An Investigation of Investor Behaviors in Financial Markets

by

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This is to certify that I have examined this copy of a M.Sc. thesis by

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To my mother and father

ABSTRACT

Portfolio selection problem considers the optimal allocation of an investor's wealth among several risky assets within a market according to some objectives and constraints. In the idealized Capital Asset Pricing Model (CAPM) world, every investor uses the same mean-variance model of Markowitz and chooses the same efficient fund. CAPM asserts that the efficient fund is identical with the market portfolio. However, due to the violations of the assumptions of the mean-variance model, the efficient fund may not be identical with the market portfolio.

The motivation of this thesis is to find the portfolio selection model that represents investor behavior in hedge funds and US stock market. The portfolio selection models we consider in this study can be classified into two categories: (i) Portfolio selection with utility models, and (ii) Portfolio selection with prospect models.

For the utility based approach, we consider the portfolio selection models in Çanakoğlu and Özekici (2010). They focus on multiperiod portfolio optimization problem for an investor with hyperbolic absolute risk aversion (HARA) utility. In this model, asset returns depend on a stochastic market depicted by a Markov chain. We follow the rolling-sample approach in order to evaluate the out-of-sample performance of the portfolio selection models across hedge funds and US stock market datasets and find the best model that represents the market portfolio for each dataset.

In the framework of prospect theory, we extend Taşkıncan (2014) by considering prospect investors within a market consisting of multiple risky assets and one risk-free asset. In our analysis, we use the piecewise linear and piecewise exponential value functions under the assumption that asset returns have multivariate normal distribution. We test the multiple risky asset prospect models for the US stock market dataset and report the results.

ÖZETÇE

Portföy seçim problemi, bir yatırımcının servetinin market içerisindeki çok sayıda riskli varlığın arasında bazı amaç ve kısıtlara göre en iyi dağılımını göz önünde bulundurur. İdealleştirilmiş Finansal Varlıkları Fiyatlama Modeli (FVFM) dünyasında her yatırımcı Markowitz'in ortalama-varyans portföy modelini kullanır ve aynı etkin fonu seçer. FVFM etkin fonun market portföyü ile aynı olduğunu ileri sürer. Fakat ortalama-varyans modelindeki varsayımların ihlali nedeniyle etkin fon ve market portföyü aynı olmayabilir.

Bu tezin motivasyonu hedge fonları ve Amerikan hisse senedi piyasasında yatırımcı davranışlarını temsil eden portföy seçim modelini bulmaktır. Bu çalışmada göz önünde bulundurduğumuz portföy modelleri iki kategoride sınıflandırılabilir: (i) Fayda modelleri ile portföy seçimi, ve (ii) Umut modelleri ile portföy seçimi.

Fayda bazlı yaklaşım için Çanakoğlu ve Özekici (2010)'nin portföy seçim modellerini dikkate almaktayız. Bu çalışmalarda HARA tipi fayda modeline sahip bir yatırımcının çoklu zamanda portföy eniyileme problemine odaklanılmaktadır. Bu modelde varlık getirileri Markov zinciriyle tarif edilmiş rassal markete dayanmaktadır. Portföy seçim modellerinin örneklem dışı performanslarını hedge fonları ve Amerikan hisse senedi piyasasında değerlendirmek için kayan örneklem yaklaşımını takip ediyoruz ve her veri seti için market portföyünü temsil eden en iyi modeli buluyoruz.

Umut modeli çerçevesinde, Taşkıncan (2014)'ı çok sayıda riskli varlık ve bir risksiz varlıktan oluşan market içerisinde umut modelini kullanan yatırımcıları göz önünde bulundurarak genişletmekteyiz. Analizimizde, varlık getirilerinin çok değişkenli normal dağılıma sahip olduğu varsayımı altında parçalı doğrusal ve parçalı üstel değer fonksiyonlarını kullanmaktayız. Amerikan hisse senedi piyasası veri seti için çok sayıda riskli varlık içeren umut modelini test ediyor ve sonuçları raporluyoruz.

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NOMENCLATURE

CAPM	:	Capital Asset Pricing Model
HARA	:	Hyperbolic Absolute Risk Aversion
F	:	The efficient fund of risky assets
M	:	The market portfolio
m	:	Number of risky assets
Y_n	:	State of the market at time n
Y	:	Market process ($Y = \{Y_n; n = 0, 1, 2, 3, \ldots\}$)
E	:	State space of Y ($E = \{i, j, \ldots\}$)
$Q(a, b)$	:	Transition matrix of market process
$R(i)$	:	Return vector of random assets given that the market state is i
$r(i)$	:	Expected value of $R(i)$
r_f	:	Return of the risk-free asset
$\sigma_{kj}(i)$	:	Covariance between kth and jth asset returns given that the market state is i
$R_k^e(i)$	:	Excess return of kth asset in state i
$r_k^e(i)$	:	Expected value of excess return of kth asset in state i
X_n	:	Wealth level of the investor at period n
u	:	Vector representing the amounts invested in risky assets
u^*	:	Vector representing the optimal amounts invested in risky assets
$E_i[\bullet]$	:	Expectation given that the initial market state is i
$Var_i[\bullet]$	:	Variance given that the initial state is i
$U(i, x)$	:	Utility function given wealth is x and market state is i
β	:	Risk tolerance factor
β_n	:	Risk tolerance factor discounted to period n
$\alpha_n(i)$	:	Optimal investment ratio in period n if the market is in state i

$v_n(i, x)$	:	Value function given that the market is in state i and the amount of money available for investment is x at period n
w_k	:	Proportion of wealth invested in kth asset
x^{ref}	:	Reference wealth level
$V(u)$	:	Value function for the investment policy u
v^+	:	Gain part of the value function v
v^-	:	Loss part of the value function v
λ^+	:	Multiplier of the gain part
λ^-	:	Multiplier of the loss part
λ	:	Loss aversion parameter
SR	:	Sharpe ratio
CEQ	:	Certainty equivalent
TO	:	Turnover

Chapter 1

INTRODUCTION

Portfolio selection problem considers the optimal allocation of an investor's wealth among several risky assets within a market according to some objectives and constraints. Long before the advances in the portfolio selection literature, the intuitive judgments of people led them to diversification among their assets. As Markowitz (1999) highlights, in The Merchant of Venice, Act I, Scene I, William Shakespeare has Antonio say:

My ventures are not in one bottom trusted,
Nor to one place; nor is my whole estate
Upon the fortune of this present year;
Therefore, my merchandise makes me not sad.

The lines of Shakespeare point out the idea of diversification in past centuries. However, no academic research has analyzed the portfolio selection problem in a mathematical framework until the famous publication of Harry Markowitz. Markowitz (1952) is the first paper in modern financial economics to mathematically formalize the idea of diversification of investments. The mean-variance portfolio theory of Markowitz provides analytical solutions for the single period portfolio selection problem where the investors consider the mean and variance of a portfolio's return. As Markowitz states, an investor wants to maximize expected return of a portfolio for a given amount of portfolio risk, or minimize risk for a given expected return level.

The concept of mean-variance efficiency in Markowitz (1952) is one of the cornerstones of Capital Asset Pricing Model (CAPM). CAPM is the idealized framework to describe the relationship between risk and return. The model was developed in 1960s by Sharpe (1964), Lintner (1965) and Mossin (1966). Perold (2004) highlights the following assumptions for CAPM:

(i) All investors are rational and risk-averse. They evaluate their portfolios in terms of expected return and standard deviation of return over the same planning horizon.

(ii) Capital markets are perfect in many aspects. There are no transaction costs or taxes. Investors can short any asset, and hold any fraction of an asset. They can borrow and lend at the risk-free rate.

(iii) All investors have access to the same set of financial securities.

(iv) Investors have the same estimates for expected returns, standard deviations of returns and correlations among asset returns.

In the idealized CAPM world, every investor uses the same mean-variance portfolio theory of Markowitz (1952). Since all investors have access to same securities, have the same estimates about them, and solve the same optimization problem, they all have a portfolio on the same efficient frontier. That portfolio is a combination of the risk-free asset and the unique efficient fund of risky assets (denoted by F). The composition of fund F is identical for all investors. Moreover, the efficient fund used by all investors is also called the market portfolio and it is denoted by M. The market portfolio can be calculated without taking the optimization problem into account. The market is already in equilibrium and the weight of an asset in the market portfolio is given by the fraction of that stock's market value relative to the total market value of all stocks.

The mean-variance criterion can be integrated with the expected utility approach in either of two ways: (i) assuming that the asset returns have normal distribution, or (ii) by using a quadratic utility function. However, both assumptions may be violated in financial markets. The fact that asset returns exhibit fat tails and asymmetry that cannot be explained by their means and variances alone makes the first assumption ineffective. Since different investors have different preferences, the utility functions of all investors cannot be restricted to quadratic form. Due to the violations of the assumptions of the mean-variance model, the efficient fund F may not be identical with the market portfolio M.

The motivation of this thesis is to find the portfolio selection model that represents investors' behavior in the market. The portfolio selection models we consider in this study can be classified into two categories: (i) Portfolio selection with utility models, and (ii) Portfolio selection with prospect models.

For the utility based approach, we consider the portfolio selection models in Çanakoğlu and Özekici (2010). They extend the utility based approach to multiperiod portfolio optimization by considering an investor with hyperbolic absolute risk aversion (HARA) utility. The main feature of the model is that asset returns depend on a stochastic market depicted by a Markov chain. They use dynamic programming to obtain an explicit characterization of the optimal policy.

Expected utility theory is the most popular approach to the problem of choice under uncertainty. However, in laboratory settings, preferences of people violate the axioms of expected utility theory. Kahneman and Tversky (1979) come up with an alternative model of risk attitudes using "prospect theory". Under prospect theory, value is assigned to gains and losses rather than to final assets. Kahneman and Tversky define gains and losses relative to a reference point and investors' risk sensitivity changes with respect to that point. If the investor's wealth level is less than the reference point, which means a loss, then the investor shows risk-seeking behavior. If the wealth level is greater than the reference point, then the investor's attitude becomes risk-averse. Taşkıncan (2014) investigates the choices of prospect investors in a market that contains one risky and one risk-free asset. In our study, we consider prospect investors within a market consisting of multiple risky assets and one risk-free asset.

In order to determine the portfolio selection model that represents the market portfolio, we need to assess the potential gains that are realized by an investor. Hence, we analyze the out-of-sample performance of the portfolio selection models we consider relative to performance of the market portfolio across hedge funds and US stock market datasets. Following the methodology of DeMiguel et al. (2009), we compare the out-of-sample performance of our portfolio models relative to that of the market portfolio under three performance measures: (i) the out-of-sample Sharpe ratio, (ii) the certainty-equivalent (CEQ) return for the expected utility of a mean-variance investor, and (iii) the turnover for each portfolio model.

This thesis is organized as follows: Chapter 2 reviews some of the literature of portfolio selection models on expected utility theory and prospect theory. In Chapter 3, we review portfolio selection with HARA utility functions in Çanakoğlu and Özekici (2010) and portfolio selection with prospect models. We describe our methodology to evaluate the out-of-sample performance of the portfolio selection models we consider in Chapter 4. Chapter 5 describes the datasets we consider in our study. In Chapter 6, we compare the performance of the portfolio selection models considered in Chapter 3 to the benchmark market portfolios for each dataset listed in Chapter 5. Finally, Chapter 7 presents the concluding remarks and suggestions for future research.

Chapter 2

LITERATURE REVIEW

2.1 Utility Models

Mean-variance optimization concept developed by Markowitz (1952) has been the cornerstone of modern portfolio theory. It is the first systematic model to explain the trade-off between risk and return. In the single-period mean-variance model, the objective is to maximize expected return of a portfolio for a given amount of portfolio risk, or minimize risk for a given expected return level. The solution of the model gives the set of efficient portfolios.

Expected utility theory, introduced by von Neumann and Morgenstern (1944), is the leading model for making decisions under uncertainty. Markowitz (1959) addresses the theory of decision making under uncertainty to interpret his approach of mean-variance portfolio optimization. Markowitz's papers initiated research on applications of risk aversion measures and utility theory to portfolio selection. The risk aversion measure suggested by Pratt (1964) and Arrow (1965) is the most commonly used one. Mossin (1968), Merton (1969), and Hakansson (1971) consider different type of utility functions in different settings for the maximization of the expected utility of the terminal wealth.

The classical mean-variance model considers a single-period investment horizon. However, investors plan their investments on a continuous basis. Mossin (1968), Samuelson (1969), Elton and Gruber (1974), Bodily and White (1982), and Li and Ng (2000) provide examples on multi-period portfolio selection models. Moreover, most of the portfolio selection models in finance literature assume that returns of the assets in different periods are not correlated. There are some models that consider asset returns as serially correlated such as Hakansson and Liu (1970), Hernández-Hernández and Marcus (1999) and Bielecki et al. (1999).

Çakmak and Özekici (2006) model the mean-variance portfolio optimization problem in a multi-period setting where the parameters of the model including the mean and covariance of the risky asset returns depend on the stochastic market, which is an indicator of ongoing economic factors. The market process is depicted by a Markov chain with finite state space and stationary transition matrix. They consider a market consisting of one risk-free asset and several risky assets and formulate the multi-period mean-variance model. They generate an auxiliary problem and solve it using dynamic programming, which gives the same efficient frontier with the mean-variance formulation. In the numerical illustration part, they show how this procedure is applied. Çelikyurt and Özekici (2007) extend the research of Çakmak and Özekici (2006) to different problems such as safety-first approach, quadratic utility functions and coefficient of variation.

In this thesis, we apply multiperiod portfolio selection for several datasets, and we follow the work of Çanakoğlu and Özekici (2010). They consider the portfolio optimization problem where the aim is to maximize the expected utility of investor's final wealth. In Çanakoğlu and Özekici (2009), the utility function of the investor is assumed to be exponential. As an extension to that work, Çanakoğlu and Özekici (2010) consider HARA utility functions including exponential, logarithmic and power utility functions. They use stochastic market approach to construct correlation among returns in different periods where random changes of market states are assumed to form a Markov chain. They use dynamic programming and derive the optimal policies and efficient frontiers.

2.2 Prospect Models

Under expected utility model, investors evaluate their final wealth positions. However, numerous experiments show that behaviors of investors deviate from the implications of expected utility theory. Kahneman and Tversky (1979) developed prospect theory in order to deal with these deviations. In prospect theory, investors value their gains and losses rather than final absolute wealth. They define the value function on deviations from a benchmark point. The fact that the value function is generally concave for gains and convex for losses implies asymmetric risk averse behavior. In addition, the value function is steeper for losses than for gains, which implies loss aversion. Moreover, investors tend to overweight small probabilities and underweight large probabilities. Tversky and Kahneman (1992) extend prospect theory by employing cumulative decision weights. The new version of the model, cumulative prospect theory, uses different weighting functions for gains and losses to transform cumulative probabilities.

Following Kahneman and Tversky's path-breaking publications, new models were developed on portfolio choice under prospect theory. Gomes (2005) uses piecewise power functions where the value function is concave for gains, convex for small losses and concave again for large losses. De Giorgi and Hens (2006) use piecewise exponential value function where their value function is more risk-averse than the piecewise power function. Jin and Zhou (2008) consider portfolio selection problem in continuous time under cumulative prospect theory. Pirvu and Schulze (2012) focus on the single-period portfolio selection problem and find optimal portfolios for multiple risky assets that have multivariate elliptical distributions.

Taşkıncan (2014) investigates the choices of prospect investors within a market consisting of one risky and one risk-free asset. He shows the differences between structures of the value functions and derives the solution of the portfolio optimization problem. He finds that there is a mean interval for the return of the risky asset where it is optimal not to buy or shortsell the risky asset. In this thesis, we extend Taşkıncan (2014) by considering prospect investors within a market consisting of multiple risky assets and one risk- free asset.

Chapter 3

PORTFOLIO SELECTION MODELS

3.1 HARA Utility Models

Consider the multi-period portfolio optimization problem where the market consists of one risk-free asset and m risky assets. The means, variances, covariances and higher moments of the risky asset returns depend on the current state of a stochastic market. We let Y_n denote the state of the market in period n and assume that $Y = \{Y_n, n = 0, 1, 2, \ldots\}$ is a Markov chain with a discrete state space E and transition matrix Q.

Suppose that the state of the market is i in period n. The risk-free asset has known return $r_f(i)$ and standard deviation $\sigma_f(i) = 0$. The random vector of risky asset returns is denoted by $R^n(i) = (R_1^n(i), R_2^n(i), \ldots, R_m^n(i))$. We assume that the random returns in consecutive periods are conditionally independent given the market states. In other words, if $k \neq n$, then $R^n(i)$ and $R^k(i)$ are independent and identically distributed random vectors. Therefore, $R(i) = R^n(i)$ denotes the random return vector in any period n to simplify the notation provided that the state of the market is i.

