.215	25.392	3.682
0.333	0.333	0.333	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.4	0.3	0.3	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.41	0.295	0.295	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.42	0.29	0.29	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.43	0.285	0.285	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.45	0.275	0.275	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.5	0.2500	0.2500	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.55	0.225	0.225	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.6	0.2	0.2	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.65	0.175	0.175	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.7	0.15	0.15	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.75	0.125	0.125	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.8	0.1	0.1	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.85	0.075	0.075	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.9	0.1	0.1	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
0.95	0.025	0.025	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592
1	0	0	7.683	7.683	7.685	7.685	7.689	7.689	7.592	7.592

Table 7.13 presents the optimal level according to the weight of the released capital between 0 and 1 for $t = 20$. From this table, it can be seen that the increase in the weight of the released capital causes an increase in optimal barrier levels and a decrease in initial surplus.

More detailed examples for the changes in the weight of released capital for the exponential claims on lower barrier model can be found in Appendix B. Table B.1 presents the optimal level according to the changes in the weight of the released capital while the weight of the expected profit is two times of the weight of the expected utility. Table B.2 presents the optimal level according to changes in the weight of the released capital while the weight of the expected utility is two times of the weight of the expected profit.

More detailed examples for the changes in the weight of expected profit for the exponential claims on lower barrier model can be found in Appendix B. Table B.3 presents the optimal level according to changes in the weight of the expected profit while the weight of the released capital is equal to the weight of the expected utility. Table B.4 presents the optimal level according to changes in the weight of the expected profit while the weight of the released capital is two times of the weight of the expected utility. Table B.5 presents the optimal level according to changes in the weight of the expected profit while the weight of the expected utility is two times of the weight of the released capital.

More detailed examples for the changes in the weight of expected utility for the exponential claims on lower barrier model can be found in Appendix A. Table B.6 presents the optimal level according to changes in the weight of the expected utility while the weight of the released capital is equal to the weight of the expected profit. Table B.7 presents the optimal level according to changes in the weight of the expected utility while the weight of the released capital is two times of the weight of the expected profit. Table B.8 presents the optimal level according to changes in the weight of the expected utility while the weight of the expected profit is two times of the weight of the released capital.

Based on the results, it can be concluded that the increase in the weight of the released capital causes an increase in the optimal barrier levels. However, the increase in the weight of the expected profit or expected utility cause a decrease in the optimal barrier levels.

In this section, particular attention is paid to the measure of the importance of the weight of the criteria. An important implication of these findings is changing on the weight of each criteria causes a difference on the optimal levels. When the weight of the expected profit and expected utility are changed, the similar pattern can be observed. However, it should be noted that the increase in the released capital causes an increase in optimal barrier levels and a decrease in initial surplus whereas the increase in the weight of the expected profit or expected utility causes a decrease in optimal barrier levels and an increase in the initial surplus. The reason is that each criteria's maximum level is different.

The maximum level of the released capital is observed at the beginning of the alternative set such as maximum barrier level and the smallest initial surplus. At the same time, the maximum levels of expected profit and expected utility are observed at the end of the alternative set such as minimum barrier level and the largest initial surplus.

When the optimal pair is observed as the largest initial surplus and lowest barrier level, it means that the optimal case is the no-reinsurance case. This level is the maximum possible values of expected profit and expected utility.

Moreover, we compare the optimal pair of initial surplus and lower barrier level for exponential claims when $t = 1$ and $t = 20$, $\theta = 0.1$ and $\xi = 0.3$. We investigate the optimal levels according to three different set of weights. We also aim to show the sensitivity of the three matrices with regard to the importance of the criteria in Table 7.14.

The results which is obtained from the first set of weight (0.99, 0.005 and 0.005) are compared with Scenario 2 in Table 7.5 . We expect that the optimal initial surplus smaller and optimal barrier level higher than or equal to those under Scenario 2 because of the increase of the importance of the released capital.

The results which is obtained from the second set of weights (0.005, 0.99 and 0.005) are compared with Scenario 3 in Table 7.6 and we expect that the optimal barrier levels would be smaller and optimal initial surplus would be higher than or equal to those of Scenario 3 because of the increasing importance of expected profit.

The results which is obtained by using the third set of weights (0.005, 0.005 and 0.99) are compared with Scenario 4 in Table 7.7. We expect that the optimal barrier levels would be smaller and optimal initial surplus would be higher than or equal to those of Scenario 4, because of the increasing importance of expected utility.

Table 7.14: The optimal pairs (u_k, k) in the TOPSIS method with Mahalanobis distance for dominant weights for exponential claims on lower barrier model

Weights of Criteria 0.99 & 0.005 & 0.005	The Vector normalized Covariance Matrix		The Original Covariance Matrix		The Correlation Matrix	
	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
t=1	5.230	5.230	45.830	0.115	5.230	5.230
t=20	7.689	7.689	49.589	2.789	7.689	7.689
Weights of Criteria 0.005 & 0.99 & 0.005	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
t=1	20.230	3.708	45.830	0.115	5.230	5.230
t=20	25.989	5.911	49.589	2.789	49.589	2.789
Weights of Criteria 0.005 & 0.005 & 0.99	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
	Optimal u_k	Optimal k	Optimal u_k	Optimal k	Optimal u_k	Optimal k
t=1	45.830	0.115	45.830	0.115	45.830	0.115
t=20	49.589	2.789	49.589	2.789	49.589	2.789

These results are consistent in good agreement with Scenario 1, 2, 3 and 4 which have shown that in the vector-normalization matrix. This matrix shows the accurate pattern of the optimal pair of initial surplus and barrier level according to changes in the weights of the criteria. We cannot observe any changes in the optimal pairs with respect to the original covariance matrix although important changes in the weight of the criteria. Moreover, in the correlation matrix, the effect of the expected profit cannot be observed as good as in the vector-normalized covariance matrix.

7.5.2. Effect of the Changes in the Parameter of the Adjustment Coefficient

We also consider the consequences of the parameter B , of the exponential utility function. We show some results when the parameter B changes between 0.015 and 0.03. We increase the parameter by 0.001 to observe the changes. The following tables show the sensitivity of the optimal pair of initial surplus and barrier level according to B parameter for four different loading factors.

Table 7.15: Changes in the parameter of the exponential utility function for exponential individual claims on lower barrier model for $t = 0.1$

t=0.1	$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.35$	
B	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.015	21.285	0.790	9.486	3.662	9.487	3.666	4.420	4.420
0.016	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.017	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.018	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.019	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.0198	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.02	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.0202	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.021	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.022	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.023	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.024	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.025	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.026	9.485	3.660	9.486	3.662	9.387	3.685	4.420	4.420
0.027	9.485	3.660	9.386	3.681	9.387	3.685	4.420	4.420
0.028	9.485	3.660	9.386	3.681	9.387	3.685	4.420	4.420
0.029	9.485	3.660	9.386	3.681	9.387	3.685	4.420	4.420
0.03	9.485	3.660	9.386	3.681	9.387	3.685	4.420	4.420

Table 7.15 presents the changes in the parameter of the exponential utility function for exponential individual claims on lower barrier model for $t = 0.1$. It can be seen that the optimal barrier level and its corresponding initial surplus do not change according to the parameter of the exponential utility function.

Table 7.16: Changes in the parameter of the exponential utility function for exponential individual claims on lower barrier model for $t = 1$

t=1	$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.35$	
B	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.015	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.016	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.017	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.018	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.019	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.0198	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.02	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.0202	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.021	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.022	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.023	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.024	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.025	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.026	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.027	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.028	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.029	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.03	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137

Table 7.16 presents the changes in the parameter of the exponential utility function for exponential individual claims on lower barrier model for $t = 1$. It can be seen that the barrier level and corresponding initial surplus do not change according to the parameter of the exponential utility function.

Table 7.17: Changes in the parameter of the exponential utility function for exponential individual claims on lower barrier model for $t = 10$

t=10	$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.35$	
B	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.015	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.016	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.017	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.018	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.019	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.0198	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.02	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.0202	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.021	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.022	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.023	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.024	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.025	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.026	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.027	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.028	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.029	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937
0.03	7.030	7.030	7.031	7.031	7.035	7.035	6.937	6.937

Table 7.17 presents the changes in the parameter of the exponential utility function for exponential individual claims on lower barrier model for $t = 10$. It can be seen that the barrier level and corresponding initial surplus do not change according to the parameter of the exponential utility function.

It is obvious that the same pattern is observed for $t = 0.5$, $t = 5$, and $t = 20$.

In the previous section, the parameter of the exponential utility function has been chosen as 0.02. We test the effect of this parameter on the optimal pair of initial surplus and barrier level. For all combinations of time and loading factors, we find the same optimal pairs. The existence of these effect implies that the optimal pair of initial surplus and barrier level do not seem to be sensitive to the value of the parameter of the utility function.

7.5.3. Maximum Possible Value of Each Attribute

We give a different comparison technique with individual outcomes under each criterion. In this context, the question under discussion measures the sensibility of the optimal pairs. We choose the pair of loading factors as $\theta = 0.1$ and $\xi = 0.3$ and $t = 1$. We assume that each criterion has the same importance (Scenario 1 in Table 5.1). We calculate the optimal initial surplus and lower barrier level for three different matrices. For the pairs of initial surplus and barrier level; the released capital, expected profit and expected utility and percentage of the maximum possible values of these criteria are compared to measure the sensibility of the optimal pairs. The values are given in Table 7.18.

Table 7.18: Percentage of the maximum possible values of the released capital, expected profit and expected utility when $t = 1$ and $\theta = 0.1$ & $\xi = 0.3$ on lower barrier model

Weights of Criteria 1/3 & 1/3 & 1/3 (Scenario 1)	Optimal Initial Surplus u_k	Optimal Barrier Level k	Released Capital	Maximum Possible Released Capital	Percentage	Expected Profit	Maximum Possible Expected Profit	Percentage	Expected Utility	Maximum Possible Expected Utility	Percentage
The Vector normalized Covariance Matrix	5.230	5.230	40.596	40.596	100%	47.247	50	48.494 %	0.650	0.853	76.201 %
The Original Covariance Matrix	45.830	0.115	0	40.596	0%	50	50	100%	0.853	0.853	100%
The Correlation Matrix	5.230	5.230	40.596	40.596	100%	47.247	50	48.494 %	0.650	0.853	76.201 %

Particular attention is paid to the sensibility of the optimal pair for three different matrices. A closer look at the relationship between the outcomes indicates that the optimal levels obtained for the original covariance matrix mostly depend on the expected profit and the expected utility. We can see that the percentage of the released capital to maximum possible released capital is 0% whereas the percentage of the expected profit and expected utility to their maximum possible values are 100%.

These results provide the confirmatory evidence that the optimal pairs obtained for the original covariance matrix do not look sensible because of the high variance impact. According to the optimal levels which are obtained by using the vector-normalized covariance matrix, the values are transformed into the same scale. In this case, the percentage of the released capital, expected profit and expected utility to their possible maximum values are close to each other in comparison with the original covariance matrix.

In the correlation matrix, the optimal pair of initial surplus and barrier level are observed as in the vector-normalized covariance matrix case.

In order to obtain a more comparable form, the analysis is extended according to Scenario 1 and Scenario 2 for the combinations of different time horizons and loading factors. Table 7.19 shows the percentage of the maximum possible value of each attribute.

Table 7.19: Maximum possible value of each attribute for exponential claims on lower barrier model

		Percentages of the maximum possible values under Scenario 1			Percentages of the maximum possible values under Scenario 2		
t	Premium Loading Factors	Released Capital	Expected Profit	Expected Utility	Released Capital	Expected Profit	Expected Utility
0.1	$\theta = 0.1, \xi = 0.15$	74%	99%	90%	100%	97%	85%
	$\theta = 0.1, \xi = 0.2$	74%	98%	89%	100%	97%	84%
	$\theta = 0.1, \xi = 0.3$	75%	97%	89%	100%	95%	84%
	$\theta = 0.2, \xi = 0.3$	100%	99%	96%	100%	99%	96%
0.5	$\theta = 0.1, \xi = 0.15$	100%	98%	85%	100%	98%	85%
	$\theta = 0.1, \xi = 0.2$	100%	97%	85%	100%	97%	85%
	$\theta = 0.1, \xi = 0.3$	100%	95%	84%	100%	95%	84%
	$\theta = 0.2, \xi = 0.3$	100%	99%	95%	100%	99%	95%
1	$\theta = 0.1, \xi = 0.15$	100%	97%	85%	100%	97%	85%
	$\theta = 0.1, \xi = 0.2$	100%	97%	85%	100%	97%	85%
	$\theta = 0.1, \xi = 0.3$	100%	95%	84%	100%	95%	84%
	$\theta = 0.2, \xi = 0.3$	100%	99%	95%	100%	99%	95%
5	$\theta = 0.1, \xi = 0.15$	100%	97%	85%	100%	97%	85%
	$\theta = 0.1, \xi = 0.2$	100%	97%	85%	100%	97%	85%
	$\theta = 0.1, \xi = 0.3$	100%	95%	84%	100%	95%	84%
	$\theta = 0.2, \xi = 0.3$	100%	99%	98%	100%	99%	98%
10	$\theta = 0.1, \xi = 0.15$	100%	97%	86%	100%	97%	86%
	$\theta = 0.1, \xi = 0.2$	100%	97%	85%	100%	97%	85%
	$\theta = 0.1, \xi = 0.3$	100%	95%	84%	100%	95%	84%
	$\theta = 0.2, \xi = 0.3$	100%	97%	86%	100%	97%	86%
20	$\theta = 0.1, \xi = 0.15$	100%	97%	86%	100%	97%	86%
	$\theta = 0.1, \xi = 0.2$	100%	97%	86%	100%	97%	86%
	$\theta = 0.1, \xi = 0.3$	100%	95%	85%	100%	95%	85%
	$\theta = 0.2, \xi = 0.3$	100%	99%	96%	100%	99%	96%

Table 7.19 presents the maximum possible value of each attribute for the exponential claims on lower barrier model. From this table, it can be seen that the optimal levels move towards to the maximum levels of the released capital under Scenario 2. In this case, the increase in the percentage of the released capital can be observed.

Summing up the results, it can be concluded that the released capital plays a significant role in determining the optimal initial surplus and barrier level.

7.6. Discussion

In this part of the study, optimal pairs of initial surplus and lower barrier level are obtained according to four different sets of weights given in Table 5.1. The main objective of this section is to give a comprehensive perspective of our approach in respect of three matrices. The matrices which are used in the TOPSIS method with Mahalanobis distance are obtained according to the original outcomes covariance matrix, the vector-normalized outcomes covariance matrix and the correlation matrix.

