ANKARA
HACI BAYRAM VELİ ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

KARIŞIK KANTOROVICH TİPLİ OPERATÖRLER İLE YAKLAŞIM

Aykut SONAY

Tez Danışmanı
Doç. Dr. Nursel ÇETİN

YÜKSEK LİSANS
MATEMATİK ANABİLİM DALI
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ETİK BEYAN

Ankara Hacı Bayram Veli Üniversitesi Tez Yazım Kurallarına uygun olarak hazırladığım bu tez çalışmasında; tez içinde sunduğum verileri, bilgileri ve dokümanları akademik ve etik kurallar çerçevesinde elde ettiğimi, tüm bilgi, belge, değerlendirme ve sonuçları bilimsel etik ve ahlak kurallarına uygun olarak sunduğumu, tez çalışmasında yararlandığım eserlerin tümüne uygun atıfta bulunarak kaynak gösterdiğimi, kullanılan verilerde herhangi bir değişiklik yapmadığımı, bu tezde sunduğum çalışmanın özgün olduğunu, bildirir, aksi bir durumda aleyhime doğabilecek tüm hak kayıplarını kabullendiğimi beyan ederim.

Aykut SONAY
29/07/2024

TEZ ONAY SAYFASI

Ankara Hacı Bayram Veli Üniversitesi Lisansüstü Eğitim Enstitüsü Matematik Anabilim Dalı Yüksek Lisans Programı öğrencisi Aykut SONAY tarafından hazırlanan Karışık Kantorovich Tipli Operatörler ile Yaklaşım başlıklı tez çalışması 29/07/2024 tarih ve 10:30 saatinde yapılan tez savunma sınavında aşağıdaki jüri tarafından OY BİRLİĞİ ile YÜKSEK LİSANS TEZİ olarak **KABUL** edilmiştir.

	Kabul	Ret
Başkan (Unvan/Ad-Soyad/Kurum): **Prof. Dr. İsmet YÜKSEL / Gazi Üniversitesi**	☒	☐
Üye (Unvan/Ad-Soyad/Kurum): **Prof. Dr. Fatih YILMAZ / Ankara Hacı Bayram Veli Üniversitesi**	☒	☐
Üye (Unvan/Ad-Soyad/Kurum): **Doç. Dr. Nursel ÇETİN / Ankara Hacı Bayram Veli Üniversitesi**	☒	☐

KARIŞIK KANTOROVICH TİPLİ OPERATÖRLER İLE YAKLAŞIM

(Yüksek Lisans Tezi)

Aykut SONAY

ANKARA HACI BAYRAM VELİ ÜNİVERSİTESİ
LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

Temmuz 2024

ÖZET

Bu tezde, Bernstein-Kantorovich tipli operatörlerin yeni bir genelleştirilmesi dikkate alınmıştır. Yeni tanımlanan karışık Kantorovich tipli operatörlerin dizisi için önce $[0,1]$ üzerinde sürekli, reel değerli fonksiyonlar uzayında düzgün yaklaşım ve daha sonra Voronovskaja tipli teoremler elde edilmiştir. Ayrıca bu operatörler için süreklilik modülü ile yaklaşım hızı verilmiştir. Son olarak, yeni operatörün bazı fonksiyonlara yaklaşımı grafiklerle gösterilmiştir.

Bilim Kodu	: 20404
Anahtar Kelimeler	: Karışık Bernstein operatörleri, Kantorovich operatörleri, karışık Kantorovich tipli operatörler, düzgün yaklaşım, Voronovskaja tipli teorem, yaklaşımın derecesi, süreklilik modülü
Sayfa Adedi	: 85
Tez Danışmanı	: Doç. Dr. Nursel ÇETİN

APPROXIMATION BY PERTURBED KANTOROVICH TYPE OPERATORS

(M.Sc. Thesis)

Aykut SONAY

ANKARA HACI BAYRAM VELİ UNIVERSITY

THE INSTITUTE OF GRADUATE STUDIES

July 2024

ABSTRACT

In this thesis, a new generelization of Bernstein-Kantorovich type operators is considered. Firstly, uniform convergence for the sequences of new defined perturbed Kantorovich type operators in the space of all real valued continuous functions on $[0,1]$ and then Voronovskaja type theorems are obtained. Also, the rate of convergence via modulus of continuity by these operators is given. Finally, the convergence of the new operator to some functions is shown by graphics.

Science Code	:	20404
Key Words	:	Perturbed Bernstein operators, Kantorovich operators, perturbed Kantorovich type operators, uniform convergence, Voronovskaja type theorem, rate of convergence, modulus of continuity
Page Number	:	85
Supervisor	:	Assoc. Prof. Dr. Nursel ÇETİN

TEŞEKKÜR

Öncelikle tez çalışmam sırasında bana, engin bilgisi, sahip olduğu deneyimi ve tüm samimiyeti ile rehberlik ederek bu çalışmamı tamamlamama vesile olan değerli hocam ve tez danışmanım Sayın Doç. Dr. Nursel ÇETİN'e en içten duygularımla teşekkürlerimi sunuyorum.

Bu çalışmam sürecinde benden hiçbir konuda destek ve yardımını esirgemeyen çalışma arkadaşlarıma şükranlarımı iletiyorum. Ayrıca her zaman ve her koşulda yanımda olan çok kıymetli eşim Gül'e ve yapmış olduğu küçük yaramazlıklara rağmen yine de sürekli olarak motivasyon ve moralimi yüksek tutmamı sağlayarak bana güç veren minik kızım Beril Ada'ya da çok teşekkür ediyorum.

Temmuz 2024 **Aykut SONAY**

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KISALTMALAR

Bu çalışmada kullanılmış kısaltmalar, açıklamaları ile birlikte aşağıda sunulmuştur.

Kısaltmalar	**Açıklamalar**
$\mathbb{N}$	Doğal sayılar
$\mathbb{R}$	Reel sayılar
$C[a,b]$	$[a,b]$ aralığında tanımlı sürekli fonksiyonların uzayı
$\|\varphi\|_{C[a,b]}$	$\max\{\|\varphi(\xi)\|; a \leq \xi \leq b\}$
$B_\eta(\varphi,\xi)$	Bernstein operatörü
$B_\eta^1(\varphi,\xi)$	Karışık Bernstein operatörü
$\mathcal{K}_\eta(\varphi,\xi)$	Kantorovich operatörü
$\mathcal{K}_\eta^{M,1}(\varphi,\xi)$	Karışık Kantorovich operatörü
$e_k(\xi)$	$\xi^k, k \in \mathbb{N} \cup \{0\}, \xi \in \mathbb{R}$
$\omega(\varphi,\delta)$	φ fonksiyonunun süreklilik modülü

1. GİRİŞ

Matematiğin birçok uygulamasında, klasik analizdeki standart fonksiyonlardan çok daha karmaşık fonksiyonlarla karşılaşırız. Bu fonksiyonlardan bazıları standart fonksiyonlar aracılığıyla kapalı biçimde ifade edilememekte, bazıları ise yalnızca kapalı olarak veya grafikleri aracılığıyla bilinmektedir. Örneğin, zamanın bir fonksiyonu olarak belirli bir noktada akımı ölçtüğümüz bir elektrik devresini düşünün: sonuç oldukça karmaşık olabilir ve en iyi şekilde bir grafikle açıklanabilir (Christensen ve Christensen, 2004).

Bir elektrik devresindeki akımı ölçen bir mühendis, bir sinyalin varlığından söz edecektir; bir matematikçi için ise bu, ölçümün çıktısının φ olarak adlandırılacak bir fonksiyon olduğu anlamına gelir. Burada $\varphi(\xi)$, ξ anındaki akıma eşittir (Christensen ve Christensen, 2004).

Sinyaller genellikle doğrudan bir fonksiyon yardımıyla verilmez; örneğin onlar sıklıkla bir ölçüm yoluyla ortaya çıkarlar. Bu, özellikle φ üzerinde bazı hesaplamalar yapmamız gerekiyorsa, onu tanımlayan φ fonksiyonundan sinyal hakkında kesin bilgi elde etmeyi zorlaştırır veya imkansız hale getirir. Bu gibi durumlarda φ'ye daha basit bir fonksiyonla yaklaşabilmek önemlidir; bunun için g olarak adlandırılacak bir fonksiyonu aşağıdaki şekilde bulmak istiyoruz:

• İlgili hesaplama g fonksiyonu üzerinde gerçekleştirilebilir;

• g üzerinde gerçekleştirilen hesaplamanın sonucu, φ ile tanımlı sinyal hakkında yararlı bilgiler vermesi bakımından g fonksiyonu, φ'ye yakındır.

Başlangıç noktamız, bir fonksiyonun başka bir fonksiyona yaklaşmasının ne anlama geldiğinin kesin bir tanımı olmalıdır. Aslında bunu tanımlamanın birkaç farklı yolu vardır ve doğru tanım, eldeki duruma bağlıdır. Şimdi Yaklaşım Teorisi'nin gerekli olduğu yere somut bir örnek verelim:

$\int_0^1 e^{-\xi^2} d\xi$ integralini hesaplamak istediğimizi kabul edelim. Elementer fonksiyonlar cinsinden $\int_0^1 e^{-\xi^2} d\xi$ integrali için bir formülün olmadığı iyi bilinmektedir. Bu nedenle $\int_0^1 e^{-\xi^2} d\xi$ integralinin değerini tahmin etmenin başka bir yolunu bulmalıyız. Yaklaşım teorisinin resme girdiği nokta işte burasıdır: bu somut örnekte g olarak adlandırılacak fonksiyonu aşağıdaki şekilde arayacağız.

•$\int_0^1 g(\xi)d\xi$ integrali hesaplanabilir; ve

•$\int_0^1 g(\xi)d\xi$ integralinin $\int_0^1 e^{-\xi^2}d\xi$ integralinin değerinden ne kadar sapacağını kontrol edebilmemiz açısından $\xi \in [0,1]$ için $g(\xi)$ fonksiyonu, $e^{-\xi^2}$'ye yakındır.

Bunu yapmanın bir yolu, integralini alabileceğimiz pozitif bir g fonksiyonu bulmaktır ve bunun için, bazı $\varepsilon > 0$ ve her $\xi \in [0,1]$ için

$$-\varepsilon \leq e^{-\xi^2} - g(\xi) \leq \varepsilon$$

veya

$$-\varepsilon + g(\xi) \leq e^{-\xi^2} \leq \varepsilon + g(\xi)$$

yazılabilir. Bu da şu anlama gelir

$$\int_0^1 \left(-\varepsilon + g(\xi)\right)d\xi \leq \int_0^1 e^{-\xi^2}d\xi \leq \int_0^1 \left(\varepsilon + g(\xi)\right)d\xi,$$

ve buradan

$$-\varepsilon + \int_0^1 g(\xi)d\xi \leq \int_0^1 e^{-\xi^2}d\xi \leq \varepsilon + \int_0^1 g(\xi)d\xi$$

olur. Böylece $\int_0^1 g(\xi)d\xi$ ifadesi bize istenilen integral için yaklaşık bir değer verir. Dolayısıyla, amacımız karmaşık bir φ fonksiyonunun bir integraline yaklaşmaksa, g fonksiyonu (eğer g'nin integralini bulabilirsek), φ'nin integralini tahmin etmeyi mümkün kılar. Bu durumda g fonksiyonunun nasıl seçileceği sorusu akla gelir. Bu örnekteki argüman genelleştirilebilir (Christensen ve Christensen, 2004). O halde, karmaşık nesnelere matematiksel bağlamda daha basit nesnelerle yaklaşmak uygundur. Tipik bir örnek, kapalı bir aralıkta keyfi sürekli fonksiyonların polinomlarla yaklaşımıdır. Weierstrass Teoremi bunun her zaman yapılabileceğini belirtir. Bu teorem açık olarak “

$I \subset \mathbb{R}$ kapalı ve sınırlı bir aralık ve φ, I üzerinde tanımlı sürekli bir fonksiyon olsun. Bu durumda her $\varepsilon > 0$ için bir P polinomu mevcuttur öyleki her $\xi \in I$ için $|\varphi(\xi) - P(\xi)| < \varepsilon$ gerçeklenir." şeklinde ifade edilir (Weierstrass, 1885). Bu teorem, reel tek değişkenli Yaklaşım Teorisindeki ilk önemli sonuçtur ve Genel Yaklaşım Teorisinin gelişiminde anahtar rol oynar. Literatürde Weierstrass teoreminin birkaç farklı ispatı bilinmektedir, ancak en göze çarpanlardan bir tanesi olasılık teorisine dayalı son derece basit ve şık bir yapıda olan 1912 yılında Bernstein tarafından yapılan ispattır. Bernstein, Weierstrass teoremini ispatlamak amacıyla "$\varphi \in C[0,1]$, $\xi \in [0,1]$ ve $\eta \in \mathbb{N}$ için

$$\wp_{\eta,\rho}(\xi) = \begin{cases} \binom{\eta}{\rho} \xi^{\rho}(1-\xi)^{\eta-\rho}, & 0 \leq \rho \leq \eta \\ 0 \quad , & \rho < 0 \text{ veya } \rho > \eta \end{cases} \tag{1.1}$$

olmak üzere φ'ye göre η. dereceden Bernstein polinomlarını

$$B_{\eta}(\varphi, \xi) = \sum_{\rho=0}^{\eta} \wp_{\eta,\rho}(\xi) \varphi\left(\frac{\rho}{\eta}\right) \tag{1.2}$$

şeklinde tanımlamış ve bu polinom dizisinin $[0,1]$ aralığında düzgün olarak φ fonksiyonuna yakınsadığını göstermiştir (Bernstein, 1912). Bilindiği gibi $\wp_{\eta,\rho}(\xi)$ temel fonksiyonları için

$$\wp_{\eta,\rho}(\xi) = (1-\xi)\wp_{\eta-1,\rho}(\xi) + \xi \wp_{\eta-1,\rho-1}(\xi), 0 \leq \rho \leq \eta \tag{1.3}$$

indirgeme bağıntısı sağlanır ve özellikle

$$\wp_{\eta,0}(\xi) = (1-\xi)\wp_{\eta-1,0}(\xi) = (1-\xi)^{\eta},$$

$$\wp_{\eta,\eta}(\xi) = \xi \wp_{\eta-1,\eta-1}(\xi) = \xi^{\eta}$$

yazılabilir. Uzun zamandır Bernstein operatörlerinin yaklaşım derecesi yoğun bir şekilde çalışılmaktadır. Birinci süreklilik modülü kullanılarak bu problemin bir çözümü Popoviciu (1934) tarafından verilmiştir. Bernstein polinomlarının bir asimptotik hata terimi ilk olarak Voronovskaja (1932) tarafından elde edilmiştir. Daha sonra bu sonuç Gonska (2007), Gonska ve Rasa (2009), Gavrea ve Ivan (2012), Tachev (2012), Gupta ve Agarval (2014), Gupta ve Tachev (2017) tarafından genişletilmiştir.

Yakın zamanda, Khosravian-Arab vd. (2018) yaklaşım derecesini iyileştirmek için modifiye Bernstein polinomlar dizisini $\varphi \in C[0,1], \xi \in [0,1]$ ve $\eta \in \mathbb{N}$ için

$$B_\eta^1(\varphi,\xi) = \sum_{\rho=0}^{\eta} \wp_{\eta,\rho}^1(\xi)\varphi\left(\frac{\rho}{\eta}\right) \tag{1.4}$$

şeklinde tanımlamışlar ve yaklaşım özelliklerini incelemişlerdir. Burada $\wp_{\eta,\rho}(\xi)$ Eş. 1.1 ile verilen Bernstein temel fonksiyonları olmak üzere

$$\wp_{\eta,\rho}^1(\xi) = \mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi), \;\; 1 \le \rho \le \eta - 1,$$

$$\wp_{\eta,0}^1(\xi) = \; \mathfrak{u}(\xi,\eta)(1-\xi)^{\eta-1}, \;\; \wp_{\eta,\eta}^1(\xi) = \mathfrak{u}(1-\xi,\eta)\xi^{\eta-1} \tag{1.5}$$

ifadesi Eş. 1.3 ün karışık formudur ve $\mathfrak{u}_0(\eta)$ ve $\mathfrak{u}_1(\eta)$,

$$\mathfrak{u}(\xi,\eta) = \mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta), \quad \eta = 0,1,\ldots,$$

olacak şekildeki bilinmeyen reel dizilerdir. $\mathfrak{u}_1(\eta) = -1, \mathfrak{u}_0(\eta) = 1$ için Eş. 1.5 ifadesi Eş. 1.3 e indirgenir, yani B_η^1 reel karışık Bernstein tipli operatörler klasik Bernstein operatörlerine dönüşür. Daha sonra, Acu ve Başcanbaz-Tunca 2020 yılında Eş. 1.4 ile verilen B_η^1 operatörlerinin kompleks formunu tanımlayarak, orijin merkezli $R > 1$ yarıçaplı diskte analitik fonksiyonlara ilişkin bu operatörler dizisi yardımıyla eşanlı yaklaşımın kesin derecesini elde etmişlerdir. Son zamanlarda, Khosravian-Arab vd. tarafından sunulan yaklaşımdan esinlenilerek, karışık Bernstein tipli operatörlerin reel ve kompleks formda bazı genelleştirmeleri yoğun olarak çalışılmıştır. Bu çalışmalara örnek olarak Gupta vd. (2019), Acu vd. (2019), Acu ve Agrawal (2019), Acu vd. (2020), Acu ve Gonska (2020), Çetin (2023), Acu vd. (2024a), Acu vd. (2024b) verilebilir.

Bilindiği gibi, Bernstein polinomları integrallenebilir fonksiyonlara yaklaşmak için uygun değildir. Klasik Kantorovich operatörleri, $[0,1]$ aralığı üzerinde tanımlı Riemann integrallenebilir fonksiyonlara yaklaşmak için Bernstein operatörlerinin integral modifikasyonudur. Bu operatörler, Kantorovich (1930) tarafından $\mathcal{K}_\eta: L_1[0,1] \to C[0,1]$,

$$\mathcal{K}_\eta(\varphi,\xi) = (\eta+1)\sum_{\rho=0}^{\eta} \wp_{\eta,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} \varphi(t)dt,\ \xi \in [0,1], \eta \in \mathbb{N} \tag{1.6}$$

şeklinde tanımlanmıştır. Burada $L_1[0,1]$, $[0,1]$ üzerinde mutlak integrallenebilir fonksiyonların uzayı ve $\wp_{\eta,\rho}(\xi)$ Eş. 1.1 ile verilen Bernstein temel fonksiyonlarıdır. Her bir $\mathcal{K}_\eta$ derecesi küçük veya eşit η olan bir operatördür ve böyle operatörlerin, $\varphi \in L_1[0,1]$ fonksiyonunun $\Phi(\xi) = \int_0^\xi \varphi(t)dt + \Phi(0)$ belirsiz integrali için Bernstein polinomları ile arasında

$$\frac{d}{d\xi} B_\eta(\Phi,\xi) = \mathcal{K}_{\eta-1}(\Phi',\xi)$$

biçiminde bir bağıntı mevcuttur.

