ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

APPLICATIONS OF GENERALIZED GUGLIELMO NUMBERS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

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ADVISOR: Prof. Dr. Yüksel SOYKAN

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Bahadır YILMAZ

ABSTRACT

Master of Science Thesis

APPLICATIONS OF GENERALIZED GUGLIELMO NUMBERS

Bahadır YILMAZ

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Prof. Dr. Yüksel SOYKAN

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In this thesis, we define generalized Gaussian Guglielmo numbers, dual generalized Guglielmo numbers, hyperbolic generalized Guglielmo numbers, and dual hyperbolic generalized Guglielmo numbers. Moreover, we define recurrence relations, generating function, sum formulas, matrix formulation, Simpson's formula, and some identities on these numbers. We summarize the chapters, given in the thesis, as follows.

In Chapter 1, we present some basic definitions, recurrence relations, Binet's formulas, matrix formulation, and sum formulas related to generalized Guglielmo numbers. In addition, we give some properties about Gaussian numbers, dual numbers, dual hyperbolic numbers and make a literature search about Gaussian numbers, dual numbers, hyperbolic and dual hyperbolic numbers.

In Chapter 2, we define Gaussian generalized Guglielmo numbers in detail, and focus on four specific cases: Gaussian triangular numbers, Gaussian triangular-Lucas numbers, Gaussian

ABSTRACT (continued)

oblong numbers, and Gaussian pentagonal numbers. In addition, we present some identities and matrices related to these sequences, as well as recurrence relations, Binet's formulas, generating functions, Simpson's formulas, and summation formulas. This chapter includes our original study.

In Chapter 3, we introduce the generalized hyperbolic Guglielmo numbers. We delve into various specific instances, including hyperbolic triangular numbers, hyperbolic triangular-Lucas numbers, hyperbolic oblong numbers, and hyperbolic pentagonal numbers. We present Binet's formulas, generating functions and summation formulas for these numbers. Furthermore, we provide Catalan's and Cassini's identities and matrices associated with these sequences. This chapter consists our original study.

In Chapter 4, we investigate the generalized dual hyperbolic Guglielmo numbers and then various special cases are explored (including dual triangular numbers, dual triangular-Lucas numbers, dual oblong numbers, and dual pentagonal numbers). Binet's formulas, generating functions, and summation formulas for these numbers are presented. Additionally, Catalan's and Cassini's identities are provided, along with matrices associated with these sequences. This chapter includes our original study.

In Chapter 5, the generalized dual hyperbolic Guglielmo numbers are introduced. Various special cases are explored (including dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers, and dual hyperbolic pentagonal numbers). Binet's formulas, generating functions and summation formulas for these numbers are presented. Moreover, Catalan's and Cassini's identities are provided, along with matrices associated with these sequences. This chapter includes our original study.

Keywords: Gaussian Generalized Guglielmo numbers, Dual Generalized Guglielmo numbers, Hyperbolic Generalized Guglielmo numbers, Dual hyperbolic Generalized Guglielmo numbers.

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ÖZET

Yüksek Lisans Tezi

GENELLEŞTİRİLMİŞ GUGLIELMO SAYILARININ UYGULAMALARI

Bahadır YILMAZ

Zonguldak Bülent Ecevit Üniversitesi
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Matematik Anabilim Dalı

Tez Danışmanı: Prof. Dr. Yüksel SOYKAN
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Bu tezde, genelleştirilmiş Gaussian Guglielmo sayılarını, dual genelleştirilmiş Guglielmo sayılarını, hiperbolik genelleştirilmiş Guglielmo sayılarını ve dual hiperbolik genelleştirilmiş Guglielmo sayılarını tanımlandı. Ayrıca, bu sayılar üzerine rekürans bağıntıları, üreteç fonksiyonu, toplam formülleri, matris formülasyonu, Simpson formülü ve bazı önemli ifadeleri tanımlandı. Tezde verilen bölümleri şu şekilde özetlenebilir.

Birinci bölümde, genelleştirilmiş Guglielmo sayılarına ilişkin temel tanımlamaları, rekürans bağıntısını, Binet formüllerini, matris formülasyonunu ve genelleştirilmiş Guglielmo sayıları ile ilgili bazı özellikleri sunuldu. Ayrıca, Gaussian sayılar, dual sayılar, dual hiperbolik sayılar hakkında bazı özellikler verildi ve Gaussian sayılar, dual sayılar, hiperbolik ve dual hiperbolik sayılarla ilgili bir literatür araştırması yapıldı.

İkinci bölümde, Gaussian genelleştirilmiş Guglielmo sayılarını detaylı olarak tanımlandı ve Gaussian üçgensel sayılar, Gaussian üçgensel-Lucas sayıları, Gaussian dikdörtgen sayılar ve

ÖZET (devam ediyor)

Gaussian beşgen sayılar olmak üzere dört özel sayı dizisine odaklanıldı. Ayrıca, bu dizilerle ilgili bazı ifadeler ve matrisler, bu dizilerle ilgili rekürans ilişkileri, Binet formülleri, üreteç fonksiyonları, Simpson formülleri ve toplam formülleri sunuyoruz. Bu bölüm, orijinal çalışmalar içerir.

Üçüncü bölümde, genelleştirilmiş hiperbolik Guglielmo sayılarını tanıtıldı. Hiperbolik üçgensel sayılar, hiperbolik üçgensel-Lucas sayıları, hiperbolik dikdörtgen sayılar ve hiperbolik beşgen sayılar gibi çeşitli özel durumları ele alıyoruz. Bu sayılar için Binet formülleri, üreteç fonksiyonları ve toplam formüllerini sunuyoruz. Ayrıca, bu dizilerle ilişkilendirilmiş Catalan ve Cassini ifadelerini ve matrisleri sunuyoruz. Bu bölüm, orijinal sonuçlar içerir.

Dördüncü bölümde, genelleştirilmiş dual Guglielmo sayılarını araştırarak çeşitli özel durumları inceliyoruz (dual üçgensel sayılar, dual üçgensel-Lucas sayıları, dual dikdörtgen sayılar ve dual beşgen sayılar dahil). Bu sayılar için Binet formülleri, üreteç fonksiyonları ve toplam formülleri sunuldu. Ek olarak, bu dizilerle ilişkilendirilmiş Catalan ve Cassini ifadelerini ile bu dizilere ait matrisleri elde ediyoruz. Bu bölüm, orijinal bilgiler içerir.

Beşinci bölümde, genelleştirilmiş dual hiperbolik Guglielmo sayıları tanıtıldı. Çeşitli özel durumlar elde edilir (dual hiperbolik üçgensel sayılar, dual hiperbolik üçgensel-Lucas sayılar, dual hiperbolik dikdörtgen sayılar ve dual hiperbolik beşgen sayılar dahil). Bu sayılar için Binet formülleri, üreteç fonksiyonları ve toplam formülleri verildi. Ayrıca, bu dizilerle ilişkilendirilmiş Catalan ve Cassini ifadelerini ve matrisleri sunuldu. Bu bölüm, orijinal çalışmalar içerir.

Anahtar Kelimeler: Gaussian Genelleştirilmiş Guglielmo sayılar, Dual Genelleştirilmiş Guglielmo sayılar, Hyperbolic Genelleştirilmiş Guglielmo sayılar, Dual hyperbolic Genelleştirilmiş Guglielmo sayılar.

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Second, I would like to express my deepest gratitude to my parents for bringing me into this world, believing in me, and placing their trust in my abilities. Additionally heartfelt thanks to my unwavering wife, who has been a constant source of support and companionship throughout my journey.

I dedicated this thesis to my beloved family and cherished children Ömer YILMAZ and Defne YILMAZ.

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

W_n	: n-th generalized Guglielmo numbers
T_n	: n-th triangular numbers
H_n	: n-th triangular-Lucas numbers
O_n	: n-th oblong numbers
p_n	: n-th pentagonal numbers
GW_n	: n-th Gaussian generalized Guglielmo numbers
GT_n	: n-th Gaussian generalized triangular numbers
GH_n	: n-th Gaussian generalized triangular-Lucas numbers
GO_n	: n-th Gaussian generalized oblong numbers
Gp_n	: n-th Gaussian generalized pentagonal numbers
DW_n	: n-th dual generalized Guglielmo numbers
DT_n	: n-th dual generalized triangular numbers
DH_n	: n-th dual generalized triangular-Lucas numbers
DO_n	: n-th dual generalized oblong numbers
Dp_n	: n-th dual generalized pentagonal numbers
HW_n	: n-th hyperbolic generalized Guglielmo numbers
HT_n	: n-th hyperbolic generalized triangular numbers
HH_n	: n-th hyperbolic generalized triangular-Lucas numbers
HO_n	: n-th hyperbolic generalized oblong numbers
Hp_n	: n-th hyperbolic generalized pentagonal numbers
$\widehat{W}_n$	: n-th dual hyperbolic generalized Guglielmo numbers
$\widehat{T}_n$	: n-th dual hyperbolic generalized triangular numbers
$\widehat{H}_n$	: n-th dual hyperbolic generalized triangular-Lucas numbers
$\widehat{O}_n$	: n-th dual hyperbolic generalized oblong numbers
$\widehat{p}_n$	: n-th dual hyperbolic generalized pentagonal numbers

CHAPTER 1

INTRODUCTION

In this chapter, we give some fundamental definitions of generalized Guglielmo numbers as well as reccurance relation, Binet's formulas, generating functions, sum formulas and some properties related to Gaussian numbers, hyperbolic numbers, dual numbers, dual hyperbolic numbers, quaternions, octonions, and sedenions that are utilized throughout the thesis.

1.1 GENERALIZED GUGLIELMO NUMBERS

First, we give the recurrence relation of generalized Guglielmo numbers.
The generalized Guglielmo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation as

$$W_n = 3W_{n-1} - 3W_{n-2} + W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero.
The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 3W_{-(n-1)} - 3W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Hence, recurrence (1.1) is true for all integer n. Soykan has conducted a study on this particular sequence, for more details, see [41].
Third order recurrence relations has been studied by many authors, for more detail see [6,11,14,16,17,30,34,36,37,39,40,46,51,52].
Next, we present Binet's formula of generalized Guglielmo numbers.

Theorem 1.1 *[41 , Theorem 1] Binet the formula of generalized Guglielmo numbers can be presented as follows:*

$$W_n = A_1 + A_2 n + A_3 n^2 \tag{1.2}$$

where A_1, A_2 *and* A_3 *are given as*

$$A_1 = W_0, \tag{1.3}$$

$$A_2 = \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \tag{1.4}$$

$$A_3 = \frac{1}{2}(W_2 - 2W_1 + W_0), \tag{1.5}$$

i.e.,

$$W_n = W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)n + \frac{1}{2}(W_2 - 2W_1 + W_0)n^2. \tag{1.6}$$

Next, we present some values of the generalized Guglielmo numbers in the following Table 1.1.

Table 1.1. A few generalized Guglielmo numbers with positive subscripts and negative subscripts.

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_0 - 3W_1 + W_2$
2	W_2	$6W_0 - 8W_1 + 3W_2$
3	$W_0 - 3W_1 + 3W_2$	$10W_0 - 15W_1 + 6W_2$
4	$3W_0 - 8W_1 + 6W_2$	$15W_0 - 24W_1 + 10W_2$
5	$6W_0 - 15W_1 + 10W_2$	$21W_0 - 35W_1 + 15W_2$
6	$10W_0 - 24W_1 + 15W_2$	$28W_0 - 48W_1 + 21W_2$
7	$15W_0 - 35W_1 + 21W_2$	$36W_0 - 63W_1 + 28W_2$
8	$21W_0 - 48W_1 + 28W_2$	$45W_0 - 80W_1 + 36W_2$
9	$28W_0 - 63W_1 + 36W_2$	$55W_0 - 99W_1 + 45W_2$
10	$36W_0 - 80W_1 + 45W_2$	$66W_0 - 120W_1 + 55W_2$
11	$45W_0 - 99W_1 + 55W_2$	$78W_0 - 143W_1 + 66W_2$
12	$55W_0 - 120W_1 + 66W_2$	$91W_0 - 168W_1 + 78W_2$
13	$66W_0 - 143W_1 + 78W_2$	$105W_0 - 195W_1 + 91W_2$

Now we define four particular cases of the sequence $\{W_n\}$ as follows: the triangular sequence $\{T_n\}_{n\geq 0}$, the triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, the oblong sequence $\{O_n\}_{n\geq 0}$ and the pentagonal sequence $\{p_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations;

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, \quad T_0 = 0, T_1 = 1, T_2 = 3, \tag{1.7}$$

$$H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, \quad H_0 = 3, H_1 = 3, H_2 = 3, \tag{1.8}$$

$$O_n = 3O_{n-1} - 3O_{n-2} + O_{n-3}, \quad O_0 = 0, O_1 = 2, O_2 = 6, \tag{1.9}$$

$$p_n = 3p_{n-1} - 3p_{n-2} + p_{n-3}, \quad p_0 = 0, p_1 = 1, p_2 = 5. \tag{1.10}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{O_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ can be extended to negative subscripts by defining,

$$\begin{aligned}
T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)}, \\
H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)}, \\
O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)}, \\
p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.7)-(1.10) hold for all integer n. Binet's formula of triangular, triangular-Lucas, oblong and pentagonal sequences can be written as

$$\begin{aligned}
T_n &= \frac{n(n+1)}{2}, \\
H_n &= 3, \\
O_n &= n(n+1), \\
p_n &= \frac{1}{2}n(3n-1).
\end{aligned}$$

Next, we have some values of the Triangular and Triangular-Lucas, oblong and pentagonal numbers as the following Table 1.2.

