ESSAYS ON GROWTH AND MACROECONOMIC DYNAMICS

A Ph.D. Dissertation

by
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Ankara
September 2015

To my mother Güli ÖZER…

ESSAYS ON GROWTH AND MACROECONOMIC DYNAMICS

Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

MEHMET ÖZER

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of
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in

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September 2015

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

ESSAYS ON GROWTH AND MACROECONOMIC DYNAMICS

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September 2015

This dissertation is composed of four essays on economic growth and macroeconomic dynamics. The first essay analyzes how the dynamic strategic interactions among agents affect the long-run distribution of wealth in terms of catching up and the transitional dynamics. It is shown that incorporating the strategic behavior among agents leads to the wealth level of the initially poor and the rich households to be the same at the stationary state. Extending the model by incorporating relative wealth concern; the resulting equilibria depends the valuation of relative wealth concern by each individual and it is proved that under some plausible conditions the catching up occurs thanks to the strategic interaction in the form of open-loop. The stability of these two models are carried out for arbitrary number of people in the economy.

In the second essay studies the effects of above mentioned strategic interaction in Ramsey model with "Easterlin hypothesis". It is shown that strategic interaction among agents in the economy leads to a change not only in the distribution of wealth but also in the transitional dynamics substantially. The obtained complex dynamics is in the form of Hopf bifurcation which is one of the main tool to explain the economic fluctuations.

Third essay of this thesis introduces Stone-Geary Preferences with an endogenous reference level of consumption in an Ak model in which reference level of consumption is an increasing function of the capital. It is shown that the resulting equilibrium presents richer dynamics under such a Stone-Geary preferences. It is proved that endogenous reference level leads to global and local indeterminacy: economies starting with different initial conditions does not necessarily converge to the same steady state and also economies starting with the same initial conditions does not necessarily follow the same transition path.

The aim of the fourth essay is to analyze the effects of a pure public good that reduces the subsistence level of consumption on the long run equilibrium and the optimal tax rate. It is shown that although the steady state amount of public good is higher for the first best allocation, the subsistence level of consumption is the same with that of the second best equilibrium. On the other hand, the capital stock and the consumption of the private good are higher for the first best equilibria. Another important result of the essay is the "government revenue-tax rate" *locus* with a dynamic threshold which depends on the total factor productivity (TFP). The optimal amount of tax rate that maximizes the revenue of the government is an increasing function of the TFP and thus revenue maximizing tax rate varies across countries.

Keywords: Ramsey Model, Growth, Strategic interaction, Open–loop, Status-seeking, Easterlin Hypothesis, Endogenous reference, Subsistence, Public good, Dynamics, Bifurcations, Indeterminacy.

ÖZET

BÜYÜME VE MAKROEKONOMİK DİNAMİKLER ÜZERİNE MAKALELER

Özer, Mehmet

Doktora, İktisat Bölümü

Tez Yöneticisi: Doç. Dr. H. Çağrı Sağlam

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Bu çalışma, büyüme ve makroekonomik dinamikler üzerine dört makaleden oluşmaktadır. İlk makale bireyler arasındaki stratejik ilişkilerin yakınsama açısından uzun dönem gelir dağılımı ve makroekonomik geçiş dinamikleri üzerine etkisini analiz etmektedir. Bireyler arasındaki stratejik davranışların başlangıçta fakir olan hanehalkı ile zengin olan hanehalkının gelir düzeylerinin durağan dengede aynı olmasına neden olduğu gösterilmiştir. Modele göreceli servet etkisinin eklenmesi sonucunda elde edilen denge, her bir bireyin göreceli servet etkisini ne kadar dikkate aldığına bağlı olup, açık döngü formundaki stratejik ilişki sayesinde, makul koşullar altında, fakir ve zengin arasında yakınsama olduğu ispatlanmıştır. Bu iki modeldeki denge analizi ise ekonomideki birey sayısından bağımsız olarak yapılmıştır.

İkinci makale, stratejik ilişkinin "Easterlin Hipotezi" dahil edilmiş Ramsey modelindeki etkileri incelenmektedir. Ekonomide yer alan bireyler arasındaki stratejik ilişkinin sadece gelir dağılımını değil aynı zamanda geçiş dinamiklerini de büyük ölçüde değiştirdiği gösterilmiştir. Elde edilen kompleks dinamik Hopf dalgalanmaları şeklinde olup, söz konusu dalgalanmalar ekonomik karasızlıkları açıklamak için kullanılan temel araçlardan biridir.

Üçüncü makale, sermaye stoğunun bir fonksiyonu olan endojen referans tüketim seviyesinin Stone-Geary Tercihlerine eklemlendiği Ak büyüme modelini analiz etmektedir. Elde edilen sonuçların standart Stone-Geary Tercihlerini içeren modellere kıyasla daha zengin dinamiklere sahip olduğu gösterilmiştir. Endojen referans tüketim seviyesinin global ve lokal belirsizliklere neden olduğu ispatlanmıştır. Bunun anlamı, faklı başlangıç koşullarından hareket eden ekonomilerin aynı durağan dengeye yakınsamayabileceği ve ayrıca aynı başlangıç koşullardan hareket eden ekonomilerin de aynı geçiş patikasını izleyemeyebileceğidir.

Dördüncü makalenin amacı bireylerin varlıklarını devam ettirmek için gerekli asgari tüketim seviyesi azaltan saf bir kamu malının uzun dönem dengesi ile optimal vergi oranı üzerine etkisini incelemektir. Durağan dengede, en iyi birinci dağılımındaki kamu malı mikarının en iyi ikinci dağılımdaki kamu malı miktarından daha yüksek olduğu ancak her iki dengede asgari tüketim seviyelerinin aynı olduğu gösterilmiştir. Diğer taraftan, özel malın tüketim miktarı ile sermaye stoğu ise en iyi birinci dağılımda daha yüksektir. Makalenin diğer önemli bir sonucu ise, toplam faktör verimliliğine (TFV) bağlı dinamik bir eşiğe sahip "kamu geliri-vergi oranı" eğrisi elde edilmesidir. Kamu gelirini azamileştiren optimal vergi oranı TFV'nin artan bir fonksiyonu olup, geliri azamileştiren vergi oranı ülkeler arasında farklılık arz etmektedir.

Anahtar Kelimeler: Ramsey Modeli, Büyüme, Stratejik ilişki, Açık-döngü, Statü-arayışı, Easterlin Hipotezi, Endojen referans, Asgari tüketim, Kamu malı, Dinamikler, Dalgalanmalar, Belirsizlik.

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It is my mother's shining example that I try to emulate in all that I do. Her patience and sacrifice will remain my inspiration throughout my life. I deeply miss you mother and no wonder this thesis is dedicated to you.

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CHAPTER 1

INTRODUCTION

The growth models (Solow, Ramsey, Ak etc.) based on two keystones; the supply and the demand. In all of the growth models, households maximize their utility by smoothing consumption subject to the income earned from the production side of the economy. Therefore, both long-run equilibrium and the transition to this equilibria heavily depend on the properties and assumptions made on the utility and the production functions. Although, main stimulus of economic growth came from the supply side; for example from the exogenous technological progress as in Solow or Ramsey economy, or endogenous technological progress as in Shumpeterian models and new growth theory, the preference side is crucial for the transitional path. In this thesis, we mainly deal with the demand side of the economy and analyze the quantitative and qualitative properties of long-run equilibriums and nonlinear transitional paths.

One of the cornerstone of the growth theory is the Ramsey growth model. Ramsey conjectures that the most patient household holds the entire capital stock of the economy in the long run. Furhermore, it is shown that if every household has the same time preference rate, then the initial wealth differences among them will perpetuate in the long run, as well (Kemp and Shimomura, 1992, Van long and Shimomura, 20004). Does this result valid for an economy in which households are aware of their market power on interest rate and wage earning? Does such a departure from competitive economy change the quantitative and qualitative feature of the standard Ramsey model? To what extent the transitional dynamics changes under such a set up when relative wealth and envy effects in preference side are taken into account?

The objective of Chapter 2 is to analyze how the dynamic strategic interactions among agents affect the long-run distribution of wealth in terms of catching up and the transitional dynamics in an economy. More specifically, we investigate the effects of the strategic interaction among agents on catching up which is not present under a competitive equilibrium framework. We ask whether a rich household that has a larger initial stock can use this as an advantage to prevent its rival from accumulating capital stock and achieve a higher long-run capital stock. In particular, we analyze whether such a head start disappears in the non-cooperative equilibrium of this class of games even with open-loop strategies. Under this framework,by taking the strategies of rivals as given, households decide on their strategies simultaneously and each household encounters a single criterion optimization problem. In this respect, adopting open-loop strategies reflect the slightest departure from the competitive equilibrium framework as it does not allow for genuine interaction between players during the game (see Sorger, 2008; Camacho, et al., 2013). However, even under this small departure from the competitive equilibrium framework, we show that considering the strategic interaction among agents in the economy changes the qualitative properties of the standard Ramsey model drastically.

In the absence of strategic interaction, poor will never be able to catch up with the rich as pointed in Van Long and Shimomura (2004). However, incorporating the strategic behavior among agents leads to the wealth level of the initially poor and the rich households to be the same at the stationary state. We extend our analysis on the dynamic implications of strategic interaction, to account for relative wealth concern. The resulting equilibria depends the valuation of relative wealth concern by each individual and we show that under some plausible conditions the catching up occurs thanks to the strategic interaction in the form of open-loop.

The second essay of this thesis studies the effects of above mentioned strategic interaction in Ramsey model with "Easterlin hypothesis". The crucial assumption of traditional measures of the preference is the independence assumption. That is to say, the utility at each point in time depends only on the consumption at that period. However, interdependence of utilities in between different time periods is known as the "habit formation" or "adjustment cost in consumption" in economics and it partly mimics the concept of "adaptation problem" defined by Friedrick and Loewenstein (1999) in sociology. Although intertemporal dependence of preferences are extensively studied in the literature, our aim is to go one step ahead and employ such a set up with heterogenous agents and strategic interaction. Heterogeneity comes from the initial wealth differences among agents as in the second chapter of this thesis. The aim of this part is to investigate the catching up and dynamic properties of the model. Deviating from the competitive equilibrium framework, we show that the strategic interaction among agents in the economy leads to a change not only in the distribution of wealth in the long run but also in the transitional dynamics substantially. Indeed, the strategic interaction not only leads to complex wealth distribution but also complex dynamics in Ramsey model with adjustment cost of consumption. The importance of this results is further emphasized if one recalls that the peculiar possibility of cyclical behaviors necessitates to extend the Ramsey growth model in two dimensions, namely capital having positive spillovers on utility and the inclusion Easterlin hypothesis (see Wirl, 1994; Wirl, et al., 2008). However, we show that when households use open loop strategies rather than being price takers, complex dynamics may emerge even without capital in utility at very low levels of adjustment costs. In this respect, we show that structurally very simple frameworks may lead to limit cycles thanks to the strategic interaction among agents in the economy.

Indeed, the complex dynamics is in the form of Hopf bifurcation. Since cyclical patterns and economic oscillations have been seen throughout the economic history, one of the main aim of the growth theory is to explain these fluctuations. Obtaining a Hopf bifurcation with such a simple set up is quite important because Hopf bifurcation can lead to limit cycles. Furthermore, limit cycles look like the long-run business cycles. We obtain the Hopf bifurcation by varying the parameter in the penalty function in the preferences and thus, our model argues that the degree of the penalty in utility can serve as an supplemental channel for clarifying the cyclical behaviours in the economy.

Third chapter of this thesis introduces Stone-Geary Preferences with an endogenous reference level of consumption in an otherwise standard Ak growth model. The reference level of consumption serves as a self-assessed reservation level as the agent cannot handle a decrease in his consumption below this level. Such a formulation allows to capture the utility that an agent takes from consumption when the consumption is not only above a constant threshold level but also above a dynamic threshold. This threshold level of consumption depends positively on the capital stock of the individuals. Manfred Max-Neef (1992) states that "What changes both over time and through economic systems is the way or the means by which the fundamental needs are satisfied and what is economically determined is not the fundamental needs such as shelter and food but the satisfiers for these needs". This actually suggest not to speak of poverty, but of poverties and thus the reference level of consumption may not constant for individuals that have different capital stocks.

We have shown that depending on the relationship between the curvature of the reference level of consumption function and the net to capital, the resulting equilibrium presents richer dynamics under such a Stone-Geary preferences. In particular, we prove that endogenous reference level of consumption posits both global and local indeterminacy: economies starting with different initial conditions does not necessarily converge to the same steady state and also economies starting with the same initial conditions does not necessarily follow the same transition path. Accordingly, we show that fluctuations due to self-fulfilling expectations may emerge thanks to Stone-Geary preferences with endogenous reference level of consumption. In this context, this model serves an additional channel for understanding the cross country income divergence and differences. There are bunch of studies in the literature that come up with local indeterminacy. However, most of them rely on either more than one control variable in addition to consumption (e.g., labor leisure choice, environmental quality, etc.) or increasing returns (see, among others, Benhabib and Farmer, 1994) that creates a wedge between social and private returns (e.g., Benhabib and Nishimura, 1996) or a variable mark-up (e.g., Woodford, 1991).

We analyze the effect of public good provision on the long run equilibrium and the optimal tax rate in the last essay of this thesis. The public good that we deal with has two specific properties: the income and price elasticity is less than one indicating a that it is a necessity good and it is a pure public good that does not have congestion effect. The standard growth models incorporate welfare enhancing public goods into the utility function either additively seperable or multiplicative forms. Thus, in such models private and public goods are taken as perfect substitutes or complements. Since the private good in usual growth models is a kind of composite good, incorporating the public good that satisfies bacis needs of indiviudals make such formulation inadequate. Therefore, we use Stone-Geary type preferences as in the fourth chapter of this thesis. The Stone-Geary preferences are traditionaly used to study the models involving subsistence level of consumption. Individuals take utility from consumption if it is above a threshold.This threshold level can be considered as the minimum level of consumption that satisfy the basic needs. As we have mentioned above, pure public goods do not have the property of congestion and they are actually goods that satisfy the basic needs. Since the absence of public goods forces individuals to consume this minimum amount from their own budget, an increase in the provision of public goods actually decreases the threshold level above which private consumption gives utility. In other words, the public good provision increases the disposable income of individuals indirectly through leaving room for private good consumption from the individual's budget.

We show that, although the steady state amount of public good is higher for the first best allocation, the subsistence level of consumption is the same with that of the second best equilibrium. On the other hand, the capital stock and the consumption of the private good are higher for the first best equilibria. There is an inverse relationship between the optimal amount of tax rate and the share of capital in the solution of second best allocation. The same result is valid between optimal tax rate and total factor productivity. However, for the social planner, the share of capital in production and total factor productivity do not matter for the allocation of public good, capital stock and consumption of private good. Another important result of the paper is the "$g-\tau$" *locus* with a dynamic threshold which depends on the total factor productivity. This means that the optimal amount of tax rate that maximizes the total revenue of the government is an increasing function of the total factor productivity and thus revenue maximizing tax rate varies across countries. Furhetmore, steady states for both first and second best allocations are stable in the saddle path sense.

CHAPTER 2

STRATEGIC INTERACTION AND CATCHING UP

The question of catching-up has always been one of the main concerns of macroeconomics. Stiglitz (1969) has shown that the income of poor will converge to that of the rich in the long-run in a Solow economy. This analysis rests on the assumption that agents do not save optimally. In contrast with this, in a dynamic general equilibrium model *a la* Ramsey-Cass-Koopmans, Kemp and Shimomura (1992) has shown that if all households have the same patiance rate, then the distribution of wealth will be history dependent so that the initial wealth inequality will persist even in the long run. However, in all of these studies, agents are thought to have no power in influencing the performance of aggregate economy and act as a price taker on all markets in a competitive equilibrium. Knowing that the number of households is finite, this contradicts with the rationality of the agents in the economy (see Pichler and Sorger, 2009). Moreover, the fact that social or economic similarities enforce individuals to constitute small number of powerful groups and agents belonging to the same economic classes show similar tendencies in choosing their decision variables, makes the consideration of the strategic interaction among agents inevitable[1].

[1] Thanks to the comment of an anonymous referee, consider as an example the labour owned enterprises in accordance with the Action Programme (1989) of the European Commission (see Guadona, 2008). Given their limited number and heterogeneity in terms of initial asset and share holdings, workers that own a share of the firm may realize their market power and act strategically in choosing their capital paths. Also for the emerging recognition of strategic interaction in growth theory, see among others, Fershtman and Muller (1984), Figueres et al. (1999), Dockner and

The objective of this paper is to analyze how the dynamic strategic interactions among agents affect the long-run distribution of wealth in terms of catching up and the transitional dynamics in an economy. More specifically, we investigate the effects of the strategic interaction among agents on catching up which is not present under a competitive equilibrium framework. We ask whether a rich household that has a larger initial stock can use this as an advantage to prevent its rival from accumulating capital stock and achieve a higher long-run capital stock. In particular, we analyze whether such a head start disappears in the non-cooperative equilibrium of this class of games even with open-loop strategies.

To do so, we consider a strategic Ramsey model in which finitely many households differ only in terms of their initial wealth. The households no longer act as price-takers but they take into account the effects of their accumulation decisions on market prices. Taking into account the inverse factor demand functions, the households play a Nash equilibrium by choosing their capital paths. We assume that the households employ open-loop strategies so that they give their accumulation decisions as simple time paths and commit themselves to stick to these preannounced paths as equilibrium strategies (see e.g. Sorger, 2002, 2008; Bethmann, 2008).

Under this framework,by taking the strategies of rivals as given, households decide on their strategies simultaneously and each household encounters a single criterion optimization problem. In this respect, adopting open-loop strategies reflect the slightest departure from the competitive equilibrium framework as it does not allow for genuine interaction between players during the game (see Sorger, 2008; Camacho, et al., 2013). However, even under this small departure from the competitive equilibrium framework, we show that considering the strategic interaction among agents in the economy changes the qualitative properties of the standard Ramsey model drastically[2].

Nishimura (2005) on capital accumulation games; Bethmann (2008) on Lucas-Uzawa model; Espino (2005), Sorger (2008) on Ramsey conjecture and Camacho, et al. (2013) on dynamics.

[2]Sorger (2008) proposes a strategic Ramsey model in which agents differ in their subjective time discount rate and analyzes Ramsey conjecture on the degeneracy of the long-run distribution of wealth. However, we assume that agents differ only in their initial wealth and analyze whether the poor can catch up with rich in the long run.

In the absence of strategic interaction, poor will never be able to catch up with the rich as pointed in Van Long and Shimomura (2004). However, incorporating the strategic behavior among agents leads to the wealth level of the initially poor and the rich households to be the same at the stationary state. We extend our analysis on the dynamic implications of strategic interaction, to account for relative wealth concern (capitalist spirit[3]). The resulting equilibria depends the valuation of relative wealth concern by each individual and we show that under some plausible conditions the catching up occurs thanks to the strategic interaction in the form of open-loop.

The rest of the paper is organized as follows. Sections 1 and 2 describes standard Ramsey model and catching up properties of it briefly, Sections 3 and 4 analyzes the catching up and provides the dynamic properties of open loop Nash equilibrium. Section 5 consider the relative wealth concern and finally, Section 6 concludes.

2.1. The Model

We consider a Ramsey (1928) economy with $N \in \mathbb{N}$ infinitely lived households and a representative firm. The firm hires capital $K(t)$ and labor $L(t)$ from the households and produces a single output $Y(t)$ that can be either consumed or saved to form future capital. The technology is represented by a neoclassical production function $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$. At any instant t, the firm chooses the variables $Y(t)$, $K(t)$, and $L(t)$ to maximize the profit $Y(t) - w(t)L(t) - r(t)K(t)$ subject to the technology $Y(t) = F(K(t), L(t))$ and the nonnegativity constraints $K(t) \geq 0$, $L(t) \geq 0$ where $w(t)$ is the real wage rate and $r(t)$ is the rental rate of capital.

The preferences of household $i \in \{1, 2, ..., N\}$ are characterized by the instantaneous utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ and the time preference rate $\rho > 0$. At any instant t, every household supply inelastically one unit of labor so that the total labor supply is N. Let $f(K(t)) = F(K(t), N)$. We assume that f and u satisfy

[3] Capitalist spirit refers to the motivation behind the perpetual acquisition of wealth not only for the sake of maximizing long-run consumption but also for the utility from accumulating wealth itself and the status associated with it (see Weber, 1958; Bakshi and Chen,1996; Corneo and Jeanne, 1997).

the following properties.

