ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

ENUMERATING ALL KNOTS UP TO SIX CROSSINGS

M.Sc. THESIS

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Department of Mathematical Engineering

Mathematical Engineering Programme

JUNE 2023

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To my family,

FOREWORD

I would like to express my deepest gratitude to my advisor Atabey Kaygun, who shared his knowledge and experience with me throughout my master's degree education and always motivated and supported me.

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SYMBOLS

R_1	: The first Reidemeister move
R_2	: The second Reidemeister move
R_3	: The third Reidemeister move
$K_1\#K_2$	: The composition of two knots
$c(K)$	: The crossing number of a knot
$lk(L_1,L_2$	: The linking number of a link
$M_{ij}(A)$	: The i,j minor of the matrix A
S^1	: The one-dimensional sphere
S^2	: The two dimensional sphere
$\mathbb{Z}$	: The integers
$\langle D\rangle$	: The Kauffman bracket of a diagram D
$w(D)$	: The writhe number of a diagram D
$X(D)$	: The Kauffman polynomial of a diagram D
$V(L)$	: The Jones polynomial of a link L
$\Delta_L(t)$	: The Alexander polynomial of a link L
$P_L(\alpha,z)$	: The HOMFLY polynomial of a link L

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ENUMERATING ALL KNOTS UP TO SIX CROSSINGS

SUMMARY

In the first chapter of our study, we introduce the knots defined on $\mathbb{R}^3$ and their representations in $\mathbb{R}^2$, namely knot diagrams. Then, basic concepts such as knot equivalence, alternate knots, composite and prime knots are introduced.

In the second chapter, we discuss knot invariants, which are tools that allow us to distinguish between two knot diagrams. Various knot invariants are given and the values of these invariants are calculated for some knots.

The third chapter introduces the idea that a knot can be encoded by sequences and introduces DT-sequences as an example. By labeling the crossings according to a pattern, it is shown that each knot diagram corresponds to a DT-sequence. Conversely, it is shown when a DT-sequence corresponds to a knot diagram.

Using the information from the second and third chapter, in the last chapter, all knots with up to crossing number of 6 were obtained with DT-sequences and these knots are classified with knot invariants. First, sequences with the crossing number of 1 and 2 were generated with DT-sequences and it was seen that the corresponding diagrams were in fact equivalent to the Unknot, which is the only knot with the crossing number of 0. Then all DT-sequences with the crossing number of 3 were generated and after eliminating the sequences that do not correspond to a diagram, i.e. non-realizable sequences, it was seen by calculating the knot invariants that only one of the remaining sequences was not equivalent to the only previous knot, the Unknot. This shows that there is only one knot with the crossing number of 3. This algorithm was continued up to knot with the crossing number of 6 and it was seen that there is only one knot with the crossing number of 4 and two knots with the crossing number of 5. All the knots found so far are prime knots, i.e. they cannot be written as the composition of two nontrivial knots. When the number of crossings is 6, five knots are found and it is shown that two of these knots are composite knots.

ALTI GEÇİŞE KADAR OLAN BÜTÜN DÜĞÜMLERİN LİSTELENMESİ

ÖZET

Bu tez, matematiksel düğümlerin üretilmesi ve sınıflandırılması üzerine yazılmıştır. Bunu yapmak için iki konuya ağırlık verilmiştir: DT-dizileri ve düğüm değişmezleri. DT-dizileri ile belirli bir geçiş sayısı için olabilecek tüm düğüm diyagramları elde edilirken, düğüm değişmezleri ile elde edilen bu diyagramların hangilerinin birbirinden farklı olduğu tespit edilmiştir. Bu sayede 6 geçişe kadar tüm düğümler sınıflandırılmıştır.

Çalışmamızın ilk bölümünde, $\mathbb{R}^3$ üzerinde düğümler tanımlanmış ve düğümlerin $\mathbb{R}^2$'deki temsilleri olan düğüm diyagramlarının nasıl çizileceği gösterilmiştir. Düğüm diyagramlarının denkliği tanıtıldıktan sonra iki diyagramın birbirine denk olması için gerek ve yeter koşulun sonlu sayıda Reidemeister hareketleri ile birbirine dönüştürülebilmesi olduğunu ifade eden teorem verilmiştir. Ardından, değişmeli düğüm, bileşke düğüm ve asal düğümler gibi özellikle düğümlerin sınıflandırılmasında ihtiyaç duyulacak temel kavramlara yer verilmiştir.

İkinci bölümde, iki düğüm diyagramını birbirinden ayrıştırmamızı sağlayan araçlar olan düğüm değişmezlerinden bahsedilmiştir. Birçok düğüm düğüm değişmezi bulunmakla birlikte bu çalışmada; geçiş sayısı, bağlantı sayısı, üç-renklendirilebilirlik, düğüm determinantı, Jones polinomu, Alexander polinomu ve HOMFLY polinomu değişmezleri tanıtılmıştır. Birçoğuna ait örnek eklenilmiş olsa da 4. bölümdeki tüm değişmezler bilgisayar programları yardımı ile hesaplanmıştır. Ayrıca bu bölümde üç yapraklı yonca düğümünün çizilen bir diyagramı ile bu diyagramın ayna görüntüsünden elde edilen diyagramın Jones polinomu sayesinde birbirlerine denk olmadığı kanıtlanmıştır.

Üçüncü bölümde, bir düğümün diziler ile kodlanabileceği fikri verilmiş ve buna örnek olarak DT-dizileri tanıtılmıştır. İlk olarak bir düğüm diyagram üzerinde keyfi bir nokta seçip burayı 1 ile etiketleyelim. Ardından, keyfi bir yön belirleyip her bir geçişte etiket numarasını bir arttıralım. Bu şekilde başlangıç noktasına geri dönene kadar her bir geçiş biri tek biri çift olmak üzere iki sayı ile etiketlenir. Eğer ele alınan diyagram değişmeli ise bütün etiketler pozitif; aksi halde değişmeli diyagramdan farklı görünen her bir geçişteki tek numralı etiket pozitif, çift numaralı etiket negatif işarete sahip olsun. Son olarak, tek sayıları eleyip sadece çift sayılar ile geçiş sırasına uygun olarak oluşturulan dizi bir DT-dizisidir. Bu şekilde her diyagrama karşılık bir dizi karşılık geleceği aşikardır. Tersine, ne zaman bir dizinin düğüm diyagramına karşılık geleceği gösterilmiştir. Bunun için önce, bir dizinin gerçeklemesinin, yani çiziminin, tanımı verilmiştir. Eğer bir dizi gerçeklemeye sahip ise bu dizi gerçeklenebilirdir, yani çizilebilirdir. Ayrıca eğer bir dizi gerçeklenebillir ise herhangi iki gerçeklemesi

birbirine denktir. Yani, bir diziden elde edilen her diyagram birbirine denktir. Bu sayede bir diziye ait yalnız bir diyagram olacağı görülmüştür ki bu sınıflandırma için oldukça önemlidir. Son olarak, bir dizinin ne zaman gerçeklenebilir olduğunu algoritmik hesaplarla öğrenebileceğimiz teorem verilmiştir. Bu teorem sayesinde bir dizinin sadece elemanlarını inceleyerek o dizinin bir düğüm diyagramına karşılık gelip gelmeyeceği öğrenilebilir ve çalışmanın son bölümü bu teoremin bir uygulaması olarak görülebilir.

İkinci ve üçüncü bölümdeki bilgileri kullanarak, son bölümde, geçiş sayısı 6'ya kadar olan tüm düğümler DT-dizileri ile elde edilmiş ve düğüm değişmezleri ile elde edilen düğümler sınıflandırılmıştır.

İlk olarak, geçiş sayısı 1 olan düğümleri temsil edebilecek tüm değişmeli diziler üretilmiştir. Geçiş sayısı 1 olan bir diyagramı temsil edebilecek diziler tek bir geçiş olduğundan sadece 1 ve 2'den oluşmalıdır. Bu durumda 1'in 2 ile eşleşmesinden başka yol yoktur, ancak üçüncü bölümde bir dizide, n geçiş sayısı olmak üzere, $2n$ modunda bir sayı kendisinin bir fazlası ile eşleşiyorsa dizinin indirgenebilir bir diyagrama sahip olduğu gösterilmiştir. Bunun anlamı, bu diziye karşılık gelen bir diyagram olsa da geçiş sayısının en fazla $n-1$ olmasıdır. Eğer 1, 2 ile eşleşiyorsa bu dizi indirgenebilir olacağından geçiş sayısı 1 olamaz. Bu da bu diziye karşılık gelen diyagramın aslında geçiş sayısı 0 olan çözük düğüme denk olması demektir. O halde geçiş sayısı 1 olan bir düğüm mevcut değildir.

Benzer şekilde, geçiş sayısı 2 olduğunda da elde edilen tüm diyagramların indirginebilir olduğu görüldüğünden, geçiş sayısı 2 olan bir düğüm de mevcut değildir.

Geçiş sayısı 3 olduğunda, tüm indirgenebilir diziler elendikten sonra yalnız bir tane değişmeli dizi elde edilmiştir. Bu dizinin gerçeklenebilir olduğu gösterilmiştir. Bu noktadan sonra, elimizdeki diziden elde edilebilecek tüm değişmeli olmayan diziler türetilmiştir. Bunlara karşılık gelen diyagramlar, değişmeli olan diyagramdaki alt ve üst geçişlerin yerlerinin değiştirilmesi ile çizilebilir. Bu sayede tüm değişmeli olmayan diyagramlar elde edilmiştir, ancak hepsinin indirgenebilir olmasından dolayı hiçbiri geçiş sayısı 3 olan bir düğümü temsil edemezler. O halde geçiş sayısı 3 olabilecek yalnız bir diyagram vardır. Ancak bu diyagram görünüşte indirgenebilir olmasa da kendinden önceki tek düğüm olan çözük düğüme denk olabileceğinden, değişmezler yardımıyla bu diyagramın çözük düğüm olamacağı gösterilmiştir. Sonuç olarak, geçiş sayısı 3 olan tek bir düğüm olduğu gösterilmiştir.

Bu yönteme geçiş sayısı 6 olan tüm düğümler üretilene ve sınıflandırılana kadar devam edilmiştir. Geçiş sayısı 4 olduğunda, olası iki tane gerçeklenebilir değişmeli dizi elde edilmiş ancak bunların aynı diyagramı temsil ettikleri görülmüştür. Bir tanesini esas alıp, değişmeli olmayan diziler türetilmiş ve elde edilen bütün diyagramların daha önce bulunan düğümlere denk olduğu görülmüştür. Buradan, geçiş sayısı 4 olan yalnız bir düğüm olduğu görülmüştür.

Geçiş sayısı 5 olduğunda, toplam 13 tane indirgenebilir olmayan değişmeli dizi elde edilmiş, bunların 7 tanesini gerçeklenebilir değil iken 6 tanesinin gerçeklenebilir olduğu görülmüştür. Ancak gerçekleneebilir diziler içinden sadece iki tanesinin farklı diyagramları temsil ettiği gösterilmiştir. Bu durumda da elde edilen her bir değişmeli olmayan dizinin daha düşük geçiş sayısına sahip düğümlere denk olduğu gösterilmiştir. Buradan, geçiş sayısı 5 olan iki tane düğüm olduğu görülmüştür.

Son olarak, geçiş sayısı 6 olduğunda toplam 78 tane indirgenebilir olmayan değişmeli diziden birbirinden ve kendinden önceki düğümlerden farklı 4 tane düğüm elde edilmiştir. Burada bulunan 4 düğümden bir tanesinin bileşke düğüm olduğu görülmüştür. Yani, kendinden önceki iki düğümün toplamı şeklinden yazılabilir. 6 geçiş sayısına sahip düğümlere kadar hiçbir değişmeli olmayan diziden yeni bir düğüm üretilememesine karşın, geçiş sayısı 6 olduğunda değişmeli olmayan bir diziden yeni bir düğüm üretilmiştir ve o da bir bileşke düğümdür.

1. FUNDAMENTALS OF KNOT THEORY

Our main purpose in this section is to introduce some basic concepts of knot theory that we will use later in the thesis.

1.1 Knots and Links

Take a string, tie a knot but do not stretch the string at the ends, join the endpoints. Thus we obtain an approximation of a mathematical knot. We come across knots often in daily life: shoe laces, carpet patterns, braids, sailor knots, etc. Although mathematical knots are similar to these, there is a slight difference. We define a mathematical knot as a *simple closed curve* in $\mathbb{R}^3$. A simple closed curve is defined as a continuous function from a closed interval $[a,b]$ into $\mathbb{R}^3$ such that the image of a equals to the image of b, and the curve has no self-intersection other than these end-points.

Definition 1.1.1 *A* ***knot*** *is a subset of* $\mathbb{R}^3$ *homeomorphic to* S^1. *A collection of disjoint knots is called a* ***link*** *and each knot is a component of the link.*

Instead of defining knots as curves, we can define them as unions of a finite numbers of line segments. We call such knots **polygonal knots**. A line segment of a polygonal knot in $\mathbb{R}^3$ is called an **edge** of the knot and an end point of the line segment is called a **vertex** of the knot.

1.2 Knot Diagrams

Now, we will see how to draw a link in $\mathbb{R}^2$. The most natural way to do this is to project the link orthogonally.

Definition 1.2.1 *Let* $p : \mathbb{R}^3 \to \mathbb{R}^2$ *be an orthogonal projection. For a link* L *in* $\mathbb{R}^3$, *we denote the projection of* L *by* $p(L)$. *For any* c *in the image* $p(L)$, *if* $p^{-1}(c)$ *has* n *points, then we say* c *is an* n*-multiple point. If* $n = 2$, c *is called double point.*

Definition 1.2.2 *For an orthogonal projection p and for a link L in* $\mathbb{R}^3$*, if the pair* (p, L) *satisfies the following conditions, we say p is a* ***regular projection*** *for L.*

1. *There are only a finite number multiple points in the image* $p(L)$*, all multiple points are double points.*
2. *No point in the preimage* $p^{-1}(c) \cap L$ *of any double point* $c \in p(L)$ *is a vertex of L.*

If $c \in p(L)$ is a double point, then we call it a **crossing**. In addition, for two points c_+, c_- in $p^{-1}(c) \cap L$, we say c_+ is an **overcrossing** and c_- is an **undercrossing**, if the z-coordinate of c_+ is greater than the z-coordinate of c_-. The line segment of L that contains the overcrossing is called the **overpass** and the line segment of L that contains the undercrossing is called **underpass**.

Although regular projection seems quite natural, it can not give us the substantial information about the link. When we see a regular projection of a link we can not determine which line segment is overpass or underpass at a crossing.

Definition 1.2.3 *A* ***diagram*** *is a regular projection of a link with line segments at crossings differentiated into overcrossings and undercrossings.*

In order to differentiate overpass and underpass, we erase a small neighborhood of each undercrossing in the regular projection.

The simplest examples of knot and link diagrams are shown in Figure 1.1 and Figure 1.2.