Let $r_k(i) = E[R_k(i)]$ denote the mean return of the kth asset in state i and $\sigma_{kj}(i) = Cov(R_k(i), R_j(i))$ denote the covariance between kth and jth asset returns in state i. The excess return of the kth asset in state i is $R_k^e = R_k(i) - r_f(i)$. It follows that

$$r_k^e(i) = E[R_k^e(i)] = r_k(i) - r_f(i) \tag{3.1}$$

and

$$\sigma_{kj} = Cov(R_k^e(i), R_j^e(i)). \tag{3.2}$$

In our notation, $r_f(i)$ is a scalar and $r(i) = (r_1(i), r_2(i), \ldots, r_m(i))$, $r^e(i) =$

$(r_1^e(i), r_2^e(i), \ldots, r_m^e(i))$ are column vectors for all i. For any column vector z, z' denotes the row vector representing its transpose.

We define the matrix

$$V(i) = E[R^e(i)R^e(i)'] = \sigma(i) + r^e(i)r^e(i)' \tag{3.3}$$

for any state i. The covariance matrix $\sigma(i)$ is positive definite for all i. Hence $V(i)$ is also positive definite.

We use X_n to denote the wealth of an investor at period n. The investor allocates the vector $u = (u_1, u_2, \ldots, u_m)$ from his wealth to risky assets $(1, 2, \ldots, m)$ and invests the remaining wealth in the risk-free asset. The stochastic evolution of the investor's wealth is given by

$$X_{n+1}(u) = u'R(Y_n) + (X_n - 1'u)\ r_f(Y_n) = r_f(Y_n)\ X_n + u'R^e(Y_n) \tag{3.4}$$

where $1 = (1, 1, \ldots, 1)$ is the column vector of all ones.

We let $E_i[Z] = E[Z|Y_0 = i]$ and $Var_i(Z) = E_i[Z^2] - E_i[Z]^2$ denote the conditional expectation and variance of any random variable Z given that the initial market state is i. We use z as a column vector and z' as a row vector.

Assumptions of the model are:

a) Borrowing and lending is unlimited at the prevailing return of the risk-free asset,

b) Investors can short-sell all assets,

c) Capital additions or withdrawals cannot be made in the investment horizon,

d) There are no transaction costs or fees.

3.1.1 Dynamic Programming with Utility Functions

A utility function U is a non-decreasing real function defined on the real numbers. As a measure of absolute risk aversion, Pratt (1964) and Arrow (1965) suggest

$$r(x) = -\frac{U''(x)}{U'(x)}. \tag{3.5}$$

Using (3.5) hyperbolic absolute risk aversion (HARA) is described by the absolute risk aversion function

$$r(x) = \frac{1}{a+bx}. \tag{3.6}$$

HARA utility functions with an identical parameter b belong to the same class. Special cases of the HARA utility functions are:

- $b = 0$ refers to constant absolute risk averison (CARA) where the utility function has the exponential form

$$U(x) = c_1 + c_2 \exp(-x/a).$$

- $b = 1$ refers to logarithmic utility function

$$U(x) = c_1 + c_2 \ln(x+a).$$

- Other values of b ($b \neq 0$ or 1) implies power utility function.

$$U(x) = c_1 + c_2 \left(\frac{(x+a)^b}{b} \right).$$

Let $U(i, x)$ denote the utility function of an investor if the state of the market is i and the wealth is x at the terminal time T. Çanakoğlu and Özekici (2010) assume that the return for the risk-free asset is same for all market states so that $r_f(i) = r_f$ for all i. They use dynamic programming to derive the optimal solution of the multiperiod portfolio selection problem. The investor maximizes his expected utility of terminal wealth X_T at terminal time T and solves

$$\max_u E[U(Y_T, X_T) \mid Y_0 = i, X_0 = x].$$

Define $g_n(i, x, u)$ as the expected utility when investment policy u is used in period n and the optimal policies are used from period $n+1$ to period T given that the market state is i and investor's amount of wealth is x at period n. Then, the problem is

$$v_n(i, x) = \max_u \; g_n(i, x, u) \tag{3.7}$$

where

$$g_n(i, x, u_n) = E[v_{n+1}(Y_{n+1}, X_{n+1}(u_n))].$$

The dynamic programming equation (3.7) is written as

$$\begin{aligned} v_n(i, x) &= \max_u E[v_{n+1}(Y_{n+1}, X_{n+1}(u))] \\ &= \max_u \sum_{j \in E} Q(i, j) \; E[v_{n+1}(j, r_f x + u' R^e(i))] \end{aligned}$$

for $n = 0, 1, \ldots, T-1$ where the boundary condition is

$$v_T(i, x) = U(i, x)$$

for all i. The optimal solution for the problem is found by solving the dynamic programming equation (DPE) recursively.

3.1.1.1 Exponential Utility Function

The utility of the investor in state i is given by the exponential function

$$U(i, x) = K(i) - C(i) exp(-x/\beta) \tag{3.8}$$

with $\beta > 0, C(i) > 0$. In the case of exponential utility function, the measure of absolute risk aversion is $1/\beta$, and β is often called the risk tolerance.

Let the utility function of the investor be the exponential function (3.8) and suppose that the risk-free return is independent of the market state. The optimal solution of the DPE is

$$v_n(i,x) = K_n(i) - C_n(i)e^{-x/\beta_n}. \tag{3.9}$$

The optimal portfolio is

$$u_n^*(i,x) = \alpha(i)\beta_{n+1} \tag{3.10}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}}, \quad K_n(i) = Q^{T-n}K(i), \quad C_n(i) = \hat{Q}^{T-n}C(i) \tag{3.11}$$

and

$$\hat{Q}(i,j) = E[\exp(-\alpha(i)'R^e(i))]Q(i,j)$$

for all $n = 0, 1, \ldots, T-1$; and $\alpha(i)$ satisfies

$$E[R_k^e(i)\exp(-\alpha(i)'R^e(i))] = 0 \tag{3.12}$$

for all assets $k = 1, 2, \ldots, m$ and all i.

Equation (3.12) implies

$$E[(R_k(i) - r_f)\ \exp\left(-\alpha(i)'R(i)\right)] = 0$$

and

$$E[R_k(i)\exp(-\alpha(i)'R(i))] = r_f E[\exp(-\alpha(i)'R(i))]$$

or

$$\frac{E[R_k(i)\exp(-\alpha(i)'R(i))]}{E[\exp(-\alpha(i)'R(i))]} = r_f$$

for all assets $k = 1, 2, \ldots, m$. The proof for the derivation of the optimal solution can be found in Çanakoğlu and Özekici (2010).

When the state of the market is i in period n, the total amount of money invested in risky assets is

$$1'u_n^*(i,x) = 1'\alpha(i)\beta_{n+1} = \frac{\beta}{r_f^{T-(n+1)}}\sum_{k=1}^{m}\alpha_k(i)$$

and the proportion of wealth invested in asset k is

$$w_k = \frac{\alpha_k(i)}{\sum_{k=1}^{m}\alpha_k(i)}. \tag{3.13}$$

The optimal solution of the problem has special properties. Equation (3.10) indicates that the amounts of money invested in the risky assets only depend on market state i and period n and they are independent of the investor's wealth level x. In contrast, equation (3.13) states that the proportion of wealth invested in a risky asset is independent of time n and wealth x. The exponential investor considers only the market state i while allocating his wealth among risky assets. However, the composition of the risky assets is random since market states change in time.

The model incorporates the stochastic market approach which makes it more realistic. But the transition matrix Q of the stochastic market does not affect the structure of the optimal portfolio. In other words, the exponential investor is myopic in the sense that he does not take future states of the market into consideration while creating his portfolio. As it is indicated by (3.12), the optimal portfolio only depends on the joint distribution of the risky asset returns.

3.1.1.2 Logarithmic Utility Function

The utility of the investor in state i is given by the logarithmic function

$$U(i,x) = \left\{ \begin{array}{ll} K(i) + C(i)\log(x+\beta), & x+\beta > 0 \\ -\infty, & x+\beta \leq 0 \end{array} \right\} \tag{3.14}$$

with $C(i) > 0$. In the case of logarithmic utility function, the measure of absolute risk aversion is $r(x) = 1/(\beta + x) > 0$ for all i. Similarly, the risk-free asset is assumed to be the same for all market states so that $r_f(i) = r_f$ for all i.

Çanakoğlu and Özekici (2010) showed that the optimal solution of the dynamic programming equation is

$$v_n(i,x) = K_n(i) + C_n(i)\log(x + \beta_n). \tag{3.15}$$

The optimal portfolio is

$$u_n^*(i,x) = \alpha(i)(r_f x + \beta_{n+1}) \tag{3.16}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}}, \quad K_n(i) = Q^{T-n}K + \left(\sum_{m=0}^{T-n-1} Q^m \widehat{Q}_\alpha Q^{T-n-1-m}\right)C, \quad C_n = Q^{T-n}C \tag{3.17}$$

and

$$\hat{Q}_\alpha(i,j) = E\left[\log(r_f\left(1 + \alpha(i)'R^e(i)\right))\right]Q(i,j)$$

for all $n = 0, 1, \ldots, T-1$; and $\alpha(i)$ satisfies

$$E\left[\frac{R_k^e(i)}{1 + \alpha(i)'R^e(i)}\right] = 0 \tag{3.18}$$

for all assets $k = 1, 2, \ldots, m$ and all i. Moreover, the proportion of wealth invested in asset k is also given by (3.13).

The optimal solution of the logarithmic utility case has a similar structure with the exponential utility case. Likewise, the risky composition of the portfolio is both myopic and memoryless. However, equation (3.16) indicates that the amounts of money invested in the risky assets depend on market state i, period n, and investor's wealth level x. The optimal policies for exponential and logarithmic cases are not identical since (3.12) and (3.18) have different solutions.

3.1.1.3 Power Utility Function

Let the utility of the investor be the power function

$$U(i,x) = K(i) + C(i)\frac{(x-\beta)^\gamma}{\gamma} \tag{3.19}$$

and Pratt-Arrow risk aversion ratio is $r(x) = (1-\gamma)/(x-\beta)$ for all i so that $b = 1/(1-\gamma)$ and $a = \beta/(\gamma-1)$ in (3.6). We assume that the utility function is well-defined for all possible values of x. For example, if $(x-\beta) < 0$ is possible, the values of $\gamma < 1$ are excluded from the analysis. It is also important to note that γ and β are the same for all market states i. Hence, the risk classification of the investor does not depend on the stochastic market. We also let the risk-free asset be the same for all market states so that $r_f(i) = r_f$ for all i.

Çanakoğlu and Özekici (2010) showed that the optimal solution of the dynamic programming equation is

$$v_n(i,x) = K_n(i) + C_n(i)\frac{(x-\beta_n)^\gamma}{\gamma}. \tag{3.20}$$

The optimal portfolio is

$$u_n^*(i,x) = \alpha(i)(r_f x - \beta_{n+1}) \tag{3.21}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}}, \quad K_n(i) = Q^{T-n}K, \quad C_n = \widehat{Q}_\alpha^{T-n}C \tag{3.22}$$

and

$$\hat{Q}_\alpha(i,j) = E\left[(r_f(1+\alpha(i)'R^e(i)))^\gamma\right]Q(i,j)$$

for all $n = 0, 1, \ldots, T-1$; and $\alpha(i)$ satisfies

$$E\left[R_k^e(i)(1+\alpha(i)'R^e(i))^{\gamma-1}\right] = 0 \tag{3.23}$$

for all assets $k = 1, 2, \ldots, m$ and all i. Moreover, the proportion of wealth invested in asset k is also given by (3.13).

Similar interpretations can be made for the optimal portfolio structure. As it is the case for exponential and logarithmic utility, the risky composition of the power utility portfolio depends only on the market state. But, the amounts of money invested in the risky assets depend on market state i, period n, and investor's wealth level x.

If $\gamma = 1$, the power utility function (3.19) is linear and the investor's aim is to maximize the expected terminal wealth. In this case, the optimal solution is trivial since the investor will invest infinite amount of money in the asset (including risk-free asset) with the highest expected return in any market state.

Another special case of power utility function is observed when $\gamma = 2$. In this case, the utility function (3.19) has a quadratic form and there is a unique solution satisfying the optimality condition (3.23) given by

$$E\left[R^e(i)\right] + E\left[R^e(i)\alpha(i)'R^e(i)\right] = 0$$

and the optimal solution is

$$\alpha(i) = -V(i)^{-1}r^e(i)$$

where the matrix of second moments is $V(i) = E\left[R^e(i)R^e(i)'\right] = \sigma(i) + r^e(i)r^e(i)'$ and $r^e(i) = E\left[R^e(i)\right]$ is the expected value of the return vector at state i.

3.1.2 Computation of the Optimal Portfolio

The mean-variance criterion proposed by Markowitz (1952) considers the expected returns and variance of returns for the computation of the optimal portfolio. However, there is an unanimous conclusion that asset returns exhibit fat tails and asymmetry that cannot be explained by their means and variances alone (Markowitz and Usmen (1996), Peiro (1999), Harvey et al. (2010), Kolm et al. (2014)). The tails of the return distributions affect the portfolio performance. Under moderate non-normality, mean-variance framework yields a good approximation for expected utility maximization. In the case of large deviations from normality, mean-variance criterion is ineffective.

Using the methodology in Jondeau and Rockinger (2006), we incorporate higher moments in our portfolio selection model by considering the Taylor series expansion of the expected utility function. Jondeau and Rockinger (2006) consider the single-period portfolio selection problem where the investor maximizes her expected utility $U(W)$ over her end-of-period wealth W. They assume that the initial wealth is equal to one and the end-of-period wealth is given by $W = 1 + r_p$ with $r_p = \alpha' R$. They approximate the expected utility by a Taylor series expansion around the expected wealth $E[W] = \overline{W}$. The infinite order Taylor series expansion of the utility function is

$$U(W) = \sum_{j=0}^{+\infty} U^{(j)}(\overline{W}) \frac{(W - \overline{W})^j}{j!} \tag{3.24}$$

where $U^j(\overline{W})$ is the jth derivative of the utility function at $\overline{W}$. We take the expectations and write

$$\begin{aligned} E[U(W)] &= U(\overline{W}) + U^{(1)}(\overline{W})E[W - \overline{W}] + \frac{1}{2}U^{(2)}(\overline{W})E[(W - \overline{W})^2] \\ &\quad + \frac{1}{3}U^{(3)}(\overline{W})E[(W - \overline{W})^3] + \frac{1}{4}U^{(4)}(\overline{W})E[(W - \overline{W})^4] + O(W^4) \end{aligned}$$

where $O(W^4)$ is the remainder for the first 4 moments.

The expected return, variance, skewness and kurtosis of the end-of-period return are defined as

$$\mu_p = E[r_p] = \alpha' r \tag{3.25}$$

$$\sigma_p^2 = E\left[(r_p - \mu_p)^2\right] = E\left[(W - \overline{W})^2\right] \tag{3.26}$$

$$\mu_p^3 = E\left[(r_p - \mu_p)^3\right] = E\left[(W - \overline{W})^3\right] \tag{3.27}$$

$$\kappa_p^4 = E\left[(r_p - \mu_p)^4\right] = E\left[(W - \overline{W})^4\right] \tag{3.28}$$

where $r = E[R]$.

Jondeau and Rockinger (2006) approximate the expected utility by the preference function

$$E[U(W)] \approx U(\overline{W}) + \frac{1}{2}U^{(2)}(\overline{W})\sigma_p^2 + \frac{1}{3!}U^{(3)}(\overline{W})s_p^3 + \frac{1}{4!}U^{(4)}(\overline{W})\kappa_p^4. \tag{3.29}$$

In order to calculate the moments of the portfolio return, we need to define the covariance, co-skewness and co-kurtosis matrices of asset returns. The $m \times m$ covariance matrix is

$$M_2 = E[(R - r)(R - r)'] = \sigma \tag{3.30}$$

consisting of elements

$$\sigma_{ij} = E\left[(R_i - r_i)(R_j - r_j)\right] \tag{3.31}$$

for $i, j = 1, \ldots, m$. The $m \times m^2$ co-skewness matrix is

$$M_3 = E[(R - r)(R - r)' \otimes (R - r)'] = s \tag{3.32}$$

where $\otimes$ stands for the Kronecker product and

$$s_{ijk} = E\left[(R_i - r_i)(R_j - r_j)(R_k - r_k)\right] \tag{3.33}$$

for $i, j, k = 1, \ldots, m$. Finally, the $m \times m^3$ co-kurtosis matrix can be expressed as

$$M_4 = E[(R - r)(R - r)' \otimes (R - r)' \otimes (R - r)'] = \kappa \tag{3.34}$$

with elements

$$\kappa_{ijkl} = E\left[(R_i - r_i)(R_j - r_j)(R_k - r_k)(R_l - r_l)\right] \tag{3.35}$$

for $i, j, k, l = 1, \ldots, m$.