$f : \mathbb{R}_+ \to \mathbb{R}_+$, is continuous, twice continuously differentiable, strictly increasing, strictly concave satisfying $f(0) = 0$, $\lim_{K \to 0} f'(K) = +\infty$ and $\lim_{K \to +\infty} f'(K) = 0$.

$u : \mathbb{R}_+ \to \mathbb{R}$, is continuous, twice continuously differentiable, strictly increasing, strictly concave satisfying $\lim_{c \to 0} u'(c) = +\infty$.

The households differ only in terms of their initial wealth levels. Agents maximize their discounted life-time utility derived from the consumption of the single good. The utility maximization problem of household i can be formalized as

$$\max_{c_i(t)} \int_0^\infty e^{-\rho t} \, u(c_i(t))dt \qquad (\mathcal{P})$$

subject to

$$\dot{k}_i(t) = r(t)k_i(t) + w(t) - c_i(t), \ \forall t \geq 0,$$

$$k_i(t) \geq 0, \ c_i(t) \geq 0, \ \forall t \geq 0,$$

$$k_i(0) = k_{i0}, \text{given}.$$

In what follows, we will analyze the competitive equilibrium and the open loop Nash equilibrium in order to identify to what extent the strategic interaction among agents in the economy affects the long run distribution of wealth in terms of catching-up.

2.2. The Competitive Equilibrium

If we assume that the households are price-takers so that they can not realize their market power and take the rental rates of capital and labor as given, the model coincides with the standard Ramsey economy. Kemp and Shimomura (1992) and Van Long and Shimomura (2004) have already shown that the initial wealth inequality will persist in the long run so that the poor individuals will never be able to catch up with the rich in such a framework. For completeness and providing a basis of comparison, the analysis of competitive equilibrium follows from their studies.

The solution to the utility maximization problem $(\mathcal{P})$ of household $i \in \{1, 2, ..., N\}$ leads to the following Euler equation:

$$\frac{\dot{c}_i(t)}{c_i(t)} = \xi(c_i(t))(r(t) - \rho),$$

where $\xi(c_i) \equiv -\frac{u'(c_i)}{c_i u''(c_i)}$ denotes the inverse of the elasticity of marginal utility.

Since the firm maximizes its profit taking the market prices as given, factors are paid their marginal products, i.e.,

$$r(t) = f'(K(t)) \text{ and } w(t) = \frac{[f(K(t)) - K(t) f'(K(t))]}{N}, \ \forall t \geq 0, \tag{1}$$

where $K(t) = \sum_{i=1}^{N} k_i(t)$. We have then the following system of $2N$ differential equations:

$$\dot{k}_i(t) = f'(K(t)) k_i(t) + \frac{f(K(t)) - K(t) f'(K(t))}{N} - c_i(t), \tag{2}$$

$$\frac{\dot{c}_i(t)}{c_i(t)} = \xi(c_i(t))(f'(K(t)) - \rho), \ \forall i \in \{1, 2, ..., N\}. \tag{3}$$

A steady state is defined by (k_i, c_i) such that the right-hand sides of the system of equations (20)-(19) equal to zero for all $i \in \{1, 2, ..., N\}$. A steady state is symmetric if $k_i = k$, and $c_i = c$, for all $i \in \{1, 2, ..., N\}$. A steady state turns out to be asymmetric if $k_i \neq k_j$, for some $i, j \in \{1, 2, ..., N\}$. The following proposition shows that if $k_i(0) \neq k_j(0)$ for some $i, j \in \{1, 2, ..., N\}$ then we will have $k_i \neq k_j$ at the steady state.

Proposition 1 *We have $k_i = k$, for all $i \in \{1, 2, ..., N\}$ at the steady state if and only if the initial wealth levels are identical, i.e., $k_i(0) = k_0$, for all $i \in \{1, 2, ..., N\}$. Moreover, there exists a continuum of steady state wealth distributions and a corresponding continuum of one-dimensional stable manifolds so that inequalities perpetuate.*

Proof. Follows directly from Van Long and Shimomura (2004) and Kemp and Shimomura (1992). ■

2.3. Open Loop Nash Equilibrium

As mentioned before, agents no longer act as price-takers but they take into account the effects of their accumulation decisions on market prices. Taking into account the inverse factor demand functions stated in (1), the households play a Nash equilibrium by choosing their *capital paths.* Households give their accumulation decisions as simple time paths and commit themselves to stick to these time paths during the entire game (i.e. they employ open-loop strategies). When choosing its path of capital, household $i \in \{1, 2, ..., N\}$ takes the chocies variables of other households as given. Accordingly, household i solves the problem:

$$\max_{c_i(t)} \int_0^{\infty} e^{-\rho t} u(c_i(t))dt \qquad (\mathcal{P}\prime)$$

subject to

$$\dot{k}_i(t) = f^{'}(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t), \ \forall t \geq 0,$$

$$k_i(t) \geq 0, \ c_i(t) \geq 0, \ \forall t \geq 0,$$

$$k_i(0) = k_{i0}, \text{given},$$

$$k_j(t), \ \forall t \geq 0, \ \forall j \in \{1, 2, ..., N\} \setminus \{i\}, \ \text{given}.$$

It is important to note that household i takes into account that it can influence $K(t)$ via $k_i(t)$ as $K(t) = \sum_{j=1}^{N} k_j(t)$. The Hamiltonian for problem $\mathcal{P}\prime$ is:

$$H(c_i(t), \ k_i(t), \ \lambda_i(t)) = e^{-\rho t} u(c_i(t)) + \lambda_i(t) \left(f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t) \right).$$

The set of necessary conditions of optimality will then be written as follows: $\forall i \in \{1, 2, ..., N\}, \forall t \geq 0$,

$$e^{-\rho t}u'(c_i(t)) = \lambda_i(t), \tag{4}$$

$$-\frac{\dot{\lambda}_i(t)}{\lambda_i(t)} = f'(K(t)) + f''(K(t))\left(k_i(t) - \frac{K(t)}{N}\right), \tag{5}$$

$$\dot{k}_i(t) = f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t). \tag{6}$$

In order to make the first order optimality conditions for $\mathcal{P}\prime$ to be sufficient, we need to assume further that the factor income of each household is a concave function of its own capital stock.

The function $k_i(t) \mapsto k_i(t)f'(K(t))$ is concave for all $i \in \{1, ..., N\}$ and for all $t \geq 0$.

Under Assumptions 1-3, the first order optimality conditions are also sufficient if the transversality condition $\lim_{t\to 0} \lambda_i(t)k_i(t) = 0$ holds for all $i \in \{1, ..., N\}$. In accordance with Assumption 2, we adopt a $CRRA$ form of utility function with an intertemporal elasticity of substitution θ under which we obtain the following system of $2N$ differential equations:
$\forall i \in \{1, 2, ..., N\}, \forall t \geq 0$,

$$\frac{\dot{c}_i(t)}{c_i(t)} = \frac{1}{\theta}\left[f'(K(t)) - \rho + f''(K(t))\left(k_i(t) - \frac{K(t)}{N}\right)\right], \tag{7}$$

$$\dot{k}_i(t) = f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t). \tag{8}$$

In the following proposition, taking into account the strategic interaction among agents, we show that the catching up prevails in the economy, so that even if the agents have initially different levels of wealth, they will eventually reach to the equal levels of wealth at the steady state.

Proposition 2 *In an open-loop Nash equilibrium, there exists a unique symmetric steady state and there are no asymmetric steady states.*

Proof. Note from (7) that we have

$$f^{'}(K) - \rho + f''(K)\left(k_i - \frac{K}{N}\right) = 0,\ \forall i \in \{1, 2, ..., N\}$$

at a steady state. Since the production function is strictly concave, i.e., $f^{'}(K) > 0$, $f''(K) < 0$, the condition that satisfies these N equations simultaneously is simply $k_i = k$, for all $i \in \{1, 2, ..., N\}$. ■

2.4. SS and The Stability Analysis

We now examine the stability properties of the symmetric steady state at which we have $f'(K) = \rho$, $c_i = \frac{f(K)}{N}, \forall i \in \{1, 2, ..., N\}$, where $K = Nk_i$. Linearizing the system of $2N$ differential equations (7)-(8) around the unique steady (Appendix A) state gives the following $2N \times 2N$ Jacobian matrix:

$$\mathbf{J}_{2N\times 2N} \equiv \begin{bmatrix} \mathbf{0}_{N\times N} & \vdots & \mathbf{A}_{N\times N} \\ \cdots\cdots & & \cdots\cdots \\ -\mathbf{I}_{N\times N} & \vdots & f'(K)\mathbf{I}_{N\times N} \end{bmatrix},$$

where $\mathbf{I}$ and $\mathbf{0}$ denote the identity and zero matrix respectively and

$$\mathbf{A}_{N\times N} = \frac{1}{\theta N} f(K) f''(K) \begin{bmatrix} (2-\frac{1}{N}) & (1-\frac{1}{N}) & \cdots & (1-\frac{1}{N}) \\ (1-\frac{1}{N}) & (2-\frac{1}{N}) & \cdots & (1-\frac{1}{N}) \\ \vdots & \vdots & \ddots & \vdots \\ (1-\frac{1}{N}) & (1-\frac{1}{N}) & \cdots & (2-\frac{1}{N}) \end{bmatrix}.$$

Note that $\mathbf{A}_{N\times N}$ is a symmetric Toeplitz (diagonal-constant) matrix with $(N-1)$ characteristic roots that are equal to $\frac{1}{\theta N} f(K) f''(K)$ and one characteristic root that is equal to $\frac{1}{\theta} f(K) f''(K)$. Without loss of generality, let $\mu_1 = \mu_2 = ... = \mu_{N-1} = \frac{1}{\theta N} f(K) f''(K)$ and $\mu_N = \frac{1}{\theta} f(K) f''(K)$. As $f''(K) < 0$, we have $\mu_i < 0$, $\forall i \in \{1, 2, ..., N\}$.

To find the eigenvalues of the Jacobian matrix $\mathbf{J}$, we need to solve $\det[\mathbf{J} - \lambda \mathbf{I}] =$

0. However, note that λ is an eigenvalue of the jacobian matrix $\mathbf{J}$ if and only if $\lambda\left(f'(K)-\lambda\right)$ is an eigenvalue of $\mathbf{A}_{N\times N}$ as

$$\det[\mathbf{J}-\lambda\mathbf{I}]=\det\left[\mathbf{A}-\boldsymbol{\lambda}\left(f'(K)-\lambda\right)\mathbf{I}\right].$$

This simply suggests that the eigenvalues of the $2N\times 2N$ Jacobian matrix can easily be characterized by the characteristic root distribution of $\mathbf{A}_{N\times N}$. Indeed, the eigenvalues of the Jacobian matrix $\mathbf{J}$ will be determined as a solution to the quadratic equations,

$$\lambda^2-f'(K)\lambda+\mu_i=0,\forall i\in\{1,2,...,N\}.$$

In particular, for each eigenvalue μ_i of matrix $\mathbf{A}$, this equation has two roots, the product of which is $\mu_i<0$. Evidently, this implies that the Jacobian matrix $\mathbf{J}$ has N positive and N negative real eigenvalues so that the symmetric steady state turns out to be stable in the saddle-point sense. Note that, the stability analysis carried on here is operative for an arbitrary N. To our knowledge, such a generalized stability analysis for an arbitrary dimension is the first in the economic literature, thanks to the strategic interaction and the Jacobian matrix in the form of Toeplitz.

2.5. Relative Wealth Effect

The assumption that the households take utility only from their own consumption has been shown to be not so realistic in a dynamic perspective. Under this assumption, as the income of an individual, and hence his consumption increase, one would expect that its welfare will increase as well. However, despite the continually rising prosperity in the developed countries, there are considerable fluctuations in the percentage of those who say they were very satisfied in terms of their welfare. Consistent with this, Ehrhardt and Veenhoven (1995) shows that the percentage of those who have attained the highest level of welfare over time is almost constant and even sometimes declining as the prosperity increase. Therefore, it is inevitable to think of a model in which the households do not take utility only from their own consumption but also from their relative position in the society.

Veblen (1922) notes that, it is not wealth but relative wealth which is important for the human being. It is argued that the higher the relative wealth the greater the social status and status is one of the crucial elements of the welfare of individuals. Bakshi and Chen (1996) provide empirical support to the spirit of the capitalism hypothesis (wealth accumulation not only for consumption) and show that the investors acquire wealth not just for its current and future consumption, but also for its reward in terms of social status. Cole *et al.* (1992), Corneo and Jeanne (1997) present that when individuals take into account their social status, the optimal consumption-saving behavior is affected systematically and the qualitative properties of the equilibrium solution path strongly diverge from the traditional models.

Accepting that wealth is more valuable than its implied consumption rewards, Van Long and Shimomura (2004) consider that the agents get utility not only from their consumption stream but also from their relative wealth level with respect to the average in the economy. Thanks to this relative wealth effect in utility, Van Long and Shimomura (2004) show that the poor will be able to catch up with the rich if the elasticity of the marginal utility of relative wealth is greater than the elasticity of the marginal utility of consumption. However, even though the agents take care about their relative wealth position in the economy so that the strategic interaction among agents inherently exists, this has not been taken into account in assessing the conclusions on catching up.

The model differs from the strategic Ramsey model by the assumption on the preference of the agents. In this set up, households take utility not only from their consumption but also from their social status represented by their relative wealth. The maximization problem of household $i \in \{1, 2, ..., N\}$ can now be formalized as follows:

$$\underset{c_i(t)}{Max} \int_0^{\infty} e^{-\rho t} \left(u(c_i(t)) + \eta_i v\left(z_i\left(t\right)\right)\right) dt \qquad (\mathcal{P}'')$$

subject to

$$\dot{k}_i(t) = f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t),\ \forall t \geq 0,$$

$$z_i\left(t\right) = \frac{k_i(t)}{\frac{K(t)}{N}},\ \forall t \geq 0,$$

$$k_i(t) \geq 0,\ c_i(t) \geq 0,\ \forall t \geq 0,$$

$$k_i(0) = k_{i0}, \text{given},$$

$$k_j(t),\ \forall t \geq 0,\ \forall j \in \{1, 2, ..., N\} \setminus \{i\},\ \text{given},$$

where z_i refers to the relative wealth of household i with respect to the average wealth in the economy. $\eta_i \in \mathbb{R}_+$ measures the weight of relative wealth (status concern) in utility. We employ an additively separable utility function between consumption and realtive wealth not only for analytical convenience but also for being consistent with Van Long and Shimomura (2004) and the recent empirical findings[4]. We adopt the following assumption on the utility from relative wealth.

$v : \mathbb{R}_+ \rightarrow \mathbb{R}$, is continuous, twice continuously differentiable, strictly increasing, strictly concave satisfying $v(0) = 0$ and $\lim_{z \rightarrow 0} v'(z) = +\infty$.

The current-value Hamiltonian associated with optimization problem $\mathcal{P}''$ is

$$\mathcal{H}(c_i(t),\ k_i(t),\ \lambda_i(t)) = \left[u(c_i(t)) + \eta_i v\left(\frac{k_i(t)}{\frac{1}{N}K(t)}\right)\right] + \lambda_i(t)\left[f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t)\right],$$

where λ_i denotes the current-value adjoint variable. Recall that household i takes into account that it can influence $K(t)$ via $k_i(t)$ as $K(t) = \sum_{j=1}^{N} k_j(t)$. A routine

[4]Compared to the multiplicative form, the separable form of the preferences is more consistent with the empirical findings on the behavior of the wealthy households since these preferences do not put any restrictions on either the substitutability or the complementarity between consumption and relative wealth (see Francis (2009) for details about the functional form of the utility function).

application of the Pontryagin's maximum principle leads to the following system of $2N$ differential equations: $\forall i \in \{1, 2, ..., N\}, \forall t \geq 0$,

$$\dot{c}_i(t) = -\frac{u'(c_i(t))}{u''(c_i(t))}\left[f'(K(t)) - \rho + f''(K(t))\left(k_i(t) - \frac{K(t)}{N}\right) + \eta_i \frac{v'(z_i(t))}{u'(c_i(t))}\frac{1-\frac{k_i(t)}{K(t)}}{\frac{1}{N}K(t)}\right], \tag{9}$$

$$\dot{k}_i(t) = f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t). \tag{10}$$

Under Assumptions 1-4, the first order optimality conditions are also sufficient if the transversality condition $\lim_{t\to 0} e^{-\rho t}\lambda_i(t)k_i(t) = 0$ holds for all $i \in \{1, ..., N\}$.

Taking into account the strategic interaction among agents, we now consider the conditions under which catching up will prevail in the economy. It is clear from equations (9)-(10) that $\eta_i = \eta$ for all $i \in \{1, ..., N\}$ turns out to be a necessary condition for the existence of a symmetric steady state. Indeed, given $\eta_i = \eta$, for all $i \in \{1, ..., N\}$, a symmetric steady state exists if and only if there exists $K > 0$ such that

$$f'(K) + \eta^1 \frac{v'(1)}{u'(\frac{f(K)}{N})}\frac{(N-1)}{K} = \rho. \tag{11}$$

In accordance with Assumption 2 and 4, adopting $CRRA$ form of utility functions, $u(c) = \frac{c^{1-\theta}}{1-\theta}$ and $v(z) = \frac{z^{1-\sigma}}{1-\sigma}$, and the standard *Cobb-Douglas* production function $f(K) = AK^\gamma, \gamma \in (0, 1)$, the following proposition provides the conditions for the existence and uniqueness of a symmetric steady state.

Proposition 3 *Let $\eta_i = \eta$, for all $i \in \{1, ..., N\}$. There exists a unique symmetric steady state if $\gamma\theta < 1$. Moreover, if $\theta \leq \sigma$ then an asymmetric steady state does not exist.*

Proof. See Van Long and Shimomura (2004) for the existence and the uniqueness of a symmetric steady state. The proof of the existence of a symmetric steady state follows from the limit properties of the left-hand side of equation (11). Given existence, if $[Ku'(f(K))]$ is monotonically decreasing in K then the uniqueness of the symmetric steady state is also ensured.

Assume now that there exists an asymmetric steady state at which $k_i > k_j$, for some $i, j \in \{1, ..., N\}$, without loss of generality. Let c_i and c_j denote the associated consumption levels at this asymmetric steady state. It is then clear from (10) that

$$f''(K)(k_i - k_j) = \frac{N\eta}{K^2}\left(\frac{v'(z_j)}{u'(c_j)}(K - k_j) - \frac{v'(z_i)}{u'(c_i)}(K - k_i)\right). \tag{12}$$

Note that the left hand side of (12) is less than zero due to the concavity of the production function. However, since $\theta \leq \sigma$, we have

$$\left(\frac{z_j}{z_i}\right)^{-\sigma} \geq \left(\frac{c_j}{c_i}\right)^{-\theta},$$

as

$$\left(\frac{k_i}{k_j}\right)^{\sigma} \geq \left(\frac{k_i}{k_j}\right)^{\theta} \geq \left(\frac{f'(K)k_i + \frac{f(K)-Kf'(K)}{N}}{f'(K)k_j + \frac{f(K)-Kf'(K)}{N}}\right)^{\theta},$$

so that the right hand side of (12)is positive: a contradiction. ■

If initially poor households attribute less weight to the relative wealth (status concern) in utility than the initially rich households, then the poor can never catch up with the rich in such a strategic Ramsey economy. Note that, as we focus on catching up, we need also the condition that the elasticity of consumption is less than that of relative wealth for avoiding the emergence of the asymmetric steady states. Moreover, we also need to verify that the unique symmetric steady state turns out to be stable at least in the saddle-point sense.