We compare the optimal pair of initial surplus and lower barrier level according to three matrices, four pairs of loading factors, and four set of weights of each criteria. From the outcome of our investigation on weight of criteria, it is possible to conclude that the vector normalized covariance matrix reflects the accurate results.

We observe that in all cases, when the time horizon increases, the optimal initial surplus and barrier levels increase. Because, the insurance company needs more initial surplus to compensate the expected total claim amount up to time t .

After a certain time period such as $t = 20$, we observe that the differences between the optimal barrier levels decrease.

The optimal initial surplus and barrier level which are obtained under the exponential claim amount assumption can be summarised in terms of the weights of the criteria. We address to not only the same weights (Scenario 1) but also the different sets of weights in which one of the criteria is more important (Scenario 2, 3, 4).

In Scenario 1, the importance of each criterion is assumed to be the same. The optimal levels that are obtained according to different importance levels are compared with the results of Scenario 1. In Scenario 2, the lowest optimal initial surplus and highest barrier level among the other scenarios are obtained.

Since the importance of the released capital is higher than the other criteria's, the optimal levels are shifted to the point where the released capital has the maximum value. In Scenario 3, the importance of the expected profit criterion is defined as the highest among the other criteria. Thus, optimal levels shifted towards the levels when the expected profit is maximum. In Scenario 4, the importance of the expected utility criterion is defined as the highest among the other criteria.

In Scenario 4 optimal levels are observed as very close to the optimal values under Scenario 1, together with the values lying between Scenarios 2 and 3. The underlying reason is that the expected utility is also maximum at the point where the expected profit is maximum. However, since the variance of expected profit is greater than that of expected utility on original data. According to vector normalized data, the variance of the expected utility is higher than the expected profit's.

The structures of the reinsurance which is described on lower barrier model are quite different from the traditional reinsurance arrangements: excess of loss and proportional reinsurance. In lower barrier model, when time horizons increases, the optimal initial surplus and optimal barrier level increase. Hence, while the largest barrier level is observed at the smallest initial surplus level, the smallest barrier level is observed at the largest initial surplus level. However, the initial surplus and corresponding retention level have a reverse relationship in excess of loss reinsurance according to lower barrier model. We observed that when the initial surplus increases, the corresponding retention level which make the ruin probability equal to 0.01 increases as well. The reinsurance arrangement which depends on the lower barrier model provides smaller optimal initial surplus for the same ruin probability level. Based on the results, it can be concluded that the required initial surplus on lower barrier model is smaller than the excess of loss reinsurance even for the same individual claim amount and the same ruin probability assumption. The existence of these effects implies that the reinsurance arrangement which is described on lower barrier model is more preferable than the excess of loss reinsurance.

We have also considered the consequence of dominance weights situations. The results are shown in Table 7.14. We aim to measure the sensitivity of our model under the covariance and the correlation matrices.

An important practical issue is to evaluate the three matrices according to the results. The analysis indicate that using the vector-normalized covariance matrix give more accurate optimal levels.

Summing up the results, it can be concluded that obtaining the optimal barrier levels under the three criteria has been a new and different approach. The originality of our work lies in the fact that although several studies have indicated that the focus is only one criteria, we used a method depending on three main criteria under the minimum fixed finite time ruin probability.

In this section, we also aim to add a new criteria to the optimal reinsurance criteria. We examine the optimal initial surplus and barrier level according to a risk measure criteria: Conditional Value at Risk (CVar). We calculate the conditional value at risk for each barrier level for all combinations of time horizons and loading factors. However, the analysis does not enable us to determine comparable values. The reason is that we investigate the lower barrier level in a set of alternatives. The barrier level k and also the differences between each barrier level are so small, hence the conditional value at risk of each barrier level does not give a comparable form and it gives the nearly very close values.

7.7. Interim Conclusion: Optimal Barrier on Lower Barrier Model

In this chapter, we have focused on lower barrier model where we reformulate for the finite time ruin probability under the reinsurance arrangement. We examine the calculation of the defective distribution function of $T_{u,k}$, the time to ruin for the process with the lower barrier, with initial surplus u and lower barrier k . In this context, we have showed how $\psi(u)$, $\psi(u, t)$ and $\psi(0, t)$ are calculated by using the exact ruin probability formulas. We have given detailed characterization about each functions.

In this chapter, we have also aimed to demonstrate how to take reinsurance premium into account lower barrier model. The net of reinsurance loading factor after reinsurance arrangement has been calculated according to the expected total claim amount paid by the reinsurer up to the time of ruin. The net of reinsurance loading factor has been added to the calculation as the new insurance loading factor.

We have considered the gain of the insurer under the fixed minimum ruin probability. From the insurers' point of view, we have calculated the released capital as the differences between the initial surplus whether in the case of reinsurance or not, whereas, we have calculated the expected profit as the differences between insurance premiums and expected total claim amount paid by the reinsurer up to the time of ruin. We have identified the third criteria as the expected utility of the insurer's surplus. In this context, we have assumed that the insurer's utility function is an exponential function with parameter B .

We have discussed our results and have made some general comments about our approach according to different scenarios and set of weights. Moreover, we have measured the sensitivity of our approach on the vector-normalized covariance matrix, the original outcomes covariance matrix and the correlation matrix by comparing the results in the dominance weight set. We have addressed sensitivity analysis not only on the weight of the criteria but also on the parameter of the exponential utility function. We have also considered the consequences of the maximum possible value of each criteria.

To our knowledge, this is the first study dealing with determining the optimal barrier level under the ruin probability constraint. The originality of our solution lies in the fact that obtaining the optimal initial surplus and lower barrier level according to three essential criteria:

released capital, expected profit and expected utility under the ruin probability constraint. This study is a novel solution of the determining the optimal barrier. We have considered the consequences of adding the reinsurance premium to the finite time ruin probability formula and also we have achieved to obtain the finite time ruin probability according to the lower barrier model numerically. The proposed method can be readily used in practice. This research is concerned with the TOPSIS method with Mahalanobis distance. However, the decision procedure can be applicable also to other decision making techniques. The findings are of direct practical relevance and the proposed method can be readily used in practice.

8. CONCLUSIONS AND FURTHER RESEARCH

In this chapter we provide an overview of the main findings of this thesis as well as some suggestions for further research.

8.1. Conclusions

The main contribution of this thesis is twofold. First, we have proposed an approximation on determining the optimal reinsurance level according to three different measures: released capital, expected profit and expected utility under the ruin probability constraint by using the classical risk model assumptions. For the purpose of determining of the optimal reinsurance level, the traditional reinsurance arrangements: excess of loss reinsurance and proportional reinsurance are used. Second, we have provided an approximation which enables us to obtain optimal barrier level and initial surplus under the ruin probability constraint on lower barrier model. In order to obtain the optimal initial surplus and reinsurance level or barrier level, we have taken the advantage of decision theory. Thus, we apply TOPSIS method with Mahalanobis distance to take the dependency between the criteria into account. To our best knowledge, this is the first study that considers three different measures together under the ruin probability constraint.

In Chapter 2 we have introduced the classical risk model and explained properties of this model. We have reviewed the approximations to estimate the aggregate claim amounts. Therefore, we have examined the ruin probability formulas and approximations for the infinite and finite time. We have also discussed their advantages, disadvantages and limitations.

In Chapter 3 we have reviewed studies which are related to the optimal reinsurance under the ruin probability constraint. We have provided three essential measures on determining optimal reinsurance: released capital, expected profit and expected utility and we have described how these measures are calculated.

In Chapter 4 we have presented the optimal reinsurance under the ruin probability constraint. Since the purpose of this study has been to propose an approach to determine the optimal reinsurance level in accordance with three measures we have used the translated gamma process approximation. We have described how these three criteria are calculated under the translated gamma process approximation when the ruin probability is fixed at a minimum level.

In Chapter 5 we have presented a comprehensive analysis in which we have examined two different types of reinsurance arrangements: excess of loss and proportional reinsurance and two types of individual claims: exponential and Pareto distribution. Different individual claim amounts have the same expected value but different variances. We have utilized TOPSIS method which is one of technique of multi attribute decision making. Since the released capital, expected profit and expected utility are based on the initial surplus and reinsurance level, we have to use a distance measure which involves the dependency. Therefore, we use Mahalanobis distance instead of Euclidean distance in TOPSIS method. Using TOPSIS method has enabled us to make two contributions. First, instead of using only the covariance matrix of original data, we also use the covariance matrix of normalized data and the correlation matrix in TOPSIS method. Second, by changing the weights of three criteria and changing the parameter of the utility function we could analyse the sensitivity of the approximation. We have constituted a set of alternative which starts with the smallest initial surplus and ends with the largest initial surplus. Each pair of initial surplus and corresponding reinsurance level which make the finite time ruin probability equals to 0.01 are calculated. The reinsurance level is minimum at the smallest initial surplus level whereas the reinsurance level is maximum at the largest initial surplus level. The analysis is four-dimensional: we have employed two types of reinsurance arrangements, two different individual claim amounts, five different time horizons, four different insurance and reinsurance loading pairs. In doing so, we have examined the possible effects of the different assumptions. For each different time horizons and different loading pairs, we have obtained the optimal initial surplus and reinsurance level. According to the results of Chapter 5, the main findings can be summarised with regard to weight of the criteria and parameter of the utility function.

We have observed that when more weight is given to released capital, there is a decrease in optimal pairs. However, when more weight is given to expected profit or expected utility, we have observed an increase in optimal pairs. We have not observed the effect of the parameter of the utility function on optimal pairs as much as the weight of the criteria. We have also noticed that the covariance matrix of the normalized data reflects the pattern of the criteria better than the other matrices. The optimal pairs are high when the individual claim amount is Pareto. In addition, the optimal initial surplus is high when the proportional reinsurance is used.

In Chapter 6 we have introduced the lower barrier model on modified surplus process. We have explained the defective distribution function of the time to ruin $T_{u,k}$ with lower barrier k and initial surplus u which is suggested by Nie et. al. [2]. This type of reinsurance is quite different from the traditional reinsurance arrangements: excess of loss and proportional reinsurance. In this reinsurance arrangement, whenever the insurance surplus process falls below the barrier level k , a capital injection is provided by the reinsurance to restore the surplus level back to k . The reinsurance premium is calculated according to the expected total claim amount paid by the reinsurer. The distribution function of time to ruin, i.e finite time ruin probability on lower barrier model is demonstrated for the exponential case. The main contribution of this chapter is that the reinsurance arrangement has been added to the finite time ruin probability formula on lower barrier model. We have described the net of reinsurance loading factor which is calculated according to net of reinsurance and premium income. Hence, we have observed the effect of reinsurance on finite time ruin probability.

In Chapter 7 we have presented a detailed analysis which considers the optimal barrier on lower barrier model. Firstly, we have examined the finite time ruin probability $\psi(u, t)$, infinite time ruin probability $\psi(u)$ and finite time ruin probability with zero initial surplus $\psi(0, t)$ formulas to obtain the defective distribution function of the time to ruin $T_{u,k}$, with lower barrier k and initial surplus u under the exponential individual claim amount assumption. In order to get the finite time ruin probability with lower barrier k and initial surplus u , the individual claim sizes are assumed as the exponential claims that is one of the distribution which has a moment generating function. Under the assumption of the reinsurance premium is paid from the insurer's premium income, by using the net of reinsurance loading factor, the reinsurance has been added to the model.

We have obtained the optimal initial surplus and corresponding barrier level according to released capital, expected profit and expected utility under the fixed ruin probability constraint. In reinsurance decision process, we use TOPSIS method with Mahalanobis distance. We have obtained and compared the optimal initial surplus and barrier level for the combinations of four different loading factors, six different time horizons and four different scenarios. The structure of the model are quite different from the excess of loss reinsurance in the classical risk model. In lower barrier model, when the time horizon increases, the optimal initial surplus and the optimal barrier level increase. We have observed the largest barrier level at the smallest initial surplus level whereas the smallest barrier level even for some cases as nearly close to zero are at the largest initial surplus level.

Moreover, we have compared the optimal levels in lower barrier model with the excess of loss reinsurance case. The reinsurance arrangement which depends on the lower barrier model provides smaller optimal initial surplus for the same ruin probability levels. The results show that the required initial surplus is much smaller than the excess of loss reinsurance even for the same individual claim amount and the same ruin probability level assumptions. As in the classical risk models, the covariance matrix of vector-normalized data reflects the pattern of the three criteria better than the covariance matrix of original data and the correlation matrix. We have also concluded that the dominant impact of the released capital is higher than the other criteria. This has happened because of the higher variance impact. We have also examined the robustness of the approximation according to varying weights of the criteria and the utility function parameter. When more weight is given to the released capital, while the optimal initial surplus decreases, the optimal barrier level increases. However when more weight is given to the expected profit or expected utility, we have observed an increase in the optimal initial surplus and decrease in the optimal barrier levels.

To our knowledge, this is the first study which deals with the optimal reinsurance according to three important criteria together under the ruin probability constraint. This thesis presents an innovative view of determining the optimal reinsurance.

The originality of our solution lies in the fact that investigation of optimal reinsurance with respect to two different measures: insurance company's ruin probability and gain. The proposed method can be readily used in practice and the findings are of direct practical relevance on determining the optimal reinsurance level.

8.2. Suggestions for Further Research

This thesis aims to provide a method to determine the optimal reinsurance level in the classical risk model and optimal barrier level on lower barrier model. On determining the optimal pairs we have used three measures: released capital, expected profit and expected utility under the ruin probability constraint. Moreover, there is a wide range of possible ways to develop this study further.

In this study, we have taken advantage of the decision theory to determine the optimal pair of initial surplus and reinsurance/barrier level. This research has been concerned with the TOPSIS method with Mahalanobis distance; however, the results can also be applicable to different decision methods which include the dependency.

Another interesting research would be to assume different premium principles. In this thesis, in order to obtain the insurance and reinsurance premium, the expected premium principle is used. However, the method is applicable to different premium principles. A numerical comparison of the optimal pairs which are obtained according to these methods can be given.

In addition, we have used the exponential utility function as the third criteria on the method. However, the optimal pairs can also be obtained by using different utility functions.

On lower barrier model, we have assumed that the individual claim has exponential distribution. This distribution is used due to the existence of moment generating function. Another interesting research topic would be the investigation of heavy tailed claim distributions on lower barrier model to obtain the optimal pairs.

This thesis is based on three criteria: released capital, expected profit and expected utility. We have discussed adding the Conditional Value at Risk (CVar) as a new component/criterion to optimal reinsurance criteria. However in this criteria, the outcomes are so close to each other and it could not be a comparable form. As we have discussed in Chapter 7 a different measure can be incorporated into the method to obtain the optimal reinsurance/ barrier level.