Table 1.2. The limited values of the unique third-order Triangular and Triangular-Lucas, oblong and pentagonal numbers with positive and negative indices.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	3	6	10	15	21	28	36	45	55	66	78	91
T_{-n}		0	1	3	6	10	15	21	28	36	45	55	66	78
H_n	3	3	3	3	3	3	3	3	3	3	3	3	3	3
H_{-n}		3	3	3	3	3	3	3	3	3	3	3	3	3
O_n	0	2	6	12	20	30	42	56	72	90	110	132	156	182
O_{-n}		0	2	6	12	20	30	42	56	72	90	110	132	156
p_n	0	1	5	12	22	35	51	70	92	117	145	176	210	247
p_{-n}		2	7	15	26	40	57	77	100	126	155	187	222	260

1.1.1 Sum formulas for generalized Guglielmo numbers

Now we give some sum formulas about generalized Guglielmo numbers that we need the rest of the thesis.

Proposition 1.2 *For the generalized Guglielmo numbers, we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{12}(n+1)\left((2n^2-2n)W_2 - 2(2n^2-5n)W_1 + (2n^2-8n+12)W_0\right).$

(b) $\sum_{k=0}^{n} W_{k+1} = \frac{1}{12}(n+1)\left((2n^2+4n)W_2 - 2(2n^2+n-6)W_1 + (2n^2-2n)W_0\right).$

(c) $\sum_{k=0}^{n} W_{k+2} = \frac{1}{12}(n+1)\left((2n^2+10n+12)W_2 - 2(2n^2+7n)W_1 + (2n^2+4n)W_0\right).$

(d) $\sum_{k=0}^{n} W_{k+3} = \frac{1}{12}(n+1)\left((2n^2+16n+36)W_2 - 2(2n^2+13n+18)W_1 + (2n^2+10n+12)W_0\right).$

(e) $\sum\limits_{k=0}^{n} W_{k-1} = \frac{1}{6}(n+1)\left((n^2-4n+6)W_2 - (2n^2-11n+18)W_1 + (n^2-7n+18)W_0\right).$

Proof. For the proof, see Soykan [41]. □

Proposition 1.3 *For the generalized Guglielmo numbers, we have the following formulas:*

(a) $\sum_{k=0}^{n} W_{2k} = \frac{1}{12}(n+1)\left((8n^2-2n)W_2 - 2(8n^2-8n)W_1 + (8n^2-14n+12)W_0\right).$

(b) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{12}(n+1)\left(W_2(8n^2+10n) - 2W_1(8n^2+4n-6) + W_0(8n^2-2n)\right).$

(c) $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{12}(n+1)\left((8n^2+22n+12)W_2 - 2(8n^2+16n)W_1 + (8n^2+10n)W_0\right).$

(d) $\sum_{k=0}^{n} W_{2k+3} = \frac{1}{12}(n+1)\left((8n^2+34n+36)W_2 - 2(8n^2+28n+18)W_1\right.$

$\left. + (8n^2+22n+12)W_0\right).$

(e) $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{12}(n+1)\left((8n^2+46n+72)W_2 - 2(8n^2+40n+48)W_1 + (8n^2+34n+36)W_0\right).$

(f) $\sum\limits_{k=0}^{n} W_{2k-1} = \frac{1}{6}(n+1)\left((4n^2-7n+6)W_2 - 2(4n^2-10n+9)W_1 + (4n^2-13n+18)W_0\right).$

Proof. The proof can be found in Soykan [41]. □

Proposition 1.4 *For the generalized Guglielmo numbers, we have the following formulas:*

(a) $\sum_{k=0}^{n} W_{-k} = \frac{1}{12}(n+1)((2n^2+4n)W_2 - 2(2n^2+7n)W_1 + (2n^2+10n+12)W_0).$

(b) $\sum_{k=0}^{n} W_{-k+1} = \frac{1}{12}(n+1)((2n^2-2n)W_2 - 2(2n^2+n-6)W_1 + (2n^2+4n)W_0).$

(c) $\sum_{k=0}^{n} W_{-k+2} = \frac{1}{12}(n+1)((2n^2-8n+12)W_2 - 2(2n^2-5n)W_1 + (2n^2-2n)W_0).$

(d) $\sum_{k=0}^{n} W_{-k+3} = \frac{1}{12}(n+1)((2n^2-14n+36)W_2 - 2(2n^2-11n+18)W_1 + (2n^2-8n+12)W_0)$

.

Proof. The proof is provided in Soykan [41]. □

Proposition 1.5 *For the generalized Guglielmo numbers, we have the following formulas:*

(a) $\sum_{k=0}^{n} W_{-2k} = \frac{1}{12}(n+1)((8n^2+10n)W_2 - 2(8n^2+16n)W_1 + (8n^2+22n+12)W_0).$

(b) $\sum_{k=0}^{n} W_{-2k+1} = \frac{1}{12}(n+1)((8n^2-2n)W_2 - 2(8n^2+4n-6)W_1 + (8n^2+10n)W_0).$

(c) $\sum_{k=0}^{n} W_{-2k+2} = \frac{1}{12}(n+1)((8n^2-14n+12)W_2 - 2(8n^2-8n)W_1$

$+(8n^2-2n)W_0).$

(d) $\sum_{k=0}^{n} W_{-2k+3} = \frac{1}{12}(n+1)((8n^2-26n+36)W_2 - 2(8n^2-20n+18)W_1 + (8n^2-14n+12)W_0).$

(e) $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{12}(n+1)((8n^2+46n+72)W_2 - 2(8n^2+40n+48)W_1 + (8n^2+34n+36)W_0).$

Proof. See Soykan [41] for validation . □

Proposition 1.6 *For the generalized Guglielmo numbers, we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^2 = \frac{n+1}{120}\Delta_1$

where

$\Delta_1 = 4W_1^2(6n^4-21n^3+11n^2+19n) - W_2^2(-6n^4+6n^3+4n^2-4n) + W_0^2(6n^4-36n^3+86n^2-116n+120) + 2W_0W_2(6n^4-21n^3+21n^2-6n) + 2W_1W_2(-12n^4+27n^3+3n^2-18n) - 2W_0W_1((12n^4-57n^3+87n^2-42n).$

(b) $\sum\limits_{k=0}^{n} W_{k-1}^2 = \frac{n+1}{120}\Delta_2$

where

$\Delta_2 = W_2^2(6n^4 - 36n^3 + 86n^2 - 116n + 120) + 4W_1^2(6n^4 - 51n^3 + 161n^2 - 251n + 270) + W_0^2(6n^4 - 66n^3 + 296n^2 - 716n + 1080) + 2W_0W_2(6n^4 - 51n^3 + 171n^2 - 306n + 360) - 2W_1W_2(12n^4 - 87n^3 + 237n^2 - 342n + 360) - 2W_0W_1(12n^4 - 117n^3 + 447n^2 - 882n + 1080).$

(c) $\sum\limits_{k=0}^{n} W_kW_{k-1} = \frac{n+1}{240}\Omega_1$

where

$\Omega_1 = W_2^2(12n^4 - 42n^3 + 42n^2 - 12n) + 8W_1^2(6n^4 - 36n^3 + 66n^2 - 36n) + W_0^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + 4W_0W_2(6n^4 - 36n^3 + 76n^2 - 76n + 60) - 2W_1W_2(24n^4 - 114n^3 + 154n^2 - 64n) - 2W_0W_1(24n^4 - 174n^3 + 454n^2 - 544n + 360).$

(d) $\sum\limits_{k=0}^{n} W_{k-1}W_{k-2} = \frac{n+1}{240}\Omega_2$

where

$\Omega_2 = 8W_1^2(6n^4 - 66n^3 + 276n^2 - 576n + 720) + W_2^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + W_0^2(12n^4 - 162n^3 + 882n^2 - 2532n + 4320) + 4W_0W_2(6n^4 - 66n^3 + 286n^2 - 646n + 900) - 2W_1W_2(24n^4 - 234n^3 + 874n^2 - 1684n + 2040) - 2W_0W_1(24n^4 - 294n^3 + 1414n^2 - 3484n + 5040).$

(f) $\sum\limits_{k=0}^{n} W_kW_{k-2} = \frac{n+1}{240}\Omega_3$

where

$\Omega_3 = W_2^2(12n^4 - 72n^3 + 132n^2 - 72n) + 8W_1^2(6n^4 - 51n^3 + 141n^2 - 141n) + W_0^2(12n^4 - 132n^3 + 552n^2 - 1152n + 1440) + 4W_0W_2(6n^4 - 51n^3 + 151n^2 - 196n + 180) - 2W_1W_2(24n^4 - 174n^3 + 394n^2 - 304n) - 2W_0W_1(24n^4 - 234n^3 + 814n^2 - 1264n + 960).$

Proof. For the proof see Soykan [41, Theorem 41]. □

1.2 MATRIX FORMULATION

Finally, we give the following proposition for the matrix A which is used in the matrix formulation throughout the thesis.

Proposition 1.7 *We define the square matrix A of order 3 as*

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. *Note that(using triangular numbers $\{T_n\}$), the following identity is true*

$$A^n = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}. \tag{1.11}$$

Proof. For the proof see [40].

1.3 SOME SPECIAL NUMBERS

In this section, we give some information about Gaussian numbers, hyperbolic numbers, dual numbers, dual hyperbolic numbers and other special numbers and some fundamental properties related to these numbers. Moreover, we give literature search on these numbers.

1.3.1 Gaussian Numbers

Gaussian numbers, generally known as Gaussian integers, are a subset of the complex numbers. A complex number is expressed in the form $a + bi$ where a and b are arbitrary real numbers, and i is the imaginary unit such that $i^2 = -1$. Gaussian integers are a specific type of complex number. In other word, z is a Gaussian integer such that $z = a + bi$ where a and b are arbitrary integers.

Next, we give some information about Gaussian sequences from the literature.

First, we give some Gaussian numbers with second-order recurrence relations.

- Horadam [26] introduced Gaussian Fibonacci numbers and defined by

  $GF_n = F_n + iF_{n-1}$

  where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.).

- Pethe and Horadam [35] introduced Gaussian generalized Fibonacci numbers by

  $GF_n = F_n + iF_{n-1}$,

  where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz [25] studied Gaussian Pell and Pell Lucas numbers by written, respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1} \end{aligned}$$

  where $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$, $Q_1 = 2$.

- Aşcı and Gürel [1] presented Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers given by, respectively,

$$\begin{aligned} GJ_n &= J_n + iJ_{n-1}, \\ Gj_n &= j_n + ij_{n-1} \end{aligned}$$

  where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$ and $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$, $j_1 = 1$.

- Taşçı [47] introduced and studied Gaussian Mersenne numbers defined by

  $GM_n = M_n + iM_{n-1}$

  where $M_n = 3M_{n-1} - 2M_{n-2}$, $M_0 = 0$, $M_1 = 1$.

- Taşçı [49] introduced and studied Gaussian balancing and Gaussian Lucas balancing numbers given by, respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + iC_{n-1} \end{aligned}$$

  where $B_n = 6B_{n-1} - BJ_{n-2}$, $B_0 = 0$, $B_1 = 1$ and $C_n = 6Cj_{n-1} - C_{n-2}$, $C_0 = 1$, $C_1 = 3$.

- Ertaş and Yılmaz [18] studied Gaussian Oresme numbers and defined them as

  $GS_n = S_n + iS_{n-1}$

  where oresme numbers are given by $S_n = S_{n-1} - \frac{1}{4}S_{n-2}$, $S_0 = 0$, $S_1 = \frac{1}{2}$.

Now, we present some Gaussian numbers with third-order recurrence relations.

- Soykan, et all [42] presented Gaussian generalized Tribonacci numbers given by

  $GW_n = W_n + iW_{n-1}$

  where $W_n = W_{n-1} + W_{n-2} + W_{n-3}$, with the initial condition W_0, W_1, W_2.

- Taşcı [48] studied Gaussian Padovan and Gaussian Pell-Padovan numbers by written, respectively,

  $$\begin{aligned} GP_n &= P_n + iP_{n-1} \\ GR_n &= R_n + iR_{n-1} \end{aligned}$$

  where $P_n = P_{n-2} + P_{n-3}$, $P_0 = 1$, $P_1 = 1$, $P_2 = 1$, and $R_n = 2R_{n-2} + R_{n-3}$, $R_0 = 1$, $R_1 = 1$, $R_2 = 1$.

- Cerda-Morales [13] defined Gaussian third-order Jacobsthal numbers as

  $GJ_n = J_n + iJ_{n-1}$

  where $J_n = J_{n-1} + J_{n-2} + 2J_{n-3}$, $J_1 = 0$, $J_2 = 1$, $J_2 = 1$.

Next, we present some special numbers such as Gaussian numbers, hyperbolic numbers, dual hyperbolic numbers used throughout the thesis and we give some properties of these numbers.