Definition 1.2.4 *A link diagram is* ***alternating*** *if an overcrossing and an undercrossing appear alternately as one goes along each component. A link is alternating if it has an alternating diagram.*

All knots and links used so far are examples of alternating link. A non-alternating knot diagram example is given in the Figure 1.3.

Definition 1.2.5 *The* ***crossing number*** *of a link L is the smallest number of crossings of any diagram of L. It is denoted by* $c(L)$*.*

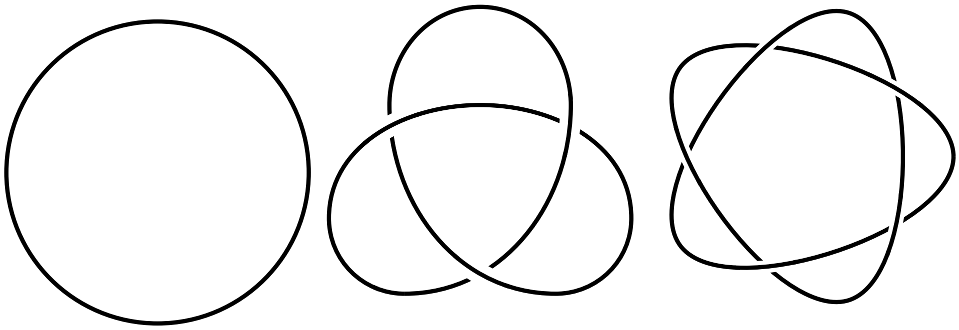


Figure 1.1 : Examples of Knots: The Unknot, The Trefoil Knot, and The Cinquefoil Knot

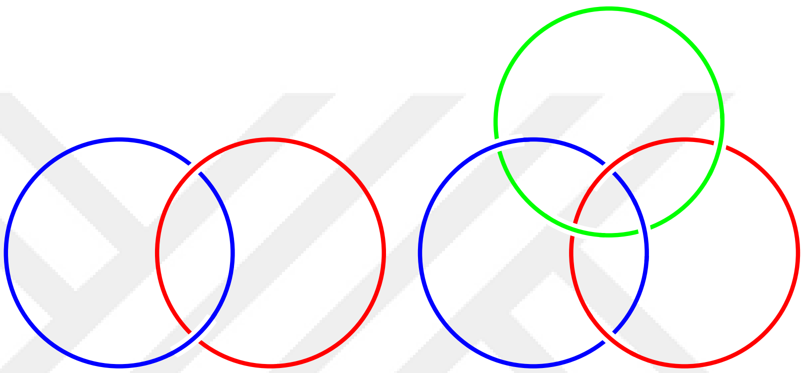


Figure 1.2 : Examples of Links: Hopf Link and Borromean Link

We can give an orientation to a knot with a choosen direction along the string. A knot that has an orientation is called **oriented knot**. If a knot has an orientation, then a diagram of the link has the induced orientation. The orientation of a knot can be shown by putting an arrow on the diagram.

1.3 Equivalence of Knots

One of the prime problems of knot theory is whether any two diagrams represent the same knot. We can get a new knot from a knot just by *continuously* moving the string without deforming it like cutting, pasting, shrinking, etc. However, this new knot will not be much different from the old one. Then, it should be quite natural to say that these two knots are *equivalent*. Now let us try to make this formal.

A **homotopy** of a space $X \subset \mathbb{R}^3$ is a continuous map

$$h : X \times [0,1] \to \mathbb{R}^3,$$

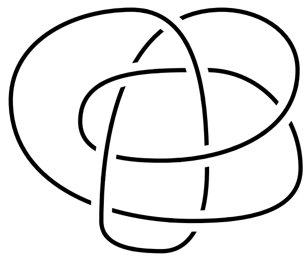


Figure 1.3 : An example of non-alternating knot which is labelled by 8_{19}

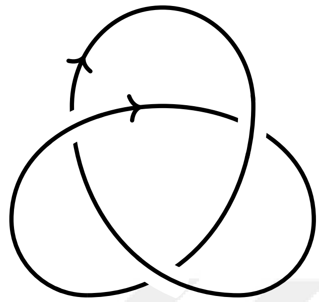


Figure 1.4 : Oriented Trefoil Knot

and for $t \in [0,1]$,

$$h_t : X \times \{t\} \to \mathbb{R}^3.$$

Here, t indicates time and the images $h_t(x)$ show the evolution of X in $\mathbb{R}^3$ when t goes from 0 to 1. Let us consider a knot. Since homotopy allows a curve to pass through itself, homotopies are not helpful for defining equivalence of knots. Because then every knot would be equivalent to unknot. To prevent this, each h_t should be one-to-one. If each h_t is one-to-one, then h is called an isotopy.

Definition 1.3.1 *Let K and K' be two knots. An* ***ambient isotopy*** *between K and K' is an isotopy $H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $H_0(K) = K$ and $H_1(K) = K'$.*

If there is an ambient isotopy between knots K and K' then we say they are **equivalent**.

As might be expected, it is quite difficult with this definition to understand whether two knots with many crossings are equivalent or not. In 1927, Kurt Reidemeister showed that two diagrams represent the equivalent knots if and only if one can be obtained from the other by a finite number of **Reidemeister moves** [1]. This moves are shown in the Figure 1.5.

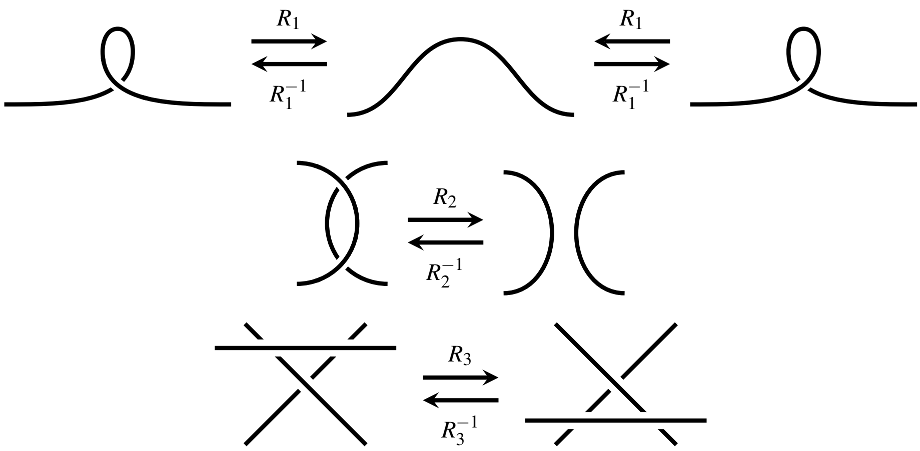


Figure 1.5 : Type I, Type II, and Type III Reidemeister Moves.

Definition 1.3.2 *A knot is called **trivial** if it is equivalent to the unknot. In general, a link is called trivial if it is equivalent to a union of disjoint circles.*

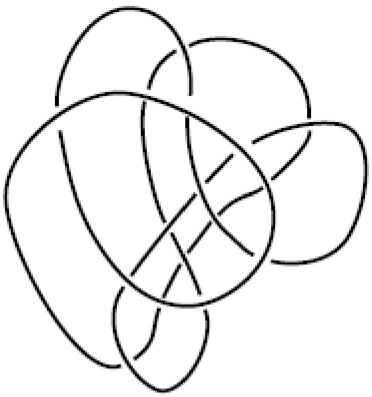


Figure 1.6 : A Diagram of the Unknot

Definition 1.3.3 *A diagram of the unknot is called **hard** if there is no R_1, R_2, R_3, R_3^{-1} moves on the diagram.*

Look at the Figure 1.6. It can be transformed to the diagram of unknot with no crossing by using R_1 and R_2 moves. However, in the Figure 1.7 there is no such moves. We have to increase the crossing number before we solve it. See Figure 1.7. The figure is taken from [4].

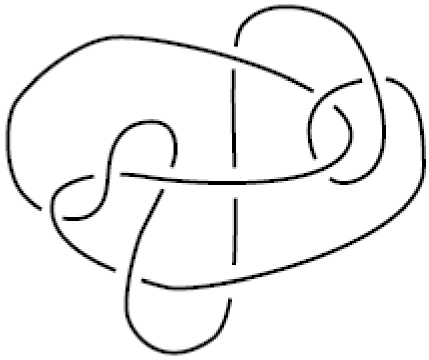


Figure 1.7 : An hard diagram of the Unknot: The Culprit

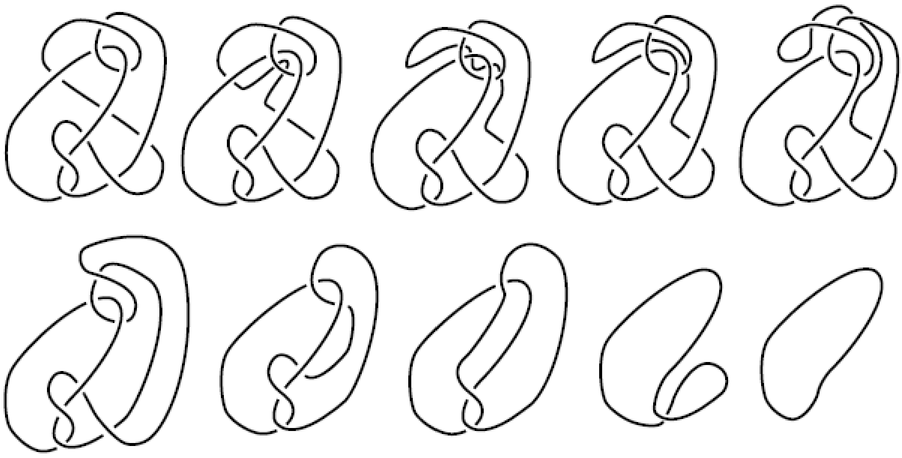


Figure 1.8 : Solving of the Culprit

1.4 Amphicheiral Links

Definition 1.4.1 *Let L be a link. The new link obtained by taking the mirror projection of L is called the* ***mirror image*** *of L.*

The mirror image diagram of a link L can simply be considered as the reversing its overcrossing to undercrossing, and vice versa, at all crossings in the diagram of L.

If a link and its mirror image are equivalent, then the link is called **amphicheiral**. The figure-eight knot is amphicheiral, it can be seen by using a series of Reidemeister moves (See [3] for the proof.) On the other hand, the trefoil knot is not aphicheiral. But this is impossible to prove with Reidemeister moves. There are endless possibilities we have to check. However, we will prove this fact thanks to the Jones polynomial in Chapter 2.

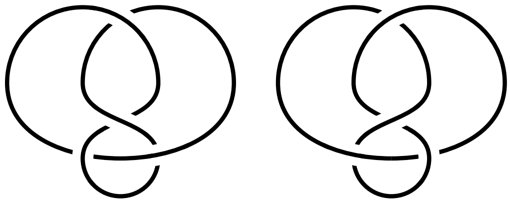


Figure 1.9 : Figure-eight knot and its mirror image

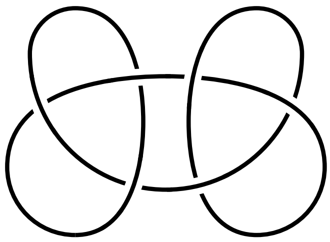


Figure 1.10 : A Composite Knot: Granny Knot

1.5 Composite and Prime Knot

Knots can be divided into two main types: composite knots and prime knots, just like integers. In Chapter 4, we are going to classify all knots up to 6 crossings. The classification will take these two types into account.

Definition 1.5.1 *Given two diagrams, take an arc that does not pass a crossing from each diagram. A new knot obtained by erasing these two arcs and joining their end points without creating a new crossing is called the* ***composition*** *of the two knots.*

The composition of two knots is denoted by $K_1 \# K_2$, where K_1 and K_2 are two knots.

Definition 1.5.2 *Let K_1 and K_2 be two non-trivial knot. If $K = K_1 \# K_2$, then K is a* ***composite knot****. If K can not be expressed as a composition of two knots, we call K a* ***prime knot****.*

2. KNOT INVARIANTS

Using the Reidemeister moves to show equivalence for knots that have many crossings is not practical. In this section, we will introduce some knot invariants that guarantees that any two knots are different by showing they have different invariants. We will also provide explicit examples to show how we can distinguish knots using invariants.

Definition 2.0.1 *Let K be the set of all links and X be any set. A mapping*

$$\varphi : K \to X$$

is called **a link invariant** *if $\varphi(k_1) = \varphi(k_2)$ when k_1 and k_2 are equivalent knots.*

In other words, a knot or a link invariant is a mapping defined on the set of all knots or links such that it takes two equivalent knots or links to the same image. As it can be seen from the definition, invariants can tell us when two links are different. Unfortunately, there is no method yet that guarantees that two links are equivalent using only these invariants without using Reidemeister moves.

2.1 Crossing Number

Proposition 2.1.1 *The crossing number of a knot or a link is invariant under the Reidemeister moves.*

Proof: Remember that the crossing number of a knot is the smallest number of crossings among all diagrams of the knot. Obviously, the crossing number is an invariant, because if k_1 and k_2 are two equivalent knots with the diagrams d_1 and d_2, by using Reidemeister moves we can convert them into the diagram that has minimum crossing number. Then, $c(k_1) = c(k_2)$ if k_1 and k_2 are equivalent. □

As an example, we know that $c(\text{Unknot}) = 0$, $c(\text{Trefoil Knot}) = 3$, $c(\text{Figure-eight Knot}) = 4$, etc. But how can we be sure, for example, c(Trefoil

Knot) = 3? How do we know it is not less than 3? This question will be answered in Chapter 4.

2.2 Linking Number

For any two components of a link, the linking number is the number that says how many times each component winds around the other.

Definition 2.2.1 *For a crossing c of an oriented link diagram D, the* ***sign*** *of c is defined as if the underpass of c goes through from the right side to the left side of the overpass of c, then* $sign(c) = 1$*; otherwise* $sign(c) = -1$.

Alternatively, we can use the right-hand rule to determine the sign of a crossing. Put your thumb on the overpass; if your other fingers go with the underpass with the same direction, then the sign is 1. Otherwise, it is -1.

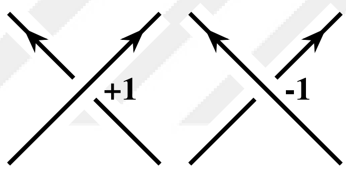


Figure 2.1 : Sign of a crossing

Definition 2.2.2 *Suppose* L_1 *and* L_2 *be two components of an oriented link L. The* ***linking number*** *is defined by*

$$lk(L_1, L_2) = \frac{1}{2} \sum_{c \in C} sign(c)$$

where C is the set of crossings which occur in both the diagrams of L_1 *and* L_2.

Thus, it can be easily seen that the linking number of a link that its components are distinct (there is no common crossing between any two component) is zero.

Now, we understand why the linking number is an invariant.

Proposition 2.2.3 *The linking number is invariant under Reidemeister moves.*

Proof: Obviously, the first Reidemeister move has no effect on the linking number because it only changes the crossings that occur in a single component of the oriented link. If we change an oriented link through two strings that are belongs to the different components by using the second Reidemeister move, then the linking number is preserved because the signs of the crossings that occur with second Reidemeister move are +1 and -1, and the sum of them is zero. The third Reidemeister move is just moves the string over a crossing from left to right of crossing, so there is no change in the sign of any crossing. □

2.3 Tricolorability

Let us meet a colorful invariant, *tricolorability*.