To do so, we characterize the Jacobian of the resulting $2N \times 2N$ system associated with (10) around a symmetric steady state (Appendix B) $(c_i, k_i)_{i \in \{1,....,N\}}$ at which $\eta_i = \eta$, $c_i = c = \frac{f(K)}{N}$, $k_i = k$ for all $i \in \{1, ..., N\}$, and $K = Nk$ satisfy (11). The Jacobian is a 2×2 block (partitioned) matrix,

$$\mathbf{J}_{2N\times 2N} \equiv \begin{bmatrix} (\rho - f'(K))\,\mathbf{I}_{N\times N} & \vdots & \mathbf{B}_{N\times N} \\ \cdots\cdots & & \cdots\cdots \\ -\mathbf{I}_{N\times N} & \vdots & f'(K)\mathbf{I}_{N\times N} \end{bmatrix},$$

at which

$$\mathbf{B}_{N\times N}=\begin{bmatrix} a & b & \cdots & b \\ b & a & \ddots & b \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{bmatrix},$$

is a symmetric Toeplitz (diagonal-constant) matrix with $(N-1)$ characteristic roots that are equal to $(a-b)$ and one characteristic root that is equal to $((N-1)\,b+a)$ where

$$\begin{aligned} a &= \frac{c}{\theta}\left(f''(K)\left(2-\frac{1}{N}\right)+\eta\frac{v'(1)}{u'(c)}\left(\frac{N-1}{K}\right)^2\left(\frac{v''(1)}{v'(1)}-\frac{2}{N-1}\right)\right), \\ b &= \frac{c}{\theta}\left(f''(K)\left(1-\frac{1}{N}\right)-\eta\frac{v'(1)}{u'(c)}\frac{N-1}{K^2}\left(\frac{v''(1)}{v'(1)}-\frac{2-N}{N-1}\right)\right). \end{aligned}$$

Since

$$\det[\mathbf{J}-\lambda\mathbf{I}]=\det\left[\mathbf{B}-\left(\lambda-f'(K)\right)\left(\rho-f'(K)-\lambda\right)\mathbf{I}\right],$$

the eigenvalues of the $2N\times 2N$ Jacobian matrix can easily be characterized by the characteristic roots μ_i, $i\in\{1,2,...,N\}$ of $\mathbf{B}_{N\times N}$. Indeed, the eigenvalues of the Jacobian matrix $\mathbf{J}$ will be determined as a solution to the quadratic equations,

$$\lambda^2-\rho\lambda+\left(f'(K)\left(\rho-f'(K)\right)+\mu_i\right)=0,\forall i\in\{1,2,...,N\},$$

where

$$\mu_i=\begin{cases} \frac{c}{\theta}\left(f''(K)+\eta\frac{v'(1)}{u'(c)}\frac{N(N-1)}{K^2}\left(\frac{v''(1)}{v'(1)}-\frac{1}{N-1}\right)\right), \forall i\in\{1,2,...,N-1\}, \\ \frac{c}{\theta}\left(f''(K)N-\eta\frac{v'(1)}{u'(c)}\frac{N(N-1)}{K^2}\right), i=N. \end{cases}$$

In particular, for each eigenvalue μ_i of matrix $\mathbf{B}$, this equation has two roots, the product of which equals to $\left(f'(K)\left(\rho-f'(K)\right)+\mu_i\right)$. Recalling that $\gamma\theta<1$, $\theta\leq\sigma$, and we have (11) at a unique symmetric steady state, one can easily show that the product of the two roots is less than zero. This implies that the Jacobian matrix $\mathbf{J}$ has N positive and N negative real eigenvalues which reveals saddle path stability with monotone convergence.

It is already clear that, the long run distribution of wealth heavily depends on the valuation of the relative position by the initially poor and the rich households. Indeed, as soon as η_i differs from η_j for some $i, j \in \{1, 2, ..., N\}$, the strategic Ramsey model with relative wealth concern would result with a complex wealth distribution characterized by a saddle path stable asymmetric steady state.

2.6. Conclusion

Considering the strategic interaction among agents changes the qualitative properties of the standard Ramsey model. In the absence of strategic interaction, poor will never be able to catch up with the rich as pointed in Van Long and Shimomura (2004). However, incorporating the strategic behavior among agents leads to the wealth level of the each individuals to be the same at the stationary state. Extending the analysis to account for relative wealth concern the strategic interaction among agents lead to a complex wealth distribution characterized by a saddle path stable asymmetric steady state.

CHAPTER 3

STRATEGIC INTERACTION AND EASTERLIN HYPOTHESIS

The income distribution and the social welfare can not be well measured due ot the assumptions on preferences and utility functions used by the common economic literature and those assumptions are not generally applicaple in reality. In addition to the other simplifying assumptions, as Hicks argued one of the most important assumption on the common measures of the utility is the independence assumption. Put differently, at each time the utility depends only on the consumption at that period according to the independence assumption.

On the contrary, in real life the utilities at different tme periods are linked to each other. The concepts of "adjustment cost in consumption" or "habit formation" in economics are used to explain the interdependence of utilities in between different time periods. The anology of these concepts in sociology is defined as the "adaptation problem" by Friedrick and Loewenstein (1999). The adaptation problem states that the change in the someones living standard (upward or downward) is penalized by the person, although the degree of the penalty can be different for an increase and a decrease in living standards. In this direction, the standard neoclassical "decisive utility" is incorporated with the Kahnemans' "experience utility" by the adjustment cost of consumption. According to Kahneman (2006), "decisive utility" is deduced from the alternatives and used to define the those different alternatives. On the contrary, "experience utility" related to the hedonistic experience that is associated with an event or outcome. In this context, although it is "decisive", the utility function incorporated with the effects of the change in the consumption levels leads to embody the hedonistic part of the experience utility. Let two alternatives (5,9,6) and (5,6,9) be the consumption sequences of a representative household for three subsequent periods. Neoclassical economy will suggest to choose the first sequence under the positive patience parameter (discount factor). However, if adjustment cost of consumption is taken into account the representative household, then, the second sequence may be optimal.

Although there is agreat controversy about the habit formation in economic literature, it is shown that a strong habit pattern exists among US households by benefiting from the monthly, quarterly and yearly data (Fearson et al. (1991)). In addition to that, as a habit formation, a reference level of consumption is taken by most of the economic studies which is determined either externally or internally to the household. Internal reference consumption is usually taken as the consumers own one period lagged consumption or average of the past consumptions, as in Alvarez-Cuadrado et al. (2004) and Carroll et al. (2000). External reference point also known as the “Keeping up with Joneses” is the average level of past consumption in the overall economy. Gali (1994) shown that presence of such preferences leads to risky share to be larger or smaller than the standard models depending on the sign of consumption externality introduced via Keeping up with Joneses effect. In his famous work Abel (1990) used both the internal and external habit formation set ups and try to explain the equity premium puzzle because habits increase the disutility resulting from large decline in consumption.

Although intertemporal dependence of preferences are extensively studied in the literature, our aim is to go one step ahead and employ such a set up with heterogenous agents and strategic interaction. Heterogeneity comes from the initial wealth differences among agents as in the second chapter of this thesis. The aim of this part is to investigate the catching up and dynamic properties of the model. Deviating from the competitive equilibrium framework, we show that the strategic interaction among agents in the economy leads to a change not only in the distribution of wealth in the long run but also in the transitional dynamics substantially. Indeed, the strategic interaction not only leads to complex wealth distribution but also complex dynamics in Ramsey model with adjustment cost of consumption. The importance of this results is further emphasized if one recalls that the peculiar possibility of cyclical behaviors necessitates to extend the Ramsey growth model in two dimensions, namely capital having positive spillovers on utility and the inclusion Easterlin hypothesis (see Wirl, 1994; Wirl, et al., 2008). However, we show that when households use open loop strategies rather than being price takers, complex dynamics may emerge even without capital in utility at very low levels of adjustment costs. In this respect, we show that structurally very simple frameworks may lead to limit cycles thanks to the strategic interaction among agents in the economy.

The paper is structured as follow. First section defines the model and analyses the optimal solution paths in the light of catching up. The second section investigates the dynamics of the system and the last section will discuss the results and conclude.

3.1. The Model

The strategic Ramsey model is now extended for considering the dynamic implications of consumption adjustment costs by assuming that the agents not only derive utility from consumption but also incur a disutility from the adjustments of consumption. As in Chapter 2, we consider a Ramsey (1928) economy with $N \in \mathbb{N}$ infinitely lived households and a representative firm. The firm hires capital $K(t)$ and labor $L(t)$ from the households and produces a single output $Y(t)$ that can be either consumed or saved to form future capital. The technology is represented by a neoclassical production function $F : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$. At any instant t, the firm chooses the variables $Y(t)$, $K(t)$, and $L(t)$ to maximize the profit $Y(t) - w(t)L(t) - r(t)K(t)$ subject to the technology $Y(t) = F(K(t), L(t))$ and the nonnegativity constraints $K(t) \geq 0$, $L(t) \geq 0$ where $w(t)$ is the real wage rate and $r(t)$ is the rental rate of capital.

The preferences of household $i \in \{1, 2, ..., N\}$ are characterized by the instantaneous utility function $u : \mathbb{R}_+ \to \mathbb{R}$ and the time preference rate $\rho > 0$. At any instant t, every household supply inelastically one unit of labor so that the total labor supply is N. Let $f(K(t)) = F(K(t), N)$. We assume that f and u satisfy the following properties.

$f : \mathbb{R}_+ \to \mathbb{R}_+$, is continuous, twice continuously differentiable, strictly increasing, strictly concave satisfying $f(0) = 0$, $\lim_{K \to 0} f'(K) = +\infty$ and $\lim_{K \to +\infty} f'(K) = 0$.

$u : \mathbb{R}_+ \to \mathbb{R}$, is continuous, twice continuously differentiable, strictly increasing, strictly concave satisfying $\lim_{c \to 0} u'(c) = +\infty$.

The households differ only in terms of their initial wealth levels. Agents maximize their discounted life-time utility derived from the consumption of the single good. The utility maximization problem of household i can be formalized as

Accordingly, the problem of household $i \in \{1, 2, ..., N\}$ recast as follows:

$$\underset{c_i(t)}{Max} \int_0^\infty e^{-\rho t}\{u(c_i(t)) - \Omega(\kappa_i(t))\}dt \qquad (\mathcal{P}')$$

subject to

$$\dot{k}_i(t) = f^{'}(K(t))k_i(t) + \frac{f(K(t)) - K(t)f^{'}(K(t))}{N} - c_i(t),\ \forall t \geq 0,$$

$$\dot{c}_i(t) = \kappa_i(t),\ \forall t \geq 0,$$

$$k_i(t) \geq 0, c_i(t) \geq 0,\ \forall t \geq 0,$$

$$k_i(0) = k_{i0},\ \text{given},$$

$$k_j(t),\ \forall t \geq 0,\ \forall j \in \{1, 2, ..., N\} \setminus \{i\},\ \text{given}.$$

The additional state equation and the account for the adjustment costs $(\Omega(\kappa_i(t)))$ constitute an important extension to the strategic Ramsey model. We adopt the following assumption on the adjustment cost function. Note that, since u satisfies is strictly concave and penalty function is strictly convex, the associated maximization problem is strictly concave. Lets give some information about $(\Omega(\kappa_i(t)))$. One of the main keystones of the economics especially the growth theory is the consumption smoothing behaviour of the individuals. There is no need to repeat the vast literature on sonsumption smoothing here. However, it is important to talk about the penalty function given above. Although, the disutility of a decrease and an increase in consumption need not to be symmetric, changes in consumption are penalized by the individuals. It can be argued that a decrease in consumption should give disutility to the individuals but what about an increase? For instance, assume a new tractor given to a farmer who uses traditional methods for farming and does not know how to utilize this tractor efficiently. Actually, he needs time to get acquainted with this tractor in order to fully utilize this new vehicle. However, note that since the first part of the utility function given above already rewards the increase in consumption, indeed, the total effect of an increase in consumption is positive for the plausible forms of $(\Omega(\kappa_i(t)))$.

$\Omega : \mathbb{R} \to \mathbb{R}$, is continuous, twice continuously differentiable, strictly increasing, strictly convex and $\Omega'(.)$ is invertible.

The current-value Hamiltonian associated with the optimization problem $\mathcal{P}'''$ writes as

$$
\begin{aligned}
H(c_i(t), k_i(t), \kappa_i(t), \lambda_i(t), \mu_i(t)) &= U(c_i(t)) - \Omega(\kappa_i(t)) + \\
&\lambda_i(t) \left(f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t) \right) + \mu_i(t)\kappa_i(t).
\end{aligned}
$$

In order to make the first order optimality conditions for $\mathcal{P}'$ to be sufficient, as in Chapter 2 we need to assume further that the factor income of each household is a concave function of its own capital.

The function $k_i(t) \mapsto k_i(t)f'(K(t))$ is concave for all $i \in \{1, ..., N\}$ and for all $t \geq 0$.

The application of the Pontryagin's maximum principle leads to the following system of $4N$ differential equations: $\forall i \in \{1, 2, ..., N\}, \forall t \geq 0$,

$$\dot{\mu}_i(t) = \mu_i(t)\rho + \lambda_i(t) - U^{'}(c_i(t)), \quad (13)$$

$$\dot{\lambda}_i(t) = \lambda_i(t)\rho - \lambda_i(t)\left(f^{'}(K(t) + \Theta\left(k_i(t)\right)\right), \quad (14)$$

$$\dot{c}_i(t) = \Lambda\left(\mu_i(t)\right), \quad (15)$$

$$\dot{k}_i(t) = f^{'}(K(t))k_i(t) + \frac{f(K(t)) - K(t)f^{'}(K(t))}{N} - c_i(t), \quad (16)$$

where $\Theta\left(k_i(t)\right) = f''(K(t))\left(k_i(t) - \frac{K(t)}{N}\right)$. Note from (15) that $\Lambda^{'}\left(\mu_i(t)\right) = \frac{1}{\Omega''(\kappa_i(t))}$ by the implicit function theorem[5]. Under Assumptions 1-5, the first order optimality conditions are not only necessary but also sufficient if the transversality conditions, $\lim_{t\to 0} e^{-\rho t}\lambda_i(t)k_i(t) = 0$, and $\lim_{t\to 0} e^{-\rho t}\mu_i(t)c_i(t) = 0$ hold for all $i \in \{1, ..., N\}$.

A steady state is defined by $(\mu_i, \lambda_i, k_i, c_i)$ such that the right-hand sides of the system of differential equations (13)-(16) equal to zero for all $i \in \{1, 2, ..., N\}$. The following proposition is devoted to the existence and uniqueness of a symmetric steady state.

Proposition 4 *There exists a unique symmetric steady state and there are no asymmetric steady states in the strategic Ramsey model augmented with the adjustment cost of consumption.*

Proof. Note from (14) that $\dot{\lambda}_i(t) = 0$ for all $i \in \{1, ..., N\}$ if and only if $\Theta\left(k_i\right) = \rho - f^{'}(K)$, for all $i \in \{1, 2, ..., N\}$. We have then $f''(K)\left(k_i - \frac{K}{N}\right) = f''(K)\left(k_j - \frac{K}{N}\right)$ for all $i, j \in \{1, 2, ..., N\}$, $i \neq j$. As $f'' < 0$, this implies $k_i = \frac{K}{N}$, for all $i \in \{1, 2, ..., N\}$ so that the emergence of an asymmetric steady state is ruled out. The existence of a unique symmetric steady state then follows easily from the monotonicity and the limit properties of the right-hand side of equation (14). ■

It is important to note that the model with adjustment cost of consumption reduces to the standard Ramsey model at the steady state thanks to the price taking assumption of the competitive equilibrium. Because of this, the qualitative prop-

[5] $\mu_i(t) = \Omega'\left(\kappa_i\left(t\right)\right)$ and as $\Omega'\left(.\right)$ is invertable, then $\Lambda\left(\mu_i(t)\right) = \kappa_i\left(t\right)$. Thus, $\Omega'\left(\Lambda\left(\mu_i(t)\right)\right) - \mu_i(t) = 0$. The first derivative will give the result.

erties of the competitive equilibrium of the standard Ramsey model will be carried over to the Ramsey model augmented with the Easterlin hypothesis. Accordingly, as Van Long and Shimomura (2004) have shown for the standard Ramsey model, there exists a continuum of steady state wealth distributions and a corresponding continuum of one-dimensional stable manifolds in the Ramsey model augmented with consumption adjustment costs. In other words, the Ramsey model augmented with consumption adjustment costs predicts that the initial wealth differences will continue to persist in a competitive equilibrium environment.

However, to what extent the qualitative features of the strategic Ramsey model will carry over to the strategic Ramsey model with adjustment cost of consumption is not yet clear. Even though, we show that there exists a symmetric steady state, we need to analyze the dynamic properties of the associated system as well. The next section presents that the strategic interaction may induce cycles a la Hopf in the strategic Ramsey model with consumption adjustment costs.

3.2. Hopf Bifurcation and Easterlin Cycles

The analysis follows from Guckenheimer et al. (1997) that serves procedures for locating Hopf bifurcations in any $n-$dimensional system of ordinary differential equations grounded on the singularity of matrices stemming from the Jacobians' algebraic transformations at a steady state.

In what follows, for the sake of dimensional simplicity, we consider an economy with $N = 2$ households which differ in terms of initial level of capital stock. Recall from the system of differential equations (13)-(16) that we have now a system of 8 differential equations in terms of capital and consumption levels of the poor and the rich household. Let J denote the Jacobian of this system of equations around the unique symmetric steady state $(c^*, k^*, \lambda^*, \mu^*)$ and let $p(\omega)$ be the associated characteristic polynomial so that

$$p(\omega) = a_0 + a_1\omega + a_2\omega^2 + ... + a_7\omega^7 + a_8\omega^8.$$

$p(\omega)$ has the non-zero root pair $\{\omega, -\omega\}$ if and only if ω is a joint root of the two equations $p(\omega)+p(-\omega)=0$ and $p(\omega)-p(-\omega)=0$. Substituting $z=\omega^2$, construct two new polynomials:

$$r_e(z) = a_0 + a_2 z + a_4 z^2 + a_6 z^3 + z^4, \quad (17)$$

$$r_o(z) = a_1 + a_3 z + a_5 z^2 + a_7 z^3. \quad (18)$$

Then p has a non-zero root pair $\{\omega, -\omega\}$ if there exist a z that satisfies $\binom{r_e(z)}{r_o(z)} = 0$. We have the Sylvester matrix of two equations (17) and (18) as the 7×7- matrix given by

$$S = \begin{bmatrix} a_0 & a_2 & a_4 & a_6 & 1 & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & a_6 & 1 & 0 \\ 0 & 0 & a_0 & a_2 & a_4 & a_6 & 1 \\ a_1 & a_3 & a_5 & a_7 & 0 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & a_7 & 0 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 & a_7 & 0 \\ 0 & 0 & 0 & a_1 & a_3 & a_5 & a_7 \end{bmatrix}$$

For $m \in \{0,1\}$, let S_m be the matrix achieved from the Sylvester matrix by deleting the first and fourth rows and the columns 1 and $m+2$. By means of the relationship among the characteristic polynomial and its corresponding matrices S, S_0, S_1, Guckenheimer, et al. (1996) provides the following result.

Theorem 5 ***(Guckenheimer, et al., 1997)*** *If S is the Sylvester matrix for the polynomials r_e and r_o,then J has exactly one pair of purely imaginary eigenvalues if* $\det[S]=0$ *and,* $\det[S_0]\det[S_1]>0$. *If* $\det[S]\neq 0$ *or* $\det[S_0]\det[S_1]<0$, *then* $p(\omega)$ *has no purely imaginary roots.*

This procedure for locating Hopf bifurcations is specifically constructed to find points at which the Jacobian of the system leads to two purely imaginary eigenvalues . If the conditions of the Theorem 5 are satisfied, the magnitude of the shared root,

$\sqrt{\det[S_1]/\det[S_0]}$ can be easily related to the period of the limit cycle created at the Hopf bifurcation point.

Then the main task is to show a parameter constellation under which the implementation of Theorem 5 yields a pair of purely imaginary eigenvalues. To do so, we adopt the standard *Cobb-Douglas* production function, $f(K) = AK^\gamma$ and $CRRA$ form of utility from consumption, $u(c) = \frac{c^{1-\theta}}{1-\theta}$. Following Wirl (1996), the adjustment cost of consumption is represented by a convex second degree penalty function, $\Omega(\kappa_i(t)) = \frac{1}{2}\psi\kappa_i(t)^2$. We consider the following fairly standard values for the parameters:

$$A = 1, \quad \rho = 0.03, \quad \theta = 0.8, \quad \gamma = 0.3,$$

under which the unique symmetric steady state of the economy reveals: $k^* = 13.4135$. Since the parameter ψ does not affect the level of the unique symmetric steady state, it turns out to be an ideal bifurcation parameter. Indeed, by Theorem 5, setting $\psi = 1.3 \times 10^{-3}$ yields precisely one pair of pure imaginary eigenvalues. These two pure imaginary roots are crucial as it enables the existence of limit cycles in the locally unstable spirals' parameter domain for the parameters which are in the neighborhood of the bifurcation point[6].