Bu tez çalışmasında, Bernstein-Kantorovich-tipli operatörlerin yeni bir çeşidi tanımlanarak, Khosravian-Arab vd. (2018) tarafından elde edilen bazı sonuçlar klasik Kantorovich operatörlerine genişletilecektir. İlk olarak, yeni tanımlanan operatörlerin dizisi için $C[0,1]$ uzayında düzgün yaklaşım incelenecek ve Voronovskaja tipli teoremler ispatlanacaktır. Daha sonra, bu operatörler için süreklilik modülü yardımıyla yaklaşım hızı elde edilecek ve operatörlerin bazı fonksiyonlara yaklaşımı grafiklerle gösterilecektir.

2. TEMEL KAVRAMLAR

Bu kısımda tezde kullanılan bazı temel tanım ve teoremlere yer verilmiştir.

2.1. Temel Tanım ve Teoremler

2.1.1. Tanım (Pozitif Lineer Operatör)

X ve Y, reel değerli fonksiyonların lineer uzayı olmak üzere, $L: X \to Y$ şeklindeki L operatörü her $\varphi, \psi \in X$ ve her $\alpha, \beta \in \mathbb{R}$ için

$$L(\alpha\varphi + \beta\psi) = \alpha L(\varphi) + \beta L(\psi)$$

koşulunu sağlıyorsa lineerdir denir. Eğer L operatörü

$$\text{her } \varphi \in X \text{ için } \varphi \geq 0 \text{ iken } L(\varphi) \geq 0$$

ise pozitiftir denir. L operatörü hem lineer hem de pozitif ise L operatörüne pozitif lineer operatör denir (Hacısalihoğlu ve Hacıyev, 1995).

2.1.1. Teorem

$L: X \to Y$ pozitif lineer bir operatör olsun. $\varphi, \psi \in X$ için

i) $\varphi \leq \psi$ ise $L(\varphi) \leq L(\psi)$

ii) $|\varphi| \in X$ iken $|L(\varphi)| \leq L(|\varphi|)$

eşitsizlikleri gerçeklenir (Hacısalihoğlu ve Hacıyev, 1995).

2.1.2. Tanım (Süreklilik Modülü)

φ, $[a, b]$ de sürekli reel değerli bir fonksiyon olsun. Her $\delta > 0$ sayısı için

$$\omega(\varphi, \delta) = \sup_{\substack{\xi, y \in [a,b] \\ |\xi - y| \leq \delta}} |\varphi(\xi) - \varphi(y)|$$

ile tanımlanan ω fonksiyonuna, φ fonksiyonunun süreklilik modülü denir (Anastassiou ve Gal, 2000).

2.1.1. Lemma

Süreklilik modülünün bazı özellikleri aşağıda verilmiştir.

i) $\omega(\varphi, \delta)$ fonksiyonu δ ya göre artandır.

ii) φ, $[a, b]$ de sürekli ve reel değerli bir fonksiyon olsun. Bu durumda

$$\lim_{\delta \to 0^+} \omega(\varphi, \delta) = 0$$

sağlanır.

iii) φ, $[a, b]$ de sürekli ve $m \in \mathbb{N}$ için

$$\omega(\varphi, m\delta) \leq m\omega(\varphi, \delta)$$

eşitsizliği gerçeklenir.

iv) φ, $[a, b]$ de sürekli olsun. Her $N > 0$ reel sayısı için

$$\omega(\varphi, N\delta) \leq (N + 1)\omega(\varphi, \delta)$$

gerçeklenir.

v) φ, $[a, b]$ de sürekli ise

$$|\varphi(t) - \varphi(\xi)| \leq \omega(\varphi, |t - \xi|)$$

sağlanır.

vi) φ, $[a, b]$ de sürekli ise

$$|\varphi(t) - \varphi(\xi)| \leq \left(1 + \frac{|t - \xi|}{\delta}\right)\omega(\varphi, \delta)$$

gerçeklenir (Anastassiou ve Gal, 2000).

2.1.3. Tanım

$[a,b]$ sonlu aralığı üzerinde tanımlı, bütün sürekli $\varphi: [a,b] \to \mathbb{R}$ fonksiyonların uzayı $C[a,b]$ olmak üzere, bu uzaydaki norm

$$\|\varphi(\xi)\|_{C[a,b]} = \max_{a \leq \xi \leq b} |\varphi(\xi)|$$

şeklindedir (Hacısalihoğlu ve Hacıyev, 1995).

2.1.4. Tanım

Bir (φ_η) fonksiyon dizisinin $C[a,b]$ normunda φ fonksiyonuna düzgün yakınsak olması için gerek ve yeter koşul her $\xi \in [a,b]$ için

$$\lim_{\eta\to\infty} \left\|\varphi_\eta(\xi) - \varphi(\xi)\right\|_{C[a,b]} = 0$$

olmasıdır (Hacısalihoğlu ve Hacıyev, 1995).

Teorem 2.1.2. (Korovkin Teoremi)

$L_\eta: C[a,b] \to C[a,b]$ lineer pozitif operetörlerin bir dizisi olmak üzere, $i = 0,1,2$ için

$$\lim_{\eta\to\infty} L_\eta\left(t^i, \xi\right) = \xi^i$$

yakınsaması $[a,b]$ aralığında düzgün ise bu durumda her $\varphi \in C[a,b]$ için

$$\lim_{\eta\to\infty} L_\eta(\varphi, \xi) = \varphi(\xi)$$

yakınsaması $[a,b]$ aralığında düzgündür (Hacısalihoğlu ve Hacıyev, 1995).

2.1.3. Teorem (Genişletilmiş Korovkin Teoremi)

0< $h \in C[a,b]$ bir fonksiyon olsun ve $\left(L_\eta\right)_{\eta\geq 1}$, $[a,b]$ aralığında

$$\lim_{\eta\to\infty} L_\eta\,(e_i) = he_i,\ i = 0,1,2$$

yakınsaması düzgün olarak gerçeklenecek şekildeki lineer pozitif operatörlerin bir dizisi olsun. Bu durumda verilen bir $\varphi \in C[a,b]$ fonksiyonu için $[a,b]$ aralığında

$$\lim_{\eta\to\infty} L_\eta(\varphi) = h\varphi$$

yakınsaması düzgündür (Gupta vd., 2019).

2.1.4. Teorem (Cauchy-Schwarz Eşitsizliği)

a_ρ ve b_ρ, $\rho = 1,\dots,\eta$ olmak üzere iki reel sayı dizisi olsun. Bu durumda

$$\sum_{\rho=1}^{\eta} |a_\rho b_\rho| \le \left(\sum_{\rho=1}^{\eta} |a_\rho|^2\right)^{\frac{1}{2}} \left(\sum_{\rho=1}^{\eta} |b_\rho|^2\right)^{\frac{1}{2}}$$

eşitsizliği gerçeklenir (Mitrinović vd., 1993).

2.1.5. Teorem (Hölder Eşitsizliği)

$p > 1$ ve $\frac{1}{p} + \frac{1}{q} = 1$ olsun. Eğer φ ve ψ, $[a,b]$ aralığında tanımlı integrallenebilen iki fonksiyon ve $|\varphi|^p, |\psi|^q$ fonksiyonları da $[a,b]$ de integrallenebilir ise

$$\int_a^b |\varphi(t)\psi(t)|dt \le \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |\psi(t)|^q dt\right)^{\frac{1}{q}}$$

eşitsizliği sağlanır (Mitrinović vd., 1993).

3. KARIŞIK KANTOROVICH TİPLİ OPERATÖRLER İLE YAKLAŞIM

Bu bölümde, önce Khosravian-Arab vd. (2018) tarafından inşâ edilen karışık Bernstein tipli operatörlerin Kantorovich tipli genelleştirmesi verilecektir. Daha sonra, yeni tanımlanan operatörün momentleri ve merkezi momentleri elde edilecektir. Ayrıca Korovkin Teoremi yardımıyla yeni tanımlanan operatörün düzgün yakınsaklığı ve Voronovskaja tipli sonuç ispatlanacaktır. Son olarak da çeşitli parametreler için karışık Kantorovich tipli operatörler ve klasik Kantorovich operatörlerinin yakınsaklığı grafik ve nümerik örneklerle gösterilecektir.

3.1. Operatörün İnşâsı

3.1.1. Tanım

$\varphi \in C[0,1]$, $\xi \in [0,1]$ ve $\eta \in \mathbb{N}$ için η. dereceden karışık Kantorovich tipli operatörleri

$$\mathcal{K}_{\eta}^{M,1}(\varphi,\xi) = (\eta+1)\sum_{\rho=0}^{\eta} \wp_{\eta,\rho}^{M,1}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} \varphi(t)dt \qquad (3.1.1)$$

şeklinde tanımlanmıştır. Burada

$$\wp_{\eta,\rho}(\xi) = \begin{cases} \binom{\eta}{\rho}\xi^{\rho}(1-\xi)^{\eta-\rho}, & 0 \le \rho \le \eta \\ 0 & , \quad \rho < 0 \; veya \; \rho > \eta \end{cases}$$

ve

$$\wp_{\eta,\rho}^{M,1}(\xi) = \mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi) \qquad (3.1.2)$$

olmak üzere $\mathfrak{u}_0(\eta)$ ve $\mathfrak{u}_1(\eta)$,

$$\mathfrak{u}(\xi,\eta) = \mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta), \eta = 0,1,..., \qquad (3.1.3)$$

olacak şekildeki bilinmeyen reel dizilerdir.

$\mathfrak{u}_1(\eta) = -1, \mathfrak{u}_0(\eta) = 1$ durumunda Eş. 3.1.1 ile verilen operatörler Eş. 1.6 ile verilen klasik Kantorovich operatörlerine indirgenir.

Eş 3.1.1 ile verilen karışık Kantorovich tipli operator, Eş. 3.1.2 yardımıyla

$$\mathcal{K}_{\eta}^{M,1}(\varphi,\xi)=(\eta+1)\sum_{\rho=0}^{\eta}\left[\mathrm{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathrm{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}\varphi(t)dt$$

$$=(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\left\{\mathrm{u}(\xi,\eta)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}\varphi(t)dt+\mathrm{u}(1-\xi,\eta)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}\varphi(t)dt\right\} \qquad (3.1.4)$$

şeklinde de yazılabilir (Gupta vd., 2019).

3.2. Operatörün Momentleri

3.2.1. Lemma

Eş 3.1.1 ile verilen $\mathcal{K}_{\eta}^{M,1}$ operatörlerinin ilk beş momenti

i) $\mathcal{K}_{\eta}^{M,1}(e_0,\xi)=2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta)$

ii) $\mathcal{K}_{\eta}^{M,1}(e_1,\xi)=\frac{1}{2(\eta+1)}\{2\xi(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta-4\mathrm{u}_1(\eta)\xi+4\mathrm{u}_0(\eta)$

$-4\xi\mathrm{u}_0(\eta)+3\mathrm{u}_1(\eta)\}$

iii) $\mathcal{K}_{\eta}^{M,1}(e_2,\xi)=\frac{1}{3(\eta+1)^2}\{3\xi^2(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta^2-3\xi[5\mathrm{u}_1(\eta)\xi+6\xi\mathrm{u}_0(\eta)$

$-6\mathrm{u}_0(\eta)-4\mathrm{u}_1(\eta)]\eta+12\mathrm{u}_1(\eta)\xi^2-18\mathrm{u}_0(\eta)\xi-18\mathrm{u}_1(\eta)\xi$

$+8\mathrm{u}_0(\eta)+7\mathrm{u}_1(\eta)+12\mathrm{u}_0(\eta)\xi^2\}$

iv) $\mathcal{K}_{\eta}^{M,1}(e_3,\xi)=\frac{1}{4(\eta+1)^3}\{4\xi^3(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta^3$

$-24\xi^3(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta^2+44\xi^3(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta$

$$-24\xi^3\big(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta)\big)+18\xi^2\big(2\mathrm{u}_1(\eta)\xi-7\mathrm{u}_1(\eta)-8\mathrm{u}_0(\eta)\big)\eta$$

$$+2\xi\big(32\mathrm{u}_0(\eta)+25\mathrm{u}_1(\eta)\big)\eta-24\mathrm{u}_1(\eta)\xi^3-36\mathrm{u}_1(\eta)\xi^2$$

$$-64\mathrm{u}_1(\eta)\xi+96\mathrm{u}_0(\eta)\xi^2-64\mathrm{u}_0(\eta)\xi+16\mathrm{u}_0(\eta)$$

$$+15\mathrm{u}_1(\eta)\}$$

v) $$\mathcal{K}_\eta^{M,1}(e_4,\xi)=\frac{1}{5(\eta+1)^4}\{10\mathrm{u}_1(\eta)\eta^4\xi^5-100\mathrm{u}_1(\eta)\eta^3\xi^5+525\mathrm{u}_1(\eta)\eta^2\xi^5$$

$$-750\mathrm{u}_1(\eta)\eta\xi^5+360\mathrm{u}_1(\eta)\xi^5+75\mathrm{u}_1(\eta)\eta^3\xi^4-400\mathrm{u}_1(\eta)\eta^2\xi^4$$

$$+325\mathrm{u}_1(\eta)n\xi^4-510\mathrm{u}_1(\eta)\xi^4-285\mathrm{u}_1(\eta)\eta^2\xi^3+77\mathrm{u}_1(\eta)\eta\xi^3$$

$$+130\xi^3-663\mathrm{u}_1(\eta)\eta\xi^2+565\mathrm{u}_1(\eta)\xi^2+5\mathrm{u}_1(\eta)\eta^4\xi^4$$

$$+110\mathrm{u}_1(\eta)\eta^3\xi^3+320\mathrm{u}_1(\eta)\eta^2\xi^2+180\mathrm{u}_1(\eta)\eta\xi$$

$$-90\mathrm{u}_1(\eta)-30\mathrm{u}_1(\eta)+70\mathrm{u}_1(\eta)\eta+10\mathrm{u}_0(\eta)\eta^4\xi^4$$

$$-100\mathrm{u}_0(\eta)\eta^3\xi^4+350\mathrm{u}_0(\eta)\eta^2\xi^4-500\mathrm{u}_0(\eta)\eta\xi^4$$

$$+240\mathrm{u}_0(\eta)\xi^4+75\mathrm{u}_0(\eta)\eta^3\xi^3-450\mathrm{u}_0(\eta)\eta^2\xi^3$$

$$+825\mathrm{u}_0(\eta)\eta\xi^3-450\mathrm{u}_0(\eta)\xi^3+270\mathrm{u}_0(\eta)\eta^2\xi^2$$

$$-810\mathrm{u}_0(\eta)\eta\xi^2+540\mathrm{u}_0(\eta)\xi^2+160\mathrm{u}_0(\eta)\eta\xi$$

$$-160\mathrm{u}_0(\eta)\xi+70\mathrm{u}_0(\eta)-40\mathrm{u}_0(\eta)+2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta)\}$$

şeklindedir (Gupta vd., 2019).

İspat

i) Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$operatöründe $\varphi(t)=e_0(t)=1$ alınırsa basit hesaplamalarla

$$\mathcal{K}_{\eta}^{M,1}(e_0,\xi) = (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} 1dt$$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]$$

$$\times\left(\frac{\rho+1}{\eta+1} - \frac{\rho}{\eta+1}\right)$$

$$= \sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]$$

$$= \mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi) \qquad (3.2.1)$$

bulunur.

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi) = \sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi) = 1 \qquad (3.2.2)$$

olduğundan, Eş. 3.2.2 ve Eş. 3.1.3 ifadesi Eş. 3.2.1 de yerine yazılırsa

$$\mathcal{K}_{\eta}^{M,1}(e_0,\xi) = \mathfrak{u}(\xi,\eta) + \mathfrak{u}(1-\xi,\eta)$$

$$= \mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta) + \mathfrak{u}_1(\eta)(1-\xi) + \mathfrak{u}_0(\eta)$$

$$= \mathfrak{u}_1(\eta) + 2\mathfrak{u}_0(\eta) \qquad (3.2.3)$$

elde edilir.

ii) Eş. 3.1.1 ile verilen $\mathcal{K}_{\eta}^{M,1}$ operatöründe $\varphi(t) = e_1(t) = t$ alınırsa basit hesaplamalarla

$$\mathcal{K}_{\eta}^{M,1}(e_1,\xi) = (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} t dt$$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]$$

$$\times \frac{1}{2}\left[\left(\frac{\rho+1}{\eta+1}\right)^2 - \left(\frac{\rho}{\eta+1}\right)^2\right]$$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]$$

$$\times \frac{1}{2}\left(\frac{\rho^2+2\rho+1-\rho^2}{(\eta+1)^2}\right)$$

$$= \sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{2\rho+1}{2(\eta+1)}$$

$$= \sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{\eta}{\eta}\frac{\rho}{\eta+1}$$

$$+ \sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{1}{2(\eta+1)}$$

$$= \frac{\eta}{\eta+1}\ \mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho}{\eta} + \frac{\eta}{\eta+1}\ \mathfrak{u}(1-\xi,\eta)$$

$$\times \sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\frac{\rho}{\eta} + \frac{1}{2(\eta+1)}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)$$

$$+ \frac{1}{2(\eta+1)}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi) \qquad (3.2.4)$$

elde edilir.