1.3.2 Hyperbolic Numbers

Hyperbolic functions and numbers find applications in various branches of engineering, such as electrical engineering (e.g., transmission lines), control systems (e.g., system dynamics), signal processing (e.g., filter design), and diverse fields of engineering physics, including special relativity, wave propagation, fluid dynamics, optics, and heat conduction. It's important to note that while hyperbolic numbers have interesting mathematical properties, their adoption in practical applications depends on the specific problem at hand and whether they offer advantages over other number systems in a given context. Hyperbolic (double, split-complex) numbers, [43], Split-complex numbers, also known as hyperbolic numbers, extend the real number system with a new element j, where $j^2 = 1$. Hyperbolic numbers are defined by,

$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}$,

Next we give some proporities on two hyperbolic numbers, $h_1 = a + jb$ and $h_2 = c + jd$, as

$$
\begin{aligned}
h_1 + h_2 &= (a + b) + j(c + d), \\
h_1.h_2 &= (ac + bd) + j(ad + bc), \\
\overline{h_1} &= a - jb \\
\frac{h_1}{h_2} &= \frac{(ac - bd) + j(cb - ad)}{c^2 - d^2}, \\
h_1 &= h_2 \text{ if only if } a = c \text{ and } b = d, \\
\langle h_1, h_2 \rangle &= (ac + bd) + j(bc + ad), \\
\text{the norm of } h_1 \text{ is } \|h_1\| &= \sqrt{|a^2 - b^2|}, \\
\text{if } \left|a^2 - b^2\right| &> 0,\ h_1 \text{ is named spacelike vector}, \\
\text{if } \left|a^2 - b^2\right| &< 0,\ h_1 \text{ is named timelike vector}, \\
\text{if } \left|a^2 - b^2\right| &= 0,\ h_1 \text{ is named null(light-like) vector.}
\end{aligned}
$$

Note that$\{\mathbb{R}^2, H, \langle,\rangle\}$ is called Lorentz plane and showed as $\mathbb{R}^2_1$. There is an isomorphism relationship between the Lorentz plane and hyperbolic numbers. For more detail see [53] Next, we present some information on hyperbolic numbers presented in literature.

- Aydın [2] presented hyperbolic Fibonacci numbers given by

  $\widetilde{F}_n = F_n + hF_{n+1}, (h^2 = 1)$

where Fibonacci numbers are given by $F_{n+2} = F_{n+1} + F_n$, with the initial conditation $F_0 = 0,\ F_1 = 1$.

- Soykan and Taşdemir [45] studied hyperbolic generalized Jacobsthal numbers given by

  $\widetilde{V}_n = V_n + hV_{n+1}$

where generalized Jacobsthal numbers are $V_{n+2} = V_{n+1} + 2V_n$ with the initial conditation $V_0 = a,\ V_1 = b$.

- Taş [50] studied hyperbolic Jacobsthal-Lucas sequence written by

  $HJ_n = J_n + hJ_{n+1}$

where Jacobsthal-Lucas numbers given by $J_{n+2} = J_{n+1} + 2J_n$ with the inintial conditation $J_0 = 2,\ J_1 = 1$.

- Dikmen and Altınsoy, [15] studied on third order hyperbolic Jacobsthal numbers given by

$$\begin{aligned} \widehat{J}_n^{(3)} &= J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &= j_n^{(3)} + hj_{n+1}^{(3)} \end{aligned}$$

  where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = 1,\ J_2^{(3)} = 1,\ j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5$.

1.3.3 Dual Numbers

Dual numbers were first introduced by W.K. Clifford in 1873. This intriguing concept has numerous applications, including screw systems, modeling plane joints, iterative methods for displacement analysis of spatial mechanisms, inertial force analysis of spatial mechanisms, and more.
Here is some general information about the applications of dual numbers.

- Engineering and Physics: Used in electrical engineering and control systems. Applied in wave analysis and signal processing. Utilized in mechanical engineering for vibration analysis, among other applications.

- Mathematics and Geometry: Alongside complex numbers, dual numbers contribute to the extension of mathematical structures. Employed in geometry to represent various transformations.

- Computer Science: Found in graphics and image processing. Used in robotics and control systems for modeling and analysis.

- Finance and Economics: Applied in risk analysis and financial engineering. Utilized in option pricing and portfolio management.

- Optimization Problems: Used for finding solutions to optimization problems. Acts as a tool in linear programming and decision-making models.

- Quantum Mechanics: Employed in quantum computers and quantum mechanics for mathematical representation.

As discussed in [20], dual numbers extend the real number system by introducing a new element ε, where $\varepsilon^2 = 0$. Dual numbers is defined as follows:

$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}$.

Let $\mathbb{D}$ =$\{d = a + \varepsilon b : a, b \in R, \varepsilon^2 = 0, \varepsilon \neq 0\}$$\subseteq \mathbb{R} \times \mathbb{R}$ is a set called dual numbers and we define following process on $\mathbb{D}$ for every $d_1 = x + x^*\varepsilon$, $d_2 = y + y^*\varepsilon \in \mathbb{D}$ as

$$\begin{aligned}
+ \quad &: \quad \mathbb{D} \times \mathbb{D} \to \mathbb{D},\ d_1 + d_2 = (x + x^*\varepsilon) + (y + y^*\varepsilon) = (x + y) + (x^* + y^*)\varepsilon, \\
\cdot \quad &: \quad \mathbb{D} \times \mathbb{D} \to \mathbb{D},\ d_1 \cdot d_2 = (x + x^*\varepsilon) \cdot (y + y^*\varepsilon) = xy + (xy^* + x^*y)\varepsilon, \\
d_1 \quad &= \quad (x + x^*\varepsilon) = (y + y^*\varepsilon) = d_2 \text{ if only if } x = x^*,\ y = y^*.
\end{aligned}$$

Using the above expressions we have the following definitions,

- $(\mathbb{D}, +)$ is an abelian grup,

  $(\mathbb{D}, +, \cdot)$ is commutative ring (where for every $d \in \mathbb{D}$ we have $d \cdot 1 = d$ so that 1 is unit element on $\cdot$ process),

  $(\mathbb{D}, +, \cdot)$ is not field because for every $d \in \mathbb{D}$ such that there is no element $d \cdot d' = d' \cdot d = 1$, the $\mathbb{D}$ is a vector space on $\mathbb{R}$,

$\widetilde{\mathbb{D}} = \{a + 0\varepsilon : a \in \mathbb{R}\}$,which is subspace of $\mathbb{D}$, is isomorph $\mathbb{R}$,

$(1, \varepsilon)$ is basis of $\mathbb{D}$,

for every $d = (x + x^{*}\varepsilon) \in \mathbb{D}$ such that $\overline{d} = (x - x^{*}\varepsilon) \in \mathbb{D}$, $\frac{1}{d} = (\frac{1}{x} + \frac{x^{*}}{x}\varepsilon) \in \mathbb{D}$, $d \cdot \overline{d} = x^{2}$,$\overline{(\overline{d})} = d$

for every $d_1 = x + x^{*}\varepsilon$, $d_2 = y + y^{*}\varepsilon \in \mathbb{D}$,$(y \neq 0)$, $\frac{d_1}{d_2} = (\frac{x}{y} + \frac{x^{*} - xy^{*}}{y^{2}}\varepsilon) \in \mathbb{D}$, $\overline{(\frac{d_1}{d_2})} = (\frac{\overline{d_1}}{\overline{d_2}})$, $\overline{(d_1 + d_2)} = (\overline{d_1} + \overline{d_2})$ and $\overline{(d_1 \cdot d_2)} = (\overline{d_1} \cdot \overline{d_2})$.

For more detail see [53].

Next, we give some information with dual sequences presented in the literature.

- Aydın [3] studied Dual Jacobsthal Quaternions as

  $QJ_{k;n} = J_{k;n} + i_1 J_{k;n+1} + i_2 J_{k;n+2} + i_3 J_{k;n+3}$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$.

- Gürses, et all [21] studied dual-generalized complex Fibonacci and Lucas numbers, respectively, as

$$\widetilde{\mathcal{F}}_n = F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3},$$
$$\widetilde{\mathcal{L}}_n = L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3},$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

1.3.4 Dual Hyperbolic Numbers

A hypercomplex system, in mathematical and geometric contexts, denotes a system that extends the principles of complex numbers. These systems exhibit intriguing algebraic properties and are often explored due to their applications in physics and engineering. Dual hyperbolic numbers can be used to represent rotations in a plane that is different from the plane represented by complex numbers. In particular, they are used in the study of the conformal group of the Minkowski plane, which plays a role in the study of special relativity. In physics, dual hyperbolic numbers can be used to represent quantities that involve both spatial and temporal components, similar to hyperbolic numbers. They can also be used in the study of quantum mechanics, where they can represent operators that act on quantum states. Overall, dual hyperbolic numbers provide a mathematical framework for dealing with quantities that exhibit both rotational and exponential behavior in a different context than hyperbolic numbers, making them a useful tool in certain areas of mathematics and physics.

A dual hyperbolic number, specifically belonging to the hyperbolic number system, is a particular kind of hypercomplex number. A dual hyperbolic number is defined by

$$q = (a_0 + ja_1) + \varepsilon(a_2 + ja_3) = a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3$$

where a_0, a_1, a_2, a_3 are real numbers.

The set of all dual hyperbolic numbers are usually denoted by

$$\mathbb{H}_{\mathbb{D}} = \{a_0 + ja_1 + \varepsilon a_2 + \varepsilon j a_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ j^2 = 1, j \neq \pm 1, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

The $\{1, j, \varepsilon, \varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}} = sp\{1, j, \varepsilon, \varepsilon j\}$ so that $\{1, j, \varepsilon, \varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$. For more detail see [5].

The next properties are true for the base elements $\{1, j, \varepsilon, \varepsilon j\}$ (commutative multiplications):

$$\begin{aligned} 1.\varepsilon &= \varepsilon, 1.j = j,\ \varepsilon^2 = \varepsilon.\varepsilon = (j\varepsilon)^2 = 0,\ j^2 = j.j = 1 \\ \varepsilon.j &= j.\varepsilon,\ \varepsilon.(\varepsilon j) = (\varepsilon j).\varepsilon = 0,\ j(\varepsilon j) = (\varepsilon j)j = \varepsilon \end{aligned}$$

where ε denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), j denotes the hyperbolic unit ($j^2 = 1$), and εj denotes the dual hyperbolic unit ($(j\varepsilon)^2 = 0$).

Next we present product of two dual hyperbolic numbers, $q = a_0 + ja_1 + \varepsilon a_2 + j\varepsilon a_3$ and $p = b_0 + jb_1 + \varepsilon b_2 + j\varepsilon b_3$, given by

$$qp = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0) + \varepsilon(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + j\varepsilon(a_0b_3 + a_1b_2 + a_2b_1 + b_0a_3)$$

and the addition of dual hyperbolic numbers is defined as componentwise.

It is known that the dual hyperbolic numbers form a commutative ring, real vector space and an algebra. Every dual hyperbolic number doesn't have an inverse so that $\mathbb{H}_{\mathbb{D}}$ is not field. For more detail see [5].

Next, we give some information related to dual hyperbolic sequences presented in the literature.

- Soykan, et all [44] presented dual hyperbolic generalized Pell numbers are given by

  $$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Cihan, et all [4] studied dual hyperbolic Fibonacci and Lucas numbers are given by, respectively,

  $$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan, et all [45] studied dual hyperbolic generalized Jacopsthal numbers given by

  $$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

- Bród, et all [8] studied dual hyperbolic generalized balancing numbers as

  $$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

1.3.5 Other Special Numbers

Now, we give some special numbers and their properties mentioned in the literature.

- Quaternion numbers, non-commutative examples of hypercomplex number systems, are a four-dimensional extension of complex numbers. They are expressed as $a_0 + ia_1 + ja_2 + ka_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and i, j, and k are the quaternion units that satisfy specific multiplication rules. For more detail see [23]. Quaternion numbers are defined by

  $$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}.$$

- Octonions is a set, every element of the set linear combinations of unit octonions $\{e_i : i = 0, 1.2, ..., 7\}$, doneted as $\mathbb{O}$. Octonions defined by,

  $$\mathbb{O} = \{\sum_{i=0}^{7} a_i e_i : a_i \in \mathbb{R}, e_0 e_i = e_i e_0 = e_i,\ \ e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k\ \}$$

where $e_e = 1$, δ_{ij} is Kroneker delta (equal to 1 if and only if $i = j$), ε_{ijk} is an anti-symmetric tensor. For more detail see [28]

- Sedenions is a set, every element of the set linear combinations of unit sedenions $\{e_i : i = 0, 1.2, ..., 15\}$, presented by $\mathbb{S}$. It can be seen from here that every sedenion can be written as

  $$\sum_{i=0}^{15} a_i e_i$$

where a_i is real number. For comprehensive details, see [39].

CHAPTER 2

GAUSSIAN GENERALIZED GUGLIELMO NUMBERS

In this chapter, we define Gaussian generalized Guglielmo numbers and present some properties such as reccurance relations, Binet's formula and generating function.

2.1 DEFINITION AND PROPERTIES

Gaussian generalized Guglielmo numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2)\}_{n\geq 0}$ are defined by

$$GW_n = 3GW_{n-1} - 3GW_{n-2} + GW_{n-3}, \tag{2.1}$$

with the initial conditions

$$GW_0 = W_0 + i(3W_0 - 3W_1 + W_2),\ GW_1 = W_1 + iW_0,\ GW_2 = W_2 + iW_1$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = 3GW_{-(n-1)} - 3GW_{-(n-2)} + GW_{-(n-3)} \tag{2.2}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) hold for all integer n. Note that for all integers n, we get

$$GW_n = W_n + iW_{n-1}. \tag{2.3}$$

At this point, the characteristic equations of (2.1) has been sighted as

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3 = 0.$$

Now, by using (2.3), we can write the Gaussian generalized Guglielmo numbers.
The first few generalized Gaussian Guglielmo numbers with positive subscript and negative subscript are presented, respectively, in the Table 2.1 and Table 2.2.