An important point that this study is based on determining the optimal reinsurance level under the ruin probability constraint. Another future research might be to model the different important measures for the insurance company under the ruin probability constraint. The method which depends on three measures under the ruin probability constraint has a great potential for other applications such as determining the optimal investment or optimal dividend payments.

A. EFFECT OF CHANGES IN WEIGHTS WHEN $T = 1$ UNDER THE EXCESS OF LOSS REINSURANCE

1.1. Changes in Released Capital When $t = 1$ for the Exponential Claims Under the Excess of Loss Reinsurance

Table A.1: Changes in weights of the released capital (w_1) and $w_2 = 2w_3$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0	0.666667	0.333333	45.850	11.778	45.816	9.348	45.772	8.379	26.550	9.927
0.0025	0.665	0.3325	45.750	8.238	45.616	7.196	45.772	8.379	26.550	9.927
0.005	0.663333	0.331667	45.650	7.438	45.516	6.766	44.672	4.968	26.550	9.927
0.01	0.66	0.33	45.650	7.438	45.516	6.766	44.572	4.853	26.550	9.927
0.05	0.633333	0.316667	45.150	5.921	45.316	6.188	44.172	4.459	26.350	6.978
0.1	0.6	0.3	44.650	5.241	45.016	5.624	43.972	4.291	25.950	5.653
0.15	0.566667	0.283333	44.050	4.715	44.816	5.347	43.872	4.213	25.550	4.991
0.2	0.533333	0.266667	43.450	4.334	44.616	5.116	43.772	4.137	25.050	4.445
0.25	0.5	0.25	42.850	4.033	44.416	4.918	43.672	4.065	24.650	4.116
0.3	0.466667	0.233333	42.150	3.746	44.116	4.664	43.572	3.995	24.150	3.784
0.333	0.444667	0.222333	41.650	3.569	44.016	4.588	43.472	3.927	23.750	3.560
0.4	0.4	0.2	40.550	3.239	43.616	4.319	43.372	3.861	23.050	3.227
0.41	0.393333	0.196667	40.450	3.213	43.616	4.319	43.372	3.861	22.950	3.184
0.42	0.386667	0.193333	40.250	3.160	43.516	4.258	43.372	3.861	22.850	3.143
0.43	0.38	0.19	40.050	3.109	43.516	4.258	43.372	3.861	22.650	3.062
0.45	0.366667	0.183333	39.650	3.012	43.316	4.144	43.272	3.798	22.450	2.985
0.5	0.333333	0.166667	38.650	2.792	43.016	3.987	43.272	3.798	21.750	2.737
0.55	0.3	0.15	37.450	2.560	42.716	3.844	43.172	3.736	21.050	2.517
0.6	0.266667	0.133333	36.150	2.338	42.216	3.631	43.072	3.675	20.150	2.264
0.65	0.233333	0.116667	34.550	2.096	41.816	3.478	42.972	3.616	19.150	2.012
0.7	0.2	0.1	32.450	1.817	41.216	3.271	42.872	3.559	17.750	1.694
0.75	0.166667	0.083333	29.850	1.512	40.416	3.027	42.772	3.502	15.850	1.292
0.8	0.133333	0.066667	26.350	1.143	39.516	2.786	42.672	3.447	14.650	1.008
0.85	0.1	0.05	23.250	0.822	38.116	2.457	42.472	3.339	14.450	0.943
0.9	0.066667	0.033333	21.850	0.655	36.216	2.067	42.272	3.234	14.350	0.901
0.95	0.033333	0.016667	21.150	0.543	33.516	1.554	41.772	2.981	14.250	0.812
1	0	0	20.750	0.405	31.516	0.961	40.272	1.977	14.250	0.812

Table A.2: Changes in weights of the released capital (w_1) and $2w_2 = w_3$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.333333	0.666667	45.850	11.778	45.816	9.348	45.772	8.379	26.550	9.927
0.0025	0.3325	0.665	43.250	4.227	45.216	5.973	45.772	8.379	26.350	6.978
0.005	0.331667	0.663333	42.650	3.945	45.016	5.624	45.772	8.379	26.250	6.505
0.01	0.33	0.66	41.950	3.672	44.716	5.227	45.772	8.379	26.050	5.882
0.05	0.316667	0.633333	37.750	2.615	43.216	4.090	45.772	8.379	24.650	4.116
0.1	0.3	0.6	30.050	1.534	42.116	3.591	45.772	8.379	22.850	3.143
0.15	0.283333	0.566667	28.050	1.318	41.316	3.304	45.772	8.379	14.250	0.812
0.2	0.266667	0.533333	27.050	1.214	40.716	3.115	45.772	8.379	14.350	0.901
0.25	0.25	0.5	25.950	1.102	40.116	2.943	45.772	8.379	14.350	0.901
0.3	0.233333	0.466667	25.050	1.011	39.716	2.837	45.772	8.379	14.450	0.943
0.333	0.222333	0.444667	24.450	0.949	39.416	2.760	45.772	8.379	14.450	0.943
0.4	0.2	0.4	23.750	0.876	38.916	2.639	45.772	8.379	14.450	0.943
0.41	0.196667	0.393333	23.650	0.865	38.816	2.615	45.772	8.379	14.450	0.943
0.42	0.193333	0.386667	23.650	0.865	38.716	2.592	45.772	8.379	14.450	0.943
0.43	0.19	0.38	23.550	0.855	38.716	2.592	45.772	8.379	14.450	0.943
0.45	0.183333	0.366667	23.450	0.844	38.516	2.546	45.772	8.379	14.450	0.943
0.5	0.166667	0.333333	23.250	0.822	38.216	2.479	45.772	8.379	14.450	0.943
0.55	0.15	0.3	23.150	0.811	37.816	2.392	45.772	8.379	14.450	0.943
0.6	0.133333	0.266667	23.050	0.800	37.516	2.328	45.772	8.379	14.450	0.943
0.65	0.116667	0.233333	22.950	0.789	37.116	2.245	45.772	8.379	14.450	0.943
0.7	0.1	0.2	22.850	0.777	36.716	2.165	45.772	8.379	14.450	0.943
0.75	0.083333	0.166667	22.650	0.754	36.316	2.086	45.772	8.379	14.450	0.943
0.8	0.066667	0.133333	22.550	0.743	35.716	1.971	45.772	8.379	14.450	0.943
0.85	0.05	0.1	22.450	0.731	35.016	1.839	44.772	5.091	14.450	0.943
0.9	0.033333	0.066667	22.250	0.707	34.316	1.707	43.372	3.861	14.450	0.943
0.95	0.016667	0.033333	21.950	0.668	33.716	1.593	42.072	3.132	14.350	0.901
1	0	0	20.750	0.405	31.516	0.961	40.272	1.977	14.250	0.812

1.2. Changes in Released Capital When $t = 1$ for the Pareto Claims Under the Excess of Loss Reinsurance

Table A.3: Changes in weights of the released capital (w_1) and $w_2 = 2w_3$ when $t = 1$ for the Pareto Claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.666667	0.333333	68.373	1030.142	68.409	1922.980	68.411	2004.297	44.655	905.073
0.0025	0.665	0.3325	67.373	79.070	68.409	1922.980	68.411	2004.297	41.455	27.969
0.005	0.663333	0.331667	66.873	55.377	68.409	1922.980	68.411	2004.297	40.055	19.588
0.01	0.66	0.33	65.873	35.262	68.409	1922.980	68.411	2004.297	39.155	16.361
0.05	0.633333	0.316667	55.873	7.966	48.309	4.370	68.411	2004.297	37.155	11.837
0.1	0.6	0.3	54.373	7.072	47.809	4.220	48.911	3.720	35.955	10.064
0.15	0.566667	0.283333	53.373	6.562	47.509	4.133	48.011	3.361	35.155	9.113
0.2	0.533333	0.266667	52.573	6.195	47.109	4.020	47.611	3.199	34.555	8.491
0.25	0.5	0.25	51.673	5.818	46.809	3.938	47.411	3.116	33.855	7.845
0.3	0.466667	0.233333	50.873	5.511	46.509	3.857	47.311	3.074	33.155	7.270
0.333	0.444667	0.222333	50.473	5.366	46.209	3.778	47.311	3.074	32.755	6.969
0.4	0.4	0.2	49.273	4.962	45.709	3.649	47.211	3.032	31.855	6.353
0.41	0.393333	0.196667	49.073	4.899	45.709	3.649	47.211	3.032	31.655	6.226
0.42	0.386667	0.193333	48.973	4.868	45.609	3.624	47.211	3.032	31.655	6.226
0.43	0.38	0.19	48.873	4.837	45.509	3.599	47.211	3.032	31.555	6.164
0.45	0.366667	0.183333	48.573	4.746	45.409	3.574	47.211	3.032	31.255	5.983
0.5	0.333333	0.166667	47.673	4.486	44.909	3.452	47.211	3.032	30.655	5.641
0.55	0.3	0.15	46.673	4.218	44.409	3.333	47.111	2.989	29.855	5.224
0.6	0.266667	0.133333	45.473	3.922	44.009	3.241	47.111	2.989	28.955	4.798
0.65	0.233333	0.116667	44.373	3.672	43.409	3.106	47.011	2.946	28.155	4.453
0.7	0.2	0.1	42.773	3.340	42.809	2.976	47.011	2.946	27.255	4.098
0.75	0.166667	0.083333	41.173	3.039	42.009	2.808	46.911	2.902	26.055	3.669
0.8	0.133333	0.066667	39.073	2.684	41.109	2.626	46.811	2.857	24.655	3.222
0.85	0.1	0.05	36.273	2.267	40.009	2.412	46.711	2.811	22.655	2.662
0.9	0.066667	0.033333	31.873	1.706	39.009	2.224	46.611	2.764	15.455	0.867
0.95	0.033333	0.016667	25.073	0.968	37.409	1.930	46.411	2.664	15.455	0.867
1	0	0	21.273	0.454	33.409	0.937	45.711	2.061	15.455	0.867

Table A.4: Changes in weights of the released capital (w_1) and $2w_2 = w_3$ when $t = 1$ for the Pareto Claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.333333	0.666667	68.373	1030.142	68.409	1922.980	68.411	2004.297	44.655	905.073
0.0025	0.3325	0.665	67.773	122.810	68.409	1922.980	68.411	2004.297	42.755	45.974
0.005	0.331667	0.663333	67.573	95.993	68.409	1922.980	68.411	2004.297	41.855	31.811
0.01	0.33	0.66	67.173	67.399	68.409	1922.980	68.411	2004.297	40.455	21.439
0.05	0.316667	0.633333	65.073	27.567	68.409	1922.980	68.411	2004.297	34.155	8.112
0.1	0.3	0.6	62.873	17.445	68.409	1922.980	68.411	2004.297	30.955	5.809
0.15	0.283333	0.566667	60.473	12.494	68.409	1922.980	68.411	2004.297	29.055	4.843
0.2	0.266667	0.533333	57.873	9.500	68.409	1922.980	68.411	2004.297	27.655	4.252
0.25	0.25	0.5	54.773	7.293	68.409	1922.980	68.411	2004.297	26.455	3.807
0.3	0.233333	0.466667	51.573	5.778	68.409	1922.980	68.411	2004.297	25.355	3.439
0.333	0.222333	0.444667	49.473	5.026	68.409	1922.980	68.411	2004.297	24.755	3.252
0.4	0.2	0.4	45.873	4.017	68.409	1922.980	68.411	2004.297	23.555	2.904
0.41	0.196667	0.393333	45.573	3.945	68.409	1922.980	68.411	2004.297	23.455	2.876
0.42	0.193333	0.386667	44.873	3.783	68.409	1922.980	68.411	2004.297	23.255	2.822
0.43	0.19	0.38	44.673	3.738	68.409	1922.980	68.411	2004.297	23.155	2.795
0.45	0.183333	0.366667	43.873	3.564	68.409	1922.980	68.411	2004.297	22.855	2.715
0.5	0.166667	0.333333	42.073	3.205	68.409	1922.980	68.411	2004.297	22.255	2.559
0.55	0.15	0.3	40.573	2.933	68.409	1922.980	68.411	2004.297	21.655	2.410
0.6	0.133333	0.266667	39.073	2.684	68.409	1922.980	56.011	7.330	21.055	2.265
0.65	0.116667	0.233333	37.673	2.468	52.309	5.824	54.111	6.123	20.455	2.125
0.7	0.1	0.2	36.273	2.267	49.109	4.621	52.611	5.330	19.955	2.012
0.75	0.083333	0.166667	34.773	2.064	46.909	3.965	51.411	4.767	19.455	1.900
0.8	0.066667	0.133333	33.273	1.874	45.109	3.500	50.411	4.333	18.955	1.791
0.85	0.05	0.1	31.573	1.671	43.109	3.040	49.411	3.921	18.455	1.682
0.9	0.033333	0.066667	29.673	1.456	41.309	2.665	48.311	3.481	17.955	1.575
0.95	0.016667	0.033333	27.273	1.199	38.809	2.187	47.211	3.032	17.455	1.466
1	0	0	21.273	0.454	33.409	0.937	45.711	2.061	15.455	0.867

1.3. Changes in Expected Profit When $t = 1$ for the Exponential Claims Under the Excess of Loss Reinsurance

Table A.5: Changes in weights of the expected profit (w_2) and $w_1 = w_3$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0.5	0	0.5	35.150	2.184	42.516	3.755	45.772	8.379	19.650	2.135
0.49875	0.0025	0.49875	34.650	2.111	42.416	3.713	45.772	8.379	19.350	2.061
0.4975	0.005	0.4975	34.450	2.082	42.316	3.671	45.772	8.379	19.150	2.012
0.495	0.01	0.495	34.050	2.027	42.216	3.631	45.772	8.379	18.950	1.965
0.475	0.05	0.475	32.250	1.792	41.616	3.406	45.772	8.379	17.850	1.716
0.45	0.1	0.45	30.150	1.545	41.016	3.207	45.772	8.379	16.550	1.439
0.425	0.15	0.425	27.650	1.276	40.116	2.943	45.772	8.379	15.250	1.159
0.4	0.2	0.4	23.750	0.876	38.916	2.639	45.772	8.379	14.450	0.943
0.375	0.25	0.375	21.150	0.543	36.816	2.185	45.772	8.379	14.250	0.812
0.35	0.3	0.35	20.750	0.405	33.916	1.631	45.772	8.379	14.250	0.812
0.3335	0.333	0.3335	20.750	0.405	39.416	2.760	41.272	2.731	14.250	0.812
0.3	0.4	0.3	40.050	3.109	42.616	3.799	42.272	3.234	23.250	3.316
0.295	0.41	0.295	40.450	3.213	42.916	3.938	42.472	3.339	23.450	3.409
0.29	0.42	0.29	40.750	3.294	43.116	4.037	42.672	3.447	23.550	3.458
0.285	0.43	0.285	41.150	3.411	43.316	4.144	42.772	3.502	23.750	3.560
0.275	0.45	0.275	41.650	3.569	43.716	4.382	43.072	3.675	24.050	3.725
0.25	0.5	0.25	42.850	4.033	44.416	4.918	43.672	4.065	24.650	4.116
0.225	0.55	0.225	43.650	4.450	44.816	5.347	43.972	4.291	25.050	4.445
0.2	0.6	0.2	44.250	4.869	45.216	5.973	44.272	4.549	25.450	4.864
0.175	0.65	0.175	44.750	5.352	45.416	6.445	44.572	4.853	25.750	5.283
0.15	0.7	0.15	45.150	5.921	45.616	7.196	44.772	5.091	26.050	5.882
0.125	0.75	0.125	45.450	6.610	45.716	7.851	44.872	5.226	26.250	6.505
0.1	0.8	0.1	45.650	7.438	45.816	9.348	45.072	5.537	26.350	6.978
0.075	0.85	0.075	45.750	8.238	45.816	9.348	45.172	5.721	26.450	7.732
0.05	0.9	0.05	45.850	11.778	45.816	9.348	45.372	6.178	26.550	9.927
0.025	0.95	0.025	45.850	11.778	45.816	9.348	45.472	6.479	26.550	9.927
0	1	0	45.850	11.778	45.816	9.348	45.772	8.379	26.550	9.927