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi)\frac{\rho}{\eta} = \sum_{\rho=0}^{\eta} \binom{\eta-1}{\rho} \xi^{\rho}(1-\xi)^{\eta-1-\rho}\frac{\rho}{\eta}$$

$$= \sum_{\rho=0}^{\eta} \frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!} \xi^{\rho}(1-\xi)^{\eta-1-\rho}\frac{\rho}{\eta}$$

$$= \frac{1}{\eta}\sum_{\rho=1}^{n} \frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!} \xi^{\rho}(1-\xi)^{\eta-1-\rho}$$

olur. Bu toplam için $\rho \to \rho + 1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi)\frac{\rho}{\eta} = \frac{1}{\eta}\sum_{\rho=0}^{\eta-1} \frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!} \xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$= \frac{\eta-1}{\eta}\xi \sum_{\rho=0}^{\eta-1} \frac{(\eta-2)!}{(\eta-2-\rho)!\,\rho!} \xi^{\rho}(1-\xi)^{\eta-2-\rho}$$

$$= \frac{\eta-1}{\eta}\xi \sum_{\rho=0}^{\eta-2} \binom{\eta-2}{\rho} \xi^{\rho}(1-\xi)^{\eta-2-\rho}$$

$$= \frac{(\eta-1)\xi}{\eta} \tag{3.2.5}$$

eşitliği elde edilir. Ayrıca

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho-1}(\xi)\frac{\rho}{\eta} = \sum_{\rho=1}^{\eta} \binom{\eta-1}{\rho-1} \xi^{\rho-1}(1-\xi)^{\eta-\rho}\frac{\rho}{\eta}$$

$$= \sum_{\rho=1}^{\eta} \frac{(\eta-1)!}{(\eta-\rho)!\,(\rho-1)!} \xi^{\rho-1}(1-\xi)^{\eta-\rho}\frac{\rho}{\eta}$$

şeklinde yazılabilir. Bu toplam için $\rho \to \rho + 1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho}{\eta} = \sum_{\rho=0}^{\eta-1} \frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\frac{\rho+1}{\eta}$$

$$= \frac{1}{\eta}\sum_{\rho=0}^{\eta-1} \frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho$$

$$+ \frac{1}{\eta}\sum_{\rho=0}^{\eta-1} \frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}$$

$$= \frac{1}{\eta}\sum_{\rho=1}^{\eta-1} \frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}$$

$$+ \frac{1}{\eta}\sum_{\rho=0}^{\eta-1} \binom{\eta-1}{\rho}\xi^{\rho}(1-\xi)^{\eta-1-\rho}$$

elde edilir. İlk toplam için $\rho \to \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho}{\eta} = \frac{1}{\eta}\sum_{\rho=0}^{\eta-2} \frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho} + \frac{1}{\eta}$$

$$= \frac{(\eta-1)}{\eta}\xi\sum_{\rho=0}^{\eta-2} \frac{(\eta-2)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-2-\rho} + \frac{1}{\eta}$$

$$= \frac{(\eta-1)}{\eta}\xi\sum_{\rho=0}^{\eta-2} \binom{\eta-2}{\rho}\xi^{\rho}(1-\xi)^{\eta-2-\rho} + \frac{1}{\eta}$$

$$= \frac{(\eta-1)\xi+1}{\eta} \tag{3.2.6}$$

şeklinde bulunur.

Eş. 3.2.5 ve Eş. 3.2.6 eşitlikleri Eş. 3.2.4 te yerine yazılırsa

$$\mathcal{K}_{\eta}^{M,1}(e_1,\xi) = \frac{\eta}{\eta+1}\,\mathfrak{u}(\xi,\eta)\frac{\eta-1}{\eta}\xi + \frac{\eta}{\eta+1}\,\mathfrak{u}(1-\xi,\eta)\left[\frac{(\eta-1)\xi+1}{\eta}\right]$$

$$+\frac{1}{2(\eta+1)}\mathfrak{u}(\xi,\eta)+\frac{1}{2(\eta+1)}\mathfrak{u}(1-\xi,\eta)$$

$$=\frac{\eta-1}{\eta+1}\xi[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)]+\frac{(\eta-1)\xi+1}{\eta+1}[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)]$$

$$+\frac{1}{2(\eta+1)}[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)]+\frac{1}{2(\eta+1)}[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)]$$

$$=\frac{(\eta-1)\xi}{\eta+1}[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)]$$

$$+\frac{1}{\eta+1}[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)]+\frac{1}{2(\eta+1)}(2\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta))$$

$$=\frac{1}{2(\eta+1)}\{2\xi\eta\big(2\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta)\big)-4\mathfrak{u}_0(\eta)\xi-4\mathfrak{u}_1(\eta)\xi+3\mathfrak{u}_1(\eta)$$

$$+4\mathfrak{u}_0(\eta)\} \qquad (3.2.7)$$

elde dilir.

iii) Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$operatöründe $\varphi(t)=e_2(t)=t^2$ alınırsa basit hesaplamalarla

$\mathcal{K}_\eta^{M,1}(e_2,\xi)$

$$=(\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}t^2dt$$

$$=(\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{1}{3}\left[\left(\frac{\rho+1}{\eta+1}\right)^3\right.$$

$$\left.-\left(\frac{\rho}{\eta+1}\right)^3\right]$$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\mathcal{p}_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\mathcal{p}_{\eta-1,\rho-1}(\xi)\right]$$

$$\times\frac{1}{3}\frac{(3\rho^2+3\rho+1)}{(\eta+1)^3}$$

$$=\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\mathcal{p}_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\mathcal{p}_{\eta-1,\rho-1}(\xi)\right]\frac{(3\rho^2+3\rho+1)}{3(\eta+1)^2}$$

$$=\frac{\eta^2}{(\eta+1)^2}\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\mathcal{p}_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\mathcal{p}_{\eta-1,\rho-1}(\xi)\right]\frac{\rho^2}{\eta^2}$$

$$+\frac{\eta}{(\eta+1)^2}\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\mathcal{p}_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\mathcal{p}_{\eta-1,\rho-1}(\xi)\right]\frac{\rho}{\eta}$$

$$+\frac{1}{3(\eta+1)^2}\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\mathcal{p}_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\mathcal{p}_{\eta-1,\rho-1}(\xi)\right]$$

$$=\frac{\eta^2}{(\eta+1)^2}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho}(\xi)\frac{\rho^2}{\eta^2}+\frac{\eta^2}{(\eta+1)^2}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho-1}(\xi)\frac{\rho^2}{\eta^2}$$

$$+\frac{\eta}{(\eta+1)^2}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho}(\xi)\frac{\rho}{\eta}+\frac{\eta}{(\eta+1)^2}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho-1}(\xi)\frac{\rho}{\eta}$$

$$+\frac{1}{3(\eta+1)^2}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho}(\xi)+\frac{1}{3(\eta+1)^2}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho-1}(\xi) \quad (3.2.8)$$

şeklinde bulunur.

$$\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho}(\xi)\frac{\rho^2}{\eta^2}=\sum_{\rho=0}^{\eta-1}\binom{\eta-1}{\rho}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\frac{\rho^2}{\eta^2}$$

$$=\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\frac{\rho^2}{\eta^2}$$

$$=\frac{1}{\eta^2}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho$$

elde edilir. Bu eşitlik için $\rho\longrightarrow\rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^2}{\eta^2}=\frac{1}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}(\rho+1)$$

$$=\frac{1}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{1}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$=\frac{1}{\eta^2}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$+\frac{(\eta-1)\xi}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-2)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-2-\rho}$$

bulunur. İlk toplamda $\rho\longrightarrow\rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^2}{\eta^2}=\frac{1}{\eta^2}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}+\frac{(\eta-1)\xi}{\eta^2}$$

$$=\frac{(\eta-2)(\eta-1)\xi^2+(\eta-1)\xi}{\eta^2}\qquad(3.2.9)$$

elde edilir. Ayrıca

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^2}{\eta^2} = \frac{1}{\eta^2}\sum_{\rho=1}^{\eta}\binom{\eta-1}{\rho-1}\xi^{\rho-1}(1-\xi)^{\eta-\rho}\rho^2$$

$$= \frac{1}{\eta^2}\sum_{\rho=1}^{\eta}\frac{(\eta-1)!}{(\eta-\rho)!\,(\rho-1)!}\xi^{\rho-1}(1-\xi)^{\eta-\rho}\rho^2$$

bulunur. Bu toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^2}{\eta^2} = \frac{1}{\eta^2}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}(\rho^2+2\rho+1)$$

$$= \frac{1}{\eta^2}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho^2$$

$$+\frac{2}{\eta^2}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho$$

$$+\frac{1}{\eta^2}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}$$

$$= \frac{1}{\eta^2}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho$$

$$+\frac{2}{\eta^2}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}+\frac{1}{\eta^2}$$

elde edilir. İlk iki toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^2}{\eta^2} = \frac{1}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}(\rho+1)$$

$$+\frac{2}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}+\frac{1}{\eta^2}$$

$$= \frac{1}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}\rho$$

$$+\frac{1}{\eta^2}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2} + \frac{2(\eta-1)\xi+1}{\eta^2}$$

$$= \frac{1}{\eta^2}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}$$

$$+\frac{(\eta-1)\xi+2(\eta-1)\xi+1}{\eta^2}$$

bulunur. İlk toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\mathcal{p}_{\eta-1,\rho-1}(\xi)\frac{\rho^2}{\eta^2} = \frac{1}{\eta^2}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-\rho-3)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-\rho-3}$$

$$+\frac{(\eta-1)\xi+2(\eta-1)\xi+1}{\eta^2}$$

$$= \frac{(\eta-2)(\eta-1)\xi^2+3(\eta-1)\xi+1}{\eta^2} \qquad (3.2.10)$$

elde edilir. Eş. 3.2.2, Eş. 3.2.5, Eş. 3.2.6, Eş. 3.2.9 ve Eş. 3.2.10 ifadeleri, Eş. 3.2.8 de yerine yazılırsa

$\mathcal{K}_{\eta}^{M,1}(e_2,\xi)$

$$= \frac{\eta^2}{(\eta+1)^2}\mathfrak{u}(\xi,\eta)\frac{(\eta-2)(\eta-1)\xi^2+(\eta-1)\xi}{n^2}$$

$$+\frac{\eta^2}{(\eta+1)^2}\mathfrak{u}(1-\xi,\eta)\frac{(\eta-2)(\eta-1)\xi^2+3(\eta-1)\xi+1}{\eta^2}$$

$$+\frac{\eta}{(\eta+1)^2}\mathfrak{u}(\xi,\eta)\frac{(\eta-1)\xi}{\eta}+\frac{\eta}{(\eta+1)^2}\mathfrak{u}(1-\xi,\eta)\frac{(\eta-1)\xi+1}{\eta}$$

$$+\frac{1}{3(\eta+1)^2}\mathfrak{u}(\xi,\eta)+\frac{1}{3(\eta+1)^2}\mathfrak{u}(1-\xi,\eta)$$

$$=\frac{\eta^2}{(\eta+1)^2}\left(\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)\right)+\frac{(\eta-2)(\eta-1)\xi^2+(\eta-1)\xi}{\eta^2}$$

$$+\frac{\eta^2}{(\eta+1)^2}\left(\mathfrak{u}_1(\eta)-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)\right)\frac{(\eta-2)(\eta-1)\xi^2+3(\eta-1)\xi+1}{\eta^2}$$

$$+\frac{\eta}{(\eta+1)^2}\left(\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)\right)\frac{(\eta-1)\xi}{\eta}$$

$$+\frac{\eta}{(\eta+1)^2}\left(\mathfrak{u}_1(\eta)-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)\right)\frac{(\eta-1)\xi+1}{\eta}$$

$$+\frac{1}{3(\eta+1)^2}\left(\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)\right)+\left(\mathfrak{u}_1(\eta)-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)\right)$$

$$=\frac{1}{3(\eta+1)^2}\{3[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)][(\eta-2)(\eta-1)\xi^2+3(\eta-1)\xi+1]$$

$$+3[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)][(\eta-1)\xi+\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)+3[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)]$$

$$\times[(\eta-2)+(\eta-1)\xi^2+3(\eta-1)\xi+1]+3[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)]$$

$$\times[(\eta-1)\xi+1]+\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)$$

$$=\frac{1}{3(\eta+1)^2}\{3\xi^2\eta^2\left(2\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta)\right)-3\xi\eta(-4\mathfrak{u}_1(\eta)-6\mathfrak{u}_0(\eta)\ +5\mathfrak{u}_1(\eta)\xi$$

$$+6\mathfrak{u}_0(\eta)\xi)-18\mathfrak{u}_1(\eta)\xi+12\mathfrak{u}_1(\eta)\xi^2-18\mathfrak{u}_0(\eta)\xi+12\mathfrak{u}_0(\eta)\xi^2+7\mathfrak{u}_1(\eta)$$

$$+8\mathfrak{u}_0(\eta)\} \tag{3.2.11}$$

olur.

iv) Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$operatöründe $\varphi(t)=e_3(t)=t^3$ alınırsa basit hesaplamalarla

$\mathcal{K}_\eta^{M,1}(e_3,\xi)$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} t^3 dt$$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{1}{4}\left[\left(\frac{\rho+1}{\eta+1}\right)^4-\left(\frac{\rho}{\eta+1}\right)^4\right]$$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{1}{4}\left[\frac{4\rho^3+6\rho^2+4\rho+1}{(\eta+1)^4}\right]$$

$$= \sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\left[\frac{4\rho^3+6\rho^2+4\rho+1}{4(\eta+1)^3}\right]$$

$$= \sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{\eta^3}{\eta^3}\frac{\rho^3}{(\eta+1)^3}$$

$$+\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{\eta^2}{\eta^2}\frac{3\rho^2}{2(\eta+1)^3}$$

$$+\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{\eta}{\eta}\frac{\rho}{(\eta+1)^3}$$

$$+\sum_{\rho=0}^{\eta}\left[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\frac{1}{4(\eta+1)^3}$$

$$= \frac{\eta^3}{(\eta+1)^3}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^3}{\eta^3}+\frac{\eta^3}{(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\frac{\rho^3}{\eta^3}$$

$$+\frac{3\eta^2}{2(\eta+1)^3}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^2}{\eta^2}+\frac{3\eta^2}{2(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\frac{\rho^2}{\eta^2}$$

$$+\frac{\eta}{(\eta+1)^3}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho}{\eta}+\frac{\eta}{(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\frac{\rho}{\eta}$$

$$+\frac{1}{4(\eta+1)^3}\mathfrak{u}(\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)+\frac{1}{4(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)$$

(3.2.12)

elde edilir.

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^3}{\eta^3}=\sum_{\rho=0}^{\eta}\binom{\eta-1}{\rho}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\frac{\rho^3}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^3$$

$$=\frac{1}{\eta^3}\sum_{\rho=1}^{\eta}\frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^2$$

olur. Bu toplamda $\rho\longrightarrow\rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^3}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}(\rho+1)^2$$

$$=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}(\rho^2+2\rho+1)$$

$$=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^2$$

$$+\frac{2}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$=\frac{1}{\eta^3}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{2}{\eta^3}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}+\frac{(\eta-1)\xi}{\eta^3}$$

bulunur. Buradaki iki toplam için de $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^3}{\eta^3}=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho+1)$$

$$+\frac{2}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}+\frac{(\eta-1)\xi}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{2(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{(\eta-1)(\eta-2)\xi^2+2(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3}$$

bulunur. Burada ilk toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\frac{\rho^3}{\eta^3} = \frac{1}{\eta^3}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!}\xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+\frac{(\eta-1)(\eta-2)\xi^2+2(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3}$$

$$=\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^3}\sum_{\rho=0}^{\eta-4}\frac{(\eta-4)!}{(\eta-4-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-4-\rho}$$

$$+\frac{3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3}$$

$$=\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3+3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3}$$

$$=\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^3}$$

$$+\frac{3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3} \tag{3.2.13}$$

elde edilir. Ayrıca

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^3}{\eta^3} = \sum_{\rho=1}^{\eta}\binom{\eta-1}{\rho-1}\xi^{\rho-1}(1-\xi)^{\eta-\rho}\frac{\rho^3}{\eta^3}$$

$$=\sum_{\rho=1}^{\eta}\frac{(\eta-1)!}{(\eta-\rho)!\,(\rho-1)!}\xi^{\rho-1}(1-\xi)^{\eta-\rho}\frac{\rho^3}{\eta^3}$$

olur. Bu toplamda $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^3}{\eta^3} = \frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}(\rho+1)^3$$

$$= \frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}(\rho^3+3\rho^2+3\rho+1)$$

$$= \frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho^3$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho^2$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho$$

$$+\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}$$

$$= \frac{1}{\eta^3}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho^2$$

$$+\frac{3}{\eta^3}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}\rho$$

$$+\frac{3}{\eta^3}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-\rho-1)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-\rho-1}+\frac{1}{\eta^3}$$

elde edilir. İlk üç toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\frac{\rho^3}{\eta^3} = \frac{1}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}(\rho+1)^2$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}(\rho+1)$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}+\frac{1}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}\rho^2$$

$$+\frac{2}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}\rho$$

$$+\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{n-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}\rho$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}+\frac{1}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}\rho$$

$$+\frac{2}{\eta^3}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}+\frac{(\eta-1)\xi}{\eta^3}$$

$$+\frac{3}{\eta^3}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-\rho-2)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-\rho-2}$$

$$+\frac{3(\eta-1)\xi\ +3(\eta-1)\xi+1}{\eta^3}$$

olur. Her toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho-1}(\xi)\frac{\rho^3}{\eta^3} = \frac{1}{\eta^3}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-\rho-3)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-\rho-3}(\rho+1)$$

$$+\frac{2}{\eta^3}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-\rho-3)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-\rho-3}$$

$$+\frac{3}{\eta^3}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-\rho-3)!\,\rho}\xi^{\rho+2}(1-\xi)^{\eta-\rho-3}+\frac{7(\eta-1)\xi+1}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-\rho-3)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-\rho-3}\rho$$

$$+\frac{1}{\eta^3}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-\rho-3)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-\rho-3}$$

$$+\frac{2(\eta-1)(\eta-2)\xi^2+3(\eta-1)(\eta-2)\xi^2+7(\eta-1)\xi+1}{\eta^3}$$

$$=\frac{1}{\eta^3}\sum_{\rho=1}^{\eta-3}\frac{(\eta-1)!}{(\eta-\rho-3)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-\rho-3}$$

$$+\frac{(\eta-1)(\eta-2)\xi^2}{\eta^3}+\frac{5(\eta-1)(\eta-2)\xi^2+7(\eta-1)\xi+1}{\eta^3}$$

bulunur. İlk toplamda $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho-1}(\xi)\frac{\rho^3}{\eta^3}$$

$$=\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^3}\sum_{\rho=0}^{\eta-4}\frac{(\eta-4)!}{(\eta-\rho-4)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-\rho-4}$$

$$+\frac{6(\eta-1)(\eta-2)\xi^2}{\eta^3}+\frac{7(\eta-1)\xi+1}{\eta^3}$$

$$=\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3+6(\eta-1)(\eta-2)\xi^2+7(\eta-1)\xi+1}{\eta^3} \qquad (3.2.14)$$

olur. Eş. 3.2.2, Eş. 3.2.5, Eş. 3.2.6, Eş. 3.2.9, Eş. 3.2.10, Eş. 3.2.13 ve Eş. 3.2.14 ifadeleri Eş. 3.2.12 de yerine yazılırsa