Table 2.1. The first few generalized Gaussian Guglielmo numbers with positive subscript.

n	GW_n
0	$W_0 + i(3W_0 - 3W_1 + W_2)$
1	$W_1 + iW_0$
2	$W_2 + iW_1$
3	$W_0 - 3W_1 + 3W_2 + iW_2$
4	$3W_0 - 8W_1 + 6W_2 + i(W_0 - 3W_1 + 3W_2)$
5	$6W_0 - 15W_1 + 10W_2 + i(3W_0 - 8W_1 + 6W_2)$
6	$10W_0 - 24W_1 + 15W_2 + i(6W_0 - 15W_1 + 10W_2)$
7	$15W_0 - 35W_1 + 21W_2 + i(10W_0 - 24W_1 + 15W_2)$

Table 2.2. The first few generalized Gaussian Guglielmo numbers with negative subscript.

n	GW_{-n}
0	$W_0 + i(3W_0 - 3W_1 + W_2)$
1	$3W_0 - 3W_1 + W_2 + i(6W_0 - 8W_1 + 3W_2)$
2	$6W_0 - 8W_1 + 3W_2 + i(10W_0 - 15W_1 + 6W_2)$
3	$10W_0 - 15W_1 + 6W_2 + i(15W_0 - 24W_1 + 10W_2)$
4	$15W_0 - 24W_1 + 10W_2 + i(21W_0 - 35W_1 + 15W_2)$
5	$21W_0 - 35W_1 + 15W_2 + i(28W_0 - 48W_1 + 21W_2)$
6	$28W_0 - 48W_1 + 21W_2 + i(36W_0 - 63W_1 + 28W_2)$
7	$36W_0 - 63W_1 + 28W_2 + i(45W_0 - 80W_1 + 36W_2)$

Gaussian triangular numbers, $GW_n : GW_n(0, 1, 3 + i) = GT_n$, are defined by

$$GT_n = 3GT_{n-1} - 3GT_{n-2} + GT_{n-3} \tag{2.4}$$

with the initial values

$$GT_0 = 0, GT_1 = 1, GT_2 = 3 + i.$$

Gaussian triangular-Lucas numbers, $GW_n(3 + 3i, 3 + 3i, 3 + 3i) = GH_n$, are defined by

$$GH_n = 3GH_{n-1} - 3GH_{n-2} + GH_{n-3} \tag{2.5}$$

with the initial values

$$GH_0 = 3 + 3i, GH_1 = 3 + 3i, GH_2 = 3 + 3i.$$

Gaussian oblong numbers , $GW_n(0, 2, 6 + 2i) = GO_n$, are defined by

$$GO_n = 3GO_{n-1} - 3GO_{n-2} + GO_{n-3} \tag{2.6}$$

with the initial values

$$GO_0 = 0, GO_1 = 2, GO_2 = 6 + 2i.$$

and Gaussian pentegonal numbers, $GW_n(2i, 1, 5+i) = Gp_n$, are defined by

$$Gp_n = 3Gp_{n-1} - 3Gp_{n-2} + Gp_{n-3} \tag{2.7}$$

with the initial values

$$Gp_0 = 2i, Gp_1 = 1, Gp_2 = 5 + i.$$

Remember that for all integers n, we have

$$\begin{aligned} GT_n &= T_n + iT_{n-1}, \\ GH_n &= H_n + iH_{n-1}, \\ GO_n &= O_n + iO_{n-1}, \\ Gp_n &= p_n + ip_{n-1}. \end{aligned}$$

The first few values of Gaussian triangular numbers, Gaussian triangular-Lucas numbers, Gaussian oblong numbers and Gaussian pentegonal numbers with positive and negative subscript are given in the Table 2.3.

Table 2.3. Special cases of Gaussian generalized Guglielmo numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7
GT_n	0	1	$3+i$	$6+3i$	$10+6i$	$15+10i$	$21+15i$	$28+21i$
GT_{-n}		i	$1+3i$	$3+6i$	$6+10i$	$10+15i$	$15+21i$	$21+28i$
GH_n	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$
GH_{-n}		$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$	$3+3i$
GO_n	0	2	$6+2i$	$12+6i$	$20+12i$	$30+20i$	$42+30i$	$56+42i$
GO_{-n}		$2i$	$2+6i$	$6+12i$	$12+20i$	$20+30i$	$30+42i$	$42+56i$
Gp_n	$2i$	1	$5+i$	$12+5i$	$22+12i$	$35+22i$	$51+35i$	$70+51i$
Gp_{-n}		$2+7i$	$7+15i$	$15+26i$	$26+40i$	$40+57i$	$57+77i$	$77+100i$

2.1.1 The Binet's Formula For The Gaussian Generalized Guglielmo Numbers

Next, we present The Binet's formula for the Gaussian generalized Guglielmo numbers.

Theorem 2.1 *The Binet's formula for the Gaussian generalized Guglielmo numbers is* $GW_n = (W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)n + \frac{1}{2}(W_2 - 2W_1 + W_0)n^2) + i(W_0 + \frac{1}{2}(-W_2 + 4W_1 - 3W_0)(n-1) + \frac{1}{2}(W_2 - 2W_1 + W_0)(n-1)^2)$.

Proof. The proof follows from (1.6) and (2.3). □

The previous Theorem gives the following results, as special cases.

Corollary 2.1 *For all integers n, we have following identities:*

(a) $GT_n = \frac{1}{2}n\,(n+1) + i(\frac{1}{2}n\,(n-1))$.

(b) $GH_n = 3 + 3i$.

(c) $GO_n = n\,(n+1) + in(n-1)$.

(d) $Gp_n = \frac{1}{2}n\,(3n-1) + i(\frac{1}{2}\,(n-1)\,(3n-4))$.

2.1.2 The Generating Function of Gaussian Generalized Guglielmo Numbers

The next theorem presents the generating function of Gaussian generalized Guglielmo numbers.

Theorem 2.2 *Let* $f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n$ *denote the generating function of Gaussian generalized Guglielmo numbers. Then,*

$$f_{GW_n}(x) = \sum_{n=0}^{\infty} GW_n x^n = \frac{GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + 3GW_0)x^2}{1 - 3x + 3x^2 - x^3}. \quad (2.8)$$

Proof. Using the definition of Gaussian Guglielmo numbers, and substracting $xf(x)$, $x^2 f(x)$ and $x^3 f(x)$ from $f(x)$ we obtain

$(1 - 3x + 3x^2 - x^3) f_{GW_n}(x) = (1-x)^3 \sum_{n=0}^{\infty} GW_n x^n$

So that we get,

$$
\begin{aligned}
(1-x)^3\sum_{n=0}^{\infty}GW_nx^n &= \sum_{n=0}^{\infty}GW_nx^n-3x\sum_{n=0}^{\infty}GW_nx^n+3x^2\sum_{n=0}^{\infty}GW_nx^n-x^3\sum_{n=0}^{\infty}GW_nx^n,\\
&= \sum_{n=0}^{\infty}GW_nx^n-3\sum_{n=0}^{\infty}GW_nx^{n+1}+3\sum_{n=0}^{\infty}GW_nx^{n+2}-\sum_{n=0}^{\infty}GW_nx^{n+3},\\
&= \sum_{n=0}^{\infty}GW_nx^n-3\sum_{n=1}^{\infty}GW_{n-1}x^n+3\sum_{n=2}^{\infty}GW_{n-2}x^n-\sum_{n=3}^{\infty}GW_{n-3}x^n,\\
&= (GW_0+GW_1x+GW_2x^2)-3(GW_0x+GW_1x^2)+3GW_0x^2\\
&\quad +\sum_{n=3}^{\infty}(GW_n-3GW_{n-1}+3GW_{n-2}-GW_{n-3})x^n,\\
&= GW_0+GW_1x+GW_2x^2-3GW_0x-3GW_1x^2+3GW_0x^2,\\
&= GW_0+(GW_1-3GW_0)x+(GW_2-3GW_1+3GW_0)x^2,
\end{aligned}
$$

and rearranging above equation, we get (2.8). □

Theorem 2.2 yields the following outcomes as special cases,

$$
\begin{aligned}
f_{GT_n}(x) &= \frac{x+ix^2}{1-3x+3x^2-x^3},\\
f_{GH_n}(x) &= \frac{(3+3i)\,x^2-(6+6i)\,x+3+3i}{1-3x+3x^2-x^3},\\
f_{GO_n}(x) &= \frac{2ix^2+2x}{1-3x+3x^2-x^3},\\
f_{Gp_n}(x) &= \frac{(2+7i)x^2+(1-6i)x+2i}{1-3x+3x^2-x^3},
\end{aligned}
$$

2.2 SOME IDENTITIES ABOUT RECCURENCE RELATIONS OF GAUSSIAN GENERALIZED GUGLIELMO NUMBERS

In this section, we present some identities involving Gaussian triangular, Gaussian triangular-Lucas, Gaussian oblong, Gaussian pentegonal numbers.

Theorem 2.3 *The following equations hold for all integer n*

$$GT_n=\frac{1}{2}GO_{n+3}-\frac{3}{2}GO_{n+2}+\frac{3}{2}GO_{n+1}, \tag{2.9}$$

$$GO_n=2GT_{n+3}-6GT_{n+2}+6GT_{n+1}, \tag{2.10}$$

$$GT_n=\frac{-2}{27}Gp_{n+2}+\frac{10}{27}Gp_{n+1}+\frac{1}{27}Gp_n, \tag{2.11}$$

$$Gp_n=2GT_{n+2}-6GT_{n+1}+7GT_n, \tag{2.12}$$

$$GO_n = \frac{-4}{27}Gp_{n+2} + \frac{20}{27}Gp_{n+1} + \frac{2}{27}Gp_n, \tag{2.13}$$

$$Gp_n = GO_{n+2} - 3GO_{n+1} + \frac{7}{2}GO_n. \tag{2.14}$$

Proof. To proof identity (2.9), we can write $GT_n = aGO_{n+3} + bGO_{n+2} + cGO_{n+1}$ and solve the system of equations we get,

$$\begin{aligned} GT_0 &= aGO_3 + bGO_2 + cGO_1, \\ GT_1 &= aGO_4 + bGO_3 + cGO_2, \\ GT_2 &= aGO_5 + bGO_4 + cGO_3. \end{aligned}$$

Then, we obtain $a = \frac{1}{2}, b = -\frac{3}{2}, c = \frac{3}{2}$. The other identities can be found similarly. □

Lemma 2.4 *([19])We assume that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$*. Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n}x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1}x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

Below, we detail the generating functions for the even and odd-indexed generalized Guglielmo sequences as provided by the theorem.

Theorem 2.5 *The generating functions of the sequence* GW_{2n} *and* GW_{2n+1} *are provided by*

$$f_{GW_{2n}}(x) = \frac{GW_0 + (GW_2 - 3GW_0)x + (6GW_0 - 8GW_1 + 3GW_2)x^2}{1 - 3x + 3x^2 - x^3}, \tag{2.15}$$

$$f_{GW_{2n+1}}(x) = \frac{GW_1 + (GW_0 - 6GW_1 + 3GW_2)x + (3GW_0 - 3GW_1 + GW_2)x^2}{1 - 3x + 3x^2 - x^3}. \tag{2.16}$$

Proof. We only proof (2.15). From Theorem 2.2 we can obtain following identities:

$$f_{GW_n}(\sqrt{x}) = \frac{GW_0 - \sqrt{x}\,(GW_1 - 3GW_0) + x\,(3GW_0 - 3GW_1 + GW_2)}{3x + 3\sqrt{x} + x^{\frac{3}{2}} + 1}.$$

$$f_{GW_n}(-\sqrt{x}) = -\frac{GW_0 + \sqrt{x}\,(GW_1 - 3GW_0) + x\,(3GW_0 - 3GW_1 + GW_2)}{3\sqrt{x} - 3x + x^{\frac{3}{2}} - 1}.$$

Thereby, using Lemma 2.4 identity (2.15) can be proved. The other identity can be found similarly. □

From Theorem 2.5, we get the following corollary.

Corollary 2.2

(a)

$$f_{GT_{2n}}(x) = \frac{(1+3i)\,x^2 + (3+i)\,x}{1 - 3x + 3x^2 - x^3} \text{ and } f_{GT_{2n+1}}(x) = \frac{ix^2 + (3+3i)\,x + 1}{1 - 3x + 3x^2 - x^3}.$$

(b)

$$f_{GH_{2n}}(x) = \frac{(3+3i)\,x^2 - (6+6i)\,x + 3 + 3i}{1 - 3x + 3x^2 - x^3} \text{ and } f_{GH_{2n+1}}(x) = \frac{(3+3i)\,x^2 - (6+6i)\,x + 3 + 3i}{1 - 3x + 3x^2 - x^3}.$$

(c)

$$f_{GO_{2n}}(x) = \frac{(2+6i)\,x^2 + (6+2i)\,x}{1 - 3x + 3x^2 - x^3} \text{ and } f_{GO_{2n+1}}(x) = \frac{2ix^2 + (6+6i)\,x + 2}{1 - 3x + 3x^2 - x^3}.$$

(d)

$$f_{Gp_{2n}}(x) = \frac{(7+15i)\,x^2 + (5-5i)\,x + 2i}{1 - 3x + 3x^2 - x^3} \text{ and } f_{Gp_{2n+1}}(x) = \frac{(2+7i)\,x^2 + (9+5i)\,x + 1}{1 - 3x + 3x^2 - x^3}.$$

From Corollary 2.2, we can obtain the following corollary which presents the identities on Gaussian Guglielmo sequences.

