INVESTIGATION OF SOME NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

BAZI DOĞRUSAL OLMAYAN KESİRSEL MERTEBEDEN DENKLEMLERİN İNCELENMESİ

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Müfit ŞAN

ABSTRACT

INVESTIGATION OF SOME NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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In this study, we take into consideration the initial value problems for the fractional differential equations in the complex plane and the Dirichlet boundary problems derived by the fractional Laplacian called as Riesz fractional derivative of functions of several variables, and its generalization.

In the first chapter we dwell on the historical developments and motivations related to the problems which we deal with in chapters 3 and 4.

In the second chapter, some well-known facts and primarily results related to the next two chapters are introduced to make the next chapters more understandable.

In Section 3, we investigate the existence and uniqueness of the solutions of initial-value problems for two different nonlinear complex fractional differential equations, one of which involves the fractional derivative and a second which involves a modification of fractional derivative. At first, the conditions for the nonlinear term of the fractional differential equations mentioned above are obtained to make the problems well-posed. In the sequel, we find the sufficient conditions for the nonlinear terms to be able to show the local existence of the solutions in the certain spaces. Moreover, by imposing extra conditions, such as Lipschitz condition, we prove the uniqueness of the solution in the appropriate spaces.

In the last chapter, the existence and uniqueness of the generalized solution of the Dirichlet boundary problems involving fractional Laplacian and, more generally, its generalization in the appropriate space are investigated. This investigation is made in sublinear, linear and superlinear cases by depending on nonlinear part of this problem. By obtaining some sufficient conditions related to nonlinear part of the problem considered according to these three cases we give some existence results for generalized solution of this problem. Furthermore, we establish uniqueness of the generalized solution for the considered problem.

Key Words: Initial-value problem, fractional differential equations, Drichlet boundary-value problem, existence and uniqueness problem, fixed point theorem, fractional Sobolev Spaces, nonlocal operator.

ÖZET

BAZI DOĞRUSAL OLMAYAN KESİRSEL MERTEBEDEN DENKLEMLERİN İNCELENMESİ

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Bu tez çalışmasında kesirsel türev operatörü ya da bu operatörün genelleşmesini içeren bazı Cauchy ve Dirichlet problemlerinin çözümlerinin uygun uzaylarda varlığı ve tekliği araştırılmıştır. Bu Cauchy problemlerinden biri,

$$\begin{array}{rcl} D^{\alpha}u(z) & = & f\left(z,u(z)\right) \\ u(0) & = & 0, \end{array}$$

şeklindedir ve burada $0<\alpha<1$, Ω, $\mathbb{C}$'de bir bölge olmak üzere $f:\Omega\times\mathbb{C}\to\mathbb{C}$ kompleks değerli ve genellikle lineer olmayan bir fonksiyon, D^{α}, tek değişkenli kompleks değerli fonksiyonlar için (2.6)'daki gibi tanımlanan kesirsel türev operatörüdür.

Tezin üçüncü bölümünde yer verilecek olan bu problemin, $\mathbb{U}$ birim disk olmak üzere $\mathbb{U}$'nun kapanışı içindeki uygun bir diskin üzerinde sürekli ve içinde analitik olan çözümlerinin var olabilmesi için, f fonksiyonu üzerinde şu koşul elde edilmiştir:

- $\mathbb{D}=\{z\in\mathbb{U}:-\pi<\arg z\leq+\pi\}$ ve $\mathbb{D}^{*}=\left\{z\in\overline{\mathbb{U}}:-\pi<\arg z\leq+\pi\right\}$ olmak üzere, $f(z,t)$ fonksiyonu, $\mathbb{D}\times\mathbb{C}$'de analitik ve $\mathbb{D}^{*}\times\mathbb{C}$'de sürekli ve $z^{\alpha}f(z,t)$ fonksiyonu ise $\mathbb{U}\times\mathbb{C}$'de analitik ve $\overline{\mathbb{U}}\times\mathbb{C}$'de süreklidir.

Sonra yukarıda verilen koşul altında bu problemin, çözümleri bu problemin çözümleriyle aynı olan aşağıdaki integral denkleme denk olduğu gösterilmiştir:

$$u(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta, u(\zeta))}{(z-\zeta)^{1-\alpha}} d\zeta.$$

Daha sonra ise bu koşula ek olarak, g, $\mathbb{U}$'da analitik ve $\overline{\mathbb{U}}$'da sürekli bir fonksiyon olmak üzere,

$$|z^{\alpha} f(z,t)| \leq c\,|t|^{n_0} + |g(z)|, \quad ((z,t) \in \overline{\mathbb{U}} \times \mathbb{C}, n_0 \geq 1, c \geq 0),$$

eşitsizliğinin de sağlaması durumunda, Schauder sabit nokta teoremi ve Schwarz lemması kullanılarak bu integral denkleminin, dolayısıyla ilgili problemin, $\overline{\mathbb{U}}$ kapalı birim diski içindeki uygun bir diskin üzerinde sürekli ve içinde analitik olan en az bir çözümünün var olduğu gösterilmiştir.

Ayrıca, yukarıda verilen ilk koşul ile birlikte $f(z,t)$ fonksiyonu, $z^{\alpha} f(z,0)|_{z=0} = 0$ koşulunu ve Lipschitz tip koşulu:

$$|f(z,\eta) - f(z,\nu)| < \frac{\kappa}{|z|^{\alpha}} |\eta - \nu| \quad (0 \leq \kappa < 1),$$

sağlaması durumunda, yukarıdaki integral denklemin, dolayısıyla ilgili problemin, $\mathbb{U}$ içinde analitik ve üzerinde sürekli olan çözümünün tekliği Banach sabit nokta teoremi ve Schwarz Lemması yardımıyla gösterilmiştir.

Buna ek olarak, bu koşullar ile ilk koşul yerine aşağıdaki koşul altında problemin $\mathbb{U}$ içinde analitik ve sınırlı olan çözümlerinin varlığı ve tekliği gösterilmiştir:

- $f(z,t)$ ve $z^{\alpha} f(z,t)$ fonksiyonları, sırasıyla $\mathbb{D} \times \mathbb{C}$'de ve $\mathbb{U} \times \mathbb{C}$'de analitiktir.

Bu çalışmada, daha önce Ibrahim ve Darus [15] tarafından $\mathbb{U}$'da birebir ve analitik çözümlerinin varlığı ve tekliği araştırılan bu Cauchy probleminin, varsaydıkları aşağıdaki koşul altında kötü konulmuş bir problem olduğu gösterilmiştir :

- $f(z,t)$, $\mathbb{U} \times \mathbb{C}$'de analitik ve süreklidir ve $\|f\| \leq M$ olacak şekilde $0 < M < \infty$ vardır.

Bunun dışında, koydukları bu koşullar altında ilgili problemin birebir ve analitik çözümlerinin olamayacağını gösteren uygun bir örnek verilmiştir. Buna ek olarak, $z^{\alpha} f(z,0)|_{z=0} = 0$ koşulu koyulmaksızın da problemin yukarıda verilen integral denkleme denk olduğu iddiasının doğru olmadığı ispatlanmıştır [Proposition 3.1.11]. Ayrıca, verdikleri varlık teoreminin ispatında [15, Theorem 4.1] yapılan hatalara yer verilmiştir [Subsection 1.2].

Bu problemde $f(z,t)$ fonksiyonunun $\mathbb{U} \times \mathbb{C}$'de analitik olması durumunda, analitik bir çözümün var olabilmesi için yukarıda verilen Cauchy probleminin nasıl modifiye edilmesi gerektiği düşünülmüş ve $D^\alpha \circ z^\alpha$, (2.11)'de verilen modifiye kesirsel türev operatörü olmak üzere,

$$\begin{array}{rcl} (D^\alpha \circ z^\alpha)u(z) & = & f\left(z, u(z)\right) \\ u(0) & = & 0, \end{array}$$

şeklindeki Cauchy problemi göz önüne alınmış ve incelenmiştır. Bu problem için de uygun koşullar elde edilmiş ve bu koşullar altında problemin yukarıda bahsedilen tipteki çözümlerinin var ve tek olduğu ispatlanmıştır.

Çok değişkenli fonksiyonlar için tanımlı kesirsel türev operatörünün bir genelleşmesi olan ve (1.8) verilen $\mathcal{L}_K$ operatörünü içeren,

$$\begin{cases} -\mathcal{L}_K u + g(x,u) = h(x) & x \in \Omega \\ u = 0 & x \in \mathbb{R}^n \backslash \Omega, \end{cases}$$

şeklindeki Dirichlet problemine tezin dördüncü bölümünde yer verilmiş olup bu problemde $n > 2s$ $(s \in (0,1))$ olacak şekilde bir doğal sayı, Ω, sınırı Lipschitz sınıfından olan $\mathbb{R}^n$'de sınırlı bir bölge, h genelleşmiş bir fonksiyon, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ ise aşağıdaki koşulu sağlayan bir Carathédory fonksiyonudur:

- Öyle bir $\mu > 0$ ve $i = 0, 1$ için $a_i(x) \geq 0$ olacak şekilde $a_i \in L_{p_i}(\Omega)$ fonksiyonları vardır ki

  $$|g(x,t)| \leq a_1(x)\,|t|^\mu + a_0(x),$$

  eşitisizliği hemen hemen her $x \in \Omega$ ve her $t \in \mathbb{R}$ için sağlanır.

Bu problem, yukarıdaki eşitsizlikteki μ parametresinin bulunduğu $0 < \mu < 1$, $\mu = 1$ ve $1 < \mu$ aralıklarına göre sırasıyla lineer altı, lineer ve lineer üstü ve $0 < \mu < \frac{n+2s}{n-2s}$, $\mu = \frac{n+2s}{n-2s}$, ve $\frac{n+2s}{n-2s} < \mu$ aralıklarına göre ise sırasıyla kritik altı, kritik ve kritik üstü olarak adlandırılacak olan durumlarda incelenmiştir . Bu durumlar dikkate alınarak yukarıdaki koşullarda yer alan p_i sayıları ve ekstra koşullar elde edilip Teorem 2.6.11 ile verilen varlık teoremi ve Lemma 2.6.9 yardımıyla problemin genelleşmiş çözümünün varlığı gösterilmiştir. $\mathcal{L}_K$ operatörü, kesirsel Laplasyanın bir genelleşmesi olduğundan, elde edilen bu sonuçlar hemen hemen aynı koşullar altında $\mathcal{L}_K$ operatörü yerine kesirsel

Laplasyan içeren yukarıdaki problem için de geçerli olduğu gösterilmiştir. Bunlara ek olarak, g fonksiyonunun uygun bir $k > 0$, her $x \in \Omega$ ve her $t_1, t_2 \in \mathbb{R}$ için

$$(g(x,t_1) - g(x,t_2))\,(t_1 - t_2) > -k\,|t_1 - t_2|^2\,,$$

eşitsizliğini sağlaması durumunda problemin genelleşmiş çözümünün mevcut ise tek olduğu gösterilmiştir.

Belirtmek gerekir ki, yukarıdaki problemlerde yer alan kompleks değişkenli ve çok değişkenli fonksiyonlar için tanımlanan kesirsel integral ve türev tanımlarını da içeren kesirsel kalkülüsün tarihsel gelişiminden tezin birinci bölümde bahsedilecektir. Bunu müteakiben, özellikle göz önüne aldığımız problemlerle yakından ilişkili ve daha önce çalışılmış kesirsel türevli denklem içeren Cauchy ve Dirichlet sınır-değer problemlerine yer verilmiştir.

Göz önüne aldığımız problemlerde yer alan kompleks değişkenli fonksiyonlar için yapılmış farklı kesirsel türev tanımlarının arasındaki farkları, kesirsel Laplasyan ve $\mathcal{L}_K$ operatörünün özelliklerini, problemlerin çözümlerinin aranacağı fonksiyon uzaylarını ve özelliklerini, bu problemlerin çözümlerinin varlığını ve tekliğini göstermek için kullanılacak olan sabit nokta ve varlık teoremlerini içeren ön bilgilere ve bilinen gerçeklere ikinci bölümde değinilmiştir.

Anahtar Kelimeler: Başlangıç-değer problemi, kesirsel mertebeden diferansiyel denklemler, Dirichlet sınır-değer problemi, varlık ve teklik problemi, sabit nokta teoremi, kesirsel Sobolev Uzayları, lokal olmayan operatörler.

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SYMBOLS AND ABBREVIATIONS

$\Re\{z\}$	Real part of a complex number z
$\mathbb{U}_R$	The disc with the center 0 and radius $R > 0$.
$\mathbb{D}_R$	$\mathbb{D}_R \equiv \{z \in \mathbb{U}_R : -\pi < \arg z \leq \pi\}$
$\mathbb{D}_R^*$	$\mathbb{D}_R^* \equiv \left\{z \in \overline{\mathbb{U}}_R : -\pi < \arg z \leq \pi\right\}$
$\mathbb{U}$	The open unit disc
$\mathbb{D}$	$\mathbb{D} \equiv \{z \in \mathbb{U} : -\pi < \arg z \leq \pi\}$
$\mathbb{D}^*$	$\mathbb{D}^* \equiv \left\{z \in \overline{\mathbb{U}} : -\pi < \arg z \leq \pi\right\}$
Q	$Q := \mathbb{R}^{2n} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$
Γ	Gamma Function
$B(.,.)$	Beta Function
${}_cI_x^\alpha$	Riemann-Liouville fractional integral
${}_cD_x^\alpha$	Riemann-Liouville fractional derivative
${}_{z_0}I_z^\alpha$	Fractional integral for the functions of complex variable
${}_{z_0}D_z^\alpha$	Fractional derivative for the functions of complex variable
$\mathcal{L}_K$	The non-local integrodifferential operator
$(-\Delta)^s$	Fractional Laplacian
$\mathcal{F}u$	The Fourier transformation of the function u
$H_\Omega^s(\mathbb{R}^n)$	Fractional Sobolev Spaces
$C^0(\Omega)$	The space of continuous functions on the set Ω
$\mathcal{B}_R$	The function space consists of the functions being analytic on $\mathbb{U}_R$ and continuous on its closure
$\mathcal{A}$	The function space consists of the functions being analytic and bounded on $\mathbb{U}$

1.INTRODUCTION

1.1. On the Fractional Integral and Derivative

The origin of fractional Calculus goes back to a letter sent from L'Hospital to Leibniz in 1695 in which L'Hospital asked him the meaning of non-integer order derivatives, especially 1/2-derivative. Right after that, there were many correspondences between Leibniz, J. Wallis, J. Bernoulli and L'Hospital in which Leibniz made some remarks about this question. The such subject also engaged the attention of Leonhard Euler. In 1798, he first evaluated the derivative $\frac{d^px^a}{dx^p}$ for the parameters $p = 2$ and $a = 6$ and he indicated that for integer n, d^n can be founded by continued differentiation but it is complicated for fraction n. The most important contribution of him to fractional calculus is that he developed the Gamma function (generalization of the factorial).

In 1812, Laplace was the first researcher who gave the detailed definition of fractional derivative. However, the domain of this operator only consists of the functions representable by the integral $\int T(t)t^{-x}dt$.

In 1819, Lacroix addressed the same question considered by Euler and a small part of his book was devoted for this question. There he first gave the formula of $\frac{d^nx^m}{dx^n}$ for the integers m, n such that $n \leq m$ as follows:

$$\frac{d^nx^m}{dx^n} = \frac{m!}{(m-n)!}x^{m-n},$$

and by using this and Gamma function he gave the answer of the question posed by Euler with the following equation:

$$\frac{d^px^m}{dx^p} = \frac{\Gamma(m+1)}{\Gamma(m-p+1)}x^{m-p},$$

for the non-integer values m and p [24].

In 1822, Joseph B. J. Fourier developed a definition of fractional operations:

$$\frac{d^px^a}{dx^p} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)d\alpha \int_{-\infty}^{\infty} \lambda^p cos[\lambda(x-\alpha) + \frac{p\pi}{2}]d\lambda,$$

where p is a negative or positive real number and the function f has the representation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)d\alpha \int_{-\infty}^{\infty} cos(\lambda(x-\alpha)d\lambda$$

is given by him. This definition is suitable for the sufficiently "good" function.

The development process of fractional calculus taking place between Leibniz and Fourier was limited to the discussion about the meaning of fractional derivative and formulating the fractional derivative in various way.

In 1823, Abel is the first who used the factional operations for finding the solution of an integral equation arising in a physical problem called as tautochrone problem. Abel dealt with the equation (omitting the problem)

$$k = \int_0^x (x-\zeta)^{-\frac{1}{2}} u(\zeta) d\zeta,$$

where the function u is to be determined. He first multiplied the both sides of the equation above with $\frac{1}{\Gamma(1/2)}$ and wrote the resulting equation as:

$$k\Gamma(1/2) = \frac{d^{-1/2}}{dx^{-1/2}}[u(x)].$$

After that, he applied $\frac{d^{1/2}}{dx^{1/2}}$ to the both sides of the equation above, and used the fact that (under suitable conditions on u) $D^{1/2}D^{-1/2}u = D^0 u = u$, therefore by using the fractional derivative of constant k he obtained

$$u(x) = \frac{k}{\pi\sqrt{x}}.$$

Actually, Abel found the solutions of the more general equation:

$$\frac{1}{\Gamma(1-\mu)} \int_0^x \frac{u(\zeta)}{(x-\zeta)^\mu} d\zeta = k(x), \ x > 0, \ 0 < \mu < 1,$$

as one can see later, the right hand side of which leads to Riemann-Liouville fractional integral of order $1-\mu$, and the inversion of which is equivalent to Riemann-Liouville fractional integral of order μ. [33]

From this aspect, the study of Abel is important for the idea of generalizing the derivative to non-integer order, although yet he didn't have such an intention.

Almost ten years after the study of Abel, J. Liouville presented his studies in succession which contributed enormously to this field and lead him as a real creator of fractional calculus. In one of them, he developed three different definitions related to fractional derivative. The first one is applicable to function $u(x)$ which can be represented as a series in the form:

$$u(x) = \sum_{k=0}^{\infty} b_k e^{a_k x}, \quad Re\{a_k\} > 0. \tag{1.1}$$

By extending the known result for derivatives of integer order $D^n e^{ax} = a^n e^{ax}$ in a natural way to the result for derivatives of arbitrary order $D^\alpha e^{ax} = a^\alpha e^{ax}$ and by applying this to all term of (1.1), he obtained

$$D^\alpha u(x) = \sum_{k=0}^{\infty} b_k a_k^\alpha e^{a_k x}.$$

The other definition in same work of him was developed for the function u in the form $u(x) = x^{-a}$ such that x and a are non-negative real number and was given as follows:

$$D^\alpha x^{-a} = \frac{(-1)^\alpha \Gamma(a+\alpha)}{\Gamma(a)} x^{-a-\alpha},$$

for an arbitrary α.

The domain of the operator (for the second definition) consists of the functions in the form x^{-a} and the existence of $D^\alpha u(x)$ (for the first definition) depends on the convergence of the related series, so these two facts represent the restrictions of two definitions above.

In addition to this, in the same paper he derived the formula:

$$I^\alpha u(x) = \frac{1}{(-1)^\alpha \Gamma(\alpha)} \int_0^\infty u(x+t) t^{\alpha-1} dt, \quad -\infty < x < \infty, \quad \Re\{\alpha\} > 0. \tag{1.2}$$

The formula obtained from (1.2) by omitting the factor $(-1)^\alpha$ is now called as Liouville form of fractional integration.

Moreover, Liouville was the first who attempted to solve differential equations involving fractional derivative. In one of his memories, he wrote: "The ordinary differential equation $d^n u/dx^n = 0$ has the complementary solution $u_c = c_0 + c_1 x + ... + c_{n-1} x^{n-1}$. Thus $d^\alpha u/dx^\alpha = 0$ (α is arbitrary) should have a corresponding complementary solution. In contrast to integer case, many researchers including Riemann (as one can see below) gave different complementary functions depending on the different definitions of fractional operators given by them. Miller and Ross [24] explained this situation as follows: "This undetermined status of one complementary function was the origin of "a longstanding controversy" in this new mathematical field, which decimated the trust in general theory of fractional operations in its youths".

G. F. B. Riemann wrote a paper in 1847, when just a student, but it was only published ten years after his death. He searched a generalization of Taylor series and obtained the definition for fractional integral in the following:

$$I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_c^x \frac{u(\zeta)}{(x-\zeta)^{1-\alpha}} d\zeta + \psi(x), \quad x > c, \tag{1.3}$$

Because of the ambiguity in the lower limit of integration c, Riemann saw fit to add his definition a complementary function $\psi(x)$ [24].

H. Holmgren (1865) was the first researcher who used the formula (1.3) without the complementary function and he proposed that the fractional derivative is considered as an operation inverse of the fractional integral. Moreover, with his study he came a long way in the application of ordinary differential equations involving fractional operations, which began with the Liouville's study mentioned above. A few years later, Letnikov, probably being unaware of the study to Holmgrem, presented the theory of fractional calculus from Holmgrem's point of view. The study of Holmgrem remained little known both to contemporaries and to later generations of researchers and therefore it was undeservedly little cited [24].

In 1869, N. Ya. Sonin [36] published his paper which led to what is now called Riemann-Liouville fractional integral. Later, A. V. Letnikov contributed to this topic with four papers between 1868 and 1872. They generalized the Cauchy's integral formula

$$u^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{u(\zeta)}{(\zeta - z)^{1+n}}d\zeta, \quad (n \in \mathbb{N}_0, z \in C), \tag{1.4}$$

to non-integer values of ρ such that $\Re\{\rho\} < 0$ and they shown that when ρ replaced by $-\alpha$ in (1.4), this generalization coincides with Riemann fractional integral given by (1.3) without complementary function, i.e, today's definition of Riemann-Liouville fractional integral:

$$ {}_cI_x^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_c^x \frac{u(\zeta)}{(x-\zeta)^{1-\alpha}}d\zeta, \quad \Re\{\alpha\} > 0, \tag{1.5}$$

By taking $c = -\infty$ and by changing variable $x - \zeta = t$ in (1.5), one can reach the formula of Liouville given in (1.2). This indicates why the formula (1.5) is called Riemann-Liouville fractional integral.

As mentioned above, Riemann-Liouville fractional derivative is a left inverse of the Riemann-Liouville integral and today's definition of fractional derivative is given as follows:

$$ {}_cD_x^{\alpha}u(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_c^x \frac{u(\zeta)}{(x-\zeta)^{\alpha}}d\zeta, \quad x > c. \tag{1.6}$$

In 1892, a work of Hadamard appeared in which he considered the fractional derivative of an analytic function via differentiation of its Taylor series (see also (2.7))

$$ {}_{z_0}D_z^{\alpha}u(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} a_n \big(z-z_0\big)^{n-\alpha}, \quad a_n = \frac{f^n(z_0)}{n!}, $$

which was known before his paper. Since he used this representation effectively as a mathematical tool, this approach is mentioned as Hadamard approach. In the same paper, he also obtained a representation of Riemann-Liouville fractional integral (see also (2.8)) in the complex plane in the following form:

$$ I_z^{\alpha}u(z) = \frac{z^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-\zeta)^{\alpha-1} f(z\zeta) d\zeta. $$

In the second half of the 20th century, some definitions of fractional integral (derivative) were developed for complex-valued functions $u(z)$ of a complex variable $z \in \mathbb{C}$ in different ways by K. Nishimoto [25], T. J. Osler [26], S. Owa [27], B. Ross [24], S.G. Samko et al. [33], etc. All of them except Samko et al. obtained their definitions by using contour integration, which are applicable only to analytic functions. In general, it is differed by the contours considered by them. Samko et al., however, introduced the fractional integral and derivative of the Lebesgue integrable complex valued functions by direct extending of Riemann-Liouville fractional integral and derivative given in real line. A part of the second chapter of this thesis is devoted for explaining in detail the definitions of Samko et al. and Owa.

The fractional integral and derivative of functions of many variable were first introduced as the potential type operators by M. Riesz in 1936. The negative fractional power of Laplace operator, i.e $(-\Delta)^{-\alpha/2}, \Re\{\alpha\} > 0:$

$$ I^{\alpha}u(x) = (-\Delta)^{-\alpha/2}u(x) = \frac{1}{\gamma(n,\alpha)} \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-\alpha}} dy, $$

is known as Riesz fractional integral or Riesz potential. The inverse of it, the positive fractional power of Laplace operator $(-\Delta)^{\alpha/2}, \Re\{\alpha\} > 0:$

$$ D^{\alpha}u(x) = (-\Delta)^{\alpha/2}u(x) = \frac{1}{\gamma(n,\alpha)} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}} dy, \tag{1.7} $$

is called Riesz fractional derivative or fractional Laplacian. The integrals above are singular and since the order of singularity of the integral in (1.7) is higher than the dimension of $\mathbb{R}^n$, it is called a hypersingular integral [33].