For instance, in the case of $m = 3$ assets, the resulting 3×9 co-skewness matrix is

$$M_3 = \left[\begin{array}{ccc|ccc|ccc} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{array}\right] = [S_{1\cdot\cdot} \quad S_{2\cdot\cdot} \quad S_{3\cdot\cdot}]$$

where "$\cdot$" denotes all elements within the given index and thus $S_{i\cdot\cdot}$ is the short notation for the $m \times m$ matrix $\{s_{ijk}\}_{j,k=1,2,3}$ for all $i = 1, 2, 3$.

Similarly, the co-kurtosis matrix M_4 of dimension $m \times m^3$ can be represented by m^2 $K_{ij\cdot\cdot}$ matrices with size $m \times m$ such that

$$M_4 = [K_{11\cdot\cdot}\ K_{12\cdot\cdot} \ldots K_{1m\cdot\cdot}|\ K_{21\cdot\cdot}\ K_{22\cdot\cdot} \ldots K_{2m\cdot\cdot}| \ldots |K_{m1\cdot\cdot} K_{m2\cdot\cdot} \ldots K_{mm\cdot\cdot}].$$

where $K_{ij\cdot\cdot}$ denotes the $m \times m$ matrix $\{\kappa_{ijkl}\}_{k,l=1,2,3}$ for all $i, j = 1, 2, 3$.

Using these notations, Jondeau and Rockinger (2006) express the moments of the portfolio return for a given portfolio as

$$\sigma_p^2 = E\left[\sum_{i=1}^{m} \alpha_i (R_i - r_i)(r_p - \mu_p)\right] = \alpha' M_2 \alpha,$$

$$s_p^3 = E\left[\sum_{i=1}^{m} \alpha_i (R_i - r_i)(r_p - \mu_p)^2\right] = \alpha' M_3 (\alpha \otimes \alpha),$$

$$\kappa_p^4 = E\left[\sum_{i=1}^{m} \alpha_i (R_i - r_i)(r_p - \mu_p)^3\right] = \alpha' M_4 (\alpha \otimes \alpha \otimes \alpha).$$

3.1.2.1 Exponential Utility Function

For the exponential utility function $U(x) = \exp(-x)$, we can write (3.29) as

$$E[U(W)] \cong \exp(-\overline{W})\left[1 + \frac{1}{2}\sigma_p^2 - \frac{1}{6}s_p^3 + \frac{1}{24}\kappa_p^4\right].$$

Loistl (1976) highlights the necessary conditions for the infinite Taylor series expansion to converge to the expected utility. Type of the utility function affects the region of convergence. The exponential utility function does not impose any constraints on the wealth level. However, logarithmic and power utility functions converge for wealth levels $0 < W < 2\overline{W}$.

Let's consider the problem

$$\max_{\alpha} E[U(W)] = \max_{\alpha} E\left[\exp\left(-(1+\alpha' R^e)\right)\right]. \tag{3.36}$$

by taking $W = 1 + \alpha' R^e$. The first order condition for (3.36) is

$$E[R_k^e \exp(-\alpha' R^e)] = 0. \tag{3.37}$$

Note that (3.37) is identical with the optimality condition of the exponential utility case (3.12). Therefore, it is sufficient to take $W = 1 + \alpha' R^e$ in the Taylor series expansion in order to find the optimal portfolio for the exponential utility case.

For the exponential utility function $U(x) = \exp(-x)$, the Taylor series approximation becomes

$$E[U(1+\alpha' R^e)] \cong \exp(-(1+\alpha' r^e)) \left[1 + \frac{1}{2}\sigma_p^2 - \frac{1}{6}s_p^3 + \frac{1}{24}\kappa_p^4\right]. \tag{3.38}$$

If we take the gradient of (3.38) with respect to α, and set it equal to zero, we find the first order conditions

$$-\left(1 + \frac{1}{2}\sigma_p^2 - \frac{1}{6}s_p^3 + \frac{1}{24}\kappa_p^4\right) r^e + M_2\alpha - \frac{1}{2}M_3(\alpha\otimes\alpha) + \frac{1}{6}M_4(\alpha\otimes\alpha\otimes\alpha) = 0. \tag{3.39}$$

3.1.2.2 Logarithmic Utility Function

When a similar analysis is done for logartihmic utility function $U(x) = log(x)$, the Taylor series approximation is

$$E[U(1+\alpha' R^e)] \cong \log(1+\alpha' r^e) - \frac{1}{2(1+\alpha' r^e)^2}\sigma_p^2 + \frac{1}{3(1+\alpha' r^e)^3}s_p^3 - \frac{1}{4(1+\alpha' r^e)^4}\kappa_p^4. \tag{3.40}$$

The first order conditions for (3.40) are

$$\left[\tfrac{1}{(1+\alpha' r^e)} + \tfrac{1}{(1+\alpha' r^e)^3}\sigma_p^2 - \tfrac{1}{(1+\alpha' r^e)^4}s_p^3 + \tfrac{1}{(1+\alpha' r^e)^5}\kappa_p^4\right] r^e - \tfrac{1}{(1+\alpha' r^e)^2}M_2\alpha + \tfrac{1}{(1+\alpha' r^e)^3}M_3(\alpha\otimes\alpha) - \tfrac{1}{(1+\alpha' r^e)^4}M_4(\alpha\otimes\alpha\otimes\alpha) = 0. \tag{3.41}$$

3.1.2.3 Power Utility Function

For the power utility function $U(x) = x^\gamma$, we obtain the Taylor series approximation

$$\begin{aligned} E[U(1+\alpha' R^e)] \cong\ & (1+\alpha' r^e)^\gamma + \tfrac{1}{2}\gamma(\gamma-1)(1+\alpha' r^e)^{\gamma-2}\sigma_p^2 \\ & + \tfrac{1}{6}\gamma(\gamma-1)(\gamma-2)(1+\alpha' r^e)^{\gamma-3}s_p^3 \\ & + \tfrac{1}{24}\gamma(\gamma-1)(\gamma-2)(\gamma-3)(1+\alpha' r^e)^{\gamma-4}\kappa_p^4 \end{aligned}$$

which give the optimality condition

$$\begin{aligned} & \gamma(1+\alpha' r^e)^{\gamma-1}r^e + \tfrac{1}{2}\gamma(\gamma-1)(\gamma-2)(1+\alpha' r^e)^{\gamma-3}\sigma_p^2 r^e + \gamma(\gamma-1)(1+\alpha' r^e)^{\gamma-2}M_2\alpha \\ & + \tfrac{1}{6}\gamma(\gamma-1)(\gamma-2)(\gamma-3)(1+\alpha' r^e)^{\gamma-4}s_p^3 r^e \\ & + \tfrac{1}{2}\gamma(\gamma-1)(\gamma-2)(1+\alpha' r^e)^{\gamma-3}M_3(\alpha\otimes\alpha) \\ & + \tfrac{1}{24}\gamma(\gamma-1)(\gamma-2)(\gamma-3)(\gamma-4)(1+\alpha' r^e)^{\gamma-5}\kappa_p^4 r^e \\ & + \tfrac{1}{6}\gamma(\gamma-1)(\gamma-2)(\gamma-3)(1+\alpha' r^e)^{\gamma-4}M_4(\alpha\otimes\alpha\otimes\alpha) = 0. \end{aligned} \tag{3.42}$$

In order to determine the optimal portfolio, we compute the sample covariance, co-skewness and co-kurtosis matrices using the equations (3.31), (3.33) and (3.35) for a given historical dataset of monthly asset returns. Using the first order conditions (3.39), (3.41) and (3.42), we compute the critical values for the optimal investment ratio vector α for the single-period portfolio selection problem. We find the critical values of α by using the "fsolve" function of MATLAB.

3.2 Portfolio Selection with Prospect Models

In Section 3.1, we considered models based on expected utility theory using HARA utility functions. However, human behavior deviates from the implications of expected utility theory. This deviation led to several alternative models of choice. Among these models, the prospect model of Tversky and Kahneman (1992) is the outstanding one. They propose cumulative prospect theory (CPT), which builds on experimental findings on human judgment and decision-making behavior.

In this section, we consider the single-period portfolio optimization problem under CPT. The market consists of one risk-free asset with known return r_f and m risky assets with random rate of returns $R = (R_1, R_2, \ldots, R_m)$. The investor allocates the vector $u = (u_1, u_2, \ldots, u_m)$ from his initial wealth x_0 to risky assets $(1, 2, \ldots, m)$ and invests the remaining amount $x_0 - u'1$ in the risk-free asset. The investor's terminal wealth at the end of the period is given by

$$X = (x_0 - u'1)(1 + r_f) + u'(1 + R)$$

or

$$X = x_0(1 + r_f) + u'R^e$$

where $R^e = R - r_f$ is the vector of excess returns.

While evaluating the final wealth, the investor considers the risk-free reference point $x^{ref} = x_0(1 + r_f)$ so that the deviation of wealth from this reference point is

$$X - x^{ref} = u'R^e.$$

Let $r_k = E[R_k]$ denote the mean rate of return of the kth asset and $\sigma_{kj} = Cov(R_k, R_j)$ denote the covariance between kth and jth asset returns. The excess return of kth asset is $R_k^e = R_k - r_f$ and $r_k^e = E\left[R_k^e\right] = r_k - r_f$.

In CPT, gains and losses are evaluated in a different way using different value functions such as linear, power, exponential or others. Tversky and Kahneman (1992) suggest the use of the piecewise power value function kinked at zero. Up to zero, the value function is convex increasing. This part of the function is the loss part in which the investor exhibits risk-seeking behavior. The part after zero is called the gain part and in this part the investor exhibits risk-averse behavior. De Giorgi and Hens (2006) propose the use of piecewise exponential value function.

We start our analysis by using the piecewise linear value function as our value function under the assumption that asset returns have multivariate normal distribution. After analyzing this model, we take the piecewise exponential value function into consideration. Similarly, we assume that asset returns have multivariate normal distribution.

3.2.1 Piecewise Linear Value Function

Portfolio optimization problem with piecewise linear value function and risk-free reference point is

$$\max_u V(u) = E[v(u'R^e)]$$

where $v(x)$ has the form

$$v(x) = \begin{cases} v^+(x) = \lambda^+ x & x \geq 0 \\ v^-(x) = \lambda^- x & x < 0 \end{cases} \tag{3.43}$$

with λ^+,$\lambda^- > 0$. Tversky and Kahneman (1992) suggest that $\lambda^- > \lambda^+$ and build their research on this assumption. Likewise, we assume that $\lambda^- > \lambda^+$ in our analysis. The shape of the piecewise linear function is illustrated in Figure 3.1.

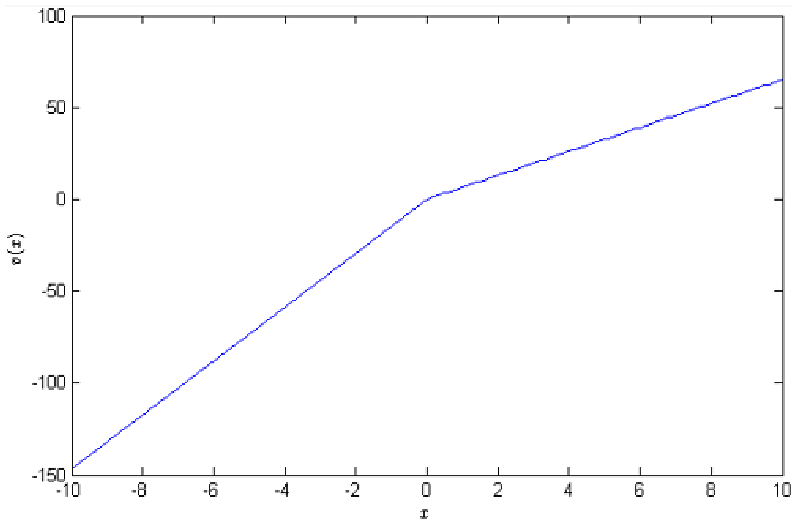

Figure 3.1: Piecewise linear value function ($\lambda^{-} = 14.7$, $\lambda^{+} = 6.52$)

Multivariate Normal Return Model

We suppose that the excess return R^e has multivariate normal distribution. For the sake of simplicity in our expressions, we will introduce some definitions regarding the standard normal distribution.

The standard normal density function is

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and the standard normal cumulative function is represented by

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

It is clear that φ is a symmetric function around zero, hence $\varphi(x) = \varphi(-x)$. In addition, Φ is the cumulative function of φ and

$$\Phi(+\infty) = \lim_{y \to +\infty} \Phi(y) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1.$$

Since φ is symmetric, $\Phi(-y) = 1 - \Phi(y)$. We also define so that

$$\Psi(y) = \int_{y}^{+\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

or

$$\Psi(y) = \frac{1}{\sqrt{2\pi}} \int_{\frac{y^2}{2}}^{+\infty} e^{-t} dt$$

or

$$\Psi(y) = \left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) \Big|_{y}^{+\infty}$$

or

$$\Psi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

or

$$\Psi(y) = \varphi(y).$$

Suppose that the excess return vector R^e has multivariate normal distribution with mean vector $\mu - r_f$ and covariance matrix σ. Then, the objective function is

$$\begin{aligned} V(u) &= E[v^-(u'R^e)1_{\{u'R^e<0\}}] + E[v^+(u'R^e)1_{\{u'R^e\geq 0\}}] \\ &= \lambda^- E[(u'R^e)1_{\{u'R^e<0\}}] + \lambda^+ E[(u'R^e)1_{\{u'R^e\geq 0\}}]. \end{aligned}$$

When the market consists of only one risky asset, we can analyze the value function for $u \geq 0$ and $u < 0$ separately and find a solution to the problem as it is done in Taşkıncan (2014). Since u is a vector in our case, we find the value function by using the information that $u'R^e$ is a normal random variable with mean μ_u and variance σ_u^2. It follows that

$$\mu_u = E[u'R^e] = u'r^e$$

and

$$\sigma_u^2 = u'\sigma u.$$

Let $V_{-}(u)$ denote the loss part and $V_{+}(u)$ denote the gain part of the objective function. The loss part of the objective function is

$$V_{-}(u) \quad = \lambda^{-} E[u' R^{e} 1_{\{u' R^{e} < 0\}}].$$

Taking $u' R^{e} = \mu_u + \sigma_u Z$ where $Z \sim N(0,1)$, we obtain

$$\begin{aligned}
V_{-}(u) &= \lambda^{-} \int_{-\infty}^{-\mu_u/\sigma_u} \frac{\mu_u + \sigma_u z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \lambda^{-} \mu_u \int_{-\infty}^{-\mu_u/\sigma_u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \lambda^{-} \sigma_u \int_{-\infty}^{-\mu_u/\sigma_u} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \lambda^{-} \mu_u \int_{-\infty}^{-\mu_u/\sigma_u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \lambda^{-} \sigma_u \left(\int_{-\infty}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \int_{-\mu_u/\sigma_u}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right) \\
&= \lambda^{-} \mu_u \Phi(-\tfrac{\mu_u}{\sigma_u}) - \lambda^{-} \sigma_u \Psi(-\tfrac{\mu_u}{\sigma_u}) \\
&= \lambda^{-} \mu_u \Phi(-\tfrac{\mu_u}{\sigma_u}) - \lambda^{-} \sigma_u \varphi(\tfrac{\mu_u}{\sigma_u}).
\end{aligned}$$

For the gain part, we write

$$V_{+}(u) \quad = \lambda^{+} E[u' R^{e} 1_{\{u' R^{e} \geq 0\}}].$$

Taking $u' R^{e} = \mu_u + \sigma_u Z$, we obtain

$$\begin{aligned}
V_{+}(u) &= \lambda^{+} \int_{-\mu_u/\sigma_u}^{+\infty} \frac{\mu_u + \sigma_u z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \lambda^{+} \mu_u \int_{-\mu_u/\sigma_u}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \lambda^{+} \sigma_u \int_{-\mu_u/\sigma_u}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \lambda^{+} \mu_u \left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \int_{-\infty}^{-\frac{\mu_u}{\sigma_u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right) + \lambda^{+} \sigma_u \int_{-\mu_u/\sigma_u}^{+\infty} \frac{z}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= \lambda^{+} \mu_u \left(1 - \Phi(-\tfrac{\mu_u}{\sigma_u}) \right) + \lambda^{+} \sigma_u \Psi(-\tfrac{\mu_u}{\sigma_u}) \\
&= \lambda^{+} \mu_u \Phi(\tfrac{\mu_u}{\sigma_u}) + \lambda^{+} \sigma_u \varphi(\tfrac{\mu_u}{\sigma_u}).
\end{aligned}$$

Finally, the objective function is

$$V(u) = V_-(u) + V_+(u) = \lambda^- \mu_u \Phi(-\tfrac{\mu_u}{\sigma_u}) - \lambda^- \sigma_u \varphi(\tfrac{\mu_u}{\sigma_u}) + \lambda^+ \mu_u \Phi(\tfrac{\mu_u}{\sigma_u}) + \lambda^+ \sigma_u \varphi(\tfrac{\mu_u}{\sigma_u}). \tag{3.44}$$

This can be analyzed further for the problem $\max_u V(u)$, but it is not in the scope of this thesis. For the single risky-asset version of this problem, Taşkıncan (2014) derives the optimal portfolio u^* that maximizes (3.44). Suppose that the excess return of the risky asset R^e has the normal distribution with parameters μ and σ. For $\lambda = \lambda^-/\lambda^+ > 1$, the optimal portfolio is:

1. If $\mu \geq 0$, then

$$u^* = \begin{cases} 0 & \mu < \sigma \bar{x}_1 \\ 0 \leq u^* \leq +\infty & \mu = \sigma \bar{x}_1 \\ +\infty & \mu > \sigma \bar{x}_1 \end{cases} \tag{3.45}$$

   where $\bar{x}_1$ satisfies $\Phi(\bar{x}_1) + \varphi(\bar{x}_1)/\bar{x}_1 = 1/(1-\lambda)$.