Corollary 2.3

(a) $(3+i)GH_{2n-2} + (1+3i)GH_{2n-4} = (3+3i)GT_{2n} - (6+6i)GT_{2n-2} + (3+3i)GT_{2n-4}$.

(b) $2iGT_{2n-4} + (6+6i)GT_{2n-2} + 2GT_{2n} = (3+i)GO_{2n-1} + (1+3i)GO_{2n-3}$.

(c) $(7+15i)GT_{2n-4} + (5-5i)GT_{2n-2} + 2iGT_{2n} = (3+i)Gp_{2n-2} + (1+3i)Gp_{2n-4}$.

(d) $(3+3i)GO_{2n-4}-(6+6i)GO_{2n-2}+(3+3i)GO_{2n}=(2+6i)GH_{2n-4}+(6+2i)GH_{2n-2}$.

(e) $(7+15i)GH_{2n-4}+(5-5i)GH_{2n-2}+2iGH_{2n}=(3+3i)Gp_{2n-4}-(6+6i)Gp_{2n-2}+(3+3i)Gp_{2n}$.

(f) $(7+15i)GO_{2n-4}+(5-5i)GO_{2n-2}+2iGO_{2n}=(2+6i)Gp_{2n-4}+(6+2i)Gp_{2n-2}$.

(g) $iGH_{2n-3}+(3+3i)GH_{2n-1}+GH_{2n+1}=(3+3i)GT_{2n-3}-(6+6i)GT_{2n-1}+(3+3i)GT_{2n+1}$.

(h) $iGH_{2n-4}+(3+3i)GH_{2n-2}+GH_{2n}=(3+3i)GT_{2n-3}-(6+6i)GT_{2n-1}+(3+3i)GT_{2n+1}$.

(i) $iGO_{2n-4}+(3+3i)GO_{2n-2}+GO_{2n}=(2+6i)GT_{2n-3}+(6+2i)GT_{2n-1}$.

(j) $iGp_{2n-3}+(3+3i)Gp_{2n-1}+Gp_{2n+1}=(2+7i)GT_{2n-3}+(9+5i)GT_{2n-1}+GT_{2n+1}$.

(k) $(3+3i)GO_{2n-3}-(6+6i)GO_{2n-1}+(3+3i)GO_{2n+1}=2iGH_{2n-3}+(6+6i)GH_{2n-1}+2GH_{2n+1}$.

(l) $2iGp_{2n-3}+(6+6i)Gp_{2n-1}+2Gp_{2n+1}=(2+7i)GO_{2n-3}+(9+5i)GO_{2n-1}+GO_{2n+1}$.

Proof. From Corollary 2.2, we obtain

$((3+i)x+(1+3i)x^2)f_{GH_{2n}}=((3+3i)-(6+6i)x+(3+3i)x^2)f_{GT_{2n}}$.

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ((3+i)x+(1+3i)x^2)\sum_{n=0}^{\infty}GH_{2n}x^n \\
&= (3+i)x\sum_{n=0}^{\infty}GH_{2n}x^n+(1+3i)x^2\sum_{n=0}^{\infty}GH_{2n}x^n \\
&= (3+i)\sum_{n=0}^{\infty}GH_{2n}x^{n+1}+(1+3i)\sum_{n=0}^{\infty}GH_{2n}x^{n+2} \\
&= (3+i)\sum_{n=1}^{\infty}GH_{2n-2}x^n+(1+3i)\sum_{n=2}^{\infty}GH_{2n-4}x^n \\
&= (3+i)(3+3i)x+\sum_{n=2}^{\infty}((3+i)GH_{2n-2}+(1+3i)GH_{2n-4})x^n.
\end{aligned}$$

whereas the RHS (right hand side) is equal to

$$
\begin{aligned}
RHS &= ((3+3i)-(6+6i)x+(3+3i)x^2)\sum_{n=0}^{\infty} GT_{2n}x^n \\
&= (3+3i)\sum_{n=0}^{\infty} GT_{2n}x^n-(6+6i)x\sum_{n=0}^{\infty} GT_{2n}x^n+(3+3i)x^2\sum_{n=0}^{\infty} GT_{2n}x^n \\
&= (3+3i)\sum_{n=0}^{\infty} GT_{2n}x^n-(6+6i)\sum_{n=0}^{\infty} GT_{2n}x^{n+1}+(3+3i)\sum_{n=0}^{\infty} GT_{2n}x^{n+2} \\
&= (3+3i)\sum_{n=0}^{\infty} GT_{2n}x^n-(6+6i)\sum_{n=1}^{\infty} GT_{2n-2}x^n+(3+3i)\sum_{n=2}^{\infty} GT_{2n-4}x^n \\
&= (3+i)(3+3i)x+\sum_{n=2}^{\infty}((3+3i)GT_{2n}-(6+6i)GT_{2n-2}+(3+3i)GT_{2n-4})x^n.
\end{aligned}
$$

Comparing the coefficients and the proof of the first identity (a) is done. We can present other identity similarly. □

We can get an identitiy related to Gaussian Guglielmo numbers and triangular numbers given below.

Theorem 2.6 *For all integers m, n the following identity holds:*

$GW_{m+n} = T_{m-1}GW_{n+2} + (T_{m-3} - 3T_{m-2})GW_{n+1} + T_{m-2}GW_n.$

Proof. First, we assume that $m, n \geqslant 0$. the Theorem 2.6 can be proved by mathematical induction on m. If $m = 0$ we get

$GW_n = T_{-1}GW_{n+2} + (T_{-3} - 3T_{-2})GW_{n+1} + T_{-2}GW_n$

which is true since $T_{-1} = 0, T_{-2} = 1, T_{-3} = 3$. We assume that the identity given holds for $m \leq k$. For $m = k + 1$, we get

$$
\begin{aligned}
GW_{(k+1)+n} &= 3GW_{n+k} - 3GW_{n+k-1} + GW_{n+k-2} \\
&= 3(T_{k-1}GW_{n+2} + (T_{k-3} - 3T_{k-2})GW_{n+1} + T_{k-2}GW_n) \\
&\quad -3(T_{k-2}GW_{n+2} + (T_{k-4} - 3T_{k-3})GW_{n+1} + T_{k-3}GW_n) \\
&\quad +(T_{k-3}GW_{n+2} + (T_{k-5} - 3T_{k-4})GW_{n+1} + T_{k-4}GW_n) \\
&= (3T_{k-1} - 3T_{k-2} + T_{k-3})GW_{n+2} + ((3T_{k-3} - 3T_{k-4} + T_{k-5}) \\
&\quad -3(3T_{k-2} - 3T_{k-3} + T_{k-4}))GW_{n+1} + (3T_{k-2} - 3T_{k-3} + T_{k-4})GW_n \\
&= T_kGW_{n+2} + (T_{k-2} - 3T_{k-1})GW_{n+1} + T_{k-1}GW_n \\
&= T_{(k+1)-1}GW_{n+2} + (T_{(k+1)-3} - 3T_{(k+1)-2})GW_{n+1} + T_{(k+1)-2}GW_n.
\end{aligned}
$$

Consequently, by mathematical induction on m, this proves Theorem 2.6. The case $m, n < 0$ can be proved similarly. $\square$

For $n \geq 0$, $m \geq 0$ and taking $GW_n = GT_n$ or $GW_n = GH_n$ or $GO_n = GW_n$ or $GW_n = Gp_n$, respectively, we get,

$$\begin{aligned} GT_{m+n} &= T_{m-1}GT_{n+2} + (T_{m-3} - 3T_{m-2})GT_{n+1} + T_{m-2}GT_n, \\ GH_{m+n} &= T_{m-1}GH_{n+2} + (T_{m-3} - 3T_{m-2})GH_{n+1} + T_{m-2}GH_n, \\ GO_{m+n} &= T_{m-1}GO_{n+2} + (T_{m-3} - 3T_{m-2})GO_{n+1} + T_{m-2}GO_n, \\ Gp_{m+n} &= T_{m-1}Gp_{n+2} + (T_{m-3} - 3T_{m-2})Gp_{n+1} + T_{m-2}Gp_n. \end{aligned}$$

2.3 SIMPSON'S FORMULA FOR GAUSSIAN GENERALIZED GUGLIELMO NUMBERS

In this section, we present Simpson's formula of generalized Gaussian Guglielmo numbers. This is a special cases of [38, Theorem 4.1]. We give the proof by calculating determinant and using Binet's formula of Gaussian generalized Guglielmo numbers.

Theorem 2.7 (Simpson's formula of generalized Gaussian Guglielmo numbers) *For all integers n, we can write following equality*

$$\begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} = -(GW_0 - 2GW_1 + GW_2)^3.$$

Proof. Using Theorem 2.1 we can obtain

$$\begin{aligned} \begin{vmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{vmatrix} &= i\left((1-i)W_0 - (2-2i)W_1 + (1-i)W_2\right)^3 \\ &= -i^3(1-i)^3(W_0 - 2W_1 + W_2)^3 \\ &= (-i - i^4)^3(W_0 - 2W_1 + W_2)^3 \\ &= -(1+i)^3(W_0 - 2W_1 + W_2)^3 \\ &= -(W_0 - 2W_1 + W_2 + i(W_0 - 2W_1 + W_2))^3 \\ &= -(GW_0 - 2GW_1 + GW_2)^3. \ \square \end{aligned}$$

From the Theorem 2.7, we get the following Corollary.

Corollary 2.4 *For all integers n, we get the following identities.*

(a) $\begin{vmatrix} GT_{n+2} & GT_{n+1} & GT_{n} \\ GT_{n+1} & GT_{n} & GT_{n-1} \\ GT_{n} & GT_{n-1} & GT_{n-2} \end{vmatrix} = 2(1-i).$

(b) $\begin{vmatrix} GH_{n+2} & GH_{n+1} & GH_{n} \\ GH_{n+1} & GH_{n} & GH_{n-1} \\ GH_{n} & GH_{n-1} & GH_{n-2} \end{vmatrix} = 0.$

(c) $\begin{vmatrix} GO_{n+2} & GO_{n+1} & GO_{n} \\ GO_{n+1} & GO_{n} & GO_{n-1} \\ GO_{n} & GO_{n-1} & GO_{n-2} \end{vmatrix} = 16(1-i).$

(d) $\begin{vmatrix} Gp_{n+2} & Gp_{n+1} & Gp_{n} \\ Gp_{n+1} & Gp_{n} & Gp_{n-1} \\ Gp_{n} & Gp_{n-1} & Gp_{n-2} \end{vmatrix} = 54(1-i).$

2.4 SUM FORMULAS FOR GAUSSIAN GENERALIZED GUGLIELMO NUMBERS

In this section, we identify some sum formulas of generalized Gaussian Guglielmo numbers.

Theorem 2.8 *For all integers $n \geq 0$, we have sum formulas given below*

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{6}(n+1)(n(n-1)W_2 - n(2n-5)W_1 + (n^2-4n+6)W_0) + \frac{1}{6}(n+1)((n^2-4n+6)W_2 - (2n^2-11n+18)W_1 + (n^2-7n+18)W_0)i.$

(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{6}(n+1)((4n^2-n)W_2 - 8(n^2-n)W_1 + (4n^2-7n+6)W_0) + \frac{1}{6}(n+1)((4n^2-7n+6)W_2 - 2(4n^2-10n+9)W_1 + (4n^2-13n+18)W_0)i.$

(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{6}(n+1)((4n^2+5n)W_2 - 2(4n^2+2n-3)W_1 + (4n^2-n)W_0) + \frac{1}{6}(n+1)((4n^2-n)W_2 - 8(n^2-n)W_1 + (4n^2-7n+6)W_0)i.$

Proof. From (2.3) we can write the following sum formulas.

$$\begin{aligned}
\sum_{k=0}^{n} GW_k &= \sum_{k=0}^{n} W_k + i\sum_{k=0}^{n} W_{k-1}, \\
\sum_{k=0}^{n} GW_{2k} &= \sum_{k=0}^{n} W_{2k} + i\sum_{k=0}^{n} W_{2k-1}, \\
\sum_{k=0}^{n} GW_{2k+1} &= \sum_{k=0}^{n} W_{2k+1} + i\sum_{k=0}^{n} W_{2k}.
\end{aligned}$$

Using Proposition 1.2 and Proposition 1.3, the proof has been done easily.□

The previous theorem gives the following Corollary.

Corollary 2.5

(a) $\sum\limits_{k=0}^{n} GT_k = \frac{1}{6}n\,(n+2)\,(n+1) + \frac{1}{6}n\,(n-1)\,(n+1)\,i.$

(b) $\sum\limits_{k=0}^{n} GH_k = 3(n+1) + 3\,(n+1)\,i.$

(c) $\sum\limits_{k=0}^{n} GO_k = \frac{1}{3}n\,(n+2)\,(n+1) + \frac{1}{3}n\,(n-1)\,(n+1)\,i.$

(d) $\sum\limits_{k=0}^{n} Gp_k = \frac{1}{2}n^2\,(n+1) + \frac{1}{2}\,(n+1)\,(-3n+n^2+4)\,i.$

Next, we give sum formulas which are given by even subscripts.

Corollary 2.6

(a) $\sum\limits_{k=0}^{n} GT_{2k} = \frac{1}{6}n\,(4n+5)\,(n+1) + \frac{1}{6}n\,(n+1)\,(4n-1)\,i.$

(b) $\sum\limits_{k=0}^{n} GH_{2k} = 3\,(n+1) + 3\,(n+1)\,i.$

(c) $\sum\limits_{k=0}^{n} GO_{2k} = \frac{1}{6}\,(8n^2+10n)\,(n+1) + \frac{1}{6}\,(8n^2-2n)\,(n+1)\,i.$

(d) $\sum\limits_{k=0}^{n} Gp_{2k} = \frac{1}{2}n\,(4n+1)\,(n+1) + \frac{1}{2}\,(n+1)\,(-5n+4n^2+4)\,i.$

Next, we give sum formulas which are given by odd subscripts.