Table A.6: Changes in weights of the expected profit (w_2) and $w_1 = 2w_3$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.666667	0	0.333333	33.55	1.959	41.716	3.441	45.772	8.379	18.95	1.965
0.665	0.0025	0.3325	32.95	1.88	41.516	3.371	45.772	8.379	18.55	1.872
0.663333	0.005	0.331667	32.65	1.842	41.416	3.337	45.772	8.379	18.45	1.849
0.66	0.01	0.33	32.25	1.792	41.216	3.271	45.772	8.379	18.15	1.782
0.633333	0.05	0.316667	29.85	1.512	40.316	2.999	45.772	8.379	16.55	1.439
0.6	0.1	0.3	26.25	1.133	39.216	2.711	45.772	8.379	15.25	1.159
0.566667	0.15	0.283333	22.65	0.754	37.416	2.307	45.772	8.379	14.35	0.901
0.533333	0.2	0.266667	20.75	0.405	33.916	1.631	45.772	8.379	14.25	0.812
0.5	0.25	0.25	20.75	0.405	31.516	0.961	40.972	2.575	14.25	0.812
0.466667	0.3	0.233333	35.55	2.244	40.916	3.176	41.772	2.981	20.45	2.345
0.444667	0.333	0.222333	37.75	2.615	42.216	3.631	42.572	3.392	21.65	2.704
0.4	0.4	0.2	40.55	3.239	43.616	4.319	43.372	3.861	23.05	3.227
0.393333	0.41	0.196667	40.85	3.323	43.816	4.447	43.472	3.927	23.25	3.316
0.386667	0.42	0.193333	41.15	3.411	43.916	4.516	43.572	3.995	23.35	3.362
0.38	0.43	0.19	41.45	3.504	44.116	4.664	43.672	4.065	23.55	3.458
0.366667	0.45	0.183333	41.95	3.672	44.316	4.828	43.772	4.137	23.85	3.613
0.333333	0.5	0.166667	42.95	4.079	44.816	5.347	44.072	4.373	24.35	3.909
0.3	0.55	0.15	43.65	4.45	45.116	5.787	44.272	4.549	24.85	4.272
0.266667	0.6	0.133333	44.25	4.869	45.316	6.188	44.472	4.745	25.25	4.64
0.233333	0.65	0.116667	44.75	5.352	45.516	6.766	44.672	4.968	25.55	4.991
0.2	0.7	0.1	45.15	5.921	45.716	7.851	44.872	5.226	25.85	5.456
0.166667	0.75	0.083333	45.45	6.61	45.716	7.851	44.972	5.373	26.05	5.882
0.133333	0.8	0.066667	45.65	7.438	45.816	9.348	45.072	5.537	26.25	6.505
0.1	0.85	0.05	45.75	8.238	45.816	9.348	45.172	5.721	26.45	7.732
0.066667	0.9	0.033333	45.85	11.778	45.816	9.348	45.372	6.178	26.55	9.927
0.033333	0.95	0.016667	45.85	11.778	45.816	9.348	45.472	6.479	26.55	9.927
0	1	0	45.85	11.778	45.816	9.348	45.772	8.379	26.55	9.927

Table A.7: Changes in weights of the expected profit (w_2) and $2w_1 = w_3$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0.333333	0	0.666667	36.75	2.437	43.216	4.09	45.772	8.379	20.35	2.318
0.3325	0.0025	0.665	36.35	2.371	43.116	4.037	45.772	8.379	20.05	2.238
0.331667	0.005	0.663333	36.15	2.338	43.116	4.037	45.772	8.379	19.95	2.212
0.33	0.01	0.66	35.85	2.29	43.016	3.987	45.772	8.379	19.75	2.16
0.316667	0.05	0.633333	34.35	2.068	42.616	3.799	45.772	8.379	18.75	1.918
0.3	0.1	0.6	32.65	1.842	42.216	3.631	45.772	8.379	17.85	1.716
0.283333	0.15	0.566667	30.75	1.614	41.716	3.441	45.772	8.379	16.85	1.502
0.266667	0.2	0.533333	28.65	1.382	41.116	3.239	45.772	8.379	15.45	1.204
0.25	0.25	0.5	25.95	1.102	40.116	2.943	45.772	8.379	14.35	0.901
0.233333	0.3	0.466667	21.25	0.562	38.616	2.569	45.772	8.379	14.25	0.812
0.222333	0.333	0.444667	20.75	0.405	37.716	2.37	45.772	8.379	14.25	0.812
0.2	0.4	0.4	39.95	3.085	41.416	3.337	45.772	8.379	23.75	3.56
0.196667	0.41	0.393333	40.45	3.213	41.816	3.478	45.772	8.379	23.95	3.668
0.193333	0.42	0.386667	40.85	3.323	42.216	3.631	40.972	2.575	24.05	3.725
0.19	0.43	0.38	41.15	3.411	42.516	3.755	41.072	2.628	24.25	3.845
0.183333	0.45	0.366667	41.85	3.637	43.116	4.037	41.972	3.081	24.45	3.975
0.166667	0.5	0.333333	43.05	4.127	44.016	4.588	42.972	3.616	25.05	4.445
0.15	0.55	0.3	43.85	4.576	44.616	5.116	43.672	4.065	25.45	4.864
0.133333	0.6	0.266667	44.45	5.042	45.116	5.787	44.072	4.373	25.75	5.283
0.116667	0.65	0.233333	44.95	5.606	45.316	6.188	44.472	4.745	26.05	5.882
0.1	0.7	0.2	45.25	6.113	45.616	7.196	44.672	4.968	26.15	6.158
0.083333	0.75	0.166667	45.55	6.957	45.716	7.851	44.872	5.226	26.35	6.978
0.066667	0.8	0.133333	45.65	7.438	45.816	9.348	45.072	5.537	26.45	7.732
0.05	0.85	0.1	45.75	8.238	45.816	9.348	45.272	5.931	26.55	9.927
0.033333	0.9	0.066667	45.85	11.778	45.816	9.348	45.372	6.178	26.55	9.927
0.016667	0.95	0.033333	45.85	11.778	45.816	9.348	45.572	6.865	26.55	9.927
0	1	0	45.85	11.778	45.816	9.348	45.772	8.379	26.55	9.927

1.4. Changes in Expected Profit When $t = 1$ for the Pareto Claims Under the Excess of Loss Reinsurance

Table A.8: Changes in weights of the expected profit (w_2) and $w_1 = w_3$ when $t = 1$ for the Pareto claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0.5	0	0.5	56.373	8.308	68.409	1922.98	68.411	2004.297	30.455	5.533
0.49875	0.0025	0.49875	55.973	8.033	68.409	1922.98	68.411	2004.297	29.955	5.274
0.4975	0.005	0.4975	55.873	7.966	68.409	1922.98	68.411	2004.297	29.755	5.174
0.495	0.01	0.495	55.673	7.837	68.409	1922.98	68.411	2004.297	29.555	5.077
0.475	0.05	0.475	54.473	7.126	68.409	1922.98	68.411	2004.297	27.955	4.371
0.45	0.1	0.45	52.873	6.329	68.409	1922.98	68.411	2004.297	26.255	3.737
0.425	0.15	0.425	50.573	5.401	68.409	1922.98	68.411	2004.297	24.655	3.222
0.4	0.2	0.4	45.873	4.017	68.409	1922.98	68.411	2004.297	23.555	2.904
0.375	0.25	0.375	39.673	2.781	68.409	1922.98	68.411	2004.297	24.855	3.283
0.35	0.3	0.35	37.473	2.439	68.409	1922.98	68.411	2004.297	27.255	4.098
0.3335	0.333	0.3335	40.173	2.865	40.809	2.566	68.411	2004.297	28.855	4.753
0.3	0.4	0.3	46.673	4.218	40.609	2.527	46.211	2.556	31.355	6.043
0.295	0.41	0.295	47.273	4.376	41.309	2.665	46.011	2.431	31.655	6.226
0.29	0.42	0.29	47.873	4.542	42.209	2.849	45.911	2.357	31.955	6.418
0.285	0.43	0.285	48.573	4.746	43.009	3.019	45.911	2.357	32.355	6.685
0.275	0.45	0.275	49.673	5.092	44.409	3.333	46.011	2.431	32.755	6.969
0.25	0.5	0.25	51.673	5.818	46.809	3.938	47.411	3.116	33.855	7.845
0.225	0.55	0.225	53.373	6.562	48.509	4.431	49.811	4.084	34.655	8.59
0.2	0.6	0.2	54.873	7.351	50.009	4.925	51.011	4.59	35.455	9.452
0.175	0.65	0.175	56.073	8.1	51.409	5.449	52.111	5.088	36.255	10.464
0.15	0.7	0.15	57.373	9.071	52.709	6.003	53.311	5.686	36.855	11.347
0.125	0.75	0.125	58.473	10.066	54.009	6.639	54.411	6.296	37.455	12.365
0.1	0.8	0.1	59.673	11.403	55.309	7.381	55.511	6.985	38.155	13.773
0.075	0.85	0.075	60.873	13.118	56.709	8.337	56.711	7.853	38.955	15.776
0.05	0.9	0.05	62.273	15.876	58.209	9.61	58.011	8.976	39.755	18.388
0.025	0.95	0.025	63.873	20.904	60.209	11.933	59.811	10.997	40.855	23.659
0	1	0	68.273	452.684	68.309	566.4	68.411	2004.297	44.655	905.073

Table A.9: Changes in weights of the expected profit (w_2) and $w_1 = 2w_3$ when $t = 1$ for the Pareto claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.666667	0	0.333333	53.073	6.421	68.409	1922.98	68.411	2004.297	28.555	4.622
0.665	0.0025	0.3325	52.473	6.151	68.409	1922.98	68.411	2004.297	28.055	4.412
0.663333	0.005	0.331667	52.173	6.023	68.409	1922.98	68.411	2004.297	27.855	4.331
0.66	0.01	0.33	51.773	5.858	68.409	1922.98	68.411	2004.297	27.455	4.174
0.633333	0.05	0.316667	49.173	4.93	68.409	1922.98	68.411	2004.297	25.355	3.439
0.6	0.1	0.3	45.173	3.852	68.409	1922.98	68.411	2004.297	23.055	2.768
0.566667	0.15	0.283333	38.873	2.652	68.409	1922.98	56.811	7.931	21.155	2.289
0.533333	0.2	0.266667	33.873	1.949	43.809	3.195	51.311	4.722	21.355	2.337
0.5	0.25	0.25	35.373	2.144	39.409	2.298	48.011	3.361	25.955	3.635
0.466667	0.3	0.233333	42.173	3.224	38.909	2.205	46.211	2.556	28.655	4.665
0.444667	0.333	0.222333	45.173	3.852	41.209	2.646	45.911	2.357	29.955	5.274
0.4	0.4	0.2	49.273	4.962	45.709	3.649	47.211	3.032	31.855	6.353
0.393333	0.41	0.196667	49.673	5.092	46.209	3.778	47.511	3.158	32.155	6.55
0.386667	0.42	0.193333	50.173	5.261	46.609	3.884	47.911	3.321	32.355	6.685
0.38	0.43	0.19	50.473	5.366	47.009	3.993	48.211	3.441	32.655	6.896
0.366667	0.45	0.183333	51.373	5.7	47.609	4.162	48.911	3.72	32.955	7.117
0.333333	0.5	0.166667	52.873	6.329	49.109	4.621	50.611	4.417	33.855	7.845
0.3	0.55	0.15	54.273	7.018	50.409	5.068	51.611	4.857	34.655	8.59
0.266667	0.6	0.133333	55.573	7.773	51.609	5.529	52.511	5.281	35.455	9.452
0.233333	0.65	0.116667	56.673	8.525	52.709	6.003	53.411	5.738	35.955	10.064
0.2	0.7	0.1	57.773	9.412	53.809	6.535	54.311	6.238	36.755	11.191
0.166667	0.75	0.083333	58.773	10.372	54.909	7.14	55.211	6.788	37.455	12.365
0.133333	0.8	0.066667	59.873	11.658	56.009	7.836	56.111	7.402	38.055	13.555
0.1	0.85	0.05	60.973	13.283	57.109	8.648	57.111	8.175	38.755	15.227
0.066667	0.9	0.033333	62.273	15.876	58.509	9.905	58.311	9.269	39.555	17.663
0.033333	0.95	0.016667	63.773	20.496	60.309	12.076	59.911	11.131	40.655	22.496
0	1	0	68.273	452.684	68.309	566.4	68.411	2004.297	44.655	905.073