$$\mathcal{K}_{\eta}^{M,1}(e_3,\xi)$$

$$=\frac{\eta^3}{(\eta+1)^3}\mathfrak{u}(\xi,\eta)\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3+3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3}$$

$$+\frac{\eta^3}{(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)$$

$$\times\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3+6(\eta-1)(\eta-2)\xi^2+7(\eta-1)\xi+1}{\eta^3}$$

$$+\frac{3\eta^2}{2(\eta+1)^3}\mathfrak{u}(\xi,\eta)\frac{(\eta-2)(\eta-1)\xi^2+(\eta-1)\xi}{\eta^2}$$

$$+\frac{3\eta^2}{2(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)\frac{(\eta-2)(\eta-1)\xi^2+3(\eta-1)\xi+1}{\eta^2}$$

$$+\frac{\eta}{(\eta+1)^3}\mathfrak{u}(\xi,\eta)\frac{(\eta-1)\xi}{\eta}+\frac{\eta}{(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)\frac{(\eta-1)\xi+1}{\eta}$$

$$+\frac{1}{4(\eta+1)^3}\mathfrak{u}(\xi,\eta)+\frac{1}{4(\eta+1)^3}\mathfrak{u}(1-\xi,\eta)$$

$$=\frac{1}{4(\eta+1)^3}\{4[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)][(\eta-1)(\eta-2)(\eta-3)\xi^3+3(\eta-1)(\eta-2)\xi^2$$

$$+(\eta-1)\xi]+4[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)][(\eta-1)(\eta-2)(\eta-3)\xi^3$$

$$+5(\eta-1)(\eta-2)\xi^2+7(\eta-1)\xi+1]$$

$$+6[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)][(\eta-1)(\eta-2)\xi^2+3(\eta-1)\xi+1]$$

$$+6[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)][(\eta-1)(\eta-2)\xi^2+3(\eta-1)\xi]+1]$$

$$+4[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)][(\eta-1)\xi]+4[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)][(\eta-1)\xi+1]$$

$$+[\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)]+[\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)]\}$$

$$=\frac{1}{4(\eta+1)^3}\{4\xi^3\big(2\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta)\big)\eta^3$$

$$-24\xi^3\big(2\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta)\big)\eta^2+44\xi^3\big(2\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta)\big)\eta-24\xi^3\big(2\mathfrak{u}_0(\eta)+\mathfrak{u}_1(\eta)\big)$$

$$+18\xi^2\big(2\mathfrak{u}_1(\eta)\xi-7\mathfrak{u}_1(\eta)-8\mathfrak{u}_0(\eta)\big)\eta+2\xi\big(32\mathfrak{u}_0(\eta)+25\mathfrak{u}_1(\eta)\big)\eta$$

$$-24\mathfrak{u}_1(\eta)\xi^3-36\mathfrak{u}_1(\eta)\xi^2-64\mathfrak{u}_1(\eta)\xi+96\mathfrak{u}_0(\eta)\xi^2-64\mathfrak{u}_0(\eta)\xi+16\mathfrak{u}_0(\eta)$$

$$+15\mathfrak{u}_1(\eta)\} \tag{3.2.15}$$

elde edilir.

v) Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$operatöründe $\varphi(t)=e_4(t)=t^4$ alınırsa basit hesaplamalarla

$\mathcal{K}_\eta^{M,1}(e_4,\xi)$

$$=(\eta+1)\sum_{\rho=0}^{\eta}\big[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\big]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}t^4dt$$

$$=(\eta+1)\sum_{\rho=0}^{\eta}\big[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\big]$$

$$\times\frac{1}{5}\left[\left(\frac{\rho+1}{\eta+1}\right)^5-\left(\frac{\rho}{\eta+1}\right)^5\right]$$

$$=(\eta+1)\sum_{\rho=0}^{\eta}\big[\mathfrak{u}(\xi,\eta)\wp_{\eta-1,\rho}(\xi)+\mathfrak{u}(1-\xi,\eta)\wp_{\eta-1,\rho-1}(\xi)\big]$$

$$\times\frac{5\rho^4+10\rho^3+10\rho^2+5\rho+1}{5(\eta+1)^5}$$

$$= \sum_{\rho=0}^{\eta} \left[\mathfrak{u}(\xi,\eta) \wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta) \wp_{\eta-1,\rho-1}(\xi) \right] \frac{5\rho^4 + 10\rho^3 + 10\rho^2 + 5\rho + 1}{5(\eta+1)^4}$$

$$= \sum_{\rho=0}^{\eta} \left[\mathfrak{u}(\xi,\eta) \wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta) \wp_{\eta-1,\rho-1}(\xi) \right] \frac{\eta^4}{\eta^4} \left[\frac{\rho^4}{(\eta+1)^4} \right]$$

$$+ \sum_{\rho=0}^{\eta} \left[\mathfrak{u}(\xi,\eta) \wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta) \wp_{\eta-1,\rho-1}(\xi) \right] \frac{\eta^3}{\eta^3} \left[\frac{2\rho^3}{(\eta+1)^4} \right]$$

$$+ \sum_{\rho=0}^{\eta} \left[\mathfrak{u}(\xi,\eta) \wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta) \wp_{\eta-1,\rho-1}(\xi) \right] \frac{\eta^2}{\eta^2} \left[\frac{2\rho^2}{(\eta+1)^4} \right]$$

$$+ \sum_{\rho=0}^{\eta} \left[\mathfrak{u}(\xi,\eta) \wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta) \wp_{\eta-1,\rho-1}(\xi) \right] \frac{\eta}{\eta} \left[\frac{\rho}{(\eta+1)^4} \right]$$

$$+ \sum_{\rho=0}^{\eta} \left[\mathfrak{u}(\xi,\eta) \wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta) \wp_{\eta-1,\rho-1}(\xi) \right] \left[\frac{1}{5(\eta+1)^4} \right]$$

$$= \mathfrak{u}(\xi,\eta) \frac{\eta^4}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{\rho^4}{\eta^4} + \mathfrak{u}(1-\xi,\eta) \frac{\eta^4}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi) \frac{\rho^4}{\eta^4}$$

$$+ \mathfrak{u}(\xi,\eta) \frac{2\eta^3}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{\rho^3}{\eta^3} + \mathfrak{u}(1-\xi,\eta) \frac{2\eta^3}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi) \frac{\rho^3}{\eta^3}$$

$$+ \mathfrak{u}(\xi,\eta) \frac{2\eta^2}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{\rho^2}{\eta^2} + \mathfrak{u}(1-\xi,\eta) \frac{2\eta^2}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi) \frac{\rho^2}{\eta^2}$$

$$+ \mathfrak{u}(\xi,\eta) \frac{\eta}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{\rho}{\eta} + \mathfrak{u}(1-\xi,\eta) \frac{\eta}{(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi) \frac{\rho}{\eta}$$

$$+ \mathfrak{u}(\xi,\eta) \frac{1}{5(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) + \mathfrak{u}(1-\xi,\eta) \frac{1}{5(\eta+1)^4} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi) \quad (3.2.16)$$

elde edilir. Ayrıca

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi)\frac{\rho^4}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta}\binom{\eta-1}{\rho}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^4$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^4$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta}\frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^3$$

bulunur. Bu toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi)\frac{\rho^4}{\eta^4}=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}(\rho+1)^3$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^3$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^2$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^2$$

$$+\frac{3}{\eta^4}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{3}{\eta^4}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}+\frac{(\eta-1)\xi}{\eta^4}$$

elde edilir. Her toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\mathscr{p}_{\eta-1,\rho}(\xi)\frac{\rho^4}{\eta^4}=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho+1)^2$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho+1)$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}+\frac{(\eta-1)\xi}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho^2+2\rho+1)$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho+1)$$

$$+\frac{3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho^2$$

$$+\frac{2}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{2}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{(\eta-1)(\eta-2)\xi^2}{\eta^4}+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^4}$$

olur. İlk iki toplamda $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^4}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!}\xi^{\rho+3}(1-\xi)^{\eta-4-\rho}(\rho+1)$$

$$+\frac{2}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!}\xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+\frac{(\eta-1)(\eta-2)\xi^2}{\eta^4}+\frac{3}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{3(\eta-1)(\eta-2)\xi^2+3(\eta-1)(\eta-2)\xi^2\ +(\eta-1)\xi}{\eta^4}$$

olarak bulunur. Üçüncü toplamda $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\frac{\rho^4}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!}\xi^{\rho+3}(1-\xi)^{\eta-4-\rho}\rho$$

$$+\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!}\xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+\frac{2(\eta-1)(\eta-2)(\eta-3)\xi^3+(\eta-1)(\eta-2)\xi^2}{\eta^4}$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!}\xi^{\rho+3}(1-\xi)^{\eta-4-\rho}+\frac{6(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta-3}\frac{(\eta-1)!}{(\eta-4-\rho)!\,(\rho-1)!}\xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+\frac{6(\eta-1)(\eta-2)(\eta-3)\xi^3+7(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^4}$$

elde edilir. İlk toplamda $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\frac{\rho^4}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-4}\frac{(\eta-1)!}{(\eta-5-\rho)!\,\rho!}\xi^{\rho+4}(1-\xi)^{\eta-5-\rho}+\frac{6(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^4}$$

$$+\frac{7(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^4}$$

$$=\frac{(\eta-1)(\eta-2)(\eta-3)(\eta-4)\xi^4+6(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^4}$$

$$\frac{+7(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^4} \qquad (3.2.17)$$

bulunur. Ayrıca

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^4}{\eta^4}=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta}\binom{\eta-1}{\rho-1}\xi^{\rho-1}(1-\xi)^{\eta-\rho}\rho^4$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta}\frac{(\eta-1)!}{(\eta-\rho)!\,(\rho-1)!}\xi^{\rho-1}(1-\xi)^{\eta-\rho}\rho^4$$

elde edilir. Bu toplam için $\rho \longrightarrow \rho+1$dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^4}{\eta^4}=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}(\rho+1)^4$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}$$

$$\times(\rho^4+4\rho^3+6\rho^2+4\rho+1)$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^4$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^3$$

$$+\frac{6}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^2$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho$$

$$+\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,\rho!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^3$$

$$+\frac{4}{\eta^4}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho^2$$

$$+\frac{6}{\eta^4}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}\rho$$

$$+\frac{4}{\eta^4}\sum_{\rho=1}^{\eta-1}\frac{(\eta-1)!}{(\eta-1-\rho)!\,(\rho-1)!}\xi^{\rho}(1-\xi)^{\eta-1-\rho}+\frac{1}{\eta^4}$$

bulunur. Yukarıdaki her toplam için $\rho \longrightarrow \rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\mathscr{p}_{\eta-1,\rho-1}(\xi)\frac{\rho^4}{\eta^4}=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}(\rho+1)^3$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}(\rho+1)^2$$

$$+\frac{6}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}(\rho+1)$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}+\frac{1}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^3$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^2$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^2$$

$$+\frac{8}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$+\frac{6}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{6}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,\rho!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}+\frac{1}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho^2$$

$$+\frac{3}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{3}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}+\frac{(\eta-1)\xi}{\eta^4}$$

$$+\frac{4}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}\rho$$

$$+\frac{8}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}+\frac{4(\eta-1)\xi}{\eta^4}$$

$$+\frac{6}{\eta^4}\sum_{\rho=1}^{\eta-2}\frac{(\eta-1)!}{(\eta-2-\rho)!\,(\rho-1)!}\xi^{\rho+1}(1-\xi)^{\eta-2-\rho}$$

$$+\frac{6(\eta-1)\xi}{\eta^4}+\frac{4(\eta-1)\xi+1}{\eta^4}$$

elde edilir. Her toplam için $\rho\longrightarrow\rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\frac{\rho^4}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho+1)^2$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho+1)$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}(\rho+1)$$

$$+\frac{8}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{6}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}+\frac{15(\eta-1)\xi+1}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho^2$$

$$+\frac{2}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{3}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{4}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{8}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{6}{\eta^4}\sum_{\rho=0}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,\rho!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}+\frac{15(\eta-1)\xi+1}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=1}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}\rho$$

$$+\frac{2}{\eta^4}\sum_{\rho=1}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}+\frac{(\eta-1)(\eta-2)\xi^2}{\eta^4}$$

$$+\frac{3}{\eta^4}\sum_{\rho=1}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}+\frac{3(\eta-1)(\eta-2)\xi^2}{\eta^4}$$

$$+\frac{3(\eta-1)(\eta-2)\xi^2}{\eta^4}+\frac{4}{\eta^4}\sum_{\rho=1}^{\eta-3}\frac{(\eta-1)!}{(\eta-3-\rho)!\,(\rho-1)!}\xi^{\rho+2}(1-\xi)^{\eta-3-\rho}$$

$$+\frac{4(\eta-1)(\eta-2)\xi^2+8(\eta-1)(\eta-2)\xi^2+6(\eta-1)(\eta-2)\xi^2+15(\eta-1)\xi+1}{\eta^4}$$

elde edilir. Her toplam için $\rho\longrightarrow\rho+1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\frac{\rho^4}{\eta^4}$$

$$= \frac{1}{\eta^4} \sum_{\rho=0}^{\eta-4} \frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!} \xi^{\rho+3}(1-\xi)^{\eta-4-\rho}(\rho+1)$$

$$+ \frac{2}{\eta^4} \sum_{\rho=0}^{\eta-4} \frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!} \xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+ \frac{3}{\eta^4} \sum_{\rho=0}^{\eta-4} \frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!} \xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+ \frac{4}{\eta^4} \sum_{\rho=0}^{\eta-4} \frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!} \xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+ \frac{25(\eta-1)(\eta-2)\xi^2 + 15(\eta-1)\xi + 1}{\eta^4}$$

$$= \frac{1}{\eta^4} \sum_{\rho=0}^{\eta-4} \frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!} \xi^{\rho+3}(1-\xi)^{\eta-4-\rho}\rho$$

$$+ \frac{1}{\eta^4} \sum_{\rho=0}^{\eta-4} \frac{(\eta-1)!}{(\eta-4-\rho)!\,\rho!} \xi^{\rho+3}(1-\xi)^{\eta-4-\rho}$$

$$+ \frac{2(\eta-1)(\eta-2)(\eta-3)\xi^3 + 3(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^4}$$

$$+ \frac{4(\eta-1)(\eta-2)(\eta-3)\xi^3 + 25(\eta-1)(\eta-2)\xi^2 + 15(\eta-1)\xi + 1}{\eta^4}$$

$$= \frac{1}{\eta^4} \sum_{\rho=1}^{\eta-4} \frac{(\eta-1)!}{(\eta-4-\rho)!\,(\rho-1)!} \xi^{\rho+3}(1-\xi)^{\eta-4-\rho} + \frac{10(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^4}$$

$$+ \frac{25(\eta-1)(\eta-2)\xi^2 + 15(\eta-1)\xi + 1}{\eta^4}$$

olur. İlk toplam için $\rho \longrightarrow \rho + 1$ dönüşümü uygulanırsa

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\frac{\rho^4}{\eta^4}$$

$$=\frac{1}{\eta^4}\sum_{\rho=0}^{\eta-5}\frac{(\eta-1)!}{(\eta-5-\rho)!\,\rho!}\xi^{\rho+4}(1-\xi)^{\eta-5-\rho}+\frac{10(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^4}$$

$$+\frac{25(\eta-1)(\eta-2)\xi^2+15(\eta-1)\xi+1}{\eta^4}$$

$$=\frac{(\eta-1)(\eta-2)(\eta-3)(\eta-4)\xi^4+10(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^4}$$

$$+\frac{25(\eta-1)(\eta-2)\xi^2\ +15(\eta-1)\xi+1}{\eta^4} \qquad (3.2.18)$$

bulunur. Eş. 3.2.2, Eş. 3.2.5, Eş. 3.2.6, Eş. 3.2.9, Eş. 3.2.10, Eş. 3.2.13, Eş.3.2.14, Eş. 3.2.17 ve Eş. 3.2.18 ifadeleri Eş. 3.2.16 da yerine yazılırsa

$$\mathcal{K}_\eta^{M,1}(e_4,\xi)$$

$$=\mathfrak{u}(\xi,\eta)\frac{\eta^4}{(\eta+1)^4}\frac{(\eta-1)(\eta-2)(\eta-3)(\eta-4)\xi^4+6(\eta-1)(\eta-2)(\eta-3)\xi^3}{\eta^4}$$

$$\frac{+(\eta-1)\xi}{\eta^4}+\mathfrak{u}(1-\xi,\eta)\frac{\eta^4}{(\eta+1)^4}\left[\frac{(\eta-1)(\eta-2)(\eta-3)(\eta-4)\xi^4}{\eta^4}\right.$$

$$\left.+\frac{10(\eta-1)(\eta-2)(\eta-3)\xi^3+25(\eta-1)(\eta-2)\xi^2\ +15(\eta-1)\xi+1}{\eta^4}\right]$$

$$+\mathfrak{u}(\xi,\eta)\frac{2\eta^3}{(\eta+1)^4}\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3+3(\eta-1)(\eta-2)\xi^2+(\eta-1)\xi}{\eta^3}$$

$$+\mathfrak{u}(1-\xi,\eta)\frac{2\eta^3}{(\eta+1)^4}\left[\frac{(\eta-1)(\eta-2)(\eta-3)\xi^3+6(\eta-1)(\eta-2)\xi^2}{\eta^3}\right.$$

$$\left.+\frac{7(\eta-1)\xi+1}{\eta^3}\right]+\mathfrak{u}(\xi,\eta)\frac{2\eta^2}{(\eta+1)^4}\frac{(\eta-2)(\eta-1)\xi^2+(\eta-1)\xi}{\eta^2}$$

$$+\mathrm{u}(1-\xi,\eta)\frac{2\eta^2}{(\eta+1)^4}\frac{(\eta-2)(\eta-1)\xi^2+3(\eta-1)\xi+1}{\eta^2}$$

$$+\mathrm{u}(\xi,\eta)\frac{\eta}{(\eta+1)^4}\frac{(\eta-1)\xi}{\eta}+\mathrm{u}(1-\xi,\eta)\frac{\eta}{(\eta+1)^4}\frac{(\eta-1)\xi+1}{\eta}$$

$$+\mathrm{u}(\xi,\eta)\frac{1}{5(\eta+1)^4}+\mathrm{u}(1-\xi,\eta)\frac{1}{5(\eta+1)^4}$$

$$=\frac{1}{5(\eta+1)^4}\{\mathrm{u}_1(10\eta^4\xi^4-100\eta^3\xi^5+525\eta^2\xi^5-750\eta\xi^5+360\xi^5+75\eta^3\xi^4$$

$$-400\eta^2\xi^4+325\eta\xi^4-510\xi^4-285\eta^2\xi^3+77\eta\xi^3+130\xi^3-663\eta\xi^2$$

$$+565\xi^2+5\eta^4\xi^4+110\eta^3\xi^3+320\eta^2\xi^2+180\eta\xi-90\xi+70\eta-29)$$

$$+\mathrm{u}_0(10\xi^4\eta^4-100\eta^3\xi^4+350\eta^2\xi^4-500\eta\xi^4+240\xi^4+75\eta^3\xi^3-450\eta^2\xi^3$$

$$+825\eta\xi^3-450\xi^3+270\eta^2\xi^2-810\eta\xi^2+540\xi^2+160\eta\xi-160\xi+70\eta$$

$$+825\eta\xi^3-450\xi^3+270\eta^2\xi^2-810\eta\xi^2+540\xi^2+160\eta\xi-160\xi+70\eta$$

$$-38)\} \qquad (3.2.19)$$

bulunur ve böylece ispat tamamlanır.