Some remarks on the existence of limit cycles due to the strategic interaction are in order. First, it must be noted that the number of agents in the economy does not bring any qualitative change in the result. Second, it is important to note that complex dynamics emerge even with the separability of objective function in $\mathcal{P}'$ contrary to the earlier attempts to explain cyclical patterns (see e.g. Dockner and Feichtinger, 1993). Third, recall that the competitive equilibrium of the Ramsey model augmented with consumption adjustment costs leads to a unique asymmetric steady state that is saddle-path sable under these parameter values. This reveals that the initially poor household can never catch up with the rich. However, deviat-

[6]Indeed, Hopf bifurcation theorem ensures the existence of limit cycles, if a pair of purely imaginary eigenvalues exist and if the velocity when crossing the imaginary axis is nonzero. See Guckenheimer and Holmes, 1983.

ing from the competitive equilibrium by taking into account the strategic interaction among agents in the economy not only leads to complex wealth distribution but also complex dynamics as well. Put differently, the strategic interaction brings out the emergence of Hopf bifurcation that leads to limit cycles in Ramsey model with consumption adjustment costs. The emergence of such cyclical patterns of consumption could provide an alternative source for explaining real business cycles (see e.g. Wirl et al., 2008).

The last but not the least, the importance of these results is further emphasized if one recalls that sufficiently large consumption adjustment costs may induce complex, in particular, cyclical policies if there exist positive contributions of capital to utility in an optimal growth framework (see e.g. Wirl, 1994). However, we show that when households use open loop strategies rather than being price takers, complex dynamics may emerge even without capital in utility at very low levels of adjustment costs. In this respect, we show that structurally very simple frameworks may lead to limit cycles thanks to the strategic interaction among agents in the economy.

3.3. Conclusion

A slight departure from the competitive economy by incorporating the open-loop strategies not only leads to a change in the distribution of wealth in the long run but also the transitional dynamics substantially. Indeed, the strategic interaction leads to the emergence of Hopf bifurcation that reveals limit cycles in Ramsey model with consumption adjustment costs. The results are robust to the number of agent in the economy and contrary to the earlier attempts to explain cyclical patterns Hopf bifurcation (a complex dynamic) emerge even under the seperable utility functions.

CHAPTER 4

ENDOGENOUS REFERENCE LEVEL OF CONSUMPTION

This paper introduces Stone-Geary Preferences with an endogenous reference level of consumption in an otherwise standard Ak growth model. The reference level of consumption serves as a self-assessed reservation level as the agent cannot handle a decrease in his consumption below this level. Such a formulation allows to capture the utility that an agent takes from consumption when the consumption is not only above a constant threshold level but also above a dynamic threshold.

We have shown that depending on the relationship between the curvature of the reference level of consumption function and the net to capital, the resulting equilibrium presents richer dynamics under such a Stone-Geary preferences. In particular, we prove that endogenous reference level of consumption posits both global and local indeterminacy: economies starting with different initial conditions does not necessarily converge to the same steady state and also economies starting with the same initial conditions does not necessarily follow the same transition path. Accordingly, we show that fluctuations due to self-fulfilling expectations may emerge thanks to Stone-Geary preferences with endogenous reference level of consumption. There are bunch of studies in the literature that come up with local indeterminacy. However, most of them rely on either more than one control variable in addition to consumption (e.g., labor leisure choice, environmental quality, etc.) or increasing returns (see, among others, Benhabib and Farmer, 1994) that creates a wedge between social and private returns (e.g., Benhabib and Nishimura, 1996) or a variable mark-up (e.g., Woodford, 1991).

The growth literature broadly benefited from Stone-Geary preferences to analyze both theoretical and empirical questions. However, little attention has been devoted to explore the constancy of reference level of consumption. To the best of our knowledge, the study by Alvarez and Diaz (2005) is the first attempt which use SGP with endogenous reference level of consumption[7]. However, they advocate that the subsistence level of consumption depends on the exogenous growth rate of the economy. Dalgaard and Strulik (2007) develop a bioeconomic Malthusian growth model in which they use reference as the subsistence level of consumption and it is endogenously linked to the body size and fertility and show that the take-off into sustained growth should be associated with increasing income, population and the body size. The concept of reference level of consumption at micro level can be extended to the concept of poverty line at macro level. The poverty line can be proxied as the estimated minimum level of income needed to secure the fundamental needs. Indeed, Manfred Max-Neef (1992) states that "What changes both over time and through economic systems is the way or the means by which the fundamental needs are satisfied and what is economically determined is not the fundamental needs such as shelter and food but the satisfiers for these needs". This actually suggest not to speak of poverty, but of poverties (see Figure 1, Ravallion, 2010).

If poverty line were a static and least-cost dietary requirement of the households, one would have expected the same level of poverty line measured in real purchasing power parity term for different countries. However, as can be clearly seen from Figure 1 (Ravallion, 2010), there is a strong positive relationship between poverty line and the income of countries. This stylized fact stems from the dynamic nature of the satisfiers of the basic needs. For instance, while the access to the electricity, irrigation, clean water, road infrastructure etc., are perceived as luxuries by a poor

[7] Actually, they use subsistence level of consumption instead of reference level.

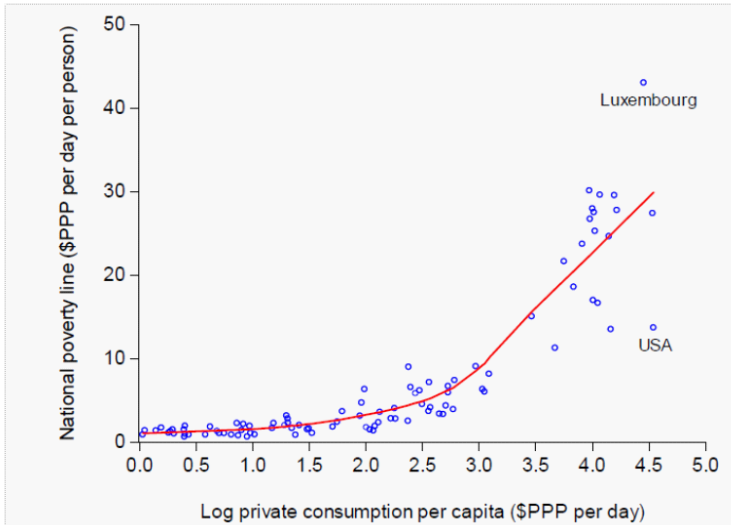

Figure 1: Poverty Lines - Income Per Capita (Source: Ravallion, 2010)

society, they are considered as necessities in a more developed economy. In this respect, Rawn et al. (2008) suggest that a more plausible way of defining necessities should include those dictated by the social norms and offer that the reference level should be considered as an increasing function of the long-run measures of output.

The paper is organized as follows. In the next section, we present the model. The existence of a stationary equilibria is analyzed systematically in Section 3. Section 4, discusses the local dynamics and provide interpretations of our findings. Finally, in section 5, we study the existence of balanced growth path.

4.1. The Model

Consider an economy inhabited by a representative agent whose preferences are represented by an instantaneous utility function of the Stone-Geary form and the time preference rate $\rho > 0$. The output is produced using capital k by a linear production function with productivity A. As such, the model generalizes Strulik (2010) by endogeneizing the reference level of consumption via wealth.

The infinitely-lived agent solves the following problem:

$$\underset{c_t}{Max} \int_0^\infty e^{-(\rho-n)t} \frac{(c(t) - \bar{c}(k(t)))^{1-\sigma}}{1-\sigma} dt \qquad (\mathcal{P})$$

subject to

$$\dot{k}(t) = (A - \delta - n) k(t) - c(t), \ \forall t \geq 0,$$
$$c(t) \geq \bar{c}(k(t)), \ k(t) \geq 0, \ \forall t \geq 0,$$
$$k(0) = k_0 > 0, \ given,$$

where σ is the intertemporal elasticity of substitution, δ is rate of depreciation, and n is the constant population growth rate. $\bar{c}(k)$ determines the reference level of consumption as a function of capital stock above which the instantaneous utility of the agent is well-defined. The value of utility function converges to finite value ensured by $\rho > n$ and net rate of return of capital is positive, i.e. $A - \delta > n$. Following Figure 1 (Ravallion, 2010) which shows clear convex relationhip between poverty line and per capita GDP, we assume that $\bar{c} : \mathbb{R}_+ \rightarrow \mathbb{R}$, is continuous, twice continuously differentiable, strictly increasing, and strictly convex satisfying that $\lim_{k \rightarrow 0} \bar{c}(k) = \underline{c} > 0$.

The first order conditions for a solution of $\mathcal{P}$ lead to the following system of differential equations:

$$\dot{c}(t) = \frac{1}{\sigma}(A - \delta - \rho - \bar{c}'(k(t)))(c(t) - \bar{c}(k(t))) + \bar{c}'(k(t))\dot{k}(t), \quad (19)$$
$$\dot{k}(t) = (A - \delta - n) k(t) - c(t), \quad (20)$$

where Euler equation now includes additional terms compared with that of a standard Ak growth model. The evolution of the consumption not only depends on the parameters of the model but also on the curvature of the reference level of consumption function, and the evolution of the capital stock. Note that, the effect of a change in capital stock on the evolution of consumption is ambiguous. Assume that $\dot{k}(t) > 0$, it means that the capital stock is increasing. Such an increase in capital stock caueses an increase in consumption as well because of the second term in the Euler equation. However, the $c(t) - \bar{c}(k(t))$ will decrease for a given level of consumption since $\bar{c}(k(t))$ is an increasing function of the $k(t)$. Furthermore, $A-\delta-\rho-\bar{c}'(k(t))$ will also decrease because of the strict convexity assumption of the $\bar{c}(k(t))$. Hence, the first term of the Euler equation will decrease. Indeed, an increase in capital stock has one negative and one positive effects on the evolution of consumption. Thus, the net effect is indeterminate.

We can now define an equilibrium path of our model economy as a trajectory of the dynamical system (19) and (20), that satisfies the initial condition $k(0) = k_0$, the transversality condition $\lim_{t\to 0} e^{-(\rho-n)t}\lambda_i(t)k_i(t) = 0$, and the nonnegativity constraints, $c(t) \geq \bar{c}(k(t))$ and $k(t) \geq 0$ for all $t > 0$.

The evolution of the economic system as specified by (19) and (20) is now dependent on the size of the reference level of consumption. As a consequence, the path for economic rates such as consumption capital ratio, the savings rate are dependent on the form of the reference level of consumption function. With this distinctive feature of the capital dependent reference level of consumption, we will first show that the dynamics of such a system can encounter multiplicity of steady states and a balanced growth path.

4.2. Stationary Equilibria

A stationary solution of the dynamic system defined by (19)-(20) is given by a feasible tuple (c, k) such that

$$(i)\ \ (A-\delta-n)\,k = c, \ \text{ and}$$

$$(ii)\ A-\delta-\rho = \bar{c}'(k) \ \text{ or } c = \bar{c}(k). \tag{21}$$

In (21), condition (i) characterizes the steady state consumption capital ratio and condition (ii) implies that there exist two sets of steady states to consider. Note that, by assumptions on $\bar{c}(k(t))$ these conditions ensure the existence of a strictly positive consumption and capital at any steady state.

Proposition 6 $(\underline{c}, 0)$ *is not a steady state.*

Proof. As $\lim_{k\to 0} \bar{c}(k) = \underline{c} > 0$, we have $c = \underline{c} > 0$. However, $k = 0$ implies $c = 0$. A contradiction. ■

The strict convexity of $\bar{c}(.)$ implies that there exists a unique steady state (c, k) at which $A - \delta - \rho = \bar{c}'(k)$ and there exist at most two steady states at which consumption equals to the reference level of level, i.e., $c = \bar{c}(k)$.

Proposition 7 *There exists a unique steady state* (c, k) *that satisfies*

$$(A - \delta - n)\, k = c, \text{ and } A - \delta - \rho = \bar{c}'(k).$$

There can be at most two steady states (c_s, k_s) *that satisfies*

$$(A - \delta - n)\, k_s = c_s, \text{ and } c_s = \bar{c}(k_s).$$

Proof. Directly comes from the strict convexity of $\bar{c}(k)$. ■

4.3. Local Dynamics

In this section, we study the stability properties of the system around the two generic steady states and show that the wealth dependent reference level of consumption raises the possibility of threshold dynamics and the possibility of multiple equilibrium paths for given initial conditions (local indeterminacy). In particular, we analyze the occurrence of local indeterminacy and endogenous cycles.

Case 8 *Let* $(c, k) \gg 0$ *be the unique steady state at which* $(A - \delta - n)\, k = c,$ *and* $A - \delta - \rho = \bar{c}'(k)$. *Linearizing (Appendix C) (19) and(20) around* (c, k)*, gives the trace* T *and the determinant* D *of the associated Jacobian matrix, which respectively present the sum and the product of the two eigenvalues of the characteristic polynomial* $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$:

$$\begin{aligned} T &= \rho - n > 0, \\ D &= \frac{1}{\sigma}\bar{c}''(k)\left(c - \bar{c}(k)\right) \geq 0. \end{aligned}$$

Depending on the sign of the discriminant $\Delta = T^2 - 4D$, *the steady state is either a source or a spiral source.*

Now we consider the cases that will characterize the trivial steady state(s) at which consumption equals to the reference level and the net rate of return to capital equals to the reference level of consumption to capital ratio.

Case 9 *Let* (c_s, k_s) *be the unique solution to* $(A - \delta - n)\, k_s = \bar{c}(k_s)$. *This necessarily implies that* $(A - \delta - n) = \bar{c}'(k_s)$. *Linearizing (19) and (20) around* (c_s, k_s), *gives the following trace* T *and the determinant* D *of the associated Jacobian matrix:*

$$\begin{aligned} T &= \frac{1}{\sigma}(n - \rho) < 0, \\ D &= 0. \end{aligned}$$

In this case, the solution to the zero eigenvalue is a constant and the dynamics are simply determined by the sign of the trace. However, there is no longer a unique equilibrium but a continuum of equilibria whose locus is the straight line through the steady state.

Case 10 *Let* $(\bar{c}(k_l), k_l)$ *and* $(\bar{c}(k_h), k_h)$ *be the two steady states at which* $(A - \delta - n)\, k = \bar{c}(k)$. *Without loss of generality, assume that* $k_l < k_h$. *We have then* $\bar{c}'(k_h) > A - \delta - n > \bar{c}'(k_l)$.

i) Linearizing (19) and (20) around $(\bar{c}(k_l), k_l)$, *gives the following trace* T *and the determinant* D *of the associated Jacobian matrix:*

$$\begin{aligned} T &= \frac{1}{\sigma}(A - \delta - \rho - \bar{c}'(k_l)) + (A - \delta - n - \bar{c}'(k_l)), \\ D &= \frac{1}{\sigma}(A - \delta - \rho - \bar{c}'(k_l))(A - \delta - n - \bar{c}'(k_l)). \end{aligned}$$

If $\bar{c}'(k_l) < A - \delta - \rho$ *then* $T > 0$, $D > 0$ *and* $\Delta > 0$ *so that the low steady state will be an unstable node. If* $\bar{c}'(k_l) > A - \delta - \rho$ *then* $D < 0$ *which implies* $\Delta > 0$ *independent from the sign of* T. *Accordingly, there exists a one-dimensional stable and a one-dimensional unstable eigenspace, hence the steady state is a saddle.*

ii) As $\vec{c}'(k_h) > A - \delta - n$ and $\rho > n$, we have $\vec{c}'(k_h) > A - \delta - \rho$ at the high steady state. In this case, we have then $T < 0$, $D > 0$ and $\Delta > 0$ so that the high steady state will be locally indeterminate (i.e., a stable node).

Note that low steady state is unstable or saddle path stable, whereas dynamics of high steady state leads to fluctuations due to self-fulfilling prophecies, i.e. local indeterminacy. First of all and most importantly, standard SGP with Ak type growth model have a counterfactual implication that economies starting with same initial conditions will converge the same steady state following the same path. Accordingly, we show that fluctuations due to self-fulfilling expectations may emerge thanks to Stone-Geary preferences with endogenous reference level of consumption. There are bunch of studies in the literature that come up with local indeterminacy. However, most of them rely on either more than one control variable in addition to consumption (e.g., labor leisure choice, environmental quality, etc.) or increasing returns (see, among others, Benhabib and Farmer, 1994) that creates a wedge between social and private returns (e.g., Benhabib and Nishimura, 1996) or a variable mark-up (e.g., Woodford, 1991). However, we show that under very simple Stone-Geary preferences with endogenous reference level of consumption, local indeterminacy may emerge. Moreover, the external effect of capital stock through endogenous reference level of consumption on utility function simply leads not only to the local indeterminacy but also multiplicity of steady states.

The simple mechanism behind the local indeterminacy is as follows: Assume that representative agent deviates from the steady state by an increase in capital stock. The increase of capital stock leads reference level of consumption (increasing convex function in capital) to rise. If marginal disutility of reference level of consumption due to the higher capital stock is greater than the net return of capital, then the consumption will increase more than capital stock so as to compensate disutility erased from higher reference level of consumption. Then, this will decrease the capital stock afterwards. A decrease in capital stock leads to a decrease in reference level of consumption and therefore consumption. Hence, capital stock will increase and so on.

Remark 11 *(Balanced Growth Path) Under strict convexity assumption of reference level of consumption function, balanced growth path does not exist. A balanced growth path can be defined for the system of (19) and (20) such that,* $c(t)$ *and* $k(t)$ *grow with a constant rate, i.e.,* $\frac{\dot{c}(t)}{c(t)} = g_c$, $\frac{\dot{k}(t)}{k(t)} = g_k$. *Under such an equilibrium definition,*

$$g_k = (A - \delta - n) - \frac{c(t)}{k(t)}, \tag{22}$$

$$g_c = \frac{1}{\sigma}\frac{(A - \delta - \rho)(c(t) - \bar{c}(k(t))) - ((A - \delta - n)k(t) - \bar{c}(k(t)))\bar{c}'(k(t))}{c(t)} \tag{23}$$

Note that from the right hand side of the (22) , constant growth rate of capital stock requires that consumption and capital should grow at the same rate, i.e. $\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = g$. *Then, simple substitution and calculation from (23) implies that in order to have constant growth rate for* $c(t)$ *and* $k(t)$, $\bar{c}'(k(t))$ *should be constant indicating a linear relationship between reference level of consumption and capital stock. In this case, the model reduces to that of Steeger (2000), analytically. The time path of growth rate of capital , which is equal to the growth rate of consumption and income, monotonically increases and converges towards its asymptotic balanced-growth equilibrium value. However, the growth rate is lower than that of Steeger (2000).*

4.4. Conclusion

This paper introduces Stone-Geary Preferences with an endogenous reference level of consumption in an otherwise standard Ak growth model. We have shown that depending on the relationship between the curvature of the reference level of consumption function and the net to capital, the resulting equilibrium presents richer dynamics under such a Stone-Geary preferences. In particular, we prove that endogenous reference level of consumption posits both global and local indeterminacy: economies starting with different initial conditions does not necessarily converge to the same steady state and also economies starting with the same initial conditions does not necessarily follow the same transition path. Accordingly, we show that fluctuations due to self-fulfilling expectations may emerge thanks to Stone-Geary preferences with endogenous reference level of consumption.

CHAPTER 5

PUBLIC GOOD AND SUBSISTENCE LEVEL OF CONSUMPTION

The public spending can be classified into two groups. These are "money tied up in redistribution" and "provision of public good". The former is pursued by progressive taxation or social aids to the poor to increase their welfare and it is desirable only if it is fairly determined. However, a common complaint about the redistribution is the wasteful result due to the cheating by politicians or political candidates. In this sense, the provision of public goods addresses a larger mass of people since it does not involve such subjective implementation. Since the provision of public goods is more fairly determined than the redistribution, the subset of people benefitting from public goods is larger than that of it benefits from the redistribution. Therefore, in this paper we focus on the effects of the provision of public goods which address a larger group of people.

There are two different aspects of the provision of public goods. Not only it is one of the key determinants of productive capability but also it is a key determinant of the well-being of individuals. Although there is quite a number of studies that analyze the effects of the provision of public goods on productive capability, the utility enhancing feature of public goods is generally ignored. The traditional approaches to poverty measurement generally focus on the private goods rather than public goods. The aim of this paper is to analyze the quantitative and qualitative effects of a pure public good that satisfy the basic needs of the individuals and hence, provides indirect utility through decreasing the subsistence level of consumption.