Table A.10: Changes in weights of the expected profit (w_2) and $2w_1 = w_3$ when $t = 1$ for the Pareto claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0.333333	0	0.666667	59.373	11.039	68.409	1922.98	68.411	2004.297	32.255	6.617
0.3325	0.0025	0.665	59.173	10.808	68.409	1922.98	68.411	2004.297	31.955	6.418
0.331667	0.005	0.663333	59.073	10.696	68.409	1922.98	68.411	2004.297	31.755	6.289
0.33	0.01	0.66	58.973	10.586	68.409	1922.98	68.411	2004.297	31.555	6.164
0.316667	0.05	0.633333	58.473	10.066	68.409	1922.98	68.411	2004.297	30.455	5.533
0.3	0.1	0.6	57.873	9.5	68.409	1922.98	68.411	2004.297	29.155	4.889
0.283333	0.15	0.566667	57.273	8.989	68.409	1922.98	68.411	2004.297	27.855	4.331
0.266667	0.2	0.533333	56.473	8.379	68.409	1922.98	68.411	2004.297	26.655	3.877
0.25	0.25	0.5	54.773	7.293	68.409	1922.98	68.411	2004.297	26.455	3.807
0.233333	0.3	0.466667	48.373	4.686	68.409	1922.98	68.411	2004.297	27.855	4.331
0.222333	0.333	0.444667	42.473	3.281	68.409	1922.98	68.411	2004.297	28.955	4.798
0.2	0.4	0.4	43.973	3.586	68.409	1922.98	68.411	2004.297	31.355	6.043
0.196667	0.41	0.393333	44.773	3.761	68.409	1922.98	68.411	2004.297	31.655	6.226
0.193333	0.42	0.386667	45.473	3.922	68.409	1922.98	68.411	2004.297	31.955	6.418
0.19	0.43	0.38	46.673	4.218	40.809	2.566	68.411	2004.297	32.355	6.685
0.183333	0.45	0.366667	48.073	4.599	40.809	2.566	68.411	2004.297	32.755	6.969
0.166667	0.5	0.333333	50.873	5.511	44.209	3.287	45.911	2.357	34.055	8.022
0.15	0.55	0.3	52.773	6.284	46.709	3.911	46.711	2.811	34.855	8.794
0.133333	0.6	0.266667	54.373	7.072	48.509	4.431	49.611	4.002	35.755	9.812
0.116667	0.65	0.233333	55.973	8.033	50.209	4.996	50.911	4.546	36.555	10.89
0.1	0.7	0.2	57.173	8.909	51.809	5.611	52.311	5.184	37.155	11.837
0.083333	0.75	0.166667	58.473	10.066	53.409	6.334	53.611	5.845	37.855	13.137
0.066667	0.8	0.133333	59.673	11.403	54.909	7.14	54.911	6.598	38.555	14.713
0.05	0.85	0.1	61.073	13.453	56.409	8.116	56.311	7.548	39.155	16.361
0.033333	0.9	0.066667	62.373	16.117	58.109	9.515	57.911	8.881	39.955	19.172
0.016667	0.95	0.033333	63.973	21.33	60.209	11.933	59.811	10.997	41.055	24.943
0	1	0	68.273	452.684	68.309	566.4	68.411	2004.297	44.655	905.073

1.5. Changes in Expected Utility When $t = 1$ for the Exponential Claims Under the Excess of Loss Reinsurance

Table A.11: Changes in weights of the expected utility (w_3) and $w_1 = w_2$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.5	0.5	0	44.45	5.042	45.716	7.851	44.872	5.226	24.75	4.192
0.49875	0.49875	0.0025	44.25	4.869	45.716	7.851	44.772	5.091	24.65	4.116
0.4975	0.4975	0.005	44.15	4.79	45.716	7.851	44.772	5.091	24.55	4.044
0.495	0.495	0.01	44.05	4.715	45.616	7.196	44.772	5.091	24.45	3.975
0.475	0.475	0.05	43.25	4.227	45.316	6.188	44.472	4.745	24.15	3.784
0.45	0.45	0.1	42.35	3.822	44.816	5.347	44.172	4.459	23.75	3.56
0.425	0.425	0.15	41.55	3.536	44.316	4.828	43.872	4.213	23.45	3.409
0.4	0.4	0.2	40.55	3.239	43.616	4.319	43.372	3.861	23.05	3.227
0.375	0.375	0.25	39.35	2.943	42.716	3.844	42.672	3.447	22.65	3.062
0.35	0.35	0.3	37.75	2.615	41.216	3.271	40.972	2.575	22.05	2.839
0.3335	0.3335	0.333	36.05	2.322	39.516	2.786	41.272	2.731	14.25	0.812
0.3	0.3	0.4	20.75	0.405	36.316	2.086	45.772	8.379	14.25	0.812
0.295	0.295	0.41	20.75	0.405	36.816	2.185	45.772	8.379	14.25	0.812
0.29	0.29	0.42	20.85	0.468	37.316	2.286	45.772	8.379	14.25	0.812
0.285	0.285	0.43	21.15	0.543	37.716	2.37	45.772	8.379	14.25	0.812
0.275	0.275	0.45	21.75	0.641	38.616	2.569	45.772	8.379	14.25	0.812
0.25	0.25	0.5	25.95	1.102	40.116	2.943	45.772	8.379	14.35	0.901
0.225	0.225	0.55	28.35	1.35	41.216	3.271	45.772	8.379	15.15	1.136
0.2	0.2	0.6	30.25	1.557	41.916	3.515	45.772	8.379	16.55	1.439
0.175	0.175	0.65	32.05	1.768	42.516	3.755	45.772	8.379	17.25	1.587
0.15	0.15	0.7	33.75	1.986	43.016	3.987	45.772	8.379	17.85	1.716
0.125	0.125	0.75	35.35	2.214	43.416	4.2	45.772	8.379	18.65	1.895
0.1	0.1	0.8	36.85	2.454	43.816	4.447	45.772	8.379	19.45	2.085
0.075	0.075	0.85	38.35	2.731	44.216	4.744	45.772	8.379	20.35	2.318
0.05	0.05	0.9	39.85	3.06	44.516	5.014	45.772	8.379	21.35	2.608
0.025	0.025	0.95	41.65	3.569	44.916	5.479	45.772	8.379	22.65	3.062
0	0	1	45.85	11.778	45.816	9.348	45.772	8.379	26.55	9.927

Table A.12: Changes in weights of the expected utility (w_3) and $w_1 = 2w_2$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.666667	0.333333	0	42.65	3.945	45.516	6.766	44.672	4.968	23.25	3.316
0.665	0.3325	0.0025	42.25	3.783	45.316	6.188	44.572	4.853	23.05	3.227
0.663333	0.331667	0.005	42.05	3.709	45.316	6.188	44.572	4.853	22.95	3.184
0.66	0.33	0.01	41.75	3.603	45.116	5.787	44.472	4.745	22.85	3.143
0.633333	0.316667	0.05	40.25	3.16	44.416	4.918	44.072	4.373	22.15	2.875
0.6	0.3	0.1	38.55	2.771	43.516	4.258	43.672	4.065	21.35	2.608
0.566667	0.283333	0.15	36.65	2.42	42.316	3.671	42.972	3.616	20.55	2.373
0.533333	0.266667	0.2	33.65	1.972	40.516	3.056	41.872	3.031	14.25	0.812
0.5	0.25	0.25	20.75	0.405	31.516	0.961	40.972	2.575	14.25	0.812
0.466667	0.233333	0.3	20.75	0.405	33.816	1.612	45.772	8.379	14.25	0.812
0.444667	0.222333	0.333	21.15	0.543	36.216	2.067	45.772	8.379	14.25	0.812
0.4	0.2	0.4	23.75	0.876	38.916	2.639	45.772	8.379	14.45	0.943
0.393333	0.196667	0.41	24.35	0.939	39.216	2.711	45.772	8.379	14.55	0.977
0.386667	0.193333	0.42	25.05	1.011	39.416	2.76	45.772	8.379	14.55	0.977
0.38	0.19	0.43	25.75	1.082	39.616	2.811	45.772	8.379	14.65	1.008
0.366667	0.183333	0.45	26.95	1.204	40.016	2.916	45.772	8.379	14.85	1.062
0.333333	0.166667	0.5	28.95	1.414	40.916	3.176	45.772	8.379	15.65	1.248
0.3	0.15	0.55	30.45	1.579	41.616	3.406	45.772	8.379	16.65	1.46
0.266667	0.133333	0.6	31.85	1.743	42.116	3.591	45.772	8.379	17.35	1.608
0.233333	0.116667	0.65	33.25	1.919	42.616	3.799	45.772	8.379	17.95	1.738
0.2	0.1	0.7	34.45	2.082	43.016	3.987	45.772	8.379	18.45	1.849
0.166667	0.083333	0.75	35.75	2.275	43.316	4.144	45.772	8.379	19.05	1.988
0.133333	0.066667	0.8	36.95	2.471	43.716	4.382	45.772	8.379	19.75	2.16
0.1	0.05	0.85	38.25	2.711	44.116	4.664	45.772	8.379	20.45	2.345
0.066667	0.033333	0.9	39.65	3.012	44.416	4.918	45.772	8.379	21.35	2.608
0.033333	0.016667	0.95	41.35	3.472	44.916	5.479	45.772	8.379	22.55	3.023
0	0	1	45.85	11.778	45.816	9.348	45.772	8.379	26.55	9.927

Table A.13: Changes in weights of the expected utility (w_3) and $2w_1 = w_2$ when $t = 1$ for the exponential claims under the excess of loss reinsurance

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.333333	0.666667	0	45.45	6.61	45.816	9.348	45.072	5.537	25.75	5.283
0.3325	0.665	0.0025	45.25	6.113	45.816	9.348	45.072	5.537	25.65	5.13
0.331667	0.663333	0.005	45.25	6.113	45.816	9.348	44.972	5.373	25.65	5.13
0.33	0.66	0.01	45.15	5.921	45.816	9.348	44.972	5.373	25.55	4.991
0.316667	0.633333	0.05	44.75	5.352	45.616	7.196	44.772	5.091	25.35	4.748
0.3	0.6	0.1	44.35	4.953	45.416	6.445	44.572	4.853	25.15	4.539
0.283333	0.566667	0.15	43.85	4.576	45.216	5.973	44.372	4.644	25.05	4.445
0.266667	0.533333	0.2	43.35	4.28	44.816	5.347	44.072	4.373	24.85	4.272
0.25	0.5	0.25	42.85	4.033	44.416	4.918	43.672	4.065	24.65	4.116
0.233333	0.466667	0.3	42.15	3.746	43.816	4.447	42.972	3.616	24.35	3.909
0.222333	0.444667	0.333	41.55	3.536	43.216	4.09	42.372	3.286	24.25	3.845
0.2	0.4	0.4	39.95	3.085	41.416	3.337	45.772	8.379	23.75	3.56
0.196667	0.393333	0.41	39.65	3.012	40.916	3.176	45.772	8.379	23.65	3.508
0.193333	0.386667	0.42	39.35	2.943	40.516	3.056	45.772	8.379	23.55	3.458
0.19	0.38	0.43	38.95	2.855	40.016	2.916	45.772	8.379	23.45	3.409
0.183333	0.366667	0.45	20.75	0.405	39.116	2.687	45.772	8.379	14.25	0.812
0.166667	0.333333	0.5	20.85	0.468	39.416	2.76	45.772	8.379	14.25	0.812
0.15	0.3	0.55	28.85	1.403	40.916	3.176	45.772	8.379	14.25	0.812
0.133333	0.266667	0.6	28.95	1.414	42.016	3.552	45.772	8.379	20.75	2.429
0.116667	0.233333	0.65	31.05	1.648	42.716	3.844	45.772	8.379	17.15	1.566
0.1	0.2	0.7	33.25	1.919	43.216	4.09	45.772	8.379	17.45	1.629
0.083333	0.166667	0.75	35.35	2.214	43.716	4.382	45.772	8.379	18.25	1.804
0.066667	0.133333	0.8	37.15	2.506	44.016	4.588	45.772	8.379	19.25	2.036
0.05	0.1	0.85	38.85	2.834	44.416	4.918	45.772	8.379	20.25	2.291
0.033333	0.066667	0.9	40.45	3.213	44.716	5.227	45.772	8.379	21.45	2.64
0.016667	0.033333	0.95	42.15	3.746	45.116	5.787	45.772	8.379	22.85	3.143
0	0	1	45.85	11.778	45.816	9.348	45.772	8.379	26.55	9.927

1.6. Changes in Expected Utility When $t = 1$ for the Pareto Claims Under the Excess of Loss Reinsurance

Table A.14: Changes in weights of the expected utility (w_3) and $w_1 = w_2$ when $t = 1$ for the Pareto claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0.5	0.5	0	57.973	9.591	55.809	7.702	56.511	7.698	35.555	9.57
0.49875	0.49875	0.0025	57.373	9.071	55.209	7.32	55.911	7.259	35.255	9.224
0.4975	0.4975	0.005	57.173	8.909	54.909	7.14	55.711	7.12	35.155	9.113
0.495	0.495	0.01	56.673	8.525	54.509	6.911	55.311	6.853	34.955	8.898
0.475	0.475	0.05	54.873	7.351	52.309	5.824	53.611	5.845	34.055	8.022
0.45	0.45	0.1	53.073	6.421	50.309	5.032	51.811	4.948	33.355	7.428
0.425	0.425	0.15	51.373	5.7	48.209	4.339	49.811	4.084	32.655	6.896
0.4	0.4	0.2	49.273	4.962	45.709	3.649	47.211	3.032	31.855	6.353
0.375	0.375	0.25	46.873	4.27	41.909	2.787	45.911	2.357	30.855	5.752
0.35	0.35	0.3	43.173	3.42	39.409	2.298	47.111	2.989	29.855	5.224
0.3335	0.3335	0.333	40.173	2.865	40.609	2.527	68.411	2004.297	28.855	4.753
0.3	0.3	0.4	39.673	2.781	68.409	1922.98	68.411	2004.297	27.255	4.098
0.295	0.295	0.41	40.573	2.933	68.409	1922.98	68.411	2004.297	27.255	4.098
0.29	0.29	0.42	41.773	3.148	68.409	1922.98	68.411	2004.297	27.055	4.023
0.285	0.285	0.43	42.873	3.359	68.409	1922.98	68.411	2004.297	26.855	3.949
0.275	0.275	0.45	46.773	4.243	68.409	1922.98	68.411	2004.297	26.555	3.842
0.25	0.25	0.5	54.773	7.293	68.409	1922.98	68.411	2004.297	26.455	3.807
0.225	0.225	0.55	57.673	9.324	68.409	1922.98	68.411	2004.297	27.155	4.06
0.2	0.2	0.6	59.573	11.279	68.409	1922.98	68.411	2004.297	28.555	4.622
0.175	0.175	0.65	60.773	12.956	68.409	1922.98	68.411	2004.297	30.055	5.324
0.15	0.15	0.7	61.873	14.978	68.409	1922.98	68.411	2004.297	31.555	6.164
0.125	0.125	0.75	62.873	17.445	68.409	1922.98	68.411	2004.297	33.055	7.193
0.1	0.1	0.8	63.773	20.496	68.409	1922.98	68.411	2004.297	34.455	8.394
0.075	0.075	0.85	64.673	24.9	68.409	1922.98	68.411	2004.297	36.055	10.195
0.05	0.05	0.9	65.573	31.894	68.409	1922.98	68.411	2004.297	37.655	12.741
0.025	0.025	0.95	66.573	47.148	68.409	1922.98	68.411	2004.297	39.655	18.018
0	0	1	68.373	1030.142	68.409	1922.98	68.411	2004.297	44.655	905.073