Definition 2.3.1 *A diagram is **tricolorable** if the following two conditions are satisfied.*

1. *At least two different colors must be used in the diagram.*
2. *Each of the arcs in the diagram can be colored with three different colors such that at each crossing all three colors meet or one color is used.*

Theorem 2.3.2 *Tricolorability is an invariant under the Reidemeister moves.*

Proof: We can prove it by illustrating.

Type I Reidemeister moves, R_1 and R_1^{-1}, are preserved under tricolorability: For R_1, since we have two arc, it must be colored with one color. If we use R_1 move, there will be one arc colored with one color. For R_1^{-1}, since we have one arc, again it must be colored with one color and if we use R_1^{-1} move, there will be one arc colored with one color.

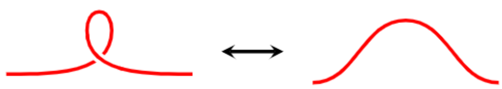


Figure 2.2 : Tricolorability is invariant under Type I Reidemeister moves.

Type II Reidemeister moves, R_2 and R_2^{-1}, are preserved under tricolorability: We have two cases. All arcs around two crossings can be colored with one color or they can be colored with three different colors. You can see that there is no possibility that the arcs around one crossing can be colored in the same color and the arcs around the other crossing can be colored in three different colors. See Figure to see that in both cases it is invariant under Type II Reidemeister moves.

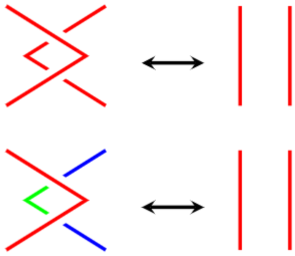


Figure 2.3 : Tricolorability is invariant under Type II Reidemeister moves.

Type III Reidemeister moves, R_3 and R_3^{-1}, are preserved under tricolorability: We have five cases. First, all arcs around three crossings can be colored with one color. Second, the arcs around one crossing can be colored with one color and the arcs around other two crossings can be colored in three different colors. This gives us three cases. Finally, the arcs around all three crossings can be colored in three different colors. Again, it can be seen by illustrating that it is invariant under Type III Reidemeister moves. □

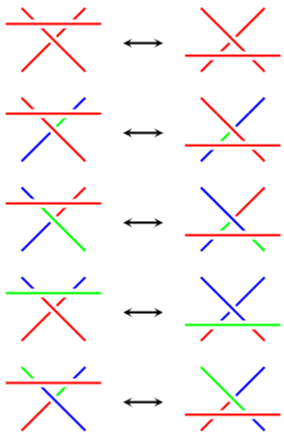


Figure 2.4 : Tricolorability is invariant under Type III Reidemeister moves.

Example 2.3.3 *Trefoil knot*(3_1) *is tricolorable while unknot and Figure-eight knot*(4_1) *is not. Unknot can be colored by just one color. There is no option for it. For Trefoil, a suitable coloring is given in the following figure. In order to see Figure-eight knot is not tricolorable you must try all possibilities. An unsuitable possibility is given in the following figure, too.*

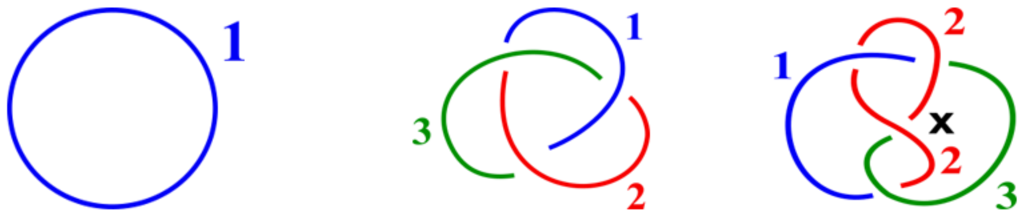


Figure 2.5 : Examples of tricolorable and non-tricolorable knot diagrams.

2.4 Knot Determinant

Our next invariant, knot determinant, depends on basic linear algebra. Somehow, we will find the corresponding matrix to a diagram. Then we will arbitrarily delete a row and a column. No matter which row and column we delete, the determinants of the remaining matrices will be equal to each other. And this determinant will be an invariant for a knot.

Definition 2.4.1 *Let D be a diagram of a knot K with n crossings. Label the crossings with* $1, 2, \ldots, n$ *and label the arcs with* $x_1, x_2, \ldots, x_n$*. For any crossing, there is only one view. It has to be the intersection of three different arcs. One arc, say* x_i*, passes over while two arcs, say* x_j *and* x_k*, pass under. Then the coloring equation is expressed by*

$$2x_i - x_j - x_k = 0.$$

The system of all coloring equations for K is called as the coloring system of K.

Definition 2.4.2 *The coloring matrix of a knot K is the coefficient matrix of the coloring system of K.*

Recall that if A is an $n \times n$ matrix, the i, j minor of A, denoted by $M_{ij}(A)$, is the determinant of the $(n-1) \times (n-1)$ matrix formed by removing the i^{th} row and the j^{th} column from A.

Definition 2.4.3 *The determinant of a knot K is defined as the absolute value of any minor of coloring matrix of K.*

Theorem 2.4.4 *The determinant of a knot is an invariant. It means it does not depend on the choice of diagram, labeling and row and column to delete.*

Proof: See page 46 in [8] for the proof. □

Example 2.4.5 *Let us calculate the determinant of the knot denoted by* 6_2*. See Figure 2.6 for a diagram of it.*

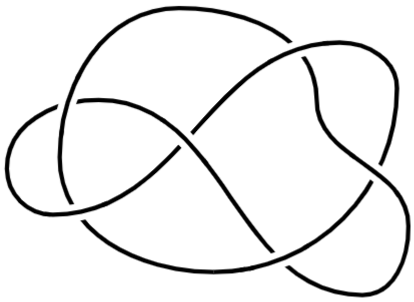


Figure 2.6 : A diagram of the knot, denoted by 6_2.

There are six arcs. Label them with a, b, c, d, e, f*. Then the coloring equations are as follows.*

$$2a - c - d = 0, \tag{2.1}$$

$$2e - a - b = 0, \tag{2.2}$$

$$2b - e - f = 0, \tag{2.3}$$

$$2f - b - c = 0, \tag{2.4}$$

$$2c - d - e = 0, \tag{2.5}$$

$$2d - a - f = 0. \tag{2.6}$$

So, the coloring matrix is

$$\begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2 & 0 & -1 \end{bmatrix}.$$

For a minor matrix, we delete the sixth row and sixth column. Thus the determinant of the minor of the coloring matrix is -11. Therefore, the determinant of 6_2 *is 11.*

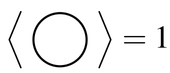
2.5 The Jones Polynomial

Using knot invariants as we shown above is not always enough to distinguish knots. However, knot polynomials that systematically associate a knot diagram to a polynomial are powerful invariants. If the corresponding polynomials of knot diagrams are different, then we can easily say that the diagrams represent different knots. Using them for knot diagrams with many crossings takes time and patience, but more than often it will yield results. Now, it is also possible to calculate these polynomial invariants in easily thanks to computer libraries. We will use one of these libraries to calculate these polynomail invariants later.

One of the most important knot polynomials is the Jones polynomial. They are discovered by Vaughan Jones and it corresponds an oriented link with a Laurent polynomial with integer coefficients. After V. Jones found his polynomial, Louis H. Kauffman developed a method to define it more easily than Jones did. So, we will start with the Kauffman bracket to define the Jones polynomial at first.

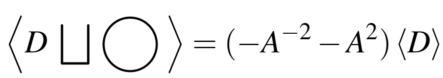
Definition 2.5.1 *The **Kauffman bracket** is a mapping which takes any diagram D in* S^2 *to* $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ *and it satisfies the following rules:*

(i) We define

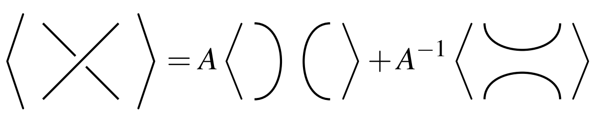


where ○ *is the minimal diagram of the unknot.*

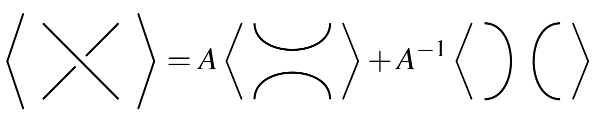
(ii) If our link contains a disjoint unknot we define



(iii) For the under and over crossings we let



and



Definition 2.5.2 *Let D be a diagram of an oriented link. The **writhe number** $w(D)$ is the sum of the signs of all crossings of D.*

Definition 2.5.3 *Let D be a diagram of an oriented link L. The **Kauffman polynomial** is defined as*

$$X(D) = (-A)^{-3w(D)} \langle D \rangle .$$

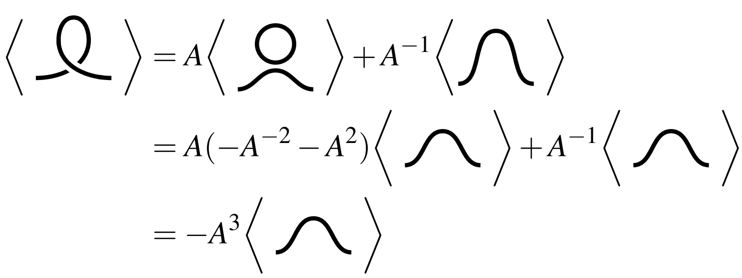
While the polynomial is an invariant, the bracket is not an invariant for links. Although the bracket of D and the bracket of any diagram that is obtained by the first and second Reidemeister moves from D are equal, the first Reidemeister move changes the brackets.

Theorem 2.5.4 *The Kauffman Polynomial is an invariant for the oriented links.*

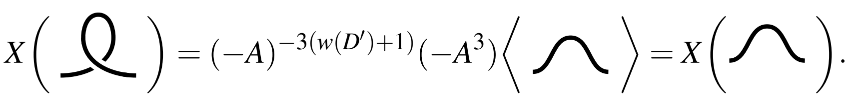
Proof: Let D be a diagram of an oriented link L. We will investigate how the Kauffman polynomial of D is affected under the Type I, II and III Reidemeister moves. Note that in order to observe the changes, we are only going to draw the neighborhood of the crossings affected by Riedemeister moves.

In an oriented link diagram, the sign of a crossing that terminates under the first Reidemeister move can be +1 or -1. Because of that we have to prove it by two steps.

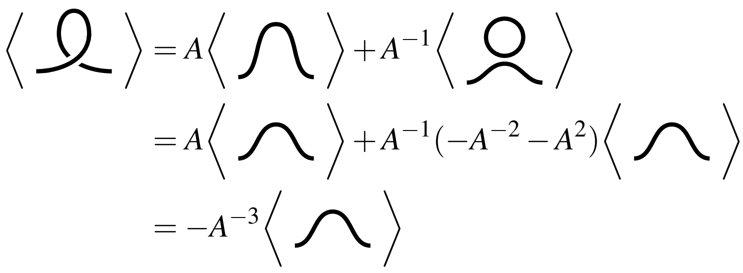
Type I Reidemeister Move:



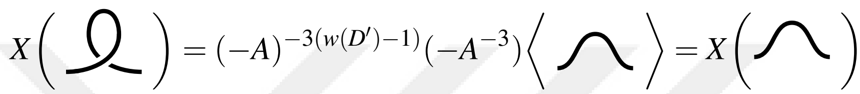
Thus, the Kauffman polynomial of an oriented link diagram under the positively directed Reidemeister is



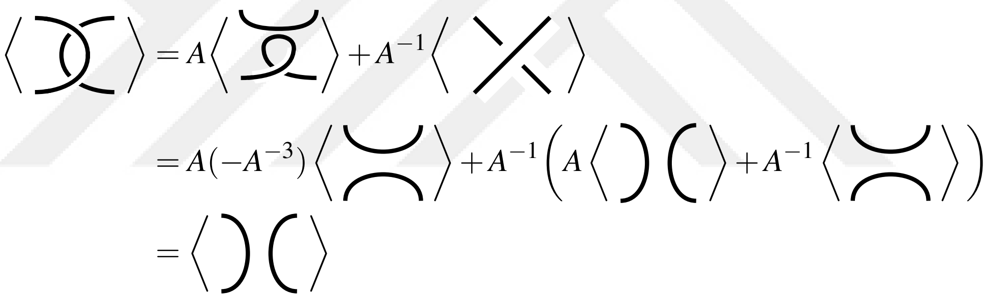
Here, note that D' is the diagram which is obtained from D by using first Reidemeister move. And,



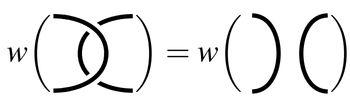
Thus, the Kauffman polynomial of an oriented link diagram under the negatively directed Reidemeister is



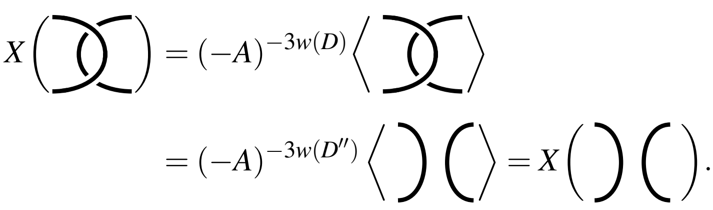
Type II Reidemeister Move:



Also we know that



Thus,



Here, note that D'' is the diagram which is obtained from D by using R_2.

Type III Reidemeister Move: Note that D''' is the diagram which is obtained from D by using R_3. It is clear that

$$\langle D\rangle = \langle D'''\rangle$$

because the number and type of crossing does not changes with R_3. Also we know that $w(D) = w(D''')$. Therefore,

$$X(D) = (-A)^{-3w(D)}\langle D\rangle = (-A)^{-3w(D''')}\langle D'''\rangle = X(D''').$$

□

Definition 2.5.5 *The **Jones polynomial** $V(L)$ of an oriented link L is obtained from the Kauffman polynomial by replacing $t^{1/2} = A^{-2}$.*

It is clearly a Laurent polynomial in the variable $t^{1/2}$ with integer coefficients. Explicitly, if D is an oriented diagram of L, then it is defined as

$$V(L) = (X(D))_{t^{1/2}=A^{-2}} = ((-A)^{-3w(D)}\langle D\rangle)_{t^{1/2}=A^{-2}}.$$

The diagrams given in the Figure 2.7 is the diagrams which exactly seem the same except in the neighborhood of one crossing. We will use them to give a relation which is related by the Jones polynomial and then we will compute the Kauffman bracket with a different approach.