2. If $\mu < 0$, then

$$u^* = \begin{cases} 0 & \mu > \sigma \bar{x}_2 \\ -\infty \leq u^* \leq 0 & \mu = \sigma \bar{x}_2 \\ -\infty & \mu < \sigma \bar{x}_2 \end{cases} \tag{3.46}$$

   where $\bar{x}_2$ satisfies $\Phi(-\bar{x}_2) - \varphi(\bar{x}_2)/\bar{x}_2 = 1/(1-\lambda)$.

3.2.2 Piecewise Exponential Value Function

Portfolio optimization problem with piecewise exponential value function and risk-free reference point is

$$\max_u V(u) = E[v(u'R^e)]$$

where $v(x)$ has the form

$$v(x) = \begin{cases} v^+(x) = \lambda^+(1 - \exp(-x/\beta)) & x \geq 0 \\ v^-(x) = -\lambda^-(1 - \exp(x/\beta)) & x < 0 \end{cases} \tag{3.47}$$

with λ^+,λ^-,$\beta > 0$. Similar to the case of HARA utility functions, we define β as the risk tolerance and $1/\beta$ is the absolute risk aversion. The shape of piecewise exponential value function is illustrated in Figure 3.2.

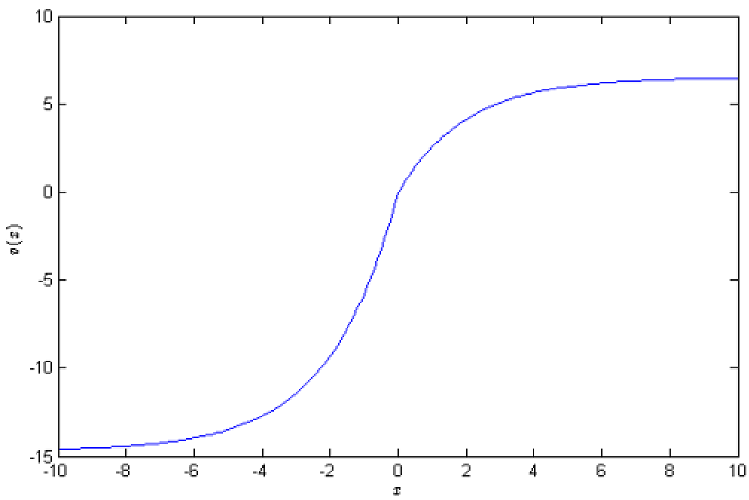

Figure 3.2: Piecewise exponential value function ($\beta = 1$, $\lambda^- = 14.7$, $\lambda^+ = 6.52$)

Multivariate Normal Return Model

Suppose that the excess return R^e has multivariate normal distribution with mean vector $\mu - r_f$ and covariance matrix σ. Then, the objective function is

$$\begin{aligned} V(u) &= E[v^-(u'R^e)1_{\{u'R^e<0\}}] + E[v^+(u'R^e)1_{\{u'R^e\geq 0\}}] \\ &= -\lambda^- E[(1 - e^{u'R^e/\beta})1_{\{u'R^e<0\}}] + \lambda^+ E[(1 - e^{-u'R^e/\beta})1_{\{u'R^e\geq 0\}}]. \end{aligned}$$

Similar to the linear value function case, we use the information that $R^{e'}u$ is a normal random variable with mean $\mu_u = E[u'R^e] = u'r^e$ and variance $\sigma_u^2 = u'\sigma u$.

Let $V_-(u)$ denote the loss part and $V_+(u)$ denote the gain part of the value function.

If we analyze the loss part, we can write

$$\begin{aligned} V_-(u) &= -\lambda^- E[(1-e^{u'R^e/\beta})1_{\{u'R^e<0\}}] \\ &= -\lambda^- E[1_{\{u'R^e<0\}}] + \lambda^- E[e^{u'R^e/\beta}\ 1_{\{u'R^e<0\}}]. \end{aligned}$$

We take $u'R^e = \mu_u + \sigma_u Z$ and obtain

$$\begin{aligned} V_-(u) &= -\lambda^- E[1_{\{\mu_u+\sigma_u Z<0\}}] + \lambda^- E[e^{(\mu_u+\sigma_u Z)/\beta} 1_{\{\mu_u+\sigma_u Z<0\}}] \\ &= -\lambda^- E[1_{\{Z<-\frac{\mu_u}{\sigma_u}\}}] + \lambda^- e^{\mu_u/\beta} E[e^{\sigma_u Z/\beta} 1_{\{Z<-\frac{\mu_u}{\sigma_u}\}}] \\ &= -\lambda^- \int_{-\infty}^{-\mu_u/\sigma_u} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \lambda^- e^{\mu_u/\beta} \int_{-\infty}^{-\mu_u/\sigma_u} \tfrac{1}{\sqrt{2\pi}} e^{\sigma_u z/\beta}\ e^{-\frac{1}{2}z^2} dz \\ &= -\lambda^- \Phi(-\tfrac{\mu_u}{\sigma_u}) + \lambda^- e^{(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2} \int_{-\infty}^{-\mu_u/\sigma_u} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-(\sigma_u/\beta))^2} dz \\ &= -\lambda^- \Phi(-\tfrac{\mu_u}{\sigma_u}) + \lambda^- e^{(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2} \int_{-\infty}^{-((\sigma_u^2/\beta)+\mu_u)/\sigma_u} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= -\lambda^- \Phi(-\tfrac{\mu_u}{\sigma_u}) + \lambda^- e^{(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2} \Phi(-\tfrac{(\sigma_u^2/\beta)+\mu_u}{\sigma_u}). \end{aligned}$$

If we analyze the gain part, we obtain

$$\begin{aligned} V_+(u) &= \lambda^+ E[(1-e^{-u'R^e/\beta})1_{\{u'R^e\geq 0\}}] \\ &= \lambda^+ E[1_{\{u'R^e\geq 0\}}] - \lambda^+ E[e^{-u'R^e/\beta} 1_{\{u'R^e\geq 0\}}]. \end{aligned}$$

Likewise, using $u'R^e = \mu_u + \sigma_u Z$ we get

$$\begin{aligned} V_+(u) &= \lambda^+ E[1_{\{\mu_u+\sigma_u Z\geq 0\}}] - \lambda^+ E[e^{-(\mu_u+\sigma_u Z)/\beta} 1_{\{\mu_u+\sigma_u Z\geq 0\}}] \\ &= \lambda^+ E[1_{\{Z\geq -\frac{\mu_u}{\sigma_u}\}}] - \lambda^+ e^{-\mu_u/\beta} E[e^{-\sigma_u Z/\beta} 1_{\{Z\geq -\frac{\mu_u}{\sigma_u}\}}] \\ &= \lambda^+ \int_{-\mu_u/\sigma_u}^{+\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \lambda^+ e^{-\mu_u/\beta} \int_{-\mu_u/\sigma_u}^{+\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\sigma_u z/\beta} e^{-\frac{1}{2}z^2} dz \\ &= \lambda^+ (1-\Phi(-\tfrac{\mu_u}{\sigma_u})) - \lambda^+ e^{-(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2} \int_{-\mu_u/\sigma_u}^{+\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z+(\sigma_u/\beta))^2} dz \\ &= \lambda^+ (1-\Phi(-\tfrac{\mu_u}{\sigma_u})) - \lambda^+ e^{-(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2} \int_{(\sigma_u^2/\beta-\mu_u)/\sigma_u}^{+\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\ &= \lambda^+ (1-\Phi(-\tfrac{\mu_u}{\sigma_u})) - \lambda^+ e^{-(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2} (1-\Phi(\tfrac{(\sigma_u^2/\beta)-\mu_u}{\sigma_u})) \\ &= \lambda^+ \Phi(\tfrac{\mu_u}{\sigma_u}) - \lambda^+ e^{-(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2} \Phi(\tfrac{\mu_u-(\sigma_u^2/\beta)}{\sigma_u}). \end{aligned}$$

Finally, the objective function is

$$\begin{aligned} V(u) &= V_{-}(u) + V_{+}(u) = & -\lambda^{-}\Phi(-\tfrac{\mu_u}{\sigma_u}) + \lambda^{-}e^{(\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2}\Phi(-\tfrac{(\sigma_u^2/\beta)+\mu_u}{\sigma_u}) \\ & & + \lambda^{+}\Phi(\tfrac{\mu_u}{\sigma_u}) - \lambda^{+}e^{(-\mu_u/\beta)+\frac{1}{2}\sigma_u^2/\beta^2}\Phi(\tfrac{\mu_u-(\sigma_u^2/\beta)}{\sigma_u}). \end{aligned} \tag{3.48}$$

Similar to the piecewise linear value function case, the optimal solution can be found for the problem $\max_u V(u)$. For the single risky asset problem, Taşkıncan (2014) finds the optimal solution as:

1. If $\mu \geq 0$, then

$$u^* = \begin{cases} 0 & \frac{f\left(\frac{\mu}{\sigma}\right)}{f\left(\frac{-\mu}{\sigma}\right)} \geq \lambda \\ \bar{u} & \frac{f\left(\frac{\mu}{\sigma}\right)}{f\left(\frac{-\mu}{\sigma}\right)} < \lambda \end{cases} \tag{3.49}$$

   where $\bar{u}$ is the unique positive value that satisfies

$$\frac{f\left(\frac{\mu+\sigma^2\bar{u}}{\sigma}\right)e^{2\mu\bar{u}}}{f\left(\frac{-\mu+\sigma^2\bar{u}}{\sigma}\right)} = \lambda \tag{3.50}$$

   and $f(x) = x\Phi(-x) - \varphi(x)$.

2. If $\mu < 0$, then

$$u^* = \begin{cases} 0 & \frac{g\left(\frac{\mu}{\sigma}\right)}{g\left(\frac{-\mu}{\sigma}\right)} \geq \lambda \\ \bar{u} & \frac{g\left(\frac{\mu}{\sigma}\right)}{g\left(\frac{-\mu}{\sigma}\right)} < \lambda \end{cases} \tag{3.51}$$

   where $\bar{u}$ is the unique positive value that satisfies

$$\frac{g\left(\frac{\mu+\sigma^2\bar{u}}{\sigma}\right)e^{2\mu\bar{u}}}{g\left(\frac{-\mu+\sigma^2\bar{u}}{\sigma}\right)} = \lambda \tag{3.52}$$

   and $g(x) = x\Phi(x) + \varphi(x)$.

   In order to compute the optimal portfolios, we maximize the value functions (3.44) and (3.48) in MATLAB by using the "fmincon" function. We report our results for prospect portfolios in Chapter 6.

Chapter 4

METHODOLOGY

We aim to evaluate the out-of-sample performance of the aforementioned portfolio selection models across several datasets and to find the best model that represents the market portfolio for each dataset. We consider the models with HARA class of utility functions such as exponential, logarithmic and power ($\gamma = 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5,\ 2$ (quadratic), 4) and prospect models with piecewise linear and piecewise exponential value function. In addition, we report the performance of the naively diversified portfolio. We define naive diversification as allocating $1/m$ fraction of the investor's wealth to each of the m risky assets.

In line with DeMiguel et al. (2009) and Daskalaki and Skiadopoulos (2011), we follow the "rolling-sample" approach in order to evaluate the out-of-sample performance of the portfolios. To assess the magnitude of the potential gains that can actually be realized by an investor, it is necessary to analyze the out-of-sample performance of the strategies from the optimizing models. Given a dataset consisting of T monthly asset returns, we choose an estimation window of $K < T$ months. In each month $t > K$, we use previous K observations to calculate the parameters (3.31), (3.33) and (3.35) for optimal portfolio computation. Using these parameters, we compute the optimal portfolio weights of the risky assets. These weights are used to compute the out-of-sample return for period $[t, t+1]$. We repeat this process by adding the asset returns of next period and removing the earliest returns, until we reach the end of the dataset. This rolling-sample approach yields $T - K$ monthly out-of-sample portfolio returns for each dataset and for each portfolio selection model that we consider in our study.

Suppose that the we have a dataset of monthly asset returns for the time period

January 2003-May 2015 ($T = 149$) and we use an estimation window of $K = 120$ months. In order to compute the optimal portfolio for January 2013, we calculate our parameters using the data during $K = 120$ months between January 2003 and December 2012. Since the asset returns in January 2013 are realized, the out-of-sample performance of the formed portfolio is evaluated. Then, for the computation of the portfolio to hold in February 2013, we calculate our parameters using the data of February 2003-January 2013 for $K = 120$ months. Similarly, we calculate the out-of-sample performance of the portfolio by using the asset returns of February 2013. We follow these steps until we reach the end of the dataset, May 2015. As a result, we obtain $T - K = 29$ monthly out-of-sample returns. In Figure 4.1 and 4.2, we illustrate the calculation of the portfolios for months January 2013 and February 2013 respectively.

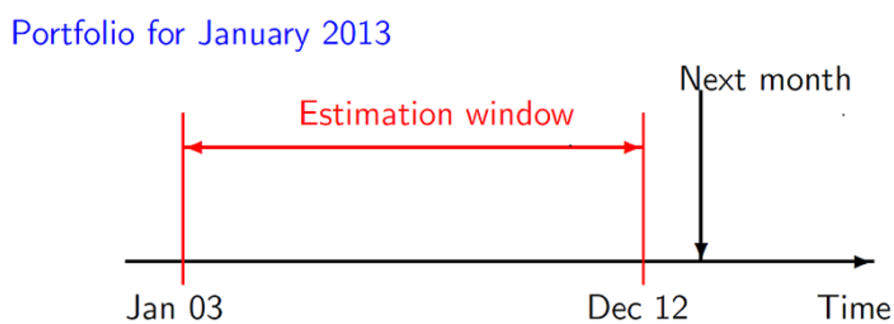

Figure 4.1: Calculation of the portfolio for January 2013

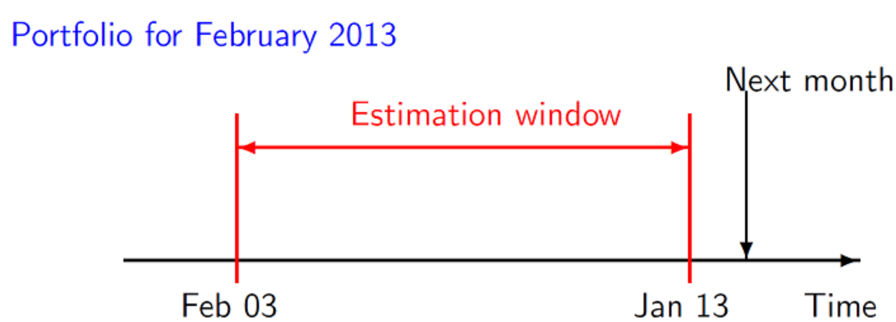

Figure 4.2: Calculation of the portfolio for February 2013

The models with HARA utility functions are multi-period portfolio selection models and they depend on an external process that represents the stochastic market. In parallel with our portfolio performance evaluation method, we suppose that the investor selects static one-period optimal portfolio and rebalance the portfolio at the start of each month. In other words, we consider the HARA utility models as single-period models. This consideration does not contradict with the explicit characterization of the optimal policy. As it is specified by (3.13), the risky composition of the portfolio only depends on the market state.