3.3. Operatörün Merkezi Momentleri

3.3.1. Lemma

Eş 3.1.1 ile verilen $K_\eta^{M,1}$ operatörünün merkezi momentleri

$$i)\ \mathcal{K}_\eta^{M,1}(t-\xi,\xi)=\frac{1-2\xi}{2(\eta+1)}\big(3\mathrm{u}_1(\eta)+4\mathrm{u}_0(\eta)\big)$$

$$ii)\ \mathcal{K}_\eta^{M,1}((t-\xi)^2,\xi)=-\frac{1}{3(\eta+1)^2}\{3\xi(\xi-1)\big(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta)\big)\eta$$

$$-8\mathrm{u}_0(\eta)-7\mathrm{u}_1(\eta)+27\mathrm{u}_1(\eta)\xi-27\mathrm{u}_1(\eta)\xi^2$$

$$+30\mathrm{u}_0(\eta)\xi-30\mathrm{u}_0(\eta)\xi^2\}$$

$$iii)\ \mathcal{K}_\eta^{M,1}((t-\xi)^4,\xi)=\frac{1}{5(\eta+1)^4}\{15\xi^2(1-\xi)^2\big(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta)\big)\eta^2$$

$$+5\xi(1-\xi)(1-2\xi)^2\big(21\mathrm{u}_1(\eta)+26\mathrm{u}_0(\eta)\big)\eta$$

$$-285\mathrm{u}_1(\eta)\xi+32\mathrm{u}_0(\eta)+31\mathrm{u}_1(\eta)+930\mathrm{u}_1(\eta)\xi^2$$

$$-290\mathrm{u}_0(\eta)\xi-1290\mathrm{u}_1(\eta)\xi^3+645\mathrm{u}_1(\eta)\xi^4$$

$$-1300\mathrm{u}_0(\eta)\xi^3+650\mathrm{u}_0(\eta)\xi^4+940\mathrm{u}_0(\eta)\xi^2\}$$

olur (Gupta vd., 2019).

İspat

$i)$ Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$ operatörünün lineerliği kullanılarak

$$\mathcal{K}_\eta^{M,1}(t-\xi,\xi)=\mathcal{K}_\eta^{M,1}(t,\xi)-\xi\mathcal{K}_\eta^{M,1}(1,\xi)$$

yazılabilir.

Eş. 3.2.7 ve Eş. 3.2.3 ifadeleri yukarıdaki eşitlikte yerine yazılırsa

$$\mathcal{K}_\eta^{M,1}(t-\xi,\xi)=\frac{1}{2(\eta+1)}\{2\xi\eta\big(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta)\big)-4\mathrm{u}_0(\eta)\xi-4\mathrm{u}_1(\eta)\xi+3\mathrm{u}_1(\eta)$$

$$+4\mathrm{u}_0(\eta)\}-\xi(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))$$

$$=\frac{1-2\xi}{2(\eta+1)}\big(3\mathrm{u}_1(\eta)+4\mathrm{u}_0(\eta)\big) \tag{3.3.1}$$

bulunur.

$ii)$ Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$ operatörünün lineerliği kullanılarak

$$\mathcal{K}_\eta^{M,1}((t-\xi)^2,\xi)=K_\eta^{M,1}(t^2,\xi)-2\xi K_\eta^{M,1}(t,\xi)+\xi^2K_\eta^{M,1}(1,\xi)$$

şeklinde yazılabilir.

Eş. 3.2.11, Eş. 3.2.7 ve Eş. 3.2.3 ifadeleri yukarıdaki eşitlikte yerine yazılırsa

$$\mathcal{K}_{\eta}^{M,1}((t-\xi)^2,\xi)=\frac{1}{3(\eta+1)^2}\{3\xi^2(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta^2-3\xi[5\mathrm{u}_1(\eta)\xi$$

$$+6\xi\mathrm{u}_0(\eta)-6\mathrm{u}_0(\eta)-4\mathrm{u}_1(\eta)]\eta+12\mathrm{u}_1(\eta)\xi^2-18\xi\mathrm{u}_0(\eta)$$

$$-18\mathrm{u}_1(\eta)\xi+8\mathrm{u}_0(\eta)+7\mathrm{u}_1(\eta)+12\xi^2\mathrm{u}_0(\eta)\}$$

$$-\frac{2\xi}{2(\eta+1)}\{2\xi(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta-4\mathrm{u}_1(\eta)\xi+4\mathrm{u}_0(\eta)$$

$$-4\xi\mathrm{u}_0(\eta)+3\mathrm{u}_1(\eta)\}+\xi^2(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))$$

$$=-\frac{1}{3(\eta+1)^2}\{3\xi(\xi-1)(2\mathrm{u}_0(\eta)+\mathrm{u}_1(\eta))\eta-8\mathrm{u}_0(\eta)$$

$$-7\mathrm{u}_1(\eta)+27\mathrm{u}_1(\eta)\xi-27\mathrm{u}_1(\eta)\xi^2$$

$$+30\mathrm{u}_0(\eta)\xi-30\mathrm{u}_0(\eta)\xi^2\} \qquad (3.3.2)$$

elde edilir.

iii) Eş. 3.1.1 ile verilen $\mathcal{K}_{\eta}^{M,1}$ operatörünün lineerliği kullanılarak

$$\mathcal{K}_{\eta}^{M,1}((t-\xi)^4,\xi)$$

$$=\mathcal{K}_{\eta}^{M,1}(t^4,\xi)-4\xi\mathcal{K}_{\eta}^{M,1}(t^3,\xi)+6\xi^2\mathcal{K}_{\eta}^{M,1}(t^2,\xi)-4\xi^3\mathcal{K}_{\eta}^{M,1}(t,\xi)$$

$$+\xi^4\mathcal{K}_{\eta}^{M,1}(1,\xi)$$

şeklinde yazılabilir. Eş. 3.2.19, Eş. 3.2.15, Eş. 3.2.11, Eş. 3.2.7 ve Eş. 3.2.3 ifadeleri yukarıdaki eşitlikte yerine yazılırsa

$$\mathcal{K}_{\eta}^{M,1}((t-\xi)^4,\xi)$$

$$=\frac{1}{5(\eta+1)^4}\{10\mathrm{u}_1(\eta)\eta^4\xi^5-100\mathrm{u}_1(\eta)\eta^3\xi^5+525\mathrm{u}_1(\eta)\eta^2\xi^5-750\mathrm{u}_1(\eta)\eta\xi^5$$

$$+360\mathrm{u}_1(\eta)\xi^5+75\mathrm{u}_1(\eta)\eta^3\xi^4-400\mathrm{u}_1(\eta)\eta^2\xi^4+325\mathrm{u}_1(\eta)\eta\xi^4$$

$$-510\mathrm{u}_1(\eta)\xi^4-285\mathrm{u}_1(\eta)\eta^2\xi^3+77\mathrm{u}_1(\eta)\eta\xi^3+130\xi^3-663\mathrm{u}_1(\eta)\eta\xi^2$$

$$+565u_1(\eta)\xi^2+5u_1(\eta)\eta^4\xi^4+110u_1(\eta)\eta^3\xi^3+320u_1(\eta)\eta^2\xi^2$$

$$+180u_1(\eta)\eta\xi-90u_1(\eta)-30u_1(\eta)+70u_1(\eta)\eta+10u_0(\eta)\eta^4\xi^4$$

$$-100u_0(\eta)\eta^3\xi^4+350u_0(\eta)\eta^2\xi^4-500u_0(\eta)\eta\xi^4+240u_0(\eta)\xi^4$$

$$+75u_0(\eta)\eta^3\xi^3-450u_0(\eta)\eta^2\xi^3+825u_0(\eta)\eta\xi^3-450u_0(\eta)\xi^3$$

$$+270u_0(\eta)\eta\xi^2-810u_0(\eta)\eta\xi^2+540u_0(\eta)\xi^2+160u_0(\eta)\eta\xi-160u_0(\eta)\xi$$

$$+70u_0(\eta)-40u_0(\eta)+2u_0(\eta)+u_1(\eta)$$

$$+\frac{1}{4(\eta+1)^3}\{4\xi^3\eta^3(2u_0(\eta)+u_1(\eta))-24\xi^3\eta^2(2u_0(\eta)+u_1(\eta))$$

$$+44\xi^3\eta(2u_0(\eta)+u_1(\eta))-24\xi^3(2u_0(\eta)+u_1(\eta))$$

$$+18\xi^2\eta(2u_1(\eta)\xi-7u_1(\eta)-8u_0(\eta))+2\xi\eta(32u_0(\eta)+25u_1(\eta))$$

$$-24u_1(\eta)\xi^3-36u_1(\eta)\xi^2-64u_1(\eta)\xi+96u_0(\eta)\xi^2-64u_0(\eta)\xi+16u_0(\eta)$$

$$+15u_1(\eta)\}+\frac{6\xi^2}{3(\eta+1)^2}\{3\xi^2\big(2u_0(\eta)+u_1(\eta)\big)\eta^2$$

$$-3x[5u_1(\eta)\xi+6\xi u_0(\eta)-6u_0(\eta)-4u_1(\eta)]\eta+12u_1(\eta)\xi^2-18\xi u_0(\eta)$$

$$-18u_1(\eta)\xi+8u_0(\eta)+7u_1(\eta)+12\xi^2u_0(\eta)\}-\frac{4\xi^3}{2(\eta+1)}\{2\xi(2u_0(\eta)$$

$$+u_1(\eta))\eta-4u_1(\eta)\xi+4u_0(\eta)-4\xi u_0(\eta)+3u_1(\eta)\}+\xi^4\big(2u_0(\eta)+u_1(\eta)\big)$$

$$=\frac{1}{5(\eta+1)^4}15\xi^2(1-\xi)^2\xi^4\big(2u_0(\eta)+\alpha_1(\eta)\big)\eta^2$$

$$+5\xi(1-\xi)(1-2\xi)^2\big(21u_1(\eta)+26u_0(\eta)\big)\eta$$

$$-285u_1(\eta)\xi+32u_0(\eta)+31u_1(\eta)+930u_1(\eta)\xi^2$$

$$-290u_0(\eta)\xi-1290u_1(\eta)\xi^3+645u_1(\eta)\xi^4$$

$$-1300u_0(\eta)\xi^3+650u_0(\eta)\xi^4+940u_0(\eta)\xi^2\} \tag{3.3.3}$$

bulunur.

3.4. Düzgün Yaklaşım

Düzgün yakınsaklığı çalışmak için $\mathfrak{u}_i(\eta), i = 0,1,$ dizilerinin

$$2\mathfrak{u}_0(\eta) + \mathfrak{u}_1(\eta) = 1 \tag{3.4.1}$$

koşulunu sağladığını dikkate alacağız.

Bilinmeyen $\mathfrak{u}_0(\eta)$ ve $\mathfrak{u}_1(\eta)$ dizileri için aşağıdaki iki durumu inceleyeceğiz.

Durum 1.

$$\mathfrak{u}_0(\eta) \geq 0, \mathfrak{u}_0(\eta) + \mathfrak{u}_1(\eta) \geq 0. \tag{3.4.2}$$

Eş. 3.4.1 koşulunu kullanırsak 0≤ $\mathfrak{u}_0(\eta) \leq 1$ ve -1≤ $\mathfrak{u}_1(\eta) \leq 1$ elde edilir. Yani $\mathfrak{u}_0(\eta)$ ve $\mathfrak{u}_1(\eta)$ dizileri sınırlıdır. Bu durumda Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$ operatörü pozitiftir.

Durum 2.

$$\mathfrak{u}_0(\eta) < 0 \text{ veya } \mathfrak{u}_0(\eta) + \mathfrak{u}_1(\eta) < 0. \tag{3.4.3}$$

$\mathfrak{u}_0(\eta) < 0$ ise $\mathfrak{u}_0(\eta) + \mathfrak{u}_1(\eta) > 1$ ve $\mathfrak{u}_0(\eta) + \mathfrak{u}_1(\eta) < 0$ ise $\mathfrak{u}_0(\eta) > 1$ olur. Bu durumda Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$ operatörü pozitif değildir.

3.4.1. Teorem

$\mathfrak{u}_0(\eta)$ ve $\mathfrak{u}_1(\eta)$, Eş. 3.4.1 ve Eş. 3.4.2 koşullarını sağlayan iki dizi olsun. $\varphi \in C[0,1]$ ise

$$\lim_{\eta\to\infty} \mathcal{K}_\eta^{M,1}(\varphi,\xi) = \varphi(\xi)$$

yakınsaması $[0,1]$ aralığında düzgün olarak gerçeklenir (Gupta vd., 2019).

İspat

$\mathcal{K}_\eta^{M,1}(e_i,\xi), i = 0,1,2,$ için iyi bilinen Korovkin Teoreminin koşullarının sağlandığını göstermeliyiz.

Lemma 3.2.1. (i) den $\mathcal{K}_\eta^{M,1}(e_0,\xi)$=1 olup

$$\lim_{\eta\to\infty}\left\|\mathcal{K}_\eta^{M,1}(e_0,\xi)-e_0(\xi)\right\|_{C[0,1]}=0$$

elde edilir.

Lemma 3.2.1. (ii) den

$\mathcal{K}_\eta^{M,1}(e_1,\xi)$

$$=\frac{1}{2(\eta+1)}\{2(\eta+1)\xi-2\xi-4\mathrm{u}_1(\eta)\xi+4\mathrm{u}_0(\eta)-4\xi\mathrm{u}_0(\eta)+3\mathrm{u}_1(\eta)\}$$

$$=\xi+\frac{1}{2(\eta+1)}\{-2\xi+\mathrm{u}_1(\eta)(-4\xi+3)+4\mathrm{u}_0(\eta)(1-\xi)\}$$

şeklinde yazılabilir. Buradan

$\left\|\mathcal{K}_\eta^{M,1}(e_1,\xi)-e_1(\xi)\right\|_{C[0,1]}$

$$=\max_{0\le\xi\le1}\left|\mathcal{K}_\eta^{M,1}(e_1,\xi)-e_1(\xi)\right|$$

$$=\max_{0\le\xi\le1}\left|\xi+\frac{1}{2(\eta+1)}\{-2\xi+\mathrm{u}_1(\eta)(3-4\xi)+4\mathrm{u}_0(\eta)(1-\xi)\}-\xi\right|$$

$$=\max_{0\le\xi\le1}\left|\frac{1}{2(\eta+1)}\{-2\xi+\mathrm{u}_1(\eta)(3-4\xi)+4\mathrm{u}_0(\eta)(1-\xi)\}\right|$$

$$\le\max_{0\le\xi\le1}\frac{1}{2(\eta+1)}\{2|\xi|+|\mathrm{u}_1(\eta)|(4\xi+3)+4|\mathrm{u}_0(\eta)|(1-\xi)\}$$

$$\le\frac{1}{2(\eta+1)}\{2+7|\mathrm{u}_1(\eta)|+4|\mathrm{u}_0(\eta)|\}$$

elde edilir.

Eş. 3.4.1 ve Eş. 3.4.2 koşullarını sağlayan $\mathrm{u}_i(\eta), i=0,1$, dizileri sınırlı yani $|\mathrm{u}_i(\eta)|\le N, N>0$, olduğundan

$$\lim_{\eta\to\infty}\left\|\mathcal{K}_\eta^{M,1}(e_1,\xi)-e_1(\xi)\right\|_{C[0,1]}=0$$

bulunur.

Lemma 3.2.1. (iii) den

$\mathcal{K}_{\eta}^{M,1}(e_2,\xi)$

$$= \frac{1}{3(\eta+1)^2}\{3\xi^2\eta^2 - 3\xi\big(5u_1(\eta)\xi + 6\xi u_0(\eta) - 6u_0(\eta) - 4u_1(\eta)\big)\eta$$

$$+12u_1(\eta)\xi^2 - 18\xi u_0(\eta) - 18u_1(\eta)\xi - 18u_1(\eta)\xi + 8u_0(\eta) + 7u_1(\eta)$$

$$+12\xi^2 u_0(\eta)\}$$

$$= \xi^2 + \frac{1}{3(\eta+1)^2}\{-6\eta\xi^2 - 3\xi^2 - 3\xi\big(5u_1(\eta)\xi + 6\xi u_0(\eta) - 6u_0(\eta) - 4u_1(\eta)\big)\eta$$

$$+12u_1(\eta)\xi^2 - 18u_0(\eta)\xi - 18u_1(\eta)\xi + 8u_0(\eta) + 7u_1(\eta) + 12\xi^2 u_0(\eta)\}$$

şeklinde yazılabilir. Buradan

$$\left\|\mathcal{K}_{\eta}^{M,1}(e_2,\xi) - e_2(\xi)\right\|_{C[0,1]}$$

$$= \max_{0\le\xi\le1}\left|\mathcal{K}_{\eta}^{M,1}(e_2,\xi) - e_2(\xi)\right|$$

$$= \max_{0\le\xi\le1}\frac{1}{3(\eta+1)^2}\{-6\eta\xi^2 - 3\xi^2 - 3\xi\big(5u_1(\eta)\xi + 6\xi u_0(\eta) - 6u_0(\eta) - 4u_1(\eta)\big)\eta$$

$$+12u_1(\eta)\xi^2 - 18u_0(\eta)\xi - 18u_1(\eta)\xi + 8u_0(\eta) + 7u_1(\eta) + 12\xi^2 u_0(\eta)\}$$

$$\le \max_{0\le\xi\le1}\frac{1}{3(\eta+1)^2}\{6\eta\xi^2 + 3\xi^2 + 15\eta\xi^2|u_1(\eta)| + 18\eta\xi^2|u_0(\eta)| + 18\eta\xi|u_0(\eta)|$$

$$+12\eta\xi|u_1(\eta)| + 12\xi^2|u_1(\eta)| + 18\xi|u_0(\eta)| + 18\xi|u_1(\eta)| + 8|u_0(\eta)|$$

$$+7|u_1(\eta)| + 12\xi^2|u_0(\eta)|\}$$

$$\le \frac{1}{3(\eta+1)^2}\{6\eta + 3 + 15\eta|u_1(\eta)| + 18\eta|u_0(\eta)| + 18\eta|u_0(\eta)| + 12\eta|u_1(\eta)|$$

$$+12|u_1(\eta)| + 18|u_0(\eta)| + 18|u_1(\eta)| + 8|u_0(\eta)| + 7|u_1(\eta)| + 12|u_0(\eta)|\}$$

$$= \frac{1}{3(\eta+1)^2}\{\eta[6 + 27|u_1(\eta)| + 36|u_0(\eta)|] + 37|u_1(\eta)| + 38|u_0(\eta)|\}$$

Eş. 3.4.1 ve Eş. 3.4.2 koşullarını sağlayan $u_i(\eta), i = 0,1$, dizileri sınırlı yani

$|\mathfrak{u}_i(\eta)| \leq N, N > 0$, olduğundan

$$\lim_{\eta\to\infty} \left\| \mathcal{K}_{\eta}^{M,1}(e_2, \xi) - e_2(\xi) \right\|_{C[0,1]} = 0$$

olur. Böylece ispat tamamlanmış olur.