The more general form of the fractional Laplacian defined in (1.7) was introduced by R. Servadei and E. Valdinoci [34] in 2012, which is given by:

$$\mathcal{L}_K u(x) := \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^n \tag{1.8}$$

with an arbitrary kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfying some certain assumptions (See Definition 2.5.1).

1.2. Brief Overview of Results Related to Differential Equations Involving Fractional Derivative

The first study on the linear ordinary differential equations involving fractional derivative dates back to Liouville (1832). After him, H. Holmgren (1867) and Letnikov (1874) looked for the solutions of some equations of this type [33].

In 1918, O'Shaughnessy was the first who discussed the methods for solving the equation

$$D^{1/2}u(x) = \frac{u}{x},$$

where $D^{1/2}$ is the Riemann-Liouville derivative given in (1.6) with $c = 0$ and $\alpha = 1/2$ [33].

In 1938, E. Pitcher and W. E. Sewell first took into consideration the nonlinear fractional differential equation

$${}_cD_x^{\alpha} u(x) = f(x, u(x)), \tag{1.9}$$

where ${}_cD_x^{\alpha}$ is fractional derivative given in (1.6), $0 < \alpha < 1$ and $x > c$, provided that the nonlinear function $f(x, u)$ is bounded in an appropriate region G lying in $\mathbb{R}^2$ and fulfills the Lipschitz condition with respect to u:

$$|f(x, u_1) - f(x, u_2)| \leq K\,|u_1 - u_2|\,, \tag{1.10}$$

where $K > 0$ doesn't depend on x. By using the compositional relation $I^{\alpha}D^{\alpha}u = u$ they reduced the solution of the equation in (1.9) to that of the following Volterra-type equation:

$$u(x) = \frac{1}{\Gamma(\alpha)}\int_c^x \frac{f(\zeta, u(\zeta))}{(x-\zeta)^{1-\alpha}} d\zeta, \quad (x > c,\ c \in \mathbb{R}) \tag{1.11}$$

and they shown the existence of the continuous solution $u(x)$ for the integral equation above. However, the compositional relation used by them is not applicable to the equation (1.9), so the proof of this theorem is not correct. In any case, if one looks from the viewpoint of the idea of reducing the solution of the equation (1.9) to that of Volterra-type integral equation in (1.11), this study is important [18].

In 1954, Al Bassam is the first who considered (Cauchy problem) the equation (1.9) subjected to initial condition ${}_cI_x^{1-\alpha}u(c) = b$ with $b \in \mathbb{R}$ and $0 < \alpha < 1$, provided that the nonlinear function $f(x,u)$ is a continuous function in a suitable domain G in $\mathbb{R}^2$ such that $\sup_{(x,u)\in G} |f(x,u)| < \infty$ and that it fulfills the Lipschitz condition (1.10). By using the relation ${}_cI_x^{\alpha}\, {}_cD_x^{\alpha}u(x) = u(x) - \frac{b(x-c)^{\alpha-1}}{\Gamma(\alpha)}$, they reduced the considered Cauchy problem to the following Volterra-type equation:

$$u(x) = \frac{b(x-c)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_c^x \frac{f(x,\zeta)}{(x-\zeta)^{1-\alpha}} d\zeta, \quad (x > c,\ c \in \mathbb{R}). \tag{1.12}$$

By applying the method of successive approximations, Al-Bassam proved the existence of the continuous solution $u(x)$ to the considered problem. Moreover, he was probably the first to state that the method of contractive mapping can be applied to establish the uniqueness of the solution $u(x)$ to the integral equation in (1.12), and formally gave such a proof. Furthermore, he stated without proof that the equivalence of the Cauchy type problem considered and the integral equation (1.12), and for this reason his results on the existence and uniqueness of the continuous solution $u(x)$ can be true only for this integral equation. It needs to note that the conditions suggested by him are not appropriate for solving the problem considered in some simple cases; for example, for $f(x,u(x)) = u(x)$ [18].

In 1996, Delbosco ve Rodino [4] considered the equation

$$D_x^{\alpha}u(x) = f(x,u(x)), \quad (0 < \alpha < 1), \tag{1.13}$$

where D_x^{α} is fractional derivative given in (1.6) with $c = 0$, provided that the following condition is satisfied:

(i) There exists a real number σ with $0 \leq \sigma < \alpha < 1$ such that the functions $f(x,t)$ and $x^{\sigma}f(x,t)$ are continuous on $(0,1] \times \mathbb{R}$ and $[0,1] \times \mathbb{R}$, respectively.

They first proved the compositional relation $I_x^{\alpha}D_x^{\alpha}u = u$ for continuous function u defined on $[0,1]$ and by using this they reduced the equation in (1.13) to the integral equation in (1.11) with $c = 0$. After that they shown, by using Schauder's fixed point theorem, that the corresponding integral equation admits at least one continuous solution defined on $[0,\delta]$ for a suitable $\delta > 0$. Moreover, they proved, by using Banach fixed point theorem, that if, additionally, the function $f(x,t)$ fulfills the Lipschitz type condition:

$$\left|f\left(x,t_1\right)-f\left(x,t_2\right)\right|<\frac{\kappa}{|x|^{\sigma}}\left|t_1-t_2\right|, \tag{1.14}$$

then the related integral equation admits a unique continuous solution defined on $[0,1]$.

In 2005, Yu C. and Gao G. [38] investigated the same problem studied by Delbosco and Rodino, provided that the nonlinear function $f(x,u)$ is continuous on $[0,T]\times\mathbb{R}$ and satisfies the condition

$$\left|f\left(x,t_1\right)-f\left(x,t_2\right)\right|\le\lambda(x)h(r),$$

where $h(r)$ is continuous on $[0,\infty)$ and $h(0)=0$, $r=\left|t_1-t_2\right|$, $\left|I^{\alpha}\lambda(x)\right|<M$ for $x\in[0,T]$. Applying Schauder's fixed point theorem, they shown that the related fractional differential equation has at least one continuous solution u defined on $[0,\delta]$ for a suitable $\delta<T$.

In 2007, Ibrahim and Darus [15] first considered the Cauchy problem

$$\begin{aligned} D_z^{\alpha}u(z)&=f(z,u(z))\\ u(0)&=0, \end{aligned} \tag{1.15}$$

where D_z^{α} is given in Definition 2.1.5 and $0<\alpha<1$, provided that the function $f(z,t)$ is an analytic and continuous function on $\mathbb{U}\times\mathbb{C}$ ($\mathbb{U}$ is the open units disc) and satisfies the condition $\|f\|\le M$ with $0<M<\infty$. They tried to establish the existence and uniqueness of the univalent (one-to-one and analytic) solutions to the problem (1.15). However, this problem is not well-posed and can not admit any analytic solution under these conditions. Indeed, we suppose that there exists an analytic solution u on $\mathbb{U}$. The above condition yields that $f\big(z,u(z)\big)$ is analytic on $\mathbb{U}$. By using this fact in the problem, it is obtained that $D_z^{\alpha}u$ is analytic on $\mathbb{U}$. However, it is a contradiction because it can be seen from (2.7) in the second chapter that both $D_z^{\alpha}u(z)$ and $u(z)$ can't simultaneously be analytic. Hence, this problem is ill-posed under the conditions above.

In addition to this, the errors made in the proof of the existence theorem [15, Theorem 4.1] are given in the following:

(1) They asserted that, under the conditions above, the problem (1.15) is equivalent to the following integral equation:

$$u(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta, u(\zeta))}{(z-\zeta)^{1-\alpha}} d\zeta. \tag{1.16}$$

Hovewer, it is proved in Proposition 3.1.11 that, without the condition $z^\alpha f(z,0)|_{z=0} = 0$, this problem can not be equivalent to the integral equation (1.16).

(2) Let us consider the following operator P which is defined by them :

$$Pu(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta, u(\zeta))}{(z-\zeta)^{1-\alpha}} d\zeta.$$

They claimed that P mapped the space $\mathbb{C}^0(\overline{\mathbb{U}})$ onto itself. Hovewer, this claim is not justified. Indeed, if one takes, in particular, $f(z,u(z)) = Au(z) + g(z)$ with $A \in \mathbb{C}$ and $u, g \in \mathcal{B}_0 \subset \mathbb{C}^0(\overline{\mathbb{U}})$, then one can drive

$$\begin{aligned} Pu(z) &= \frac{1}{\Gamma(\alpha)} \int_0^z \frac{a_0 + \sum_{n=1}^\infty (Ab_n + a_n)\zeta^n}{(z-\zeta)^{1-\alpha}} d\zeta \\ &= \frac{a_0 z^\alpha}{\Gamma(1+\alpha)} + \sum_{n=1}^\infty (Ab_n + a_n) \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} z^{n+\alpha}, \end{aligned}$$

where the power series expansions $u(z) = \sum_{n=1}^\infty b_n z^n$ and $g(z) = \sum_{n=0}^\infty a_n z^n$ with $z \in \mathbb{U}$ and all $a_n, b_n \in \mathbb{C}$ were used.

As one can see from the above, the function $Pu(z)$ is neither analytic nor continuous on $\mathbb{U}$, since $Pu(z)$ has a branch point at $z = 0$. Consequently, P is not well-defined.

(3) They investigated the solution of the integral equation (1.16), therefore the problem (1.15), in the space of continuous functions, but the fractional derivative in Definition 2.1.5 used by them can not be applicable to the continuous functions.

(4) They supposed that the problem has at least one univalent solution. However, the solutions of the problem need not to be univalent in related domain, unless some more conditions on the function f are imposed. Indeed, if $f(z,t) := cz^{-\alpha}t$ with $c = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}$ for fixed $n \in \mathbb{N} - \{1\}$ in the problem, then the problem admits the non-univalent solutions $u(z) = c^* z^n$ for all $c^* \in \mathbb{C}$.

(5) They claimed that the solutions of the problem exist on $\mathbb{U}$, although they only shown that the existence of the solutions for the problem is on the compact subset $|z| \leq l$, where $l < 1$. It needs to be shown that the domain of solutions can be extended to the whole open unit disk.

In chapter 3 of this thesis, we also consider the problem (1.15). However, unlike them, we use Definition 2.1.2 as a definition of fractional derivative and we impose different conditions on the function $f(z,t)$, one of which is:

> (ii) $f(z,t)$ is analytic on $\mathbb{D}\times\mathbb{C}$ and continuous on $\mathbb{D}^*\times\mathbb{C}$, and $z^\alpha f(z,t)$ is analytic on $\mathbb{U}\times\mathbb{C}$ and continuous on $\overline{\mathbb{U}}\times\mathbb{C}$.

This condition is sufficient for the well-posedness of the problem considered and thus we can investigate the solutions of the problem (1.15) which are analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$. Moreover, by using this condition we obtain some results [Lemma 3.1.2-3.1.3], by applying which to this problem we can reduce the solutions of this problem to the solutions of the integral equation (1.16). By using Schauder fixed point theorem, we show that the integral equation mentioned above admits at least one solution which is analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$, provided that, in addition to the condition (ii), the following inequality holds:

> (iii) There exist a natural number $n_0 \geq 1$, a non-negative real number c and a function g which is analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$ such that the following inequality holds for all $(z,t)\in\overline{\mathbb{U}}\times\mathbb{C}$:
>
> $$\left|z^\alpha f\left(z,t\right)\right| \leq c\left|t\right|^{n_0} + \left|g(z)\right|.$$

If one considers the condition (i), a question can arises as follows: Why isn't this problem investigated under the following more general condition than (ii)?:

> (iv) There exists a real number σ with $0\leq\sigma\leq\alpha<1$ such that $f(z,t)$ is analytic on $\mathbb{D}\times\mathbb{C}$ and continuous on $\mathbb{D}^*\times\mathbb{C}$, and $z^\sigma f(z,t)$ is analytic on $\mathbb{U}\times\mathbb{C}$ and continuous on $\overline{\mathbb{U}}\times\mathbb{C}$.

The reason for this not happening is that the problem considered is ill-posed under the condition (iv) when $0\leq\sigma<\alpha<1$, because of the same reason mentioned above for the problem (1.15) investigated under the condition posed by Ibrahim and Darus.

On the other hand, Delbosco and Rodino [4] obtained an existence theorem for the equation (1.13) under the condition (i). However, the condition (i) is not enough to reveal an existence theorem for the case $\sigma := \alpha$ and, for this case, with the help of Lipschitz type condition in (1.14) they obtained the existence and uniqueness theorem by using Banach fixed point theorem [4, Theorem 3.5]. If one takes into consideration the Lipschitz condition and the condition (iii) we posed, it is clear that the condition (iii) is weaker than the Lipschitz condition.

Moreover, in chapter 3 we show, by using contractive method and Schwarz's Lemma, that if, in addition to (ii), the function $f(z,t)$ fulfills the condition $z^{\alpha}f(z,0)|_{z=0}=0$ and the following Lipschitz type condition:

$$|f(z,\eta)-f(z,\nu)|<\frac{\kappa}{|z|^{\alpha}}|\eta-\nu|, \quad (0\leq\kappa<1),$$

then the problem (1.15) has a unique solution being analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$. The using of Schwarz's Lemma provides us to be able to select the function f from a wider range [Remark 3.1.10].

We indicated above that the Cauchy problem (1.15) can not have any analytic solution, when the function $f(z,t)$ is an analytic function on $\mathbb{U}\times\mathbb{C}$. The motivation of the other problem considered in this chapter is that: How the equation in the Cauchy problem is modified such that the obtained problem can be well-posed and the analytic solution of the problem can be investigated, when $f(z,t)$ is an analytic function on $\mathbb{U}\times\mathbb{C}$? The answer of this question is closely related to the definition of modified fractional derivative given in (2.11) and thus we investigate the following Cauchy problem

$$\begin{gathered}\left(D^{\alpha}\circ z^{\alpha}\right)u(z)=f(z,u(z))\\ u(0)=0,\end{gathered}$$

provided that the function $f(z,t)$ satisfies the some conditions, one of which is:

(v) The function $f(z,t)$ is analytic on $\mathbb{U}\times\mathbb{C}$ and continuous on $\overline{\mathbb{U}}\times\mathbb{C}$,

The problem (2.15) is well-posed under the condition (v) and the solutions of this problem which are same type with those of the first problem considered can be investigated. The results for the first problem mentioned above will be obtained for this problem by posing extra conditions.

Moreover, by obtaining sufficient conditions on the function $f(z,t)$ we establish the existence and uniqueness of bounded analytic solution for these two Cauchy problems above.

Furthermore, we consider these two Cauchy problem with the more general initial data $u(0)=b$ with $b\in\mathbb{C}$ and we obtain some results which are the similar to those mentioned above.

Now, we give some studies on the Dirichlet problem driven by fractional Laplacian operator $(-\Delta)^s$ given in (1.7) with $0 < s = \alpha/2 < 1$ or, more general, the fractional integrodifferential operator $\mathcal{L}_K$ in (1.8), which are closely related to the problems considered by us .

In 2010, M. Webb [38] investigated the existence of the solutions of the following problem:

$$\begin{cases} (-\Delta)^s u + \mathbf{b}(x).\nabla u + c(x)u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where Ω is a bounded Lipschitz domain, $f \in L_2(\Omega)$, $\mathbf{b} = (b_1, ..., b_n)^T \in W^{1,\infty}(\Omega)$, $c \in L_\infty(\Omega)$ and, $s \in (0, 1)$, if $b = 0$ or $s \in [\frac{1}{2}, 1)$, if $b \neq 0$. He proved by using Lax-Milgram theorem that there exist a unique generalized solution $u \in H^s_\Omega(\mathbb{R}^n)$ of this problem provided that $c - \frac{1}{2}\nabla.b \geq 0$.

Servadei and Valdinoci [34] was the first who considered the Dirichlet problem driven by operator $\mathcal{L}_K$ given by

$$\begin{cases} \mathcal{L}_K u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where f is a Carathédory function satisfying certain conditions. They first introduced the space $\mathcal{X}_0$ and after that, under certain conditions they proved, by using Mountain Pass theorem and variational methods, that there exists at least one generalized solution in $\mathcal{X}_0$ of the problem above.

In 2015, Zhangab and Ferrara [42] considered the following problem as follows:

$$\begin{cases} -\mathcal{L}_K u + \lambda f(x, u) + \mu g(x, u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $s \in (0, 1)$, λ, μ are parameters, Ω is a open bounded subset of $\mathbb{R}^n (n > 2s)$, with Lipschitz boundary, and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are the Carathédory functions. They shown by using variational methods that there exist two generalized solutions in the space $\mathcal{X}_0$ of the problem considered, provided that the parameters and the functions f, g satisfy certain conditions.

In 2015, Raghavendra and Rasmista [28] establised, by using monotone operator theory, the existence of the generalized solution in X_0 of the following problem:

$$\begin{cases} -\mathcal{L}_K u + \mu g(x)u + h(u) = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.17}$$

where μ is a parameter, Ω is a open bounded subset of $\mathbb{R}^n$ $(n > 2s)$, with Lipschitz boundary, provided that the functions $f, g : \Omega \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions:

- $f \in L^2(\Omega)$ and $g \in L_\infty(\Omega)$;
- $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $K > 0$ and $h(0) = 0$;
- The following inequality holds for all $\zeta_1, \zeta_2 \in \mathbb{R}$:

$$(h(\zeta_1) - h(\zeta_2), \zeta_1 - \zeta_2) \geq 0.$$

In chapter 4 in this thesis, we consider the problem

$$\begin{cases} -\mathcal{L}_K u + g(x, u) = h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $\mathcal{L}_K$ is the non-local integrodifferential operator given in Definition 2.5.1, u is an unknown function, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $g(x, 0) = 0$ for any $x \in \mathbb{R}^n \backslash \Omega$ and h is a generalized function.
We investigate the existence of the generalized solution of this problem in subcritical, critical and supercritical cases depending on the parameter $\mu > 0$ in the following condition:

- There exist $\mu > 0$ and the functions $a_i \in L_{p_i}(\Omega)$ with $a_i(x) \geq 0$ for $i = 0, 1$, such that the following inequality holds for a.e $x \in \Omega$ and for all $t \in \mathbb{R}$:

$$|g(x,t)| \leq a_1(x) \, |t|^\mu + a_0(x),$$

  where the numbers $p_i > 1$

In addition to the above condition, we obtain sufficient conditions for the function g to prove the existence of generalized solution of this problem in this cases. For the proof, we apply compactness method and use Theorem 2.6.11. We note that the conditions we imposed are weaker than the conditions given by Raghavendra and Rasmita. Moreover, we establish uniqueness of the generalized solution for this problem.

2.PRELEMINARIES

In this chapter, we give, with or without proof, various known facts and preliminary results which are required for the next two chapters. In chapter 3, we deal with the problems involving complex fractional differential equations, and in chapter 4 we consider a Dirichlet boundary problem for the non-local operator. So, we first introduce those which are related to the chapter 3, and then related to other chapter. We begin with giving the definitions of the fractional integral and derivative used in the problems considered.

2.1. The Definitions of Fractional Derivative and Integral

The fractional integral and derivative in the real line called as Riemann-Liouville integral and derivative, are given as follows:

Definition 2.1.1. [33] Let the function $u(x)$ be defined on (0,1). The integral

$$ {}_{c}I_{x}^{\alpha}u\left(x\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_{c}^{x}\frac{u\left(\zeta\right)}{\left(x-\zeta\right)^{1-\alpha}}d\zeta,\quad 0<c<x<1, \tag{2.1}$$

and

$$ {}_{c}D_{x}^{\alpha}u\left(x\right):=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dx}\int_{c}^{x}\frac{u\left(\zeta\right)}{\left(x-\zeta\right)^{\alpha}}d\zeta,\quad 0<c<x<1, \tag{2.2}$$

are called the fractional integral and derivative of order $0<\alpha<1$, respectively, where Γ is the well-known Gamma function, i.e.

$$\Gamma(z)=\int_{0}^{\infty}e^{-t}t^{z-1}dt,\quad \Re\{z\}>0.$$

There are many approaches to extend the above definitions in the real line to the definitions in the complex plane. Some of them were given by Sonin [36], Letnikov [19], Nishimoto [25], Owa [27] and Samko et. al. [33]. In this thesis we use the definitions given in [33] to obtain our main results in chapter 3. In the following, we show the differences between the definitions given in [36] and the definitions given in [33], which explains why we use them. We understand these differences better by taking into consideration an important result and definitions given in [19],[36] and [33].

One of the approaches mentioned above is to generalize the integral in (2.1) to the integral

$$ {}_{z_0}I_{z}^{\alpha}u\left(z\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_{z_0}^{z}\frac{u\left(\zeta\right)}{\left(z-\zeta\right)^{1-\alpha}}d\zeta,\quad (0<\alpha<1), \tag{2.3}$$

where $u(z)$ is defined in a certain domain G of the complex plane and the integration is along the straight line from point z_0 to point z, which lies entirely in G.
As such the integral above isn't defined uniquely and doesn't make sense for all $z \in G$. In order that the integral makes sense for all $z \in G$, the domain G needs to be starlike respect to the point z_0. Henceforward, such domain G is called as admissible domain and represented as G_{z_0}.
The integral in (2.3) can be uniquely defined by fixing the point z and choosing the principal value of the many-valued function $(z-\zeta)^{1-\alpha}$. This means the following. By the reason fact that the integration is over the line segment $[z_0, z]$, ζ has to lie in it. In this case, one of the values of $\arg(z-\zeta)$ has to equal to the value of $\arg(z-z_0)$, i.e.

$$\arg(z-\zeta) = \arg(z-z_0).$$

If one fixes the $\arg(z-z_0)$ in $(-\pi, \pi]$, then the unique branch of the many valued function $(z-\zeta)^{1-\alpha}$ is determined and one can write

$$(z-\zeta)^{1-\alpha} = |z-\zeta|^{1-\alpha} e^{i(1-\alpha)\arg(z-z_0)}, \quad \arg(z-z_0) \in (-\pi, \pi]. \tag{2.4}$$

Since $\arg(z-z_0)$ in $(-\pi, \pi]$, the fractional integral $I_{z_0}^{\alpha} u\,(z)$ is defined on the domain G with the cut along the ray which is parallel to the real axis and goes from the point z_0 to $-\infty + i\Im m\big(z_0\big)$ [33].

Consequently, from the above considerations, the definition of the fractional integral in the complex plane is given as following and by using this definition one can easily derive the fractional derivative in the complex plane.

Definition 2.1.2. [33] Let a function $u(z)$ be defined on an admissible domain G of complex plane containing the points z_0 and z. Then, the fractional integral of order α $\big(0 < \alpha < 1\big)$ of $u(z)$ is defined by

$${}_{z_0}I_z^{\alpha} u\,(z) := \frac{1}{\Gamma\,(\alpha)} \int_{z_0}^{z} \frac{u\,(\zeta)}{(z-\zeta)^{1-\alpha}} d\zeta, \quad \big(0 < \alpha < 1\big), \tag{2.5}$$

with the integration along the straight line interval connecting points z_0 and z as a rule, and with the principal value (2.4).

Definition 2.1.3. [33] Let a function $u(z)$ be defined on an admissible domain G of complex plane containing the points z_0 and z. Then, the fractional derivative of order α $\big(0 < \alpha < 1\big)$ of $u(z)$ is defined by

$${}_{z_0}D_z^{\alpha} u\,(z) = \frac{1}{\Gamma\,(1-\alpha)} \frac{d}{dz} \int_{z_0}^{z} \frac{u\,(\zeta)}{(z-\zeta)^{\alpha}} d\zeta, \quad \big(0 < \alpha < 1\big), \tag{2.6}$$

with the integration along the straight line interval connecting points z_0 and z as a rule, and with the principal value (2.4).

Throughout this thesis, we assume the following. For $z_0 := 0$ we set ${}_{z_0}I_z^\alpha = I_z^\alpha$ and ${}_{z_0}D_z^\alpha = D_z^\alpha$. We select the admissible domain G defined above as $\mathbb{U}_R$ which is the open disc with the center $z_0 = 0$ and radius $R > 0$ or as the set $\overline{\mathbb{U}}_R$ which is the closure of $\mathbb{U}_R$. Let the sets

$$\mathbb{D}_R := \left\{ z \in \mathbb{U}_R : -\pi < \arg z \leq \pi \right\}$$

and

$$\mathbb{D}_R^* := \left\{ z \in \overline{\mathbb{U}}_R : -\pi < \arg z \leq \pi \right\},$$

which are obtained by cutting, respectively, $\mathbb{U}_R$ and $\overline{\mathbb{U}}_R$ along the ray which starts at $z_0 = 0$ and goes to $-\infty$.
Moreover, we set $\mathbb{U} = \mathbb{U}_R$, $\overline{\mathbb{U}} := \overline{\mathbb{U}}_R$, $\mathbb{D} := \mathbb{D}_R$ and $\mathbb{D}^* := \mathbb{D}_R^*$ when $R := 1$.