Corollary 2.7

(a) $\sum\limits_{k=0}^{n} GT_{2k+1} = \frac{1}{6}(n+1)(4n^2+11n+6) + \frac{1}{6}n(4n+5)(n+1)i.$

(b) $\sum\limits_{k=0}^{n} GH_{2k+1} = 3(n+1) + 3(n+1)i.$

(c) $\sum\limits_{k=0}^{n} GO_{2k+1} = \frac{1}{6}(n+1)(8n^2+22n+12) + \frac{1}{6}(8n^2+10n)(n+1)i.$

(d) $\sum\limits_{k=0}^{n} Gp_{2k+1} = \frac{1}{6}(n+1)(12n^2+21n+6) + \frac{1}{6}(12n^2+3n)(n+1)i.$

2.4.1 Sums of Squares

Theorem 2.9 *For all integers m and j with W_0, W_1, W_2 are the initial values of (1.1), we have the following sum formulas for generalized Gaussian Guglielmo numbers*

$$\sum_{k=0}^{n} GW_k^2 = \frac{n+1}{120}\Psi$$

where Ψ, ζ, γ and ϑ are $\Psi = \zeta - \gamma + i\vartheta$,

$\zeta = W_2^2(12n^4 - 42n^3 + 42n^2 - 12n) + 8W_1^2(6n^4 - 36n^3 + 66n^2 - 36n) + W_0^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + 4W_0W_2(6n^4 - 36n^3 + 76n^2 - 76n + 60) - 2W_1W_2(24n^4 - 114n^3 + 154n^2 - 64n) - 2W_0W_1(24n^4 - 174n^3 + 454n^2 - 544n + 360)$,

$\gamma = W_2^2(6n^4 - 36n^3 + 86n^2 - 116n + 120) + 4W_1^2(6n^4 - 51n^3 + 161n^2 - 251n + 270) + W_0^2(6n^4 - 66n^3 + 296n^2 - 716n + 1080) + 2W_0W_2(6n^4 - 51n^3 + 171n^2 - 306n + 360) - 2W_1W_2(12n^4 - 87n^3 + 237n^2 - 342n + 360) - 2W_0W_1(12n^4 - 117n^3 + 447n^2 - 882n + 1080)$,

$\vartheta = W_2^2(12n^4 - 42n^3 + 42n^2 - 12n) + 8W_1^2(6n^4 - 36n^3 + 66n^2 - 36n) + W_0^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + 4W_0W_2(6n^4 - 36n^3 + 76n^2 - 76n + 60) - 2W_1W_2(24n^4 - 114n^3 + 154n^2 - 64n) - 2W_0W_1(24n^4 - 174n^3 + 454n^2 - 544n + 360)$.

$$\sum_{k=0}^{n} GW_kGW_{k-1} = \frac{1}{240}(n+1)((\lambda_1 - \lambda_2) + i(\Gamma_1 + 2\Gamma_2))$$

where λ_1, λ_1, Γ_1 and Γ_2 are

$\lambda_1 = W_2^2(12n^4 - 42n^3 + 42n^2 - 12n) + 8W_1^2(6n^4 - 36n^3 + 66n^2 - 36n) + W_0^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + 4W_0W_2(6n^4 - 36n^3 + 76n^2 - 76n + 60) - 2W_1W_2(24n^4 - 114n^3 + 154n^2 - 64n) - 2W_0W_1(24n^4 - 174n^3 + 454n^2 - 544n + 360)$,

$\lambda_2 = 8W_1^2(6n^4 - 66n^3 + 276n^2 - 576n + 720) + W_2^2(12n^4 - 102n^3 + 342n^2 - 612n + 720) + W_0^2(12n^4 - 162n^3 + 882n^2 - 2532n + 4320) + 4W_0W_2(6n^4 - 66n^3 + 286n^2 - 646n + 900) -$

$2W_1W_2(24n^4-234n^3+874n^2-1684n+2040)-2W_0W_1(24n^4-294n^3+1414n^2-3484n+5040)$,

$\Gamma_1 = W_2^2(12n^4-72n^3+132n^2-72n)+8W_1^2(6n^4-51n^3+141n^2-141n)+W_0^2(12n^4-132n^3+552n^2-1152n+1440)+4W_0W_2(6n^4-51n^3+151n^2-196n+180)-2W_1W_2(24n^4-174n^3+394n^2-304n)-2W_0W_1(24n^4-234n^3+814n^2-1264n+960)$,

$\Gamma_2 = W_2^2(6n^4-36n^3+86n^2-116n+120)+4W_1^2(6n^4-51n^3+161n^2-251n+270)+W_0^2(6n^4-66n^3+296n^2-716n+1080)+2W_0W_2(6n^4-51n^3+171n^2-306n+360)-2W_1W_2(12n^4-87n^3+237n^2-342n+360)-2W_0W_1(12n^4-117n^3+447n^2-882n+1080)$.

Proof: From (2.3) we can write the following sum formulas.

$$\sum_{k=0}^{n} GW_k^2 = \sum_{k=0}^{n} W_k^2 - \sum_{k=0}^{n} W_{k-1}^2 + 2i\sum_{k=0}^{n} W_kW_{k-1}.$$

$$\sum_{k=0}^{n} GW_kGW_{k-1} = \sum_{k=0}^{n}(W_kW_{k-1} - W_{k-1}W_{k-2}) + i\sum_{k=0}^{n}(W_kW_{k-2} + W_{k-1}^2).$$

Using Proposition 1.6, the proof can be done. □

The previous theorem provides the following corollary.

Corollary 2.8

(a) $\sum\limits_{k=0}^{n} GT_k^2 = \frac{1}{4}n^2(n+1)^2 + i(\frac{1}{20}(n+1)n(n-1)(2n+1)(n+2))$.

(b) $\sum\limits_{k=0}^{n} GH_k^2 = 18i(n+1)$.

(c) $\sum\limits_{k=0}^{n} GO_k^2 = n^2(n+1)^2 + i(\frac{1}{5}(n+1)n(n-1)(2n+1)(n+2))$.

(d) $\sum\limits_{k=0}^{n} Gp_k^2 = \frac{1}{4}(3n-4)(n+1)(-n+3n^2+4)+i(\frac{1}{60}(n+1)n(n-1)(-45n+54n^2-26))$.

(e) $\sum\limits_{k=0}^{n} GT_kGT_{k-1} = \frac{1}{4}n^2(n+1)(n-1) + i(\frac{1}{30}n(n-1)(n+1)(3n^2-7))$.

(f) $\sum\limits_{k=0}^{n} GH_kGH_{k-1} = 18i(n+1)$.

(g) $\sum\limits_{k=0}^{n} GO_kGO_{k-1} = n^2(n-1)(n+1) + i(\frac{2}{15}n(n-1)(n+1)(3n^2-7))$.

(h) $\sum\limits_{k=0}^{n} Gp_kGp_{k-1} = \frac{1}{4}(3n-7)(n+1)(-4n+3n^2+8) + i(\frac{1}{30}(n+1)(27n^4-117n^3+167n^2-107n+120)$.

2.5 MATRIX FORMULATION OF GAUSSIAN GENERALIZED GUGLIELMO NUMBERS

In this section, we present by introducing the following lemma, which serves as a foundational result for our study. Later, we proceed to define some fundemental formulas related to matrix formulation of GW_n.

Lemma 2.10 *For $n \geq 0$ the following identity is true*

$$\begin{pmatrix} GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} \tag{2.17}$$

Proof. The identity (2.17) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_2 \\ GW_1 \\ GW_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus, the following identity is true:

$$\begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix}$$
$$= \begin{pmatrix} 3GW_{k+2} - 3GW_{k+1} + GW_k \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.$$

Consequently, by mathematical induction on n, the proof is completed. □

We define

$$N_{GW} = \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \tag{2.18}$$

$$E_{GW} = \begin{pmatrix} GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+1} & GW_n & GW_{n-1} \\ GW_n & GW_{n-1} & GW_{n-2} \end{pmatrix}. \tag{2.19}$$

Now, we have the following theorem with N_{GW} and E_{GW}.

Theorem 2.11 *Using (1.11) N_{GW} and E_{GW}, we get*

$$A^n N_{GW} = E_{GW}.$$

Proof. Note that using (1.11), we get

$$
\begin{aligned}
A^n N_{GW} &= \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} GW_2 & GW_1 & GW_0 \\ GW_1 & GW_0 & GW_{-1} \\ GW_0 & GW_{-1} & GW_{-2} \end{pmatrix}, \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} &= GW_2T_{n+1} + GW_1(T_{n-1} - 3T_n) + GW_0T_n, \\
a_{12} &= GW_1T_{n+1} + GW_0(T_{n-1} - 3T_n) + GW_{-1}T_n, \\
a_{13} &= GW_0T_{n+1} + GW_{-1}(T_{n-1} - 3T_n) + GW_{-2}T_n, \\
a_{21} &= GW_2T_n + GW_1(T_{n-2} - 3T_{n-1}) + GW_0T_{n-1}, \\
a_{22} &= GW_1T_n + GW_0(T_{n-2} - 3T_{n-1}) + GW_{-1}T_{n-1}, \\
a_{23} &= GW_0T_n + GW_{-1}(T_{n-2} - 3T_{n-1}) + GW_{-2}T_{n-1}, \\
a_{31} &= GW_2T_{n-1} + GW_1(T_{n-3} - 3T_{n-2}) + GW_0T_{n-2}, \\
a_{32} &= GW_1T_{n-1} + GW_0(T_{n-3} - 3T_{n-2}) + GW_{-1}T_{n-2}, \\
a_{33} &= GW_0T_{n-1} + GW_{-1}(T_{n-3} - 3T_{n-2}) + GW_{-2}T_{n-2}.
\end{aligned}
$$

Using the Theorem 2.6, the proof is done. □

By taking, $GW_n = GT_n$ with GT_0, GT_1, GT_2 in (2.18) and (2.19), $GW_n = GH_n$ with GH_0, GH_1, GH_2 in (2.18) and (2.19), $GW_n = GO_n$ with GO_0, GO_1, GO_2 in (2.18) and (2.19), $GW_n = Gp_n$ with Gp_0, Gp_1, Gp_2 in (2.18) and (2.19) respectively, we get:

$$
N_{GT} = \begin{pmatrix} 3+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1+3i \end{pmatrix}, \quad E_{GT} = \begin{pmatrix} GT_{n+2} & GT_{n+1} & GT_n \\ GT_{n+1} & GT_n & GT_{n-1} \\ GT_n & GT_{n-1} & GT_{n-2} \end{pmatrix}
$$

$$
N_{GH} = \begin{pmatrix} 3+3i & 3+3i & 3+3i \\ 3+3i & 3+3i & 3+3i \\ 3+3i & 3+3i & 3+3i \end{pmatrix}, \quad E_{GH} = \begin{pmatrix} GH_{n+2} & GH_{n+1} & GH_n \\ GH_{n+1} & GH_n & GH_{n-1} \\ GH_n & GH_{n-1} & GH_{n-2} \end{pmatrix}
$$

$$N_{GO} = \begin{pmatrix} 6+2i & 2 & 0 \\ 2 & 0 & 2i \\ 0 & 2i & 2+6i \end{pmatrix} \quad , \quad E_{GO} = \begin{pmatrix} GO_{n+2} & GO_{n+1} & GO_{n} \\ GO_{n+1} & GO_{n} & GO_{n-1} \\ GO_{n} & GO_{n-1} & GO_{n-2} \end{pmatrix}$$

$$N_{Gp} = \begin{pmatrix} 5+i & 1 & 2i \\ 1 & 2i & 2+7i \\ 2i & 2+7i & 7+15i \end{pmatrix} \quad , \quad E_{Gp} = \begin{pmatrix} Gp_{n+2} & Gp_{n+1} & Gp_{n} \\ Gp_{n+1} & Gp_{n} & Gp_{n-1} \\ Gp_{n} & Gp_{n-1} & Gp_{n-2} \end{pmatrix}.$$

From Theorem 5.10, we can write the following corollary.

Corollary 2.9 *The following identities are holds.*

(a) $A^n N_{GT} = E_{GT}$.

(b) $A^n N_{GH} = E_{GH}$.

(b) $A^n N_{GO} = E_{GO}$.

(c) $A^n N_{Gp} = E_{Gp}$.

CHAPTER 3

HYPERBOLIC GENERALIZED GUGLIELMO NUMBERS

In this section, we define hyperbolic generalized Guglielmo numbers then we present generating functions and Binet formulas for them.

3.1 DEFINITION AND PROPERTIES

We now investigate hyperbolic generalized Guglielmo numbers over $\mathbb{H}$. The nth hyperbolic generalized Guglielmo number is

$$HW_n = W_n + jW_{n+1} \tag{3.1}$$

with the initial values HW_0, HW_1, HW_2. (3.1) can be written to negative subscripts by defining,

$$HW_{-n} = W_{-n} + jW_{-n+1} \tag{3.2}$$

so that (3.1) is true for all integers n.

Now we define some extraordinary cases of hyperbolic generalized Guglielmo numbers named the nth hyperbolic triangular numbers, the nth hyperbolic triangular-Lucas numbers, the nth hyperbolic oblong numbers and the nth hyperbolic pentegonal numbers and give them below.