Following DeMiguel et al. (2009) and Daskalaki and Skiadopoulos (2011), we use three measures in order to evaluate the performance of the considered portfolio models: Sharpe ratio (SR), certainty-equivalent (CEQ) return and portfolio turnover (PT).

4.1 Performance Measures

4.1.1 Sharpe Ratio

The Sharpe ratio, introduced by Sharpe (1965), is the ratio of the excess expected return of an investment to the standard deviation of excess returns. The measure is motivated by mean-variance analysis and Sharpe-Lintner Capital Asset Pricing Model. We measure the out-of-sample Sharpe ratio of portfolio selection model c, defined as the sample mean of out-of-sample excess returns $\widehat{\mu}_c$, divided by their sample standard deviation $\widehat{\sigma}_c$ given by

$$\widehat{SR}_c = \frac{\widehat{\mu}_c}{\widehat{\sigma}_c}. \tag{4.1}$$

To determine the effect of estimation error, we report the in-sample Sharpe ratio for the mean-variance strategy. That is the case with no estimation error and it gives the highest Sharpe ratio of all the strategies considered. We use the entire time series of asset returns by taking the estimation window $M = T$. The in-sample Sharpe ratio is defined as

$$\widehat{SR}_{IS} = \frac{\widehat{\mu}'_{IS}\widehat{w}}{\sqrt{\widehat{w}'\Sigma_{IS}\widehat{w}}} \tag{4.2}$$

in which $\widehat{\mu}_{IS}$ and Σ_{IS} are the in-sample mean and variance estimates and $\widehat{w}$ is the vector of portfolio weights computed using these estimates.

We compare the Sharpe ratio of each portfolio selection model to the Sharpe ratio of a benchmark portfolio. For each dataset, we select the corresponding market index as the benchmark. In order to test whether the Sharpe ratios of two portfolio selection models are statistically different, we employ the most commonly used method was introduced by Jobson and Korkie (1981) and corrected by Memmel (2003), which we refer as JKM-test. DeMiguel et al. (2009) examines the computation of the test statistic in detail. For two portfolios g and h, with mean estimates $\widehat{\mu}_g$ and $\widehat{\mu}_h$, variance estimates $\widehat{\sigma}_g$ and $\widehat{\sigma}_h$, and estimated covariance $\widehat{\sigma}_{gh}$ over a sample size of $T-K$, the hypothesis

$$H_0 : \frac{\widehat{\mu}_g}{\widehat{\sigma}_g} = \frac{\widehat{\mu}_h}{\widehat{\sigma}_h}$$

is tested against the two-sided alternative. The test statistic is defined as

$$\widehat{z}_{JKM} = \frac{\widehat{\sigma}_h\widehat{\mu}_g - \widehat{\sigma}_g\widehat{\mu}_h}{\sqrt{\widehat{\vartheta}}} \tag{4.3}$$

where

$$\widehat{\vartheta} = \frac{1}{T-K}\left(2\widehat{\sigma}_g^2\widehat{\sigma}_h^2 - 2\widehat{\sigma}_g\widehat{\sigma}_h\widehat{\sigma}_{gh} + \frac{1}{2}\widehat{\mu}_g^2\widehat{\sigma}_h^2 + \frac{1}{2}\widehat{\mu}_h^2\widehat{\sigma}_g^2 - \frac{\widehat{\mu}_g\widehat{\mu}_h}{\widehat{\sigma}_g\widehat{\sigma}_h}\widehat{\sigma}_{gh}^2\right)$$

and $\widehat{z}_{JKM}$ is asymptotically distributed as standard normal. However, this test may not work if asset returns are far from normally distributed or autocorrelated. A more generally valid test method for the comparison of the Sharpe ratios was introduced by Ledoit and Wolf (2008). This procedure is robust to nonnormal and serially correlated portfolio returns. Ledoit and Wolf (2008) propose to construct a studentized time series bootstrap confidence interval for the difference of the Sharpe ratios and conclude that the two ratios are different if the obtained interval does not contain zero. Since the proposed method is complex to implement, the authors made the programming code available at `http://www.econ.uzh.ch/faculty/wolf/publications.html`.

To assess the statistical difference of two portfolios in terms of Sharpe ratio, we calculate the p-value of the JKM-test. If the p-value of the test is greater than the significance level, we do not reject the null hypothesis and conclude that two portfolios are identical in terms of their Sharpe ratios.

4.1.2 Certainty Equivalent Return

Certainty-equivalent return for a risky portfolio is the risk-free return such that the investor is indifferent between that risk-free return and the stochastic return of the risky portfolio. The CEQ return of portfolio selection model c is calculated as

$$\widehat{CEQ}_c = \widehat{\mu}_c - \frac{\gamma}{2}\widehat{\sigma}_c^2 \tag{4.4}$$

where γ is the risk aversion parameter. This definition of CEQ in (4.4) is the expected utility of a mean-variance investor. As DeMiguel et al. (2009) proposes, it is approximately the CEQ of an investor with quadratic utility. Despite this fact, we stick with the emprical literature and use (4.4) as the certainty equivalent for portfolio selection model c. To assess the statistical difference of two portfolios, we calculate the p-value of the test.

We follow DeMiguel et al. (2009) in our test statistic computation. Let $\upsilon = (\mu_h, \mu_g, \sigma_h, \sigma_g)$ denote the vector of moments and $\widehat{\upsilon}$ be the empirical counterpart from a sample of size $T - K$. Let

$$f(\upsilon) = \left(\widehat{\mu}_g - \frac{\gamma}{2}\widehat{\sigma}_g^2\right) - \left(\widehat{\mu}_h - \frac{\gamma}{2}\widehat{\sigma}_h^2\right)$$

be the difference in the certainty equivalent of two strategies g and h. The asymptotic distribution of $f(\upsilon)$ is

$$\sqrt{T-K}(f(\widehat{\upsilon})-f(\upsilon)) \longrightarrow N(0, \frac{\partial f'}{\partial \upsilon}\Theta\frac{\partial f}{\partial \upsilon}) \tag{4.5}$$

where Θ stands for

$$\Theta = \begin{pmatrix} \sigma_g^2 & \sigma_{g,h} & 0 & 0 \\ \sigma_{g,h} & \sigma_h^2 & 0 & 0 \\ 0 & 0 & 2\sigma_g^4 & 2\sigma_{g,h}^2 \\ 0 & 0 & 2\sigma_{g,h}^2 & 2\sigma_h^4 \end{pmatrix}.$$

4.1.3 Turnover

Turnover quantifies the amount of trading required to implement each portfolio selection strategy. The portfolio turnover PT_c for a strategy c is the average absolute change in the portfolio weights over $T-K$ periods and across m assets, which is defined as

$$PT_c = \frac{1}{T-K}\sum_{t=1}^{T-K}\sum_{j=1}^{m}(|w_{c,j,t+1} - w_{c,j,t+}|) \tag{4.6}$$

where $w_{c,j,t}$ and $w_{c,j,t+1}$ are the optimal weights of asset j under strategy c at time t and $t+1$ respectively. Note that $w_{c,j,t+1}$ is the weight before rebalancing at time $t+1$. Hence, $|w_{c,j,t+1} - w_{c,j,t+}|$ is the amount of trade needed for asset j at the rebalancing point $t+1$.

4.2 Robustness Checks

In our benchmark case, we assume that:

i) The investor does not classify the historical performance of the securities with respect to market states or regimes. In other words, the investor does not consider the market as regime switching.

ii) The estimation window is rolling.

iii) The length of estimation window is $K = 120$ months.

iv) The investor holds the portfolio for one month.

In order to compare the performance of the considered portfolio models with the market portfolio, the benchmark case would not be sufficient to reach a conclusion for a given dataset. To check the robustness of our result, we change the aforementioned assumptions one by one and report Sharpe ratio, CEQ return and turnover for each case.

In Table 4.1, we report the choices for the benchmark case and the alternative case. Then, we explain each choice in detail.

Table 4.1: Benchmark case and alternative choices for robustness-check experiments

Choice	**Benchmark**	**Tag**	**Alternative**	**Tag**
Stochastic market approach	Not regime-switching	NotRS	Regime-switching	RS
Estimation window type	Rolling	Rolling	Expanding	Expanding
Estimation window length	120 months	M120	60 months	M60
Investment horizon period	1 month	hp1	12 months	hp12

4.2.1 Regime-switching Market

For the regime-switching application, we classify the states of the market by considering the overall economic activity in US. Our discrete state space E consists of two states: expansion and recession. We consider the following steps to compute the optimal portfolio that the investor will hold for period $[t, t+1]$.

1. For a given dataset, we classify the state of the market for each month in our time-horizon T. In order to do this, we use Business Cycle Expansions and Contractions data of National Bureau of Economic Research (NBER) and we classify each month into two market states: expansion and recession. We explain our classification procedure further in Chapter 5.

2. Suppose that the market state is expansion (recession) at month $t-1$. For the calculation of parameters (3.31), (3.33) and (3.35), we consider the historical asset returns at months that are just after the expansion (recession) months.

4.2.2 Estimation Window Type

In our benchmark case, we use rolling-sample approach, in which the earliest observation is dropped out when a new observation is added. In the expanding window approach, we do not remove any observations when we add a new observation

4.2.3 Estimation Window Length

In the benchmark case, the length of the estimation window is $K = 120$ months. In our experiment, we consider the case when the estimation window is $K = 60$ months.

4.2.4 Investment Horizon Period

We hold the optimal portfolio for one month in our benchmark case, since we work with monthly asset returns data. Following DeMiguel et al. (2009), we consider a holding period of one year and report the results.

Chapter 5

DATA

In our study, we consider the emprical datasets in Table 5. We provide the descriptive statistics for each dataset in Appendix A.

Table 5.1: List of datasets considered

#	Dataset and source	m	Time Period	Abbrevation
1	HFRI Indices Source: Hedge Fund Research Website	4	01/1990-02/2014	HFRI
2	Credit Suisse Hedge Fund Indices Source: CS Hedge Fund Indices Website	9	05/1994-11/2014	CSHF
3	Five industry portfolios Source: Kenneth French Website	5	01/1990-02/2014	I5
4	Ten industry portfolios Source: Kenneth French Website	10	01/1990-02/2014	I10

Alternative investments in general, and hedge funds in particular, have become more and more important for the institutional investors, banks and the traditional fund houses in recent years. Bali et al. (2013) reports that the hedge fund industry had $39 billion of assets in 1990, whereas the assets under management had reached $2.375 trillion in 2013. In such an immensely growing market, monitoring the historical performance of hedge funds is significant for the investors' optimal portfolio selection. However, unlike mutual funds, it is not mandatory for individual hedge funds to report their performance data to the public. In addition, a representative hedge fund does not exist since each hedge fund manager has different trading strategies. These drawbacks create a need for hedge fund index providers who collect performance data from different hedge funds, construct hedge fund performance indices and make them available for public. Each data provider has its own classification system, which assigns an individual hedge fund to a sub-strategy. The databases do not represent the entire hedge fund universe, because some hedge funds may prefer not to disclose their performance to a database. Moreover, each different data provider may cover different segments of the market, which causes selection bias. There are also other types of biases that hedge fund indices suffer, such as survivorship bias and backfill bias (see Bali et al. (2013)). Considering the different index construction methodologies of hedge fund providers and various data biases, we decided to test our portfolio selection models across two hedge funds datasets, each obtained from different data providers. We employ hedge fund indices of two leading data providers, Hedge Fund Research (HFR) and Credit Suisse (CS), in our numerical analysis.

5.1 HFRI Hedge Fund Indices

HFR reports performance data for noninvestable indices which are called HFRI and for investable indices which are named as HFRX. HFRI indices, which we employ in our numerical work, reflect the performance of hedge funds that are not open to new investors. HFRI indices are equally-weighted performance indices. HFR justify this reporting style by claiming that equal weighting reduces any bias toward larger funds. HFR divides the hedge fund into four main strategy categories and report hedge fund indices for each category. These main strategy categories are event driven, equity hedge, relative value and global macro. Event-driven funds attempts to take the advantage of extraordinary corporate events such as mergers and restructurings. Equity hedge funds take long positions in undervalued equities and short positions overvalued equities. Relative value funds purchase a security that is expected to appreciate, while simultaneously selling short a related security that is expected to depreciate. Global macro funds place directional bets based on their analysis of macroeconomic events. In practice, an investor cannot invest money in a hedge fund index since the index is only a numerical representation of the underlying strategy. In our study, we consider these four hedge fund indices as risky assets and suppose that an investor can invest in each of the indices. As our market portfolio, we employ the

HFRI Fund Weighted Composite Index which includes all HFRI Index constituents and accounts for over 2200 funds. The data we consider span from January 1990 to February 2014. Detailed information about HFRI Hedge Fund Indices can be found on HFRI website `https://www.hedgefundresearch.com`.

5.2 Credit Suisse Hedge Fund Indices

In contrast with HFR hedge fund indices, CS hedge fund indices are asset-weighted. In other words, the index performance is measured by weighting each hedge fund's return with the total assets that the fund has under management. The noninvestable hedge fund indices provided by CS include a composite index named the Credit Suisse Hedge Fund Index and 10 primary strategy categories. We employ this composite index as our market portfolio. The strategy indices are event driven, long/short equity hedge, equity market neutral, dedicated short bias, convertible arbitrage, fixed income arbitrage, global macro, emerging markets, managed futures, and multi-strategy.

CS Hedge Fund Index reflects the performance of more than 400 funds and it rebalances quarterly. The historical weights of the index show that dedicated short bias strategies' weight is approximately zero over the period 1994-2014. Hence, we exclude dedicated short bias index from our analysis and we consider the remaining nine hedge fund indices as our risky assets (plus the market portfolio, CS Hedge Fund Index). The monthly returns we use in our analysis range from May 1994 to November 2014. Further information on CS hedge fund indices can be accessed from `https://www.hedgeindex.com/`.

5.3 Industry Portfolios

The industry dataset contains the excess returns on 10 industry portfolios in the United States, which are created by Kenneth French. The 10 industries considered are Consumer-Discretionary, Consumer-Staples, Manufacturing, Energy, High-Tech, Telecommunication, Wholesale and Retail, Health, Utilities and Others. The data consist of monthly excess returns from January 1990 to February 2014. As the market portfolio, we consider the US equity market portfolio, MKT, which is the value-weighted return on all NYSE, AMEX and NASDAQ stocks.

In addition to 10 industry portfolio, we also consider 5 industry portfolios in the United States. The 5 industries considered are Consumer, High-Tech, Health, Manufacturing and Others. The main reason for considering 5 industry portfolios is to check the robustness of our portfolio selection models. Industry portfolios dataset can be reached from `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

5.4 Other Data

In all datasets, we employ the one-month US treasury-bill rate as the risk-free rate. In order to classify the states of the market for our regime-switching application, we consider the US Business Cycle Expansions and Contractions data of NBER. The organization determines the recession dates by analyzing the historical and on-going business activity. Using the information we obtain from NBER, we classify the market in two states: expansion and contraction. The data can be accessed from `https://www.nber.org/cycles.html`.

Chapter 6

RESULTS FROM THE EMPIRICAL DATASETS

6.1 HARA Utility Models

In this chapter, we compare the performance of the portfolio selection models considered in Chapter 3 to the benchmark market portfolios for each dataset in Table 5. For each dataset, we report the in-sample and out-of-sample Sharpe ratios, the CEQ returns, and the turnover values considering our robustness-check experiments. In all tables, the portfolio selection strategies are listed in rows, while the columns correspond to the experiments.

In order to determine which portfolio selection strategy leads to the market portfolio in terms of Sharpe ratio, we calculate the p-value for the difference between the Sharpe ratio of each strategy and that of the market portfolio. For the calculation of p-values, we use the test statistic (4.3) of Jobson and Korkie (1981). The null hypothesis of the test claims that the portfolio selection model and the market portfolio are identical in terms of Sharpe ratio. Hence, we favor portfolio strategies with higher p-values. If a strategy has high p-values across all experiments, that portfolio selection model is good at representing the market portfolio. We follow a similar procedure to find the identical portfolio selection model with the market portfolio in terms of CEQ return. To calculate the p-value, we use the test statistic (4.5).

We report the p-values obtained from tests (4.3) and (4.5) in parentheses in each Sharpe ratios table and CEQ returns table respectively. If the p-value is higher than the significance level = 0.05, we write the value in bold.

We evaluate each dataset separately in order to find the best model that represents the market portfolio.