Besley and Ghatak (2011) argue that when public goods are taken into account, individuals that seem to enjoy similar amount of private goods actually have very different standards of living. More importantly, public good provision is crucial for the low income groups which have a limited consumption of private goods. Recently, policy makers start to concentrate on the delivery and the level of public goods as it becomes more prominant for the poor. For example, in the World Bank Financing for Development Post-2015, it is argued that under provision of global public goods, such as controlling contagious diseases and preserving environment, impact the poor disproportinately and therefore they should be taken into account in regional and national development policies. Why is the level and the delivery of public good is more important for the poor? The rich satisfies its needs by choosing among different alternatives of private goods, whereas the poor does not have this opportunity. Then the next question arises: What characteristics of the public goods make them more important for the poor? Although some of the public goods such as education, law and order, infrastructure play a dual role for both private resourse allocation and social welfare, public goods such as free tap water, public transportation, public gardens, child vaccination programs satisy the basic necessities of the individuals. Since poor is more constrained than the rich to satisfy these basic needs it is more important for the poor to receive the optimal amount of public goods.

By using the data of 26 provinces in Thailand, Panudulkitti (2000) shows that the price elasticity of demand for public good is less than one in absolute value. Similarly, Chu (2001) shows that when the environmental factors[8] are excluded, the income elasticity of demand for the public goods is equal to nearly 0.5, when environmental factors are included it is around 0.8. In other words, the public goods can be considered as "necessities".

The other important feature of the public goods is the "the congestion effect". The traditional literature on public goods argues that given the quantity of public

[8] Hamilton (1983) argues that a good proxy for the environmental factors is the average income of the society.

services and goods, the available amount for an individual declines as the others use those facilities. More concretely, the amount of available public goods and services that enters the utility function takes the following form: $G^* = G.N^\omega$ where G denotes the total supply of public goods for a given province and N is the population size. $0 \leq \omega \leq 1$ measures the degree of congestion and it is greater than or equal to zero. If it is zero, there is no congestion (pure public good) and if ω equal to 1 the congestion is equal to that of usual private good. In this sense, assuming ω equals to one indicates that what literature deals with is a kind of quasi-public good. However, Oates (1988) argues that the empirical evidence suggesting a statistically significant congestion effect is subject to measurement problems and public goods actually have the property of publicness (pure public good). Similarly, Barro (1990) and Futagami et al. (1993) handle government expenditure as a pure public good, they do not take into account of the congestion effect associated with public goods.

In this study, we discover the effects of pure public goods that satisfy the basic needs of the individuals on the economy. The standard growth models incorporate welfare enhancing public goods into the utility function either additively seperable or multiplicative forms. Thus, in such models private and public goods are taken as perfect substitutes or complements. Since the private good in usual growth models is a kind of composite good, incorporating the public good that satisfies bacis needs of indiviudals make such formulation inadequate. Therefore, we use Stone-Geary type preferences. The Stone-Geary preferences are traditionaly used to study the models involving subsistence level of consumption. Individuals take utility from consumption if it is above a threshold. This threshold level can be considered as the minimum level of consumption that satisfy the basic needs. As we have mentioned above, pure public goods do not have the property of congestion and they are actually goods that satisfy the basic needs. Since the absence of public goods forces individuals to consume this minimum amount from their own budget, an increase in the provision of public goods actually decreases the threshold level above which private consumption gives utility. In other words, the public good provision increases the disposable income of individuals indirectly through leaving room for private good consumption from the individual's budget.

In this study, we analyze the effects of public good in an otherwise standard neoclassical growth model and show that, although the steady state amount of public good is higher for the first best allocation, the subsistence level of consumption is the same with that of the second best equilibrium. On the other hand, the capital stock and the consumption of the private good are higher for the first best equilibria. There is an inverse relationship between the optimal amount of tax rate and the share of capital in the solution of second best allocation. The same result is valid between optimal tax rate and total factor productivity. However, for the social planner, the share of capital in production and total factor productivity do not matter for the allocation of public good, capital stock and consumption of private good. Another important result of the paper is the "$g - \tau$ *locus*" with a dynamic threshold which depends on the total factor productivity. This means that the optimal amount of tax rate that maximizes the total revenue of the government is an increasing function of the total factor productivity and thus revenue maximizing tax rate varies across countries. Furhetmore, steady states for both first and second best allocations are stable in the saddle path sense.

The Chapter is organized as follows: Section 5.2 characterises the second best equilibrium, analyzes the Laffer similar curve and provides numerical results of steady state. Section 5.3 studies the first best allocation. Section 5.4 compares the results in Section 5.2 and Section 5.3 . In section 5.5, we conclude.

5.1. The Model

The closed economy is populated by a government and a representative household which ownes a firm. The representative household supplies the factors of production, capital and labor. The labor is normalized to one and there is no population growth. The public good is supplied by the government in order to satisfy the basic necessities of the household by reducing the threshold level of consumption above which the household starts to take utility. In other words, public good provides utility indirectly in contrast to prevalent literature which assumes public good as a direct source of utility. Accordingly, we assume Stone-Geary preferences in which the minimum level of consumption is affected negatively from the level of public good provision.

5.1.1. Household

Taking the interest and the wage rate as given, the representative household maximizes the discounted life-time utility subjet to the usual budget constraint:

$$\underset{\{c_t\}}{Max} \int_0^\infty e^{-\rho t} \frac{\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{1-\sigma}}{1-\sigma} dt \tag{P}$$

subject to

$$\begin{aligned} \dot{k}\left(t\right) &= \left(1-\tau\right)\left(r(t)k\left(t\right) + w(t)\right) - c(t) - \delta k\left(t\right), \\ c\left(t\right) &\geq \bar{c}\left(g\left(t\right)\right),\ k\left(t\right) \geq 0,\ \forall t \geq 0, \\ k(0) &= k_0 > 0,\ given. \end{aligned}$$

$c\left(t\right)$ is the consumption of private good, $g\left(t\right)$ is the stock of the public good. ρ is the discount factor and σ measures the relative risk aversion. $0 \leq \tau < 1$ is the tax rate imposed on income of the agent and taken as given by the agent, as well. The δ is the usual depreciation rate of capital. $r(t)$ and $w(t)$ are interest and the wage rate, respectively. The subsistence level of consumption, the level of consumption above which the household starts to take utility, is a decreasing and convex function of the public good that satisfies

$$\lim_{g(t)\to 0} \bar{c}\left(g\left(t\right)\right) = \overline{\gamma}, \text{ and } \lim_{g(t)\to\infty} \bar{c}\left(g\left(t\right)\right) = 0.$$

The Hamiltonian associated with (P) is as follows:

$$H = \frac{\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{1-\sigma}}{1-\sigma} + \lambda\left(t\right)\left[\left(1-\tau\right)\left(r(t)k\left(t\right) + w(t)\right) - c(t) - \delta k\left(t\right)\right].$$

The first order conditions of optimality are given by

$$\frac{\partial H}{\partial c\left(t\right)} = \left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{-\sigma} - \lambda\left(t\right) = 0, \tag{24}$$

$$\frac{\partial H}{\partial k\left(t\right)} = \lambda\left(t\right)\left(\left(1-\tau\right)r(t) - \delta\right) = -\lambda\left(t\right) + \rho\dot{\lambda}\left(t\right), \tag{25}$$

$$\frac{\partial H}{\partial \lambda\left(t\right)} = \left(1-\tau\right)\left(r(t)k\left(t\right) + w(t)\right) - c(t) - \delta k\left(t\right) = \dot{k}\left(t\right). \tag{26}$$

Together with the transversality condition $\lim_{t\to\infty} \lambda\left(t\right)k\left(t\right) = 0$, the first order conditions are necessary and sufficient for optimality. Taking the time derivative of (24) leads to the following differantial equation for the co-state:

$$\dot{\lambda}\left(t\right) = -\sigma\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{-\sigma-1}\dot{c}\left(t\right) + \dot{g}\left(t\right)\bar{c}'\left(g\left(t\right)\right) \tag{27}$$

The Euler equation can then be derived by substituting equation (24) and (27) into (25) as

$$\dot{c}\left(t\right) = \frac{1}{\sigma}\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)\left(\left(1-\tau\right)r(t) - \delta - \rho\right) + \dot{g}\left(t\right)\bar{c}'\left(g\left(t\right)\right). \tag{28}$$

Note that the evolution of consumption depends not only on the interest rate, depreciation rate and the discount factor but also on the change in the level of public good provision and the slope of the function that determines the subsistence level of consumption. Note that if $\dot{g}\left(t\right)$ is positive, i.e., $g\left(t\right)$ is increasing, the subsistence level of consumption will be reduced due to $\bar{c}'\left(g\left(t\right)\right) < 0$, but $c\left(t\right) - \bar{c}\left(g\left(t\right)\right)$ will also increase.

5.1.2. Firm

Taking the interest and the wage rate as given, the representative firm maximizes its profit in a perfectly competitive environment;

$$\underset{\{K(t),L(t)\}}{Max} F(K(t), L(t)) - r(t)K(t) - w(t)L(t)$$

The price of private good is normalized to 1. Assuming a homogenous of degree one production function, we have:

$$F\left(\frac{K(t)}{L(t)}, 1\right) = f(k(t)) \tag{29}$$

Thus, the equilibrium rate of return on capital and wage rate can be obtained by

$$r(t) = f'(k(t)), \tag{30}$$

$$w(t) = f(k(t)) - f'(k(t))k(t). \tag{31}$$

5.1.3. Goverment

The government collects taxes and provides a public good that reduces the subsistence level of consumption. The public good is subject to depreciation, the rate of which is equal to that of the private capital[9]. We consider that the government runs a balanced budget in each period and it can not borrow.

The dynamic equation that governs the evolution of the stock of the public good is given by

$$\dot{g}(t) = E(t) - \delta g(t), \tag{32}$$

where $E(t)$ is the provision of public good so that

$$E(t) = \tau(r(t)k(t) + w(t)). \tag{33}$$

[9]Note that, if $\delta = 1$, then the public good turns out to be a flow variable. Among others, Barro (1990), Turnovsky and Fisher (1995), Corsetti and Roubini (1996) and Fiasci (1999) use public good as a flow variable, whereas Futugami et. al.(1993), Turnovsky (1997, 2004), Dasgupta (1999) take public good as a stock variable along the lines of Arrow and Kurz (1970).

5.2. Competitive Equilibrium

Definition 12 *1: A competitive equilibrium is given by the set of prices* $\{r(t), w(t)\}$ *and the quantities* $\{c(t), k(t), g(t), \bar{c}(g(t))\}$ *such that;*

i) Given $\{r(t),\ w(t),\ g(t),\ \tau\}$ *and the initial capital-labour ratio* $k(0)$*, representative agent maximizes discounted life-time utility,*

ii) The pair $\{K(t), L(t)\}$ *solves the profit maximization problem of the firm given the factor prices* $\{r(t),\ w(t)\}$,

iii) The government 's budget is balanced in each period, i.e, $E(t) = \tau (r(t)k(t) + w(t))$,

iv) All markets clear.

From the first order conditions of the representative household, the representative firm, and the budget constraint of government, the dynamics of the economy will be governed by the following three equations:

$$\dot{c}(t) = \frac{1}{\sigma}(c(t) - \bar{c}(g(t)))((1-\tau) f'(k(t)) - \delta - \rho) + (\tau f(k(t)) - \delta g(t)) \bar{c}'(g(t)), \quad (34)$$

$$\dot{k}(t) = (1-\tau) f(k(t)) - c(t) - \delta k(t), \quad (35)$$

$$\dot{g}(t) = \tau f(k(t)) - \delta g(t). \quad (36)$$

5.3. The Second Best Equilibrium

To characterize the second best equilibrium, consider the problem of the government that chooses optimal tax rate τ to maximize the representative agent's utility subject to the dynamic equations (34)-(36). In other words, the dynamic equations associated with the competitive equilibrium constitute now the constraints of the government's optimization problem. Formally, we consider the following problem:

$$\underset{\{c_t\}}{Max} \int_0^\infty e^{-\rho t} \frac{(c(t) - \bar{c}(g(t)))^{1-\sigma}}{1-\sigma} dt \quad (P')$$

subject to

$$\dot{c}(t) = \frac{1}{\sigma}\left(c(t) - \bar{c}(g(t))\right)\left((1-\tau) f'(k(t)) - \delta - \rho\right) + \left(\tau f(k(t)) - \delta g(t)\right) \bar{c}'(g(t)),$$

$$\begin{aligned}
\dot{k}(t) &= (1-\tau) f(k(t)) - c(t) - \delta k(t), \\
\dot{g}(t) &= \tau f(k(t)) - \delta g(t), \\
c(t) &\geq \bar{c}(g(t)), \ k(t) \geq 0, \ \forall t \geq 0, \\
k(0) &> 0, \ \textit{given}.
\end{aligned}$$

The associated Hamiltonian of the problem (P') is as follows:

$$H(c(t), k(t), g(t), \lambda_c(t), \lambda_k(t), \lambda_g(t)) = \frac{\left(c(t) - \bar{c}(g(t))\right)^{1-\sigma}}{1-\sigma} + \lambda_c(t)\left[\frac{\left(c(t) - \bar{c}(g(t))\right)}{\sigma}\left((1-\tau) f'(k(t)) - \delta - \rho\right) + \left(\tau f(k(t)) - \delta g(t)\right) \bar{c}'(g(t))\right] + \lambda_k(t)\left[(1-\tau) f(k(t)) - c(t) - \delta k(t)\right] + \lambda_g(t)\left[\tau f(k(t)) - \delta g(t)\right].$$

The associated first order conditions are given by:

$$\frac{\partial H}{\partial \tau} = -\lambda_c(t)\frac{1}{\sigma}\left(c(t) - \bar{c}(g(t))\right) f'(k(t)) + \lambda_c(t) f(k(t)) \bar{c}'(g(t)) - \lambda_k(t) f(k(t)) + \lambda_g(t) f(k(t)) = 0 \quad (37)$$

$$\frac{\partial H}{\partial c(t)} = \left(c(t) - \bar{c}(g(t))\right)^{-\sigma} + \lambda_c(t)\frac{1}{\sigma}\left((1-\tau) f'(k(t)) - \delta - \rho\right) - \lambda_k(t) = -\dot{\lambda}_c(t) + \rho\lambda_c(t) \quad (38)$$

$$\frac{\partial H}{\partial k(t)} = \lambda_c(t)\left(c(t) - \bar{c}(g(t))\right)\frac{1}{\sigma}(1-\tau) f''(k(t)) + \lambda_c(t) f'(k(t)) \bar{c}'(g(t)) +$$

$$\lambda_k(t)(1-\tau) f'(k(t)) - \lambda_k(t)\delta + \lambda_g(t)\tau f'(k(t)) = -\dot{\lambda}_k(t) + \rho\lambda_k(t) \quad (39)$$

$$\frac{\partial H}{\partial g(t)} = -\left(c(t) - \bar{c}(g(t))\right)^{-\sigma} \bar{c}'(g(t)) -$$

$$\lambda_c(t)\bar{c}'(g(t))\frac{1}{\sigma}\left((1-\tau) f'(k(t)) - \delta - \rho\right) - \lambda_c(t)\delta\bar{c}'(g(t)) +$$

$$\lambda_c(t)\left[\tau f(k(t)) - \delta g(t)\right]\bar{c}''(g(t)) - \lambda_g(t)\delta = -\dot{\lambda}_g(t) + \rho\lambda_g(t) \quad (40)$$

In addition to these first order conditions, the second best equilibrium path should also satisfy the limiting transversality conditions, $\lim_{t\to\infty} e^{-\rho t}\lambda_k(t)k(t) = 0$ and $\lim_{t\to\infty} e^{-\rho t}\lambda_g(t)g(t) = 0$.

5.3.1. Steady States:

A steady state equilibrium is defined by the system of equations: $\dot{c}(t) = 0$, $\dot{k}(t) = 0$, $\dot{g}(t) = 0$, $\dot{\lambda}_c(t) = 0$, $\dot{\lambda}_k(t) = 0$, $\dot{\lambda}_g(t) = 0$. Note from the Euler equation,

$$\dot{c}(t) = \frac{1}{\sigma}\left(c(t) - \bar{c}(g(t))\right)\left((1-\tau) f'(k(t)) - \delta - \rho\right) + \left(\tau f(k(t)) - \delta g(t)\right)\bar{c}'(g(t)),$$

that we have either $(c - \bar{c}(g)) = 0$ or $((1-\tau) A\alpha k^{\alpha-1} - \delta - \rho) = 0$ at the steady state. Between these multiple steady state condiyions, the first one refers to the trivial steady state at which consumption equals to the subsistence level.

We first elaborate on the seady state at which the consumption is above the subsistence level, i.e, $c^* > \bar{c}(g^*)$ and $((1-\tau) f'(k^*) - \delta - \rho) = 0$, so that

$$\begin{aligned} k^* &= (f')^{-1}\left(\frac{\delta+\rho}{1-\tau}\right), \\ c^* &= (1-\tau) f(k(t)) - \delta (f')^{-1}\left(\frac{\delta+\rho}{1-\tau}\right), \\ g^* &= \frac{\tau}{\delta}(f')^{-1}\left(\frac{\delta+\rho}{1-\tau}\right), \\ \bar{c}(g^*) &= \frac{\overline{\gamma}}{\left(1+\frac{\tau}{\delta}(f')^{-1}\left(\frac{\delta+\rho}{1-\tau}\right)\right)^{\beta}}. \end{aligned}$$

Recall that f' is a monotonically decreasing function. Hence, $(f')^{-1}$ is also a strictly decreasing function. It is then clear that an increase in the tax rate increases the steady state level of capital stock. However, to consider the effects of an increase in tax rate on consumption and the level of public good provision, we need to specify the functional forms of the production and the subsistence functions:

$$\begin{aligned} f(k(t)) &= Ak(t)^{\alpha}, \ \alpha \in (0,1), \\ \bar{c}(g(t)) &= \frac{\overline{\gamma}}{(1+g(t))^{\beta}}, \ \beta \in (0,1), \overline{\gamma} > 0. \end{aligned}$$

The steady state values of the aggregate variables such as capital stock, consumption, public good, and the subsistence level of consumption can then be recast as follows:

$$k^* = \left(\frac{A\alpha(1-\tau)}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}, \tag{41}$$

$$c^* = (1-\tau) A \left(\frac{A\alpha(1-\tau)}{\delta+\rho}\right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{A\alpha(1-\tau)}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}, \tag{42}$$

$$g^* = \frac{\tau A}{\delta}\left(\frac{A\alpha(1-\tau)}{\delta+\rho}\right)^{\frac{\alpha}{1-\alpha}}, \tag{43}$$

$$\bar{c}^* = \frac{\overline{\gamma}}{\left(1+\frac{\tau A}{\delta}\left(\frac{A\alpha(1-\tau)}{\delta+\rho}\right)^{\frac{\alpha}{1-\alpha}}\right)^{\beta}}. \tag{44}$$

Note that the effects of the tax rate on the steady state values of the public good, capital stock and the consumption can be analyzed analytically. It is clear that

$\frac{\partial k^*}{\partial \tau} < 0, \forall\ \tau\ \epsilon\ (0,1)$, and we have

$$\begin{aligned} \frac{\partial g^*}{\partial \tau} &> 0, \text{ if } \tau < \frac{A}{1+A}, \\ \frac{\partial g^*}{\partial \tau} &< 0, \text{ if } \tau > \frac{A}{1+A}, \end{aligned} \tag{45}$$

and

$$\begin{aligned} \frac{\partial c^*}{\partial \tau} &> 0, \text{ if } \frac{\delta A \alpha}{\delta + \rho} > 2, \\ \frac{\partial c^*}{\partial \tau} &< 0, \text{ if } \frac{\delta A \alpha}{\delta + \rho} < 2. \end{aligned} \tag{46}$$

Equation (45) implies an inverted U-shaped relationship between public good and the tax rate. This result is in line with the Laffer Curve that suggests that an increase in the tax rate increases the tax revenue until a threshold level. However, a further increase in the tax rate above the threshold decreases the collected revenue. Since in our model, the revenue collected by taxes is directly spent for the provision of the public good as the government budget is assumed to be balanced, we obtain actually a "$g-\tau$" *locus*. The main mechanism in the traditional Laffer Curve is the labor supply effects of the changes in the tax rate. However, in our model even though the labor is supplied inelastically, there is a inverted-U relationship between the tax rate and the government revenue. More importantly, the "$g-\tau$" *locus* derived from this analysis is a dynamic curve which depends on TFP. The higher the TFP, the higher is the tax rate above which government revenue starts to decrease. In other words, the peak of the inverted-U curve shifts to right as TFP increases.