We now give some basic results related to the fractional integral and derivative of the function $u(z) = z^n$ $(n \in \mathbb{N}_0)$ on $\mathbb{U}_R$, which are useful for computing the fractional derivative of any analytic function in the complex plane. These results are

$$I_z^\alpha z^n = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} z^{n+\alpha} \quad \left(n \in \mathbb{N}_0; 0 < \alpha < 1\right),$$

and

$$D_z^\alpha z^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} z^{n-\alpha} \quad \left(n \in \mathbb{N}_0; 0 < \alpha < 1\right),$$

which can be obtained by using the change of variable and the definition of Beta function:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^p (1-t)^q dt.$$

As one can easily see from the above, neither $I_z^\alpha z^n$ nor $D_z^\alpha z^n$ are analytic or continuous on $\mathbb{U}_R$, because resulting functions have branch point at $z = 0$. But they are analytic and continuous on $\mathbb{D}_R$.

More generally, by using the result just above and by applying the fractional derivative to an analytic function $u(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{U}_R$, one can get the following representation:

$$D_z^\alpha u(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} a_n z^{n-\alpha}, \quad a_n = \frac{f^n(0)}{n!}, \tag{2.7}$$

which is not analytic on $\mathbb{U}_R$. This shows that the fractional derivative (integral) of any analytic (or continuous) function defined on a certain domain need not to be analytic (or continuous) on the same domain.

From here, the question is raised as follows: How are the fractional derivative and integral operators modified such that this obtained modified fractional derivative and integral operators map any analytic (or continuous) function to an analytic (or continuous) function? The answer is following: If one first changes of the variable such that $\zeta = z_0 + t\left(z - z_0\right)$ with $0 \leq t \leq 1$ in the integrals (2.5) and (2.6) which are over the line segment, then the following integrals can be considered which are equivalent the integrals (2.5) and (2.6), respectively:

$$ {}_{z_0}I_z^{\alpha} u\left(z\right) \;=\; \frac{\left(z-z_0\right)^{\alpha}}{\Gamma\left(\alpha\right)} \int_0^1 \frac{u\left(z_0 + t\left(z - z_0\right)\right)}{\left(1-t\right)^{1-\alpha}} dt \tag{2.8} $$

and

$$ {}_{z_0}D_z^{\alpha} u\left(z\right) := \frac{1}{\Gamma\left(1-\alpha\right)} \frac{d}{dz} \left[\left(z-z_0\right)^{1-\alpha} \int_0^1 \frac{u\left(z_0 + t\left(z - z_0\right)\right)}{\left(1-t\right)^{\alpha}} dt \right]. \tag{2.9} $$

This approach given by (2.8) and (2.9) is known as Hadamard approach and it can be seen from above that the choice of the branch depends on the fixed point z_0 at which the integrals above are many valued. In that case, it is sufficient to remove this branch point to reach the required modification. Hence, by multiplying the ${}_{z_0}I_z^{\alpha} u\left(z\right)$ with $\left(z-z_0\right)^{-\alpha}$ and the integrand of ${}_{z_0}D_z^{\alpha} u\left(z\right)$ with $\left(z-z_0\right)^{\alpha}$,

$$ \left(\left(z-z_0\right)^{-\alpha} {}_{z_0}I_z^{\alpha}\right) u\left(z\right) \;=\; \frac{1}{\Gamma\left(\alpha\right)} \int_0^1 \frac{u\left(z_0 + t\left(z - z_0\right)\right)}{\left(1-t\right)^{1-\alpha}} dt, \tag{2.10} $$

and

$$ \left({}_{z_0}D_z^{\alpha} \circ \left(z-z_0\right)^{\alpha}\right) u\left(z\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \frac{d}{dz} \left[\left(z-z_0\right) \int_0^1 \frac{t^{\alpha} u\left(z_0 + t\left(z - z_0\right)\right)}{\left(1-t\right)^{\alpha}} dt \right]. \tag{2.11} $$

are easily obtained, which are modified fractional integral and fractional derivative, respectively.

At the present, we explain the extension of well-known Cauchy integral formula to non-integer values ρ and the fact that this generalization is equivalent to fractional integral in (2.5), when u is an analytic function and $-1 < \rho = -\alpha < 0$ to understand Definitions 2.1.4 and 2.1.5 in the following better.

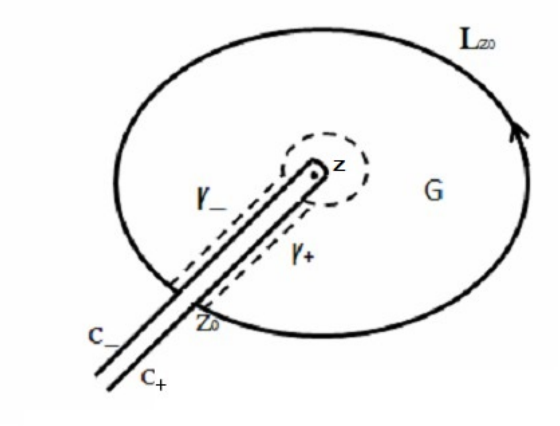

Fig. 1. Contour of integration

The Cauchy integral formula is as follows:

$$u^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{u(\zeta)}{(\zeta - z)^{1+n}}d\zeta, \quad (n \in \mathbb{N}_0, z \in C), \tag{2.12}$$

where u is analytic in a simply connected domain G and C any simple closed contour lying entirely within G. The generalization mentioned above occurs in the following way: $n!$ in the integral (2.12) may be replaced by $\Gamma(\rho + 1)$. However, the same easy way can't be applied to integrand of (2.12), since the function $(\zeta - z)^{1+\rho}$ has a branch point (not pole) at $\zeta = z$ anymore and, the multivalued function $(\zeta - z)^{1+\rho}$ shows up. To single out its one-valued branch, first of all, the plane is cut by the ray passes from z through z_0 to ∞, and then the single valued function

$$(\zeta - z)^{-(\rho+1)} = |\zeta - z|^{-(\rho+1)}\, e^{-i(\rho+1)\arg(\zeta - z)}, \tag{2.13}$$

is obtained. It is assumed that the principal value of $\arg(\zeta - z)$ is chosen, i.e. $\arg(\zeta - z) = 0$, when $\zeta - z > 0$. Since the cut may prove to be parallel to the real axis and lie to the right of the point z, this choice is specified by the condition

$$\arg(\zeta - z)|_{\zeta \in C_+} \in (-2\pi, 0], \tag{2.14}$$

where C_+ is the edge of the cut. [33]

Hence, the function $\frac{u(\zeta)}{(\zeta - z)^{1+\rho}}$ is analytic in the domain which is bounded by the curves γ_-, γ_+, γ, C' as shown at Fig. 1. By using well-known Cauchy-Goursat theorem,

$$\frac{\Gamma(1+\rho)}{2\pi i}\int_{C' \cup \gamma_- \cup \gamma \cup \gamma_+} \frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta = 0$$

is obtained. From here,

$$\frac{\Gamma(1+\rho)}{2\pi i}\int_{C'} \frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta = \frac{\Gamma(1-\alpha)}{2\pi i}\left[\int_{\gamma_-} ... + \int_{\gamma} ... + \int_{\gamma_+} ...\right]$$

can be written (omitting the integrands of integral in the right side of above equation), where all curves are traversed in the positive sense.

From above, if $\gamma_- \to C_-$, $\gamma_+ \to C_+$ and $\gamma \to \{z\}$, then $C' \to C$ and

$$\frac{\Gamma(1+\rho)}{2\pi i}\int_C \frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta = \frac{\Gamma(1-\alpha)}{2\pi i}\left[\lim_{\gamma_-\to C_-}\int_{\gamma_-}... + \lim_{\gamma\to\{z\}}\int_{\gamma}... + \lim_{\gamma_+\to C_+}\int_{\gamma_+}...\right]$$

is obtained. The second integral in the right-hand side of the above equation is equivalent to zero when $\rho < 0$. Indeed, by using the polar coordinates in the related integral, this claim can be justified as follow:

$$\lim_{\gamma\to\{z\}}\int_{\gamma}\frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta = \lim_{\epsilon\to 0}\epsilon^{-\rho}\int_0^{2\pi} u(z+\epsilon e^{i\theta})e^{-i\rho\theta}d\theta = 0.$$

Now, the following are considered to evaluate the other two integrals in the above equality:

$$(\zeta - z)^{-(\rho+1)} = (-1)^{-(\rho+1)}(z-\zeta)^{-(\rho+1)} = e^{-i(\rho+1)\pi}(z-\zeta)^{-(\rho+1)},$$

can be easily seen.

Since $\arg(\zeta - z) = 0$ on C_+ and $\arg(\zeta - z) = -2\pi$ on C_-, from (2.13) and the equality above one can write

$$(\zeta - z)^{-(\rho+1)} = e^{-i(\rho+1)\pi}(z-\zeta)^{-(\rho+1)} \text{ on } C_+ \tag{2.15}$$

and

$$(\zeta - z)^{-(\rho+1)} = e^{+i(\rho+1)\pi}(z-\zeta)^{-(\rho+1)} \text{ on } C_-. \tag{2.16}$$

By using (2.15) and (2.16),

$$\begin{aligned}\left(\lim_{\gamma_-\to C_-}\int_{\gamma_-} + \lim_{\gamma_+\to C_+}\int_{\gamma_+}\right)\frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta &= \left(\int_{z_0}^{z} + \int_{z}^{z_0}\right)\frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta \\ &= \left(e^{+i(\rho+1)\pi} - e^{-i(\rho+1)\pi}\right)\int_{z_0}^{z}\frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta \\ &= 2isin((\rho+1)\pi)\int_{z_0}^{z}\frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta\end{aligned}$$

is concluded. From the equation above

$$\frac{\Gamma(1+\rho)}{2\pi i}\int_C \frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta = \frac{1}{\Gamma(-\rho)}\int_{z_0}^{z}\frac{u(\zeta)}{(\zeta - z)^{1+\rho}}d\zeta$$

is obtained, where the following identity is used [24]:

$$\frac{\Gamma(\rho+1)sin((\rho+1)\pi)}{\pi} = \frac{1}{\Gamma(-\rho)}.$$

Consequently, if $u(z)$ is an analytic in a domain G, then from above equation for $0 < -\rho = \alpha < 1$

$$ {}_{z_0}I_z^{\alpha}u(z) = \frac{\Gamma(1-\alpha)}{2\pi i}\int_C \frac{u(\zeta)}{(\zeta - z)^{1-\alpha}}d\zeta, \tag{2.17}$$

is satisfied, where ${}_{z_0}I_z^{\alpha}$ is given in (2.3) and the principal value of the function $(z-\zeta)^{1+\rho}$ is fixed by (2.13) and (2.14), while the closed contour C lying in the domain G is any one passes through the point z_0 and goes round the point z in positive direction.

As distinct from (2.8)-(2.9) (which are equivalent to (2.4)-(2.5), respectively), the choice of the unique branch in (2.17) depends on the variable $z = \zeta$.
In consideration of equality in (2.17) for the analytic functions when $0 < \alpha < 1$, by taking $z_0 := 0$ and $z \in \mathbb{R}^+$ in (2.3), Owa [27] gave the following definitions:

Definition 2.1.4. The fractional integral of order α is defined for a function u by

$$I_z^{\alpha}u(z) := \frac{1}{\Gamma(\alpha)}\int_0^z \frac{u(\zeta)}{(z-\zeta)^{1-\alpha}}d\zeta, \quad 0 < \alpha < 1, \tag{2.18}$$

where u is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2.1.5. The fractional derivative of order α is defined for a function u by

$$D_z^{\alpha}u(z) := \frac{1}{\Gamma(\alpha)}\frac{d}{dz}\int_0^z \frac{u(\zeta)}{(z-\zeta)^{\alpha}}d\zeta, \quad 0 < \alpha < 1 \tag{2.19}$$

where u is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $z - \zeta > 0$.

It needs to note that although the integrals over the line segment connecting the point 0 and $z \in \mathbb{R}^+$, according to the identity (2.17), $I_z^{\alpha}u$ and $D_z^{\alpha}u$ in (2.18) and (2.19) can exist on the certain domain inside any curve C which passes through the point 0 and contains a line segment connecting the points 0 and any real number z, and lies in the simply connected region on which the function u is analytic.

In the light of the explanation above, we summarize the difference between the Definitions 2.1.2-2.1.3 and Definition 2.1.4-2.1.5 as follows:

(1) The operators D_z^α and I_z^α given in Definitions 2.1.4-2.1.5 are valid only for analytic functions, while D_z^α and I_z^α given in Definitions 2.1.2-2.1.3 are applicable to integrable functions.

(2) Let u be an analytic function on $\mathbb{U}$. From 2.7 it can be said that $D_z^\alpha u$ (D_z^α given in Definition 2.1.3) is analytic on $\mathbb{D}$. However, from the explanations above, $D_z^\alpha u$ (D_z^α given in Definition 2.1.5) exists on the domain inside any curve C which lies entirely in $\mathbb{U}$, and so it can't be analytic on $\mathbb{D}$, because there isn't such curve C in $\mathbb{U}$ that contains the domain $\mathbb{D}$. It should be noted that $D^\alpha u$ can be analytically continued to $\mathbb{D}$.

To avoid such difficulties in (2) and to deal with the larger class of functions, i.e. integrable functions, we use the Definition 2.1.3 as a fractional derivative in Cauchy problem in the next chapter.

2.2. A Method to Show the Existence and Uniqueness of the Solution for the Problem Involving Fractional Differential Equation

In the above, we explained which definition of the fractional derivative and integral in the complex plane for the problem considered by us in the next chapter and, also why we will use this definition.

Now, we clarify which method will be used to show the existence and uniqueness of the solution of the related problem in the appropriate function spaces. The method used frequently in the theory of differential equations is that:

Step 1. At first, under the certain conditions, the problem involving a differential equation is transformed into the integral equation, which is equivalent to the problem by using some results and then;

Step 2. Using the certain fixed point theorem in accordance with the imposed conditions, the existence and uniqueness of the fixed point of the integral operator are investigated, which are equivalent to the solution of the related problem.

2.2.1. The Function Spaces

$C^0(\overline{\mathbb{U}}_R)$ is the well-known space consisting of the continuous functions on $\overline{\mathbb{U}}_R$, where R is a real number in $(0, 1]$. This space is a Banach space when endowed with supremum norm.

One of the appropriate function spaces referred above is the space $\mathcal{B}_R$ is the class of the functions which are analytic on $\mathbb{U}_R$ and continuous on $\overline{\mathbb{U}}_R$. It is clear that $\mathcal{B}_R \subset C^0(\overline{\mathbb{U}}_R)$. We set $\mathcal{B} := \mathcal{B}_R$ for $R = 1$. The other one is the space $\mathcal{A}$ which contains the analytic and bounded functions on $\mathbb{U}$.

It was indicated without proof in [32] and [14] that the spaces $\mathcal{A}$ and $\mathcal{B}$, respectively, are the Banach spaces when they are endowed with sup-norm. Here, we only prove that for any $R \in (0,1]$ the space $\mathcal{B}_R$ is a Banach space when endowed with the sup-norm. The fact that the space $\mathcal{A}$ is a Banach space (with sup-norm) can be proved in the similar way. For the proof, we first consider the following well-known Morera Theorem:

Theorem 2.2.1. [29] Let $u(z)$ be continuous in a domain G. If

$$\int_C u(z)dz = 0,$$

along every simple closed contour C contained in G, then $u(z)$ is analytic in G.

Lemma 2.2.2. Let R be a real number in $(0,1]$. The space $\mathcal{B}_R$ is a Banach space when endowed with supremum norm.

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be Cauchy sequence in the space $\mathcal{B}_R$. Then, $\{u_n\}_{n=1}^{\infty}$ is also Cauchy sequence in $C^0(\overline{\mathbb{U}}_R)$, since $\mathcal{B}_R \subset C^0(\overline{\mathbb{U}}_R)$. From here, there is a $u \in C^0(\overline{\mathbb{U}}_R)$ such that $u_n \to u$ in $C^0(\overline{\mathbb{U}}_R)$. This implies that $\{u_n(z)\}_{n=1}^{\infty}$ converges uniformly to $u(z)$ on $\overline{\mathbb{U}}_R$ and on all compact subsets of $\overline{\mathbb{U}}_R$. By using this and the fact that for all $n \in \mathbb{N}$ the functions $u_n(z)$ are analytic on $\mathbb{U}_R$, we have

$$\int_C u(z)dz = \int_C \lim_{n\to\infty} u_n(z)dz = \lim_{n\to\infty} \int_C u_n(z)dz = 0,$$

along any simple closed curve C contained in $\mathbb{U}_R$. As a consequence of Morera theorem, it is obtained that the function $u(z)$ is analytic on $\mathbb{U}_R$. Hence, $u \in \mathcal{B}_R$. This gives the desired proof for the lemma.

2.2.2. Semigroup Property of Fractional Integral

One of the results mentioned in the step 1 above is related to the semigroup property of the fractional integral ${}_{z_0}I_z^{\alpha}$ as follows:

Lemma 2.2.4. [33] Let a function $u(z)$ be locally integrable (continuous) in an admissible domain G. Then for almost all (for all) $z \in G$ the semigroup property holds

$${}_{z_0}I_z^{\alpha}\,{}_{z_0}I_z^{\beta}u(z) =_{z_0} I_z^{\alpha+\beta}u(z), \qquad (\alpha > 0, \beta > 0).$$

2.2.3. Fixed Point Theorems

One of the certain fixed point theorems mentioned in the step 2 is given as a corollary of Schauder's fixed point theorem (only for the existence) and Arzelá-Ascoli Theorem in the following. Before this we give the following definitions:

Definition 2.2.5. [41] Let X be Banach space and $M \subset X$. M is relatively compact iff every sequence in M contains a convergent subsequence.

Definition 2.2.6. [41] Let X, Y be Banach spaces, and $T : D(T) \to Y$ an operator. T is called compact iff:
(i) T is continuous;
(ii) T maps bounded sets into relatively compact sets.

Theorem 2.2.7. [41] (Schauder's fixed point theorem) If M is a close bounded convex subset of a Banach space X and $T : M \to M$ is a compact operator, then T has a fixed point in M.

Theorem 2.2.8. [11] (Arzelá-Ascoli Theorem) Let G be compact subset of $\mathbb{C}$, and let M be a family of all continuous complex-valued functions on G that is uniformly bounded. Then the following are equivalent.

(i) The family M is equicontinuous on G, i.e. for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that $\sup_{u \in M} |u(z_1) - u(z_2)| < \epsilon$ whenever $z_1, z_2 \in G$ and $|z_1 - z_2| < \delta(\epsilon)$. Here $\delta(\epsilon)$ is independent of z_1, z_2 and u.

(ii) Each sequence of functions in M has a subsequence that converges uniformly on G.

We now suppose that G is a compact subset of $\mathbb{C}$, M is a close bounded convex subset of $X := \big\{u : G \to \mathbb{C} \text{ continuous} : G \subset \mathbb{C} \text{ compact}\big\}$ and $T : M \to M$ is a compact operator. In view of Definition 2.2.5 and Definition 2.2.6, for the compactness of the related operator T, it is sufficient to show that T is a continuous and $T(M)$ is a relatively compact set in M, i.e. every sequence in $T(M)$ contains a convergent subsequence. Consequently, from here and from Arzelá-Ascoli Theorem, the following corollary can be given.

Corollary 2.2.9. Let M be a close bounded convex subset of a Banach space $X := \big\{u : G \to \mathbb{C} \text{ continuous} : G \subset \mathbb{C} \text{ compact}\big\}$. If $T : M \to M$ is a continuous operator and $T(M)$ is a equicontinuous set on G, then T has a fixed point in M.

The other fixed point theorem used by us in the next chapter is the Banach fixed point theorem. As a result of this theorem, not only existence but also uniqueness of the fixed point of the related operator can be obtained.

Theorem 2.2.10. [41] If (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, i.e there is a β $(0 \leq \beta < 1)$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \beta d(x, y),$$

is satisfied, then T has a unique fixed point.

2.2.4. Schwarz's Lemma for one variable

We use the complex analysis tools such as Schwarz's Lemma to show the existence and uniqueness of the fixed point of the related operator:

Lemma 2.2.11. [5] If $u(z)$ is analytic on $\mathbb{U}_R$ and satisfies the conditions $u(0) = 0$ and $|u(z)| \leq r$ for all $z \in \mathbb{U}_R$, then

$$|u(z)| \leq \frac{r\,|z|}{R} \tag{2.20}$$

for all $z \in \mathbb{U}_R$. In addition, if the function $u(z)$ above is continuous on the boundary of $\partial\mathbb{U}_R$ and satisfies the equality $|u(z)| \leq r$ on ∂U_R, then the inequality (2.20) is true on $\overline{\mathbb{U}}_R$.

Now we give preliminary results related to chapter 4. In that chapter we investigate the existence and uniqueness of the solution for problems involving some nonlocal operators in the appropriate function spaces, so we give the definitions related to these function spaces and operators which can be found in [6], [34] and [38].

2.3. Fractional Sobolev Spaces

Here, as well as Fractional Sobolev Spaces, we mention briefly the well-known Lebesgue spaces $L^p(\Omega)$, $1 \leq p < \infty$ and its some properties. These spaces consist of the measurable functions defined on $\Omega \subseteq \mathbb{R}^n$ for which

$$\int_\Omega |u(x)|^p \, dx < \infty,$$

and are Banach spaces when endowed with the following norm:

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

The space $L^{\infty}(\Omega)$ is the function space of measurable functions u that are essentially bounded on Ω. It is a Banach space when endowed with the norm:

$$\|u\|_{L^{\infty}(\Omega)} = ess\sup_{x\in\Omega} |u(x)| .$$

The space $L^p(\Omega)$ is a separable space for any $p \in [1, \infty)$ and a reflexive space for any $p \in [1, \infty)$. Moreover, the dual space of $L_p(\mathbb{R}^n)$ is the space $L_q(\mathbb{R}^n)$, where $1 \leq p < \infty$ and $q = p/(p-1)$ [1].

Now, let us give the definition of fractional Sobolev spaces and their some properties in the following.

Definition 2.3.1. Let $\Omega \subseteq \mathbb{R}^n$ open set. For any real $s \in (0, 1)$ and $1 \leq p < \infty$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \left\{ u \in L_p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L_p(\Omega \times \Omega) \right\}$$

and is also Banach space when equipped with the norm:

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L_p(\Omega)} + \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} . \tag{2.21}$$

The definition of $W^{s,p}(\Omega)$ given above is not valid for $s \geq 1$. Because, if u is a measurable function satisfying

$$\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy < \infty$$

then it is indicated that u has to be constant function [6].

The space $H^s(\Omega)$ is equivalent to the space $W^{s,p}(\Omega)$ when $p = 2$, and is a Hilbert space when endowed with the following inner product and with the norm (2.21):

$$< u, v >_{H^s(\Omega)} = < u, v >_{L^2(\Omega)} + \int_{\Omega\times\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dxdy, \ (u, v \in H^s(\Omega)).$$

The alternative definition of fractional Sobolev Space $W^{s,2}(\mathbb{R}^n)$ is given by the help of Fourier transformation as follows:

Definition 2.3.2. The space $\hat{H}^s(\mathbb{R}^n)$ is defined as follows:

$$\hat{H}^s(\mathbb{R}^n) := \big\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\mathcal{F}u(\xi)|^2 d\xi < +\infty \big\},$$

where $s \geq 0$ and $\mathcal{F}u$ is the Fourier transformation of u defined by

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix.\xi} u(x) dx, \quad (i = \sqrt{-1}, \quad x.\xi = \sum_{i=1}^{n} x_i \xi_i).$$

The following result shows that the space $\hat{H}^s(\mathbb{R}^n)$ coincides the fractional Sobolev space $H^s(\mathbb{R}^n)$ when $s \in (0,1)$.

Lemma 2.3.3. Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$. Then,

$$[u]^2_{H^s(\mathbb{R}^n)} = 2c^{-1}_{n,s} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi$$

is provided, where $c_{n,s} := \big(\int_{\mathbb{R}^n} \frac{1-cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \big)^{-1}$.

The difference between the spaces $\hat{H}^s(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ is that Definition 2.3.2 is valid also for any $s \geq 1$.