6.1.1 HFRI

6.1.1.1 Sharpe Ratios

The first row of Table 6.1 contains the Sharpe ratio of the market portfolio for each experiment. As it is stated in Chapter 5, we employ HFRI Fund Weighted Index as our market portfolio and calculate the Sharpe ratio using the monthly return data. The second row corresponds to the Sharpe ratio of $1/m$ naive-diversification strategy. This strategy is independent of data and requires no estimation or optimization.

The third row of the table reports the Sharpe ratio of in-sample mean-variance strategy. This strategy gives the optimal portfolio with no estimation error and the highest Sharpe ratio of all strategies considered. To calculate the in-sample Sharpe ratio for the mean-variance strategy, we use the entire time series of asset returns by taking the estimation window M = T = 290 for HFRI dataset. DeMiguel et al. (2009) highlight that the difference between in-sample Sharpe ratio for the mean-variance strategy and out-of-sample Sharpe ratio for the mean-variance strategy is a sign of estimation error. In the remaining rows, we report the out-of-sample Sharpe ratios for mean-variance model and exponential, logarithmic and power utility models. For our benchmark case (NotRS_Rolling_M120_hp1), we take the length of estimation window as $M = 120$. As a result, we obtain $T - M = 170$ months of out-of-sample returns.

For the HFRI dataset, we observe that there is a substantial difference between in-sample Sharpe ratio for the mean-variance strategy and that of out-of-sample mean-variance strategy. For our benchmark case, in-sample Sharpe ratio of mean-variance strategy is 0.4866, while out-of-sample Sharpe ratio for the mean-variance strategy is 0.2923. Moreover, the out-of-sample Sharpe ratio for the sample-based mean-variance strategy is less than that for the $1/m$ strategy. The estimation error is very large and it erodes the gains from optimal diversification.

For the benchmark case, out-of-sample Sharpe ratio for $1/m$ naive-diversification strategy is 0.2468 which is close to the market portfolio's Sharpe ratio of 0.1820. However, when we test the equality of Sharpe ratios of naive-diversification strategy and market portfolio, the p-value we obtain is approximately zero. Note that the p-values are given in parenthesis for each case. Moreover, they are expressed in boldface if the p-value is greater than 0.05 significance level. The reason for this result is that the test statistic (4.3) is high since the covariance between out-of-sample returns of $1/m$ diversification and market portfolio is high. Hence, we reject the null hypothesis for this test.

For our benchmark case, the identical strategies to the market portfolio in terms of out-of-sample Sharpe ratio are power utility models with parameters $\gamma = 0.5, 0.7$, 0.9 with p-values 0.30, $0.31, 0.31$ respectively. With the p-value of 0.29, out-of-sample Sharpe ratio of logarithmic utility portfolio is also identical with the market portfolio. Out-of-sample Sharpe ratio of the market portfolio is 0.1820, whereas out-of-sample Sharpe ratios of power utility models with parameters $\gamma = 0.5, 0.7, 0.9$ are $0.2549, 0.2549$ and 0.2449 respectively. In addition, out-of-sample Sharpe ratio of the logarithmic utility portfolio is 0.2569.

If we consider all seven experiments, we see similar results with our benchmark case. Power utility models with parameters $\gamma = 0.5, 0.7, 0.9$ and the logarithmic utility model are good at estimating the market portfolio in terms of Sharpe ratio. For all of these models, we do not reject the null hypothesis of test (4.3) for six out of seven experiments.

6.1.1.2 Certainty Equivalent Return

In Table 6.2, we report CEQ returns for HFRI dataset. The table shows that the in-sample mean-variance strategy has the highest CEQ return. All portfolio selection models (including $1/m$ naive-diversification rule) are identical with the market portfolio in terms of CEQ return across all experiments. For the benchmark experiment, the models with closest CEQ values to the market portfolio are mean-variance, power $\gamma = 1.3$, and power $\gamma = 1.5$.

6.1.1.3 Turnover

Table 6.3 reports absolute TO values for HFRI dataset. TO value for the benchmark market portfolio is zero. In table 6.3, we see that TO values for $1/m$ naive diversification model are the lowest among all portfolio selection models in all experiments. We also observe that NotRS_Rolling_M60_hp1 experiment gives the highest TO values. Since fewer months of historical data are used to estimate the model's parameters in this experiment and rolling-sample approach is used, the portfolio weights fluctuate a lot. For the benchmark experiment, the lowest TO values are given by mean-variance, power $\gamma = 1.5$ and $\gamma = 4$ strategies.

Table 6.1: Sharpe ratios and p-values for HFRI dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	RS_ Expanding_ M60_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.1820	0.1820	0.1820	0.1820	0.1820	0.4383	0.4383
EW (1/m)	0.2468	0.2461	0.2461	0.2461	0.2461	0.5579	0.5579
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)
mv (in-sample)	0.4866	0.4866	0.4866	0.4866	0.4866	-	-
mv	0.2923	0.3041	0.3219	0.3779	0.4245	0.5748	0.3111
	(0.11)	**(0.21)**	(0.00)	(0.04)	(0.00)	**(0.48)**	**(0.67)**
expo	0.2673	0.0234	0.2887	0.3129	0.3961	0.5365	0.1387
	(0.22)	**(0.13)**	(0.03)	**(0.17)**	(0.00)	**(0.64)**	**(0.37)**
log	0.2569	0.1043	0.2716	0.2973	0.3687	0.5139	0.1816
	(0.29)	**(0.44)**	**(0.08)**	**(0.23)**	(0.00)	**(0.73)**	**(0.42)**
power 0.5	0.2549	0.0866	0.2657	0.2950	0.3578	0.5075	0.2203
	(0.30)	**(0.34)**	**(0.10)**	**(0.23)**	(0.01)	**(0.76)**	**(0.48)**
power 0.7	0.2549	0.0348	0.2701	0.3011	0.3404	0.5062	0.2753
	(0.31)	**(0.15)**	**(0.08)**	**(0.20)**	(0.01)	**(0.77)**	**(0.59)**
power 0.9	0.2449	0.1740	0.2734	0.2144	0.2828	0.5476	0.1296
	(0.31)	**(0.93)**	(0.01)	**(0.74)**	**(0.06)**	**(0.48)**	**(0.37)**
power 1.1	0.2826	0.3055	0.2902	0.3387	0.3580	0.5962	0.3450
	(0.15)	**(0.17)**	(0.01)	**(0.09)**	(0.00)	**(0.47)**	**(0.73)**
power 1.3	0.2865	0.3256	0.3029	0.3590	0.3768	0.5791	0.3542
	(0.13)	**(0.12)**	(0.00)	**(0.06)**	(0.00)	**(0.49)**	**(0.76)**
power 1.5	0.2889	0.3291	0.3112	0.3701	0.3927	0.5744	0.3501
	(0.12)	**(0.12)**	(0.00)	(0.04)	(0.00)	**(0.49)**	**(0.75)**
power 4	0.2760	0.1056	0.3010	0.3328	0.4114	0.5513	0.1608
	(0.17)	**(0.47)**	(0.01)	**(0.11)**	(0.00)	**(0.58)**	**(0.40)**

Table 6.2: CEQ returns and p-values for HFRI dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	RS_ Expanding_ M60_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.0033	0.0033	0.0033	0.0033	0.0033	0.0394	0.0394
EW (1/m)	0.0037	0.0037	0.0037	0.0037	0.0037	0.0462	0.0462
	(0.19)	**(0.19)**	**(0.19)**	**(0.19)**	**(0.19)**	**(0.13)**	**(0.13)**
mv (in-sample)	0.0055	0.0055	0.0055	0.0055	0.0055	-	-
mv	0.0033	0.0048	0.0035	0.0051	0.0043	0.0381	0.0215
	(0.99)	**(0.35)**	**(0.74)**	**(0.24)**	**(0.35)**	**(0.94)**	**(0.49)**
expo	0.0031	0.0001	0.0032	0.0046	0.0040	0.0345	0.0085
	(0.88)	**(0.37)**	**(0.92)**	**(0.40)**	**(0.51)**	**(0.79)**	**(0.33)**
log	0.0030	0.0024	0.0030	0.0044	0.0038	0.0328	0.0113
	(0.83)	**(0.70)**	**(0.80)**	**(0.49)**	**(0.65)**	**(0.73)**	**(0.32)**
power 0.5	0.0030	0.0021	0.0030	0.0043	0.0037	0.0322	0.0135
	(0.81)	**(0.62)**	**(0.75)**	**(0.52)**	**(0.71)**	**(0.71)**	**(0.34)**
power 0.7	0.0030	0.0006	0.0030	0.0044	0.0036	0.0320	0.0172
	(0.81)	**(0.41)**	**(0.79)**	**(0.49)**	**(0.78)**	**(0.71)**	**(0.39)**
power 0.9	0.0030	0.0049	0.0034	0.0044	0.0036	0.0364	0.0082
	(0.80)	**(0.50)**	**(0.87)**	**(0.57)**	**(0.70)**	**(0.83)**	**(0.38)**
power 1.1	0.0032	0.0042	0.0032	0.0048	0.0038	0.0374	0.0211
	(0.96)	**(0.52)**	**(0.95)**	**(0.33)**	**(0.59)**	**(0.91)**	**(0.44)**
power 1.3	0.0032	0.0043	0.0034	0.0049	0.0040	0.0377	0.0225
	(0.97)	**(0.48)**	**(0.90)**	**(0.27)**	**(0.49)**	**(0.92)**	**(0.48)**
power 1.5	0.0032	0.0045	0.0035	0.0050	0.0041	0.0379	0.0230
	(0.99)	**(0.44)**	**(0.81)**	**(0.25)**	**(0.43)**	**(0.93)**	**(0.50)**
power 4	0.0032	0.0049	0.0033	0.0048	0.0041	0.0358	0.0105
	(0.92)	**(0.88)**	**(0.98)**	**(0.34)**	**(0.44)**	**(0.84)**	**(0.36)**

Table 6.3: TO value for HFRI dataset

Strategy	NotRS_Rolling_M120_hp1	NotRS_Rolling_M60_hp1	NotRS_Expanding_M120_hp1	RS_Expanding_M120_hp1	RS_Expanding_M60_hp1	NotRS_Rolling_M120_hp12	RS_Expanding_M120_hp12
Market	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EW (1/m)	0.01	0.01	0.01	0.01	0.01	0.08	0.08
mv (in-sample)	-	-	-	-	-	0.00	0.00
mv	0.19	0.94	0.05	0.19	0.09	0.93	1.18
expo	0.21	3.90	0.05	0.23	0.10	0.96	1.48
log	0.22	2.23	0.06	0.22	0.11	0.96	1.28
power 0.5	0.23	2.40	0.06	0.21	0.11	0.96	1.16
power 0.7	0.24	3.41	0.07	0.22	0.12	0.96	1.03
power 0.9	0.30	2.58	0.13	0.37	0.13	0.95	1.43
power 1.1	0.27	0.72	0.05	0.18	0.10	1.05	1.05
power 1.3	0.23	0.77	0.05	0.18	0.09	1.01	1.07
power 1.5	0.21	0.85	0.04	0.18	0.09	0.98	1.10
power 4	0.20	4.66	0.05	0.22	0.09	0.95	1.45

6.1.2 CSHF

6.1.2.1 Sharpe Ratios

We report the Sharpe ratios for CSHF dataset in Table 6.4. For the CSHF dataset, there is a significant difference between in-sample Sharpe ratio for the mean-variance strategy and that of out-of-sample Sharpe ratio of mean-variance strategy, which is a similar result to the HFRI dataset. For the benchmark case, in-sample Sharpe ratio of mean-variance strategy is 0.4907, whereas out-of-sample Sharpe ratio of mean-variance strategy is less than half, only 0.2710. Similar to our result for HFRI dataset, we conclude that the effect of estimation error is large on out-of-sample performance.

Contrary to our results for HFRI dataset, out-of-sample Sharpe ratios for $1/m$ naive-diversification strategy are the same with Sharpe ratios of the market portfolio across all experiments.

For the benchmark case, all portfolio selection models are identical with the market portfolio in terms of Sharpe ratio. Moreover, out-of-sample Sharpe ratios of some strategies such as mean-variance strategy and power $\gamma = 1.1$ (out-of-sample Sharpe ratios are 0.2710 and 0.2425, respectively) are very close to the Sharpe ratio of the market portfolio of 0.2577. If we consider all experiments, the models that are identical with the market portfolio in terms of Sharpe ratio are: (i) $1/m$ naive-diversification strategy, (ii) power $\gamma = 1.1$, and (iii) power $\gamma = 4$.

6.1.2.2 Certainty Equivalent Return

The comparison of CEQ returns for CSHF dataset are reported in Table 6.5. The highest CEQ returns belong to in-sample mean-variance strategy. Portfolio selection models with exponential, logarithmic and power with $\gamma = 0.5,\ 0.7,\ 1.5,$and 4 utility functions are identical with the market portfolio in terms of CEQ returns across all experiments. For the benchmark experiment, the models with closest CEQ values to the market portfolio are power $\gamma = 0.5$, power $\gamma = 0.7$, and power $\gamma = 0.9$ models.

6.1.2.3 Turnover

In Table 6.6, we report TO values for CSHF dataset. Similar to the results for HFRI dataset, $1/m$ strategy gives the minimum TO values in all experiments and NotRS_Rolling_ M60_hp1 experiment has the highest TO values. If we compare TO values across portfolio selection models for the benchmark experiment, we see that mean-variance strategy, power $\gamma = 4$ and exponential models give the lowest TO values.

Table 6.4: Sharpe ratios and p-values for CSHF dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.2577	0.2577	0.2577	0.2577	0.6200	0.6200
EW (1/m)	0.2401	0.2401	0.2401	0.2401	0.5778	0.5778
	(0.22)	**(0.22)**	**(0.22)**	**(0.22)**	**(0.26)**	**(0.26)**
mv (in-sample)	0.4907	0.4907	0.4907	0.4907	-	-
mv	0.2710	0.0647	0.2760	0.1764	1.2527	1.9455
	(0.90)	**(0.14)**	**(0.82)**	**(0.48)**	**(0.08)**	(0.03)
expo	0.2367	0.1201	0.2643	0.0118	1.2604	0.3907
	(0.85)	**(0.29)**	**(0.94)**	(0.05)	**(0.07)**	**(0.70)**
log	0.2209	0.0104	0.2507	0.1930	1.2545	1.2626
	(0.74)	(0.04)	**(0.93)**	**(0.61)**	(0.05)	**(0.30)**
power 0.5	0.2100	0.0744	0.2446	0.1139	1.2360	-0.0199
	(0.66)	**(0.14)**	**(0.87)**	**(0.25)**	(0.05)	(0.01)
power 0.7	0.1954	-0.0633	0.2358	0.1374	1.2010	0.1947
	(0.56)	(0.01)	**(0.79)**	**(0.40)**	(0.05)	**(0.06)**
power 0.9	0.1924	-0.0990	0.2134	-0.0538	1.0140	0.4141
	(0.50)	(0.00)	**(0.51)**	(0.00)	**(0.09)**	**(0.15)**
power 1.1	0.2425	0.1239	0.2797	0.2390	1.2820	0.3480
	(0.89)	**(0.32)**	**(0.78)**	**(0.88)**	(0.03)	**(0.28)**
power 1.3	0.2654	-0.0939	0.2812	0.1136	1.2837	-0.2001
	(0.94)	(0.01)	**(0.76)**	**(0.25)**	(0.04)	(0.00)
power 1.5	0.2742	-0.0141	0.2794	0.0210	1.2785	0.9950
	(0.88)	(0.02)	**(0.78)**	(0.05)	(0.05)	**(0.54)**
power 4	0.2479	0.1106	0.2715	0.1582	1.2663	1.0193
	(0.93)	**(0.25)**	**(0.87)**	**(0.40)**	**(0.07)**	**(0.32)**