One of the main issues in the public policy that led to a great controversy is whether a decrease in the tax rate will push up the economic activity to an extent that government's revenue will actually increase. To obtain such self-financing tax cuts, static models with sufficiently high labor supply elasticities have been advocated. Bruce and Turnovsky (1999) argue that if the intertemporal elasticity

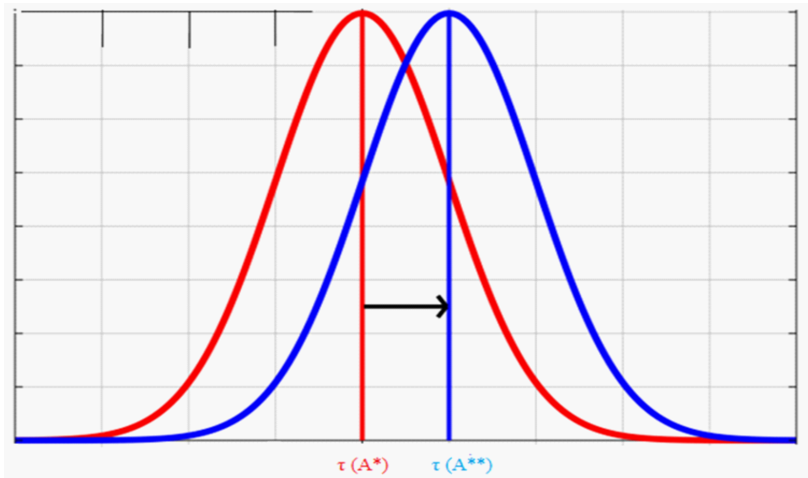

Figure 2: "$g - \tau$" *Locus*

of substitution (IES) is high enough (much higher than 1), then a decrease in the tax rate may improve the tax revenue. However, such a high IES is not empirically plausible. Moreover, Trabant and Uhlig (2011, 2012) show that, under both perfectly competitive market and monopolistic competition, the Laffer curves for labour and capital taxation for the U.S., EU-14, and the individual European countries may have different peaks. However, they assume that the structural parameters for preferences and technologies are the same for all countries and differences arise from the government debt and spending. On the contrary, the dynamic threshold in our model directly depends on the technology parameter even under an ineleastic labor supply. This interesting result suggests that "$g - \tau$" *locus* is not the same for different countries that have different TFP and also not constant for a country in different periods, as well. In additon to these, (46) implies that the tax rate have an inverted-U effect on the level of consumption. Accordingly, if total factor productivity and the share of capital in production is high enough, then an increase in tax rate causes an increase in consumption.

5.3.2. Stability and Comparative Statics

In order to characterize the local stability of the dynamic system, we need to linearize (**??**-40) and the budget constraints of the government around the steady state (41)-

(44) at which $c^* > \bar{c}(g^*)$. As the stability analysis of such a 7×7 system is hard to carry out, we first define an auxiliary variable $\vartheta(t) = \lambda_g(t) - \lambda_k(t)$ to obtain a $4 - tuple$ system in$\{c(t), k(t), g(t), \vartheta(t)\}_{t=0}^{t=\infty}$ and adopt the following benchmark parameterization in accordance with the literature:

$$A = 1,\ \alpha = 0.3,\ \delta = 0.06,\ \rho = 0.02,\ \beta = 0.5,\ \overline{\gamma} = 1.$$

Accordingly, we obtain the steady state values

$k^* = 5.85,$	$c^* = 1.21,$	$g^* = 2.3,$	$\bar{c}^* = 0.54,$	$\tau^* = 8.14\%,$

around which the linearization of the 4×4 system leads to 2 negative and 2 positive eigenvalues $(-1.07448, 0.542425, 0.293903, -0.0768328)$ that imply saddle path stability.

The depreciation rate and the discount factor are standard in the literature. The important parameters are β, $\overline{\gamma}$, α and the total factor productivity parameter A. The effect of one unit change in public good on subsistence level of consumption is measured with parameter β. The higher (lower) the β the the lower (higher) the subsistence level of consumption for a given level of public good. The $\overline{\gamma}$ is the highest subsistence level of consumption when there is no public good provision in the economy. Thus, it is important to look at the steady state values of the aggregate variables for different values of β and $\overline{\gamma}$. The equilibrium levels of the aggregate variables for different sets of β and $\overline{\gamma}$ are given in the table below:

β	k^*	c^*	g^*	$\bar{c}^*$	τ^*
0.3	6.05	1.25	1.70	0.74	5.96%
0.5	5.85	1.21	2.31	0.54	8.14%
0.7	5.78	1.19	2.49	0.41	8.83%

Table 1: The effects of β on SBA

The higher the β the lower (higher) the equilibrium level of capital stock (public good). Also, the equilibrium level of tax rate increases with an increase in β. Actually, this is not a surprising result since when β increases the contribution of one unit increase in public good to the utility increases through reducing the subsistence level of consumption more. Let us look at the autonomous part of the subsistence level of consumption.

$\overline{\gamma}$	k^*	c^*	g^*	$\overline{c}^*$	τ^*
0.5	6.25	1.29	1.11	0.34	3.85%
1	5.85	1.21	2.30	0.54	8.14%
1.2	5.71	1.18	2.71	0.62	9.65%

Table 2: The effects of $\overline{\gamma}$ on SBA

Sharif (1986) states that subsistence is not only the minimum physical needs but it is minimum standard of mental and physical survival. In other words, the autonomous part of the subsistence level of consumption in our model may differ across countries and even within a country between different periods. Model suggests that the optimal amount of tax has a positive relationship with autonomous part of subsistence level of consumption. Therefore, the lower (higher) the $\bar{c}$ the lower the optimal amount of tax rate and public good provision and also the lower the equilibrium subsistence level of consumption.

α	k^*	c^*	g^*	$\overline{c}^*$	τ^*
0.2	2.70	0.91	2.31	0.54	11.41%
0.3	5.85	1.21	2.30	0.54	8.14%
0.5	37.29	3.73	2.34	0.55	2.30%

Table 3: The effects of α on SBA

A	k^*	c^*	g^*	$\bar{c}^*$	τ^*
0.5	1.73	0.35	2.12	0.57	21.57%
1	5.85	1.21	2.30	0.54	8.14%
1.5	11.03	2.27	2.34	0.54	4.56%

Table 4: The effects of TFP on SBA

Since in our model we did not differantiate the human and physical capital, the capital stock in our model can be considered as the combination of these two kinds of capital. Beginning from the seminal paper of Mankiw, Romer Weil (1990), the growth accounting exercises have tried to estimate the capital share in the production function. Caselli (2004) and Caselli and Wilson (2004) argue that not only the quantity of capital in production varies across countries but also the composition of capital changes. The richer countries tend to employ more technology intensive capital than the poor countries. Therefore, it is expected that as development increases the capital share (total of physical and human capital) in production increases as well. Our study have shown that, there ise an inverse relationship between the optimal tax rate and the share of capital in production. The higher the capital share the lower the optimal amount of tax. The more interesting results is about the optimal provision of public good and subsistence level of consumption. Although, the optimal amount of taxes vary greatly among countries with different capital share in production, the optimal amount of public good provision is almost same and so does subsistence level of consumption.

Hall and Jones (1999) have shown that across countries the per labor output varies extremely. They argue that this difference can only be partly explained by the differences in physical and human capital. The most important variation is came from the total factor productivity. Thus, the total factor productivities are different in great extent across countries[10]. Our model suggest that, the optimal tax rate at steady state have an inverse relationship with the total factor productivity. The larger (smaller) the total factor productivity, the lower (higher) the optimal tax rate.

[10] Among others, see also Jorgenson and Kuroda (1990), Dollar and Wolf (1993), Harrigan (1997), and Aghion and Howitt (2007).

Although, the steady state capital stock and consumption is highly different for the countries with different levels of total factor productivity, the optimal amount of public good provision is almost same. In addition to that, the subsistence level of consumption is not varying to much for different levels of total factor productivity.

5.4. Social Planner Problem

The social planner chooses the optimal amount of consumption, investment on capital stock and the public good. The social planner internalizes the externality coming from the endogenous subsistence and chooses directly the paths of consumption, capital and public good. The social planner problem maximizes discounted life-time utility of representative household subject to the stock of capital and the public good.

$$\underset{c_t}{Max} \int_0^\infty e^{-\rho t} \frac{\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{1-\sigma}}{1-\sigma} dt \qquad (P'')$$

subject to

$$\begin{aligned}
f\left(k\left(t\right)\right) &= c(t) + i_g\left(t\right) + i_k\left(t\right), \\
\dot{k}\left(t\right) &= i_k\left(t\right) - \delta k\left(t\right), \ \forall t \geq 0 \\
\dot{g}\left(t\right) &= i_g\left(t\right) - \delta_g g\left(t\right) \\
c\left(t\right) &\geq \bar{c}\left(g\left(t\right)\right), \ k\left(t\right) \geq 0, \ \forall t \geq 0, \\
k(0) &= k_0 > 0, \ given.
\end{aligned}$$

The associated Hamiltonian is given by

$$H = \frac{\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{1-\sigma}}{1-\sigma} + \vartheta\left(t\right)\left[f\left(k\left(t\right)\right) - c(t) - i_{g(t)} - i_{k(t)}\right] + \mu\left(t\right)\left[i_{k(t)} - \delta k\left(t\right)\right] + \phi\left(t\right)\left[i_{g(t)} - \delta_g g\left(t\right)\right].$$

Given the initial level of public good and the capital stock, the associated first order conditions are as follows:

$$\begin{aligned}
\frac{\partial H}{\partial c} &= (c(t) - \bar{c}(g(t)))^{-\sigma} - \vartheta(t) = 0 && (47)\\
\frac{\partial H}{\partial i_k} &= -\vartheta(t) + \mu(t) = 0 && (48)\\
\frac{\partial H}{\partial i_g} &= -\vartheta(t) + \phi(t) = 0 && (49)\\
\frac{\partial H}{\partial k} &= \vartheta_t f'(k(t)) - \delta\mu(t) = -\dot{\mu}(t) + \rho\mu(t) && (50)\\
\frac{\partial H}{\partial g} &= -(c(t) - \bar{c}(g(t)))^{-\sigma}\bar{c}'(g(t)) - \phi(t)\delta_g = -\dot{\phi}(t) + \rho\phi(t) && (51)
\end{aligned}$$

From (47) and (48), we have:

$$\vartheta(t) = \mu(t) = \phi(t). \tag{52}$$

From (47), (50) and (51), we have:

$$-\bar{c}'(g(t)) = f'(k(t)). \tag{53}$$

Taking the time derivative of (47) and considering (50), we obtain the following Euler equation:

$$\dot{c}(t) = \frac{1}{\sigma}(c(t) - \bar{c}(g(t)))(f'(k(t)) - \delta - \rho) + \dot{g}(t)\bar{c}'(g(t)). \tag{54}$$

Note from (53) that along the optimal path the evolution of the public spending depends on the level of capital stock, i.e.,

$$\dot{g}(t)\bar{c}'(g(t)) = -\dot{k}(t)(f''(k(t))). \tag{55}$$

Combining (55) with the budget constraints of (P''), we obtain that

$$\dot{k}(t) = \frac{f(k(t)) - c(t) - \delta k(t) - \delta(\bar{c}')^{-1}(f'(k(t)))}{1 + \frac{f''(k(t))}{\bar{c}'(f'(k(t)))}}. \tag{56}$$

After necessary substitution of equations (53), (54), (55) and (56), the dynamics of the system is governed by the following two differantial equations which depend on the capital stock and consumption level. The system with three controls $\{c(t), i_k(t), i_g(t)\}$ and two state variables $\{k(t), g(t)\}$ is reduced to a system with two variables $\{c(t), k(t)\}$:

$$\dot{c}(t) = \frac{1}{\sigma}(c(t) - \bar{c}(g(t)))(f'(k(t)) - \delta - \rho) - \frac{f(k(t)) - c(t) - \delta k(t) - \delta(\bar{c}')^{-1}(f'(k(t)))}{1 + \frac{f''(k(t))}{\bar{c}'(f'(k(t)))}} f'(k(t)) \quad (57)$$

$$\dot{k}(t) = \frac{f(k(t)) - c(t) + \delta k(t) + \delta(\bar{c}')^{-1}(f'(k(t)))}{1 + \frac{f''(k(t))}{\bar{c}'(f'(k(t)))}} \quad (58)$$

Note that the socially optimal path can not be replicated by a second best policy in which the steady state values of the aggregate variables depend on tax rate. Turnovsky (1997a, 2000) shows that besides income taxes, when lump sum taxes and consumption taxes are also introduced, the first best could be implemented. Put differently, in order to reach the first best in a decentralized economy, the tax policy should consist of two parts, one of which is time varying and the other is constant. Hence, in our model there is no sufficient policy instruments to reach the first best allocation as our tax policy is just the income tax. The steady state capital levels are the same for two types of allocation (the first and the second best) if and only if the tax rate is set to be zero. However, in that case the model will not have any externalities anyway.

5.4.1. Comparative Statics

Since our model is based on neoclassical production function, the aggregate variables do not change in the long run. Therefore, a steady state equilibrium in the first best allocation can be defined as $\dot{c}(t) = 0$, $\dot{k}(t) = 0$, $\dot{g}(t) = 0$.

Lets leave aside trivial steady state at which consumption is equal to subsistence level for now and focus on the steady state at which life time discounted utility is strictly positive, i.e. the $c(t) > \bar{c}(g(t))\ \forall t$. One can easily notice from equation (57), $f'(k^*)$ is equal to $\delta + \rho$ and therefore steady state level of capital stock and investment in capital are:

$$
\begin{aligned}
k^* &= f'^{-1}(\delta + \rho) \qquad (59) \\
i_k^* &= \delta f'^{-1}(\delta + \rho) \qquad (60)
\end{aligned}
$$

From (53), (56) and the budget contsraints and we obtain:

$$
\begin{aligned}
g^* &= (\bar{c}')^{-1}(-f'(k^*)) \qquad (61) \\
i_g^* &= \delta (\bar{c}')^{-1}(-f'(k^*)) \qquad (62) \\
c^* &= f(k^*) - \delta\left(f'^{-1}(\delta + \rho) + (\bar{c}')^{-1}(-f'(k^*))\right) \qquad (63)
\end{aligned}
$$

Note that since f' is a decreasing function due to the concavity of production function, so does f'^{-1}. Thus an increase in depreciation rate and impatience decrease the steady state level of capital stock. The functional forms are specified as in the second best problem, $U(c(t), \bar{c}(g(t))) = \ln(c(t) - \bar{c}(g(t)))$, $f(k(t)) = Ak(t)^{\alpha}$, $\bar{c}(g(t)) = \frac{\overline{\gamma}}{(1+g(t))^{\beta}}$. Under these specific forms of functions, the steady state values of capital stock, consumption, public good, investment in public good, investment in capital and subsistence level of consumption are as follows:

$k^* = \left(\frac{A\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$
$c^* = A\left(\frac{A\alpha}{\delta+\rho}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\left(\frac{A\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} + \left(\left(\frac{\bar{c}\beta}{\delta+\rho}\right)^{1+\beta} - 1\right)\right)$
$g^* = \left(\frac{\bar{c}\beta}{\delta+\rho}\right)^{\frac{1}{1+\beta}} - 1$
$i_k^* = \delta\left(\frac{A\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$
$i_g^* = \delta\left(\left(\frac{\bar{c}\beta}{\delta+\rho}\right)^{\frac{1}{1+\beta}} - 1\right)$
$\overline{c}^* = \frac{\bar{c}}{\left(\frac{\bar{c}\beta}{\delta+\rho}\right)^{\frac{\beta}{1+\beta}}}$

Under this set of parameters, the steady state values of aggregate variables are:

$k^* = 6.61$	$c^* = 1.22$	$g^* = 2.39$	$\overline{c}^* = 0.54$

Note that compared with the second best allocation, the amount subsistence level of consumption is the same. Furthermore, the steady state level of consumption is almost equal to that of second best allocation. The interesting and one of the most important result of this study is such that although the amount of public good is higher for the first best allocation, the subsistence level of consumption is the same for both types of equilibrium. On the other hand, capital stock in the first best allocation is higher in the first best allocation.

Furhermore, the system admits two eigenvalues one of which is positive and the other is negative, implying a saddle path stability.

As in the second best allocation the total factor productivity the share of capital in production, depreciation rate and discount factor are standard with the literature. The important parameters are β and $\overline{\gamma}$. The equilibrium levels of aggregate variables for different sets of β and $\overline{\gamma}$ are given below:

β	k^*	c^*	g^*	$\overline{c}^*$
0.3	6.61	1.25	1.76	0.74
0.5	6.61	1.22	2.39	0.54
0.7	6.61	1.21	2.58	0.41

Table 5: The effects of β on FBA

Note that for the second best equilibrium the higher the β the lower (higher) the equilibrium level of capital stock (public good) and the equilibrium level of tax rate increases with an increase in β. However, for the first best equilibrium the steady state level of capital stock is independent of the β. Whatever the β, the capital stock is 6.61 for the set of parameters given above. In addition to that, the subsistence level of consumption is same with that of the second best equilibrium although public good provision is slightly higher for the first best allocation.

$\overline{\gamma}$	k^*	c^*	g^*	$\overline{c}^*$
0.5	6.61	1.29	1.13	0.34
1	6.61	1.21	2.30	0.54
1.2	6.61	1.20	2.83	0.62

Table 6: The effects of $\overline{\gamma}$ on FBA

The model suggests that an increase in autonomous part of the subsistence level of consumption decreases the steady state consumption and subsistence level whereas leads to an increase in public good provision.

The last but not the least important result of the social planner problem is about the capital share in production and total factor productivity. As stated in the solution of second best allocation, there is an inverse relationship between the optimal amount of tax rate and the share of capital and the total factor productivity. However, for the social planner, the share of capital in production and total factor productivity does not matter for the allocation of public good, capital stock and consumption of private good.

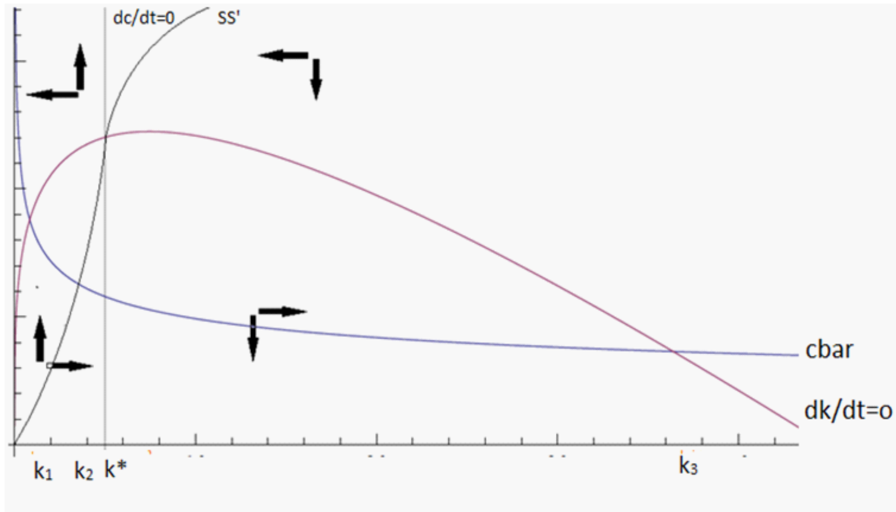

Figure 3: Phase Diagram

Let us analyze the trivial steady state(s) where consumption equals to the subsistence level. The fourth constraint $c(t) \geq \bar{c}(g(t))$ of the problem (P") is based on a technical assumption that agents do not want to lower their consumption below the subsistence level. Note that, when consumption is below the subsistence level, the life-time utility goes to minus infinity and therefore when agents' consumption levels equal to subsistence level at a point in time, agents do not want to lower consumption even for a higher future consumption. The current value Hamiltonian that we deal with so far is valid when consumption is strictly higher than the subsistence level, i.e. $c(t) - \bar{c}(g(t)) > 0$. Actually, if $c(t) - \bar{c}(g(t)) > 0$, the optimal solution to the current value Hamiltonian for non-binding case (solved above) and that with the constraint qualifications coincides. However, if the optimal solution lies on or below the $c(t) - \bar{c}(g(t)) = 0$ curve, then this path is not an optimal path of the problem that we solved above. If the system admit one stationary point then there exists another stationary point with higher capital stock. This result is directly comes from the properties of $\dot{k}(t)$ and $\bar{c}(g(t))$.