In chapter 4, we are interested in the space $H^s_\Omega(\mathbb{R}^n) = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ on } \mathbb{R}^n \backslash \Omega\}$ when $n > 2s$. This space is a Hilbert space when endowed with the inner product

$$< u, v >_{H^s_\Omega(\mathbb{R}^n)} = \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dxdy \tag{2.22}$$

for all $u, v \in H^s_\Omega(\mathbb{R}^n)$, and with the norm induced by the scalar product in (2.22):

$$\|u\|_{H^s_\Omega(\mathbb{R}^n)} := \left(\int_Q \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dxdy \right)^{1/2}, \tag{2.23}$$

where

$$Q := \mathbb{R}^{2n} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega) \text{ with } \mathcal{C}\Omega := \mathbb{R}^n \backslash \Omega. \tag{2.24}$$

The norm in (2.23) and the norm in (2.20) with $\Omega := \mathbb{R}^n$ and $p := 2$ are equivalent on the space $H^s_\Omega(\mathbb{R}^n)$, i.e.

$$\|.\|_{W^{s,p}(\mathbb{R}^n)} \cong \|.\|_{H^s_\Omega(\mathbb{R}^n)} \quad \text{on} \quad H^s_\Omega(\mathbb{R}^n),$$

since the following inequality holds for any $u \in H^s_\Omega(\mathbb{R}^n)$, when $n > 2s$ and Ω is a open bounded domain with sufficiently smooth boundary:

$$\|u\|^2_{L_2(\Omega)} \leq c \int_Q \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dxdy,$$

where $c = c(n, s, p)$ is a positive constant.

The space $H^s_\Omega(\mathbb{R}^n)$ is a reflexive space, since this space is a Hilbert space. Moreover, the separability of this space follows from the that $W^{s,2}(\mathbb{R}^n)$ is a separable space and the fact that any subset of a separable metric space is also separable.

In that chapter, the other space in which the existence of the solutions of a problem is investigated is the space $H^s_\Omega(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ with $p \geq 1$. This space is a Banach space when equipped with the norm:

$$\|u\|_{H^s_\Omega(\mathbb{R}^n)\cap L_p(\mathbb{R}^n)} = \|u\|_{H^s_\Omega(\mathbb{R}^n)} + \|u\|_{L_p(\Omega)},$$

where the norm $\|.\|_{H^s_\Omega(\mathbb{R}^n)}$ is defined by (2.23).
Furthermore, this space is separable and reflexive Banach space [20].

The dual of this space is defined by the space $H^{-s}(\mathbb{R}^n) + L_q(\mathbb{R}^n)$ with $q = p/(p-1)$, and the duality for the pair $(H^s_\Omega(\mathbb{R}^n) \cap L_p(\mathbb{R}^n), H^{-s}(\mathbb{R}^n) + L_q(\mathbb{R}^n))$ is denoted by

$$\langle u, v\rangle_* = \int_\Omega uv_1 dx + \int_Q \frac{(u(x)-u(y))(v_2(x)-v_2(y))}{|x-y|^{n+2s}} dxdy,$$

where $v = (v_1, v_2)$ such that $v_1 \in L_q(\mathbb{R}^n)$ and $v_2 \in H^{-s}(\mathbb{R}^n)$.

The norm of dual space of $H^{-s}(\mathbb{R}^n) + L_q(\mathbb{R}^n)$ is denoted by

$$\|v\|_{H^{-s}_\Omega(\mathbb{R}^n)+L_q(\mathbb{R}^n)} := \sup_{\|u\|_{H^s_\Omega(\mathbb{R}^n)\cap L_p(\mathbb{R}^n)}=1} \frac{|\langle u, v\rangle_*|}{\|u\|_{H^s_\Omega(\mathbb{R}^n)\cap L_p(\mathbb{R}^n)}}, \quad (q = p/(p-1)).$$

2.3.1. The Embeddings in Fractional Sobolev Spaces

Before giving the embedding theorems in Fractional Sobolev Spaces, it would be appropriate to introduce the definitions of continuous and compact embeddings.

Definition 2.3.4. [10] Let X and Y are Banach spaces such that $X \subset Y$. If the identity mapping is a continuous operator from the space X to the space Y (i.e., a constant $c > 0$ exists such that for all $u \in X$ we have $\|u\|_Y \leq c\|u\|_X$), then we say that the Banach space X is continuously embedded into the space Y. The fact is denoted by the symbol

$$X \hookrightarrow Y.$$

Definition 2.3.5. [10] We say that the Banach space X is compactly embedded into the space Y if

(i) $X \subset Y$;

(ii) any sequence $\{u_n\}_{n=1}^{\infty} \subset X$ of elements of the Banach space X which converges weakly in the space X to $u_0 \in X$ (i.e., $u_n \rightharpoonup u_0$ in X) converges strongly in the space Y to u_0 (i.e., $u_n \to u_0$ in Y).
The fact that the space X is compactly embedded into the space Y is denoted by the symbol

$$X \hookrightarrow\hookrightarrow Y.$$

Theorem 2.3.6. [1] Let Ω be a bounded subset of $\mathbb{R}^n$ and $1 \leq p \leq q \leq \infty$. Then

$$L_q(\Omega) \hookrightarrow L_p(\Omega).$$

Theorem 2.3.7. [35] Let $s \in (0,1)$ and $1 \leq p < \infty$, and let Ω be bounded Lipschitz domain. Then the following statements are true:
(i) If $sp < n$, then the following continuous embedding holds for all for all $q \in [1, \frac{np}{n-sp}]$:

$$W^{s,p}(\Omega) \hookrightarrow L_q(\Omega)$$

(ii) If $sp < n$, then the following compact embedding holds for all $q \in [1, \frac{np}{n-sp})$:

$$W^{s,p}(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)$$

From the above theorems it is obvious that for $n > 2s$ the following continuous and compact embeddings hold:

$$H^s_\Omega(\mathbb{R}^n) \hookrightarrow L^\Omega_q(\mathbb{R}^n) \ \text{ for all } \ q \in \big[1, \frac{2n}{n-2s}\big],$$

and

$$H^s_\Omega(\mathbb{R}^n) \hookrightarrow\hookrightarrow L^\Omega_q(\mathbb{R}^n) \ \text{ for all } \ q \in \big[1, \frac{2n}{n-2s}\big),$$

where

$$L^\Omega_q(\mathbb{R}^n) := \{u \in L_q(\mathbb{R}^n) : u = 0 \text{ on } \mathbb{R}^n \backslash \Omega\}.$$

2.4. Fractional Laplacian $(-\Delta)^s$

The facts used below can be found in [6] and [36].

Before introducing fractional Laplacian, it is appropriate to mention the space $C_0^\infty(\mathbb{R}^n)$ which is frequently used in subsection. This space consists of the functions have continuous derivative of all orders, and 'compact support' describes functions which vanish of outside of some bounded set.

The fractional Laplacian $(-\Delta)^s$ is defined as follows:

Definition 2.4.1. For any $u \in C_0^\infty(\mathbb{R}^n)$ and $s \in (0,1)$, $(-\Delta)^s$ defined as

$$(-\Delta)^s u(x) = PV\ c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \tag{2.25}$$

where PV is a principal value of the integral and $c_{n,s}$ is a dimensional constant that depends on n and s, precisely given by

$$c_{n,s} = \pi^{-(2s+\frac{n}{2})} \frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(-s)}. \tag{2.26}$$

If $s \in (0, \frac{1}{2})$, the integral (2.25) is not singular and it is finite. Indeed, by separating the integral (2.25) in two parts: $B(x,\epsilon)$ and $\mathbb{R}^n - B(x,\epsilon)$ for $\epsilon > 0$, and by using Lipschitz continuity of u on $B(x,\epsilon)$ and boundedness of u on $\mathbb{R}^n - B(x,\epsilon)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy &\le \int_{B(x,\epsilon)} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy + \int_{\mathbb{R}^n - B(x,\epsilon)} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}} dy \\ &\le c \int_{B(x,\epsilon)} \frac{1}{|x-y|^{n+2s-1}} dy + 2\|u\|_\infty \int_{\mathbb{R}^n - B(x,\epsilon)} \frac{1}{|x-y|^{n+2s}} dy \\ &\le c_1 \int_0^\epsilon \frac{1}{r^{2s}} dr + 2\|u\|_\infty c_2 \int_\epsilon^\infty \frac{1}{r^{1+2s}} dr < \infty, \end{aligned} \tag{2.27}$$

where the spherical coordinates are used in the last integrals. If $s \in \left[\frac{1}{2}, 1\right)$, it is clear that the second integral in (2.27) is finite. But, unless there exists a positive real number C and $\eta > 0$ such that

$$|u(x) - u(y)| \le C|x-y|^{1+\eta} \text{ on } B(x,\epsilon),$$

the first integral in (2.27) can not be integrable. This shows that why the integral in (2.25) is considered with principal value.

Now, a different and important representation of (2.25) is given. For this, if one changes the variable from $y - x$ to z and after that z to $-z$ in (2.25),

$$(-\Delta)^s u(x) = PV\ c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}} dz = PV\ c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(x-z)}{|z|^{n+2s}} dz,$$

is obtained. From here, it is clear that

$$(-\Delta)^s u(x) = -\frac{1}{2} PV\ c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{|z|^{n+2s}} dz. \tag{2.28}$$

The integral in (2.28) is Lebesgue integrable on $\mathbb{R}^n$. Indeed, by help of Taylor's theorem to the second order in the integrands above

$$\frac{|u(x+y) + u(x-y) - 2u(x)|}{|y|^{n+2s}} \le \frac{1}{|y|^{n+2s-2}} \max_{m=2} \max_{y \in \mathbb{R}^n} |D^m u(y)| \quad (y := z)$$

is obtained, where $\max_{m=2}\max_{y\in\mathbb{R}^n}|D^m u(y)|<\infty$ for any $u\in C_0^\infty(\mathbb{R}^n)$. By using the above inequality in the neighborhood of the point 0 and the boundedness of u out of this neighborhood, it can be seen that the integral (2.28) is Lebesgue integrable in $y\in\mathbb{R}^n$. Hence, PV in (2.25) doesn't need anymore. Consequently, the following Lemma can be given.

Lemma 2.4.2. Let $s\in(0,1)$ and let $(-\Delta)^s$ be fractional Laplacian operator defined by (2.21). Then, for any $u\in C_0^\infty(\mathbb{R}^n)$

$$(-\Delta)^s u(x) = -\frac{1}{2}\,c_{n,s}\int_{\mathbb{R}^n}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dz, \quad \forall x\in\mathbb{R}^n. \tag{2.29}$$

It can be shown that the integrand of (2.25) is integrable on $\mathbb{R}^{2n}$. Then, by applying the Fourier transform to the both sides of (2.25) can be obtained the following Lemma:

Lemma 2.4.3. Let $s\in(0,1)$ and $(-\Delta)^s$ defined by (2.25). Then,

$$(-\Delta)^s u = c_{n,s}\mathcal{F}^{-1}\left(|\xi|^{2s}\left(\mathcal{F}(u)\right)\right)$$

holds for all $u\in C_0^\infty(\mathbb{R}^n)$, where $c_{n,s}$ given in (2.26).

Now, we give the following equality showing the relation between fractional Laplacian and fractional Sobolev spaces.

Lemma 2.4.4. Let $s\in(0,1)$ and let $u\in H^s(\mathbb{R}^n)$. Then

$$[u]^2_{H^s(\mathbb{R}^n)} = 2c_{n,s}^{-1}\left\|(-\Delta)^{s/2}u\right\|^2_{L^2(\mathbb{R}^n)}$$

where $c_{n,s}$ defined by (2.26).

Lemma 2.4.5. Let $u,w\in H^s_\Omega(\mathbb{R}^n)$. Then the equality

$$\langle(-\Delta)^s u, w\rangle_{L_2(\mathbb{R}^n)} = \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{n+2s}}dydx \tag{2.30}$$

holds, where $c_{n,s}$ defined by (2.26).

Proof. From the definition of fractional Laplacian and the well-known Fubuni theorem we have

$$\begin{aligned}\langle(-\Delta)^s u, w\rangle_{L_2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}(-\Delta)^s u w dx\\ &= \int_{\mathbb{R}^n}\left(c_{n,s}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}dy\right)w(x)dx\\ &= c_{n,s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))}{|x-y|^{n+2s}}w(x)dydx.\end{aligned} \tag{2.31}$$

If we interchange the variables as follows: $x \to y$ and $y \to x$, and if we use again the Fubuni theorem, then by little calculation we obtain

$$\langle (-\Delta)^s u, w \rangle_{L_2(\mathbb{R}^n)} = c_{n,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))}{|x - y|^{n+2s}} (-w(y)) dy dx. \tag{2.32}$$

Consequently, (2.30) follows from (2.31) and (2.32).

2.5. The Non-local Operator $\mathcal{L}_K$ and the Spaces $\mathcal{X}$ and $\mathcal{X}_0$

The facts given in the following can be found in [33] and [37]. One of the problems we investigate in the last chapter is driven by the following non-local integrodifferential operator:

Definition 2.5.1 The non-local operator $\mathcal{L}_K$ is defined as follows:

$$\mathcal{L}_K u(x) := \frac{1}{2} \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K(y) dy, \quad x \in \mathbb{R}^n,$$

where the kernel $K : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$ has the following properties:

- $mK \in L_1(\mathbb{R}^n)$, where $m(x) = \min \{|x|^2, 1\}$;
- $\exists \theta > 0 : K(x) \geq \theta \, |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \backslash \{0\}$ and $s \in (0,1)$;
- $K(x) = K(-x)$ for any $x \in \mathbb{R}^n$.

It can be seen that the particular case of $\mathcal{L}_K$ when $K(x) := x^{-(n+2s)}$ yield the fractional Laplacian (without normalization factor $c_{n,s}$) given in (2.25) .

Fractional Sobolev spaces are not adequate for investigating a problem driven by the operator $\mathcal{L}_K$, so the following function spaces $\mathcal{X}$ and $\mathcal{X}_0$ were introduced.

Definition 2.5.2. $\mathcal{X}$ is the linear space of Lebesgue measurable functions u from $\mathbb{R}^n$ to $\mathbb{R}$ such that $u|_\Omega \in L^2(\Omega)$ and the map $(x, y) \to (u(x) - u(y))\sqrt{K(x-y)}$ is in $L^2(Q, dxdy)$, where Q is as in (2.24).

The space $\mathcal{X}$ is endowed with the norm defined by

$$\|u\|_{\mathcal{X}} := \|u\|_{L^2(\Omega)} + \left(\int_Q (u(x) - u(y))^2 K(x-y) dx dy \right)^{1/2}.$$

The space $\mathcal{X}_0 := \{u \in \mathcal{X} : u \equiv 0 \text{ in } \mathbb{R}^n \backslash \Omega\}$ is an Hilbert space when it is endowed with the inner product given by

$$< u, v >_{\mathcal{X}_0} := \int_Q (u(x) - u(y))(v(x) - v(y)) K(x-y) dx dy$$

and with the norm defined by

$$\|u\|_{\mathcal{X}_0} := \left(\int_Q (u(x)-u(y))^2 K(x-y)dxdy \right)^{1/2}.$$

From [37] and the fact that $\mathcal{X}_0$ is a Hilbert space, one can say that this space separable and reflexive.

Lemma 2.5.4. There are continuous and compact embeddings, respectively, as follows:

$$\mathcal{X}_0 \hookrightarrow L_q^{\Omega}(\mathbb{R}^n), \qquad q \in \big[1, \frac{2n}{n-2s}\big],$$

and

$$\mathcal{X}_0 \hookrightarrow\hookrightarrow L_q^{\Omega}(\mathbb{R}^n), \qquad q \in \big[1, \frac{2n}{n-2s}\big).$$

The following Lemma can be proved by help of same way used in the proof of Lemma 2.4.5.

Lemma 2.5.5. Let $u, w \in X_0$. Then, the following equality holds:

$$\langle -\mathcal{L}_K u, w \rangle = \frac{1}{2} \int_{\mathbb{R}^{2n}} (u(x)-u(y)) (v(x)-v(y)) K(x-y)dydx.$$

2.6. An Existence Theorem and Some Related Definitions and Results

In chapter 4, we apply weak compactness method to show the existence of generalized solution of the considered problem. For this, we give the related definitions and theorems we used in that chapter. It is appropriate to begin with the definition of weak convergence:

Definition 2.6.1. [10] Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of elements of the Banach space X. Let $u \in X$ and let X^* be dual space of X. We say that u is the weak limit of the sequence $\{u_n\}_{n=1}^{\infty}$ and we denote this by

$$u_n \underset{X}{\rightharpoonup} u_0,$$

if for every continuous linear functional $u^* \in X^*$ we have $\lim_{n\to\infty} < u^*, u_n >=< u^*, u >$.

Theorem 2.6.2. [10] A weakly convergent sequence in Banach spaces is bounded.

Theorem 2.6.3. [10] Assume that X is a reflexive Banach space and let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in X. Then there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ and $u_0 \in X$ such that $u_{n_k} \rightharpoonup u_0$ in X.

Definition 2.6.4. [29] Let (Ω, Σ, μ) be a measure space. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on Ω and u a measurable function on Ω. The sequence $\{u_n\}_{n=1}^{\infty}$ is said to converge in measure on Ω to u provided for each $\epsilon > 0$,

$$\lim_{n \to \infty} \mu\left(\{x \in \Omega : |u(x) - u_n(x)| < \epsilon\}\right) = 0.$$

Theorem 2.6.5. [29] Let Ω be a domain in $\mathbb{R}^n$ and (Ω, Σ, μ) be a measure space. Moreover, let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions in the space $L_p(\Omega)$ such that $u_n \to u$ in $L_p(\Omega)$. Then, $u_n \to u$ in measure.

Theorem 2.6.6. [29] Let (Ω, Σ, μ) be a space with a finite measure. If a sequence of μ-measurable functions $\{u_n\}_{n=1}^{\infty}$ converges to u in measure μ, then there exists its subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ that converges to u almost everywhere.

Definition 2.6.7. [10] Let Ω be domain in $\mathbb{R}^n$ and let $g = g(x, \xi)$ be a function defined for almost all $x \in \Omega$ and for all $\xi \in \mathbb{R}^m$. We say that the function g has the Carathèdory property if:
(i) for all $\xi \in \mathbb{R}^m$, the function $g_\xi(x) = g(x; \xi)$ is measurable on Ω.
(ii) for almost all $x \in \Omega$, the function $g_x(\xi) = g(x; \xi)$ is continuous on $\mathbb{R}^m$.

Lemma 2.6.8. [2] Let (Ω, Σ, μ) be a measure space. The function g preserves almost everywhere convergence of sequences of measurable functions if and only if g is continuous.

Lemma 2.6.9. [19] Let Ω $(n \geq 1)$ be a open bounded domain in $\mathbb{R}^n$ and let g, g_n $(n \geq 1)$ be functions in $L_q(\Omega)$ with $1 < q < \infty$ such that

$$\|g_n\|_{L_q(\Omega)} \leq c \;\text{ and }\; g_n \to g \;\text{ a.e in }\; \Omega.$$

Then,

$$g_n \rightharpoonup g \;\text{ in }\; L_q(\Omega).$$

Theorem 2.6.10. [23] Let X be a reflexive Banach space and let Y be a Banach space. Then, every bounded linear operator $f : X \to Y$ is weakly compact.

We use the following theorem in order to prove existence of the solution for the problem given in chapter 4.

Theorem 2.6.11. [7] Let X be a separable, reflexive Banach space. Further, let X^* be dual space. The value of $y^* \in X^*$ on an element $x \in X$ we denote by $\langle y^*, x \rangle$.
We consider the equation

$$f(u) = h, \tag{2.33}$$

where $f : X \to X^*$ is an operator which satisfies following the following conditions:
(i) (Coercivity Relation.) For any $u \in X$, we have the relation

$$\frac{\langle f(u), u \rangle}{\|u\|_X} \to \infty, \text{ when } \ \|u\|_X \to \infty.$$

(ii) (Weak Compactness of the Operator f.) If for any $u_n \rightharpoonup u$ in X, then for any $v \in X$

$$\lim_{n\to\infty} \langle f(u_{n_k}), v \rangle = \langle f(u), v \rangle,$$

where $\{u_{n_k}\}_{k=1}^{\infty}$ subsequence of $\{u_n\}_{n=1}^{\infty}$.

Then for any element $h \in X^*$, the equation (2.33) has at least one solution $u \in X$.

2.7. Some Inequalities Involving Hölder and Young Inequalities

Lastly, we give some well-known inequalities which will be frequently used in chapter 4.

Lemma 2.7.1. (Generalized Hölder Inequality) [9] Let $1 \le p_1, ..., p_m \le \infty$, $\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$ and let $u_k \in L_{p_k}(\Omega)$ for $k = 1, 2, ..., m$. Then $u_1 ... u_m \in L_1(\Omega)$ and one has

$$\int_\Omega |u_1(x) ... u_m(x)| \, dx \le \prod_{k=1}^{m} \|u_k\|_{L_{p_k}(\Omega)}.$$

Lemma 2.7.2. (Young Inequality) [9] For $1 < p < \infty$, q the conjugate of p, and any two positive numbers a and b,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

or

$$ab \le \epsilon a^p + c(\epsilon) b^q, \quad (\epsilon > 0).$$

Lemma 2.7.3. [1] If $1 \le p < \infty$ and $a, b \ge 0$, then

$$(a + b)^p \le 2^{p-1} (a^p + b^p).$$

3. INITIAL VALUE PROBLEMS FOR THE NONLINEAR COMPLEX FRACTIONAL DIFFERENTIAL EQUATIONS

In this chapter, we consider some initial value problems for nonlinear complex fractional differential equations. One of these problems is as follows:

$$\begin{array}{rcl} D_z^{\alpha}u(z) & = & f\left(z, u(z)\right) \\ u(0) & = & 0\ , \end{array} \tag{3.1}$$

where D_z^{α} is the fractional derivative given in Definition 2.1.3, α is a fixed real number in $(0,1)$ and $f : \Omega \times \mathbb{C} \to \mathbb{C}$ ($\Omega = \overline{\mathbb{U}}$ or $\Omega = \mathbb{U}$) is a nonlinear function.

We investigate the existence and uniqueness of the solution of the problem (3.1) in the different spaces, i.e spaces $\mathcal{A}_0$ and $\mathcal{B}_R^0$, $R \in (0,1]$. The conditions on the function f depends on these spaces, namely: If this investigation is made in the space $\mathcal{B}_R^0$ with $0 < R \leq 1$, then we assume that the function $f : \overline{\mathbb{U}} \times \mathbb{C} \to \mathbb{C}$ satisfies the following condition:

> (3i) The function $f(z,t)$ is analytic on $\mathbb{D} \times \mathbb{C}$ and continuous on $\mathbb{D}^* \times \mathbb{C}$, and $z^{\alpha}f(z,t)$ is analytic on $\mathbb{U} \times \mathbb{C}$ and continuous on $\overline{\mathbb{U}} \times \mathbb{C}$,

and if this investigation is made in the space $\mathcal{A}_0$, then we suppose that the function $f : \mathbb{U} \times \mathbb{C} \to \mathbb{C}$ fulfills the condition as follows:

> (3ii) The function $f(z,t)$ be analytic on $\mathbb{D} \times \mathbb{C}$ and $z^{\alpha}f(z,t)$ be analytic on $\mathbb{U} \times \mathbb{C}$.

The conditions (3i) ((3ii)) together with the conditions given later will be sufficient for the well-posedness of the problem (3.1).

The other problem considered in this chapter is as follows:

$$\begin{gathered} (D_z^{\alpha} \circ z^{\alpha})u\left(z\right) = f\left(z, u(z)\right) \\ u(0) = 0, \end{gathered} \tag{3.2}$$

where $D^{\alpha} \circ z^{\alpha}$ is modified fractional derivative given in (2.11), α is a fixed real number in $(0,1)$ and $f : \Omega \times \mathbb{C} \to \mathbb{C}$ ($\Omega = \overline{\mathbb{U}}$ or $\Omega = \mathbb{U}$) is a nonlinear function.

We investigate the existence and uniqueness of the solution of the problem (3.2) in the spaces $\mathcal{B}_R^0$, provided that the function $f : \overline{\mathbb{U}} \times \mathbb{C} \to \mathbb{C}$ satisfies the condition:

> (3i*) The function $f(z,t)$ is analytic on $\mathbb{U} \times \mathbb{C}$ and continuous on $\overline{\mathbb{U}} \times \mathbb{C}$.

Moreover, this investigation is also made in the space $\mathcal{A}_0$, if the function $f : \mathbb{U}\times\mathbb{C} \to \mathbb{C}$ fulfills the condition as follows:

(3ii*) The function $f(z,t)$ is analytic on $\mathbb{U} \times \mathbb{C}$.

It can be seen later that the problem (3.2) is well-posed under the condition (3i*) ((3ii*)) together with the conditions given later.

Furthermore, in this chapter we consider the problems (3.1) and (3.2) with the nonhomogenous initial data and we investigate the existence and uniqueness of the solutions of these problems in the appropriate spaces.