Table 6.5: CEQ returns and p-values for CSHF dataset

Strategy	NotRS_Rolling_M120_hp1	NotRS_Rolling_M60_hp1	NotRS_Expanding_M120_hp1	RS_Expanding_M120_hp1	NotRS_Rolling_M120_hp12	RS_Expanding_M120_hp12
Market	0.0028	0.0040	0.0040	0.0040	0.0498	0.0498
EW (1/m)	0.0036	0.0036	0.0036	0.0036	0.0450	0.0450
	0.00	**0.09**	**0.09**	**0.09**	**0.08**	**0.08**
mv (in-sample)	0.0054	0.0054	0.0054	0.0054	-	-
mv	0.0046	-0.5072	0.0033	0.0037	0.0584	0.0550
	(0.30)	(0.00)	**(0.56)**	**(0.90)**	**(0.68)**	**(0.85)**
expo	0.0042	0.0072	0.0032	-0.0168	0.0594	0.0341
	(0.46)	**(0.75)**	**(0.52)**	**(0.24)**	**(0.63)**	**(0.95)**
log	0.0040	-0.0031	0.0031	0.0080	0.0592	0.0659
	(0.52)	**(0.37)**	**(0.46)**	**(0.36)**	**(0.62)**	**(0.68)**
power 0.5	0.0039	0.0027	0.0031	0.0039	0.0587	-0.0211
	(0.57)	**(0.87)**	**(0.43)**	**(0.97)**	**(0.63)**	**(0.10)**
power 0.7	0.0037	-0.0109	0.0030	0.0064	0.0576	0.0160
	(0.62)	**(0.08)**	**(0.38)**	**(0.69)**	**(0.66)**	**(0.13)**
power 0.9	0.0034	-0.0147	0.0027	-0.0037	0.0509	0.0357
	(0.71)	(0.03)	**(0.18)**	(0.05)	**(0.95)**	**(0.22)**
power 1.1	0.0048	0.0030	0.0034	0.0076	0.0590	0.0265
	(0.31)	**(0.96)**	**(0.60)**	**(0.27)**	**(0.61)**	**(0.23)**
power 1.3	0.0048	-0.0737	0.0034	-0.0008	0.0592	-0.2130
	(0.26)	(0.00)	**(0.60)**	**(0.82)**	**(0.62)**	**(0.07)**
power 1.5	0.0048	-0.0039	0.0034	-0.0003	0.0590	0.0758
	(0.26)	**(0.23)**	**(0.58)**	**(0.38)**	**(0.63)**	**(0.58)**
power 4	0.0043	0.0057	0.0033	0.0038	0.0598	0.0413
	(0.41)	**(0.82)**	**(0.55)**	**(0.94)**	**(0.63)**	**(0.72)**

Table 6.6: TO values for CSHF dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.00	0.00	0.00	0.00	0.00	0.00
EW (1/m)	0.01	0.01	0.01	0.01	0.09	0.09
mv (in-sample)	-	-	-	-	-	-
mv	1.10	75.10	0.25	1.25	1.79	7.20
expo	1.14	6.90	0.30	26.28	2.05	40.01
log	1.21	15.16	0.32	7.23	2.12	6.13
power 0.5	1.29	7.85	0.34	2.53	2.10	10.01
power 0.7	1.42	33.48	0.35	2.36	2.04	3.71
power 0.9	1.41	32.43	0.41	3.63	1.89	1.88
power 1.1	1.54	88.71	0.28	1.76	1.68	16.02
power 1.3	1.29	31.13	0.25	6.08	1.70	18.19
power 1.5	1.17	8.88	0.24	3.49	1.72	23.61
power 4	1.11	7.86	0.28	1.59	1.97	9.13

6.1.3 I5

6.1.3.1 Sharpe Ratios

If we check Table 6.7 for I5 dataset, we see that the effect of estimation error is large again. For the benchmark experiment, using mean-variance strategy give in-sample Sharpe ratio of 0.2286, which is higher than out-of-sample Sharpe ratio of that strategy, 0.0941. Similar to our results for HFRI dataset, $1/m$ naive-diversification strategy is not identical with the market portfolio in terms of Sharpe ratio for five experiments out of seven.

For the I5 database, we observe that all portfolio selection strategies except $1/m$ naive-diversification are identical with the market portfolio in terms of Sharpe ratio across all experiments. However, in the industry dataset, there are only $m = 5$ assets. As DeMiguel et al. (2009) indicate, choosing smaller number of assets implies fewer parameters to be estimated and, therefore, less room for estimation error.

For the benchmark case, the models that have closest Sharpe ratios to the Sharpe ratio of the market portfolio are exponential, mean-variance and logarithmic models with Sharpe ratios 0.0908, 0.0941 and 0.0956 respectively, where the Sharpe ratio of the market portfolio is 0.0668.

6.1.3.2 Certainty Equivalent Return

The comparison of CEQ returns for CSHF dataset are reported in Table 6.8. The highest CEQ returns belong to in-sample mean-variance strategy. Portfolio selection models with exponential, logarithmic and power with $\gamma = 0.5,\ 0.7,\ 1.5$,and 4 utility functions are identical with the market portfolio in terms of CEQ returns across all experiments. For the benchmark experiment, power $\gamma = 0.5$ and power $\gamma = 1.3$ models have the closest CEQ returns to the CEQ return of the market portfolio.

6.1.3.3 Turnover

In Table 6.9, TO for I5 dataset are reported. Similarly, $1/m$ naive diversification gives the minimum TO values in all experiments. Among all experiments, NotRS_Rolling_M120_hp12 experiment has the highest TO values. For the benchmark case, the lowest TO values belong to power $\gamma = 0.5$, power $\gamma = 0.7$,and power $\gamma = 0.9$ models.

Table 6.7: Sharpe ratios and p-values for I5 dataset

Strategy	NotRS_Rolling_M120_hp1	NotRS_Rolling_M60_hp1	NotRS_Expanding_M120_hp1	RS_Expanding_M120_hp1	RS_Expanding_M60_hp1	NotRS_Rolling_M120_hp12	RS_Expanding_M120_hp12
Market	0.0668	0.0668	0.0668	0.0668	0.0668	0.1979	0.1979
EW (1/m)	0.1016	0.1013	0.1013	0.1013	0.1013	0.3034	0.3034
	0.01	0.02	0.02	0.02	0.01	**0.08**	**0.08**
mv (in-sample)	0.2286	0.2286	0.2286	0.2286	0.2286	-	-
mv	0.0941	0.0311	0.0737	0.0703	0.0790	-0.0616	0.2399
	0.78	**0.74**	**0.91**	**0.97**	**0.86**	**0.50**	**0.76**
expo	0.0908	0.0312	0.0620	0.0765	0.0844	-0.0908	0.1977
	0.80	**0.74**	**0.94**	**0.92**	**0.81**	**0.46**	**0.97**
log	0.0956	0.0690	0.0613	0.0294	0.1161	-0.0529	0.2005
	0.77	**0.98**	**0.94**	**0.70**	**0.57**	**0.52**	**0.99**
power 0.5	0.1029	0.0944	0.0639	0.1060	0.0910	0.0017	0.2052
	0.72	**0.79**	**0.97**	**0.70**	**0.78**	**0.60**	**0.96**
power 0.7	0.1122	0.1159	0.0673	-0.0594	0.0739	0.0602	0.2134
	0.65	**0.63**	**0.99**	**0.23**	**0.92**	**0.71**	**0.92**
power 0.9	0.1359	0.1259	0.0763	0.0790	0.0492	0.1682	0.2819
	0.49	**0.58**	**0.89**	**0.92**	**0.90**	**0.93**	**0.74**
power 1.1	0.1259	-0.0128	0.0985	-0.0398	-0.0433	0.1675	0.3072
	0.54	**0.46**	**0.60**	**0.30**	**0.30**	**0.93**	**0.49**
power 1.3	0.1226	0.0798	0.0893	-0.0673	-0.0676	0.0843	0.2900
	0.58	**0.90**	**0.70**	**0.21**	**0.23**	**0.75**	**0.54**
power 1.5	0.1052	0.0461	0.0830	0.0876	0.0903	0.0138	0.2737
	0.70	**0.84**	**0.79**	**0.84**	**0.78**	**0.62**	**0.60**
power 4	0.0908	0.0201	0.0650	0.0736	0.0800	-0.0902	0.2049
	0.81	**0.66**	**0.98**	**0.94**	**0.85**	**0.47**	**0.96**

Table 6.8: CEQ returns and p-values for I5 dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	RS_ Expanding_ M60_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.0018	0.0020	0.0020	0.0020	0.0018	0.0194	0.0020
EW (1/m)	0.0034	0.0034	0.0034	0.0034	0.0034	0.0404	0.0041
	(0.02)	(0.04)	(0.04)	(0.04)	(0.02)	**(0.10)**	(0.03)
mv (in-sample)	0.0075	0.0075	0.0075	0.0075	0.0075	-	-
mv	-0.0848	-0.0503	0.0025	0.0018	0.0030	-0.3129	-0.0002
	(0.04)	**(0.06)**	**(0.88)**	**(0.98)**	**(0.87)**	**(0.15)**	**(0.89)**
expo	-0.1290	-0.0328	0.0019	0.0028	0.0035	-0.4566	-0.0002
	(0.01)	**(0.13)**	**(0.97)**	**(0.92)**	**(0.78)**	**(0.09)**	**(0.89)**
log	-0.0787	-0.0082	0.0019	-0.0044	0.0066	-0.3633	0.0099
	(0.05)	**(0.54)**	**(0.97)**	**(0.51)**	**(0.53)**	**(0.13)**	**(0.51)**
power 0.5	-0.0388	0.0004	0.0020	-0.0135	-0.0174	-0.2634	-0.0093
	(0.21)	**(0.91)**	**(1.00)**	**(0.51)**	**(0.41)**	**(0.21)**	**(0.43)**
power 0.7	-0.0157	0.0054	0.0022	-0.0675	-8.7299	-0.1952	-0.0089
	(0.49)	**(0.80)**	**(0.96)**	(0.00)	(0.00)	**(0.31)**	**(0.48)**
power 0.9	0.0046	-0.0055	0.0027	-2.6027	-0.0073	-0.1354	-1.2855
	(0.88)	**(0.74)**	**(0.85)**	(0.00)	**(0.53)**	**(0.48)**	(0.00)
power 1.1	-0.0182	-0.0814	0.0036	-0.0244	-0.0233	-0.0148	-0.0597
	(0.47)	(0.01)	**(0.58)**	**(0.07)**	**(0.07)**	**(0.77)**	(0.01)
power 1.3	-0.0021	-0.0083	0.0032	-0.1855	-0.1839	-0.0655	-7.3123
	(0.85)	**(0.57)**	**(0.68)**	(0.00)	(0.00)	**(0.53)**	(0.00)
power 1.5	-0.0257	-0.0206	0.0029	-0.0073	-0.0057	-0.1441	-0.1770
	(0.33)	**(0.25)**	**(0.76)**	**(0.61)**	**(0.67)**	**(0.33)**	(0.00)
power 4	-0.1278	-0.0468	0.0021	0.0025	0.0032	-0.4389	-0.0022
	(0.01)	**(0.06)**	**(0.99)**	**(0.95)**	**(0.82)**	**(0.10)**	**(0.78)**

Table 6.9: TO value for I5 dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	RS_ Expanding_ M60_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EW (1/m)	0.02	0.02	0.02	0.02	0.02	0.15	0.15
mv (in-sample)	-	-	-	-	-	-	-
mv	5.41	6.68	0.25	2.11	1.50	24.59	2.76
expo	5.69	5.44	0.27	1.78	1.35	25.36	2.76
log	34.73	5.47	0.28	15.51	1.85	23.45	2.92
power 0.5	4.93	6.23	0.28	2.52	3.00	22.58	3.11
power 0.7	4.00	12.49	0.28	2.95	7.67	24.36	3.46
power 0.9	3.68	17.89	0.27	5.71	2.46	65.86	78.59
power 1.1	4.16	10.52	0.21	4.07	3.05	14.75	2.70
power 1.3	6.34	28.83	0.22	5.72	4.16	53.04	2.73
power 1.5	6.01	10.20	0.23	3.30	2.36	27.32	2.76
power 4	5.53	5.63	0.27	1.78	1.36	25.48	2.71

6.1.4 I10

6.1.4.1 Sharpe Ratios

From Table 6.10, we observe that the results for I10 dataset are similar to the results for I5 dataset in terms of Sharpe ratio. For benchmark case, in-sample Sharpe ratio of mean-variance strategy is 0.2521, while out-of-sample Sharpe ratio for the mean-variance strategy is 0.0466. The effect of estimation error on performance for I10 dataset is larger than that for I5 dataset.

Similar with the I5 dataset, $1/m$ naive-diversification strategy does not represent the market portfolio in terms of Sharpe ratio in any of the experiments for I10 dataset. For our benchmark experiment, power $\gamma = 1.1,\ 1.3$, and 1.5 models are good at representing the market portfolio since they have higher p-values than the other portfolio selection models. These power models are also identical with the market portfolio in terms of Sharpe ratio across all experiments.

For the benchmark case, the models with closest Sharpe ratios to that of the market portfolio are power $\gamma = 1.1$, power $\gamma = 1.3$, and power $\gamma = 1.5$ models with Sharpe ratios 0.0542, 0.0561 and 0.0664 respectively, where the Sharpe ratio of the market portfolio is 0.0668.

6.1.4.2 Certainty Equivalent Return

We report the comparison of CEQ returns for I10 dataset in Table 6.11. Similarly, the highest CEQ returns belong to in-sample mean-variance strategy. None of our portfolio selection models are identical with the market portfolio in terms of CEQ returns across all experiments. For the benchmark experiment, the models with closest CEQ values to the market portfolio are power $\gamma = 0.9$, power $\gamma = 1.1$, and power $\gamma = 1.3$ models.

6.1.4.3 Turnover

In Table 6.12, we report portfolio TO for I10 dataset. Similar to all other datasets, $1/m$ strategy gives the minimum TO values in all experiments. For this dataset, NotRS_Rolling_M120_hp12 experiment gives the highest TO values across all experiments. For the benchmark case, the lowest TO values are given by logarithmic, power $\gamma = 0.5$ and power $\gamma = 1.1$ models.

Table 6.10: Sharpe ratios and p-values for I10 dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	RS_ Expanding_ M60_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.0668	0.0668	0.0668	0.0668	0.0668	0.1979	0.1979
EW (1/m)	0.1151	0.1148	0.1148	0.1148	0.1148	0.3454	0.3454
	(0.01)	(0.02)	(0.02)	(0.01)	(0.02)	(0.02)	(0.02)
mv (in-sample)	0.2521	0.2521	0.2521	0.2521	0.2521	-	-
mv	0.0466	0.0351	0.0416	0.1311	0.0952	0.2458	0.0009
	(0.88)	**(0.75)**	**(0.73)**	**(0.55)**	**(0.80)**	**(0.90)**	**(0.27)**
expo	-0.0017	0.0781	0.0317	0.1124	0.0952	0.2207	-0.0286
	(0.55)	**(0.91)**	**(0.64)**	**(0.68)**	**(0.80)**	**(0.95)**	**(0.23)**
log	0.0011	-0.0276	0.0306	0.1200	0.1408	0.1857	-0.0321
	(0.57)	**(0.39)**	**(0.63)**	**(0.63)**	**(0.52)**	**(0.97)**	**(0.23)**
power 0.5	0.0075	0.0699	0.0338	0.0499	0.0369	0.1671	-0.1262
	(0.61)	**(0.98)**	**(0.66)**	**(0.91)**	**(0.79)**	**(0.94)**	**(0.16)**
power 0.7	0.0128	-0.1160	0.0384	-0.0163	0.0461	0.1553	-0.2423
	(0.64)	**(0.08)**	**(0.71)**	**(0.45)**	**(0.86)**	**(0.91)**	**(0.18)**
power 0.9	0.0210	0.0493	0.0509	0.1083	0.0309	0.1435	0.1700
	(0.70)	**(0.87)**	**(0.83)**	**(0.65)**	**(0.75)**	**(0.89)**	**(0.89)**
power 1.1	0.0542	0.0211	0.0704	0.0237	-0.0201	0.1941	0.1459
	(0.93)	**(0.65)**	**(0.96)**	**(0.70)**	**(0.38)**	**(0.99)**	**(0.73)**
power 1.3	0.0561	-0.0776	0.0584	-0.0761	0.0825	0.2103	0.0910
	(0.95)	**(0.18)**	**(0.91)**	**(0.16)**	**(0.89)**	**(0.97)**	**(0.49)**
power 1.5	0.0664	-0.0704	0.0506	0.0892	0.0155	0.2311	0.0549
	(0.97)	**(0.20)**	**(0.83)**	**(0.82)**	**(0.66)**	**(0.93)**	**(0.38)**
power 4	0.0052	0.0827	0.0344	0.0092	0.0781	0.2338	-0.1040
	(0.60)	**(0.87)**	**(0.66)**	**(0.65)**	**(0.92)**	**(0.93)**	**(0.17)**