Yukarıdaki sonuç Durum 2 için genişletilebilir. Bu sonucu ispatlamak için Korovkin Teoreminin genişletilmiş formu olan Teorem 2.1.3 ü kullanacağız.

3.4.2. Teorem

$\varphi \in C[0,1]$ olsun. Eş. 3.4.1 ve Eş. 3.4.3 koşullarını sağlayan $\mathfrak{u}_1(\eta)$ ve $\mathfrak{u}_0(\eta)$ sınırlı dizileri için

$$\lim_{\eta\to\infty} \mathcal{K}_{\eta}^{M,1}(\varphi, \xi) = \varphi(\xi)$$

yakınsaması [0,1] aralığında düzgün olarak gerçeklenir (Gupta vd., 2019).

İspat

$\mathcal{K}_{\eta,1}^{M,1}(\varphi, \xi)$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[-\mathfrak{u}_1(\eta)\xi\, p_{\eta-1,\rho}(\xi) - \mathfrak{u}_1(\eta)\, p_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} \varphi(t)\, dt$$

ve

$$\mathcal{K}_{\eta,2}^{M,1}(\varphi, \xi) = (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}_0(\eta)\, p_{n-1,\rho}(\xi) + (-\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta))\, p_{\eta-1,\rho-1}(\xi)\right]$$

$$\times \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} \varphi(t)\, dt$$

ile gösterelim.

Eş. 3.1.1 ile tanımlanan $\mathcal{K}_{\eta}^{M,1}$ operatörleri

$$\mathcal{K}_{\eta}^{M,1}(\varphi,\xi) = \mathcal{K}_{\eta,2}^{M,1}(\varphi,\xi) - \mathcal{K}_{\eta,1}^{M,1}(\varphi,\xi) \tag{3.4.4}$$

şeklinde yazılabilir.

Şimdi $\mathcal{K}_{\eta,1}^{M,1}$ve $\mathcal{K}_{\eta,2}^{M,1}$ operatörlerinin ilk üç momentini hesaplayalım.

$$\mathcal{K}_{\eta,1}^{M,1}(e_0,\xi)$$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[-\mathfrak{u}_1(\eta)\xi \wp_{\eta-1,\rho}(\xi) - \mathfrak{u}_1(\eta)\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} dt$$

$$= \sum_{\rho=0}^{\eta}\left[-\mathfrak{u}_1(\eta)\xi \wp_{\eta-1,\rho}(\xi) - \mathfrak{u}_1(\eta)\wp_{\eta-1,\rho-1}(\xi)\right]$$

$$= -\mathfrak{u}_1(\eta)\xi\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi) - \mathfrak{u}_1(\eta)\sum_{\rho=1}^{\eta}\wp_{\eta-1,\rho-1}(\xi)$$

$$= -\mathfrak{u}_1(\eta)\xi - \mathfrak{u}_1(\eta)$$

$$= -\mathfrak{u}_1(\eta)(\xi+1) \tag{3.4.5}$$

elde edilir.

$$\mathcal{K}_{\eta,1}^{M,1}(e_1,\xi)$$

$$= (\eta + 1) \sum_{\rho=0}^{\eta} \left[-\mathfrak{u}_1(\eta)\xi \wp_{\eta-1,\rho}(\xi) - \mathfrak{u}_1(\eta)\wp_{\eta-1,\rho-1}(\xi) \right] \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} t dt$$

$$= \frac{1}{2(\eta+1)} \sum_{\rho=0}^{\eta} \left[-\mathfrak{u}_1(\eta)\xi \wp_{\eta-1,\rho}(\xi) - \mathfrak{u}_1(\eta)\wp_{\eta-1,\rho-1}(\xi) \right](2\rho + 1)$$

$$= \frac{1}{2(\eta+1)} \left[-2\mathfrak{u}_1(\eta)\xi \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\rho - 2\mathfrak{u}_1(\eta) \sum_{\rho=1}^{\eta} \wp_{\eta-1,\rho-1}(\xi)\,\rho \right]$$

$$+ \frac{1}{2(\eta+1)} K_{\eta,1}^{M,1}(e_0, \xi)$$

elde edilir. Eş. 3.2.5, Eş. 3.2.6, Eş. 3.4.5 ve Lemma 3.2.1. (i) den

$$\mathcal{K}_{\eta,1}^{M,1}(e_1, \xi)$$

$$= \frac{1}{2(\eta+1)} \{ -2\mathfrak{u}_1(\eta)\xi[(\eta-1)\xi] - 2\mathfrak{u}_1(\eta)[(\eta-1)\xi + 1] - \mathfrak{u}_1(\eta)(\xi+1) \}$$

$$= \frac{1}{2(\eta+1)} \{ -2\mathfrak{u}_1(\eta)\xi^2\eta + 2\mathfrak{u}_1(\eta)\xi^2 - 2\mathfrak{u}_1(\eta)\eta\xi + 2\mathfrak{u}_1(\eta)\xi - 2\mathfrak{u}_1(\eta)\xi - \mathfrak{u}_1(\eta) \}$$

$$= -\frac{1}{2}\mathfrak{u}_1(\eta) \left[\frac{2\xi(\xi+1)\eta}{\eta+1} + \frac{(1-\xi)(2\xi+3)}{\eta+1} \right] \tag{3.4.6}$$

bulunur.

$$\mathcal{K}_{\eta,1}^{M,1}(e_2, \xi)$$

$$= (\eta + 1) \sum_{\rho=0}^{\eta} \left[-\mathfrak{u}_1(\eta)\xi \wp_{\eta-1,\rho}(\xi) - \mathfrak{u}_1(\eta)\wp_{\eta-1,\rho-1}(\xi) \right] \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} t^2 dt$$

$$= -\frac{1}{3(\eta+1)^2}\mathfrak{u}_1(\eta)\sum_{\rho=0}^{\eta}\left[\xi \wp_{\eta-1,\rho}(\xi) + \wp_{\eta-1,\rho-1}(\xi)\right](3\rho^2+3\rho+1)$$

$$= -\frac{1}{3(\eta+1)^2}\mathfrak{u}_1(\eta)\left\{3\xi\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\,\rho^2 + 3\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\,\rho^2\right.$$

$$\left. +3\xi\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\,\rho + 3\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\,\rho + \xi\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi) + \sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\right\}$$

elde edilir. Bu eşitlik için Eş. 3.2.2, Eş. 3.2.5, Eş. 3.2.6, Eş. 3.2.9 ve Eş. 3.2.10 kullanılırsa

$\mathcal{K}_{\eta,1}^{M,1}(e_2,\xi)$

$$= -\frac{1}{3(\eta+1)^2}\mathfrak{u}_1(\eta)\{3\xi[(\eta-2)(\eta-1)\xi^2 + (\eta-1)\xi] + 3[(\eta-2)(\eta-1)\xi^2$$

$$+3(\eta-1)\xi+1] + 3\xi(\eta-1)\xi + 3[(\eta-1)\xi+1] + \xi + 1\}$$

$$= -\frac{1}{3(\eta+1)^2}\mathfrak{u}_1(\eta)\{3\xi^3\eta^2 - 9\xi^3\eta + 6\xi^3 + 3\eta\xi^2 - 3\xi^2 + 3\xi^2\eta^2 - 9\eta\xi^2 + 6\xi^2$$

$$+9\eta\xi - 9\xi + 3 + 3\xi^2\eta - 3\xi^2 + 3\eta\xi - 3\xi + 3 + \xi + 1\}$$

$$= -\frac{1}{3}\mathfrak{u}_1(\eta)\left[3\xi^2(\xi+1)\frac{\eta^2}{(\eta+1)^2} + 3\xi(3\xi+4)(1-\xi)\frac{\eta}{(\eta+1)^2} + \frac{6\xi^3 - 11\xi + 7}{(\eta+1)^2}\right] \tag{3.4.7}$$

bulunur. Diğer taraftan

$\mathcal{K}_{\eta,2}^{M,1}(e_0,\xi)$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}_0(\eta)\wp_{\eta-1,\rho}(\xi)+(-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta))\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} dt$$

$$= \mathfrak{u}_0(\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)+(-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta))\sum_{\rho=1}^{\eta}\wp_{\eta-1,\rho-1}(\xi)$$

$$= \mathfrak{u}_0(\eta)-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)$$

$$= 2\mathfrak{u}_0(\eta)-\mathfrak{u}_1(\eta)\xi \tag{3.4.8}$$

elde edilir.

$\mathcal{K}_{\eta,2}^{M,1}(e_1,\xi)$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}_0(\eta)\wp_{\eta-1,\rho}(\xi)+(-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta))\wp_{\eta-1,\rho-1}(\xi)\right]\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} tdt$$

$$= \frac{1}{2(\eta+1)}\sum_{\rho=0}^{\eta}\left[\mathfrak{u}_0(\eta)\wp_{\eta-1,\rho}(\xi)+(-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta))\wp_{\eta-1,\rho-1}(\xi)\right](2\rho+1)$$

$$= \frac{1}{2(\eta+1)}\left\{2\mathfrak{u}_0(\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\rho+2\,(-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta))\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\rho\right.$$

$$\left.+\mathfrak{u}_0(\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)+(-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta))\sum_{\rho=1}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\right\}$$

$$= \frac{1}{2(\eta+1)}\{2\mathfrak{u}_0(\eta)[(\eta-1)\xi]+2(-\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta))[(\eta-1)\xi+1]$$

$$+2\mathfrak{u}_0(\eta)-\mathfrak{u}_1(\eta)\xi\}$$

$$= \left\{ \frac{\eta}{\eta+1} 2\xi\big(-\mathfrak{u}_1(\eta)\xi + 2\mathfrak{u}_0(\eta)\big) + \frac{2\mathfrak{u}_1(\eta)\xi^2 - 4\mathfrak{u}_0(\eta)\xi - 3\mathfrak{u}_1(\eta)\xi + 4\mathfrak{u}_0(\eta)}{\eta+1} \right\}$$

(3.4.9)

elde edilir. Diğer yandan

$\mathcal{K}_{\eta,2}^{M,1}(e_2,\xi)$

$$= (\eta+1)\sum_{\rho=0}^{\eta}\left[\mathfrak{u}_0(\eta)\wp_{\eta-1,\rho}(\xi) + (-\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta))\wp_{\eta-1,\rho-1}(\xi)\right] \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} t^2 dt$$

$$= \frac{1}{3(\eta+1)^2}\sum_{\rho=0}^{\eta}\left[\mathfrak{u}_0(\eta)\wp_{\eta-1,\rho}(\xi) + (-\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta))\wp_{\eta-1,\rho-1}(\xi)\right]$$

$$\times (3\rho^2 + 3\rho + 1)$$

$$= \frac{1}{3(\eta+1)^2}\left\{ 3\mathfrak{u}_0(\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\rho^2 + 3(-\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta))\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\rho^2 \right.$$

$$+3\mathfrak{u}_0(\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi)\rho + 3(-\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta))\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi)\rho$$

$$\left. +\mathfrak{u}_0(\eta)\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho}(\xi) + (-\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta))\sum_{\rho=0}^{\eta}\wp_{\eta-1,\rho-1}(\xi) \right\}$$

olur. Eş. 3.2.9, Eş. 3.2.10, Eş. 3.2.5, Eş. 3.2.6 ve Eş. 3.2.2 den

$\mathcal{K}_{\eta,2}^{M,1}(e_2,\xi)$

$$= \frac{1}{3(\eta+1)^2}\{3\mathfrak{u}_0(\eta)[(\eta-2)(\eta-1)\xi^2 + (\eta-1)\xi] + 3(-\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta))$$

$$\times [(\eta-2)(\eta-1)\xi^2 + 3(\eta-1)\xi + 1] + 3\mathfrak{u}_0(\eta)(\eta-1)\xi + 3(-\mathfrak{u}_1(\eta)\xi$$

$$+\mathfrak{u}_0(\eta))[(\eta-1)\xi + 1] + 2\mathfrak{u}_0(\eta) - \mathfrak{u}_1(\eta)\xi\}$$

$$= \frac{1}{3}\Bigg\{3\xi^2(-\mathrm{u}_1(\eta)\xi + 2\mathrm{u}_0(\eta))\frac{\eta^2}{(\eta+1)^2} - 3\xi(-3\mathrm{u}_1(\eta)\xi^2 + 6\mathrm{u}_0(\eta)\xi + 4\mathrm{u}_1(\eta)\xi$$

$$-6\mathrm{u}_0(\eta))\frac{\eta}{(\eta+1)^2}$$

$$+\frac{-6\mathrm{u}_1(\eta)\xi^3 + 12\mathrm{u}_0(\eta)\xi^2 + 12\mathrm{u}_1(\eta)\xi^2 - 18\mathrm{u}_0(\eta)\xi - 7\mathrm{u}_1(\eta)\xi + 8\mathrm{u}_0(\eta)}{(\eta+1)^2}\Bigg\} \qquad (3.4.10)$$

bulunur.

Durum 2 den $(\mathrm{u}_0(\eta) < 0, \mathrm{u}_1(\eta) > 0)$ veya $(\mathrm{u}_0(\eta) > 0, \mathrm{u}_1(\eta) < 0)$ yazılabilir. Dolayısıyla Teorem 2.1.3 ile verilen Korovkin Teoreminin genişletilmiş formu $\mathcal{K}_{\eta,1}^{M,1}$ ve $\mathcal{K}_{\eta,2}^{M,1}$ operatörlerine uygulanabilir.

Yukarıdaki bağıntılar ve Teorem 2.1.3 kullanılırsa $l_1 = \lim_{\eta\to\infty} \mathrm{u}_1(\eta)$ olmak üzere

$$\lim_{\eta\to\infty} \mathcal{K}_{\eta,1}^{M,1}(\varphi,\xi) = -l_1(1+\xi)\varphi(\xi),$$

$$\lim_{\eta\to\infty} \mathcal{K}_{\eta,2}^{M,1}(\varphi,\xi) = (1 - l_1(1+\xi))\varphi(\xi),$$

olur. Buradan

$$\lim_{\eta\to\infty} \mathcal{K}_{\eta}^{M,1}(\varphi,\xi) = \varphi(\xi)$$

elde edilir.

3.5. Voronovskaja Tipli Teorem

3.5.1. Teorem

$\mathrm{u}_i(\eta), i = 0,1,$ Eş. 3.4.1, Eş. 3.4.2 ve $l_i = \lim_{\eta\to\infty} \mathrm{u}_i(\eta)\,, i = 0,1,$ koşullarını sağlayan yakınsak bir dizi olsun. Eğer $\varphi'' \in C[0,1]$ ise, bu durumda

$\lim_{\eta\to\infty} \eta\left(\mathcal{K}_{\eta}^{M,1}(\varphi,\xi) - \varphi(\xi)\right)$

$$= \frac{1}{2}(1-2\xi)(3l_1+4l_0)\varphi'(\xi) + \frac{1}{2}\xi(1-\xi)(2l_0+l_1)\varphi''(\xi),$$

yakınsaması [0,1] aralığında düzgündür (Gupta vd., 2019).

İspat

$\varphi''(\xi)$ mevcut olduğundan sabit $\xi \in [0,1]$ noktasındaki

$$\varphi(t) = \varphi(\xi) + (t-\xi)\varphi'(\xi) + (t-\xi)^2\frac{\varphi''(\xi)}{2} + \theta(t,\xi)(t-\xi)^2$$

Taylor formülüne Eş. 3.1.1 ile verilen $\mathcal{K}_\eta^{M,1}$ operatörü uygulanırsa

$$\mathcal{K}_\eta^{M,1}(\varphi,\xi) - \varphi(\xi) = \mathcal{K}_\eta^{M,1}(t-\xi,\xi)\varphi'(\xi) + \frac{1}{2}\mathcal{K}_\eta^{M,1}((t-\xi)^2,\xi)\varphi''(\xi)$$
$$+\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)$$

elde edilir. Burada $\theta \in C[0,1]$ ve $\lim\limits_{t\to\xi}\theta(t,\xi) = 0$’dır. Dolayısıyla

$$\eta\left[\mathcal{K}_\eta^{M,1}(\varphi,\xi) - \varphi(\xi)\right] = \varphi'(\xi)\eta\mathcal{K}_\eta^{M,1}(t-\xi,\xi) + \frac{1}{2}\varphi''(\xi)\eta\mathcal{K}_\eta^{M,1}((t-\xi)^2,\xi)$$
$$+\eta\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)$$

olup

$$\lim_{\eta\to\infty}\eta\left[\mathcal{K}_\eta^{M,1}(\varphi,\xi) - \varphi(\xi)\right]$$

$$= \varphi'(\xi)\lim_{\eta\to\infty}\eta\mathcal{K}_\eta^{M,1}(t-\xi,\xi) + \frac{1}{2}\varphi''(\xi)\lim_{\eta\to\infty}\eta\mathcal{K}_n^{M,1}((t-\xi)^2,\xi)$$

$$+\lim_{\eta\to\infty}\eta\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)$$

bulunur. Eş.3.3.2 ve Eş.3.3.1 den

$$\lim_{\eta\to\infty}\eta\mathcal{K}_\eta^{M,1}(t-\xi,\xi) = \frac{1}{2}(1-2\xi)(3l_1+4l_0)$$

ve

$$\lim_{\eta\to\infty} \eta\mathcal{K}_\eta^{M,1}((t-\xi)^2,\xi) = \xi(1-\xi)(2l_0+l_1)$$

olur. Böylece

$$\lim_{\eta\to\infty} \eta\left[\mathcal{K}_\eta^{M,1}(\varphi,\xi)-\varphi(\xi)\right]$$

$$= \frac{1}{2}(1-2\xi)(3l_1+4l_0)\varphi'(\xi) + \frac{1}{2}\xi(1-\xi)(2l_0+l_1)\varphi''(\xi)$$

$$+ \lim_{\eta\to\infty} \eta\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)$$

bulunur. Bu durumda

$$\lim_{\eta\to\infty} \eta\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi) = 0$$

olduğunu göstermek yeterlidir.