Lastly, we here show that the equivalence of the problem (3.2) and the following problem:

$$\begin{array}{rcl} D^{\alpha}\left(u(z)\right) & = & f\left(z, z^{-\alpha}u(z)\right) \\ z^{-\alpha}u(z)|_{z=0} & = & 0, \end{array} \tag{3.3}$$

and by obtaining sufficient conditions in the sequel, we prove the existence and the uniqueness of the multivalued solution to the problem (3.3).

3.1. Existence and Uniqueness of the Solution of the Problem (3.1)

We begin with proving the compositional relations in the following which help us to define the solution of the problem (3.1) and, therefore, to show the existence and uniqueness of the solution of this problem.

Lemma 3.1.1. Let $0 < \alpha < 1$. Suppose that u is a continuous and integrable function in $\mathbb{D}_R$ for an arbitrary fixed $R > 0$, then the fractional differential equation

$$D_z^{\alpha}u(z) = 0 \quad (z \in \mathbb{D}_R) \tag{3.4}$$

has the solutions which are only in the form $u(z) = cz^{\alpha-1}$ with $c \in \mathbb{C}$.

Proof. For the proof, it is shown that there is a contradiction. It is obvious that $u(z) = cz^{\alpha-1}$ $(c \in \mathbb{C})$ are the solutions of (3.4). We suppose that there exists a different solution v of (3.4). Hence, $v(z) - cz^{\alpha-1}$ are also the solutions of (3.4), since D_z^{α} is a linear operator. By using $D_z^{\alpha} = DI_z^{1-\alpha}$ in (3.4), we have $I_z^{1-\alpha}(v(z) - cz^{\alpha-1}) = c^*$ for an arbitrary $c^* \in \mathbb{C}$. From here, we get

$$I_z^{1-\alpha}(v(z) - cz^{\alpha-1}) = \int_0^z \frac{[v(\zeta) - (c + \frac{c^*}{\Gamma(\alpha)})\zeta^{\alpha-1}]}{(z-\zeta)^{\alpha}} d\zeta = 0$$

for all $z \in \mathbb{D}_R$ and for all $c, c^* \in \mathbb{C}$. This implies that $v(z) = cz^{\alpha-1}$. Hence, $u(z) = cz^{\alpha-1}$ are unique solutions of (3.4).

Lemma 3.1.2. Under the conditions of Lemma 3.1.1, the following assertions are provided.

(i) $D_z^\alpha I_z^\alpha u(z) = u(z)$ for all $z \in \mathbb{D}_R$.

(ii) If $D_z^\alpha u$ is continuous and integrable on $\mathbb{D}_R$, and $z^\alpha D_z^\alpha u$ is continuous and integrable on $\mathbb{U}_R$, then the equality

$$I_z^\alpha D_z^\alpha u(z) = u(z) + cz^{\alpha-1} \quad (c \in \mathbb{C})$$

holds for all $z \in \mathbb{D}_R$.

Proof. (i) Let u be continuous and integrable on $\mathbb{D}_R$. By using that $D_z^\alpha = DI_z^{1-\alpha}$ and semi group property of the fractional integral we get

$$D_z^\alpha I_z^\alpha u(z) = DI_z^{1-\alpha} I_z^\alpha u(z) = D_z I_z u(z) = u(z) \tag{3.5}$$

for all $z \in \mathbb{D}_R$. The last equality in (3.5) is satisfied for any continuous function u, since the integral I_z is over the line segment.

(ii) At first, set

$$v\left(z\right) := I_z^\alpha \left(D_z^\alpha u(z)\right) \tag{3.6}$$

for all $z \in \mathbb{D}_R$. Let us show that $u(z) = v(z) + cz^{\alpha-1}$. If D_z^α is applied to both sides of (3.6) and, after that, if the equality in (i) is used for the right side of the obtained equality, then the equality

$$D_z^\alpha v(z) = D_z^\alpha u(z).$$

is obtained. Hence, if the linearity of the fractional derivative is considered, then, from the above equality, it is derived that

$$D_z^\alpha \left(\left(u - v\right)\left(z\right)\right) = 0.$$

From the fact that $D_z^\alpha u(z)$ is continuous and integrable on $\mathbb{D}_R$ in (3.6), it follows that $u - v$ is continuous and integrable on $\mathbb{D}_R$. Now, if Lemma 3.1.1 is used in the last equation, then it is clear that $u(z) = v(z) + cz^{\alpha-1}$. Consequently, by using this in (3.6) the equality

$$I_z^\alpha D_z^\alpha u(z) = u(z) + cz^{\alpha-1}$$

holds for all $z \in \mathbb{D}_R$.

Remark 3.1.3. (i) The unique continuous solution on $\mathbb{U}_R$ among solutions of the equation in (3.4) is $u = 0$. Therefore, if one takes $u \in \mathbb{C}^0(\mathbb{U}_R)$ in hypotheses in Lemma 3.1.1 and Lemma 3.1.2, then c in Lemma 3.1.2 (ii) has to be equal to zero. So, it is clear that I_z^α is inverse of the operator D_z^α with the domain $\mathbb{C}^0(\mathbb{U}_R)$.

(ii) The equalities in Lemma 3.1.2 are also valid on $\overline{\mathbb{U}}_R$ and $\mathbb{D}_R^*$.

Now, we suppose that the condition (3i) ((3ii)) is satisfied and the problem (3.1) has a solution $u \in \mathcal{B}$ ($u \in \mathcal{A}$). Then, $f(z, u(z))$ is continuous and integrable on $\mathbb{D}^*$ (on $\mathbb{D}$). By considering this fact in the problem (3.1), it can be seen that $D_z^\alpha u(z)$ is continuous and integrable on $\mathbb{D}$. Hence, if I_z^α is applied to the both sides of the equation in the problem (3.1) and, after that if Lemma 3.1.2 (ii) and Remark 3.1.3 (i) are used in the obtained equation, then the integral equation

$$u(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta, u(\zeta))}{(z-\zeta)^{1-\alpha}} d\zeta \tag{3.7}$$

holds for all $z \in \overline{\mathbb{U}}$ ($z \in \mathbb{U}$). In addition to this, when $u(0) = 0$, the equality (3.7) holds for $z = 0$, provided that the condition $z^\alpha f(z, 0)|_{z=0} = 0$ is satisfied.
Consequently, the following Lemma can be deduced from the above explanations.

Lemma 3.1.4. Let the condition (3i) ((3ii)) and the condition $z^\alpha f(z, 0)|_{z=0} = 0$ are satisfied. If $u \in \mathcal{B}^0$ ($\mathcal{A}_0$), then u is a solution of the problem (3.1) if and ony if, u satisfies the the Volterra-type integral equation in (3.7).

It should be noted that if $z^\alpha f(z, 0)|_{z=0} \neq 0$, the contradiction can be obtained as follows:

$$0 = u(0) = \lim_{z\to 0} u(z) = \frac{1}{\Gamma(\alpha)} \lim_{z\to 0} \int_0^1 \frac{(zt)^\alpha\, f(zt, u(zt))}{t^\alpha (1-t)^{1-\alpha}} dt \neq 0.$$

Theorem 3.1.5. Let the condition (3i) be satisfied. Moreover, we assume that there exist a fixed natural number $n_0 \geq 1$, a non-negative real number c and a function $g \in \mathcal{B}^0$ such that the following inequality holds for all $(z, t) \in \overline{\mathbb{U}} \times \mathbb{C}$:

$$|z^\alpha f(z, t)| \leq c\, |t|^{n_0} + |g(z)|\,. \tag{3.8}$$

Then there exists a $R \in (0, 1]$ such that the problem in (3.1) has at least one solution which is analytic on $\mathbb{U}_R$ and continuous on $\overline{\mathbb{U}}_R$.

Proof. Suppose that $u \in \mathcal{B}^0$. It is clear that the hypothesis of Lemma 3.1.4 are satisfied. As a consequences of this Lemma, the problem (3.1) is equivalent to the integral equation (3.7). If the operator P is defined as

$$Pu\left(z\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_0^z \frac{f\left(\zeta, u\left(\zeta\right)\right)}{\left(z-\zeta\right)^{1-\alpha}} d\zeta,$$

then P is an operator from $\mathcal{B}^0$ to $\mathcal{B}^0$. Hence, the fixed points of P in $\mathcal{B}^0$ coincide the solutions of the problem (3.1).

Thus, it is sufficient to prove the existence of the fixed points of the operator P. For the proof, it is shown that that the all conditions of Corollary 2.2.9 are satisfied.

Consider first the bounded, closed and convex subset B_r of $\mathcal{B}^0_R$ given as $B_r = \left\{u \in \mathcal{B}^0_R : \|u\|_{\mathcal{B}^0_R} \leq r\right\}$ with the fixed $r > 0$ and $R \in (0, 1]$, and let us see that there exist a suitable R such that

$$P(B_r) \subseteq B_r \tag{3.9}$$

is satisfied.

Now, let $R = 1$ and $u \in B_r$. Then, $|u(z)| \leq r$ for all $z \in \overline{\mathbb{U}}$. Moreover, since $g \in \mathcal{B}_0$, there exists a real number $M > 0$ such that $|g(z)| \leq M$ for all $z \in \overline{\mathbb{U}}$. From Schwarz's Lemma, it is deduced that $|u(z)| \leq r\,|z|$ on $\overline{\mathbb{U}}$ for all $u \in B_r$, and $|g(z)| \leq M\,|z|$ on $\overline{\mathbb{U}}$. If this inequalities are used in the inequality (3.8), then the chain of inequalities

$$\left|z^\alpha f\left(z, u(z)\right)\right| \leq c\left|u(z)\right|^{n_0} + \left|g(z)\right| \leq cr^{n_0}\left|z\right|^{n_0} + M\left|z\right| \quad (\forall z \in \overline{\mathbb{U}}, \forall u \in B_r \subset \mathcal{B}_0)$$

can be obtained.

From the above inequality and (3.8), it is derived that

$$\left|Pu\left(z\right)\right| \leq \frac{1}{\Gamma\left(\alpha\right)} \int_0^{|z|} \frac{\left|\zeta\right|^\alpha \left|f\left(\zeta, u\left(\zeta\right)\right)\right|}{\left|\zeta\right|^\alpha \left|\zeta - z\right|^{1-\alpha}} \left|d\zeta\right| \leq \frac{1}{\Gamma\left(\alpha\right)} \int_0^{|z|} \frac{cr^{n_0}\left|\zeta\right|^{n_0} + M\left|\zeta\right|}{\left|\zeta\right|^\alpha \left|\zeta - z\right|^{1-\alpha}} \left|d\zeta\right|,$$

for all $0 < |z| \leq R \leq 1$ and for all $u \in B_r \subset \mathcal{B}^0_R$.

If the variable ζ by $z\xi$ in the last inequality is changed, then

$$\begin{aligned} \left|Pu\left(z\right)\right| &\leq \frac{1}{\Gamma\left(\alpha\right)} \sup_{z \in \overline{\mathbb{U}}_R} \left[\int_0^1 \frac{cr^{n_0}\left|z\xi\right|^{n_0}}{\xi^\alpha\left(1-\xi\right)^{1-\alpha}} d\xi + M \int_0^1 \frac{\left|z\xi\right|}{\xi^\alpha\left(1-\xi\right)^{1-\alpha}} d\xi\right] \\ &\leq cr^{n_0}R^{n_0}\frac{\Gamma\left(n_0+1-\alpha\right)}{\Gamma\left(n_0+1\right)} + MR\Gamma\left(2-\alpha\right) \end{aligned}$$

is obtained for all $0 < |z| \leq R \leq 1$. Therefore, it can be easily seen that $P(B_r) \subseteq B_r$ holds for a suitable $0 < R = R(r, n_0, M, \alpha) < 1$.

It remains to show that P is a continuous operator on B_r and $P(B_r)$ is an equicontinuous set of $\mathcal{B}_R^0$.

For the continuity of P on B_r, it is supposed that $\{u_n\}_{n=1}^{\infty} \subset B_r$ is a sequence with $u_n \overset{\mathcal{B}_R^0}{\rightarrow} u$ as $n \to \infty$. Then, it is clear that u_n converges uniformly to $u \in B_r$, since B_r is a closed subset of $\mathcal{B}_R^0$. By using the uniform continuity of $z^{\alpha} f(z,t)$ on $\overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r$ ($\overline{\mathbb{U}}_r := \{\nu \in \mathbb{C} : |\nu| \leq r\}$) and uniform convergence of u_n to the function u on $\overline{\mathbb{U}}_R$, one can conclude that

$$\begin{aligned}\|Pu_n - Pu\|_{\mathcal{B}_R^0} &= \sup_{z\in\overline{\mathbb{U}}_R} \left| \frac{1}{\Gamma(\alpha)} \int_0^z \frac{[f(\zeta, u_n(\zeta)) - f(\zeta, u(\zeta))]}{(z-\zeta)^{1-\alpha}} d\zeta \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \sup_{\xi z\in\overline{\mathbb{U}}_R} \int_0^1 \frac{|(\xi z)^{\alpha} f(\xi z, u_n(\xi z)) - (\xi z)^{\alpha} f(\xi z, u(\xi z))|}{\xi^{\alpha}(1-\xi)^{1-\alpha}} d\xi \to 0\end{aligned}$$

as $n \to \infty$.

Now, let us show that $P(B_r)$ is an equicontinuous set of $\mathcal{B}_R^0$. Since all $u \in B_r$ are uniformly continuous on $\overline{\mathbb{U}}_R$ and $z^{\alpha} f(z,t)$ is uniformly continuous on $\overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r$, then $z^{\alpha} f(z, u(z))$ is also uniformly continuous on $\overline{\mathbb{U}}_R$. Therefore, for given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$|z_1^{\alpha} f(z_1, u(z_1)) - z_2^{\alpha} f(z_2, u(z_2))| < \frac{\epsilon}{\Gamma(1-\alpha)},$$

for all $z_1, z_2 \in \overline{\mathbb{U}}_R$ satisfying $|z_1 - z_2| < \delta$.

By using above inequality and by changing variable such that $\zeta = \xi z$ for $0 \leq t \leq 1$ when $|z_1 - z_2| < \delta$, one can conclude that

$$\begin{aligned}|Pu(z_1) - Pu(z_2)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{|(\xi z_1)^{\alpha} f(\xi z_1, u(\xi z_1)) - (\xi z_2)^{\alpha} f(\xi z_2, u(\xi z_2))|}{\xi^{\alpha}(1-\xi)^{1-\alpha}} d\xi \\ &< \Gamma(1-\alpha) \frac{\epsilon}{\Gamma(1-\alpha)} = \epsilon,\end{aligned}$$

since $|\xi z_1 - \xi z_2| < \delta$. So, it is obtained that $P(B_r)$ is an equicontinuous set of $\mathcal{B}_R^0$.

Consequently, by a direct application of Corollary 2.2.9, one can say that the operator P has at least one fixed point in $\mathcal{B}_R^0$ for a $R \in (0,1]$ given above, and it is also a solution of the problem (3.1).

Remark 3.1.6. As showed in [4], [40], Schauder fixed point theorem is applicable to prove the existence of local continuous solution for the equation (1.3) for any $f(x,t)$ satisfying the condition: $f(x,t)$ is continuous on $[0,1] \times \mathbb{R}$, or the condition: $x^{\sigma} f(x,t)$

$(0 < \sigma < \alpha < 1)$ and $f(x,t)$ are continuous on $[0,1] \times \mathbb{R}$ and $(0,1] \times \mathbb{R}$, respectively. However, by using this theorem with same technique applied in [4], [40], the existence of local desired solution of the problem (1.1)-(1.2) can be proved only for a subclass of functions f satisfying the conditions (3i) and $z^{\alpha} f(z,0)|_{z=0} = 0$. In Theorem 3.1.5, for a larger class of functions f satisfying these conditions, we proved the existence of desired solution for the considered problem by using this theorem. Indeed, from the condition (3i) one can suppose that

$$|z^{\alpha} f(z,t)| \leq M \quad (\forall (z,t) \in \overline{\mathbb{U}} \times \overline{\mathbb{U}}_r),$$

and by using this inequality one can write

$$\sup_{z \in \overline{\mathbb{U}}_R} |Pu(z)| \leq \frac{M}{\Gamma(\alpha)} \int_0^{|z|} \frac{1}{|\zeta|^{\alpha} |z-\zeta|^{1-\alpha}} |d\zeta| \leq M\Gamma(1-\alpha) \tag{3.10}$$

for all $u \in B_r$ and for all $z \in \overline{\mathbb{U}}_R$ with an arbitrary $R \in (0,1]$. Hence, it must be $M\Gamma(1-\alpha) \leq r$ in order that the condition $P(B_r) \subseteq B_r$ is satisfied. This indicates that the function $f(z,t)$ has to satisfy the following inequality:

$$|z^{\alpha} f(z,r)| \leq \frac{r}{\Gamma(1-\alpha)} \quad (\forall z \in \overline{\mathbb{U}}), \tag{3.11}$$

which means that f increases not faster than a linear function of r. Moreover, this inequality is a particular case of the inequality in Theorem 3.1.5, when $c := \frac{1}{\Gamma(1-\alpha)}$, $n_0 = 1$ and $g(z) \equiv 0$. On the other hand, the solution of the considered problem exists on whole $\mathbb{U}$, since the inequality (3.10) holds for all $z \in \overline{\mathbb{U}}_R$ with an arbitrary $R \in (0,1]$.

Remark 3.1.7. Theorem 3.1.5 does not indicate that the problem (3.1) admits a unique solution in $\mathcal{B}_R^0$ with any $R \in (0,1]$. Indeed, if $f(z,u) := \frac{z^{-\alpha}}{\Gamma(2-\alpha)} u$ with the fixed $\alpha \in (0,1)$ in the problem (3.1), then the problem admits the solutions $u(z) = c_* z$ for all $c_* \in \mathbb{C}$.

Now, we give in the following theorem which implies not only existence but also uniqueness of the desired solution for the considered problem.

Theorem 3.1.8. Let the condition (3i) be satisfied and let $z^{\alpha} f(z,0)|_{z=0} = 0$. Moreover, assume that there exists a constant $\kappa < 1/\Gamma(2-\alpha)$ such that

$$|f(z,\eta) - f(z,\nu)| < \frac{\kappa}{|z|^{\alpha}} |\eta - \nu| \tag{3.12}$$

for all $z \in \mathbb{D}^*$ and for all $\eta,\nu \in \mathbb{C}$. Then the problem (3.1) has a unique solution in $\mathcal{B}_0$.

Proof. It is supposed that $u \in \mathcal{B}_0$. Since the conditions of Lemma 3.1.4 are satisfied, the integral equation (3.7) can be considered. If we define the operator P as follows:

$$Pu\left(z\right) \;=\; \frac{1}{\Gamma\left(\alpha\right)}\int_0^z \frac{f\left(\zeta,u(\zeta)\right)}{\left(z-\zeta\right)^{1-\alpha}}d\zeta,$$

then it is clear that P is an operator from the space $\mathcal{B}_0$ to itself. The operator P is well defined. Indeed, if the inequality in (3.12) is taken into account, then it is obtained that the inequality

$$\left|z^{\alpha}f(z,\eta)\right| < \kappa\left|\eta\right| + \left|z^{\alpha}f(z,0)\right|$$

for all $z \in \overline{\mathbb{U}}$ and for all $\eta \in \mathbb{C}$. By using this inequality

$$\left|Pu\left(z\right)\right| = \left|\frac{1}{\Gamma\left(\alpha\right)}\int_0^z \frac{f\left(\zeta,u(\zeta)\right)}{\left(z-\zeta\right)^{1-\alpha}}d\zeta\right| \le \Gamma\left(1-\alpha\right)\left(\kappa\left\|u\right\|_{\mathcal{B}_0} + \sup_{z\in\overline{\mathbb{U}}}\left|z^{\alpha}f(z,0)\right|\right)$$

can be obtained.

Hence, the fixed points of the operator P coincide with the solutions of the problem (3.1). Then, it is sufficient to show that this operator has a unique fixed point in $\mathcal{B}_0$. To do this by virtue of Banach fixed point theorem, it is enough to see that the operator P is a contraction. Now, let us suppose that u and u_0 are the arbitrary elements of $\mathcal{B}_0$. Then, $u-u_0 \in \mathcal{B}_0$ and it is clear that $\left|(u-u_0)(z)\right| \le \left\|u-u_0\right\|_{\mathcal{B}_0}$ for all $z \in \overline{\mathbb{U}}$. By using Schwarz's Lemma for this inequality, it follows that

$$\left|\frac{(u-u_0)(z)}{z}\right| \le \left\|u-u_0\right\|_{\mathcal{B}_0} \quad \text{for all } z \in \overline{\mathbb{U}}.$$

If the last inequality with (3.12) is considered, then the following chain of inequalities can be obtained:

$$\begin{aligned}
\left|Pu\left(z\right)-Pu_0\left(z\right)\right| &\le \frac{1}{\Gamma\left(\alpha\right)}\int_0^{|z|}\frac{\left|f\left(\zeta,u(\zeta)\right)-f\left(\zeta,u_0(\zeta)\right)\right|}{\left|z-\zeta\right|^{1-\alpha}}\left|d\zeta\right| \\
&< \frac{\kappa}{\Gamma\left(\alpha\right)}\int_0^{|z|}\frac{\left|u(\zeta)-u_0(\zeta)\right|}{\left|\zeta\right|^{\alpha}\left|z-\zeta\right|^{1-\alpha}}\left|d\zeta\right| \\
&\le \frac{\kappa}{\Gamma\left(\alpha\right)}\int_0^{|z|}\frac{\left|\zeta\right|\left|\frac{(u-u_0)(\zeta)}{\zeta}\right|}{\left|\zeta\right|^{\alpha}\left|z-\zeta\right|^{1-\alpha}}\left|d\zeta\right| \\
&\le \frac{\kappa\left|z\right|\left\|u-u_0\right\|_{\mathcal{B}_0}}{\Gamma\left(\alpha\right)}\int_0^1 \xi^{1-\alpha}\left(1-\xi\right)^{\alpha-1}d\xi \\
&\le \kappa\Gamma\left(2-\alpha\right)\left\|u-u_0\right\|_{\mathcal{B}_0},
\end{aligned}$$

where we used the change of variable such as $\zeta = z\xi$ with $\xi \in [0,1]$. Since $\kappa < 1/\Gamma(2-\alpha)$, the above inequality implies that P is a contraction operator. As a consequence of Banach fixed point theorem, one can say that there exists a unique fixed point of the operator P in the space $\mathcal{B}_0$. Consequently, the problem (3.1) has a unique solution u in $\mathcal{B}_0$.

Following corollary can be similarly proved as above theorem.

Corollary 3.1.9. Suppose that the condition (3ii) and the conditions $z^{\alpha} f(z,0)|_{z=0} = 0$, $\sup_{z\in\mathbb{U}} |z^{\alpha} f(z,0)| < \infty$ are satisfied. Moreover, assume that the inequality (3.12) is provided for all $z \in \mathbb{D}$ and for all $\eta, \nu \in \mathbb{C}$, when $\kappa < 1/\Gamma(2-\alpha)$. Then the problem (3.1) has a unique solution u in $\mathcal{A}_0$.

Remark 3.1.10. If we didn't use the technique related to Schwarz's Lemma in the proof of Theorem 3.1.8, then we would have

$$|Pu(z) - Pu_0(z)| < \kappa\Gamma(1-\alpha)\,\|u - u_0\|_{\mathcal{B}_0}$$

for any $u, u_0 \in \mathcal{B}_0$. For P to be contraction operator we require that $\kappa < 1/\Gamma(1-\alpha)$. The following example shows that how such technique contributed the function f to be from wider class of functions satisfying the hypothesis of Theorem 3.1.8. Let $f(z,u) := \frac{z^{-\alpha}}{2\Gamma(2-\alpha)}\left[u + z + \frac{2+\alpha}{2-\alpha}z^2\right]$ in the problem (3.1). In this case, this problem admits a unique solution (which is equal to $u(z) = z + z^2$) for all $\alpha \in (0,1)$. But, for the same function f satisfying the inequality (3.12) with $\kappa < 1/\Gamma(1-\alpha)$, there exists a unique solution of the problem for only all $\alpha \in (0,1/2)$, since $\frac{1}{2\Gamma(2-\alpha)} < \frac{1}{\Gamma(1-\alpha)}$ is satisfied for only all $\alpha \in (0,1/2)$.

The condition $z^{\alpha} f(z,0)|_{z=0} = 0$ in Theorem 3.1.5, Theorem 3.1.8, Corollary 3.1.9 is needed for the equivalence of the Cauchy problem (3.1) and the Volterra type equation in (3.7) and was ignored by Ibrahim and Darus in their work [15]. In the following it is shown that this condition is indispensable by proving that, without this condition, the problem (3.1) has no analytic solution.

Proposition 3.1.11. The condition $z^{\alpha} f(z,0)|_{z=0} = 0$ is necessary for the existence of analytic solution to the problem (3.1).