Table A.15: Changes in weights of the expected utility (w_3) and $w_1 = 2w_2$ when $t = 1$ for the Pareto claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.666667	0.333333	0	55.673	7.837	54.209	6.746	55.411	6.919	33.555	7.591
0.665	0.3325	0.0025	54.873	7.351	53.409	6.334	54.711	6.475	33.255	7.348
0.663333	0.331667	0.005	54.373	7.072	53.109	6.189	54.511	6.355	33.055	7.193
0.66	0.33	0.01	53.973	6.861	52.509	5.913	54.111	6.123	32.855	7.042
0.633333	0.316667	0.05	51.373	5.7	50.009	4.925	52.011	5.041	31.655	6.226
0.6	0.3	0.1	48.673	4.776	47.209	4.048	49.311	3.881	30.655	5.641
0.566667	0.283333	0.15	45.473	3.922	43.709	3.173	46.711	2.811	29.255	4.935
0.533333	0.266667	0.2	41.273	3.057	38.709	2.168	46.011	2.431	27.755	4.291
0.5	0.25	0.25	35.373	2.144	39.409	2.298	48.011	3.361	25.955	3.635
0.466667	0.233333	0.3	35.073	2.104	44.509	3.356	52.111	5.088	23.955	3.017
0.444667	0.222333	0.333	37.573	2.453	68.409	1922.98	68.411	2004.297	23.155	2.795
0.4	0.2	0.4	45.873	4.017	68.409	1922.98	68.411	2004.297	23.555	2.904
0.393333	0.196667	0.41	47.073	4.322	68.409	1922.98	68.411	2004.297	23.755	2.96
0.386667	0.193333	0.42	48.373	4.686	68.409	1922.98	68.411	2004.297	24.055	3.045
0.38	0.19	0.43	49.473	5.026	68.409	1922.98	68.411	2004.297	24.255	3.103
0.366667	0.183333	0.45	51.273	5.661	68.409	1922.98	68.411	2004.297	24.755	3.252
0.333333	0.166667	0.5	54.473	7.126	68.409	1922.98	68.411	2004.297	26.055	3.669
0.3	0.15	0.55	56.573	8.451	68.409	1922.98	68.411	2004.297	27.455	4.174
0.266667	0.133333	0.6	58.273	9.871	68.409	1922.98	68.411	2004.297	28.755	4.709
0.233333	0.116667	0.65	59.573	11.279	68.409	1922.98	68.411	2004.297	30.155	5.376
0.2	0.1	0.7	60.773	12.956	68.409	1922.98	68.411	2004.297	31.355	6.043
0.166667	0.083333	0.75	61.873	14.978	68.409	1922.98	68.411	2004.297	32.555	6.825
0.133333	0.066667	0.8	62.973	17.738	68.409	1922.98	68.411	2004.297	33.955	7.933
0.1	0.05	0.85	63.973	21.33	68.409	1922.98	68.411	2004.297	35.355	9.337
0.066667	0.033333	0.9	65.073	27.567	68.409	1922.98	68.411	2004.297	36.855	11.347
0.033333	0.016667	0.95	66.273	41.144	68.409	1922.98	68.411	2004.297	38.955	15.776
0	0	1	68.373	1030.142	68.409	1922.98	68.411	2004.297	44.655	905.073

Table A.16: Changes in weights of the expected utility (w_3) and $2w_1 = w_2$ when $t = 1$ for the Pareto claims under the excess of loss reinsurance

t=1 Pareto			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.333333	0.666667	0	60.173	12.063	57.409	8.893	57.611	8.606	37.155	11.837
0.3325	0.665	0.0025	59.573	11.279	56.809	8.413	57.111	8.175	36.855	11.347
0.331667	0.663333	0.005	59.373	11.039	56.509	8.189	56.911	8.011	36.855	11.347
0.33	0.66	0.01	58.973	10.586	56.109	7.904	56.511	7.698	36.655	11.039
0.316667	0.633333	0.05	57.373	9.071	54.209	6.746	54.911	6.598	35.955	10.064
0.3	0.6	0.1	55.973	8.033	52.409	5.868	53.311	5.686	35.455	9.452
0.283333	0.566667	0.15	54.673	7.237	50.709	5.178	51.811	4.948	34.855	8.794
0.266667	0.533333	0.2	53.073	6.421	48.809	4.525	50.211	4.249	34.355	8.299
0.25	0.5	0.25	51.673	5.818	46.809	3.938	47.411	3.116	33.855	7.845
0.233333	0.466667	0.3	49.773	5.125	44.009	3.241	45.911	2.357	33.155	7.27
0.222333	0.444667	0.333	48.273	4.657	41.509	2.706	46.311	2.612	32.655	6.896
0.2	0.4	0.4	43.973	3.586	68.409	1922.98	68.411	2004.297	31.355	6.043
0.196667	0.393333	0.41	43.273	3.44	68.409	1922.98	68.411	2004.297	31.055	5.866
0.193333	0.386667	0.42	42.873	3.359	68.409	1922.98	68.411	2004.297	30.855	5.752
0.19	0.38	0.43	42.673	3.32	68.409	1922.98	68.411	2004.297	30.655	5.641
0.183333	0.366667	0.45	42.873	3.359	68.409	1922.98	68.411	2004.297	30.355	5.48
0.166667	0.333333	0.5	57.273	8.989	68.409	1922.98	68.411	2004.297	29.655	5.125
0.15	0.3	0.55	60.273	12.203	68.409	1922.98	68.411	2004.297	29.255	4.935
0.133333	0.266667	0.6	61.573	14.368	68.409	1922.98	68.411	2004.297	29.655	5.125
0.116667	0.233333	0.65	62.573	16.623	68.409	1922.98	68.411	2004.297	30.855	5.752
0.1	0.2	0.7	63.373	19.015	68.409	1922.98	68.411	2004.297	32.455	6.755
0.083333	0.166667	0.75	64.073	21.774	68.409	1922.98	68.411	2004.297	34.155	8.112
0.066667	0.133333	0.8	64.773	25.514	68.409	1922.98	68.411	2004.297	35.655	9.69
0.05	0.1	0.85	65.473	30.917	68.409	1922.98	68.411	2004.297	37.055	11.67
0.033333	0.066667	0.9	66.173	39.485	68.409	1922.98	68.411	2004.297	38.655	14.966
0.016667	0.033333	0.95	66.973	58.848	68.409	1922.98	68.411	2004.297	40.455	21.439
0	0	1	68.373	1030.142	68.409	1922.98	68.411	2004.297	44.655	905.073

B. EFFECT OF CHANGES IN WEIGHTS WHEN $T = 1$ ON LOWER BARRIER MODEL

2.1. Changes in Released Capital When $t = 1$ for the Exponential Claims on Lower Barrier Model

Table B.1: Changes in weights of the released capital (w_1) and $w_2 = 2w_3$ when $t = 1$ for the exponential claims on lower barrier model

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.6667	0.3333	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.0025	0.665	0.3325	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.005	0.6633	0.3317	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.01	0.66	0.33	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.05	0.6333	0.3167	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.1	0.6	0.3	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.15	0.5667	0.2833	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.2	0.5333	0.2667	5.227	5.227	5.228	5.228	5.23	5.23	26.537	0.169
0.25	0.5	0.25	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.3	0.4667	0.2333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.333	0.4447	0.2223	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.4	0.4	0.2	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.41	0.3933	0.1967	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.42	0.3867	0.1933	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.43	0.38	0.19	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.45	0.3667	0.1833	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.5	0.3333	0.1667	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.55	0.3	0.15	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.6	0.2667	0.1333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.65	0.2333	0.1167	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.7	0.2	0.1	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.75	0.1667	0.0833	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.8	0.1333	0.0667	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.85	0.1	0.05	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.9	0.0667	0.0333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.95	0.0333	0.0167	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
1	0	0	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137

Table B.2: Changes in weights of the released capital (w_1) and $2w_2 = w_3$ when $t = 1$ for the exponential claims on lower barrier model

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0	0.3333	0.6667	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.0025	0.3325	0.665	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.005	0.3317	0.6633	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.01	0.33	0.66	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.05	0.3167	0.6333	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.1	0.3	0.6	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.15	0.2833	0.5667	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.2	0.2667	0.5333	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.25	0.25	0.5	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.3	0.2333	0.4667	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169
0.333	0.2223	0.4447	41.927	1.011	42.028	0.995	42.33	0.947	26.537	0.169
0.4	0.2	0.4	33.327	2.161	33.528	2.136	33.83	2.1	20.037	2.056
0.41	0.1967	0.3933	31.927	2.331	32.128	2.307	32.43	2.272	18.737	2.339
0.42	0.1933	0.3867	30.527	2.499	30.628	2.488	31.03	2.441	17.337	2.639
0.43	0.19	0.38	28.927	2.689	29.128	2.666	29.43	2.633	15.837	2.958
0.45	0.1833	0.3667	25.527	3.088	25.728	3.066	26.23	3.011	11.937	3.777
0.5	0.1667	0.3333	5.227	5.227	5.228	5.228	13.13	4.5	5.137	5.137
0.55	0.15	0.3	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.6	0.1333	0.2667	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.65	0.1167	0.2333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.7	0.1	0.2	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.75	0.0833	0.1667	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.8	0.0667	0.1333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.85	0.05	0.1	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.9	0.0333	0.0667	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.95	0.0167	0.0333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
1	0	0	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137

2.2. Changes in Expected Profit When $t = 1$ for the Exponential Claims on Lower Barrier Model

Table B.3: Changes in weights of the expected profit (w_2) and $w_1 = w_3$ when $t = 1$ for the exponential claims on lower barrier model

t=1 Exponential			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0.5	0	0.5	35.327	1.912	35.528	1.887	35.93	1.837	21.137	1.811
0.4988	0.0025	0.4988	35.127	1.938	35.328	1.913	35.73	1.863	21.137	1.811
0.4975	0.005	0.4975	35.127	1.938	35.328	1.913	35.63	1.875	21.037	1.833
0.495	0.01	0.495	35.027	1.95	35.228	1.925	35.63	1.875	21.037	1.833
0.475	0.05	0.475	34.527	2.012	34.728	1.988	35.13	1.938	20.737	1.901
0.45	0.1	0.45	34.127	2.062	34.328	2.038	34.73	1.988	20.537	1.945
0.425	0.15	0.425	33.727	2.111	33.928	2.087	34.23	2.051	20.237	2.012
0.4	0.2	0.4	33.327	2.161	33.528	2.136	33.83	2.1	20.037	2.056
0.375	0.25	0.375	32.927	2.209	33.028	2.198	33.33	2.162	19.737	2.122
0.35	0.3	0.35	32.327	2.282	32.428	2.271	32.63	2.247	19.437	2.187
0.3335	0.333	0.3335	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.3	0.4	0.3	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.295	0.41	0.295	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.29	0.42	0.29	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.285	0.43	0.285	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.275	0.45	0.275	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.25	0.5	0.25	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.225	0.55	0.225	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.2	0.6	0.2	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.175	0.65	0.175	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.15	0.7	0.15	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.125	0.75	0.125	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.1	0.8	0.1	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.075	0.85	0.075	5.227	5.227	5.228	5.228	5.53	5.208	5.137	5.137
0.05	0.9	0.05	6.627	5.115	7.128	5.076	8.23	4.986	5.137	5.137
0.025	0.95	0.025	10.327	4.774	12.728	4.533	13.43	4.468	5.937	4.99
0	1	0	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169

Table B.4: Changes in weights of the expected profit (w_2) and $w_1 = 2w_3$ when $t = 1$ for the exponential claims on lower barrier model

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*	u_{M*}	M^*
0.6667	0	0.3333	19.627	3.767	20.328	3.69	21.33	3.581	9.637	4.254
0.665	0.0025	0.3325	18.727	3.869	19.328	3.804	20.53	3.673	9.037	4.377
0.6633	0.005	0.3317	18.227	3.925	18.928	3.85	20.13	3.719	8.737	4.439
0.66	0.01	0.33	17.527	4.004	18.228	3.929	19.53	3.788	8.237	4.54
0.6333	0.05	0.3167	12.327	4.57	13.728	4.426	15.63	4.228	5.137	5.137
0.6	0.1	0.3	5.227	5.227	5.228	5.228	8.03	5.004	5.137	5.137
0.5667	0.15	0.2833	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.5333	0.2	0.2667	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.5	0.25	0.25	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.4667	0.3	0.2333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.4447	0.333	0.2223	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.4	0.4	0.2	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.3933	0.41	0.1967	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.3867	0.42	0.1933	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.38	0.43	0.19	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.3667	0.45	0.1833	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.3333	0.5	0.1667	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.3	0.55	0.15	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.2667	0.6	0.1333	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.2333	0.65	0.1167	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.2	0.7	0.1	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.1667	0.75	0.0833	5.227	5.227	5.228	5.228	5.23	5.23	5.137	5.137
0.1333	0.8	0.0667	5.227	5.227	5.228	5.228	5.93	5.178	5.137	5.137
0.1	0.85	0.05	6.327	5.14	6.828	5.101	7.83	5.021	5.137	5.137
0.0667	0.9	0.0333	8.627	4.938	9.128	4.895	10.03	4.817	5.137	5.137
0.0333	0.95	0.0167	12.827	4.518	13.728	4.426	14.43	4.36	6.837	4.818
0	1	0	45.827	0.119	45.828	0.117	45.83	0.115	26.537	0.169

Table B.5: Changes in weights of the expected profit (w_2) and $2w_1 = w_3$ when $t = 1$ for the exponential claims on lower barrier model

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.3333	0	0.6667	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.3325	0.0025	0.6650	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.3317	0.0050	0.6633	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.33	0.010	0.66	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.3167	0.0500	0.6333	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.30	0.10	0.60	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.2833	0.1500	0.5667	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.2667	0.2000	0.5333	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.25	0.25	0.50	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.2333	0.3000	0.4667	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.2223	0.3330	0.4447	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.20	0.40	0.40	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1967	0.4100	0.3933	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1933	0.4200	0.3867	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.19	0.43	0.38	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1833	0.4500	0.3667	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1667	0.5000	0.3333	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1500	0.5500	0.3000	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1333	0.6000	0.2667	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1167	0.6500	0.2333	45.827	0.119	45.828	0.117	5.230	5.230	26.537	0.169
0.10	0.70	0.20	5.227	5.227	5.228	5.228	5.230	5.230	26.537	0.169
0.0833	0.7500	0.1667	5.227	5.227	5.228	5.228	5.230	5.230	26.537	0.169
0.0667	0.8000	0.1333	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.05	0.85	0.10	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.0333	0.9000	0.0667	5.227	5.227	5.428	5.213	6.530	5.131	5.137	5.137
0.0167	0.9500	0.0333	7.527	5.038	14.528	4.340	12.730	4.542	5.137	5.137
0	1	0	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169