$\mathcal{K}_\eta^{M,1}$ pozitif operatörleri için Teorem 2.1.4 ile verilen Cauchy-Schwarz eşitsizliği uygulanırsa,

$$\eta\left|\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)\right| \le \eta\mathcal{K}_\eta^{M,1}(|\theta(t,\xi)(t-\xi)^2|,\xi)$$

$$\le \sqrt{\mathcal{K}_\eta^{M,1}(\theta^2(t,\xi),\xi)}\sqrt{\eta^2\mathcal{K}_\eta^{M,1}((t-\xi)^4,\xi)}$$

sonucuna ulaşılır. $\theta^2(\xi,\xi)=0$ ve $\theta^2(\cdot,\xi)\in C[0,1]$ olduğundan, Teorem 3.4.1 yardımıyla $\xi\in[0,1]$ için

$$\lim_{\eta\to\infty} \mathcal{K}_\eta^{M,1}(\theta^2(t,\xi),\xi) = 0$$

yakınsaması düzgündür. Diğer taraftan Lemma 3.3.1. (iii) den

$$\lim_{\eta\to\infty} \eta^2\mathcal{K}_\eta^{M,1}((t-\xi)^4,\xi) = 3\xi^2(1-\xi)^2(2l_0+l_1)$$

olur. Böylece

$$\lim_{\eta\to\infty} \eta\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi) = 0$$

elde edilir ve ispat tamamlanır.

Şimdi Eş. 3.1.1 ile tanımladığımız $\mathcal{K}_\eta^{M,1}$ operatörü pozitif olmadığında, yani $\mathfrak{u}_0(\eta)$ ve $\mathfrak{u}_1(\eta)$ dizileri Eş. 3.4.1 ve Eş. 3.4.3 koşullarını sağladığında, Teorem 3.5.1 deki sonuçları aşağıdaki şekilde genişletebiliriz.

3.5.2. Teorem

$\mathfrak{u}_i(\eta), i = 0{,}1$, dizileri Eş. 3.4.1, Eş. 3.4.3 ve $l_i = \lim_{\eta\to\infty} \mathfrak{u}_i(\eta)\,, i = 0{,}1$, koşullarını sağlayan sınırlı yakınsak diziler olsun. Eğer $\varphi \in C[0{,}1]$ ve φ'' belirli bir $\xi \in [0{,}1]$ noktasında mevcut ise,

$$\lim_{\eta\to\infty} \eta\left[\mathcal{K}_\eta^{M,1}(\varphi,\xi) - \varphi(\xi)\right] = \frac{1}{2}(1-2\xi)(3l_1 + 4l_0)\varphi'(\xi) + \frac{1}{2}\xi(1-\xi)(2l_0 + l_1)\varphi''(\xi) \qquad (3.5.1)$$

gerçeklenir. Ayrıca, $\varphi'' \in C[0{,}1]$ ise Eş. 3.5.1 ifadesi $[0{,}1]$ aralığında düzgün olarak sağlanır (Gupta vd., 2019).

İspat

Teorem 3.5.1'in ispatında olduğu gibi Taylor formülüne Eş. 3.1.1 ile tanımlanan $\mathcal{K}_\eta^{M,1}$ operatörünü uygulamak

$$\lim_{\eta\to\infty} \eta\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi) = 0 \qquad (3.5.2)$$

olduğunu göstermek için yeterlidir.

Eş. 3.4.1 ve Eş. 3.4.3 koşulları altında $\mathcal{K}_\eta^{M,1}$ pozitif lineer operatör olmadığından burada Cauchy-Schwarz eşitsizliğini kullanamayız.

$\varepsilon > 0$ verilsin. $|t-\xi| < \delta$ ise bu durumda $|\theta(t,\xi)| < \varepsilon$ olacak şekilde bir $\delta > 0$ sayısı mevcuttur.

$$A = \left\{\rho : \left|\frac{\rho}{\eta} - \xi\right| < \delta, \rho = 0,1,2, \dots, \eta\right\}$$

ve

$$B = \left\{\rho : \left|\frac{\rho}{\eta} - \xi\right| \geq \delta, \rho = 0,1,2, \dots, \eta\right\}$$

ile gösterelim.

$\mathfrak{u}_i(\eta), i = 0,1,$ dizileri sınırlı olduğundan $|\mathfrak{u}(\xi,\eta)| < N$ şeklinde bir $N > 0$ sabiti vardır. Eş. 3.1.4 ten

$$\left|\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)\right| = \left|(\eta+1)\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\left\{\mathfrak{u}(\xi,\eta)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} \theta(t,\xi)(t-\xi)^2 dt\right.\right.$$

$$\left.\left.+\mathfrak{u}(1-\xi,\eta)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} \theta(t,\xi)(t-\xi)^2 dt\right\}\right|$$

$$\leq (\eta+1)\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\left\{|\mathfrak{u}(\xi,\eta)|\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} |\theta(t,\xi)|(t-\xi)^2 dt\right.$$

$$\left.+|\mathfrak{u}(1-\xi,\eta)|\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} |\theta(t,\xi)|(t-\xi)^2 dt\right\}$$

$$\leq N(\eta+1)\left\{\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} |\theta(t,\xi)|(t-\xi)^2 dt\right.$$

$$+\sum_{\rho=0}^{\eta-1} \mathcal{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} |\theta(t,\xi)|(t-\xi)^2\,dt \Bigg\} \tag{3.5.3}$$

elde edilir.

$\rho \in A$ olsun. Bu durumda $|\theta(t,\xi)| < \varepsilon$ olur. Dolayısıyla

$$\left|\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)\right| \le N(\eta+1)\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} |\theta(t,\xi)|(t-\xi)^2\,dt$$

$$+N(\eta+1)\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} |\theta(t,\xi)|(t-\xi)^2\,dt$$

$$\le \varepsilon N(\eta+1)\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi)$$

$$\times\left(\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} (t-\xi)^2\,dt + \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} (t-\xi)^2\,dt\right)$$

bulunur.

$$\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} (t-\xi)^2\,dt$$

$$=\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} t^2\,dt - 2\xi\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} t\,dt + \xi^2\sum_{\rho=0}^{\eta} \mathcal{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} dt$$

$$= \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{1}{3}\left[\left(\frac{\rho+1}{\eta+1}\right)^3 - \left(\frac{\rho}{\eta+1}\right)^3\right]$$

$$-2\xi \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{1}{2}\left[\left(\frac{\rho+1}{\eta+1}\right)^2 - \left(\frac{\rho}{\eta+1}\right)^2\right]$$

$$+\xi^2 \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \left[\left(\frac{\rho+1}{\eta+1}\right) - \left(\frac{\rho}{\eta+1}\right)\right]$$

$$= \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{3\rho^2+3\rho+1}{3(\eta+1)^3} - \xi \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{2\rho+1}{(\eta+1)^2}$$

$$+\xi^2 \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{1}{\eta+1}$$

$$= \frac{1}{(\eta+1)^3} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\rho^2 + \frac{1}{(\eta+1)^3} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\rho + \frac{1}{3(\eta+1)^3}$$

$$-\frac{2\xi}{(\eta+1)^2} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\rho - \frac{\xi}{(\eta+1)^2} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) + \frac{\xi^2}{\eta+1}$$

$$= \frac{(\eta-2)(\eta-1)\xi^2 + (\eta-1)\xi}{(\eta+1)^3} + \frac{(\eta-1)\xi}{(\eta+1)^3} + \frac{1}{3(\eta+1)^3} - \frac{2\xi^2(\eta-1)}{(\eta+1)^2}$$

$$-\frac{\xi}{(\eta+1)^2} + \frac{\xi^2}{\eta+1}$$

$$= \frac{3(\eta-2)(\eta-1)\xi^2 + 6(\eta-1)\xi + 1 - 6\xi^2(\eta-1)(\eta+1) - 3\xi(\eta+1)}{3(\eta+1)^3}$$

$$+\frac{\xi^2(\eta+1)^2}{(\eta+1)^3} \qquad (3.5.4)$$

elde edilir. Ayrıca

$$\sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} (t-\xi)^2\, dt$$

$$= \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} t^2\, dt - 2\xi \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} t\, dt + \xi^2 \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} dt$$

$$= \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{1}{3}\left[\left(\frac{\rho+2}{\eta+1}\right)^3 - \left(\frac{\rho+1}{\eta+1}\right)^3\right]$$

$$-2\xi \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{1}{2}\left[\left(\frac{\rho+2}{\eta+1}\right)^2 - \left(\frac{\rho+1}{\eta+1}\right)^2\right]$$

$$+\xi^2 \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \left[\left(\frac{\rho+2}{\eta+1}\right) - \left(\frac{\rho+1}{\eta+1}\right)\right]$$

$$= \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{3\rho^2+9\rho+7}{3(\eta+1)^3} - \xi \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{2\rho+3}{(\eta+1)^2}$$

$$+\xi^2 \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) \frac{1}{\eta+1}$$

$$= \frac{1}{(\eta+1)^3} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\rho^2 + \frac{3}{(\eta+1)^3} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\rho + \frac{7}{3(\eta+1)^3} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)$$

$$- \frac{2\xi}{(\eta+1)^2} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi)\rho - \frac{3\xi}{(\eta+1)^2} \sum_{\rho=0}^{\eta} \wp_{\eta-1,\rho}(\xi) + \frac{\xi^2}{\eta+1}$$

$$= \frac{(\eta-2)(\eta-1)\xi^2 + (\eta-1)\xi}{(\eta+1)^3} + \frac{3(\eta-1)\xi}{(\eta+1)^3} + \frac{7}{3(\eta+1)^3} - \frac{2\xi^2(\eta-1)}{(\eta+1)^2}$$

$$- \frac{3\xi}{(\eta+1)^2} + \frac{\xi^2}{(\eta+1)}$$

$$= \frac{3(\eta-2)(\eta-1)\xi^2 + 12(\eta-1)\xi + 7 - 6\xi^2(\eta-1)(\eta+1) - 9\xi(\eta+1)}{3(\eta+1)^3}$$

$$+ \frac{3\xi^2(\eta+1)^2}{3(\eta+1)^3} \tag{3.5.5}$$

bulunur. Eş. 3.5.4 ve Eş. 3.5.5 den

$$\left|\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)\right|$$

$$\leq \varepsilon N(\eta+1)\frac{6(\eta-2)(\eta-1)\xi^2 + 18(\eta-1)\xi + 8 - 12\xi^2(\eta^2-1) - 12\xi(\eta+1)}{3(\eta+1)^3}$$

$$\times 6\xi^2(\eta+1)^2$$

$$\leq \frac{2\varepsilon N}{3(\eta+1)}[3\xi\eta(1-\xi) + 15\xi^2 - 15\xi + 4] \tag{3.5.6}$$

eşitsizliği elde edilir.

$\rho \in B$ olsun. $M = \sup_{0\leq t\leq 1} |\theta(t,\xi)|\,(t-\xi)^2$ ile gösterelim. O halde

$$|\theta(t,\xi)|(t-\xi)^2 \leq \frac{M}{\delta^4}\left(\frac{\rho}{\eta} - \xi\right)^4$$

olur. Eş. 3.5.3 den

$$\left|\mathcal{K}_\eta^{M,1}(\theta(t,\xi)(t-\xi)^2,\xi)\right|$$

$$\leq \frac{2MN}{\delta^4}\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\left(\frac{\rho}{\eta} - \xi\right)^4$$

$$= \frac{2MN}{\delta^4}\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\left\{\frac{\rho^4}{\eta^4} - 4\frac{\rho^3}{\eta^3}\xi + 6\frac{\rho^2}{\eta^2}\xi^2 - 4\frac{\rho}{\eta}\xi^3 + \xi^4\right\}$$

$$= \frac{2MN}{\delta^4}\left\{\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\frac{\rho^4}{\eta^4} - 4\xi\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\frac{\rho^3}{\eta^3} + 6\xi^2\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\frac{\rho^2}{\eta^2}\right.$$

$$-4\xi^3 \sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\frac{\rho}{\eta} + \xi^4 \sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi)\Bigg\}$$

$$= \frac{2MN}{\delta^4}\Bigg\{\frac{(\eta-1)(\eta-2)(\eta-3)(\eta-4)\xi^4 + 6(\eta-1)(\eta-2)(\eta-3)\xi^3 + (\eta-1)\xi}{\eta^4}$$

$$-\frac{4\xi[(\eta-1)(\eta-2)(\eta-3)\xi^3 + 3(\eta-1)(\eta-2)\xi^2 + (\eta-1)\xi]}{\eta^3}$$

$$+\frac{6\xi^2[(\eta-1)(\eta-2)\xi^2 + (\eta-1)\xi]}{\eta^2} - \frac{4\xi^3[(\eta-1)\xi]}{\eta} + \xi^4\Bigg\}$$

$$= \frac{2MN}{\delta^4\eta^4}\{3\xi^2(1-\xi)^2\eta^2 + \xi(1-\xi)(26\xi^2 - 16\xi + 1)\eta$$

$$+(2\xi-1)(12\xi^2 - 12\xi + 1)\xi\} \qquad (3.5.7)$$

üst sınırı elde edilir.

Eş. 3.5.6 ve Eş. 3.5.7 yardımıyla, Eş. 3.5.2 gösterilmiş olur. Bu da Teorem 3.5.2’nin ispatını tamamlar.

3.6. Yaklaşım Oranı

3.6.1. Teorem

Eğer $\varphi(\xi)$ fonksiyonu $\xi \in [0,1]$ için sınırlı, $\mathfrak{u}_0(\eta)$, $\mathfrak{u}_1(\eta)$ dizileri Eş. 3.4.1 koşulunu sağlar ve $\mathfrak{u}_1(\eta)$ sınırlı bir dizi ise bu durumda $\eta > 5$ için

$$\left\|\mathcal{K}_\eta^{M,1}(\varphi) - \varphi\right\|_{C[0,1]} \leq \frac{3}{2}[3|\mathfrak{u}_1(\eta)| + 1]\omega\left(\varphi, \frac{1}{\sqrt{\eta}}\right),$$

eşitsizliği gerçeklenir (Gupta vd., 2019).

İspat

Lemma 3.2.1. (i), Eş. 3.4.1 koşulu ve Eş. 3.1.4 gösteriminden

$$\mathcal{K}_{\eta}^{M,1}(\varphi,\xi)-\varphi(\xi)$$

$$=(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\left\{\mathfrak{u}(\xi,\eta)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}\varphi(t)\,dt+\mathfrak{u}(1-\xi,\eta)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}\varphi(t)\,dt\right\}$$

$$-\varphi(\xi)(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\left\{\mathfrak{u}(\xi,\eta)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}dt+\mathfrak{u}(1-\xi,\eta)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}dt\right\}$$

şeklinde yazılabilir. Basit hesaplamalarla

$$\left|\mathcal{K}_{\eta}^{M,1}(\varphi,\xi)-\varphi(\xi)\right|$$

$$=\left|(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\left\{\mathfrak{u}(\xi,\eta)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}[\varphi(t)-\varphi(\xi)]\,dt\right.\right.$$

$$\left.\left.+\mathfrak{u}(1-\xi,\eta)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}[\varphi(t)-\varphi(\xi)]\,dt\right\}\right|$$

$$\leq(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\left\{|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}[\varphi(t)-\varphi(\xi)]\,dt\right.$$

$$\left.+|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}[\varphi(t)-\varphi(\xi)]\,dt\right\}$$

$$=(\eta+1)|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}[\varphi(t)-\varphi(\xi)]\,dt$$

$$+(\eta+1)|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}[\varphi(t)-\varphi(\xi)]\,dt$$

$$\leq(\eta+1)|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}\omega\,(\varphi,|t-\xi|)dt$$

$$+(\eta+1)|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}\omega\,(\varphi,|t-\xi|)dt$$

elde edilir. Süreklilik modülünün iyi bilinen

$$\omega(\varphi,|t-\xi|)\leq(1+\sqrt{\eta}|t-\xi|)\omega(\varphi,\frac{1}{\sqrt{\eta}})$$