Proof. This proposition is proved by showing that there exist a contradiction. For this, let α be fixed in $(0,1)$ and let $f(z,u) := cz^{-\alpha}u + dz^{-\alpha}$ with any $d \in \mathbb{C} - \{0\}$,

$c \in \mathbb{C}$ in the problem (3.1). We suppose that the problem admits an analytic solution u on $\mathbb{U}$. Then, u is represented by a power-series expansion such as $u(z) = \sum_{k=1}^{\infty} a_k z^k$ on $\mathbb{U}$. Since u satisfies this problem, then the following equality can be obtained:

$$\sum_{k=1}^{\infty} a_k \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} z^{k-\alpha} = \sum_{k=1}^{\infty} c a_k z^{k-\alpha} + d z^{-\alpha} \quad (z \in \mathbb{D}).$$

However, it is a contradiction since the above equality is not provided. Therefore, there is no analytic solution of this problem unless the condition $z^{\alpha} f\left(z,0\right)|_{z=0} = 0$ is satisfied.

The existence and uniqueness results given above can be obtained for the problem (3.1) with non-homogenous initial data. However, it is sufficient to focus on some differences in the following.

Remark 3.1.12. Consider the following problem which is a more general form of the problem (3.1):

$$\begin{array}{rcl} D_z^{\alpha} u(z) &=& f(z, u(z)) \\ u(0) &=& b \quad (b \in \mathbb{C}). \end{array} \tag{3.13}$$

By replacing $u(z)$ by $v(z) + b$ in the above, we get the problem:

$$\begin{array}{rcl} D_z^{\alpha} v(z) &=& h(z, v(z) + b) \\ v(0) &=& 0. \end{array}$$

where $h(z, v(z) + b) = -\frac{b}{\Gamma(1-\alpha)} z^{-\alpha} + f(z, v(z) + b)$. If we suppose that the condition (3i) and the condition

$$\left|z^{\alpha} h\left(z,t\right)\right| \leq c \left|t - b\right|^{n_0} + \left|g(z)\right|, \quad (n_0 \in \mathbb{N}, c \geq 0, g \in \mathcal{B}^0)$$

instead of the inequality (3.7) in Theorem 3.1.5 are satisfied, then it can be shown, in the similar way in the proof of Theorem 3.1.5, that the problem above has at least one solution $v \in \mathcal{B}_R^0$ for a suitable $0 < R \leq 1$. So, from the equivalence of problems above, it can be obtained that the problem (3.13) admits at least one solution $u \in \mathcal{B}_R$ with $u(0) = b$.

Furthermore, if one changes the condition $z^{\alpha} f(z,0)|_{z=0} = 0$ in Theorem 3.1.8 (Corollary 3.1.9) with $z^{\alpha} f(z,b)|_{z=0} = \frac{b}{\Gamma(1-\alpha)}$ by keeping the other conditions of this theorem (corollary) same, then one can prove, by using same method in the proof of Theorem 3.1.8, that the problem (3.13) admits unique solution $u \in \mathcal{B}$ with $u(0) = b$ ($u \in \mathcal{A}$ with $u(0) = b$).

3.2. Existence and Uniqueness of the Solution of the Problems (3.2) and (3.3)

As in the previous subsection, we first show in the following that the compositional relations are satisfied.

Lemma 3.2.1. Let α be fixed in $(0,1)$. Suppose that $u(z)$ is continuous on $\mathbb{U}$, then the fractional differential equation

$$(D_z^{\alpha} \circ z^{\alpha})u(z) = 0 \quad (z \in \mathbb{U}) \tag{3.14}$$

has a unique solution $u = 0$.

Proof. The following equality follows from the equation in (3.14) and the fact $D_z^{\alpha} = DI_z^{1-\alpha}$ for $\alpha \in (0,1)$:

$$(I_z^{1-\alpha} \circ z^{\alpha})u(z) = z\int_0^1 \frac{t^{\alpha}u(zt)}{(1-t)^{\alpha}}dt = c \quad (c \in \mathbb{C},\ z \in \mathbb{U})$$

This equality only holds for $u = 0$, since u is continuous on $\mathbb{U}$. Therefore, $u = 0$ is the unique continuous solution of the equation in (3.14).

Lemma 3.2.2. Let the conditions of Lemma 3.2.1 be fulfilled. Then, the following assertions are provided.

(i) $(D_z^{\alpha} \circ z^{\alpha})(z^{-\alpha} \circ I_z^{\alpha})u(z) = u(z)$ for all $z \in \mathbb{U}$.

(ii) If $(D_z^{\alpha} \circ z^{\alpha})u$ is continuous on $\mathbb{U}$, then $(z^{-\alpha} \circ I_z^{\alpha})(D_z^{\alpha} \circ z^{\alpha})u(z) = u(z)$ for all $z \in \mathbb{U}$.

Proof. (i) Let u be continuous on $\mathbb{U}$. By using semi group property of the fractional integrals, one can write the following equality for all $z \in \mathbb{U}$:

$$(D_z^{\alpha} \circ z^{\alpha})(z^{-\alpha} \circ I_z^{\alpha})u(z) = D_z^{\alpha}I_z^{\alpha}u(z) = DI_z^{1-\alpha}I_z^{\alpha}u(z) = DIu = u(z).$$

The last equality in the above holds, since I is an integral over the line segment.

(ii) It is supposed that the functions u and $(D_z^{\alpha} \circ z^{\alpha})u$ are continuous on $\mathbb{U}$. Let us define the function v as follows:

$$v(z) := \left(z^{-\alpha}I_z^{\alpha}\right)(D_z^{\alpha} \circ z^{\alpha})u(z)$$

for all $z \in \mathbb{U}$.

Our aim is to show $u = v$ on $\mathbb{U}$. Now, if $D_z^\alpha \circ z^\alpha$ is applied to the both sides of the above equality, then from (i) it follows that

$$(D_z^\alpha \circ z^\alpha)v(z) = (D_z^\alpha \circ z^\alpha)u(z)$$

for all $z \in \mathbb{U}$. Hence, the desired result follows from the linearity of the fractional derivative and Lemma 3.2.1.

It needs to note that the compositional relations in Lemma 3.2.2 are also valid on $\overline{\mathbb{U}}$. Now, by using this relations, we try to define the solution of the problem (3.2). For this, we first assume that the condition (3i*) ((3ii*)) is satisfied and the problem (3.2) has a solution $u \in \mathcal{B}$ ($u \in \mathcal{A}$). By using this assumptions, it follows that $f(z, u(z))$ is continuous on $\overline{\mathbb{U}}$ (on $\mathbb{U}$). Considering this fact in this problem, it is easily seen that $(D_z^\alpha \circ z^\alpha)u(z)$ is continuous on $\mathbb{U}$. By applying $z^{-\alpha} I_z^\alpha$ to the equation in the problem (3.2) and, after that, by using Lemma 3.2.2, the following integral equation is obtained:

$$u\left(z\right) = \frac{z^{-\alpha}}{\Gamma\left(\alpha\right)} \int_0^z \frac{f\left(\zeta, u(\zeta)\right)}{\left(z-\zeta\right)^{1-\alpha}} d\zeta, \quad (z \in \overline{\mathbb{U}}, (z \in \mathbb{U})). \tag{3.15}$$

Moreover, since $u(0) = 0$ in the problem (3.2), the equation above has to satisfy this initial condition. This is only possible if the function f satisfies the condition $f(0, 0) = 0$.

Therefore, the following Lemma can be deduced from the above explanations.

Lemma 3.2.3. Let the condition (3i*) ((3ii*)) and the condition $f(0, 0) = 0$ are satisfied. If $u \in \mathcal{B}^0$ ($\mathcal{A}_0$), then u is a solution of the problem (3.2) if and only if u satisfies the Volterra-type integral equation in (3.15).

It needs to note that, if the function f does not satisfy the condition $f(0, 0) = 0$, then one can obtain the following contradiction:

$$0 = u(0) = \lim_{z\to 0} u(z) = \frac{1}{\Gamma(\alpha)} \lim_{z\to 0} \int_0^1 \frac{f\left(zt, u\left(zt\right)\right)}{\left(1-t\right)^{1-\alpha}} dt \neq 0.$$

Theorem 3.2.4. Let the condition (3i*) be satisfied. Moreover, we assume that we assume that there exist a fixed natural number $n_0 \geq 1$, a non-negative real number c and a function $g \in \mathcal{B}^0$ such that the following inequality holds for all $(z, t) \in \overline{\mathbb{U}} \times \mathbb{C}$:

$$\left|f\left(z, t\right)\right| \leq c \left|t\right|^{n_0} + \left|g(z)\right|. \tag{3.16}$$

Then there exists a suitable $R \in (0,1]$ such that the problem (3.2) has at least one solution which is analytic on $\mathbb{U}_R$ and continuous on $\overline{\mathbb{U}}_R$.

Proof. Let $u \in \mathcal{B}_0$. From our assumptions, it is easily seen that the hypotheses of Lemma 3.2.4 are justified. Then, one can consider the solutions in $\mathcal{B}_0$ of the problem (3.2) same as those of the equation (3.15).

Now, we define the operator T as follows:

$$Tu\,(z) = \frac{z^{-\alpha}}{\Gamma\,(\alpha)} \int_0^z \frac{f\,(\zeta, u(\zeta))}{(z-\zeta)\,.^{1-\alpha}} d\zeta$$

It is clear that T is an operator from $\mathcal{B}_0$ to $\mathcal{B}_0$, and the fixed points of this operator are at the same time the solutions of the problem (3.2). So, it is sufficient for our aim to show the existence of the fixed points of the operator T. We show this by proving that the hypothesis of Corollary 2.2.9 are fulfilled.

Let us at consider $B_r = \left\{u \in \mathcal{B}_R^0 : \|u\|_{\mathcal{B}_R^0} \leq r\right\}$ with a fixed $r > 0$ and $0 < R \leq 1$ which is a closed, bounded and convex subset of $\mathcal{B}_R^0$. It is first shown that there exists a suitable $R \in (0,1]$ such that the following inclusion is satisfied:

$$T(B_r) \subseteq B_r. \tag{3.17}$$

Now, let $R = 1$ and $u \in B_r$. Then, $|u(z)| \leq r$ for all $z \in \overline{\mathbb{U}}$. Moreover, since $g \in \mathcal{B}^0$, there exists a real number $M > 0$ such that $|g(z)| \leq M$ for all $z \in \overline{\mathbb{U}}$. From these facts it is obtained, by using Schwarz's Lemma (see Lemma 2.11), that

$$|u(z)| \leq r\,|z| \text{ and } |g(z)| \leq M\,|z| \text{ for all } z \in \overline{\mathbb{U}}.$$

If this inequalities are used in the inequality (3.16), then it can be seen that the inequality

$$|f\,(z, u(z))| \leq cr^{n_0}\,|z|^{n_0} + M\,|z|$$

is satisfied for all $z \in \overline{\mathbb{U}}$ and for all $u \in B_r \subset \mathcal{B}_0$.

From the above inequality, the equality

$$|Tu\,(z)| \leq \frac{|z|^{-\alpha}}{\Gamma\,(\alpha)} \int_0^{|z|} \frac{|f\,(\zeta, u\,(\zeta))|}{|\zeta - z|^{1-\alpha}}\,|d\zeta| \leq \frac{1}{\Gamma\,(\alpha)} \int_0^{|z|} \frac{cr^{n_0}\,|\zeta|^{n_0} + M\,|\zeta|}{|\zeta - z|^{1-\alpha}}\,|d\zeta|\,,$$

holds for all $0 < |z| \leq R \leq 1$ and for all $u \in B_r \subset \mathcal{B}_R^0$.

If one changes the variable ζ by $z\xi$ in the last inequality, then one can write the following chain of inequalities for all $0<|z|\leq R\leq 1:$

$$\begin{aligned}|Tu\left(z\right)| &\leq \frac{1}{\Gamma\left(\alpha\right)}\sup_{z\in\overline{\mathbb{U}}_R}\left[\int_0^1\frac{cr^{n_0}\left|z\xi\right|^{n_0}}{\left(1-\xi\right)^{1-\alpha}}d\xi+M\int_0^1\frac{\left|z\xi\right|}{\left(1-\xi\right)^{1-\alpha}}d\xi\right]\\ &\leq cr^{n_0}R^{n_0}\frac{\Gamma\left(n_0+1\right)}{\Gamma\left(n_0+1+\alpha\right)}+\frac{MR}{\Gamma\left(2+\alpha\right)}.\end{aligned}$$

From here, one can find a suitable $0<R=R(\alpha,n_0,M,c,r)\leq 1$ such that $|Tu(z)|\leq r$ is satisfied for all $0<|z|\leq R\leq 1$. Therefore, by taking the norms in $\mathcal{B}_R^0$, it is easily seen that $T(B_r)\subset B_r$.

Now, let us show the continuity of T from $\mathcal{B}_R^0$ to $\mathcal{B}_R^0$. Let $\{u_n\}_{n=1}^{\infty}\subset B_r$ be a sequence with $u_n\overset{\mathcal{B}_R^0}{\to}u$ as $n\to\infty$. Then, u_n converges uniformly to $u\in\mathcal{B}_R^0$ on $\overline{\mathbb{U}}_R$. From here and from the fact that $f(z,t)$ is uniformly continuous on $\overline{\mathbb{U}}_R\times\overline{\mathbb{U}}_r$ ($\overline{\mathbb{U}}_r:=\{\nu\in\mathbb{C}:|\nu|\leq r\}$), it follows that

$$\begin{aligned}\|Tu_n-Tu\|_{\mathcal{B}_R^0} &= \sup_{z\in\overline{\mathbb{U}}_R}\left|\frac{z^{-\alpha}}{\Gamma\left(\alpha\right)}\int_0^z\frac{\left[f\left(\zeta,u_n(\zeta)\right)-f\left(\zeta,u(\zeta)\right)\right]}{\left(z-\zeta\right)^{1-\alpha}}d\zeta\right|\\ &\leq \frac{1}{\Gamma\left(\alpha\right)}\sup_{\xi z\in\overline{\mathbb{U}}}\int_0^1\frac{\left|f\left(\xi z,u_n(\xi z)\right)-f\left(\xi z,u(\xi z)\right)\right|}{\left(1-\xi\right)^{1-\alpha}}d\xi\to 0\end{aligned}$$

as $n\to\infty$.

So, the remaining is to show the equicontinuity of $T(B_r)$ in $\mathcal{B}_R^0$. For all $u\in B_r$, u is uniformly continuous on $\overline{\mathbb{U}}_R$ and $f(z,t)$ is uniformly continuous on $\overline{\mathbb{U}}_R\times\overline{\mathbb{U}}_r$, then $f(z,u(z))$ is uniformly continuous on $\overline{\mathbb{U}}_R$. Thus, for given $\epsilon>0$ there exists a $\delta=\delta(\epsilon)>0$ such that

$$|f(z_1,u(z_1))-f(z_2,u(z_2))|<\Gamma(1+\alpha)\epsilon,$$

for all $z_1,z_2\in\overline{\mathbb{U}}_R$ satisfying $|z_1-z_2|<\delta$. By using the above inequality and by changing variable such that $\zeta=\xi z$ for $0\leq t\leq 1$ when $|z_1-z_2|<\delta$, we have

$$\begin{aligned}|Tu\left(z_1\right)-Tu\left(z_2\right)| &= \left|\frac{z_1^{-\alpha}}{\Gamma\left(\alpha\right)}\int_0^{z_1}\frac{f\left(\zeta,u(\zeta)\right)}{\left(z_1-\zeta\right)^{1-\alpha}}d\zeta-\frac{z_2^{-\alpha}}{\Gamma\left(\alpha\right)}\int_0^{z_2}\frac{f\left(\zeta,u(\zeta)\right)}{\left(z_2-\zeta\right)^{1-\alpha}}d\zeta\right|\\ &\leq \frac{1}{\Gamma\left(\alpha\right)}\int_0^1\frac{\left|f\left(\xi z_1,u\left(\xi z_1\right)\right)-f\left(\xi z_2,u\left(\xi z_2\right)\right)\right|}{\left(1-\xi\right)^{1-\alpha}}d\xi\\ &< \frac{1}{\Gamma(1+\alpha)}\Gamma(1+\alpha)\epsilon=\epsilon,\end{aligned}$$

since $|\xi z_1-\xi z_2|<\delta$. So, $T(B_r)$ is equicontinuous set in $\mathcal{B}_R^0$ and therefore T is a compact operator from $\mathcal{B}_R^0$ to $\mathcal{B}_R^0$.

Consequently, as a result of Corollary 2.2.9, there exists at least one fixed point of P in $\mathcal{B}_R^0$ which is also a solution of (3.2).

Theorem 3.2.5. Let the condition (3i*) be satisfied and let $f(0,0) = 0$. Moreover, assume that there exists a constant $\kappa < \Gamma(2+\alpha)$ such that

$$|f(z,\eta) - f(z,\nu)| < \kappa |\eta - \nu| \tag{3.18}$$

for all $z \in \overline{\mathbb{U}}$ and for all η,$\nu \in \mathbb{C}$. Then the problem (3.3) has a unique (multi-valued) solution u in $\mathcal{B}_{-\alpha}^0\left(\overline{\mathbb{U}}\right)$.

Proof. Suppose that $u \in \mathcal{B}_{-\alpha}\left(\overline{\mathbb{U}}\right)$. Then there exists a function v in $\mathcal{B}^0$ such that $v(z) = z^{-\alpha}u(z)$. If $u(z)$ is replaced with $z^{\alpha}v(z)$ in problem (3.3), then the problem (3.2) is obtained. Since the hypotheses of Lemma 3.2.3 are satisfied, the integral equation in (3.15) with $u = v$ can be considered instead of the problem (3.2).

Now, let us define the operator T as follows

$$Tv(z) = \frac{z^{-\alpha}}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta, v(\zeta))}{(z-\zeta)^{1-\alpha}} d\zeta$$

and investigate the existence of the fix points of this operator, which are at the same time the solutions of the problem (3.2). $T : \mathcal{B}^0 \to \mathcal{B}^0$ is well-defined. In fact, if we take into account the assumption in (3.18), then we have $|f(z,\eta)| < \kappa|\eta| + |f(z,0)|$ for all $z \in \overline{\mathbb{U}}$ and for all $\eta \in \mathbb{C}$. Hence, the following inequality justifies the our claim:

$$|Tv(z)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta, v(\zeta))}{(z-\zeta)^{1-\alpha}} d\zeta \right| \leq \frac{1}{\Gamma(1+\alpha)} \left(\kappa \|v\|_{\mathcal{B}^0} + \sup_{z \in \overline{\mathbb{U}}} |f(z,0)| \right)$$

We now show that the operator T is a contraction. Let v and v_0 be the arbitrary elements of $\mathcal{B}^0$. Then, $v - v_0 \in \mathcal{B}^0$ and it is clear that $|(v-v_0)(z)| \leq \|v - v_0\|_{\mathcal{B}^0}$ for all $z \in \overline{\mathbb{U}}$. From Schwarz's Lemma it is obtained that

$$\left| \frac{(v-v_0)(z)}{z} \right| \leq \|v - v_0\|_{\mathcal{B}^0} \quad \text{for all} \quad z \in \overline{\mathbb{U}}.$$

By using the last inequality and the equality (3.18) the following chain of inequalities can be written:

$$\begin{aligned}
|Tv(z) - Tv_0(z)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^{|z|} \frac{|f(\zeta, v(\zeta)) - f(\zeta, v_0(\zeta))|}{|z-\zeta|^{1-\alpha}} |d\zeta| \\
&< \frac{\kappa}{\Gamma(\alpha)} \int_0^{|z|} \frac{|v(\zeta) - v_0(\zeta)|}{|z-\zeta|^{1-\alpha}} |d\zeta| \\
&< \frac{\kappa}{\Gamma(\alpha)} \int_0^{|z|} \frac{|\zeta| \left| \frac{(v-v_0)(\zeta)}{\zeta} \right|}{|z-\zeta|^{1-\alpha}} |d\zeta| \\
&\leq \frac{\kappa}{\Gamma(2+\alpha)} \|v - v_0\|_{\mathcal{B}^0}.
\end{aligned}$$

Since $\kappa < \Gamma(2+\alpha)$, then T is a contraction operator. The conditions of Banach fixed point theorem are provided and hence there exists a unique fixed point of the operator T in the space $\mathcal{B}^0$. Consequently the problem (3.3) has a unique solution u in $\mathcal{B}_{-\alpha}\left(\overline{\mathbb{U}}\right)$.

Following two corollaries can be proved as in Theorem 3.2.5.

Corollary 3.2.6. Let the condition (3i*) be satisfied and let $f(0,0) = 0$. Moreover, assume that the inequality (3.18) is provided for all $(z,\eta), (z,\nu) \in \overline{\mathbb{U}} \times \mathbb{C}$ when $\kappa < \Gamma(2+\alpha)$. Then the problem (3.2) has a unique solution u in $\mathcal{B}_0$.

Corollary 3.2.7. Let the condition (3ii*) and the conditions $f(0,0) = 0$, $\sup_{z\in\mathbb{U}} |f(z,0|$ $< \infty$ be satisfied. Moreover, if $f(z,t)$ satisfies the inequality (3.18) for all $(z,\eta), (z,\nu) \in \mathbb{U} \times \mathbb{C}$ when $\kappa < \Gamma(2+\alpha)$, then the problem (3.2) has a unique solution in $\mathcal{A}_0$.

Remark 3.2.8. Let us consider the equation in the problem (3.2) with a more general initial data $u(0) = b$ with $b \in \mathbb{C}$:

$$\begin{gathered}
(D_z^\alpha \circ z^\alpha)u(z) = f(z, u(z)) \quad (z \in \mathbb{U}) \\
u(0) = b.
\end{gathered} \tag{3.19}$$

If we set $v(z) = u(z) - b$ in the problem above, then we have

$$\begin{gathered}
(D_z^\alpha \circ z^\alpha)v(z) = g(z, v(z) + b) \quad (z \in \mathbb{U}) \\
v(0) = 0,
\end{gathered} \tag{3.20}$$

where $g(z, v(z) + b) = -b\Gamma(1+\alpha) + f(z, v(z) + b)$. By using Lemma 3.2.3 (ii), it can be easily seen that the problem (3.20) is equivalent to the following Volterra-type equation:

$$v(z) = \frac{z^{-\alpha}}{\Gamma(\alpha)} \int_0^z \frac{g(\zeta, v(\zeta) + b)}{(z-\zeta)^{1-\alpha}} d\zeta$$

for all $z \in \overline{\mathbb{U}}$. Now we define the operator T^* in the following form:

$$T^*v(z) = \frac{z^{-\alpha}}{\Gamma(\alpha)} \int_0^z \frac{g(\zeta, v(\zeta) + b)}{(z-\zeta)^{1-\alpha}} d\zeta.$$

If it is supposed that the condition (3i*) and the condition:

$$|g\left(z,t\right)| \leq c\left|t-b\right|^{n_0} + \left|g(z)\right|, \quad (n_0 \in \mathbb{N}, c \geq 0, g \in \mathcal{B}^0)$$

for all $(z,t) \in \overline{\mathbb{U}} \times \mathbb{C}$ are fulfilled, it can be shown, by using the same way in the proof of Theorem 3.2.4, that there exists a $0 < R \leq 1$ such that the operator T^* has at least one fixed point in $\mathcal{B}_0^R$. Since the fixed points of the related operator coincide the solutions of the problem (3.20) and the problem is equivalent to the problem (3.19), therefore the problem (3.19) has at least one solution $u \in \mathcal{B}^R$ with $u(0) = b,\ b \in \mathbb{C}$.

Moreover, if it is assumed that the condition $f(0,b) = b\Gamma(1+\alpha)$ and the inequality (3.18) for all $(z,\eta),(z,\nu) \in \mathbb{U} \times \mathbb{C}$ are satisfied in addition to (3i*) (the conditions (3ii*) and $\sup_{z\in\mathbb{U}} |f(z,b)| < \infty$), then, by following same method in the proof of Theorem 3.2.5, it can be shown that there exists unique fixed point $v \in \mathcal{B}_0$ (or $v \in \mathcal{A}_0$) of the operator T^*. Because of the reasons mentioned above, the problem (3.19) has a unique solution $u \in \mathcal{B}$ ($v \in \mathcal{A}$) with $u(0) = b,\ b \in \mathbb{C}$.

4. DIRICHLET BOUNDARY PROBLEMS INVOLVING NON-LOCAL ELLIPTIC OPERATOR

In this chapter, we consider the following Dirichlet boundary problem

$$\begin{cases} -\mathcal{L}_K u + g(x,u) = h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{4.1}$$

where $\mathcal{L}_K$ is the non-local integrodifferential operator given in Definition 2.5.1, u is an unknown function, $\Omega \subset \mathbb{R}^n$, $n > 2s$, $(s \in (0,1))$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $g(x,0) = 0$ for any $x \in \mathbb{R}^n \backslash \Omega$ and h is a generalized function with compactly supported on $\overline{\Omega}$.