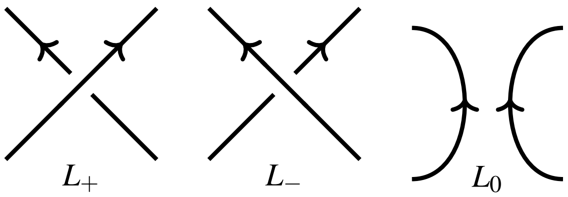
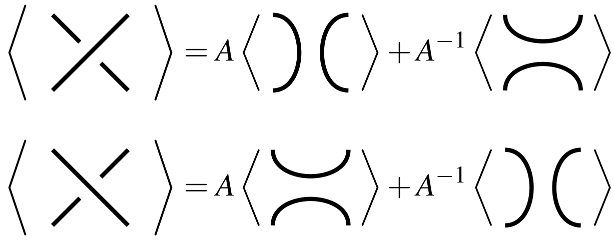


Figure 2.7 : Three same diagrams except in the neighborhoods of one crossing

Theorem 2.5.6 *The Jones polynomial V(L) satisfies the skein relation given below:*

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0).$$

Proof: First, let us find a relation between them in the Kauffman bracket.



Multiply the first equation by A, the second equation by A^{-1} and then substract the second from first. We will get the relation below:

$$A\langle L_+\rangle - A^{-1}\langle L_-\rangle = (A^2 - A^{-2})\langle L_0\rangle$$

Now, let us multiply this relation by $(-A)^{-3w(L_0)}$. Then, we get the following relation by using $w(L_+) - 1 = w(L_0) = w(L_-) + 1$.

$$A^4 V(L_+) - A^{-4} V(L_-) = (A^2 - A^{-2}) V(L_0).$$

The final result will be obtained by substituting $A^{-2} = t^{1/2}$. $\square$

2.5.1 An Example: The trefoil knot and its mirror image

In Chapter 1, we claimed that the trefoil knot is not amphicheiral, i.e. the trefoil knot and its mirror image are not equivalent. Now, we have a strong tool to do that: *the Jones polynomial*.

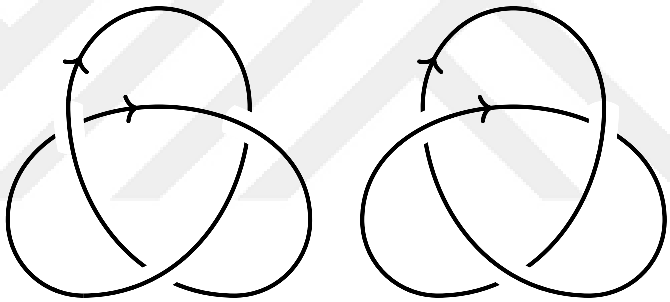
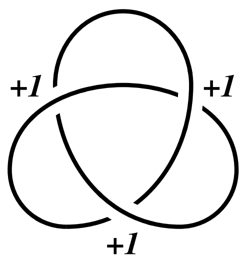


Figure 2.8 : The left trefoil and the right trefoil

Example 2.5.7 *We will show that the trefoil knot,* 3_1*, is not amphicheiral by using the Jones polynomial. For convenience, we call them left trefoil and right trefoil and we denote them by* 3_1 *and* -3_1*. Let us start with the Jones polynomial of the right trefoil knot:*

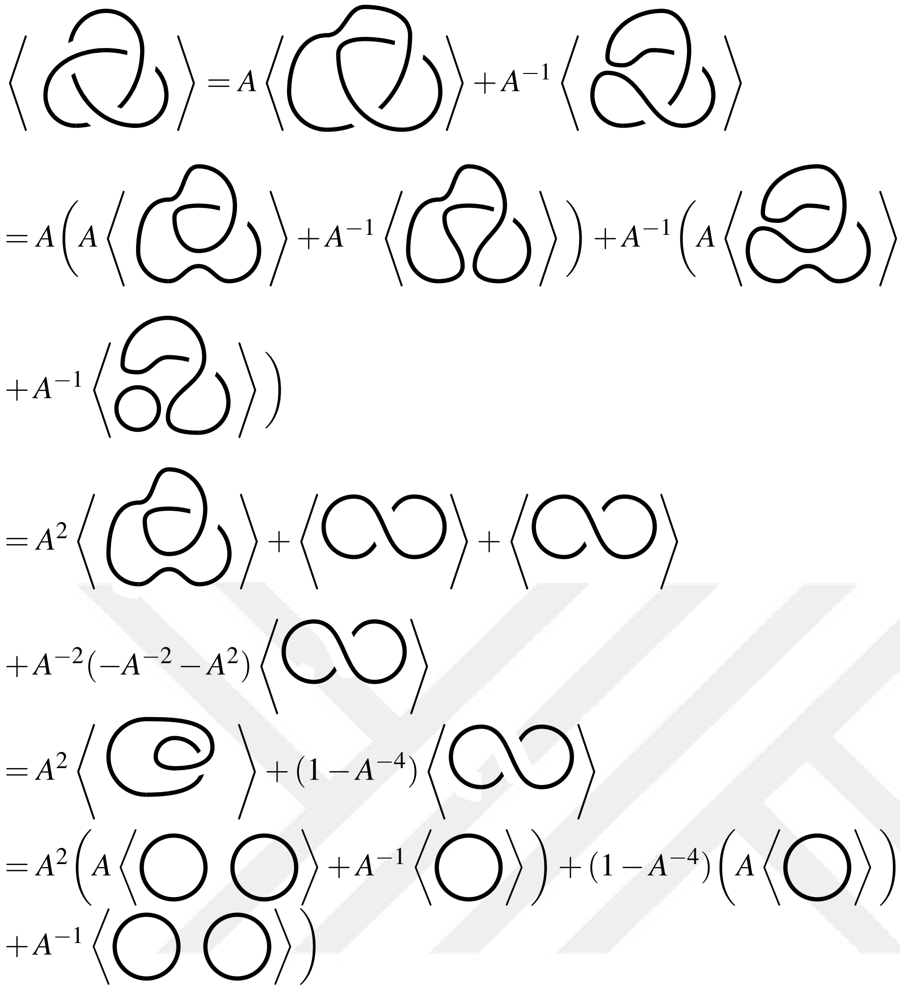
$$V_{3_1}(t) = [(-A)^{-3w(3_1)}\langle 3_1\rangle]_{A^{-2}=t^{1/2}}.$$

Let us compute the writhe and the Kauffman bracket of the right trefoil.

$w\Big($ 

$\Big) = 1 + 1 + 1 = 3$

and,



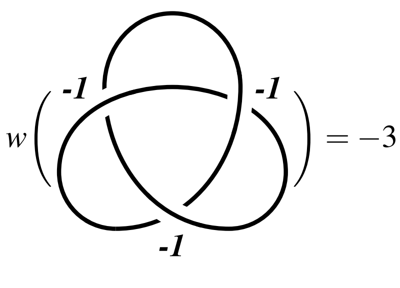
$$= -A - A^5 + A - A^{-3} + A^{-7} + A^{-3} + A - A^{-3} - A$$
$$= A^{-7} - A^{-3} - A^5$$

Therefore,

$$V_{3_1}(t) = -t^{9/4}(t^{7/4} - t^{3/4} - t^{-5/4}) = -t^4 + t^3 + t.$$

Now, let us calculate the Jones polynomial of the left trefoil knot:

$$V_{-3_1}(t) = [(-A)^{-3w(3_1')}\langle 3_1' \rangle]_{A^{-2}=t^{1/2}}.$$



The polynomial can be found as follows, similar to the previous one.

$$V_{-3_1}(t) = -t^{-4} + t^{-3} + t^{-1}.$$

We would like to note that the Jones polynomial of the mirror link of any link can be found by replacing t by t^{-1}.

2.6 The Alexander Polynomial

The Alexander polynomial, $\Delta_L(t)$, is the first (Laurent) polynomial invariant invented by J. Alexander in 1928. His method was essentially similar to the calculation of the knot determinant. However, in 1969, John Conway improved a new method for calculating the Alexander polynomial using only two rules. See the paper [7] for proof that the Alexander polynomial is invariant.

First, let us give the original definition of the polynomial introduced by Alexander.

Definition 2.6.1 *Let D be an oriented diagram of a knot K.*

- *If a crossing of D is right-handed with the arcs i, j, k as shown in the diagram on the left in figure 2.9, then the polynomial coloring equation at the crossing is*

$$(1-t)x_i - x_j + tx_k = 0.$$

- *If a crossing of D is left-handed with the arcs i, j, k as shown in the diagram on the right in figure 2.9, then the polynomial coloring equation at the crossing is*

$$(1-t)x_i + tx_j - x_k = 0.$$

The minor of the corresponding matrix to the system of polynomial coloring equations is called Alexander matrix of K and the Alexander polynomial of K is defined as the determinant of the Alexander matrix.

When we introduced the Jones polynomial, we mentioned the skein relation. Like the Jones polynomial, the Alexander polynomial also satisfies the following relations.

Rule 1: $\Delta(unknot) = 1.$

Rule 2: $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0).$

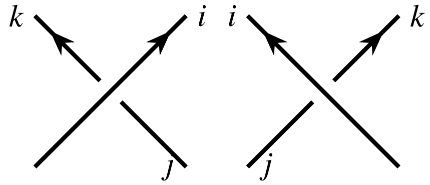


Figure 2.9 : Right-handed and left-handed crossing

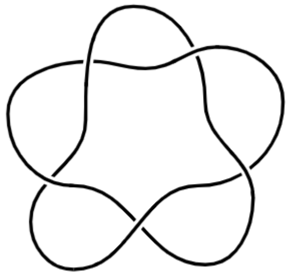


Figure 2.10 : A diagram of the knot, denoted by 5_1

Although Alexander polynomial is a knot invariant, it depends on the choice of the diagram and labelings. Two polynomials obtained from two different choices of diagram or labellings will differ by a multiple of $\pm t^k$, where $k \in \mathbb{Z} \cup \{0\}$. It means two Alexander polynomial are equivalent if they are $\pm t^k$ multiples of each other. See page 50 in [8] for the proof.

Example 2.6.2 *Let us calculate the Alexander polynomial for the knot,* 5_1*, given in Figure 2.10.*

If we label the arcs with a,b,c,d,e, then the polynomial coloring equations are

$$(1-t)a + td - c = 0, \tag{2.7}$$

$$(1-t)b + te - d = 0, \tag{2.8}$$

$$(1-t)c + ta - e = 0, \tag{2.9}$$

$$(1-t)d + tb - a = 0, \tag{2.10}$$

$$(1-t)e + tc - b = 0. \tag{2.11}$$

Hence, the corresponding matrix of the system of polynomial coloring equations is

$$\begin{bmatrix} 1-t & 0 & -1 & t & 0 \\ 0 & 1-t & 0 & -1 & t \\ t & 0 & 1-t & 0 & -1 \\ -1 & t & 0 & 1-t & 0 \\ 0 & -1 & t & 0 & 1-t \end{bmatrix}.$$

Any minor of this matrix gives us the Alexander polynomial. If we delete the fifth row and fifth column and calculate the determinant of it, we find $\Delta_{5_1}(t) = t^4 - t^3 + t^2 - t + 1$.

If we calculate the polynomial using the skein relation it would be $\Delta_{5_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$. However we can see that it is t^2 multiples of the other.

2.7 The HOMFLY Polynomial

Our last polynomial invariant is a generalization of the Jones polynomial and the Alexander polynomial. It was found by Hoste, Ocneau, Millett, Freyd, Lickorish and Yetter a little after invention of Jones polynomial. They called it HOMFLY, using their initials.

This polynomial can be calculated using a skein relation like the previous polynomial invariants. The HOMFLY polynomial, $P_L(\alpha, z)$, is a Laurent polynomial in two variables and it satisfies the following two rules.

Rule 1: $P(unknot) = 1.$

Rule 2: $\alpha P(L_+) - \alpha^{-1} P(L_-) = zP(L_0).$

If we set $\alpha = t^{-1}$ and $z = t^{1/2} - t^{-1/2}$ in Rule 2, we will get the same relation as the Jones polynomial. If we set $\alpha = 1$ and $z = t^{1/2} - t^{-1/2}$, the relation turns into the skein relation of the Alexander polynomial.

3. KNOT ENCODINGS

Since the process of finding the minimal diagram is not easy, the crossing number as an invariant is not useful. However, it enables to classify the knots according to the crossing number. For example, there is exactly one knot that has three crossings. Peter Guthrie Tait started the first attempt to classify knots according to the crossing numbers, in 1885. He published a table of knots with up to ten crossings. See also the Knot Atlas [2].

Let us consider the knots given in Figure 1.1 and Figure 1.9. The Trefoil knot is denoted as 3_1 because it is the *first* and the only knot with *three* crossings. Similarly, the Cinquefoil knot, 5_1, is the *first* knot with *five* crossings and the figure-eight knot, 4_1, is the *first* and only knot with *four* crossings. The second knot with five crossings is given in the Figure 3.1. But how do we know, for example, the trefoil knot is the only knot with three crossings?

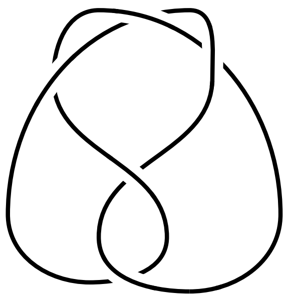


Figure 3.1 : The knot 5_2

3.1 Encoding of a Knot

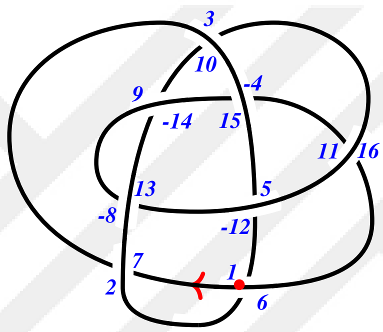
Given a diagram D of an n-crossing knot, follow the following steps to encode a knot:

1. Starting a random crossing, label the crossings from 1 to $2n$ in a chosen direction along K. (Realize that every crossing has two value that one of them is odd and other one is even.)

2. If the diagram is alternating then it is okay but if the diagram is non-alternating then change the sign of the even labels of crossings which look different from the alternating one.

3. Encode the knot with a sequence $\{a_1, a_3, ..., a_{2i-1}, ..., a_{2n-1}\}$ where a is the involutory function from $\{1, 2, ..., 2n\}$ to itself such that $a(i) = a_i$ is the even number at the point where i is.

Example 3.1.1 *Let us encode the knot given in Figure 1.3. Let us draw the knot and choose a basepoint and an orientation. Then, label the crossings by starting from the basepoint in the chosen orientation.*



Then, we have $a_1 = 6$*,* $a_3 = 10$*,* $a_5 = -12$*,* $a_7 = 2$*,* $a_9 = -14$*,* $a_{11} = 16$*,* $a_{13} = -8$*,* $a_{15} = -4$*. Thus, the knot is encoded with*

$$\{6, 10, -12, 2, -14, 16, -8, -4\}.$$

3.2 Realization of a Sequence

Obviously, all knots can be expressed as a sequence. However in order to tabulate we need the answer of a question: Do all sequences correspond to a knot? Although the answer is no, we know how we eliminate the sequences which do not correspond to a knot.