The nth hyperbolic triangular numbers $HT_n = T_n + jT_{n+1}$, with the initial values as

$$\begin{aligned} HT_0 &= T_0 + jT_1, \\ HT_1 &= T_1 + jT_2, \\ HT_2 &= T_2 + jT_3. \end{aligned}$$

The nth hyperbolic triangular-Lucas numbers $HH_n = H_n + jH_{n+1}$ with the initial values as

$$\begin{aligned} HH_0 &= H_0 + jH_1, \\ HH_1 &= H_1 + jH_2, \\ HH_2 &= H_2 + jH_3. \end{aligned}$$

The nth hyperbolic oblong numbers $HO_n = O_n + jO_{n+1}$ with the initial values as

$$\begin{aligned} HO_0 &= O_0 + jO_1, \\ HO_1 &= O_1 + jO_2, \\ HO_2 &= O_2 + jO_3. \end{aligned}$$

The nth hyperbolic pentegonal numbers $Hp_n = p_n + jp_{n+1}$ with the initial values as

$$\begin{aligned} Hp_0 &= p_0 + jp_1, \\ Hp_1 &= p_1 + jp_2, \\ Hp_2 &= p_2 + jp_3. \end{aligned}$$

For hyperbolic triangular numbers (taking $W_n = T_n$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$) we derive

$$\begin{aligned} HT_0 &= j \\ HT_1 &= 1 + 3j \\ HT_2 &= 3 + 6j. \end{aligned}$$

for hyperbolic triangular-Lucas numbers (taking $W_n = H_n$, $H_0 = 3$, $H_1 = 3$, $H_2 = 3$) we get

$$\begin{aligned} HH_0 &= 3 + 3j, \\ HH_1 &= 3 + 3j, \\ HH_2 &= 3 + 3j. \end{aligned}$$

for hyperbolic oblong numbers (taking $W_n = O_n$, $O_0 = 0$, $O_1 = 2$, $O_2 = 6$) we have

$$\begin{aligned} HO_0 &= 2j, \\ HO_1 &= 2 + 6j, \\ HO_2 &= 6 + 12j, \end{aligned}$$

and for hyperbolic pentegonal numbers (taking $W_n = p_n$, $p_0 = 0$, $p_1 = 1$, $p_2 = 5$) we deduce

$$\begin{aligned} Hp_0 &= j, \\ Hp_1 &= 1 + 5j, \\ Hp_2 &= 5 + 12j. \end{aligned}$$

So, using (3.1) the following identity can be expressed for non-negative integers n,

$$HW_n = 3HW_{n-1} - 3HW_{n-2} + HW_{n-3}. \tag{3.3}$$

Hence for the negative integers n the sequence $\{HW_n\}_{n\geq 0}$ can be deduced as

$$HW_{-n} = 3HW_{-(n-1)} - 3HW_{-(n-2)} + HW_{-(n-3)},$$

for $n = 1, 2, 3, ...$ by using (3.2). Consequently, recurrence (3.3) holds for all integer n. In the Table 3.1, we present the first few hyperbolic generalized Guglielmo numbers with positive subscript and negative subscript.

Table 3.1. A few hyperbolic generalized Guglielmo numbers

n	HW_n	HW_{-n}
0	HW_0	HW_0
1	HW_1	$3HW_0 - 3HW_1 + HW_2$
2	HW_2	$6HW_0 - 8HW_1 + 3HW_2$
3	$HW_0 - 3HW_1 + 3HW_2$	$10HW_0 - 15HW_1 + 6HW_2$
4	$3HW_0 - 8HW_1 + 6HW_2$	$15HW_0 - 24HW_1 + 10HW_2$
5	$6HW_0 - 15HW_1 + 10HW_2$	$21HW_0 - 35HW_1 + 15HW_2$
6	$10HW_0 - 24HW_1 + 15HW_2$	$28HW_0 - 48HW_1 + 21HW_2$

Remember that

$$\begin{aligned} HW_0 &= W_0 + jW_1, \\ HW_1 &= W_1 + jW_2, \\ HW_2 &= W_2 + jW_3. \end{aligned}$$

A few hyperbolic triangular numbers, hyperbolic triangular-Lucas numbers, hyperbolic oblong numbers and hyperbolic pentegonal numbers with positive subscript and negative subscript are given, respectively, in the following Table 3.2, Table 3.3, Table 3.4 and Table 3.5.

Table 3.2. hyperbolic triangular numbers

n	HT_n	HT_{-n}
0	j	
1	$1+3j$	0
2	$3+6j$	1
3	$6+10j$	$3+j$
4	$10+15j$	$6+3j$
5	$15+21j$	$10+6j$

Table 3.3. hyperbolic triangular-Lucas numbers

n	HH_n	HH_{-n}
0	$3+3j$	
1	$3+3j$	$3+3j$
2	$3+3j$	$3+3j$
3	$3+3j$	$3+3j$
4	$3+3j$	$3+3j$
5	$3+3j$	$3+3j$

Table 3.4. hyperbolic oblong numbers

n	HO_n	HO_{-n}
0	$2j$	
1	$2+6j$	
2	$6+12j$	2
3	$12+20j$	$6+2j$
4	$20+30j$	$12+6j$
5	$30+42j$	$20+12j$

Table 3.5. hyperbolic pentegonal numbers

n	Hp_n	Hp_{-n}
0	j	
1	$1+5j$	2
2	$5+12j$	$7+2j$
3	$12+22j$	$15+7j$
4	$22+35j$	$26+15j$
5	$35+51j$	$40+26j$

Now, we will give Binet's formula for the hyperbolic generalized Guglielmo numbers and in the rest of the study the following notations are used:

$$\widehat{\alpha} = 1 + j, \tag{3.4}$$

$$\widehat{\beta} = j. \tag{3.5}$$

Observe that the following identities are obtained:

$$\begin{aligned} \widehat{\alpha}^2 &= 2 + 2j, \\ \widehat{\beta}^2 &= 1, \\ \widehat{\alpha}\widehat{\beta} &= 1 + j. \end{aligned}$$

3.1.1 The Binet's Formula For The Hyperbolic Generalized Guglielmo Numbers

Theorem 3.1 *(Binet's Formula) For any integer n, the nth hyperbolic generalized Guglielmo number is*

$$HW_n = (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2 \tag{3.6}$$

where $\widehat{\alpha}$, $\widehat{\beta}$ *are shown as (3.4)-(3.5).*

Proof. Using Binet's formula given below

$W_n = A_1 + A_2 n + A_3 n^2,$

of the generalized Guglielmo numbers, A_1, A_2, A_2 are given (1.3), (1.4), (1.5). We get

$$\begin{aligned} HW_n &= W_n + jW_{n+1} \\ &= (A_1(j+1) + j(A_2 + A_3)) + ((1+j)A_2 + 2jA_3)n + A_3\,(j+1)\,n^2 \\ &= (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3 n^2. \ \square \end{aligned}$$

As special cases, for any integer n, the Binet's Formula of nth hyperbolic triangual number is given below

$$HT_n = \frac{1}{2}(\beta + (\alpha + 2\beta)n + \alpha n^2), \tag{3.7}$$

the Binet's Formula of nth hyperbolic triangular-Lucas number is given below

$$HH_n = 3\widehat{\alpha}, \tag{3.8}$$

the Binet's Formula of nth hyperbolic oblong number given below

$$HO_n = \beta + (\alpha + 2\beta)n + \alpha n^2, \tag{3.9}$$

and the Binet's Formula of nth hyperbolic pentegonal number given below

$$Hp_n = \frac{1}{2}(2\beta + (6\beta - \alpha)\,n + 3\alpha n^2). \tag{3.10}$$

3.1.2 The Generating Function of Hyperbolic Generalized Guglielmo Numbers

The next step is to provide the generating function for the hyperbolic generalized Guglielmo numbers.

Theorem 3.2 *The generating function for the hyperbolic generalized Guglielmo numbers is*

$$f_{HW}(x) = \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + 3HW_0)x^2}{(1 - 3x + 3x^2 - x^3)}. \tag{3.11}$$

Proof. Let

$$f_{HW}(x)=\sum_{n=0}^{\infty}HWx^{n}$$

be generating function of the hyperbolic generalized Guglielmo numbers. Then, using the definition of the hyperbolic generalized Guglielmo numbers, and substracting $xg(x)$ and $x^2g(x)$ from $g(x)$, we get
$(1-3x+3x^2-x^3)f_{HW}(x)=(1-x)^3\sum_{n=0}^{\infty}HWx^n$.

So that

$$\begin{aligned}
(1-x)^3\sum_{n=0}^{\infty}HWx^{n} &= \sum_{n=0}^{\infty}HWx^{n}-3x\sum_{n=0}^{\infty}HWx^{n}+3x^{2}\sum_{n=0}^{\infty}HWx^{n}-x^{3}\sum_{n=0}^{\infty}HWx^{n}\\
&= \sum_{n=0}^{\infty}HWx^{n}-3\sum_{n=0}^{\infty}HWx^{n+1}+3\sum_{n=0}^{\infty}HWx^{n+2}-\sum_{n=0}^{\infty}HWx^{n+3}\\
&= \sum_{n=0}^{\infty}HWx^{n}-3\sum_{n=1}^{\infty}HWx^{n}+3\sum_{n=2}^{\infty}HWx^{n}-\sum_{n=3}^{\infty}HWx^{n}\\
&= (HW_0+HW_1x+HW_2x^2)-3(HWx+HW_1x^2)+3HW_0x^2\\
&\quad +\sum_{n=3}^{\infty}(HW_n-3HW_{n-1}+3HW_{n-2}-HW_{n-3})x^n\\
&= HW_0+HW_1x+HW_2x^2-3HW_0x-3HW_1x^2+3HW_0x^2\\
&= HW_0+(HW_1-3HW_0)x+(HW_2-3HW_1+3HW_0)x^2.
\end{aligned}$$

Note that using the recurrence relation $HW_n=3HW_{n-1}-3HW_{n-2}+HW_{n-3}$ and rearranging above equation, we get (3.11). □
As special cases, the generating functions of the hyperbolic triangular, triangular-Lucas, oblong and pentegonal numbers are written by

$$\begin{aligned}
f_{HW_n}(x) &= \frac{j+x}{(1-3x+3x^2-x^3)},\\
f_{HH_n}(x) &= \frac{(3+3j)+(-6-6j)x+(3+3j)\,x^2}{(1-3x+3x^2-x^3)},\\
f_{HO_n}(x) &= \frac{2j+2x}{(1-3x+3x^2-x^3)},\\
f_{Hp_n}(x) &= \frac{j+(1+2j)x+2x^2}{(1-3x+3x^2-x^3)},
\end{aligned}$$

respectively. □

3.2 GETTING BINET'S FORMULA FROM GENERATING FUNCTION OF HYPERBOLIC GENERALIZED GUGLIELMO NUMBERS

Our next step involves exploring Binet formula of hyperbolic generalized Guglielmo number $\{HW_n\}$ utilizing generating function $f_{HW_n}(x)$.

Theorem 3.3 *(Binet formula of hyperbolic generalized Guglielmo numbers)*

$$HW_n = (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3 n^2. \tag{3.12}$$

Proof. We write

$$\sum_{n=0}^{\infty} HWx^n = \frac{HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + 3HW_0)x^2}{(1 - 3x + 3x^2 - x^3)} \tag{3.13}$$

$$= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3}. \tag{3.14}$$

So that

$$\begin{aligned} \sum_{n=0}^{\infty} HW_n x^n &= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3} \\ &= \frac{d_1(1-x)^2 + d_2(1-x) + d_3}{(1-x)^3}. \end{aligned}$$

Then, we get

$$HW_0 + (HW_1 - 3HW_0)x + (HW_2 - 3HW_1 + 3HW_0)x^2 = (d_1 + d_2 + d_3) + (-2d_1 - d_2)x + d_1x^2.$$

If we equalize the coefficients of the same degree terms of x in the above equation, we obtain

$$\begin{aligned} HW_0 &= d_1 + d_2 + d_3, \\ HW_1 - 3HW_0 &= -2d_1 - d_2, \\ HW_2 - 3HW_1 + 3HW_0 &= d_1. \end{aligned} \tag{3.15}$$

If we solve system of equations (3.15), we get

$$\begin{aligned} d_1 &= 3HW_0 - 3HW_1 + HW_2. \\ d_2 &= 5HW_1 - 3HW_0 - 2HW_2. \\ d_3 &= HW_0 - 2HW_1 + HW_2. \end{aligned}$$

Thus (3.14) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} HW_n x^n &= d_1 \sum_{n=0}^{\infty} x^n + d_2 \sum_{n=0}^{\infty} (n+1) x^n + d_3 \sum_{n=0}^{\infty} \frac{n^2+3n+2}{2} x^n, \\ &= \sum_{n=0}^{\infty} (d_1 + d_2(n+1) + d_3 \frac{n^2+3n+2}{2}) x^n, \\ &= \sum_{n=0}^{\infty} (HW_0 + \frac{1}{2}(-HW_2 + 4HW_1 - 3HW_0) n + \frac{1}{2}(HW_2 - 2HW_1 + HW_0) n^2) x^n.\end{aligned}$$

As a result, we get the following identity

$$HW_n = \widehat{A_1} + \widehat{A_2} n + \widehat{A_3} n^2$$

where

$$\begin{aligned}\widehat{A_1} &= HW_0, \\ \widehat{A_2} &= \frac{1}{2}(-HW_2 + 4HW_1 - 3HW_0), \\ \widehat{A_3} &= \frac{1}{2}(\widehat{HW}_2 - 2\widehat{HW}_1 + HW_0).\end{aligned}$$

Take note that the following equalities holds:

$$\begin{aligned}\widehat{A_1} &= HW_0 \\ &= W_0 + jW_1 \\ &= (1+j)W_0 + j(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) + j(\frac{1}{2}(W_2 - 2W_1 + W_0)) \\ &= \widehat{\alpha} A_1 + \widehat{\beta}(A_2 + A_3).\end{aligned} \tag{3.16}$$

$$\begin{aligned}\widehat{A_2} &= \frac{1}{2}(-HW_2 + 4HW_1 - 3HW_0) \\ &= \frac{1}{2}((-3W_0 + 4W_1 - W_2) + j(-W_0 + W_2) \\ &= (1+j)(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) \\ &\quad +2j(\frac{1}{2}(W_2 - 2W_1 + W_0)) \\ &= (\widehat{a} A_2 + 2\widehat{\beta} A_3).\end{aligned} \tag{3.17}$$

$$\begin{aligned}\widehat{A_3} &= \frac{1}{2}(HW_2 - 2HW_1 + HW_0) \\ &= \frac{1}{2}((W_2 - 2W_1 + W_0) + j(W_2 - 2W_1 + W_0) \\ &= \widehat{a} A_3.\end{aligned} \tag{3.18}$$

Using (3.16), (3.17) and (3.18), we get

$$HW_n = (A_1 \widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha} A_2 + 2\widehat{\beta} A_3) n + \widehat{\alpha} A_3 n^2. \ \square$$

3.3 SOME IDENTITIES FOR HYPERBOLIC GENERALIZED GUGLIELMO NUMBERS

We now provide some special identities concerning the hyperbolic generalized Guglielmo sequence $\{HW_n\}$. The following theorem gives the Simpson's formula for the hyperbolic generalized Guglielmo numbers.