It is remarkable that the Dirichlet boundary condition in the problem is not only on the boundary of Ω but also on whole $\mathbb{R}^n \backslash \Omega$. It stems from the definition of the non-local operator.

On the other hand, since the operator $\mathcal{L}_K$ is a general type of the fractional Laplacian operator $-(-\Delta)^s$, we also reach some existence and uniqueness results for the related boundary problem involving $-(-\Delta)^s$.

We preliminarily suppose that the following conditions are satisfied to investigate the existence of the generalized solution of the problem (4.1):

(4i) There exist $\mu > 0$ and the functions $a_i \in L_{p_i}(\Omega)$ with $a_i(x) \geq 0$ for $i = 0, 1$, such that the following inequality holds for a.e $x \in \Omega$ and for all $t \in \mathbb{R}$:

$$|g(x,t)| \leq a_1(x)\,|t|^{\mu} + a_0(x), \tag{4.2}$$

where the numbers p_i are defined in the sequel.

Now, the proof the existence of the solution of the problem (4.1) by using Theorem 2.6.11.

4.1 Existence of the Generalized Solution of (4.1)

We first assume that the conditions (4i) is satisfied throughout this chapter to establish some existence results related to the problem (4.1).

4.1.1 Formulation of the Problem (4.1)

Let $h \in (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ be a given function such that $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$. The solutions of the investigated problem are to be understood in the sense of following.

Definition 4.1.1. A function $u \in \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ is called the generalized solution of the problem (4.1), if it satisfies the following equality for all $v \in \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$:

$$\frac{1}{2}\int_{\mathbb{R}^{2n}}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy+\int_{\Omega}g(x,u)vdx=\int_{\Omega}hvdx. \quad (4.3)$$

We investigate the generalized solution of this integral equation according to the cases depending on the value of the parameter μ arising in (4i). Since, from the Lemma 2.5.4, there exists the embedding $\mathcal{X}_0 \hookrightarrow L^{\Omega}_{\mu+1}(\mathbb{R}^n)$ when $0 < \mu \leq \frac{n+2s}{n-2s}$, the cases $0 < \mu < \frac{n+2s}{n-2s}$, $\mu = \frac{n+2s}{n-2s}$, and $\mu > \frac{n+2s}{n-2s}$ are called subcritical, critical and supercritical cases, respectively. Thus, if the solution of the problem (4.1) exits, it will be in the space $\mathcal{X}_0$ in the subcritical and critical cases, whereas it will be in the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ in the supercritical case.

On the other hand, the embedding mentioned above holds for both the subcritical and critical cases, so we consider simultaneously the problem for this cases in the following.

4.1.2 Existence of the Generalized Solution of the Problem (4.1) in the Subcritical and Critical Cases

Here, it is supposed that there is a relation between the parameters μ and the numbers p_0, p_1 mentioned in (4i) as follows:

$$\text{(4ii)} \quad \begin{cases} p_1 := \frac{2n}{(\mu-1)(2s-n)+4s},\ p_0 := \frac{2n}{n+2s} & ,\ \mu \in (0,1) \\ p_1 := \infty,\ p_0 := \frac{\mu+1}{\mu} & ,\ \mu \in \left[1, \frac{n+2s}{n-2s}\right]. \end{cases}$$

Thus, it can be said that all integrals in the equation (4.3) make sense under the conditions (4i) with the relation (4ii) .

Before proceeding to prove the existence of the solution of the problem (4.1) in the subcritical and critical cases by using Theorem 2.6.11, we define the operator and function spaces given in this theorem by being dependent on the related problem as follows:

$$X :\equiv \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n), \qquad X^* :\equiv (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$$

and

$$f(u) := -\mathcal{L}_K u + g(x,u). \quad (4.4)$$

Theorem 4.1.2. Suppose that, additionally to the condition (4i) , the following conditions be satisfied:

(4iii) If $\mu \in \left(1, \frac{n+2s}{n-2s}\right]$ in (4.2), then there exist the number $c_1 > 0$, a function $a_2 \in L_1(\Omega)$ and $0 < \alpha < 2$ such that the following inequality holds for a.e $x \in \Omega$ and for all $t \in \mathbb{R}$:

$$tg(x,t) \geq -c_1 |t|^{\alpha} - a_2(x). \tag{4.5}$$

(4iv) If $\mu = 1$ in (4.2), then function the inequality in (4.5) is satisfied for $\alpha = 2$, $a_2 \in L_1(\Omega)$ and $c_1 > \frac{\lambda_1}{2}$, where λ_1 is the first eigenvalue of the operator $-\mathcal{L}_K$.

Then, the operator f defined in (4.4) is bounded, coercive and weakly compact on $\mathcal{X}_0$.

Proof. (a) It is at first shown that the operator f is coercive on $\mathcal{X}_0$. For this, let us consider the dual pair $\langle f(u), u\rangle$ given as follows:

$$\langle f(u), u\rangle = \frac{1}{2}\int_{\mathbb{R}^{2n}} (u(x) - u(y))^2 K(x-y)dxdy + \int_{\Omega} g(x,u)udx \tag{4.6}$$

Let $0 < \mu < 1$ in the inequality (4.2). By using this inequality in (4.6)

$$\begin{aligned}\langle f(u), u\rangle &\geq \frac{1}{2}\|u\|^2_{\mathcal{X}_0} - \int_{\Omega} |g(x,u)|\,|u|\,dx \\ &\geq \frac{1}{2}\|u\|^2_{\mathcal{X}_0} - \int_{\Omega} a_1(x)\,|u|^{\mu+1}\,dx - \int_{\Omega} a_0(x)\,|u|\,dx,\end{aligned}$$

is obtained. If, by considering the relation (4iii), we apply the Hölder inequality to the last two terms in right hand side of the above inequality, we have

$$\langle f(u), u\rangle \geq \frac{1}{2}\|u\|^2_{\mathcal{X}_0} - \|a_1\|_{L_{p_1}(\Omega)} \|u\|^{\mu+1}_{L_{\frac{2n}{n-2s}}(\Omega)} - \|a_0\|_{L_{\frac{2n}{n+2s}}(\Omega)} \|u\|_{L_{\frac{2n}{n-2s}}(\Omega)}.$$

By using the embedding $\mathcal{X}_0 \hookrightarrow L^{\Omega}_{\frac{2n}{n-2s}}(\mathbb{R}^n)$ in the last inequality, we get

$$\langle f(u), u\rangle \geq \frac{1}{2}\|u\|^2_{\mathcal{X}_0} - c_*^{\mu+1}\|a_1\|_{L_{p_1}(\Omega)} \|u\|^{\mu+1}_{\mathcal{X}_0} - c_*\|a_0\|_{L_{\frac{2n}{n+2s}}(\Omega)} \|u\|_{\mathcal{X}_0},$$

where the constant c_* come from the related embedding.

Applying $\epsilon-$Young inequality with the indices $2/(\mu+1)$ and $2/(1-\mu)$ to the second term in the right hand side of above inequality and by a little calculation we have

$$\langle f(u), u\rangle \geq (\frac{1}{2} - \epsilon)\|u\|^2_{\mathcal{X}_0} - c(\epsilon)b^{\frac{2}{1-\mu}} - c_*\|a_0\|_{L_{\frac{2n}{n+2s}}(\Omega)} \|u\|_{\mathcal{X}_0}, \tag{4.7}$$

where $\epsilon < 1/2$ is chosen and $b := c_*^{\mu+1} \|a_1\|_{L_{p_1}(\Omega)}$.

Now, let $\mu = 1$ in (4.2). Since the condition (4iv) is satisfied, the following chain of inequalities can be written

$$\begin{aligned}
\langle f(u), u\rangle &\geq \langle -\mathcal{L}_K u, u\rangle + \int_\Omega g(x,u)u dx \\
&\geq \frac{1}{2}\|u\|_{X_0}^2 - c_1 \int_\Omega |u|^2 dx - \int_\Omega a_0(x)dx \\
&\geq \frac{1}{2}\|u\|_{X_0}^2 - c_1 \|u\|_{L_2^\Omega(\mathbb{R}^n)}^2 - \|a_0\|_{L_2(\Omega)}
\end{aligned}$$

Since λ is the first eigenvalue of the operator $-\mathcal{L}_K$, one can write for all $u \in X_0$,

$$\|u\|_{X_0}^2 \geq \lambda_1 \|u\|_{L_2^\Omega(\mathbb{R}^n)}^2.$$

From here and from the last inequality above, it can easily seen that

$$\langle f(u), u\rangle \geq \left(\frac{1}{2} - \frac{c_1}{\lambda_1}\right)\|u\|_{X_0}^2 - \|a_0\|_{L_1(\Omega)}. \tag{4.8}$$

Finally, let $\mu \in \left(1, \frac{n+2s}{n-2s}\right]$. From the condition (4iii), it is clear that

$$\begin{aligned}
\langle f(u), u\rangle &\geq \frac{1}{2}\|u\|_{X_0}^2 + \int_\Omega g(x,u)u dx \\
&\geq \frac{1}{2}\|u\|_{X_0}^2 - c_1 \int_\Omega |u|^\alpha dx - \int_\Omega a_2(x)dx \\
&\geq \frac{1}{2}\|u\|_{X_0}^2 - c_1 \|u\|_{L_\alpha(\Omega)}^\alpha - \int_\Omega a_2(x)dx \\
&\geq \frac{1}{2}\|u\|_{X_0}^2 - c_1 c_2^\alpha \|u\|_{X_0}^\alpha - \|a_2\|_{L_1(\Omega)},
\end{aligned}$$

where c_2 comes from the embedding $X_0 \hookrightarrow L_\alpha^\Omega(\mathbb{R}^n)$.

By applying $\epsilon-$Young inequality with the indices $2/\alpha$ and $2/(2-\alpha)$ to the second term in the last inequality in the above, the following inequality can be obtained

$$\langle f(u), u\rangle \geq (\frac{1}{2} - \epsilon)\|u\|_{X_0}^2 - c(\epsilon)(c_1 c_2^\alpha)^{2/(2-\alpha)} - \|a_2\|_{L_1(\Omega)} \tag{4.9}$$

where $\epsilon < \frac{1}{2}$ is chosen.

Consequently, from the inequalities (4.7), (4.8) and (4.9) one can easily check that

$$\frac{\langle f(u), u\rangle}{\|u\|_{\mathcal{X}_0}} \nearrow \infty \text{ when } \|u\|_{\mathcal{X}_0} \nearrow \infty.$$

Hence, this implies the coerciveness of the operator f on $\mathcal{X}_0$.

(b) Now, let us show that the operator f is bounded from the space $\mathcal{X}_0$ to its dual space $(\mathcal{X}_0)^*$. From the equality (4.3)

$$|\langle f(u), v\rangle| \leq \frac{1}{2}\int_Q |u(x)-u(y)|\,|v(x)-v(y)|\,|K(x-y)|\,dxdy + \int_\Omega |g(x,u)|\,dx.$$

can be easily obtained. By taking consideration the condition (4i) in the above inequality it follows that

$$\begin{aligned} |\langle f(u), v\rangle| \leq{} & \frac{1}{2}\int_Q |u(x)-u(y)|\,|v(x)-v(y)|\,|K(x-y)|\,dxdy \\ & + \int_\Omega a_1(x)\,|u|^\mu\,|v|\,dx + \int_\Omega a_0(x)\,|v|\,dx. \end{aligned} \tag{4.10}$$

If Cauchy-Schwarz inequality is applied to first term in the right side of inequality (4.10), and if, by considering the relation (4ii), Hölder inequality with the indices p_1, p_2, p_3 and p_0, p_0^* (p_0^* is the conjugate of p_0) as following is applied to other terms, respectively,

$$\begin{cases} p_2 := \frac{2n}{\mu(n-2s)},\ p_3 = \frac{2n}{n-2s},\ p_0^* := \frac{2n}{n-2s} & ,\ \mu \in (0,1) \\ p_2 := \frac{\mu+1}{\mu},\ p_3 = \mu+1,\ p_0^* := \mu+1 & ,\ \mu = [1, \frac{n+2s}{n-2s}] \end{cases}$$

then one can write the following inequality

$$|\langle f(u), v\rangle| \leq \frac{1}{2}\|u\|_{\mathcal{X}_0}\|v\|_{\mathcal{X}_0} + \|a_1\|_{L_{p_1}(\Omega)}\|u\|^\mu_{L_{\mu p_2}(\Omega)}\|v\|_{L_{p_3}(\Omega)} + \|a_0\|_{L_{p_0}(\Omega)}\|v\|_{L_{p_0^*}(\Omega)}.$$

Using the embedding $\mathcal{X}_0 \hookrightarrow L^\Omega_{\mu p_2}(\mathbb{R}^n)$ in the last inequality and by a little calculation,

$$\frac{|\langle f(u), v\rangle|}{\|v\|_{\mathcal{X}_0}} \leq \frac{1}{2}\|u\|_{\mathcal{X}_0} + c_*^{\mu+1}\|a_1\|_{L_{p_1}(\Omega)}\|u\|^\mu_{\mathcal{X}_0} + c_*\|a_0\|_{L_{p_0}(\Omega)}\|u\|_{\mathcal{X}_0},$$

can be obtained, where c_* comes from this embedding. This inequality holds for all $u, v \in \mathcal{X}_0$, so one can write

$$\|f(u)\|_{(\mathcal{X}_0)^*} \leq \hat{c}(\|u\|_{\mathcal{X}_0}),$$

where the function $\hat{c} : R_0^+ \to R_0^+$ is defined as follows:

$$\hat{c}(t) = \frac{t}{2} + c_*^{\mu+1}\|a_1\|_{L_{p_1}(\Omega)}|t|^\mu + c_*\|a_0\|_{L_{p_0}(\Omega)}|t|.$$

Since $\hat{c}$ is a continuous and non-decreasing function, the inequality

$$\|f(u)\|_{(\mathcal{X}_0)^*} \leq \hat{c}(r),$$

holds for all $u \in B(0;r) := \left\{u \in \mathcal{X}_0 : \|u\|_{\mathcal{X}_0} \leq r\right\}$. This shows the boundedness of f from $\mathcal{X}_0$ to $(\mathcal{X}_0)^*$.

(c) Finally, let us prove that $f : \mathcal{X}_0 \longrightarrow (\mathcal{X}_0)^*$ is a weakly compact operator. For this, let $\{u_m\}_{m=1}^{\infty} \subset \mathcal{X}_0$ and $u_0 \in \mathcal{X}_0$ such that

$$u_m \underset{\mathcal{X}_0}{\rightharpoonup} u_0$$

and let us show that there is a subsequence $\{u_{m_k}\}_{k=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ such that

$$-\mathcal{L}_K u_{m_k} + g(x, u_{m_k}) \underset{(\mathcal{X}_0)^*}{\rightharpoonup} -\mathcal{L}_K u_0 + g(x, u_0).$$

$\mathcal{L}_K$ is a weakly compact operator, since $\mathcal{L}_K$ is a linear bounded operator and $\mathcal{X}_0$ is a reflexive space. Hence, it remains to prove the weakly compactness of g. Before proceeding, it should be noted the following embeddings holds:

$$\mathcal{X}_0 \hookrightarrow L_p^{\Omega}(\mathbb{R}^n), \quad L_q^{\Omega}(\mathbb{R}^n) \hookrightarrow (\mathcal{X}_0)^*,$$

where the numbers p and q are defined as follows:

$$\begin{cases} p := \frac{2n}{n-2s}, \ q := \frac{2n}{n+2s}, & \mu \in (0,1) \\ p := \mu + 1, \ q := \frac{\mu+1}{\mu}, & \mu \in \left[1, \frac{n+2s}{n-2s}\right]. \end{cases}$$

At first, let us show that $g(x, .)$ is a bounded operator from $L_p^{\Omega}(\mathbb{R}^n)$ to $L_q^{\Omega}(\mathbb{R}^n)$. For this, let $u \in \mathcal{X}_0$. By using the condition (4i) and $g(x, 0) = 0$ for all $x \in \mathbb{R}^n \backslash \Omega$, it is easily seen that

$$\int_{\mathbb{R}^n} |g(x, u(x))|^q \, dx \leq \int_{\Omega} (a_1(x) \, |u|^{\mu} + a_0(x))^q \, dx.$$

Applying the inequality in Lemma 2.7.3 to integrand in the right hand side of above inequality,

$$\int_{\mathbb{R}^n} |g(x, u)|^q \, dx \leq 2^{q-1} \left(\int_{\Omega} (a_1(x))^q \, |u|^{q\mu} \, dx + \int_{\Omega} (a_0(x))^q \, dx \right)$$

is obtained.

If, keeping the relation (4ii) in mind, Hölder inequality with the indices k_0, k_1 as following is applied to the first integral of the right side of the last inequality

$$\begin{cases} k_0 := \frac{n+2s}{(\mu-1)(2s-n)+4s}, \ k_1 = \frac{n+2s}{\mu(n-2s)} & , \ \mu \in (0,1) \\ k_0 := \infty, \ k_1 = 1 & , \ \mu \in \left[1, \frac{n+2s}{n-2s}\right], \end{cases}$$

then the following inequality is obtained:

$$\int_{\mathbb{R}^n} |g(x, u(x))|^q \, dx \leq 2^{q-1} \left(\|a_1\|_{L_{p_1}(\Omega)}^q \, \|u\|_{L_p^{\Omega}(\mathbb{R}^n)}^{q\mu} + \|a_0\|_{L_q(\Omega)}^q \right),$$

where p_1 in (4ii). Therefore, this inequality yields the boundedness of g from $L_p^{\Omega}(\mathbb{R}^n)$ to $L_q^{\Omega}(\mathbb{R}^n)$.

On the other hand, considering the embeddings $\mathcal{X}_0 \hookrightarrow L_p^{\Omega}(\mathbb{R}^n)$ and $(L_p^{\Omega}(\mathbb{R}^n))^* \hookrightarrow (\mathcal{X}_0)^*$, the fact $u_m \underset{L_p^{\Omega}(\mathbb{R}^n)}{\rightharpoonup} u_0$ follows from the assumption given above, i.e., weak convergence of $\{u_m\}_{m=1}^{\infty}$ to u_0 in $\mathcal{X}_0$. So, $\{u_m\}_{m=1}^{\infty}$ is bounded in $L_p^{\Omega}(\mathbb{R}^n)$. Since $g(x,.)$ is bounded from $L_p^{\Omega}(\mathbb{R}^n)$ to $L_q^{\Omega}(\mathbb{R}^n)$, $\{g(x,u_m)\}_{m=1}^{\infty}$ is bounded sequence in $L_q^{\Omega}(\mathbb{R}^n)$. Furthermore, $L_q^{\Omega}(\mathbb{R}^n)$ is a reflexive space, so one can extract a subsequence $\{g(x,u_{m_k})\}_{m=1}^{\infty}$ such that

$$g(x,u_{m_k}) \rightharpoonup w \ \text{ in } \ L_q^{\Omega}(\mathbb{R}^n). \tag{4.11}$$

Moreover, the weak convergence of $\{u_m\}_{m=1}^{\infty}$ in the reflexive space $\mathcal{X}_0$ yield its boundedness. From here and from the continuous embedding in Lemma 2.5.4, there exists a subsequence, still denoted by $\{u_m\}_{m=1}^{\infty}$ such that $u_m \overset{L_\gamma^{\Omega}(\mathbb{R}^n)}{\to} u_0$ for $\gamma \in [1, \frac{2n}{n-2s})$. From here, by using Theorems 2.6.5 and 2.6.6, it can be obtained that $u_m \to u_0$ in measure, and after that it can be revealed that there exists a subsequence $\{u_{m_k}\}_{k=1}^{\infty}$ such that $u_{m_k} \overset{a.e}{\to} u_0$ in $\mathbb{R}^n$, since Ω is a set of finite measure and for all $m \in \mathbb{N}_0$, $u_m \equiv 0$ on $\mathbb{R}^n \backslash \Omega$. From here and from the fact that g is continuous in second variable, it follows, by Theorem 2.6.8, that

$$g(x,u_{m_k}) \to g(x,u_0) \ \text{ a.e on } \ \mathbb{R}^n. \tag{4.12}$$

If one considers (4.11) and (4.12), then it is sufficient to show the equality

$$w = g(x,u_0)$$

to be

$$g(x,u_{m_k}) \rightharpoonup g(x,u_0) \ \text{ in } \ L_q^{\Omega}(\mathbb{R}^n).$$

In fact, this equality holds as a consequence of Lemma 2.6.9, because one can easily check above that the hypothesis of this Lemma are fulfilled.

Consequently, one can say that there exists a subsequence, still denoted by $\{u_{m_k}\}_{k=1}^{\infty}$ such that

$$g(x,u_{m_k}) \underset{L_q(\mathbb{R}^n)}{\rightharpoonup} g(x,u_0).$$

Moreover, since the embedding $L_q(\mathbb{R}^n) \hookrightarrow (\mathcal{X}_0)^*$ holds, it follows that

$$g(x,u_{m_k}) \underset{(\mathcal{X}_0)^*}{\rightharpoonup} g(x,u_0).$$

Therefore, the weak compactness of $f(u) = -\mathcal{L}_K u + g(x,u)$ from $\mathcal{X}_0$ to $(\mathcal{X}_0)^*$ is obtained.

Theorem 4.1.3. Let the hypothesis of the Theorem 4.1.2 are fulfilled. Then, for any $h \in (\mathcal{X}_0)^*$ the problem (4.1) admits at least one solution in the space $\mathcal{X}_0$.

Proof From Theorem 4.1.2, it is understood that the operator f is coercive and weak compact from $\mathcal{X}_0$ to $(\mathcal{X}_0)^*$. Hence, as a consequence of Theorem 2.6.11, for all $h \in (\mathcal{X}_0)^*$ with $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$, there exists at least one solution in $\mathcal{X}_0$ of the equation

$$f(u) = h,$$

where f given in (4.1). This implies that the equation

$$\langle -\mathcal{L}_K u + g(x,u), v\rangle = \langle h, v\rangle \quad (\forall v \in \mathcal{X}_0)$$

is solvable for all $h \in (\mathcal{X}_0)^*$ with $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$, i.e. the problem (4.1) admits at least one solution in the space $\mathcal{X}_0$ for all $h \in (\mathcal{X}_0)^*$ with $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$.

As a special case, if $-\mathcal{L}_K = (-\Delta)^s$ is taken in Theorem 4.1.3, then one can give the following corollary and it can be proved by using the similar way in Theorems 4.1.2-4.1.3.

Corollary 4.1.4. Let the conditions (4i)-(4iv) (by taking the first eigenvalue λ_1 of $(-\Delta)^s$ instead of $-\mathcal{L}_K$ in (4iv)) of the Theorem 4.1.2 are fulfilled. Then, under these conditions, for any $h \in H^{-s}(\mathbb{R})$ with $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$, the problem

$$\begin{cases} (-\Delta)^s u + g(x,u) = h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

has at least one solution $u \in H^s_\Omega(\mathbb{R})$.

Remark 4.1.5. If one considers the problem (1.17) with $h(t) := t\,|t|^\mu - |t|^\alpha \ln(1+|t|)$, then the Lipschitz condition imposed by Raghavendra and Rasmita [28] (see page 12) can't be satisfied. However, as a consequences of Theorem 4.1.3, one can say that the problem in the following examples, which involves a more general form of the function h has at least one solution in $\mathcal{X}_0$.

Example 4.1.6. Let $f \in (\mathcal{X}_0)^*$. As a consequences of Theorem 4.1.3, there exists a solution in $\mathcal{X}_0$ of the following problem

$$\begin{cases} -\mathcal{L}_K u + d_1(x)u\,|u|^{\beta} - d_0(x)\,|u|^{\alpha}\ln(1+|u|) = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\backslash\Omega, \end{cases}$$

where β, α are the real numbers with $0 < \alpha \leq \beta$, $d_1 \in L_{p_1}(\Omega)$ and $d_0 \in L_{p_0}(\Omega)$, provided that the following conditions are fulfilled:

(a) If $-1 < \alpha, \beta < 0$, then $p_1 = p_0 = \frac{2n}{\beta(2s-n)+4s}$,

(b) If $-1 < \alpha \leq \beta = 0$, then $p_1 = p_0 = \infty$ and $d_1(x) - |d_0(x)| \geq -k$, with $k > \frac{\lambda_1}{2}$, where λ is the first eigenvalue of the operator $-\mathcal{L}_K$.

(c) If $0 < \alpha \leq \beta \leq \frac{n+2s}{n-2s}$ with $\beta > 1$, then $d_1(x) - |d_0(x)| \geq 0$ for all $x \in \Omega$ and $p_1 = p_0 = \infty$.