We define $S = \{a_1, a_2, a_3, ..., a_{2n}\}$ where $a : i \rightarrow a_i$ is a parity reversing involution of the interval $\{1, 2, 3, ..., 2n\}$. For the integers of $\{i, i+1, ..., j-1, j\}$ in modulo $2n$, we use the notation $[i, j]$. Now suppose that S corresponds to a diagram of a knot K. If there

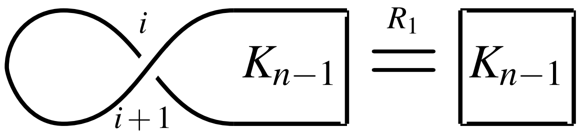


Figure 3.2 : A crossing satisfying $a_i = i+1$ is removing from the knot.

is an element $a_i \in S$ such that $a_i = i+1 (\text{mod } 2n)$, the diagram can be reduced using R_1 move. Look at the Figure3.2 to see. We call such a sequence **reducible sequence**.

Although the sequences having $a_i = i+1$ correspond to a knot, we ignore them since they are reducible. If we are looking for all n-crossing knots, we will eliminate the reducible ones. In general, knot tabulating considers only prime knots. The following definition tell us which sequences belong to a composite knot.

Definition 3.2.1 *Let $n \geqslant 3$. If there is a proper subinterval $[i, j]$ (mod $2n$) of $\{1,2,3,...2n\}$ is mapped onto itself by the involution $a : k \to a_k$, then the subsequence $\{a_1, a_3, ..., a_{2n-1}\}$ gives a composite knot.*

For example, the DT-sequence $\{4,6,2,10,12,8\}$ gives us a composite knot because the sequence $S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ has the proper subinterval $\{1,2,3,4,5,6\}$ such that it is mapped onto itself by the function a.

From now on, we assume that our sequences do not gives a composite knot, that is, there is no proper subinterval like in the definition. The following definition tells us how a sequence corresponds to a diagram.

Definition 3.2.2 *A piecewise linear mapping $k : [0,2n] \to \mathbb{R}^2$ satisfying the following conditions is called a **realization** of the sequence S.*

1. *$k(0) = k(2n)$: for it to be closed,*

2. *$k(i) = k(a_i)$: for $i = 1,2,...,2n$: for crossings to occur (we let i be over while a_i is under),*

3. *k is a homeomorphism on $[0,2n] \setminus \{0,1,2,...,2n\}$: for undercrossings and overcrossings to occur,*

4. *the arc* $k([i-\frac{1}{2}, i+\frac{1}{2}])$ *and the arc* $k([a_i-\frac{1}{2}, a_i+\frac{1}{2}]$ *intersect at* $k(i)$ *: for underpasses and overpasses to occur.*

If the sequence S has a realization, we call S **realizable**. We need to find the answer to this question: How do we determine that a sequence is realizable? Before we answer it, we will give an important theorem whose proof can be found in [6].

Theorem 3.2.3 *If S is realizable, any two realizations of S are equivalent.*

This is important because if we find a diagram corresponding to a sequence, it must be unique. That is, each sequence corresponds to at most one knot. However, a knot diagram may not be represented by just one sequence. We can relabel a diagram by putting 1 to an other under or over crossing and also we can change the direction. Some of these changes may give us a new sequence but all represent the same diagram. The number of sequences with relabeling is $4n$: there is $2n$ choices of crossings and 2 directions for an n-crossing diagram.

In the oriented plane, notice that an overpass crosses an underpass either from right to left or vice versa. Let us define it by a function:

$$f(i) = \begin{cases} 1, & \text{if } k([a_i-1, a_i+1]) \text{ crosses } k([i-1, i+1]) \text{ from right to left;} \\ -1, & \text{if } k([a_i-1, a_i+1]) \text{ crosses } k([i-1, i+1]) \text{ from left to right.} \end{cases}$$

Note that f is not the sign function. The sign function considers a crossing while f considers two points of a crossing. It is even seen that $f(i) = -f(a_i)$. Also, we can choose for the realization k so that $f(1) = 1$. We denote the oriented realization of S with (S, f) and we call (S, f) realizable if it has a realization. Note that if S is realizable, then orientation f is unique. [6]

For any sequence $S = \{a_1, a_2, ..., a_{2n}\}$ and for all $i \in \{1, 2, ..., 2n\}$, we define

$$\phi_i : \{1, 2, ..., 2n\} \to \{-1, 1\}$$

such that

$$\phi_i(i) = 1,$$

and

$$\phi_i(j) = \begin{cases} -\phi_i(j-1), & \text{if } a_j \in \{i, ..., a_i\}; \\ \phi_i(j-1), & \text{otherwise.} \end{cases}$$

Theorem 3.2.4 *If (S, f) is realizable, it must satisfy the following two conditions.*

1. $\phi_i(s)\phi_i(a_s) = 1$ *if* $a_s \notin [i, a_i]$,

2. $\phi_i(s)\phi_i(a_s)f(i) = f(s)$ *if* $a_s \in [i, a_i]$,

where $i < a_i < s$ and $a_s < s$ for all i and s in $\{1, 2, ..., 2n\}$. These conditions are necessary and sufficient conditions for (S, f) to be realizable.

Lemma 3.2.5 *Let B be a proper subset of $\{1, 2, ..., 2n\}$ such that $a_i \in B$ for all $i \in B$. If $C = \{1, 2, ..., 2n\} \setminus B$, some pair i, a_i in B separates some pair s, a_s in C in the cyclic order mod $2n$.*

Proof: B can be expressed as the union of intervals $B_1, B_2, ..., B_k$ such that there exists i is in B_h while a_i is not. Otherwise the sequence would turn into a composite knot. Then we have

$$\{1, 2, ..., 2n\} = B_1 \cup C_1 \cup B_2 \cup C_2 \cup ... \cup B_k \cup C_k,$$

where C_h's are successive intervals in mod $2n$. Let r be the minimal positive number such that some i in B_h or C_h with a_i in B_{h+r} or C_{h+r}. We let $i \in B_h$ and $a_i \in B_{h+r}$. We know there exists s in C_h with $a_s \notin C_h$ and since r is the minimal, $a_s \notin \bigcup_{h+1}^{h+r-1} C_j$. Therefore $a_s \notin [i, a_i]$. However $s \in [i, a_i]$ because C_h comes after B_h and before B_{h+r}. So, i and a_i cyclically separate s and a_s. □

Lemma 3.2.6 *Let A_h and B_h be inductively definable subsets of $\{1, 2, ..., 2n\}$. Assume that $A_1 = \{1, a_1\}$, $B_1 = \emptyset$ and $i_1 = 1$. For $h > 1$, we define*

$$A_h = A_{h-1} \cup \{s : i_{h-1} \text{ and } a_{i_{h-1}} \text{ separate } s \text{ and } a_s\},$$

$$B_h = B_{h-1} \cup \{i_{h-1}, a_{i_{h-1}}\},$$

$$i_h := \text{ the least member of } A_h \setminus B_h \text{ if } A_h \setminus B_h \neq \emptyset.$$

Then the sequence $i_1, i_2, ...$ ends with i_n.

Proof: At every step, A_h gets bigger with some s and a_s. So h can take at most n values. Since B is a proper subset of {1,2,...,n} such that for all i in B, $a_i \in B$, we can use Lemma 3.2.5. That is, there is some $i \in B$ and $s \notin B$ such that i and a_i separate s and a_s. If $i < a_i$, $i \in \{i_1, i_2, ..., i_{h-1}\}$ due to definition of B_h. Then we get s and a_s are in A_h because of its definition. That means $s, a_s \in A_h \setminus B_h$. Hence for $h \leqslant n$, there exists an i_h. □

The orientation function f given in Theorem 3.2.4 cannot be calculated from a sequence. In the next definition we define a new canonical orientation function f that can be calculated inductively from a sequence.

Definition 3.2.7 *For s and a_s in $A_h \setminus A_{h-1}$, $[i_{h-1}, a_{i_{h-1}}]$ contains only one of s and a_s. Assume that $s \notin [i_{h-1}, a_{i_{h-1}}]$ and $a_s \in [i_{h-1}, a_{i_{h-1}}]$. The canonical orientation function of a sequence is a map*

$$f : \bigcup_{h=1}^{n} A_h \setminus A_{h-1} = \{1, 2, ..., 2n\} \to \{-1, 1\}$$

such that for s and a_s in $A_h \setminus A_{h-1}$,

$$f(1) = 1 \text{ and } f(a_1) = -1;$$

$$f(s) = \phi_{i_h}(s)\phi_{i_h}(a_s)f(i_h) \text{ and } f(a_s) = -f(s).$$

Theorem 3.2.8 *S is realizable if and only if (S, f) is realizable where f is the canonical orientation function.*

Proof: Let S be realizable. Then there is a unique orientation function g that satisfies the condition of Theorem 3.2.4. Clearly, $f(1) = g(1)$ and $f(a_1) = g(a_1)$. Then $f(j) = g(j)$ for all j in A_{h-1}. If $s \in A_h \setminus A_{h-1}$ which does not belong $[i_h, a_{i_{h-1}}]$, then $f(s) = g(s)$ and $f(a_s) = g(a_s)$. By induction $f = g$. The other side of the theorem is trivial. □

4. GENERATING KNOTS WITH DT-SEQUENCES

In this section, we will generate all possible knots with crossing number of up to 6. Although our main goal is to find prime knots, the method will also allow us to find composite knots. After that there will be a lot of knots that look different. Fortunately, we will separate them using the knot invariants we introduced before.

Our method is quite simple:

We know that Unknot is the only knot with crossing number of 0.

Let the following process be applied for n to take the values 1,2,...,6 respectively.

- Generate all alternating DT-sequences with n elements except reducible sequences.
- Delete the sequences that is not realizable.
- Find non-alternating sequences of realizable sequences and delete sequences consisting of negative elements of another sequence. That is, we ignore the reflections.
- Draw the diagrams for the remaining sequences.
- If a diagram can be reduced by R_1 or R_2, delete them. This is because they have at most $n-1$ crossing number. They are equivalent to a knot that was found before.
- For each remaining diagram, use the knot invariants and compare them with the previous knots. If the diagram has a distinguishing invariant, it must be a new knot. If not, then it is probably equivalent to a knot preceding it.

If a diagram does not have a distinguishing invariant, we will able to show that it is equivalent to a previous knot by using the Reidemeister moves. However, in theory, it is possible that it can represent a different knot although it does not have a distinguishing invariant. This is because our invariants introduced in the thesis are not *perfect invariants.* That is, two different knots can have a same value of an invariant.

4.1 Knots with Crossing Number of 1

Since it is clear that the only knot with $c(K) = 0$ is Unknot which is denoted by 0_1, let us start by proving that there can be no knot with $c(K) = 1$.

If $c(K) = 1$, then the minimal diagram of K has 1 crossing. So, all DT-sequences that are likely to represent a knot diagram are $\{2\}$ and $\{-2\}$. We can ignore the negative of a DT-Sequence since we do not see reflection of a knot as different from that knot. Actually, the diagrams corresponding to these sequences can be different, it means one of them does not become other using Reidemeister moves, but they are just mirror images of each other in $\mathbb{R}^3$. We know the sequence $\{2\}$ is a reducible sequence since 1 goes 2. It can also be easily seen by drawing that $\{2\}$ corresponds to a diagram Unknot. See Figure 4.1:

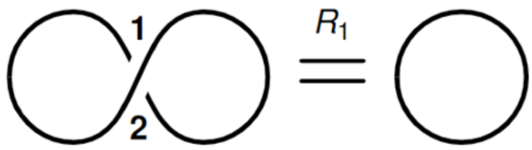


Figure 4.1 : $\{2\}$ corresponds to a diagram of Unknot.

Thus, there is no knot with $c(K) = 1$.

4.2 Knots with Crossing Number of 2

The list of all DT-sequences for knots with 2-crossings is $\{2,4\}$, $\{-2,4\}$, $\{2,-4\}$, $\{-2,-4\}$, $\{4,2\}$, $\{-4,2\}$, $\{4,-2\}$, $\{-4,-2\}$. If we eliminate reflections, we are left with $\{2,4\}$, $\{-2,4\}$, $\{4,2\}$, $\{-4,2\}$. All of these DT-sequeunces are reducible. That is each two crossings diagram is the diagram of Unknot. Hence, there is no knot with $c(K) = 2$.

4.3 Knots with Crossing Number of 3

Suppose that $c(K) = 3$. If $a_1 = \pm 2$ or $a_1 = \pm 6$, then the crossing labelled by 1 can be removed. It means K is Unknot because there is no knot with $c(K) = 2$ and $c(K) = 1$. So, the only choice is $a_1 = \pm 4$. Similarly, $a_3 = \pm 6$ and $a_5 = \pm 2$. Then the only possible alternating DT-sequence for K is $\{4, 6, 2\}$. Let us check whether it is realizable. We use Theorem 3.2.4 for it.

- If $i = 1$, we have $a_i = 4$, $s \in \{5, 6\}$.
  - If $s = 5$, we have $a_s = 2$ which belongs to $[i, a_i]$. Thus we have to check the second condition of the theorem. From the definition of ϕ_i, we have $\phi_1(1) = \phi_1(2) = \phi_1(3) = -\phi_1(4) = \phi_1(5) = -\phi_1(6) = 1$. Now we know that $f(s)$ is computed by $\phi_i(s)\phi_i(a_s)f(i)$, for all s and a_s in $A_h \setminus A_{h-1}$. From the definition, $A_1 = \{1, 4\}$ and $B = \emptyset$ and $i_1 = 1$. Now, for $h = 2$, we have $A_2 = \{1, 4\} \cup \{2, 3, 5, 6\}$, $B_2 = \{1, 4\}$. That is enough for computing $f(5)$ as $a_s = 2 \in A_2 \setminus A_1$. Thus $f(5) = \phi_i(5)\phi_i(2)f(1)$ and it trivially satisfies the second condition for $s = 5$.
  - If $s = 6$, we have $a_s = 3$ and again it belongs to $[i, a_i]$. As $a_s = 3 \in A_2 \setminus A_1$, $f(6) = \phi_1(6)\phi_1(3)f(1)$.
- If $i = 2$, we have only $a_i = 5$ and $s = 6$. As $a_s = 3$ belongs to $[i = 2, a_i = 5]$, the second condition must be checked. Before we found $f(6) = -1$ and $f(2) = -1$ since $f(5) = 1$. Also, we have $\phi_2(3) = \phi_2(6) = 1$. Thus $f(s) = \phi_i(s)\phi_i(a_s)f(i)$, for $i = 2$ and $s = 6$.

Therefore, we guaranteed the sequence is realizable. The corresponding diagram for $\{4, 6, 2\}$ is given in Figure 4.2.