özelliği kullanılırsa

$$\left|\mathcal{K}_\eta^{M,1}(\varphi,\xi)-\varphi(\xi)\right|$$

$$\leq(\eta+1)|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}(1+\sqrt{\eta}|t-\xi|)\omega(\varphi,\frac{1}{\sqrt{\eta}})\,dt$$

$$+(\eta+1)|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(1+\sqrt{\eta}|t-\xi|)\omega(\varphi,\frac{1}{\sqrt{\eta}})\,dt$$

$$=(\eta+1)|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}\omega(\varphi,\frac{1}{\sqrt{\eta}})\,dt$$

$$+(\eta+1)|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}\sqrt{\eta}|t-\xi|\,\omega(\varphi,\frac{1}{\sqrt{\eta}})dt$$

$$+(\eta+1)|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)dt$$

$$+(\eta+1)|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}\sqrt{\eta}|t-\xi|\,\omega(\varphi,\frac{1}{\sqrt{\eta}})dt$$

$$=|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)+\sqrt{\eta}(\eta+1)|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)$$

$$\times\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}|t-\xi|\,dt+|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)+\sqrt{\eta}(\eta+1)$$

$$+\sqrt{\eta}(\eta+1)|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}|t-\xi|\,dt$$

$$=|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\left[1+\sqrt{\eta}(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}|t-\xi|\,dt\right]$$

$$+|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\left[1+\sqrt{\eta}(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}|t-\xi|\,dt\right]$$

bulunur. Teorem 2.1.5 ile verilen Hölder eşitsizliğinden

$$\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} |t-\xi|\, dt \le \left(\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} (t-\xi)^2\, dt \right)^{1/2} \left(\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} dt \right)^{1/2}$$

$$= \left(\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} (t-\xi)^2\, dt \right)^{1/2} \frac{1}{\sqrt{\eta+1}}$$

olur. Böylece

$$\left|\mathcal{K}_\eta^{M,1}(\varphi,\xi)-\varphi(\xi)\right|$$

$$\le |\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\,\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\left[1+\sqrt{\eta}(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\left(\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} (t-\xi)^2\, dt \right)^{1/2}\right.$$

$$\left.\times\frac{1}{\sqrt{\eta+1}}\right]+|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\,\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\left[1+\sqrt{\eta}(\eta+1)\sum_{\rho=0}^{\eta-1}\wp_{\eta-1,\rho}(\xi)\right.$$

$$\left.\times\left(\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} (t-\xi)^2\, dt \right)^{1/2} \frac{1}{\sqrt{\eta+1}}\right]$$

elde edilir. Teorem 2.1.4 ile verilen Cauchy-Shwarz eşitsizliğinden

$$\left|\mathcal{K}_\eta^{M,1}(\varphi,\xi)-\varphi(\xi)\right|$$

$$\le |\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\,\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\left[1+\sqrt{\eta}(\eta+1)\frac{1}{\sqrt{\eta+1}}\right.$$

$$\times\sum_{\rho=0}^{\eta-1}\sqrt{\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}(t-\xi)^2\,dt}\sqrt{\mathcal{p}_{\eta-1,\rho}(\xi)}\Bigg]$$

$$+|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\Bigg[1+\sqrt{\eta}(\eta+1)\frac{1}{\sqrt{\eta+1}}$$

$$\times\sum_{\rho=0}^{\eta-1}\sqrt{\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt}\sqrt{\mathcal{p}_{\eta-1,\rho}(\xi)}\Bigg]$$

$$\leq|\mathfrak{u}_1(\eta)\xi+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\Bigg[1+\frac{\sqrt{\eta}}{\sqrt{\eta+1}}(\eta+1)$$

$$\times\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\right)^{\frac{1}{2}}\Bigg]$$

$$+|\mathfrak{u}_1(\eta)(1-\xi)+\mathfrak{u}_0(\eta)|\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\Bigg[1+\frac{\sqrt{\eta}}{\sqrt{\eta+1}}(\eta+1)$$

$$\times\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\right)^{\frac{1}{2}}\Bigg]$$

bulunur. Burada

$$1+\frac{\sqrt{\eta}}{\sqrt{\eta+1}}(\eta+1)\leq 1+(\eta+1)$$

olduğu açıktır. Dolayısıyla

$$\left|\mathcal{K}_\eta^{M,1}(\varphi,\xi)-\varphi(\xi)\right|$$

$$\leq |\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta)|\omega\left(\varphi, \frac{1}{\sqrt{\eta}}\right)\left[1 + (\eta+1)\left(\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} (t-\xi)^2\, dt\right)\right]^{\frac{1}{2}}$$

$$+|\mathfrak{u}_1(\eta)(1-\xi) + \mathfrak{u}_0(\eta)|\omega\left(\varphi, \frac{1}{\sqrt{\eta}}\right)$$

$$\times\left[1 + (\eta+1)\left(\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi) \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}} (t-\xi)^2\, dt\right)\right]^{\frac{1}{2}}$$

bulunur.

Eş. 3.4.1 den

$$|\mathfrak{u}_1(\eta)\xi + \mathfrak{u}_0(\eta)| \leq |\mathfrak{u}_1(\eta)||\xi| + |\mathfrak{u}_0(\eta)| \leq |\mathfrak{u}_1(\eta)| + |\mathfrak{u}_0(\eta)|$$

$$= |\mathfrak{u}_1(\eta)| + \left|\frac{1-\mathfrak{u}_1(\eta)}{2}\right| \leq |\mathfrak{u}_1(\eta)| + \frac{1}{2} + \frac{|\mathfrak{u}_1(\eta)|}{2}$$

$$= \frac{3}{2}|\mathfrak{u}_1(\eta)| + \frac{1}{2}$$

elde edilir. Benzer şekilde

$$|\mathfrak{u}_1(\eta)(1-\xi) + \mathfrak{u}_0(\eta)| \leq \frac{3}{2}|\mathfrak{u}_1(\eta)| + \frac{1}{2}$$

bulunur. Böylece

$$\left|\mathcal{K}_\eta^{M,1}(\varphi,\xi) - \varphi(\xi)\right|$$

$$\leq \frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi, \frac{1}{\sqrt{\eta}}\right)\left[1 + (\eta+1)\left(\sum_{\rho=0}^{\eta-1} \wp_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} (t-\xi)^2\, dt\right)\right]^{\frac{1}{2}}$$

$$+\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)\left[1+(\eta+1)\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}\right]$$

$$=\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)+\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)(\eta+1)$$

$$\times\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}+\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)$$

$$+\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)(\eta+1)\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}$$

$$=2\,\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)+\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)(\eta+1)$$

$$\times\left[\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}+\left(\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}\right]$$

$$\leq 2\,\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)+\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)(\eta+1)$$

$$\times\left\{\left[\sum_{\rho=0}^{\eta-1}\mathcal{p}_{\eta-1,\rho}(\xi)\left(\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}}+\int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}\right)(t-\xi)^2dt\right]^{\frac{1}{2}}\right.$$

$$+\left[\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi)\left(\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+1}{\eta+1}} + \int_{\frac{\rho+1}{\eta+1}}^{\frac{\rho+2}{\eta+1}}\right)(t-\xi)^2 dt\right]^{\frac{1}{2}}\Bigg\}$$

$$\leq 2\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)+2\frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)(\eta+1)\left[\sum_{\rho=0}^{\eta-1}\mathscr{p}_{\eta-1,\rho}(\xi)\right.$$

$$\left.\times\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt\right]^{\frac{1}{2}}$$

elde edilir. Sonuç olarak

$$\left|\mathcal{K}_\eta^{M,1}(\varphi,\xi)-\varphi(\xi)\right|$$

$$\leq \frac{3|\mathfrak{u}_1(\eta)|+1}{2}\omega\left(\varphi;\frac{1}{\sqrt{\eta}}\right)\left[2+2(\eta+1)\left(\sum_{\rho=0}^{\eta-1}\mathscr{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt\right)^{\frac{1}{2}}\right]$$

bulunur. Buradan

$$\sum_{\rho=0}^{\eta-1}\mathscr{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t-\xi)^2\,dt$$

$$=\sum_{\rho=0}^{\eta-1}\mathscr{p}_{\eta-1,\rho}(\xi)\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}}(t^2-2t\xi+\xi^2)dt$$

$$=\sum_{\rho=0}^{\eta-1}\mathscr{p}_{\eta-1,\rho}(\xi)\left\{\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}}t^2\,dt-2\xi\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}}t\,dt+\xi^2\int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}}dt\right\}$$

$$= \sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi) \left\{ \frac{1}{3}\left[\left(\frac{\rho+2}{\eta+1}\right)^3 - \left(\frac{\rho}{\eta+1}\right)^3\right] - 2\xi \left[\left(\frac{\rho+2}{\eta+1}\right)^2 - \left(\frac{\rho}{\eta+1}\right)^2\right]\right.$$

$$\left. + \xi^2 \left[\left(\frac{\rho+2}{\eta+1}\right) - \left(\frac{\rho}{\eta+1}\right)\right]\right\}$$

$$= \sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi) \left\{ \frac{1}{3(\eta+1)^3}(6\rho^2 + 12\rho + 8) - \frac{2\xi}{(\eta+1)^2}(4\rho+4) + \frac{2\xi^2}{\eta+1}\right\}$$

$$= \frac{1}{3(\eta+1)^3}\left\{6\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi)\rho^2 + 12\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi)\,\rho + 8\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi)\right\}$$

$$- \frac{2\xi}{(\eta+1)^2}\left\{4\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi)\,\rho + 4\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi)\right\} + \frac{2\xi^2}{\eta+1}\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi)$$

olur. Eş. 3.2.9, Eş. 3.2.5, Eş. 3.2.2 ifadeleri yukarıdaki eşitlikte yerine yazılırsa

$$\sum_{\rho=0}^{\eta-1} \mathscr{p}_{\eta-1,\rho}(\xi) \int_{\frac{\rho}{\eta+1}}^{\frac{\rho+2}{\eta+1}} (t-\xi)^2\, dt$$

$$= \frac{1}{3(\eta+1)^3}[6(\eta-2)(\eta-1)\xi^2 + 18(\eta-1)\xi + 8] - \frac{2\xi^2}{\eta+1}[4(\eta-1)\xi + 4]$$

$$+ \frac{2\xi^2}{\eta+1}$$

$$= \frac{2(5-\eta)}{(\eta+1)^3}\left[\xi^2 - \xi + \frac{4}{3(5-\eta)}\right] := g(\xi)$$

olur ve $\eta > 5$ için

$$\max_{\xi\in[0,1]} g(\xi) = g\left(\frac{1}{2}\right) = \frac{3\eta+1}{6(\eta+1)^3}$$

bulunur. Böylece

$$\left|\mathcal{K}_{\eta}^{M,1}(\varphi,\xi)-\varphi(\xi)\right| \leq \frac{3}{2}(3|\mathrm{u}_1(\eta)|+1)\omega\left(\varphi,\frac{1}{\sqrt{\eta}}\right)$$

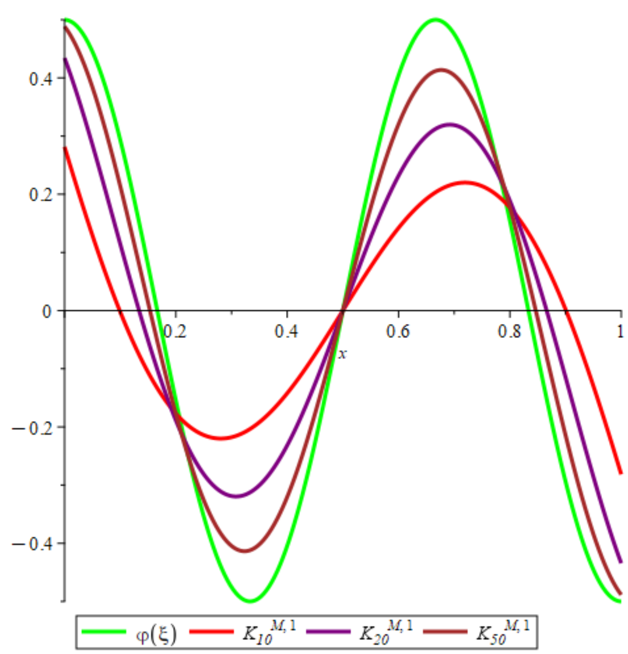
sağlanır ve ispat tamamlanır.

3.7. Nümerik Örnekler

Şimdi Eş. 3.1.1 ile tanımlanan $\mathcal{K}_{\eta}^{M,1}$ operatör dizisinin bazı fonksiyonlara yaklaşımını grafik ve nümerik örneklerle verelim.

3.7.1. Örnek

$\varphi(\xi)=\frac{1}{2}\cos(3\pi\xi)$, $\mathrm{u}_1(\eta)=\frac{1}{2\eta+1}$, $\mathrm{u}_0(\eta)=\frac{\eta}{2\eta+1}$ olsun. Şekil 3.7.1. de sırasıyla $\eta=10, \eta=20$ ve $\eta=50$ değerleri için $\mathcal{K}_{\eta}^{M,1}$ operatörlerinin φ fonksiyonuna yaklaşımı gösterilmiştir.

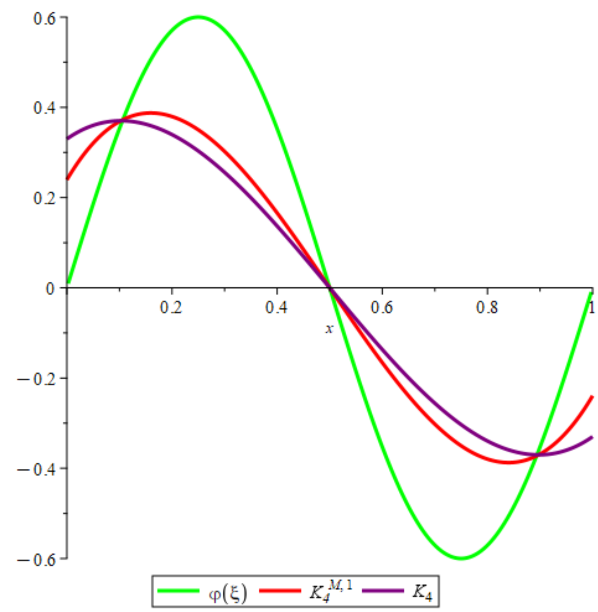


Şekil 3.7.1. $\eta=10,20,50$ için $\mathcal{K}_{\eta}^{M,1}$ operatörlerinin φ fonksiyonuna yaklaşımı

Şekil 3.7.1'de η'nin artan değerleri için $\mathcal{K}_{\eta}^{M,1}$ operatörünün φ fonksiyonuna daha iyi bir yaklaşım sağladığı görülmektedir.

3.7.2. Örnek

$\varphi(\xi) = \frac{3}{5}\sin(2\pi\xi)$, $\eta = 4$, $\mathfrak{u}_1(\eta) = \frac{1}{2\eta+1}$, $\mathfrak{u}_0(\eta) = \frac{\eta}{2\eta+1}$ için $\mathcal{K}_{\eta}$ klasik Kantorovich ve $\mathcal{K}_{\eta}^{M,1}$ karışık Kantorovich operatörleri yardımıyla φ fonksiyonuna yaklaşımı Şekil 3.7.2 de gösterilmiştir.



Şekil 3.7.2. $\eta = 4$ için $\mathcal{K}_{\eta}^{M,1}$ ve $\mathcal{K}_{\eta}$ operatörlerinin φ fonksiyonuna yaklaşımı

Şekil 3.7.2'de $\eta = 4$ için $\mathcal{K}_{\eta}^{M,1}$ operatörünün $\mathcal{K}_{\eta}$ operatörüne göre φ fonksiyonuna daha iyi yaklaştığı açıkça görülmektedir.

3.7.3. Örnek

$\varphi(\xi) = 3^{\xi}\sin(\frac{1}{2}\pi\xi)$, $u_1(\eta) = \frac{1}{2\eta+1}$, $u_0(\eta) = \frac{\eta}{2\eta+1}$, $\xi = 0.3$ ve η nın farklı değerleri için $\mathcal{K}_\eta^{M,1}$ karışık Kantorovich operatörleri ve $\mathcal{K}_\eta$ klasik Kantorovich operatörlerinin φ fonksiyonu ile farkının mutlak değeri Tablo 3.7.1 de gösterilmiştir.

η	$\lvert\mathcal{K}_\eta(\varphi, 0.3) - \varphi(0.3)\rvert$	$\lvert\mathcal{K}_\eta^{M,1}(\varphi, 0.3) - \varphi(0.3)\rvert$
5	0.1403590862	0.0551297813
25	0.0338901537	0.0136546300
50	0.0173938082	0.0122668963
100	0.0088134383	0.0035900642
500	0.0017817693	0.0007282181

Tablo 3.7.1. η'nin artan değerleri için $\xi = 0.3$ noktasında $\mathcal{K}_\eta$ ve $\mathcal{K}_\eta^{M,1}$ operatörlerinin hata tahminleri

Tablo 3.7.1. de $\xi = 0.3$ noktasında $\mathcal{K}_\eta^{M,1}$ operatörünün φ fonksiyonuna yaklaşımının $\mathcal{K}_\eta$ operatöründen daha iyi olduğu görülür.

4. SONUÇ

Yaklaşım teorisinin en popüler, pratik, önemli ve sıkça kullanılan teoremlerinden biri olan Weierstrass Yaklaşım Teoremi'nin en basit ve en zarif ispatlarından biri 1912 yılında Bernstein tarafından binom dağılımı kullanılarak elde edilen Bernstein operatörleri yardımıyla verilmiştir. Lineer pozitif operatörlerin iyi özelliklerinin olmasının yanı sıra bu tip operatörlerin oluşturulmasının daha basit, kullanımı ve analizinin daha kolay olması, geçmişten günümüze Bernstein operatörlerinin çeşitli genelleştirmelerinin tanımlanmasına sebep olmuştur. Diğer taraftan, lineer pozitif operatörlerin yakınsama oranının çok yavaş olması, hesaplama bakımından bu tip operatörleri her zaman çekici göstermemektedir. Bu sorunu çözmek amacıyla, yaklaşım derecesini iyileştirmek için literatürde bazı çalışmalar yapılmıştır. Yakın zamanda, Khosravian-Arab vd. (2018) Bernstein operatörlerinden daha iyi özelliklere sahip bazı operatörleri inşâ etmek için yeni ve basit bir yaklaşım sunarak, literatürde karışık Bernstein tipli operatörler olarak bilinen klasik Bernstein operatörlerinin üç modifikasyonunu inşâ etmişlerdir. Daha sonra, Khosravian-Arab vd. nin yaklaşımından esinlenerek, bazı bilim insanları karışık Bernstein tipli operatörlerin Kantorovich, Stancu, Durrmeyer, genuine Bernstein-Durrmeyer, Baskakov gibi çeşitli genelleştirmelerini yoğun olarak çalışmışlardır. Bu tez çalışmasında, Gupta vd. (2019) tarafından tanımlanan Bernstein-Kantorovich-tipli operatörlerin yeni bir çeşidi dikkate alınarak, Khosravian-Arab vd. (2018) tarafından elde edilen bazı sonuçlar klasik Kantorovich operatörlerine genişletilmiştir. Öncelikle, karışık Kantorovich tipli operatörler dizisinin sürekli fonksiyonlar uzayında düzgün yaklaşımı incelenmiş ve daha sonra Voronovskaja tipli teoremler ispatlanmıştır. Ayrıca, yeni tanımlanan operatör için süreklilik modülü kullanılarak yaklaşım hızı elde edilmiş ve operatörlerin bazı fonksiyonlara yaklaşımı grafiklerle gösterilmiştir. 2019 yılında Gupta vd. nin elde ettiği bazı sonuçların, Çetin (2023) tarafından tanımlanan karışık Stancu tipli operatörler ile Acu vd. (2024b) tarafından tanımlanan karışık Baskakov tipli operatörlerin Kantorovich tipinde genelleştirmelerinin tanımlanması ve yeni tanımlanan operatör dizilerinin yaklaşım özelliklerinin incelenmesi problemlerinin çözümüne ilişkin bilgiler sunduğu düşünülmektedir.

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