2.3. Changes in Expected Utility When $t = 1$ for the Exponential Claims on Lower Barrier Model

Table B.6: Changes in weights of the expected utility (w_3) and $w_1 = w_2$ when $t = 1$ for the exponential claims on lower barrier model

t=1 Exponential			$\theta = 0.1$ & $\xi = 0.15$		$\theta = 0.1$ & $\xi = 0.2$		$\theta = 0.1$ & $\xi = 0.3$		$\theta = 0.2$ & $\xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.5	0.5	0	12.027	4.601	12.228	4.585	12.730	4.542	7.837	4.621
0.4988	0.4988	0.0025	10.327	4.774	10.628	4.749	11.230	4.697	6.937	4.799
0.4975	0.4975	0.0050	9.627	4.843	9.928	4.818	10.530	4.768	6.437	4.896
0.495	0.495	0.010	8.627	4.938	8.928	4.914	9.530	4.865	5.837	5.009
0.475	0.475	0.050	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.45	0.45	0.10	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.425	0.425	0.150	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.40	0.40	0.20	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.375	0.375	0.250	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.35	0.35	0.30	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.3335	0.3335	0.333	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.30	0.30	0.40	41.727	1.042	41.928	1.011	42.230	0.963	26.537	0.169
0.295	0.295	0.410	42.927	0.849	43.028	0.832	43.330	0.780	26.537	0.169
0.29	0.29	0.42	44.027	0.651	44.228	0.611	44.530	0.548	26.537	0.169
0.285	0.285	0.430	45.127	0.405	45.328	0.347	45.630	0.239	26.537	0.1697
0.275	0.275	0.450	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.25	0.25	0.50	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.225	0.225	0.550	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.20	0.20	0.60	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.175	0.175	0.650	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.15	0.15	0.70	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.125	0.125	0.750	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.10	0.10	0.80	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.075	0.075	0.850	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.05	0.05	0.90	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.025	0.025	0.950	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0	0	1	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169

Table B.7: Changes in weights of the expected utility (w_3) and $w_1 = 2w_2$ when $t = 1$ for the exponential claims on lower barrier model

t=1			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.6667	0.3333	0.0000	10.727	4.734	11.028	4.709	11.430	4.677	7.337	4.720
0.6650	0.3325	0.0025	9.127	4.891	9.328	4.876	9.830	4.836	6.337	4.915
0.6633	0.3317	0.0050	8.427	4.957	8.628	4.942	9.130	4.903	5.937	4.990
0.6600	0.3300	0.0100	7.327	5.055	7.628	5.032	8.130	4.995	5.337	5.101
0.6333	0.3167	0.0500	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.60	0.30	0.10	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.5667	0.2833	0.1500	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.5333	0.2667	0.2000	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.50	0.25	0.25	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.4667	0.2333	0.3000	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.4447	0.2223	0.3330	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.40	0.20	0.40	33.327	2.161	33.528	2.136	33.830	2.100	20.037	2.056
0.3933	0.1967	0.4100	34.727	1.988	34.928	1.963	35.230	1.926	21.337	1.766
0.3867	0.1933	0.4200	36.027	1.824	36.228	1.799	36.530	1.760	22.537	1.487
0.380	0.190	0.430	37.327	1.657	37.428	1.644	37.730	1.604	23.637	1.218
0.3667	0.1833	0.4500	39.627	1.347	39.828	1.319	40.130	1.277	25.837	0.568
0.3333	0.1667	0.5000	45.027	0.431	45.228	0.377	45.530	0.279	26.537	0.169
0.3000	0.1500	0.5500	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.2667	0.1333	0.6000	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.2333	0.1167	0.6500	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.20	0.10	0.70	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1667	0.0833	0.7500	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1333	0.0667	0.8000	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.10	0.05	0.85	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0667	0.0333	0.9000	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0333	0.0167	0.9500	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0	0	1	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169

Table B.8: Changes in weights of the expected utility (w_3) and $2w_1 = w_2$ when $t = 1$ for the exponential claims on lower barrier model

t=1			$\theta = 0.1 \ \& \ \xi = 0.15$		$\theta = 0.1 \ \& \ \xi = 0.2$		$\theta = 0.1 \ \& \ \xi = 0.3$		$\theta = 0.2 \ \& \ \xi = 0.3$	
w_1	w_2	w_3	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*	u_{M^*}	M^*
0.3333	0.6667	0	13.327	4.465	13.628	4.437	14.130	4.392	8.537	4.479
0.3325	0.6650	0.0025	11.627	4.643	12.028	4.606	12.530	4.563	7.437	4.700
0.3317	0.6633	0.005	10.927	4.714	11.228	4.688	11.830	4.636	7.037	4.779
0.33	0.66	0.01	9.827	4.823	10.228	4.789	10.830	4.738	6.337	4.915
0.3167	0.6333	0.05	5.227	5.227	5.328	5.220	6.030	5.170	5.137	5.137
0.30	0.60	0.10	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.2833	0.5667	0.150	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.2667	0.5333	0.20	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.25	0.50	0.25	5.227	5.227	5.228	5.228	5.230	5.230	5.137	5.137
0.2333	0.4667	0.30	5.227	5.227	5.228	5.228	5.230	5.230	26.537	0.169
0.2223	0.4447	0.333	44.827	0.480	45.028	0.431	45.330	0.346	26.537	0.169
0.20	0.40	0.40	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1967	0.3933	0.41	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1933	0.3867	0.42	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.19	0.38	0.43	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1833	0.3667	0.45	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1667	0.3333	0.50	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.15	0.30	0.55	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1333	0.2667	0.60	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.1167	0.2333	0.65	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.10	0.20	0.70	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0833	0.1667	0.75	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0667	0.1333	0.80	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.05	0.10	0.85	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0333	0.0667	0.90	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0.0167	0.0333	0.9500	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169
0	0	1	45.827	0.119	45.828	0.117	45.830	0.115	26.537	0.169

BIBLIOGRAPHY

- [1] Nie, C. *On Lower Barrier Insurance Risk Processes*. PhD thesis, University of Melbourne, Faculty of Business and Economics, **2012**.
- [2] Nie, C., Dickson, D.C.M., and Li, S. Minimizing the ruin probability through capital injections. *Annals of Actuarial Science*, 5(2):195–209, **2011**.
- [3] Kaas, R., Goovaerts, M., Dhaene, J., and Denuit, M. *Modern Actuarial Risk Theory-Using R*. Springer-Verlag Berlin Heidelberg, **2008**.
- [4] Hardy, M.R. *Approximating the Aggregate Claims Distribution*. Encyclopedia of Actuarial Science 1, **2006**.
- [5] Pentikainen, T. On the approximation of the total amount of claims. *ASTIN Bulletin*, 9(3):281–289, **1977**.
- [6] Pentikainen, T. Approximative evaluation of the distribution function of aggregate claims. *ASTIN Bulletin*, 17(1):15–39, **1987**.
- [7] Hipp, C. Approximation of aggregate claims distributions by compound poisson distributions. *Insurance: Mathematics and Economics*, 4(4):227 – 232, **1985**.
- [8] Hipp, C. Improved approximations for the aggregate claims distribution in the individual model. *ASTIN Bulletin*, 16(2):89–100, **1986**.
- [9] Papush, D.E., Patrik, G.S., and Podgaitis, F. Approximations of the aggregate loss distribution. In: *Casualty Actuarial Society Forum, Casualty Actuarial Society, Virginia Arlington*, pages 175–186, **2001**.
- [10] Beard, R.E., Pentikinen, T., and Pesonen, E. *Risk Theory*. Monographs on Statistics and Applied Probability. Chapman and Hall, London, 3 edition, **1984**.
- [11] Dickson, D.C.M. *Insurance Risk and Ruin*. Cambridge University Press, Cambridge, **2005**.

- [12] Bowers, N.L. Expansion of probability density functions as a sum of gamma densities with applications in risk theory. *Transactions of the Society of Actuaries*, XVIII(52):125–139, **1966**.
- [13] Haldane, J.B.S. The approximate normalization of a class of frequency distributions. *Biometrika*, 29:392404, **1938**.
- [14] Wilson, E.B. and Hilferty, M. The distribution of chi-square. *Proceedings of the National Academy of Science*, 17:684–688, **1931**.
- [15] Gerber, H.U. *An Introduction to Mathematical Risk Theory*. Huebner Foundation Monograph 8, Wharton School, University of Pennsylvania, **1979**.
- [16] Pollaczek, F. Über eine aufgabe der wahrscheinlichkeitstheorie. *Mathematische Zeitschrift*, 32(1):64–100, **1930**.
- [17] Khintchine, A. Mathematical theory of a stationary queue. *Matematicheskii Sbornik*, 39(4):73–84, **1932**.
- [18] Beekman, J.A. *Two Stochastic Processes*. A Halsted Press book. Almqvist & Wiksell International, **1974**.
- [19] Beekman, J.A. A series for infinite time ruin probabilities. *Insurance: Mathematics and Economics*, 4(2):129 – 134, **1985**.
- [20] Asmussen, S. *Ruin Probabilities*. World Scientific Publishing Company, Singapore, **2000**.
- [21] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. *Actuarial Mathematics*. Society of Actuaries, **1987**.
- [22] DeVyllder, F. *Advanced Risk Theory : A self-contained introduction / F. Etienne de Vylder ; forewords by Hans Buhlmann and Hans U. Gerber*. Bruxelles : Editions de l'Universite de Bruxelles, **1996**.
- [23] Grandell, J. Simple approximations of ruin probability. *Insurance: Mathematics and Economics*, 26:157–173, **2000**.

- [24] Beekman, J.A. A ruin function approximation. *Transactions of Society of Actuaries*, 21(59):41–48, **1969**.
- [25] De Vylder, F. A practical solution to the problem of ultimate ruin probability. *Scandinavian Actuarial Journal*, (2):114–119, **1978**.
- [26] Embrechts, P., Klüppelberg, C., and Mikosch, T. *Modelling Extremal Events for Insurance and Finance*. Springer-Verlag Berlin Heidelberg, **2001**.
- [27] Burnecki, K. and Teuerle, M. *Statistical Tools for Finance and Insurance, 2nd Edition*, chapter Ruin Probability in Finite Time, pages 341–379. Springer, Berlin, **2005**.
- [28] Asmussen, S. Approximations for the probability of ruin within finite time. *Scandinavian Actuarial Journal*, (1):31–57, **1984**.
- [29] Seal, H.L. Numerical calculation of the probability of ruin in the Poisson/Exponential Case. *Mitt. Verein. Schweiz. Versich. Math*, 72:77–100, **1972**.
- [30] Takács, L. *Combinatorial Methods in The Theory of Stochastic Processes*. Wiley series in probability and mathematical statistics. Probability and mathematical statistics. Wiley, **1967**.
- [31] Prabhu, N.U. On the ruin problem of collective risk theory. *The Annals of Mathematical Statistics*, 32:757–764, **1961**.
- [32] Segerdahl, C.O. When does ruin occur in the collective theory of risk. *Scandinavian Actuarial Journal*, (1-2):22–36, **1955**.
- [33] Iglehard, D.L. Diffusion approximations in collective risk theory. *Journal of Applied Probability*, 6:285–292, **1969**.
- [34] Grandell, J. *Aspects of risk theory*. Springer series in statistics: Probability and its applications. Springer-Verlag, United States of America, **1991**.
- [35] Dufresne, F.S., Gerber, H.U., and Shiu, E.S.W. Risk theory with the gamma process. *ASTIN Bulletin*, 21(2):177–192, **1991**.
- [36] Dickson, D.C.M. and Waters, H.R. Gamma processes and finite time survival probabilities. *ASTIN Bulletin*, 23(2):259–272, **1993**.

- [37] Waters, H.R. Excess of loss reinsurance limits. *Scandinavian Actuarial Journal*, (1):37–43, **1979**.
- [38] Waters, H.R. Some mathematical aspects of reinsurance. *Insurance: Mathematics and Economics*, 2(1):17 – 26, **1983**.
- [39] Centeno, M.L. Measuring the effects of reinsurance by the adjustment coefficient. *Insurance: Mathematics and Economics*, 5(2):169 – 182, **1986**.
- [40] Centeno, M. L. On combining quota-share and excess of loss. *ASTIN Bulletin*, 15(1):49–63, **1985**.
- [41] Centeno, M.L. Measuring the effects of reinsurance by the adjustment coefficient in the sparre anderson model. *Insurance: Mathematics and Economics*, 30(1):37 – 49, **2002**.
- [42] Hesselager, O. Some results on optimal reinsurance in terms of the adjustment coefficient. *Scandinavian Actuarial Journal*, 1-2(1990):80–95, **1990**.
- [43] Centeno, M.L. Excess of loss reinsurance and the probability of ruin in finite horizon. *Astin Bulletin*, 27(1):59–70, **1997**.
- [44] Dickson, D.C.M. and Waters, H.R. Reinsurance and ruin. *Insurance: Mathematics and Economics*, 19:61–80, **1996**.
- [45] Liang, Z. and Guo, J. Optimal proportional reinsurance and ruin probability. *Stochastic Models*, 23(2):333–350, **2007**.
- [46] Guerra, M. and Centeno, M.L. Optimal reinsurance policy: The adjustment coefficient and the expected utility criteria. *Insurance: Mathematics and Economics*, 42(2):529 – 539, **2008**.
- [47] Kolkovsko, E.T. Minimizing the ruin probability of risk processes with reinsurance. *International Journal of Pure and Applied Mathematics*, 1(46):83–93, **2008**.
- [48] Centeno, M.L. and Simoes, O. Optimal reinsurance. *Revista de la Real Academia de Ciencias Exactas Fsicas y Naturales Serie A-Matematicas*, 103(2):387–405, **2009**.
- [49] De Finetti, B. *Il Problema Dei Pieni*. Istituto italiano degli attuari, **1940**.