The non-alternating DT-sequences without reflections are $\{-4, 6, 2\}$, $\{4, -6, 2\}$ and $\{4, 6, -2\}$. If you draw them by alternating the crossings, you can see that they are equivalent to Unknot. See Figure 4.3 for one of them. Thus, there is only one possible

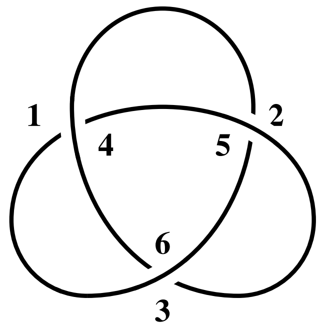


Figure 4.2 : The corresponding diagram of $\{4,6,2\}$

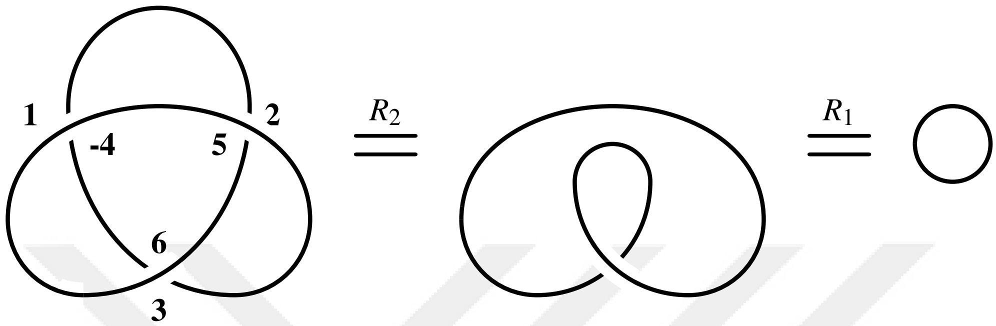


Figure 4.3 : The corresponding diagram of $\{-4,6,2\}$

knot with $c(K) = 3$. It is the corresponding knot to the sequence $\{4,6,2\}$. However, we can not guarantee that it does not become Unknot by using Reidemeister moves.

4.3.1 Invariants of the Diagram of {4,6,2}

Let us compute our invariants for $\{4,6,2\}$ by using a computer program.

DT-sequence	{4,6,2}
Tricolorability	Yes
Knot Determinant	3
Jones Polynomial	$-t^{-4}+t^{-3}+t^{-1}$
Alexander Polynomial	$t^{-1}-1+t$
HOMFLY Polynomial	$-\alpha^4+\alpha^2z^2+2\alpha^2$

Table 4.1 : Invariants of the corresponding diagram to $\{4,6,2\}$.

These results show us that we have a new knot different from Unknot. We call this knot Trefoil and we will denote it by 3_1.

4.4 Knots with Crossing Number of 4

Suppose that $c(K) = 4$. For the sequences of K, we have $a_1 \in \{\pm4,\pm6\}$, $a_3 \in \{\pm6,\pm8\}$, $a_5 \in \{\pm2,\pm8\}$ and $a_7 \in \{\pm2,\pm4\}$. So, we have two possible DT-sequence that represent an alternating diagram. They are $\{4,6,8,2\}$ and $\{6,8,2,4\}$. Although they look different as a sequence, they represent the same diagram. However, we first show the realizability of the sequences.

Our knowledge is $A_1 = \{1,4\}$, $B_1 = \emptyset$, $i_1 = 1$; for $h = 2$, $A_2 = \{1,4\} \cup \{2,3,6,7\}$, $B_2 = \{1,4\}$ and $i_2 = 2$; for $h = 3$, $A_3 = \{1,2,3,4,6,7\} \cup \{5,8\}$, $B_3 = \{1,4\} \cup \{2,7\}$; and each value of $\phi_i(j)$ for all i.

- If $i = 1$, we have $a_i = 4$ and $s \in \{6,7,8\}$
  - If $s = 6$, since $a_s = 3 \in [1,4]$ we have to check the second condition. Since 6 and 3 in $A_2 \setminus A_1$, $f(6) = \phi_1(6)\phi_1(3)f(1) = 1.1.1 = 1$. Then the condition is satisfied trivially.
  - If $s = 7$, since $a_s = 2 \in [1,4]$ we have to check the second condition. Since 7 and 2 in $A_2 \setminus A_1$, $f(7) = \phi_1(7)\phi_1(2)f(1) = (-1).1.1 = -1$. Then the condition is satisfied trivially.
  - If $s = 8$, since $a_s = 5 \notin [1,4]$ we have to check the first condition. It is satisfied as $\phi_1(8)\phi_1(5) = (-1)(-1) = 1$.
- If $i = 2$, we have $a_i = 7$ and $s = 8$. Since $a_s = 5 \in [2,7]$ we have to check the second condition. Since 8 and 5 are in $A_3 \setminus A_2$, $f(8) = \phi_2(8)\phi_2(5)f(2) = 1.(-1).1 = -1$. The condition is satisfied trivially.
- If $i = 3$, we have $a_i = 6$ and $s \in \{7,8\}$.
  - If $s = 7$, since $a_s = 2 \notin [3,6]$ we have to check the first condition. Thus $\phi_3(7)\phi_3(2) = (-1).(-1) = 1$.
  - If $s = 8$, the second condition must be checked but it is not trivial for this time because 8 and 5 are not in $A_4 \setminus A_3$. Since we found $f(8) = -1$ before and we have $\phi_3(8)\phi_3(5)f(3) = 1.1.(-1) = -1$, the condition is satisfied.

Thus the sequence $\{4,6,8,2\}$ is realizable. Let us draw it at first (Figure 4.4).

Next, delete all numbers on the diagram and relabel it putting 1 at the labeled 2 and 7. Then you will get the diagram given in Figure 4.5.

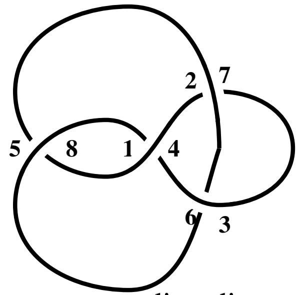


Figure 4.4 : The corresponding diagram of $\{4,6,8,2\}$.

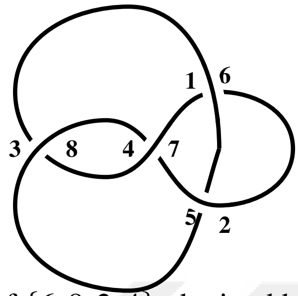


Figure 4.5 : The diagram of $\{6,8,2,4\}$ obtained by relabeling the diagram of $\{4,6,8,2\}$.

Then, it is enough to check the sequences derived from $\{4,6,8,2\}$ that correspond to non-alternating diagrams. If the sequence has one negative number, then it is clear that it has a removable crossing. Furthermore, two crossings can be removed due to R_2 move. It means, $\{-4,6,8,2\}$, $\{4,-6,8,2\}$, $\{4,6,-8,2\}$ and $\{4,6,8,-2\}$ represent a diagram of Unknot. Now, $\{-4,-6,8,2\}$, $\{-4,6,-8,2\}$ and $\{-4,6,8,-2\}$ are left. $\{-4,-6,8,2\}$ and $\{-4,6,8,-2\}$ represent a diagram of Unknot. We can easily see it using R_2, again. The remaining sequence represents the following diagram.

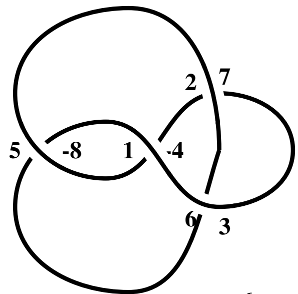


Figure 4.6 : The diagram of $\{-4,6,-8,2\}$.

It may not be easy to see that the remaining non-alternating diagram is different from the alternating one by using Reidemeister moves. However we will apply the invariants and we will see the result.

4.4.1 Invariants of the Diagram of {6,8,2,4}

Let us compute our invariants for $\{6,8,2,4\}$.

DT-sequence	{6,8,2,4}
Tricolorability	No
Knot Determinant	5
Jones Polynomial	$t^{-2}-t^{-1}+1-t+t^{2}$
Alexander Polynomial	$-t^{-1}+3-t$
HOMFLY Polynomial	$\alpha^{2}-z^{2}-1+\alpha^{-2}$

Table 4.2 : Invariants of the corresponding diagram to $\{6,8,2,4\}$.

Obviously, this sequence generates a new knot different from 0_1 and 3_1. Let us call it Figure-eight knot and denote it by 4_1.

4.4.2 Invariants of the Diagram of {-4,6,-8,2}

Let us compute our invariants for $\{-4,6,-8,2\}$.

DT-sequence	{-4,6,-8,2}
Tricolorability	Yes
Knot Determinant	3
Jones Polynomial	$-t^{-4}+t^{-3}+t^{-1}$
Alexander Polynomial	$t^{-1}-1+t$
HOMFLY Polynomial	$-\alpha^{4}+\alpha^{2}z^{2}+2\alpha^{2}$

Table 4.3 : Invariants of the corresponding diagram to $\{-4,6,-8,2\}$.

Although this table of invariants suggests to us that this sequence corresponds to a diagram of Trefoil, it can not guarantee that. However, we can see that it is Trefoil by using Reidemeister moves. See Figure 4.7.

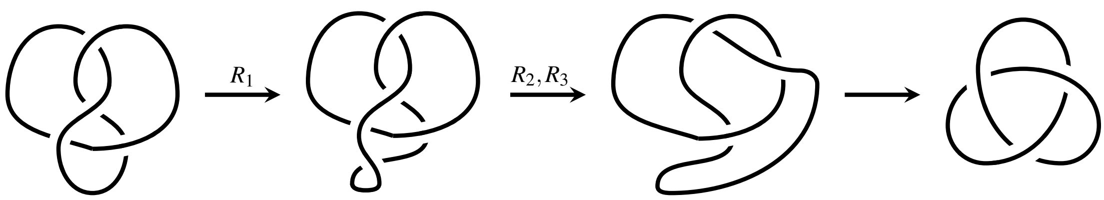


Figure 4.7 : $\{-4,6,-8,2\}$ corresponds to a diagram of -3_1.

4.5 Knots with Crossing Number of 5

Suppose that $c(K) = 5$. In this situation, we have $a_1 \in \{\pm4, \pm6, \pm8\}$, $a_3 \in \{\pm6, \pm8, \pm10\}$, $a_5 \in \{\pm2, \pm8, \pm10\}$, $a_7 \in \{\pm2, \pm4, \pm10\}$ and $a_9 \in \{\pm2, \pm4, \pm6\}$. Let us list all possible alternating DT-sequences for K.

$\alpha_1 = \{4,6,8,10,2\}$	$\alpha_2 = \{4,8,2,10,6\}$	$\alpha_3 = \{4,8,10,2,6\}$
$\alpha_4 = \{4,10,8,2,6\}$	$\alpha_5 = \{6,8,2,10,4\}$	$\alpha_6 = \{6,8,10,2,4\}$
$\alpha_7 = \{6,8,10,4,2\}$	$\alpha_8 = \{6,10,8,2,4\}$	$\alpha_9 = \{6,10,8,4,2\}$
$\alpha_{10} = \{8,6,2,10,4\}$	$\alpha_{11} = \{8,6,10,2,4\}$	$\alpha_{12} = \{8,6,10,4,2\}$
$\alpha_{13} = \{8,10,2,4,6\}$		

Table 4.4 : All possible alternating DT-sequences where $c(K) = 5$.

Some of these sequences do not correspond to a diagram. They are $\alpha_1, \alpha_2, \alpha_4, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{13}$. It is not hard to see this. We will show it for α_1 and we hope you will trust us for the others. We will first show the sequence is not realizable and then we will see geometrically.

For α_1, take $i = 1$ and $s = 8$. Since $a_s = 5$ is not in $[1,4]$, the first condition must be checked. Since $\phi_1(8)\phi_1(5) = 1.(-1) = -1$, α_1 is not realizable. For α_2, if you take $i = 1$ and $s = 9$, you can see it is not realizable.

Let us now see geometrically in Figure 4.8 why α_1 is not realizable, i.e it does not correspond to a knot diagram. Start to draw a curve when you label the crossings from 1 to 10. You will see that starting point and ending point of the curve do not connect.

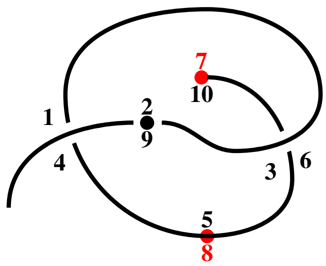


Figure 4.8 : A sequence that does not correspond to a knot diagram.

It is impossible to connect the points which is labelled by 7 and 8 without passing through another point. Since this curve is not closed, it can not be a diagram of any knot.

Let us now show α_3 is realizable. Our knowledge is that $A_1 = \{1,4\}$, $B_1 = \emptyset$ and $i_1 = 1$; for $h = 2$, $A_2 = \{1,4\} \cup \{2,3,7,8\}$, $B_2 = \{1,4\}$ and $i_2 = 2$; for $h = 3$, $A_3 = \{1,2,3,4,7,8\} \cup \{5,6,9,10\}$, $B_3 = \{1,4\} \cup \{2,7\}$; and each value of ϕ_i for all i.

- If $i = 1$, we have $a_i = 4$ and $s \in \{7,8,9\}$.
  - If $s = 7$, we get $a_s = 2 \in [1,4]$ and $f(7) = \phi_1(7)\phi_1(2)f(1) = 1$ since $7 \in A_2 \setminus A_1$.
  - If $s = 8$, we get $a_s = 3 \in [1,4]$ and $f(8) = \phi_1(8)\phi_1(3)f(1) = -1$ since $8 \in A_2 \setminus A_1$.
  - If $s = 9$, we get $a_s = 6 \notin [1,4]$ and $\phi_1(9)\phi_1(6) = 1$.
- If $i = 2$, we have $a_i = 7$ and $s \in \{8,9\}$.
  - If $s = 8$, we get $a_s = 3 \in [2,7]$. Since we have already found $f(8) = -1$ and we have $\phi_2(8)\phi_2(3)f(2) = -1$, the second condition is satisfied.
  - If $s = 9$, we get $a_s = 6 \in [2,7]$. Since $9 \in A_3 \setminus A_2$, $f(9) = \phi_2(9)\phi_2(6)f(2) = 1$.
- If $i = 3$, we have $a_i = 8$, $s = 9$ and $a_s = 6 \in [3,8]$. We know $f(9) = 1$. Also $\phi_3(9)\phi_(6)f(3) = 1$. So, the second condition is satisfied.
- If $i = 6$, we have $a_i = 9$, $s = 10$ and $a_s = 5 \notin [6,9]$. Since $\phi_6(10)\phi_6(5) = 1$, the first condition is satisfied.

Therefore $\alpha_3 = \{4,8,10,2,6\}$ is realizable. We now draw the corresponding diagram of α_3.

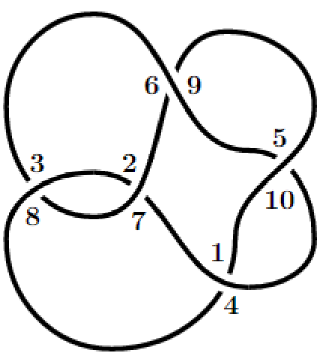


Figure 4.9 : The corresponding diagram of $\{4,8,10,2,6\}$.

The sequences $\alpha_5, \alpha_7, \alpha_8$ and α_{11} are just a result of relabelling of α_3. Only one sequence is left with a chance to generate a new knot, α_6. Let us see that it is realizable.