Indeed, if we get $g(x,t) := d_1(x)t\,|t|^{\beta} - d_0(x)\,|t|^{\alpha}\,ln(1+|t|)$ in the problem (4.1), then we can show that the hypotheses of Theorem 4.1.3 are satisfied. It is clear that g is Carathéodory function and $g(x,0) = 0$ for any $x \in \mathbb{R}^n\backslash\Omega$. Moreover, if we set

$$m(\tau) = \tau - ln(1+\tau),$$

then the minimum value of the function m is at $\tau = 0$. This at first indicates that $\tau \geq ln(1+\tau)$ for all $\tau \geq 0$, and then that $|t| \geq ln(1+|t|)$ for all $t \in \mathbb{R}^n$. In addition to this, for any a, b with $a \geq b \geq 0$ and for all $t \in \mathbb{R}^n$ the inequality $|t|^{a} + 1 \geq |t|^{b}$ is satisfied. From these two facts one can write the following chain of inequalities for the function g :

$$\begin{aligned} |g(x,t)| &\leq |d_1(x)|\,|t|^{\beta+1} + |d_0(x)|\,|t|^{\alpha}\,ln(1+|t|) \\ &\leq |d_1(x)|\,|t|^{\beta+1} + |d_0(x)|\,|t|^{\alpha+1} \\ &\leq (|d_1(x)| + |d_0(x)|)\,|t|^{\beta+1} + |d_0(x)| \end{aligned}$$

If one pays attention to last inequality and the conditions (a), (b) and (c), then it can be seen that the hypothesis (4i) with the relation (4ii) is fulfilled. Finally, by considering the conditions (b) and (c), one can easily show that the hypotheses (4iii) and (4iv) are justified.

4.1.3 Existence of Solution of the Problem (4.1) in the Supercritical Case

Here, we investigate the existence of the solutions of the problem (4.1) in the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$, when $\mu > \frac{n+2s}{n-2s}$. We suppose that the numbers p_0, p_1 given in (4ii) are as follows:

$$\text{(4ii*)} \left\{ p_1 := \infty, \ p_0 := \tfrac{\mu+1}{\mu} \right.$$

Theorem 4.1.7. Let, in addition to the condition (4i) with $\mu > \frac{n+2s}{n-2s}$ and the relation (4ii*), the following condition is fulfilled:

(4iii*) There exist the numbers $k_0 > 0$, a function k_1 in the space $L_1(\Omega)$ such that the following inequality holds when $\mu > \frac{n+2s}{n-2s}$:

$$tg(x,t) \geq k_0 \left|t\right|^{\mu+1} - k_1(x) \tag{4.13}$$

for almost $x \in \Omega$ and for all $t \in \mathbb{R}^n$.

Then, the operator f defined in (4.4) is bounded, coercive and weakly compact on $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$.

Proof. Let us again take into account the equality in (4.6) to show that the operator f is coercive on $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$. Considering (4.13) in (4.6), at first the inequality

$$\langle f(u), u \rangle \geq \frac{\left\|u\right\|_{X_0}^2}{2} + k_0 \left\|u\right\|_{L^{\Omega}_{\mu+1}(\mathbb{R}^n)}^{\mu+1} - k_1(x)$$

is obtained. From here, since $\mu + 1 > 2$, the following inequality can be derived:

$$\langle f(u), u \rangle \geq \frac{\left\|u\right\|_{X_0}^2}{2} + k_0 \left\|u\right\|_{L^{\Omega}_{\mu+1}(\mathbb{R}^n)}^{2} - \left\|k_1\right\|_{L_1(\Omega)} - k_0 \tag{4.14}$$

where we used the inequality:

$$\left\|u\right\|_{L^{\Omega}_{\mu+1}(\mathbb{R}^n)}^{2} \leq \left\|u\right\|_{L^{\Omega}_{\mu+1}(\mathbb{R}^n)}^{\mu+1} - 1.$$

By a little calculation, the following inequality follows from (4.14):

$$\langle f(u), u \rangle \geq \beta \left(\left\|u\right\|_{X_0}^2 + \left\|u\right\|_{L^{\Omega}_{\mu+1}(\mathbb{R}^n)}^{2} \right) - \left\|k_1\right\|_{L_1(\Omega)} - k_0.$$

where β is denoted by $\beta := \min\left\{\frac{1}{2}, k_0\right\}$
If one takes into account the norm on the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$, then from just above and from the inequality for $p = 2$ in Lemma 2.7.3 one can write

$$\langle f(u), u \rangle \geq \frac{\beta}{2} \left\|u\right\|_{X_0 \cap L_{\mu+1}(\mathbb{R}^n)}^2 - \left\|k_1\right\|_{L_1(\Omega)} - k_0.$$

Consequently, the last inequality implies that

$$\frac{\langle f(u), u \rangle}{\left\|u\right\|_{\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)}} \nearrow \infty, \quad \text{when} \quad \left\|u\right\|_{\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)} \nearrow \infty.$$

Therefore, this indicates that the operator f is coercive on $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$.

Now, let us show that $f : \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n) \longrightarrow (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ is a bounded operator. If, by considering the condition (4i), Cauchy-Schwarz inequality is applied to first term in the right side of (4.10), then the following inequality is obtained:

$$|\langle f(u), v\rangle| \leq \frac{1}{2}\left\|u\right\|_{\mathcal{X}_0}\left\|v\right\|_{\mathcal{X}_0} + \int_\Omega a_1(x)\left|u\right|^\mu\left|v\right| dx + \int_\Omega a_0(x)\left|v\right| dx.$$

By applying Hölder inequality with the indices $r_1, \frac{\mu+1}{\mu}, \mu+1$ and $r_0, \mu+1$ to the integrals of the right side of the last inequality, respectively, then we get

$$\begin{aligned}|\langle f(u), v\rangle| &\leq \frac{1}{2}\left\|u\right\|_{\mathcal{X}_0}\left\|v\right\|_{\mathcal{X}_0} + \left\|a_1\right\|_{L_{r_1}(\Omega)}\left\|u\right\|^\mu_{L_{\mu+1}(\Omega)}\left\|v\right\|_{L_{\mu+1}(\Omega)} + \left\|a_0\right\|_{L_{r_0}(\Omega)}\left\|v\right\|_{L_{\mu+1}(\Omega)}\\ &\leq \frac{1}{2}\left\|u\right\|_{\mathcal{X}_0}\left\|v\right\|_{\mathcal{X}_0} + \left\|a_1\right\|_{L_\infty(\mathbb{R}^n)}\left\|u\right\|^\mu_{L^\Omega_{\mu+1}(\mathbb{R}^n)}\left\|v\right\|_{L^\Omega_{\mu+1}(\mathbb{R}^n)}\\ &+ \left\|a_0\right\|_{L^\Omega_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)}\left\|v\right\|_{L^\Omega_{\mu+1}(\mathbb{R}^n)}. \end{aligned} \tag{4.15}$$

From the above inequality, it is obtained that

$$\frac{|\langle f(u), v\rangle|}{\left\|v\right\|_{\mathcal{X}_0\cap L_{\mu+1}(\mathbb{R}^n)}} \leq \frac{1}{2}\left\|u\right\|_{\mathcal{X}_0\cap L_{\mu+1}(\mathbb{R}^n)} + \left\|a_1\right\|_{L_\infty(\mathbb{R}^n)}\left\|u\right\|^\mu_{\mathcal{X}_0\cap L_{\mu+1}(\mathbb{R}^n)} + \left\|a_0\right\|_{L^\Omega_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)}, \tag{4.16}$$

where the continuous embeddings $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n) \hookrightarrow \mathcal{X}_0$ and $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n) \hookrightarrow L^\Omega_{\mu+1}(\mathbb{R}^n)$ are used.

The inequality above is satisfied for any $u, v \in \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$, so if the function $\hat{c} : R^+_0 \to R^+_0$ is defined as follows:

$$\hat{c}(t) = \frac{t}{2} + \left\|a_1\right\|_{L_\infty(\mathbb{R}^n)}\left|t\right|^\mu + \left\|a_0\right\|_{L^\Omega_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)}\left|t\right|,$$

then the inequality (4.16) can be written as:

$$\left\|f(u)\right\|_{(\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)} \leq \hat{c}(\left\|u\right\|_{\mathcal{X}_0\cap L_{\mu+1}(\mathbb{R}^n)}).$$

The function $\hat{c}$ is a non-decreasing continuous function, thus it follows that

$$\left\|f(u)\right\|_{(\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)} \leq \hat{c}(r),$$

for all $u \in B(0; r) := \left\{u \in \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n) : \left\|u\right\|_{\mathcal{X}_0\cap L_{\mu+1}(\mathbb{R}^n)} \leq r\right\}$.

Hence, this inequality implies the boundedness of f from $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ to $(\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$.

Finally, it remains to show that the operator f defined in (4.4) is weakly compact from $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ to $(\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$. This is shown by proving separately the weak compactness of the operator $-L_\mathcal{K}$ and the nonlinear function $g(x, t)$.

The weak compactness of the linear operator $-L_{\mathcal{K}}$ follows from the boundedness of it from $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ to $(\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu+1}}(\mathbb{R}^n)$ (as one can see from the inequalities (4.15) and (4.16)), and from the fact that the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ is a reflexive space.

For the weak compactness of the function g, it needs to show that if $u_m \underset{\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)}{\rightharpoonup} u_0$ for any $\{u_m\}_{m=1}^{\infty} \subset \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$, there is a subsequence $\{u_{m_k}\}_{k=1}^{\infty} \subset \{u_m\}_{m=1}^{\infty}$ such that

$$g(x, u_{m_k}) \underset{(\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)}{\rightharpoonup} g(x, u_0).$$

Before proceeding, it is appropriate to emphasize here that the following embeddings are used in the sequel:

$$\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n) \hookrightarrow L^{\Omega}_{\mu+1}(\mathbb{R}^n), \quad L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n) \hookrightarrow (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n) \tag{4.17}$$

The weak compactness of the operator g is proved by the help of Lemma 2.6.9. For this reason, it is showed that the conditions of this Lemma are satisfied. One of this conditions is related to the boundedness of the function g from $L^{\Omega}_{\mu+1}(\mathbb{R}^n)$ to $L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$. For this, let $u \in L^{\Omega}_{\mu+1}(\mathbb{R}^n)$ and let us consider the following inequality:

$$\int_{\mathbb{R}^n} |g(x,u)|^{\frac{\mu+1}{\mu}}\, dx \leq \int_{\Omega} \left(a_1(x)\, |u|^{\mu} + a_0(x)\right)^{\frac{\mu+1}{\mu}} dx,$$

where the condition (4i) with the relation (4ii*) was used. From the above inequality, at first, the inequality

$$\int_{\mathbb{R}^n} |g(x,u)|^{\frac{\mu+1}{\mu}}\, dx \leq 2^{1/\mu} \left(\int_{\Omega} (a_1(x))^{\frac{\mu+1}{\mu}}\, |u|^{\mu+1}\, dx + \int_{\Omega} (a_0(x))^{\frac{\mu+1}{\mu}}\, dx \right)$$

is obtained, after that, if one applies Hölder inequality with the indices $k_0 := \infty, k_1 := 1$ to the first integral of the right side of the above inequality, then one has the inequality:

$$\int_{\mathbb{R}^n} |g(x,u)|^{q_1}\, dx \leq 2^{1/\mu} \left(\|a_1\|^{\frac{\mu+1}{\mu}}_{L_{\infty}(\Omega)}\, \|u\|^{\mu+1}_{L_{\frac{\mu+1}{\mu}}(\Omega)} + \|a_0\|^{\frac{\mu+1}{\mu}}_{L_{\frac{\mu+1}{\mu}}(\Omega)} \right),$$

which states that the the function g is bounded from $L^{\Omega}_{\mu+1}(\mathbb{R}^n)$ to $L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$.

Now, assume that $u_m \underset{\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)}{\rightharpoonup} u_0$. From here and the the embedding (4.13), it follows that

$$u_m \underset{L^{\Omega}_{\mu+1}(\mathbb{R}^n)}{\rightharpoonup} u_0 \tag{4.18}$$

and

$$u_m \underset{\mathcal{X}_0}{\rightharpoonup} u_0. \tag{4.19}$$

From (4.18), it is obtained that $\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in $L^{\Omega}_{\mu+1}(\mathbb{R}^n)$. Then, $\{g(x,u_m)\}_{m=1}^{\infty}$ is bounded sequence in $L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$, since g is bounded from $L^{\Omega}_{\mu+1}(\mathbb{R}^n)$ to $L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$. From the reflexivity of the space $L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ it follows that there exist a subsequence $\{g(x,u_{m_k})\}_{k=1}^{\infty}$ of $\{g(x,u_m)\}_{m=1}^{\infty}$ and $g^* \in L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ such that

$$g(x,u_{m_k}) \rightharpoonup g^* \;\text{ in }\; L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n) \tag{4.20}$$

On the other hand, from (4.19) and from the fact that $\mathcal{X}_0$ is a reflexive space, it yields that $\{u_m\}_{m=1}^{\infty}$ is bounded in $\mathcal{X}_0$. By using this with the compact embedding in Lemma 2.5.4, one can reveal that there exists a subsequence, still denoted by $\{u_m\}_{m=1}^{\infty}$ such that $u_m \overset{L^{\Omega}_{\gamma}(\mathbb{R}^n)}{\rightarrow} u_0$ for $\gamma \in [1, \frac{2n}{n-2s})$. From here, by using Theorems 2.6.5 and 2.6.6, respectively, it can first obtained that $u_m \rightarrow u_0$ in measure, and then that there exists a subsequence $\{u_{m_k}\}_{k=1}^{\infty}$ such that

$$u_{m_k} \rightarrow u_0 \;\text{ a.e. on }\; \mathbb{R}^n,$$

since Ω is a set of finite measure and for all $m \in \mathbb{N}_0$, $u_m \equiv 0$ on $\mathbb{R}^n \backslash \Omega$.
Moreover, from here and the continuity of the operator $g(x,.) : \mathbb{R} \to \mathbb{R}$ it reveals that

$$g(x,u_{m_k}) \rightarrow g(x,u_0) \;\text{ a.e. on }\; \mathbb{R}^n. \tag{4.21}$$

Thus, as one can see from (4.20) and (4.21), if one can show the equality:

$$g^* = g(x,u_0),$$

then it is obtained that

$$g(x,u_{m_k}) \rightharpoonup g(x,u_0) \;\text{ in }\; L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n) \tag{4.22}$$

Indeed, this equality is satisfied as a consequences of Lemma 2.6.9, since it was showed above that the conditions of this lemma are fulfilled. Hence, from (4.22) and from the embedding in (4.17), the desired result

$$g(x,u_{m_k}) \rightharpoonup g(x,u_0) \;\text{ in }\; (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$$

is obtained, i.e g is a weak compact operator.

Therefore, the weak compactness of f follows from the weak compactness of the operators $-L_{\mathcal{K}}$ and g.

Theorem 4.1.8. Let the hypothesis of the Theorem 4.1.7 are fulfilled. Then, for any $h \in (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ the problem (4.1) admits at least one solution in the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$.

Proof From the Theorem 4.1.7 it can be seen that the corresponding operator f to the problem (4.1) satisfy the all conditions of the Theorem 2.6.11, i.e f is coercive and weak compact on the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$. Hence, as a consequence of this theorem, one can say that for any $h \in (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ with $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$, there exists a solution $u \in \mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ for the equation

$$f(u) = h.$$

Therefore, the problem (4.1) has at least one solution in the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ for any $h \in (\mathcal{X}_0)^* + L_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ with $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$.

Since $(-\Delta)^s$ is equal to the operator $-L_{\mathcal{K}}$ when $K(x) = |x|^{-(n+2s)}$, the existence result mentioned above can be given for the problem (3.1) involving fractional Laplacian $(-\Delta)^s$ as follows:

Corollary 4.1.9. Let all hypotheses of the Theorem 4.1.7 are fulfilled. Then, under these conditions for any $h \in H^{-s}(\mathbb{R}) + L^{\Omega}_{\frac{\mu+1}{\mu}}(\mathbb{R}^n)$ with $h \equiv 0$ on $\mathbb{R}^n \backslash \Omega$, the problem

$$\begin{cases} (-\Delta)^s u + g(x,u) = h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

has at least one solution $u \in H^s_{\Omega}(\mathbb{R}) \cap L_{\mu+1}(\mathbb{R}^n)$.

Proof. One can prove this corollary by following the same way used in Theorems 4.1.4 and 4.1.5.

4.2. Uniqueness of the Generalized Solution of the Problem (4.1)

In this subsection, we show in the following that if the problem has a solution in the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n), \mu > 0$, then it must be unique.

Theorem 4.2.1. Let c be defined as in the condition (4iii). Suppose that there exists a real number k with $0 < k < \frac{1}{2c}$ for which the following inequality holds for all $x \in \Omega$ and for all $t_1, t_2 \in \mathbb{R}$:

$$\left(g(x,t_1) - g(x,t_2)\right)(t_1 - t_2) > -k\left|t_1 - t_2\right|^2,$$

where the function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is given in the problem (4.1).

In this case, if the generalized solution of the problem (4.1) exits, then it is unique.

Proof. For the proof, it is shown that there is a contradiction and this is in the following way: It is first supposed that the problem (4.1) has two different generalized solutions u and v in the space $\mathcal{X}_0 \cap L_{\mu+1}(\mathbb{R}^n)$ and after that it is showed that $u = v$. Now, if one considers that u and u satisfy the problem (4.1) and defines w as $w = u-v$, then the problem

$$\begin{cases} -\mathcal{L}_K w + g(x,u) - g(x,v) = 0 & \text{in } \Omega \\ w = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

can be obtained.

Hence, if it is shown that there isn't any nonzero generalized solution w of the problem (4.23), then it means that $u = v$ and thus the desired result is obtained.

Now, it is supposed that there exists a solution $w \neq 0$ of the problem (4.23). If the equation in (4.23) is first multiplied with w and then is integrated over $\mathbb{R}^n$, then one can obtain the following equality:

$$\frac{1}{2}\int_{\mathbb{R}^{2n}} (w(x) - w(y))^2 K(x-y)dxdy + \int_{\Omega} (g(x,u) - g(x,v))(u-v)dx = 0.$$

From here and from the hypothesis of this theorem, one can write

$$0 > \frac{1}{2}\|w\|^2_{\mathcal{X}_0} - c\|u-v\|^2_{L_2(\Omega)} \geq \left(\frac{1}{2} - cc*\right)\|w\|^2_{\mathcal{X}_0} \geq 0,$$

where $c*$ comes from the embedding $X_0 \hookrightarrow L_2^{\Omega}(\mathbb{R}^n)$. It is easily seen that this is contradiction. Therefore, the problem (4.23) can not have any nonzero generalized solution w, namely $w = 0$ is a unique generalized solution of this problem. This gives us the desired contradiction.

5. CONCLUSIONS

In this thesis, we showed the some mistakes made in the proof of existence and uniqueness theorems for the following initial value problem, which was given in [15]:

$$\begin{array}{rcl} D^{\alpha}u(z) & = & f\left(z, u(z)\right) \\ u(0) & = & 0. \end{array}$$

Later we proved that the above problem with the non-homogenous initial data has at least one solution u in the space $\mathcal{B}_R$ wirh $0 < R < 1$ provided that the nonlinear function f satisfies the following conditions:

(i) The function $f(z,t)$ is analytic on $\mathbb{D} \times \mathbb{C}$ and continuous on $\mathbb{D}^* \times \mathbb{C}$, and $z^{\alpha} f(z,t)$ is analytic on $\mathbb{U} \times \mathbb{C}$ and continuous on $\overline{\mathbb{U}} \times \mathbb{C}$,

(ii) There exist a natural number $n_0 \geq 1$, a non-negative real number c and a function g which is analytic on $\mathbb{U}$ and continuous on $\overline{\mathbb{U}}$ such that the following inequality holds for all $(z,t) \in \overline{\mathbb{U}} \times \mathbb{C}$:

$$\left| z^{\alpha} f\left(z,t\right) - \frac{b}{\Gamma(1-\alpha)} \right| \leq c \left| t - b \right|^{n_0} + \left| g(z) \right| .$$

Moreover, we established not only existence but also uniqueness of the problem mentioned above in the space $\mathcal{B}_R$ if the function f fulfills the following conditions in addition to condition (i):

(iii) $z^{\alpha} f(z,b)|_{z=0} = \frac{b}{\Gamma(1-\alpha)}$.

(iv) There exists a constant $\kappa < 1/\Gamma\left(2-\alpha\right)$ such that

$$\left| f\left(z,\eta\right) - f\left(z,\nu\right) \right| < \frac{\kappa}{|z|^{\alpha}} \left| \eta - \nu \right|$$

for all $z \in \mathbb{D}^*$ and for all η,$\nu \in \mathbb{C}$.

When, in addition to the condition (iii) and (iv), the condition $\sup_{z \in \mathbb{U}} \left| z^{\alpha} f(z,b) \right| < \infty$ is satisfied, we proved that the considered problem has a unique bounded analytic solution on $\mathbb{U}$.

Furthermore, unlike [15] we used a method related to Schwarz Lemma, which provided us to prove the existence and uniqueness of desired solution of this problem for the functions f in the more general class of analytic functions.

In addition to the above problem, we considered the following initial value problem for modified complex fractional differential equation:

$$(D_z^\alpha \circ z^\alpha)u(z) = f(z, u(z)) \quad (z \in \mathbb{U})$$
$$u(0) = b.$$

By imposing some condition similar to the conditions given above, we prove the existence and uniqueness of the solution for this problem, which is in $\mathcal{B}_R$ or $\mathcal{A}$.

In the last section of this thesis, we proved the existence of the generalized solution in the following Dirichlet boundary problem involving non-local operator $\mathcal{L}_K$ in the subcritial, critical and supercritical cases by using the compactness method:

$$\begin{cases} -\mathcal{L}_K u + g(x,u) = h(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases},$$

provided that the nonlinear function g satisfy the conditions (i*), (ii*) and (iii*) in the subcritial and critical cases and the conditions (i*) and (iv*) in the supercritical case:

(i*) There exist $\mu > 0$ and the functions $a_i \in L_{p_i}(\Omega)$ with $a_i(x) \geq 0$ for $i = 0, 1$, such that the following inequality holds for a.e $x \in \Omega$ and for all $t \in \mathbb{R}$:

$$|g(x,t)| \leq a_1(x) |t|^\mu + a_0(x), \tag{5.1}$$

where the numbers p_i are defined as follows:

$$\begin{cases} p_1 := \frac{2n}{(\mu-1)(2s-n)+4s},\ p_0 := \frac{2n}{n+2s} & ,\ \mu \in (0,1) \\ p_1 := \infty,\ p_0 := \frac{\mu+1}{\mu} & ,\ \mu \in \left[1, \frac{n+2s}{n-2s}\right] \\ p_1 := \infty,\ p_0 := \frac{\mu+1}{\mu} & ,\ \mu > \frac{n+2s}{n-2s}. \end{cases},$$

(ii*) If $\mu \in \left(1, \frac{n+2s}{n-2s}\right]$ in (5.1), then there exist the number $c_1 > 0$, a function $a_2 \in L_1(\Omega)$ and $0 < \alpha < 2$ such that the following inequality holds for a.e $x \in \Omega$ and for all $t \in \mathbb{R}$:

$$tg(x,t) \geq -c_1 |t|^\alpha - a_2(x). \tag{5.2}$$

(iii*) If $\mu = 1$ in (5.1), then function the inequality in (5.2) is satisfied for $\alpha = 2$, $a_2 \in L_1(\Omega)$ and $c_1 > \frac{\lambda_1}{2}$, where λ_1 is the first eigenvalue of the operator $-\mathcal{L}_K$.

(iv*) There exist the numbers $k_0 > 0$, a function k_1 in the space $L_1(\Omega)$ such that the following inequality holds when $\mu > \frac{n+2s}{n-2s}$:

$$tg(x,t) \geq k_0 |t|^{\mu+1} - k_1(x)$$

for almost $x \in \Omega$ and for all $t \in \mathbb{R}^n$.

Moreover, we showed that, if the generalized solution of the problem just above exists, then it is unique, provided that the gunction g in the considered problem satisfies the following condition:

(v*) There exists a real number k with $0 < k < \frac{1}{2c}$ for which the inequality

$$(g(x, t_1) - g(x, t_2))\,(t_1 - t_2) > -k\,|t_1 - t_2|^2\,,$$

holds for all $x \in \Omega$ and for all $t_1, t_2 \in \mathbb{R}$.

The obtained results for the problem involving $\mathcal{L}_K$ mentioned above are also valid for the problem involving $(-\Delta)^s$, since the fractional Laplacian $(-\Delta)^s$ is a special form of the non-local operator $\mathcal{L}_K$,

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