- [50] Bühlmann, H. *Mathematical Methods in Risk Theory*. Grundlehren der mathematischen Wissenschaft: A series of comprehensive studies in mathematics. Springer, **1970**.
- [51] Dickson, D.C.M. and Waters, H.R. Relative reinsurance retention levels. *ASTIN Bulletin*, 27:207–227, **1997**.
- [52] Ignatov, Z.G., Kaishev, V.K., and Krachunov, R.S. Optimal retention levels, given the joint survival of cedent and reinsurer. *Scandinavian Actuarial Journal*, 2004(6):401–430, **2004**.
- [53] Kaluszka, M. Truncated stop loss as optimal reinsurance agreement in one-period models. *ASTIN Bulletin*, 35(2):337–349, **2005**.
- [54] Dickson, D.C.M. and Waters, H.R. Optimal dynamic reinsurance. *ASTIN Bulletin*, 36(2):415–432, **2006**.
- [55] Vladimir, K.K. and Dimitrina, S.D. Excess of loss reinsurance under joint survival optimality. *Insurance: Mathematics and Economics*, 39(3):376 – 389, **2006**.
- [56] Christian, H. and Plum, M. Optimal investment for insurers. *Insurance: Mathematics and Economics*, 27(2):215 – 228, **2000**.
- [57] Liu, C.S. and Yang, H. Optimal investment for a insurer to minimize its probability of ruin. *North American Actuarial Journal*, 8(2):11–31, **2004**.
- [58] Schmidli, H. Optimal proportional reinsurance policies in a dynamic setting. *Scandinavian Actuarial Journal*, 2001(1):55–68, **2001**.
- [59] Schmidli, H. On minimizing the ruin probability by investment and reinsurance. *Annals of Applied Probability*, 12:890–907, **2002**.
- [60] Taksar, M. and Markussen, C. Optimal dynamic reinsurance policies for large insurance portfolios. *Finance and Stochastics*, 7(1):97–121, **2003**.
- [61] Hipp, C. and Vogt, M. Optimal dynamic XL reinsurance. *ASTIN Bulletin*, 33(2):193–207, **2003**.

- [62] Castillo, M.T. and Parrocha, G. Stochastic control theory for optimal investment. *Working Paper, Department of Actuarial Studies, University of New South Wales, Australia, 2003.*
- [63] Gaier, J. and Grandits, P. Ruin probabilities in the presence of regularly varying tails and optimal Investment. *Insurance: Mathematics and Economics*, 30(2):211–217, **2002.**
- [64] Hipp, C. Stochastic Control with Application in Insurance. In *Stochastic Methods in Finance*, volume 1856 of *Lecture Notes in Mathematics*, pages 127–164. Springer Berlin Heidelberg, **2004.**
- [65] Schmidli, H. Asymptotics of ruin probabilities for risk processes under optimal reinsurance and investment policies: The large claim case. *Queueing Systems*, 46(1-2):149–157, **2004.**
- [66] Irgens, C. and Paulsen, J. Optimal control of risk exposure, reinsurance and investments for insurance portfolios. *Insurance: Mathematics and Economics*, 35(1):21 – 51, **2004.**
- [67] Ma, J., Bai L., and Liu, J. Minimizing the probability of ruin under interest force. *Applied Mathematical Sciences*, 17:843–851, **2008.**
- [68] Kasumo, C. Minimizing Probability of Ultimate Ruin by Proportional Reinsurance and Investment. Master’s thesis, University of Dar es salaam, **2011.**
- [69] Joseph, A. Minimizing the Probability of Ultimate Ruin by Excess of Loss Reinsurance and Investments. Master’s thesis, University of Dar es salaam, **2013.**
- [70] De Finetti, B. Su un’impostazione alternativa della teoria collettiva del rischio. 2:433–443, **1957.**
- [71] Gerber, H.U. *Entscheidungskriterien für den zusammengesetzten Poisson-Prozess.* **1969.** Diss. Math. ETH Zrich, Nr. 4383, 0000. Ref.: Buhlmann, H. ; Korref.: Huber, P.
- [72] Dickson, D.C.M. and Waters, H.R. Some optimal dividends problems. *ASTIN Bulletin*, 34(1):49–74, **2004.**

- [73] Bäuerle, N. Approximation of optimal reinsurance and dividend payout policies. *Mathematical Finance*, 14:99–113, **2004**.
- [74] Azcue, P. and Muler, N. Optimal reinsurance and dividend distribution policies in the Cramer-Lundberg model. *Mathematical Finance*, 15(2):261–308, **2005**.
- [75] Schmidli, H. Optimization in non-life insurance. *Stochastic Models*, 22:689–722, **2006**.
- [76] Thonhauser, S. and Albrecher, H. Dividend maximization under consideration of the time value of ruin. *Insurance: Mathematics and Economics*, 41(1):163 – 184, **2007**.
- [77] Beveridge, C.J., Dickson, D.C.M., and Wu, X. Optimal dividends under reinsurance. *Bulletin de l'Association Suisse des Actuaires*, 1(2):149–166, **2008**.
- [78] Liang, Z. and Yao, J. Optimization of Dividend and Reinsurance Strategies Under Ruin Probability Constraint. *ArXiv e-prints*, **2010**.
- [79] Meng, H. and Siu, T.K. On optimal reinsurance, dividend and reinvestment strategies. *Economic Modelling*, 28(12):211 – 218, **2011**.
- [80] Zhou, M. and Yuen, K.C. Optimal reinsurance and dividend for a diffusion model with capital injection: Variance premium principle. *Economic Modelling*, 29(2):198 – 207, **2012**.
- [81] Wu, Y. Optimal reinsurance and dividend strategies with capital injections in Cramer-Lundberg approximation model. *Bulletin of the Malaysian Mathematical Sciences Society*, 36(1):193–210, **2013**.
- [82] Schnieper, R. Portfolio optimization. *ASTIN Bulletin*, 30(1):195–248, **2000**.
- [83] Forbes, S.W. Growth performances of nonlife insurance companies: 1955-1966. *The Journal of Risk and Insurance*, 37(3):341–360, **1970**.
- [84] Lee, C.F. and Forbes, S.W. *Dividend policies of non-life insurance companies : a theoretical and empirical analysis*. College of Commerce and Business Administration, University of Illinois at Urbana-Champaign, **1978**.

- [85] Oyatoye, E.O. and Arogundade, K.K. Optimization models for insurance portfolio optimization in the presence of background risk. *British Journal of Management Economics*, 1(2):114–127, **2011**.
- [86] Bernoulli, D. Exposition of a New Theory on the Measurement of Risk. *Econometrica*, 22(1):23–36, **1954**.
- [87] Von Neumann, J. and Morgenstern, O. *Theory of Games and Economic Behavior*. Princeton University Press, **1947**.
- [88] Hwang, C.L and Yoon, K. *Multiple Attribute Decision Making: Methods and Applications: A State-of-the-Art Survey*. Lecture notes in economics and mathematical systems. Springer-Verlag, New York, **1981**.
- [89] Rao, R.V. *Introduction to Multiple Attribute Decision-making (MADM) Methods*. Springer Series in Advanced Manufacturing. Springer, London, **2007**.
- [90] Dodgson, J., Spackman, M., Pearman, A., and Phillips, L. *Multi-Criteria Analysis: A Manual*. Department for Communities and Local Government, **2000**.
- [91] Saaty, T.L. A scaling method for priorities in hierarchical structures. *Journal of Mathematical Psychology*, 15(3):234–281, **1977**.
- [92] Saaty, T.L. *The Analytic Hierarchy Process*. McGraw-Hill, **1980**.
- [93] Chou, A.T.W., Kalaba, R.E., and Spingarn, K. A comparison of two methods for determining the weights of belonging to fuzzy sets. *Journal of Optimization Theory and Applications*, 27(4):531–538, **1979**.
- [94] Nijkamp, P. Stochastic quantitative and qualitative multicriteria analysis for environmental design. *Papers of the Regional Science Association*, 39:175–199, **1977**.
- [95] Churchman, C.W., Ackoff, R.L., and Arnoff, E.L. *Introduction to Operations Research*. John Wiley & Sons, **1957**.
- [96] Bernardo, J.J. and Blin, J.M. A Programming Model of Consumer Choice Among Multi-Attributed Brands. *Journal of Consumer Research*, 4(2):111–118, **1977**.

- [97] Churchman, C.W. and Ackoff, R.L. An approximate measure of value. *Journal of the Operations Research Society of America*, 2(2):172–187, **1954**.
- [98] MacCrimmon, K.R. *Decision making Among Multiple-attribute Alternatives: A Survey and Consolidated Approach*. ARPA order. No. 189-1. Rand Corporation, **1968**.
- [99] Klee, A.J. The role of decision models in the evaluation of competing environmental health alternatives. *Management Science*, 18(2):B–52–B–67, **1971**.
- [100] Saaty, T.L. *Decision Making for Leaders: The Analytic Hierarchy Process for Decisions in a Complex World*. The analytic hierarchy process series. RWS Publications, **1990**.
- [101] Saaty, T.L. How to make a decision: The analytic hierarchy process. *European Journal of Operational Research*, 48(1):9 – 26, **1990**.
- [102] Saaty, T.L. Decision making with the analytic hierarchy process. *Int J Serv Sci*, 1(1):83–97, **2008**.
- [103] Benayoun, R., Roy, B., and Sussman, N. *Manual de Reference du Program ELECTRE, Note de Synthese et Formation*. Number 25 in Princeton University. Econometric Research Program. Research memorandum. Direction Scientifique SEMA, **1966**.
- [104] Tzeng, G.H. and Huang, J.J. *Multiple Attribute Decision Making: Methods and Applications*. Chapman & Hall. **2011**.
- [105] Opricovic, S. *Multicriteria Optimization in Civil Engineering*. Faculty of Civil Engineering, Belgrade, Serbian, **1998**.
- [106] Wang, T.C. and Hsu, J.C. Evaluation of the business operation performance of the listing companies by applying TOPSIS method. In *SMC (2)*, pages 1286 – 1291. IEEE, **2004**.
- [107] Wu, D. and Olson, D.L. A TOPSIS data mining demonstration and application to credit scoring. *IJDWM*, 2(3):16–26, **2006**.
- [108] Shih, H.S., Shyur, H.J., and Lee, E.S. An Extension of TOPSIS for Group Decision Making. *Mathematical and Computer Modelling*, 45(7-8):801–813, **2007**.

- [109] Jahanshahloo, G.R., Lotfi, F.H., and Davoodi, A.R. Extension of TOPSIS for decision-making problems with interval data: Interval efficiency. *Mathematical and Computer Modelling*, 49(5-6):1137–1142, **2009**.
- [110] Bulgurcu, B. Application of TOPSIS technique for financial performance evaluation of technology firms in istanbul stock exchange market. *Procedia - Social and Behavioral Sciences*, 62:1033 – 1040, **2012**.
- [111] Zhu, X., Wang, F., Liang, C., Li, J., and Sun, X. Quality credit evaluation based on TOPSIS: Evidence from air-conditioning market in China. *Procedia Computer Science*, 9:1256 – 1262, **2012**.
- [112] Hosseini, S.H.S., Ezazi, M.E., and Heshmati, M.R. Top companies ranking based on financial ratio with AHP-TOPSIS combined approach and indices of Tehran Stock Exchange - a comparative study. *International Journal of Economics and Finance*, 5(3):126–133, **2013**.
- [113] Behzadian, M., Otaghsara, S.K., Yazdani, M., and Ignatius, J. A state-of-the-art survey of TOPSIS applications. *Expert Systems with Applications*, 39(17):13051 – 13069, **2012**.
- [114] Kim, G., Park, C.S., and Yoon, K.P. Identifying investment opportunities for advanced manufacturing systems with comparative-integrated performance measurement. *International Journal of Production Economics*, 50(1):23 – 33, **1997**.
- [115] Parkan, C. and Wu, M.L. On the equivalence of operational performance measurement and multiple attribute decision making. *International Journal of Production Research*, 35(11):2963–2988, **1997**.
- [116] Zanakakis, S.H., Solomon, A., Wishart, N., and Dublish, S. Multi attribute decision making: A simulation comparison of select methods. *European Journal of Operational Research*, 107(3):507 – 529, **1998**.
- [117] Yeh, C.H. A problem-based selection of multi-attribute decision-making methods. *International Transactions in Operational Research*, 9(2):169–181, **2002**.
- [118] Velasquez, M. and Hester, P.T. An analysis of multi-criteria decision making methods. *International Journal of Operations Research*, 10(2):56–66, **2013**.

- [119] Dunford, N. and Schwartz, J.T. *Linear Operators: General Theory*. Linear Operators. Interscience Publishers, **1958**.
- [120] Riesz, F. Untersuchungen ber systeme integrierbarer funktionen. *Mathematische Annalen*, 69(4):449–497, **1910**.
- [121] Wang, Z.X. and Wang, Y.Y. Evaluation of the provincial competitiveness of the Chinese High-Tech Industry using an improved TOPSIS method. *Expert Syst. Appl.*, 41(6):2824–2831, **2014**.
- [122] Antucheviciene, J., Zavadskas, E.K., and Zakarevicius, A. Multiple criteria construction management decisions considering relations between criteria. *Technological and Economic Development of Economy*, 16(1):109–125, **2010**.
- [123] De Maesschalck, R. and Massart, D.L. The Mahalanobis distance. *Chemometrics and Intelligent Laboratory Systems*, 50(1):1–18, **2000**.
- [124] Mahalanobis, P.C. On the generalised distance in statistics. *Proceedings of the National Institute of Science, India*, 2(1):49–55, **1936**.
- [125] Garca, A.J., Ibarra, M.G., and Rico, P.L. Improvement of TOPSIS technique through integration of Malahanobis distance: A case study. Anaheim, California, **2009**. 14th Annual International Conference on Industrial Engineering Theory, Applications and Practice.
- [126] Ching-Hui, C., Jyh-Jiuan, L., Jyh-Horng, L., and Miao-Chen, C. Domestic open-end equity mutual fund performance evaluation using extended TOPSIS method with different distance approaches. *Expert Systems with Applications*, 37(6):4642 – 4649, **2010**.
- [127] Lahby, M, Cherkaoui, L., and Adib, A. New multi access selection method based on mahalanobis distance. *Applied Mathematical Sciences*, 6(53-56):2745–2760, **2012**.
- [128] Vega, A., Aguarn, J., Garca-Alcaraz, J., and Moreno-Jimnez, J.M. Notes on dependent attributes in TOPSIS. *Procedia Computer Science*, 31(0):308 – 317, **2014**.
- [129] Miller, I. and Miller, M. *John E. Freund's Mathematical Statistics with Applications*. Pearson/Prentice Hall, **1998**.

- [130] Wasserman, L. *All of Statistics: A Concise Course in Statistical Inference*. Springer Texts in Statistics. Springer, **2004**.
- [131] Abramowitz, M. and Stegun, I.A. *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*. Applied mathematics series. Dover Publications, **1964**.
- [132] Panjer, H.H. Recursive evaluation of a family of compound distributions. *ASTIN Bulletin*, 12:22–26, **1981**.
- [133] Nie, C., Dickson, D.C.M., and Li, S. The finite time ruin probability in a risk model with capital injections. *Scandinavian Actuarial Journal*, 2015(4):301–318, **2015**.
- [134] Pafumi, G. On the time value of ruin, H.U. Gerber and E.S.W. Shiu. *North American Actuarial Journal*, 2(1):75–76, **1998**.

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