Our knowledge is $A_1 = \{1,6\}$, $B_1 = \emptyset$ and $i_1 = 1$; for $h = 2$, $A_2 = \{1,6\} \cup \{2,3,4,5,7,8,9,10\}$ and $B_2 = \{1,6\}$; and each value of ϕ_i for all i.

- If $i = 1$, we have $a_i = 6$ and $s \in \{7,8,9,10\}$.
  - If $s = 7$, we get $a_s = 2 \in [1,6]$ and $f(7) = \phi_1(7)\phi_1(2)f(1) = 1$ since 7 is in $A_2 \setminus A_1$.
  - If $s = 8$, we get $a_s = 3 \in [1,6]$ and $f(8) = \phi_1(8)\phi_1(3)f(1) = -1$ since 8 is in $A_2 \setminus A_1$.
  - If $s = 9$, we get $a_s = 4 \in [1,6]$ and $f(9) = \phi_1(9)\phi_1(4)f(1) = 1$ since 9 is in $A_2 \setminus A_1$.
  - If $s = 10$, we get $a_s = 5 \in [1,6]$ and $f(10) = \phi_1(10)\phi_1(5)f(1) = -1$ since 10 is in $A_2 \setminus A_1$.
- If $i = 2$, we have $a_i = 7$ and $s \in \{8,9,10\}$.
  - If $s = 8$, we get $a_s = 3 \in [2,7]$. Since $f(8) = -1$ and $\phi_2(8)\phi_2(3)f(2) = -1$, the second condition is satisfied.
  - If $s = 9$, we get $a_s = 4 \in [2,7]$. Since $f(9) = 1$ and $\phi_2(9)\phi_2(4)f(2) = 1$, the second condition is satisfied.
  - If $s = 10$, we get $a_s = 5 \in [2,7]$. Since $f(10) = -1$ and $\phi_2(10)\phi_2(5)f(2) = -1$, the second condition is satisfied.
- If $i = 3$, we have $a_i = 8$ and $s \in \{9,10\}$.
  - If $s = 9$, we get $a_s = 4 \in [3,8]$. Since $f(9) = 1$ and $\phi_3(9)\phi_3(4)f(3) = 1$, the second condition is satisfied.
  - If $s = 10$, we get $a_s = 5 \in [3,8]$. Since $f(10) = -1$ and $\phi_3(10)\phi_3(5)f(3) = -1$, the second condition is satisfied.
- If $i = 4$, we have $a_i = 9$, $s = 10$ and $a_s = 5$ belongs to $[4,9]$. Since $f(10) = -1$ and $\phi_4(10)\phi_4(5)f(4) = -1$, the second condition is satisfied.

Therefore, $\alpha_6 = \{6, 8, 10, 2, 4\}$ is realizable.

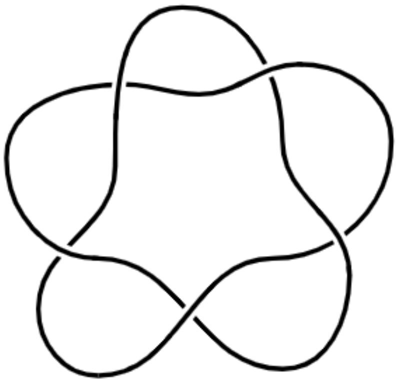


Figure 4.10 : The corresponding diagram of $\{6, 8, 10, 2, 4\}$.

We will see that α_3 and α_6 generate different diagrams and also both are different from the diagrams we found before.

Now, let us check the non-alternating DT-sequences of α_3 and α_6. We can see that the sequences that contains one negative number will turn into less crossing diagrams using R_1 and R_2 moves. Then, it is enough to check the sequences that contain two negative numbers because the sequences with 3 and 4 negative numbers are the reflections of the sequences with 2 and 1 negative numbers, respectively. And, of course, the sequence with five negative numbers is the reflection of the our starting sequence with non-negative numbers.

The non-alternating sequences derived from α_6 can be reduced less crossing diagrams with R_1 and R_2 moves. Let us specify this sequence and then scratch them in Figure 4.11.

$\alpha_{6,1} = \{-6, -8, 10, 2, 4\}$	$\alpha_{6,2} = \{-6, 8, -10, 2, 4\}$
$\alpha_{6,3} = \{-6, 8, 10, -2, 4\}$	$\alpha_{6,4} = \{-6, 8, 10, 2, -4\}$
$\alpha_{6,5} = \{6, -8, -10, 2, 4\}$	$\alpha_{6,6} = \{6, -8, 10, -2, 4\}$
$\alpha_{6,7} = \{6, -8, 10, 2, -4\}$	$\alpha_{6,8} = \{6, 8, -10, -2, 4\}$
$\alpha_{6,9} = \{6, 8, -10, 2, -4\}$	$\alpha_{6,10} = \{6, 8, 10, -2, -4\}$

Table 4.5 : Non-alternating sequences derived from $\{6, 8, 10, 2, 4\}$.

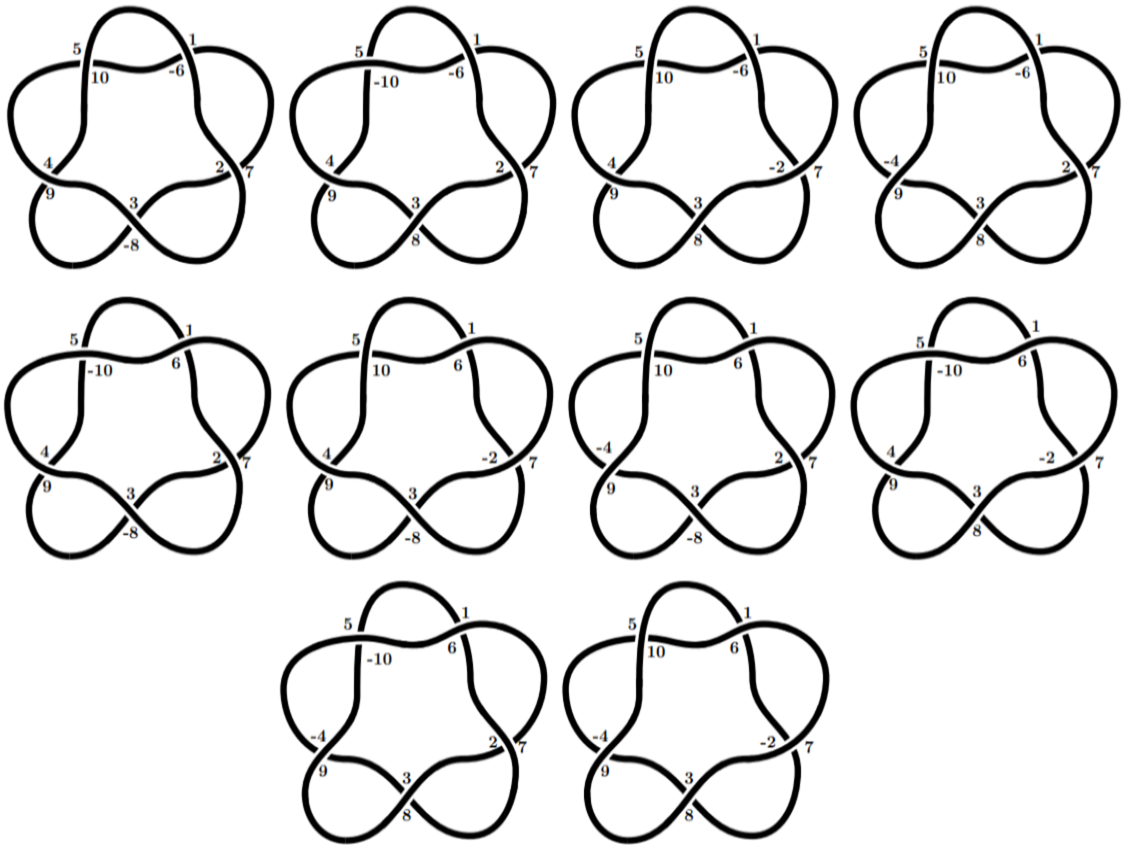


Figure 4.11 : The corresponding non-alternating diagrams derived from the sequence $\{6, 8, 10, 2, 4\}$.

Now, let us repeat this for α_3. There will not be a new knot with crossing number of 5 but we will see that the corresponding diagram of $\alpha_{3,6}$ do not have a removable crossing.

$\alpha_{3,1} = \{-4, -8, 10, 2, 6\}$	$\alpha_{3,2} = \{-4, 8, -10, 2, 6\}$
$\alpha_{3,3} = \{-4, 8, 10, -2, 6\}$	$\alpha_{3,4} = \{-4, 8, 10, 2, -6\}$
$\alpha_{3,5} = \{4, -8, -10, 2, 6\}$	$\alpha_{3,6} = \{4, -8, 10, -2, 6\}$
$\alpha_{3,7} = \{4, -8, 10, 2, -6\}$	$\alpha_{3,8} = \{4, 8, -10, -2, 6\}$
$\alpha_{3,9} = \{4, 8, -10, 2, -6\}$	$\alpha_{3,10} = \{4, 8, 10, -2, -6\}$

Table 4.6 : Non-alternating sequences derived from $\{4, 8, 10, 2, 6\}$

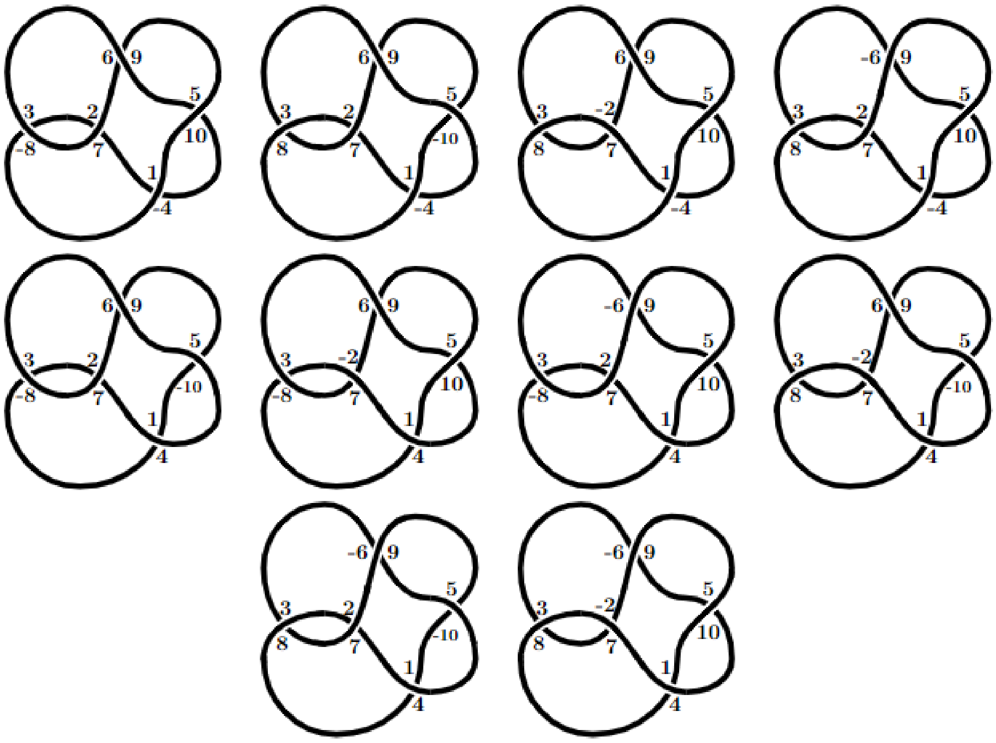


Figure 4.12 : The corresponding non-alternating diagrams derived from the sequence $\{4,8,10,2,6\}$.

4.5.1 Invariants of the Diagram of {6,8,10,2,4}

Let us compute our invariants for $\alpha_6 = \{6,8,10,2,4\}$.

DT-sequence	{6,8,10,2,4}
Tricolorability	No
Knot Determinant	5
Jones Polynomial	$-t^{-7}+t^{-6}-t^{-5}+t^{-4}+t^{-2}$
Alexander Polynomial	$t^{-2}-t^{-1}+1-t+t^{2}$
HOMFLY Polynomial	$-\alpha^6 z^2+\alpha^4 z^4-2\alpha^6+4\alpha^4 z^2+3\alpha^4$

Table 4.7 : Invariants of the corresponding diagram to $\{6,8,10,2,4\}$.

Hence, we found our first knot with $c(K)=5$ since it is different from other knots we found before. It is called as Cinquefoil knot and denoted by 5_1.

4.5.2 Invariants of the Diagram of {4,8,10,2,6}

Let us compute our invariants for $\alpha_3 = \{4, 8, 10, 2, 6\}$.

DT-sequence	{4,8,10,2,6}
Tricolorability	No
Knot Determinant	7
Jones Polynomial	$-t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}$
Alexander Polynomial	$2t^{-1} - 3 + 2t$
HOMFLY Polynomial	$-\alpha^6 + \alpha^4 z^2 + \alpha^4 + \alpha^2 z^2 + \alpha^2$

Table 4.8 : Invariants of the corresponding diagram to $\{4, 8, 10, 2, 6\}$

This means that we have a new knot with $c(K) = 5$. This new knot is called as 3-twist knot and denoted by 5_2.

4.5.3 Invariants of the Diagram of {4,-8,10,-2,6}

Let us compute our invariants for $\alpha_{3,6} = \{4, -8, 10, -2, 6\}$.

DT-sequence	{4,-8,10,-2,6}
Tricolorability	No
Knot Determinant	5
Jones Polynomial	$t^{-2} - t^{-1} + 1 - t + t^2$
Alexander Polynomial	$-t^{-1} + 3 - t$
HOMFLY Polynomial	$\alpha^2 - z^2 - 1 + \alpha^{-2}$

Table 4.9 : Invariants of the corresponding diagram to $\{4, -8, 10, -2, 6\}$.

Our invariants could not distinguish this diagram from 4_1. Figure 4.13 proves that it is a diagram of Figure-eight knot.

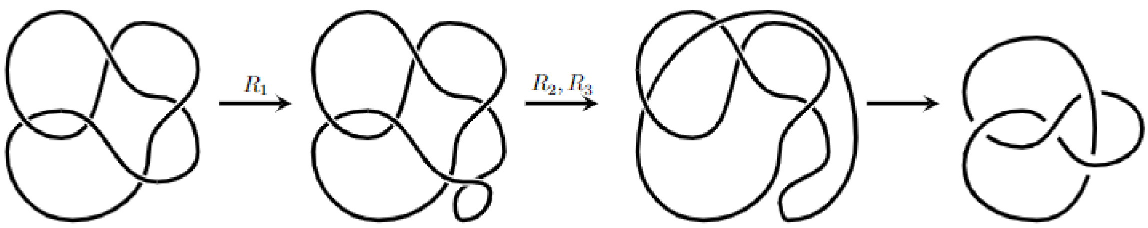


Figure 4.13 : $\{4, -8, 10, -2, 6\}$ corresponds to a diagram of 4_1.