We plot the phase diagram of the social planner problem and subsistence level of consumption in the same plane. It is obvious that when subsistence curve lies above the $\dot{k}(t)$ there is no steady state. Therefore, we assume that $\bar{c}(g(t)) < \dot{k}(t)$ for some $k(t) \epsilon (0, \infty)$. k^* is the steady state level of capital stock where consumption is above the subsistence level. The SS' curve indicates the saddle path on which economy converges to the k^*. When $k \epsilon (k_2, k^*)$, the SS' curve lies above the subsistence and consumption and the capital stock adjusts so that it reaches the SS' curve and along this curve the economy converges to the steady state. However, when $k \epsilon (0, k_1)$, the subsistence level of consumption is above the SS' curve. Since the technical assumption that agents do not want to lower their consumption below the subsistence level, agents consume at each point as much as the subsistence level and eventually eat up the entire capital stock. When $k \epsilon (k_1, k_2)$ is the case, then agents' consumption is equal to subsistence level for a while and then hits to the SS' curve and converges to the steady state k^*.

5.5. Conclusion

In this study, we have analyzed the quantitative and qualitative effects of a pure public good which do not have any congestion effect and satisfy the basic needs of the individuals. We have incorporated such a public good through the Stone-Geary preferences. First of all, we have analyzed the second best allocation and shown that the optimal amount of tax rate depends negatively on the share of capital in production and total factor productivity. Furthermore, the higher the return of public good on subsistence level of consumption, the lower the steady state value of the capital stock. The same kind of relationship is valid between the autonomous part of the subsistence level of consumption and steady state level of capital stock. Another important result of the paper is the "$g-\tau$" *locus* with a dynamic threshold which depends on the total factor productivity in the second best allocation. This means that the optimal amount of tax rate that maximizes the total revenue of the government is an increasing function of the total factor productivity. It means that different countries that have different total factor productivity have different optimal amount of tax rate. The next step was the analyzing the social planner solution. We have shown that, although the steady state amount of public good is higher for the first best allocation, the subsistence level of consumption is the same with that of the second best equilibrium. On the other hand, the capital stock and the consumption of the private good are higher for the first best equilibria. Furhermore, the steady states for both the first and the second best allocations are stable in the saddle path sense.

CHAPTER 6

CONCLUSION

The The growth models (Solow, Ramsey, Ak etc.) based on two keystones; the supply and the demand. In all of the growth models, households maximize their utility by smoothing consumption subject to the income earned from the production side of the economy. Therefore, both long-run equilibrium and the transition to this equilibria heavily depend on the properties and assumptions made on the utility and the production functions. Although, main stimulus of economic growth came from the supply side; for example from the exogenous technological progress as in Solow or Ramsey economy, or endogenous technological progress as in Shumpeterian models and new growth theory, the demand side is crucial for the transitional path. In this thesis, we mainly deal with the demand side of the economy and analyze the quantitative and qualitative properties of long-run equilibriums and nonlinear transitional paths.

The objective of Chapter 2 is to analyze how the dynamic strategic interactions among agents affect the long-run distribution of wealth in terms of catching up and the transitional dynamics in an economy. More specifically, we investigate the effects of the strategic interaction among agents on catching up which is not present under a competitive equilibrium framework. We ask whether a rich household that has a larger initial stock can use this as an advantage to prevent its rival from accumulating capital stock and achieve a higher long-run capital stock. In particular, we analyze whether such a head start disappears in the non-cooperative equilibrium of this class of games even with open-loop strategies. Under this small departure from the competitive equilibrium framework, we show that considering the strategic interaction among agents in the economy changes the qualitative properties of the standard Ramsey model drastically.

In the absence of strategic interaction, poor will never be able to catch up with the rich. However, incorporating the strategic behavior among agents leads to the wealth level of the initially poor and the rich households to be the same at the stationary state. We extend our analysis on the dynamic implications of strategic interaction, to account for relative wealth concern. The resulting equilibria depends the valuation of relative wealth concern by each individual and we show that under some plausible conditions the catching up occurs thanks to the strategic interaction in the form of open-loop.

The second essay of this thesis studies the effects of above mentioned strategic interaction in Ramsey model with "Easterlin hypothesis". Deviating from the competitive equilibrium framework, we show that the strategic interaction among agents in the economy leads to a change not only in the distribution of wealth in the long run but also in the transitional dynamics substantially. Indeed, the strategic interaction not only leads to complex wealth distribution but also complex dynamics in Ramsey model with adjustment cost of consumption. In this respect, we show that structurally very simple frameworks may lead to limit cycles thanks to the strategic interaction among agents in the economy. The complex dynamics is in the form of Hopf bifurcation. We obtain the Hopf bifurcation by varying the parameter in the penalty function in the preferences and thus, our model argues that the degree of the penalty in utility can serve as an supplemental channel for clarifying the cyclical behaviours in the economy.

Third chapter of this thesis introduces Stone-Geary Preferences with an endogenous reference level of consumption in an otherwise standard Ak growth model. This threshold level of consumption depends positively on the capital stock of the individuals. We have shown that depending on the relationship between the curvature of the reference level of consumption function and the net to capital, the resulting equilibrium presents richer dynamics under such a Stone-Geary preferences. In particular, we prove that endogenous reference level of consumption posits both global and local indeterminacy: economies starting with different initial conditions does not necessarily converge to the same steady state and also economies starting with the same initial conditions does not necessarily follow the same transition path. Accordingly, we show that fluctuations due to self-fulfilling expectations may emerge thanks to Stone-Geary preferences with endogenous reference level of consumption.

We analyze the effect of public good provision on the long run equilibrium and the optimal tax rate in the last essay of this thesis. Since the absence of public goods forces individuals to consume minimum amount from their own budget, an increase in the provision of public goods actually decreases the threshold level above which private consumption gives utility. We show that, although the steady state amount of public good is higher for the first best allocation, the subsistence level of consumption is the same with that of the second best equilibrium. On the other hand, the capital stock and the consumption of the private good are higher for the first best equilibria. There is an inverse relationship between the optimal amount of tax rate and the share of capital in the solution of second best allocation. The same result is valid between optimal tax rate and total factor productivity. However, for the social planner, the share of capital in production and total factor productivity do not matter for the allocation of public good, capital stock and consumption of private good. Another important result of the essay is the "$g - \tau$" *locus* with a dynamic threshold which depends on the total factor productivity. This means that the optimal amount of tax rate that maximizes the total revenue of the government is an increasing function of the total factor productivity and thus revenue maximizing tax rate varies across countries. Furhetmore, steady states for both first and second best allocations are stable in the saddle path sense.

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APPENDICES

A. APPENDIX A

Linearization of the Euler equations of the model with strategic interaction:

$$\dot{c_i}(t) = \frac{1}{\theta} c_i(t) \left[f^{'}(K(t)) - \rho + f''(K(t)) \left(k_i(t) - \frac{K(t)}{N} \right) \right]$$

$$\frac{\partial \left(\dot{c_i}(t) \right)}{\partial c_i} = \frac{1}{\theta} \left[f^{'}(K(t)) - \rho + f''(K(t)) \left(k_i(t) - \frac{K(t)}{N} \right) \right]$$

$$\frac{\partial \left(\dot{c_i}(t) \right)}{\partial c_j} = \frac{1}{\theta} \left[f^{'}(K(t)) - \rho + f''(K(t)) \left(k_i(t) - \frac{K(t)}{N} \right) \right] \ \forall j \in \{1, 2, ..., N\} \setminus \{i\}$$

$$\frac{\partial \left(\dot{c_i}(t) \right)}{\partial k_i} = \frac{1}{\theta} c_i(t) \left[f''(K(t)) \left(2 - \frac{1}{N} \right) + f'''(K(t)) \left(k_i(t) - \frac{K(t)}{N} \right) \right]$$

$$\frac{\partial \left(\dot{c_i}(t) \right)}{\partial k_j} = \frac{1}{\theta} c_i(t) \left[f''(K(t)) \left(1 - \frac{1}{N} \right) + f'''(K(t)) \left(k_i(t) - \frac{K(t)}{N} \right) \right]$$

At the symmetric steady state at which we have $f'(K) = \rho$, $c_i = \frac{f(K)}{N}, \forall i \in \{1, 2, ..., N\}$, where $K = N k_i$,hence

$$\frac{\partial\left(\dot{c}_i(t)\right)}{\partial c_i} \mid_{\{c_i^*, c_j^*, k_i^*, k_j^*\}} = 0,$$
$$\frac{\partial\left(\dot{c}_i(t)\right)}{\partial c_j} \mid_{\{c_i^*, c_j^*, k_i^*, k_j^*\}} = 0,$$
$$\frac{\partial\left(\dot{c}_i(t)\right)}{\partial k_i} \mid_{\{c_i^*, c_j^*, k_i^*, k_j^*\}} = \frac{1}{\theta N} f(K) f''(K) \left(2 - \frac{1}{N}\right),$$
$$\frac{\partial\left(\dot{c}_i(t)\right)}{\partial k_j} \mid_{\{c_i^*, c_j^*, k_i^*, k_j^*\}} = \frac{1}{\theta N} f(K) f''(K) \left(1 - \frac{1}{N}\right),$$

The linearization of the dynamic equations of capital stock

Since the evolution of capital stocks are same for all three models, first of all we linearize these differential equations.

$$\dot{k}_i(t) = f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t).$$

$$\begin{aligned} \frac{\partial\left(\dot{k}_i(t)\right)}{\partial c_i} &= -1 \\ \frac{\partial\left(\dot{k}_i(t)\right)}{\partial c_j} &= 0 \ \forall j \in \{1, 2, ..., N\} \setminus \{i\} \end{aligned}$$

$$\frac{\partial(\dot{k}_i(t))}{\partial k_i} = f'(K) + f''(K)\left(\frac{K(t)}{N} - k_i\right)$$

$$\frac{\partial\left(\dot{k}_i(t)\right)}{\partial k_j} = f''(K)\left(\frac{K(t)}{N} - k_i\right) \ \forall j \in \{1, 2, ..., N\} \setminus \{i\}$$

Since at steady state $k_i^* = k_j^*$, $c_1^* = c_2^*$, $\forall\ i,\ j \in \{1, 2, ..., N\}$ we have

$$\frac{\partial(\dot{k}_i(t))}{\partial k_i} \mid_{\{c_i^*, c_j^*, k_i^*, k_j^*\}} = f^{'}(K^*),$$

$$\frac{\partial\left(\dot{k}_i(t)\right)}{\partial k_j} \mid_{\{c_i^*, c_j^*, k_i^*, k_j^*\}} = 0.$$

Thus the associated Jacobian matrix will be as follows:

$$J \equiv \begin{bmatrix} 0 & 0 & \cdots & 0 & \varsigma & \xi & \cdots & \xi \\ 0 & 0 & \cdots & 0 & \xi & \varsigma & \cdots & \xi \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \xi & \xi & \cdots & \varsigma \\ -1 & 0 & \cdots & 0 & f'(K) & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & f'(K) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & f'(K) \end{bmatrix}_{2N*2N}$$

where

$$\begin{aligned} \varsigma &= \frac{1}{\theta N} f(K) f''(K)(2 - \frac{1}{N}), \\ \xi &= \frac{1}{\theta N} f(K) f''(K)(1 - \frac{1}{N}). \end{aligned}$$

$$J \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{2N*2N}$$ and where $\frac{1}{\theta} f(K) f''(K) = a$ and $\frac{1}{N} * (1 - \frac{1}{N}) = x$

$$A_{11} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{N*N} \qquad A_{12} = \begin{bmatrix} a(x + \frac{1}{N}) & ax & \cdots & ax \\ ax & a(x + \frac{1}{N}) & \cdots & ax \\ \vdots & \vdots & \ddots & \vdots \\ ax & ax & \cdots & a(x + \frac{1}{N}) \end{bmatrix}_{N*N}$$

$$A_{21} = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & -1 \end{bmatrix}_{N*N} \qquad A_{22} = \begin{bmatrix} f'(K) & 0 & \cdots & 0 \\ 0 & f'(K) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'(K) \end{bmatrix}_{N*N}$$

Note that A_{21} and A_{22} are scalar matrix of -1 and $f'(K)$, respectively. The row reduced echelon form of matrix A_{12} is a lower triangular matrix of the form:

$$A_{12} = \begin{bmatrix} \frac{a}{N} & -\frac{a}{N} & 0 & \cdots & 0 \\ 0 & \frac{a}{N} & -\frac{a}{N} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{a}{N} & -\frac{a}{N} \\ 0 & 0 & 0 & \cdots & a\frac{N^2-N+1}{N} \end{bmatrix}_{N*N}$$

The eigenvalues of Jacaobian matrix can be obtained from $\det\left[J-\lambda I\right]$. The associated $J-\lambda I$ can be written as:

$$J-\lambda I \equiv \begin{bmatrix} -\lambda & 0 & \cdots & 0 & \frac{a}{N} & -a & \cdots & 0 \\ 0 & -\lambda & \cdots & 0 & 0 & \frac{a}{N} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda & 0 & 0 & \cdots & a\frac{N^2-N+1}{N} \\ -1 & 0 & \cdots & 0 & f'(K)-\lambda & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & f'(K)-\lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & f'(K)-\lambda \end{bmatrix}_{2N*2N}$$

and

$$\det\left[J-\lambda I\right] = \det\left[A_{11} - A_{12}A_{22}^{-1}A_{21}\right]\det\left[A_{22}\right] =$$
$$\left(\frac{a}{N} - \lambda f'(K) + \lambda^2\right)^{N-1}\left(a\frac{N^2-N+1}{N} - \lambda f'(K) + \lambda^2\right) = 0.$$

Then the associated eigenvalues are $(N-1)$ of eigenvalues equalt to $(\frac{f'(K)+\sqrt{f'(K)-4\frac{a}{N}}}{2})$ and $(N-1)$ are equal to $(\frac{f'(K)-\sqrt{f'(K)-4\frac{a}{N}}}{2})$, 1 eigenvalue is equal to $(\frac{f'(K)+\sqrt{f'(K)-4a\frac{N^2-N+1}{N}}}{2})$,the last one is $(\frac{f'(K)-\sqrt{f'(K)-4a\frac{N^2-N+1}{N}}}{2})$. Since, $a = \frac{1}{\theta}f(K)f''(K)$ is negative, clearly the $2N*2N$ Jacobian matrix have n negative and n positive eigenvalues, indicating the saddle path stability.

B. APPENDIX B

Linerization and stability analysis of strategic Ramsey model with relative capital:

$$\dot{c}_i(t) = -\frac{u'(c_i(t))}{u''(c_i(t))}\left[f'(K(t)) - \rho + f''(K(t))\left(k_i(t) - \frac{K(t)}{N}\right) + \eta_i \frac{v'(z_i(t))}{u'(c_i(t))}\frac{1-\frac{k_i(t)}{K(t)}}{\frac{1}{N}K(t)}\right],$$

$$\dot{k}_i(t) = f'(K(t))k_i(t) + \frac{f(K(t)) - K(t)f'(K(t))}{N} - c_i(t).$$

Linearizing the above equations around the unique steady state (c_i^*, k_i^*) that is characterized by $c_i^* = \frac{f(K^*)}{N}$ and $k_i^* = k^*$ and $K^* = N * k^*$

$$f'(K^*) + \eta^1 \frac{v'(1)}{u'(\frac{f(K^*)}{N})}\frac{(N-1)}{K^*} = \rho$$

$$\frac{\dot{c}_i(t)}{\partial c_i} = \frac{c_i}{\theta}\left(-\eta_i \frac{v'(z_i(t))}{(u'(c_i(t)))^2} u''(c_i(t)) \frac{1-\frac{k_i(t)}{K(t)}}{\frac{1}{N}K(t)}\right) \text{ and since } -\frac{u'(c_i(t))}{u''(c_i(t))}\frac{1}{c_i(t)} = \frac{1}{\theta}$$

$$\frac{\dot{c}_i(t)}{\partial c_i} = \left(\eta_i \frac{v'(1)}{u'(\frac{f(K^*)}{N})}\frac{(N-1)}{K^*}\right) = \rho - f'(K^*)$$

$$\frac{\dot{c}_i(t)}{\partial c_j} \ for \ \forall \ i \neq j$$

$$\frac{\dot{c}_i(t)}{\partial k_i} = \left(f''(K^*)\left(2 - \frac{1}{N}\right) + \eta_i \frac{v''(1)}{u'(\frac{f(K^*)}{N})}\left(\frac{N-1}{K^*}\right)^2 - 2\eta_i \frac{v'(1)}{u'(\frac{f(K^*)}{N})}\frac{N-1}{(K^*)^2}\right) = u$$

$$\frac{\dot{c}_i(t)}{\partial k_j} = \left(f''(K^*) \left(1 - \frac{1}{N}\right) + \eta_i \frac{v''(1)}{u'(\frac{f(K^*)}{N})} \frac{N-1}{(K^*)^2} + \eta_i \frac{v'(1)}{u'(\frac{f(K^*)}{N})} \frac{2-N}{(K^*)^2} \right) \quad for \ \forall \ i \neq j$$

Moreover, the linearization of capital stock dynamics will give the same results in previous chapter. Let denote right hand side of equation $\frac{\dot{c}_i(t)}{\partial k_i}$ as u and right hand sideof equation $\frac{\dot{c}_i(t)}{\partial k_j}$ as v. Then, the associated Jacobian matrix will be;

$$J \equiv \begin{bmatrix} \rho - f'(K^*) & 0 & \cdots & 0 & u & v & \cdots & v \\ 0 & \rho - f'(K^*) & \cdots & 0 & v & u & \cdots & v \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho - f'(K^*) & v & v & \cdots & u \\ -1 & 0 & \cdots & 0 & f'(K) & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & f'(K) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & f'(K) \end{bmatrix}_{2N*2N}$$

Since

$$\begin{aligned} u'\left(\frac{f(K^*)}{N}\right) &= \left(\frac{A}{N}\right)^{-\theta} (K^*)^{-\theta\gamma}, \\ v'(1) &= 1, \\ v''(1) &= -\sigma \\ f''(K^*) &= A\gamma(\gamma-1)(K^*)^{\gamma-2} \end{aligned}$$

$$\begin{aligned} u &= a = A\gamma(\gamma-1)(K^*)^{\gamma-2}\left(2 - \frac{1}{N}\right) - \eta\left(\frac{A}{N}\right)^{\theta}(K^*)^{\theta\gamma-2}(N-1)(\sigma(N-1)+2) \\ v &= b = A\gamma(\gamma-1)(K^*)^{\gamma-2}\left(1 - \frac{1}{N}\right) - \eta\left(\frac{A}{N}\right)^{\theta}(K^*)^{\theta\gamma-2}((\sigma+1)(N-1)-1) \end{aligned}$$

$$\text{Claim:}\quad det\left(\begin{bmatrix}\lambda\left(A_{11}-I\right) & A_{12}\\ A_{21} & \lambda\left(A_{22}-I\right)\end{bmatrix}_{2N*2N}\right)=\det\left(A_{12}+\left(x-\lambda\right)\left(c-\lambda\right)\right)$$

and where

$$A_{11}=\begin{bmatrix}x & 0 & \cdots & 0\\ 0 & x & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & x\end{bmatrix}_{N*N}\qquad A_{12}=\begin{bmatrix}u & v & v & \cdots & v\\ v & u & v & \cdots & v\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ v & v & \ddots & u & v\\ v & v & v & \cdots & u\end{bmatrix}_{N*N}$$

$$A_{21}=\begin{bmatrix}-1 & 0 & \cdots & 0\\ 0 & -1 & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -1\end{bmatrix}_{N*N}\qquad A_{22}=\begin{bmatrix}c & 0 & \cdots & 0\\ 0 & c & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & c\end{bmatrix}_{N*N}$$

Note that A_{11}, A_{21} and A_{22} are scalar matrix of x, -1 and c, respectively. A_{12} has $(N-1)$ eigenvalues of $u-v$ and 1 eigenvalues of $(N-1)v+u$. From eq. $\lambda^2-(x+c)\lambda+(xc+\mu_i)=0.\lambda_1+\lambda_2=x+c, \lambda_1\lambda_2=xc+\mu_i$.Since,

$$u=\left(f''(K^*)\left(2-\frac{1}{N}\right)+\eta_i\frac{v''(1)}{u'(\frac{f(K^*)}{N})}\left(\frac{N-1}{K^*}\right)^2-2\eta_i\frac{v'(1)}{u'(\frac{f(K^*)}{N})}\frac{N-1}{(K^*)^2}\right)$$

$$v=\left(f''(K^*)\left(1-\frac{1}{N}\right)+\eta_i\frac{v''(1)}{u'(\frac{f(K^*)}{N})}\frac{N-1}{(K^*)^2}+\eta_i\frac{v''(1)}{u'(\frac{f(K^*)}{N})}\frac{2-N}{(K^*)^2}\right)\quad for\ \forall\ i\neq j$$

$$x=\rho-f'(K^*)\ and\ c=f'(K^*)$$

$(N-1)v+u$ is always negative. However, the sign of $u-v$ is not clear.

$$u-v=\frac{f(K^*)f''(K^*)}{\theta N}+\left(\rho-f'(K^*)\right)\left[f'(K^*)+\frac{f(K^*)\left(\frac{v''(1)}{v'(1)}\frac{N-2}{N}-1\right)}{\theta NK^*}\right]$$

Since at steady state, we have $f'(K^*) + \eta^1 \frac{v'(1)}{u'(\frac{f(K^*)}{N})} \frac{(N-1)}{K^*} = \rho, \rho > f'(K^*)$ and also $\frac{f(K^*)f''(K^*)}{\theta N} < 0$. Thus, in order to have $u - v < 0$, the term in the brackets should be less than zero. Subsitttuting open form of production and utility functions, the term in brackets will be;

$$AK^{\gamma} \left[\gamma\theta + \frac{v''(1)}{v'(1)} \frac{N-2}{N} - 1 \right]$$

and since $\frac{v''(1)}{v'(1)} \frac{N-2}{N} < 0$, $\gamma\theta < 1$ is sufficient in order to have $u - v < 0$.