Table 6.11: CEQ returns and p-values for I10 dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	RS_ Expanding_ M60_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.0018	0.0020	0.0020	0.0018	0.0020	0.0194	0.0194
EW (1/m)	0.0041	0.0041	0.0041	0.0041	0.0041	0.0492	0.0492
	(0.02)	(0.03)	(0.03)	(0.02)	(0.03)	(0.01)	(0.01)
mv (in-sample)	0.0085	0.0085	0.0085	0.0085	0.0085	-	-
mv	-0.0313	-0.0556	0.0008	0.0057	-0.0002	-10.4324	-0.0326
	(0.16)	(0.05)	**(0.74)**	**(0.82)**	**(0.89)**	(0.02)	**(0.23)**
expo	-0.0199	-13.2344	0.0002	-0.0086	-0.0002	-3.2444	-0.0523
	(0.17)	(0.00)	**(0.64)**	**(0.65)**	**(0.89)**	(0.05)	**(0.18)**
log	-0.0145	-0.1187	0.0001	-0.0085	0.0099	-1.1776	-0.0575
	(0.24)	(0.00)	**(0.63)**	**(0.67)**	**(0.51)**	**(0.10)**	**(0.17)**
power 0.5	-0.0117	-4.6945	0.0003	-0.0436	-0.0093	-0.7865	-0.0840
	(0.29)	(0.00)	**(0.67)**	**(0.11)**	**(0.43)**	**(0.14)**	**(0.11)**
power 0.7	-0.0098	-1.3439	0.0006	-0.3870	-0.0089	-0.6182	-0.1907
	(0.35)	(0.00)	**(0.71)**	(0.00)	**(0.48)**	**(0.17)**	**(0.07)**
power 0.9	-0.0072	-0.0462	0.0013	-0.0134	-1.2855	-0.4578	-0.0122
	(0.43)	**(0.09)**	**(0.85)**	**(0.52)**	(0.00)	**(0.21)**	**(0.69)**
power 1.1	-0.0026	-0.0814	0.0023	-0.0488	-0.0597	-0.3098	0.0070
	(0.70)	(0.01)	**(0.94)**	**(0.06)**	(0.01)	**(0.33)**	**(0.71)**
power 1.3	-0.0100	-0.7811	0.0017	-36.9969	-7.3123	-0.7429	-0.0057
	(0.47)	(0.00)	**(0.92)**	(0.00)	(0.00)	**(0.17)**	**(0.47)**
power 1.5	-0.0338	-4.5162	0.0013	-0.0866	-0.1770	-2.4947	-0.0154
	(0.18)	(0.00)	**(0.84)**	(0.04)	(0.00)	**(0.06)**	**(0.36)**
power 4	-0.0223	-0.1975	0.0004	-0.1142	-0.0022	-5.6767	-0.0688
	(0.16)	(0.00)	**(0.67)**	(0.00)	**(0.78)**	(0.03)	**(0.12)**

Table 6.12: TO value for I10 dataset

Strategy	NotRS_ Rolling_ M120_hp1	NotRS_ Rolling_ M60_hp1	NotRS_ Expanding_ M120_hp1	RS_ Expanding_ M120_hp1	RS_ Expanding_ M60_hp1	NotRS_ Rolling_ M120_hp12	RS_ Expanding_ M120_hp12
Market	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EW (1/m)	0.03	0.03	0.03	0.03	0.03	0.16	0.16
mv (in-sample)	-	-	-	-	-	-	-
mv	20.64	12.69	0.44	8.51	19.08	74.30	3.68
expo	13.14	20.70	0.45	6.58	22.38	46.55	5.79
log	8.40	13.10	0.45	5.71	11.69	32.67	6.51
power 0.5	7.99	28.09	0.44	8.98	25.76	28.86	6.90
power 0.7	9.14	28.12	0.43	15.67	50.87	27.02	58.11
power 0.9	18.02	22.74	0.42	4.14	22.86	25.53	3.97
power 1.1	6.42	12.31	0.41	6.33	8.98	81.54	3.23
power 1.3	8.66	21.13	0.42	28.69	15.23	38.38	3.13
power 1.5	11.89	28.68	0.43	7.52	8.32	45.05	3.24
power 4	18.49	11.75	0.45	22.78	54.41	57.77	7.43

6.2 Prospect Models

In this section, we test the multiple risky asset prospect models in Chapter 3 for the I5 dataset. We consider prospect models with piecewise linear value function and piecewise exponential value function, and assume that excess returns of risky assets have multivariate normal distribution. We comment on the effect of prospect models on portfolio selection and compare our results with the findings in Taşkıncan (2014).

6.2.1 *Piecewise Linear Value Function*

In this subsection, we consider the prospect model with piecewise linear value function. We compute the optimal portfolios for different loss aversion λ values. For our numerical example, we use I5 data during $K = 120$ months between January 1990 and December 1999. Using the data, we calculate sample mean and covariance. The estimated parameters for the multivariate normal distribution of the excess return are

$$r^e = \begin{bmatrix} 0.0066 \\ 0.0144 \\ 0.0098 \\ 0.0056 \\ 0.0083 \end{bmatrix}$$

and

$$\sigma = \begin{bmatrix} 1.7 & 1.5 & 1.5 & 1.1 & 1.8 \\ 1.5 & 2.8 & 1.5 & 1.2 & 1.8 \\ 1.5 & 1.5 & 2.6 & 1.0 & 1.7 \\ 1.1 & 1.2 & 1.0 & 1.1 & 1.4 \\ 1.8 & 1.8 & 1.7 & 1.4 & 2.4 \end{bmatrix} \times 10^{-3}.$$

Optimal portfolios in Table 6.13 show that up to some critical value $\lambda_c \cong 2.1$ value, the solutions are unbounded (i.e, $u_i^* = \pm\infty$), which means that one should buy or sell a risky asset as much as he can. For loss aversion coefficient values greater that λ_c, the optimal solutions are zero. In other words, the investor invests all his money in the risk-free asset. Our results are parallel with the single risky asset prospect model of Taşkıncan (2014). From (3.45), we see that $u^* = +\infty$ up to a critical level of λ. For greater λ values, u^* becomes zero.

Table 6.13: Optimal portfolios calculated using piecewise linear value function

	Loss aversion values (λ)						
	1.1	1.5	1.9	2.0	2.1	2.2	2.3
$u^*_{consumer}$	2,98E+84	-1,76E+84	-2,34E+84	-2,87E+83	-9,93E-13	-1,00E-12	-4,16E-13
u^*_{hitec}	6,30E+84	1,18E+85	7,75E+84	5,25E+83	1,53E-12	1,96E-12	1,02E-12
u^*_{health}	5,85E+84	2,53E+84	2,31E+84	3,09E+83	6,05E-13	6,44E-13	1,09E-13
u^*_{manuf}	-3,10E+82	-6,54E+84	-3,43E+84	-1,80E+83	-1,86E-14	-9,09E-14	-2,76E-13
u^*_{other}	-9,40E+83	2,64E+84	-3,66E+83	-3,85E+82	1,29E-13	-3,72E-13	3,62E-13

6.2.2 Piecewise Exponential Value Function

Our aim is to show the effect of using piecewise exponential value function on optimal portfolio selection. We use the same I5 data in Section (6.2.1) for parameter estimation. Here, we calculate optimal portfolios for different risk tolerance β and loss aversion λ values.

For risk tolerance $\beta = 1$, our results in Table 6.14 show that up to critical $\lambda_c \cong 2.1$, the optimal portfolio changes with respect to λ. However, an important observation is that the composition of the risky portfolio is the same for different λ values. Moreover, this risky composition of the portfolios is identical with that of mean-variance portfolios, which we reported in Section 6.1. For loss aversion values $\lambda > \lambda_c$, the optimal solution is to make no investments in risky assets and invest all the money in the risk-free asset. Our results are similar to the results in Taşkıncan (2014). For the single risky asset case, the optimal solution (3.50) is 0 or $\bar{u}$ depending on λ. However in our multiple risky asset case, u^* changes depending on λ.

We change the risk tolerance to $\beta = 0.5$ to test its effect on optimal solutions. For a given $\lambda < \lambda_c$, let the optimal solutions for $\beta = 1$ and $\beta = 0.5$ cases be $u^*_{\beta=1}$ and $u^*_{\beta=0.5}$ respectively, where both are 5×1 column vectors. We see that $2 \times u^*_{\beta=0.5} = u^*_{\beta=1}$,which implies that $u^*_{\beta=1}$ and $u^*_{\beta=0.5}$ are linearly dependent vectors. However, the composition of risky assets for both cases are the same.

Table 6.14: Optimal portfolios calculated using piecewise exponential value function for $\beta = 1$

	Loss aversion values (λ)						
	1.1	1.5	1.9	2.0	2.1	2.2	2.3
$u^*_{consumer}$	-121,90	-19,45	-3,90	-1,56	-3,03E-12	-1,56E-13	-1,09E-12
u^*_{hitec}	241,75	38,58	7,74	3,10	4,48E-12	1,38E-12	8,96E-13
u^*_{health}	97,82	15,61	3,13	1,25	2,31E-12	2,76E-13	2,79E-13
u^*_{manuf}	-26,05	-4,16	-0,83	-0,33	-4,57E-13	2,54E-15	2,03E-13
u^*_{other}	-3,10	-0,49	-0,10	-0,04	-5,54E-14	-6,09E-13	3,09E-13
$w^*_{consumer}$	-0,65	-0,65	-0,65	-0,65	-	-	-
w^*_{hitec}	1,28	1,28	1,28	1,28	-	-	-
w^*_{health}	0,52	0,52	0,52	0,52	-	-	-
w^*_{manuf}	-0,14	-0,14	-0,14	-0,14	-	-	-
w^*_{other}	-0,02	-0,02	-0,02	-0,02	-	-	-

Chapter 7

CONCLUSION

The main objective of this thesis is to find the portfolio selection model that represents investor behavior in hedge funds and US stock market. We focus on two categories of portfolio selection models in our study: (i) Portfolio selection with utility models, and (ii) Portfolio selection with prospect models.

For the utility based approach, we consider the portfolio selection models in Çanakoğlu and Özekici (2010). They consider multiperiod portfolio optimization problem for an investor with hyperbolic absolute risk aversion (HARA) utility. The main feature of the model is that asset returns depend on a stochastic market depicted by a Markov chain. Dynamic programming is used to find the optimal portfolios. No distributional assumptions on asset returns are required as higher moments are incorporated in the portfolio selection models by considering the Taylor series expansion of the expected utility function.

We follow the rolling-sample approach in order to evaluate the out-of-sample performance of the portfolio selection models across HFRI, CSHF, I5 and I10 datasets and find the best model that represents the market portfolio for each dataset. We also conduct experiments to verify the robustness of our results. Our results show that, for the HFRI dataset, identical strategies to the market portfolio in terms of out-of-sample Sharpe ratio are power utility models with parameters $\gamma = 0.5$-0.9. Numerical results for the CSHF dataset imply that power utility models with parameters $\gamma = 1.1$ and 4 are good at representing the market portfolio in terms of Sharpe ratio. For the I5 dataset, exponential, mean-variance and logarithmic models are identical with the market portfolio in terms of Sharpe ratio. Finally, for the I10 dataset, portfolio selection models that represent the market portfolio in terms of Sharpe ratio are power $\gamma = 1.1$-1.5 models. As an extension to our research, different datasets can be analyzed to find the portfolio selection models that represent investor behavior for those markets.

In the framework of prospect theory, we extend Taşkıncan (2014) by considering prospect investors within a market consisting of multiple risky assets and one risk-free asset. In our analysis, we use the piecewise linear and piecewise exponential value functions under the assumption that asset returns have multivariate normal distribution. We test the multiple risky asset prospect models for the I5 dataset and report the results.

Optimal portfolios for piecewise linear value function show that up to some critical loss aversion level, the solutions are unbounded and for greater loss aversion values, the optimal solutions are zero. For the piecewise exponential value function, up to some critical loss aversion value, the optimal portfolio changes with respect to loss aversion level. However, the composition of the risky portfolio is the same for different loss aversion levels. In addition, this risky composition of the portfolios is identical with that of mean-variance portfolios. For loss aversion values greater than the critical level, it is optimal to make no investments in risky assets.

Future research on this topic can be the further analysis of the optimal solutions for the portfolio selection problem with piecewise linear and piecewise exponential value functions. We concentrated on the case where excess returns of the risky assets have multivariate normal distribution, but other cases with different distributions may be considered.

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Appendix A

DESCRIPTIVE STATISTICS FOR DATASETS

Table A.1: Descriptive statistics for the monthly return of HFRI indices

	Event Driven	Equity Hedge	Global Macro	Relative Value	Market
Mean	0.94%	1.03%	0.94%	0.81%	0.89%
StDev	1.94%	2.62%	2.15%	1.25%	1.99%
Skewness	-1.312	-0.260	0.573	-2.137	-0.689
Kurtosis	4.172	1.894	1.065	13.903	2.553
Minimum	-8.90%	-9.46%	-6.40%	-8.03%	-8.70%
Maximum	5.13%	10.88%	7.88%	5.72%	7.65%
Median	1.24%	1.21%	0.72%	0.88%	1.08%
Q1	0.02%	-0.55%	-0.49%	0.35%	-0.18%
Q3	2.03%	2.54%	2.03%	1.45%	2.08%

Table A.2: Descriptive statistics for the monthly return of CSHF indices

	Convertible Arbitrage	Emerging Markets	Equity Market Neutral	Event Driven	Fixed Income Arbitrage	Global Macro	Long/Short Equity	Managed Futures	Multi-Strategy	Market
Mean	0.64%	0.59%	0.41%	0.75%	0.47%	0.98%	0.84%	0.46%	0.69%	0.75%
StDev	1.86%	3.66%	2.68%	1.67%	1.58%	2.64%	2.76%	3.36%	1.43%	2.06%
Skewness	-2.647	-1.193	-12.411	-2.310	-4.000	0.110	-0.046	0.034	-1.906	-0.158
Kurtosis	16.854	7.130	188.824	12.102	28.773	4.855	3.670	-0.072	7.973	3.173
Minimum	-12.59%	-23.03%	-40.45%	-11.77%	-14.04%	-11.55%	-11.43%	-9.35%	-7.35%	-7.55%
Maximum	5.81%	15.34%	3.66%	4.22%	4.33%	10.60%	13.01%	9.95%	4.28%	8.53%
Median	0.95%	1.09%	0.57%	1.00%	0.69%	1.00%	0.87%	0.29%	0.82%	0.82%
Q1	-0.03%	-1.00%	-0.06%	0.12%	0.18%	-0.05%	-0.79%	-1.94%	0.13%	-0.20%
Q3	1.50%	2.41%	1.27%	1.70%	1.18%	2.00%	2.33%	2.90%	1.61%	1.72%

Table A.3: Descriptive statistics for the monthly return of I5 indices

	Cnsmr	HiTec	Hlth	Manuf	Other	Market
Mean	0.93%	0.96%	1.06%	0.98%	0.86%	0.87%
StDev	4.04%	6.33%	4.47%	4.21%	5.30%	4.41%
Skewness	-0.430	-0.460	-0.100	-0.700	-0.680	-0.680
Kurtosis	1.350	1.280	0.330	1.800	2.140	1.140
Minimum	-14.85%	-22.65%	-12.26%	-17.85%	-21.35%	-17.15%
Maximum	13.35%	19.98%	16.47%	13.58%	16.47%	11.35%
Median	1.26%	1.64%	1.34%	1.33%	1.51%	1.45%
Q1	-1.29%	-2.18%	-1.94%	-1.34%	-2.30%	-1.75%
Q3	3.48%	4.60%	3.93%	3.68%	3.82%	3.80%

Table A.4: Descriptive statistics for the monthly return of I10 indices

	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other	Market
Mean	0.96%	0.87%	1.05%	1.04%	1.15%	0.72%	0.97%	1.06%	0.83%	0.86%	0.87%
StDev	3.83%	7.12%	4.86%	5.34%	7.28%	5.26%	4.68%	4.47%	4.01%	5.30%	4.41%
Skewness	-0.330	0.226	-0.629	-0.054	-0.327	-0.332	-0.282	-0.098	-0.608	-0.685	-0.679
Kurtosis	1.009	5.358	1.971	1.011	0.979	1.289	0.629	0.334	0.903	2.139	1.140
Minimum	-12.99%	-32.63%	-20.75%	-17.23%	-26.01%	-16.22%	-15.12%	-12.26%	-12.65%	-21.35%	-17.15%
Maximum	14.63%	42.62%	17.51%	19.03%	20.75%	21.34%	13.32%	16.47%	11.72%	16.47%	11.35%
Median	1.17%	1.07%	1.48%	0.93%	1.53%	1.33%	1.18%	1.34%	1.24%	1.51%	1.46%
Q1	-1.29%	-2.88%	-1.45%	-2.40%	-2.63%	-1.99%	-1.92%	-1.94%	-1.18%	-2.30%	-1.75%
Q3	3.61%	4.58%	4.16%	4.69%	5.24%	3.88%	4.08%	3.93%	3.38%	3.82%	3.80%

VITA

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