4.6 Knots with Crossing Number of 6

Let us specify the conditions for a_i where $c(K) = 6$. Our conditions are

- $a_1 \in \{4, 6, 8, 10\}$,
- $a_3 \in \{6, 8, 10, 12\}$,
- $a_5 \in \{2, 8, 10, 12\}$,
- $a_7 \in \{2, 4, 10, 12\}$,
- $a_9 \in \{2, 4, 6, 12\}$,
- $a_{11} \in \{2, 4, 6, 8\}$.

There are 78 possible sequences under these conditions. We will find the sequences which corresponds to a knot. In this time, there will be sequences that correspond to a composite knot. If we ignore them, we are left with 72 sequences. For the primes, there are three different classes. We will first give the sequences that are not realizable by omitting calculations. Non-realizability of them can be easily seen, as in the previous examples. After that, we will give the realizable sequences and their relabelings by separating three classes. Finally, we will give the sequences that correspond to a composite knot.

The non-realizable sequences are given in the table 4.10.

4.6.1 First Class of Sequences with Crossing Number of 6

After eliminating 6 composite sequences and 42 non-realizable sequences, it can be shown that $\{4, 8, 12, 10, 2, 6\}$ is realizable. And some of the remaining sequences are relabelings of it. We give those sequences in the table 4.11. They form the first class of sequences that represent the same diagram with crossing number of 6. This diagram is given in Figure 4.14.

$\{4,6,8,10,12,2\}$	$\{4,6,10,2,12,8\}$	$\{4,6,10,12,2,8\}$	$\{4,6,12,10,2,8\}$
$\{4,8,2,10,12,6\}$	$\{4,8,10,12,6,2\}$	$\{4,8,12,10,6,2\}$	$\{4,10,2,12,6,8\}$
$\{4,10,8,2,12,6\}$	$\{4,10,8,12,6,2\}$	$\{4,10,12,2,6,8\}$	$\{4,12,8,10,2,6\}$
$\{4,12,10,2,6,8\}$	$\{6,8,2,10,12,4\}$	$\{6,8,10,4,12,2\}$	$\{6,8,10,12,2,4\}$
$\{6,10,2,4,12,8\}$	$\{6,10,8,4,12,2\}$	$\{6,12,2,10,4,8\}$	$\{6,12,8,10,2,4\}$
$\{6,12,8,10,4,2\}$	$\{6,12,10,4,2,8\}$	$\{8,6,2,10,12,4\}$	$\{8,6,10,4,12,2\}$
$\{8,6,12,10,4,2\}$	$\{8,10,2,4,12,6\}$	$\{8,10,12,2,4,6\}$	$\{8,10,12,4,6,2\}$
$\{8,12,10,4,6,2\}$	$\{10,6,2,12,4,8\}$	$\{10,6,8,2,12,4\}$	$\{10,6,8,12,2,4\}$
$\{10,6,8,12,4,2\}$	$\{10,6,12,2,4,8\}$	$\{10,6,12,4,2,8\}$	$\{10,8,2,4,12,6\}$
$\{10,8,2,12,6,4\}$	$\{10,8,12,4,6,2\}$	$\{10,12,2,4,6,8\}$	$\{10,12,8,2,4,6\}$
$\{10,12,8,2,6,4\}$	$\{10,12,8,4,2,6\}$		

Table 4.10 : Non-realizable sequences where $c(K) = 6$.

$\{4,8,12,10,2,6\}$	$\{6,10,2,12,4,8\}$	$\{6,10,8,12,4,2\}$
$\{8,6,10,2,12,4\}$	$\{8,12,10,2,6,4\}$	$\{10,8,12,4,2,6\}$

Table 4.11 : First Class of Sequences with Crossing Number of 6.

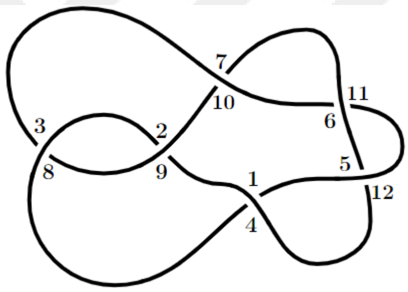


Figure 4.14 : The corresponding diagram of $\{4,8,12,10,2,6\}$.

Let us put the invariants of the corresponding diagram of $\{4,8,12,10,2,6\}$ in the table.

DT-sequence	{4,8,12,10,2,6}
Tricolorability	Yes
Knot Determinant	9
Jones Polynomial	$t^2 - t - 2t^{-1} + t^{-2} - t^{-3} + t^{-4} + 2$
Alexander Polynomial	$-2t^{-1} + 5 - 2t$
HOMFLY Polynomial	$\alpha^4 - \alpha^2 z^2 - \alpha^2 - z^2 + \alpha^{-2}$

Table 4.12 : Invariants of the corresponding diagram to $\{4,8,12,10,2,6\}$.

Therefore, we found our first knot with the crossing number of 6. This is the Stevedore's knot denoted by 6_1.

4.6.2 Second Class of Sequences with Crossing Number of 6

We are now left with 24 sequences for primes. The sequence $\{4,8,10,12,2,6\}$ is realizable and eleven sequences are relabelings of it. They are given in the following table.

$\{4,8,10,12,2,6\}$	$\{6,8,10,2,12,4\}$	$\{6,8,10,12,4,2\}$
$\{6,8,12,10,2,4\}$	$\{6,10,8,12,2,4\}$	$\{6,10,12,2,4,8\}$
$\{8,6,10,12,2,4\}$	$\{8,10,2,12,4,6\}$	$\{8,10,12,2,6,4\}$
$\{8,10,12,4,2,6\}$	$\{8,12,10,2,4,6\}$	$\{10,8,12,2,4,6\}$

Table 4.13 : Second class of sequences with crossing number of 6.

Choosing the sequence $\{4,8,10,12,2,6\}$ as a representative, we draw the corresponding diagram of the sequences in the class and we put the invariants of it in the table.

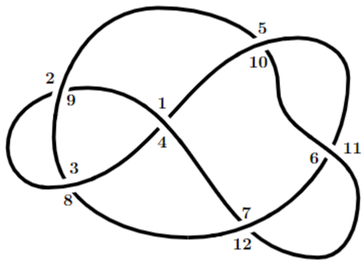


Figure 4.15 : The corresponding diagram of $\{4,8,10,12,2,6\}$.

DT-sequence	{4,8,10,12,2,6}
Tricolorability	No
Knot Determinant	11
Jones Polynomial	$t+2t^{-1}-2t^{-2}+2t^{-3}-2/t^4+t^{-5}-1$
Alexander Polynomial	$-t^{-2}+3t^{-1}-3+3t-t^2$
HOMFLY Polynomial	$\alpha^4z^2-\alpha^2z^4+\alpha^4-3\alpha^2z^2-2\alpha^2+z^2+2$

Table 4.14 : Invariants of the corresponding diagram to $\{4,8,10,2,6\}$.

This gives us a new knot with the crossing number of 6. It is called as The Miller Institute knot and denoted by 6_2.

4.6.3 Third Class of Sequences with Crossing Number of 6

Now, 12 sequences are left for primes. The sequences of third class are given in the following table. One can show that each of them is realizable but it is enough to show one of them is.

$\{4,8,10,2,12,6\}$	$\{4,10,8,12,2,6\}$	$\{6,8,12,10,4,2\}$
$\{6,10,8,2,12,4\}$	$\{6,10,12,4,2,8\}$	$\{6,12,10,2,4,8\}$
$\{8,6,10,12,4,2\}$	$\{8,6,12,10,2,4\}$	$\{8,10,2,12,6,4\}$
$\{8,12,10,4,2,6\}$	$\{10,8,2,12,4,6\}$	$\{10,8,12,2,6,4\}$

Table 4.15 : Third class of sequences with crossing number of 6.

Each of them represents the diagram below. We choose the sequence $\{4,8,10,2,12,6\}$ as a representative.

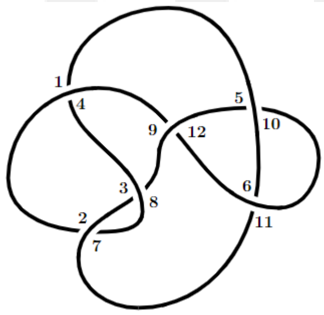


Figure 4.16 : The corresponding diagram of $\{4,8,10,2,12,6\}$.

The table of invariants is included below.

DT-sequence	{4,8,10,2,12,6}
Tricolorability	No
Knot Determinant	13
Jones Polynomial	$-t^3+2t^2-2t-2t^{-1}+2t^{-2}-t^{-3}+3$
Alexander Polynomial	$t^{-2}-3t^{-1}+5-3t+t^2$
HOMFLY Polynomial	$-\alpha^2z^2+z^4-\alpha^2+3z^2+3-\alpha^{-2}z^2-\alpha^{-2}$

Table 4.16 : Invariants of the corresponding diagram to $\{4,8,10,2,12,6\}$.

Again, we get a new knot that is denoted by 6_3.

4.6.4 Fourth Class of Sequences with Crossing Number of 6

The final class consists of the following elements. There are no more sequences for primes. Following 6 sequences are represent a composite knot but do not forget that realizability theorem is not for composite knots. Just draw the sequence $\{4,6,2,10,12,8\}$ and realize that other five sequences are a relabeling of it.

$\{4,6,2,10,12,8\}$	$\{4,12,2,10,6,8\}$	$\{4,12,8,10,6,2\}$
$\{10,6,2,4,12,8\}$	$\{10,6,8,4,12,2\}$	$\{10,12,8,4,6,2\}$

Table 4.17 : Fourth class of sequences with crossing number of 6.

The representative sequence $\{4,6,2,10,12,8\}$ corresponds to the following diagram.

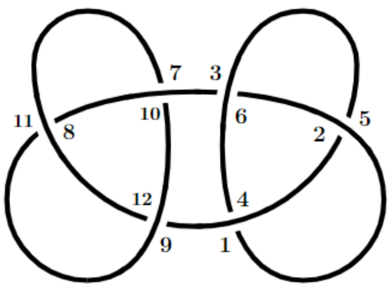


Figure 4.17 : The corresponding diagram of $\{4,6,2,10,12,8\}$.

This diagram is the composition of right Trefoil and right Trefoil. We found our first knot that is not prime. Let us check the our invariants.

DT-sequence	$\{4,6,2,10,12,8\}$
Tricolorability	Yes
Knot Determinant	9
Jones Polynomial	$t^{-2}+2t^{-4}-2t^{-5}+t^{-6}+2t^{-7}+t^{-8}$
Alexander Polynomial	$t^{-2}-2t^{-1}+3-2t+t^{2}$
HOMFLY Polynomial	$\alpha^{8}-2\alpha^{6}z^{2}+\alpha^{4}z^{4}+4\alpha^{6}-4\alpha^{4}z^{2}+4\alpha^{4}$

Table 4.18 : Invariants of the corresponding diagram of $\{4,6,2,10,12,8\}$.

As you can see, the invariants say that we have a new knot. However, it can be expressed by the knots preceding it. Thus, we denote it by $3_1\#(3_1)$ and it is called as Granny knot.

4.6.5 Non-alternating Sequences with Crossing Number of 6

We finally check the non-alternating sequences derived from the representatives of four classes.

4.6.5.1 Non-alternating DT-sequences derived from {4,8,12,10,2,6}

There are 6 non-alternating sequences which have one negative. All are reducible.

The number of non-alternating sequences having two negatives is $\binom{6}{2} = 15$. However, all diagrams obtained from these sequences have removable crossings, except for the corresponding diagram of $\{4,-8,12,10,-2,6\}$. However, if you increase the number of crossing at first, you will see that its crossing number is less than 6.

The corresponding diagrams to $\binom{6}{3} = 20$ non-alternating diagrams having three negatives are all reducible.

4.6.5.2 Non-alternating DT-sequences derived from {4,8,10,12,2,6}

Although the number of non-alternating DT-sequences is the same as the previous one, the number of diagrams that do not have removable crossings is higher. Following table gives us the non-alternating sequences correspond to the diagrams that do not have removable crossings. It easily can be seen that their crossing number is less than 6.

Having one negative	$\{-4,8,10,12,2,6\}$
Having two negatives	$\{4,-8,10,12,-2,6\}$
Having three negatives	{-4,-8,10,12,-2,6}, {4,8,10,-12,-2,-6}

Table 4.19 : Non-alternating sequences derived from $\{4,8,10,12,2,6\}$ that do not correspond to a reducible diagram.

4.6.5.3 Non-alternating DT-sequences derived from {4,8,10,2,12,6}

Like the others the sequences derived from $\{4,8,10,2,12,6\}$ does not generate a new knot. All diagrams that emerged from these sequences are either removable crossings or can be reduced after increasing the number of crossing. Following table contains non-alternating DT-sequences whose diagrams can be reduced after increasing the number of crossing.

Having one negative	$\{-4,8,10,2,12,6\},\{4,8,10,2,-12,6\}$
Having two negatives	{-4,8,10,2,-12,6}, {4,-8,10,-2,12,6}, {4,8,-10,2,12,-6}
Having three negatives	{-4,-8,10,-2,12,6},{-4,8,-10,2,12,-6}, {4,-8,10,-2,-12,6}, {-4,8,10,2,12,6}

Table 4.20 : Non-alternating sequences derived from $\{4,8,10,2,12,6\}$ that do not correspond to a reducible diagram.

4.6.5.4 Non-alternating DT-sequences derived from {4,6,2,10,12,8}

So far, we have not found a non-alternating knot. We saw that all non-alternating sequences we found either did not correspond to a knot diagram, or their diagrams were equivalent to a previous alternating diagram. But now, this will be change. All non-alternating sequences derived from $\{4,6,2,10,12,8\}$ except $\{-4,-6,-2,10,12,8\}$ create a reducible diagram. Let us draw and compute the invariants of $\{-4,-6,-2,10,12,8\}$.

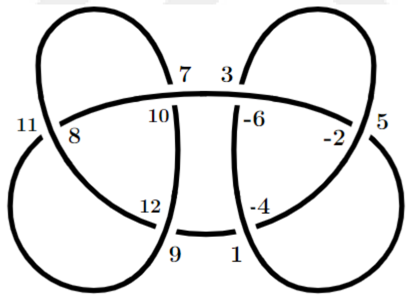


Figure 4.18 : The corresponding diagram of $\{-4,-6,-2,10,12,8\}$

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DT-sequence	$\{-4,-6,-2,10,12,8\}$
Tricolorability	Yes
Knot Determinant	9
Jones Polynomial	$-t^3+t^2-t-t^{-1}+t^{-2}-t^{-3}+3$
Alexander Polynomial	$t^{-2}-2t^{-1}+3-2t+t^2$
HOMFLY Polynomial	$-\alpha^2z^2+z^4+2\alpha^2-4z^2+5-\alpha^{-2}z^2+2\alpha^{-2}$

Table 4.21 : Invariants of the corresponding diagram of $\{-4,-6,-2,10,12,8\}$.

Since the table contains distinguishing invariants, this sequences gives us a knot that we could not find before. This knot is called Square knot and it is the composition of left Trefoil and right Trefoil. So, we denote it by $3_1\#(-3_1)$.

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