Thus, for each μ_i the system has one negative and one positive eigenvalues. This implies that, the Jacobian matrix has N negative and N positive eigenvalues.

C. APPENDIX C

Case 1: The unique steady state

The Euler and capital accumulation equations are as follows:

$$\begin{aligned} \dot{c}(t) &= \frac{1}{\sigma}\left(A-\delta-\rho-\bar{c}'(k(t))\right)\left(c(t)-\bar{c}(k(t))\right)+\bar{c}'(k(t))\,\dot{k}(t), \\ \dot{k}(t) &= (A-\delta-n)\,k(t)-c(t), \end{aligned}$$

The substitution of capital accumulation rule into Euler will give the following equation:

$$\dot{c}(t)=\frac{1}{\sigma}\left(A-\delta-\rho-\bar{c}'(k(t))\right)\left(c(t)-\bar{c}(k(t))\right)+\bar{c}'(k(t))\,(A-\delta-n)\,k(t)-c(t),$$

The partial derivative of Euler and capital accumulation rule gives;

$$\begin{aligned} \frac{\partial\dot{c}(t)}{\partial c(t)} &= \frac{1}{\sigma}\left(A-\delta-\rho-\bar{c}'(k(t))\right)-\bar{c}'(k(t)) \\ \frac{\partial\dot{c}(t)}{\partial k(t)} &= -\frac{1}{\sigma}\bar{c}''(k(t))\left(c(t)-\bar{c}(k(t))\right)+\bar{c}''(k(t))\left((A-\delta-n)\,k(t)-c(t)\right)+\bar{c}'(k(t))\,(A-\delta \\ \frac{\partial\dot{k}(t)}{\partial c(t)} &= -1 \\ \frac{\partial\dot{k}(t)}{\partial k(t)} &= (A-\delta-n) \end{aligned}$$

Since at steady state we have $(A-\delta-n)\,k^{*}=c^{*}, \textit{and } A-\delta-\rho=\bar{c}'(k^{*})$. Subsituting these two conditions into above linearized system, we will have;

$$
\begin{aligned}
\frac{\partial \dot{c}(t)}{\partial c(t)} &\mid_{\{c^{*},k^{*}\}} = -(A-\delta-\rho) \\
\frac{\partial \dot{c}(t)}{\partial k(t)} &\mid_{\{c^{*},k^{*}\}} = -\frac{1}{\sigma}\bar{c}''(k^{*})(c^{*}-\bar{c}(k^{*})) + (A-\delta-\rho)(A-\delta-n) \\
\frac{\partial \dot{k}(t)}{\partial c(t)} &\mid_{\{c^{*},k^{*}\}} = -1 \\
\frac{\partial \dot{k}(t)}{\partial k(t)} &\mid_{\{c^{*},k^{*}\}} = (A-\delta-n)
\end{aligned}
$$

The associated Jacobian will be;

$$
J \equiv \begin{bmatrix} -(A-\delta-\rho) & -\frac{1}{\sigma}\bar{c}''(k^{*})(c^{*}-\bar{c}(k^{*})) + (A-\delta-\rho)(A-\delta-n) \\ -1 & (A-\delta-n) \end{bmatrix}_{2*2}
$$

Thus, the trace and determinant of the J is as follows:

$$
\begin{aligned}
T &= \rho - n > 0, \\
D &= \frac{1}{\sigma}\bar{c}''(k)\left(c-\bar{c}(k)\right) \geq 0.
\end{aligned}
$$

Case 2: The unique subsistence steady state

Let (c_s, k_s) be the unique solution to $(A-\delta-n)k_s = \bar{c}(k_s)$. This necessarily implies that $(A-\delta-n) = \bar{c}'(k_s)$. Substituting these two conditions into (1), we have;

$$
\begin{aligned}
\frac{\partial \dot{c}(t)}{\partial c(t)} &\mid_{\{c^{*},k^{*}\}} = -(A-\delta-\rho) \\
\frac{\partial \dot{c}(t)}{\partial k(t)} &\mid_{\{c^{*},k^{*}\}} = (A-\delta-\rho)(A-\delta-n) \\
\frac{\partial \dot{k}(t)}{\partial c(t)} &\mid_{\{c^{*},k^{*}\}} = -1 \\
\frac{\partial \dot{k}(t)}{\partial k(t)} &\mid_{\{c^{*},k^{*}\}} = (A-\delta-n)
\end{aligned}
$$

The associated Jacobian is:

$$J \equiv \begin{bmatrix} -(A-\delta-\rho) & (A-\delta-\rho)(A-\delta-n) \\ -1 & (A-\delta-n) \end{bmatrix}_{2*2}$$

Thus, trace and determinant of the J is as follows:

$$\begin{aligned} T &= \frac{1}{\sigma}(n-\rho) < 0, \\ D &= 0. \end{aligned}$$

In this case, the solution to the zero eigenvalue is a constant and the dynamics are simply determined by the sign of the trace. However, there is no longer a unique equilibrium but a continuum of equilibria whose locus is the straight line through the steady state.

Case 3: Multiple subsistence steady state

Let $(\bar{c}(k_l), k_l)$ and $(\bar{c}(k_h), k_h)$ be the two steady states at which $(A-\delta-n)k = \bar{c}(k)$. Without loss of generality, assume that $k_l < k_h$. We have then $\bar{c}'(k_h) > A-\delta-n > \bar{c}'(k_l)$.

i) Substituting conditions of $(\bar{c}(k_l), k_l)$ into (1)gives the following steady state values;

$$\begin{aligned} \frac{\partial \dot{c}(t)}{\partial c(t)} &\mid_{\{c^*,k^*\}} = \frac{1}{\sigma}(A-\delta-\rho-\bar{c}'(k^*)) - \bar{c}'(k^*) \\ \frac{\partial \dot{c}(t)}{\partial k(t)} &\mid_{\{c^*,k^*\}} = \bar{c}(k^*)(A-\delta-n) \\ \frac{\partial \dot{k}(t)}{\partial c(t)} &\mid_{\{c^*,k^*\}} = -1 \\ \frac{\partial \dot{k}(t)}{\partial k(t)} &\mid_{\{c^*,k^*\}} = (A-\delta-n) \end{aligned}$$

The associated Jacbian, T and the determinant D of the associated Jacobian matrix are as follows:

$$J \equiv \begin{bmatrix} \frac{1}{\sigma}\left(A-\delta-\rho-\overline{c}'\left(k_l\right)\right)-\overline{c}'\left(k_l\right) & \overline{c}\left(k_l\right)\left(A-\delta-n\right) \\ -1 & \left(A-\delta-n\right) \end{bmatrix}_{2*2}$$

$$\begin{aligned} T &= \frac{1}{\sigma}\left(A-\delta-\rho-\overline{c}'\left(k_l\right)\right)+\left(A-\delta-n-\overline{c}'\left(k_l\right)\right), \\ D &= \frac{1}{\sigma}\left(A-\delta-\rho-\overline{c}'\left(k_l\right)\right)\left(A-\delta-n-\overline{c}'\left(k_l\right)\right). \end{aligned}$$

If $\overline{c}'\left(k_l\right) < A-\delta-\rho$ then $T>0$, $D>0$ and $\Delta>0$ so that the low steady state will be an unstable node. If $\overline{c}'\left(k_l\right) > A-\delta-\rho$ then $D<0$ which implies $\Delta>0$ independent from the sign of T. Accordingly, there exists a one-dimensional stable and a one-dimensional unstable eigenspace, hence the steady state is a saddle.

ii) As $\overline{c}'\left(k_h\right) > A-\delta-n$ and $\rho>n$, we have $\overline{c}'\left(k_h\right) > A-\delta-\rho$ at the high steady state. In this case, we have then $T<0$, $D>0$ and $\Delta>0$ so that the high steady state will be locally indeterminate (i.e., a stable node).

D. APPENDIX D

The first order conditions of the second best equilibrium are as follows:

$$H\left(c\left(t\right),k\left(t\right),g\left(t\right),\lambda_c\left(t\right),\lambda_k\left(t\right),\lambda_g\left(t\right)\right)=\frac{\left(c\left(t\right)-\bar{c}\left(g\left(t\right)\right)\right)^{1-\sigma}}{1-\sigma}+$$

$$\lambda_c\left(t\right)[\frac{1}{\sigma}\left(c\left(t\right)-\bar{c}\left(g\left(t\right)\right)\right)\left(\left(1-\tau\right)f'\left(k\left(t\right)\right)-\delta-\rho\right)+$$

$$\left(\tau f\left(k\left(t\right)\right)-\delta g\left(t\right)\right)\bar{c}'\left(g\left(t\right)\right)]+$$

$$\lambda_k\left(t\right)\left[\left(1-\tau\right)f\left(k\left(t\right)\right)-c(t)-\delta k\left(t\right)\right]+\lambda_g\left(t\right)\left[\tau f\left(k\left(t\right)\right)-\delta g\left(t\right)\right].$$

The associated first order conditions are given by:

$$\frac{\partial H}{\partial \tau}=-\lambda_c\left(t\right)\frac{1}{\sigma}\left(c\left(t\right)-\bar{c}\left(g\left(t\right)\right)\right)f'\left(k\left(t\right)\right)+$$

$$\lambda_c\left(t\right)f\left(k\left(t\right)\right)\bar{c}'\left(g\left(t\right)\right)-\lambda_k\left(t\right)f\left(k\left(t\right)\right)+\lambda_g\left(t\right)f\left(k\left(t\right)\right)=0 \quad (2)$$

$$\frac{\partial H}{\partial c(t)}=\left(c\left(t\right)-\bar{c}\left(g\left(t\right)\right)\right)^{-\sigma}+$$

$$\lambda_c\left(t\right)\frac{1}{\sigma}\left(\left(1-\tau\right)f'\left(k\left(t\right)\right)-\delta-\rho\right)-\lambda_k\left(t\right)=-\dot{\lambda}_c\left(t\right)+\rho\lambda_c\left(t\right) \quad (3)$$

$$\frac{\partial H}{\partial k(t)}=\lambda_c\left(t\right)\left(c\left(t\right)-\bar{c}\left(g\left(t\right)\right)\right)\frac{1}{\sigma}\left(1-\tau\right)f''\left(k\left(t\right)\right)+$$

$$\lambda_c\left(t\right)f'\left(k\left(t\right)\right)\bar{c}'\left(g\left(t\right)\right)+\lambda_k\left(t\right)\left(1-\tau\right)f'\left(k\left(t\right)\right)-$$

$$\lambda_k\left(t\right)\delta+\lambda_g\left(t\right)\tau f'\left(k\left(t\right)\right)=-\dot{\lambda}_k\left(t\right)+\rho\lambda_k\left(t\right) \quad (4)$$

$$\frac{\partial H}{\partial g(t)} = -\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{-\sigma} \bar{c}'\left(g\left(t\right)\right) -$$

$$\lambda_c\left(t\right) \bar{c}'\left(g\left(t\right)\right) \frac{1}{\sigma}\left(\left(1-\tau\right) f'\left(k\left(t\right)\right) - \delta - \rho\right) - \lambda_c\left(t\right) \delta \bar{c}'\left(g\left(t\right)\right) +$$

$$\lambda_c\left(t\right)\left[\tau f\left(k\left(t\right)\right) - \delta g\left(t\right)\right] \bar{c}''\left(g\left(t\right)\right) - \lambda_g\left(t\right) \delta = -\dot{\lambda}_g + \rho \lambda_g\left(t\right) \quad (5)$$

From (2), we have

$$\lambda_c\left(t\right) = -\frac{\left(\lambda_g\left(t\right) - \lambda_k\left(t\right)\right) f\left(k\left(t\right)\right)}{\left(\frac{1}{\sigma}\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right) f'\left(k\left(t\right)\right) - f\left(k\left(t\right)\right) \bar{c}'\left(g\left(t\right)\right)\right)}. \quad (6)$$

Let $\lambda_g\left(t\right) - \lambda_k\left(t\right) = \lambda\left(t\right)$, then $\dot{\lambda}_g\left(t\right) - \dot{\lambda}_k\left(t\right) = \dot{\lambda}\left(t\right)$. Substracting (4) from (5), we have

$$\rho\left(\lambda_g\left(t\right) - \lambda_k\left(t\right)\right) - \left(\dot{\lambda}_g\left(t\right) - \dot{\lambda}_k\left(t\right)\right) =$$

$$-\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{-\sigma} \bar{c}'\left(g\left(t\right)\right) - \lambda_c\left(t\right) \times$$

$$\{\bar{c}'\left(g\left(t\right)\right) \frac{1}{\sigma}\left(\left(1-\tau\right) f'\left(k\left(t\right)\right) - \delta - \rho\right) + \delta \bar{c}'\left(g\left(t\right)\right) - \left[\tau f\left(k\left(t\right)\right) - \delta g\left(t\right)\right] \times \bar{c}''\left(g\left(t\right)\right) -$$

$$\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right) \frac{1}{\sigma}\left(1-\tau\right) f''\left(k\left(t\right)\right) + \left[\tau f\left(k\left(t\right)\right) - \delta g\left(t\right)\right] \bar{c}''\left(g\left(t\right)\right)\}$$

$$-\lambda_g\left(t\right)\left(\tau f'\left(k\left(t\right)\right) + \delta\right) - \lambda_k\left(t\right)\left(\left(1-\tau\right) f'\left(k\left(t\right)\right) - \delta\right) \quad (7)$$

Substituting (6), $\lambda_g\left(t\right) - \lambda_k\left(t\right) = \lambda\left(t\right)$ and $\dot{\lambda}_g\left(t\right) - \dot{\lambda}_k\left(t\right) = \dot{\lambda}\left(t\right)$ into (7), then we have:

$$\rho \lambda\left(t\right) - \dot{\lambda}\left(t\right) = -\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right)^{-\sigma} \bar{c}'\left(g\left(t\right)\right) -$$

$$\frac{\lambda\left(t\right) f\left(k\left(t\right)\right)}{\left(\frac{1}{\sigma}\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right) f'\left(k\left(t\right)\right) - f\left(k\left(t\right)\right) \bar{c}'\left(g\left(t\right)\right)\right)}$$

$$\frac{\lambda\left(t\right) f\left(k\left(t\right)\right)}{\left(\frac{1}{\sigma}\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right) f'\left(k\left(t\right)\right) - f\left(k\left(t\right)\right) \bar{c}'\left(g\left(t\right)\right)\right)} \times$$

$$\{\bar{c}'\left(g\left(t\right)\right) \frac{1}{\sigma}\left(\left(1-\tau\right) f'\left(k\left(t\right)\right) - \delta - \rho\right) + \delta \bar{c}'\left(g\left(t\right)\right) - \left[\tau f\left(k\left(t\right)\right) - \delta g\left(t\right)\right] \times \bar{c}''\left(g\left(t\right)\right) -$$

$$\left(c\left(t\right) - \bar{c}\left(g\left(t\right)\right)\right) \frac{1}{\sigma}\left(1-\tau\right) f''\left(k\left(t\right)\right) + \left[\tau f\left(k\left(t\right)\right) - \delta g\left(t\right)\right] \bar{c}''\left(g\left(t\right)\right)\}$$

$$\begin{aligned}\rho\lambda(t) - \dot{\lambda}(t) = &-\left(c(t) - \bar{c}(g(t))\right)^{-\sigma}\bar{c}'(g(t)) - \\ &\frac{\lambda(t) f(k(t))}{\left(\frac{1}{\sigma}(c(t) - \bar{c}(g(t))) f'(k(t)) - f(k(t))\bar{c}'(g(t))\right)} \\ &\{\bar{c}'(g(t))\frac{1}{\sigma}((1-\tau) f'(k(t)) - \delta - \rho) + \delta\bar{c}'(g(t)) - \\ &[\tau f(k(t)) - \delta g(t)]\bar{c}''(g(t)) - (c(t) - \bar{c}(g(t)))\frac{1}{\sigma}(1-\tau) f''(k(t)) + \\ &[\tau f(k(t)) - \delta g(t)]\bar{c}''(g(t))\} - \lambda(t)[\tau f'(k(t)) + \delta] - \lambda_k(t) f'(k(t)) \quad (8)\end{aligned}$$

Note that (8) shows the evolution of $\lambda(t)$ which depends on the $k(t)$, $c(t)$, $g(t)$, $\lambda(t)$ and the $\lambda_k(t)$. We should write this equation in terms of $\{k(t), c(t), g(t), \lambda(t)\}$. Thus, from (3), we have:

$$\lambda_k(t) = (c(t) - \bar{c}(g(t)))^{-\sigma} + \lambda_c(t)\left(\frac{1}{\sigma}((1-\tau) f'(k(t)) - \delta - \rho) - \rho\right) + \dot{\lambda}_c(t) \quad (9)$$

Taking the time derivative of (6) ,we have:

$$\frac{\left(\lambda(t) f'(k(t))\dot{k}(t) + \dot{\lambda}(t) f(k(t))\right) Z - \frac{\partial Z}{\partial t}(\lambda_g(t) - \lambda_k(t)) f(k(t))}{Z^2} \quad (10)$$

where

$$Z = \left(\frac{1}{\sigma}(c(t) - \bar{c}(g(t))) f'(k(t)) - f(k(t))\bar{c}'(g(t))\right)$$

Combining (10), (9), (8) and the evolution of capital stock and public good from the budget constraints of the government, we have:

$$\dot{\lambda}(t) = \Psi\{c(t), k(t), g(t), \lambda(t)\}$$

Thus, the dynamics of the second best allocation is now governed by the following four equations:

$$
\begin{aligned}
\dot{c}(t) &= \frac{1}{\sigma}(c(t)-\bar{c}(g(t)))((1-\tau)f'(k(t))-\delta-\rho)+(\tau f(k(t))-\delta g(t))\bar{c}'(g(t)), \\
\dot{k}(t) &= (1-\tau)f(k(t))-c(t)-\delta k(t), \\
\dot{g}(t) &= \tau f(k(t))-\delta g(t), \\
\dot{\lambda}(t) &= \Psi\{c(t), k(t), g(t), \lambda(t)\}.
\end{aligned}
$$

Linearizing these four dynamic equations around the steady state where $c^* > \bar{c}(g^*)$ and constructing the associated Jacobian matrix will give the following four eigenvalues. Two of them are negative and the other two of them are positive indicating saddle path stability.