Theorem 3.4 *(Simpson's formula for hyperbolic generalized Guglielmo numbers) For all integers n we have,*

$$\begin{vmatrix} HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+1} & HW_n & HW_{n-1} \\ HW_n & HW_{n-1} & HW_{n-2} \end{vmatrix} = \begin{vmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{vmatrix}. \tag{3.19}$$

Proof. For the proof we use mathematical induction. For $n = 0$ identity (3.19) is true. Now we obtain (3.19) is true for $n = k$. We prove that (3.19) is true for $n = k + 1$. Thus,we write the identity given below,

$$\begin{vmatrix} HW_{k+2} & HW_{k+1} & HW_k \\ HW_{k+1} & HW_k & HW_{k-1} \\ HW_k & HW_{k-1} & HW_{k-2} \end{vmatrix} = \begin{vmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{vmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{vmatrix} HW_{k+3} & HW_{k+2} & HW_{k+1} \\ HW_{k+2} & HW_{k+1} & HW_k \\ HW_{k+1} & HW_k & HW_{k-1} \end{vmatrix} &= \begin{vmatrix} 3HW_{k+2} - 3HW_{k+1} + HW_k & HW_{k+2} & HW_{k+1} \\ 3HW_{k+1} - 3HW_k + HW_{k-1} & HW_{k+1} & HW_k \\ 3HW_k - 3HW_{k-1} + HW_{k-2} & HW_k & HW_{k-1} \end{vmatrix} \\ &= 3\begin{vmatrix} HW_{k+2} & HW_{k+2} & HW_{k+1} \\ HW_{k+1} & HW_{k+1} & HW_k \\ HW_k & HW_k & HW_{k-1} \end{vmatrix} - 3\begin{vmatrix} HW_{k+1} & HW_{k+2} & HW_{k+1} \\ HW_k & HW_{k+1} & HW_k \\ HW_{k-1} & HW_k & HW_{k-1} \end{vmatrix} \\ &\quad + \begin{vmatrix} HW_k & HW_{k+2} & HW_{k+1} \\ HW_{k-1} & HW_{k+1} & HW_k \\ HW_{k-2} & HW_k & HW_{k-1} \end{vmatrix} \\ &= \begin{vmatrix} HW_{k+2} & HW_{k+1} & HW_k \\ HW_{k+1} & HW_k & HW_{k-1} \\ HW_k & HW_{k-1} & HW_{k-2} \end{vmatrix}. \end{aligned}$$

Thus, the proof is completed.□

From Theorem 3.19 we get following corollary.

Corollary 3.1

(a) $$\begin{vmatrix} HT_{n+2} & HT_{n+1} & HT_n \\ HT_{n+1} & HT_n & HT_{n-1} \\ HT_n & HT_{n-1} & HT_{n-2} \end{vmatrix} = -4(j+1).$$

(b) $$\begin{vmatrix} HH_{n+2} & HH_{n+1} & HH_n \\ HH_{n+1} & HH_n & HH_{n-1} \\ HH_n & HH_{n-1} & HH_{n-2} \end{vmatrix} = 0.$$

(c) $$\begin{vmatrix} HO_{n+2} & HO_{n+1} & HO_n \\ HO_{n+1} & HO_n & HO_{n-1} \\ HO_n & HO_{n-1} & O_{n-2} \end{vmatrix} = -32(j+1).$$

(d) $$\begin{vmatrix} Hp_{n+2} & Hp_{n+1} & Hp_n \\ Hp_{n+1} & Hp_n & Hp_{n-1} \\ Hp_n & Hp_{n-1} & Hp_{n-2} \end{vmatrix} = -108(j+1).$$

Now, we define Catalan's identity of hyperbolic generalized Guglielmo numbers.

Theorem 3.5 *(Catalan's identity) For all integers n and m, the following identity holds*

$$\begin{aligned} HW_{n+m}HW_{n-m} - HW_n^2 &= -2m^2(\widehat{\alpha}(A_2^2 - 2A_1A_3 + A_2A_3 + 2nA_2A_3) \\ &\quad -A_3^2(\widehat{\alpha} - 2n\widehat{\alpha} + m^2\widehat{\alpha} - 2n^2\widehat{\alpha} - 2)). \end{aligned} \tag{3.20}$$

Proof. By using the Binet Formula we get

$$HW_n = (A_1\widehat{\alpha} + \widehat{\beta}(A_2 + A_3)) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2.$$

The proof is completed. □

As special cases of the above theorem, we give Catalan's identity of hyperbolic triangular, Lucas-triangular, Oblong, pentagonal numbers.

Next, we present Catalan's identity of hyperbolic triangular numbers.

Corollary 3.2 *(Catalan's identity for the hyperbolic triangular numbers) For all integers n and m, the following identity holds:*

$$HT_{n+m}HT_{n-m} - HT_n^2 = \frac{1}{2}m^2(-\widehat{\alpha} - 4n\widehat{\alpha} + m^2\widehat{\alpha} - 2n^2\widehat{\alpha} - 2)$$

Proof. Taking $HW_n = HT_n$ in Theorem 3.5, we achieve the desired result. □

Now we show Catalan's identity of hyperbolic triangular-Lucas numbers.

Corollary 3.3 *(Catalan's identity for the hyperbolic Lucas-triangular numbers) For all integers n and m, the following identity holds:*

$$HH_{n+m}HH_{n-m} - HH_n^2 = 0.$$

Proof. Taking $HW_n = HH_n$ in Theorem 3.5, we acquire the desired result.. □

Following this, we provide Catalan's identity of hyperbolic oblong numbers.

Corollary 3.4 *(Catalan's identity for the hyperbolic oblong numbers) For all integers n and m, the following identity holds:*

$$HO_{n+m}HO_{n-m} - HO_n^2 = 2m^2\left(-\widehat{\alpha} - 4n\widehat{\alpha} + m^2\widehat{\alpha} - 2n^2\widehat{\alpha} - 2\right).$$

Proof. Taking $HW_n = HO_n$ in Theorem 3.5, we obtain the result we've been seeking. □

In the next step, we detail Catalan's identity of hyperbolic pentegonal numbers.

Corollary 3.5 *(Catalan's identity for the hyperbolic pentagonal numbers) For all integers n and m, the following identity holds:*

$$Hp_{n+m}Hp_{n-m} - Hp_n^2 = \frac{1}{2}m^2\left(11\widehat{\alpha} - 12n\widehat{\alpha} + 9m^2\widehat{\alpha} - 18n^2\widehat{\alpha} - 18\right)$$

Proof. Taking $HW_n = Hp_n$ in Theorem 3.5, we get the result we have been seeking. □

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the hyperbolic generalized Guglielmo numbers. Hence,we present corollary given below.

Corollary 3.6 *(Cassini's identity for the hyperbolic generalized Guglielmo numbers) For all integers n, the following identities holds:*

(a) $HT_{n+1}HT_{n-1} - HT_n^2 = -\widehat{\alpha}n^2 - 2\widehat{\alpha}n - 1.$

(b) $HH_{n+1}HH_{n-1} - HH_n^2 = 0.$

(c) $HO_{n+1}HO_{n-1} - HO_n^2 = -4\left(n^2\widehat{\alpha} + 2n\widehat{\alpha} + 1\right).$

(d) $Hp_{n+1}Hp_{n-1} - Hp_n^2 = -9\widehat{\alpha}n^2 - 6\widehat{\alpha}n + 10\widehat{\alpha} - 9.$

Theorem 3.6 *Let n and m be integers, T_n is triangular numbers, the following identity is true:*

$$HW_{m+n} = T_{m-1}HW_{n+2} + (T_{m-3} - 3T_{m-2})HW_{n+1} + T_{m-2}HW_n. \tag{3.21}$$

Proof. First we give the proof for $n \geqslant 0$. For $n \geqslant 0$ the Theorem 3.6 can be proved by mathematical induction on m. If $m = 0$ we get

$$HW_n = T_{-1}HW_{n+2} + (T_{-3} - 3T_{-2})HW_{n+1} + T_{-2}HW_n$$

which is true by seeing that $T_{-1} = 0, T_{-2} = 1, T_{-3} = 3$. We assume that the identity given holds for $m = k$. For $m = k + 1$, we get

$$\begin{aligned}
HW_{(k+1)+n} &= 3HW_{n+k} - 3HW_{n+k-1} + HW_{n+k-2} \\
&= 3(T_{k-1}HW_{n+2} + (T_{k-3} - 3T_{k-2})HW_{n+1} + T_{k-2}HW_n) \\
&\quad -3(T_{k-2}HW_{n+2} + (T_{k-4} - 3T_{k-3})HW_{n+1} + T_{k-3}HW_n) \\
&\quad +(T_{k-3}HW_{n+2} + (T_{k-5} - 3T_{k-4})HW_{n+1} + T_{k-4}HW_n) \\
&= (3T_{k-1} - 3T_{k-2} + T_{k-3})HW_{n+2} + ((3T_{k-3} - 3T_{k-4} + T_{k-5}) \\
&\quad -3(3T_{k-2} - 3T_{k-3} + T_{k-4}))HW_{n+1} + (3T_{k-2} - 3T_{k-3} + T_{k-4})HW_n \\
&= T_kHW_{n+2} + (T_{k-2} - 3T_{k-1})HW_{n+1} + T_{k-1}HW_n \\
&= T_{(k+1)-1}HW_{n+2} + (T_{(k+1)-3} - 3T_{(k+1)-2})HW_{n+1} + T_{(k+1)-2}HW_n.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 3.6. For $n < 0$, the proof can be done similarly. □

3.4 LINEAR SUM FORMULAS FOR HYPERBOLIC GENERALIZED GUGLIELMO NUMBERS

Below, we detail the summation formulas of the hyperbolic generalized Guglielmo numbers with positive and negative subscripts.
First, we will introduce the formulas that allow us to find the sum of hyperbolic generalized Guglielmo numbers.

Theorem 3.7 *For $n \geq 0$, hyperbolic generalized Guglielmo numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{W}_k = \frac{1}{6}(n+1)((-n+jn^2+2jn+n^2)W_2+(6j+5n-2jn^2-jn-2n^2)W_1+(-4n+jn^2-jn+n^2+6)W_0)$.

(b) $\sum_{k=0}^{n} \widehat{W}_{2k} = \frac{1}{6}(n+1)((-n+4jn^2+5jn+4n^2)W_2+(6j+8n-8jn^2-4jn-8n^2)W_1+(-7n+4jn^2-jn+4n^2+6)W_0)$.

(c) $\sum_{k=0}^{n} \widehat{W}_{2k+1} = \frac{1}{6}(n+1)((6j+5n+4jn^2+11jn+4n^2)W_2+(6-8jn^2-16jn-8n^2-4n)W_1+(-n+4jn^2+5jn+4n^2)W_0)$.

Proof.

(a) Note that using (3.1), we get

$$\sum_{k=0}^{n} HW_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1}$$

and using Proposition 1.2, the proof can be easily conducted.

(b) Note that using (3.1), we get

$$\sum_{k=0}^{n} HW_{2k} = \sum_{k=0}^{n} W_{2k} + j\sum_{k=0}^{n} W_{2k+1}$$

and using Proposition 1.3, the proof can be easily accomplished.

(c) Note that using (3.1), we get

$$\sum_{k=0}^{n} HW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + j\sum_{k=0}^{n} W_{2k+2}$$

and using Proposition 1.3, the proof can be done easily. □

As a special case of the Theorem 3.7 (a), we present following corollary.

Corollary 3.7

(a) $\sum_{k=0}^{n} HT_k = \frac{1}{6}(n+1)(6j+(5j+2)n+(j+1)n^2)$.

(b) $\sum_{k=0}^{n} HH_k = (3j+3)(n+1)$.

(c) $\sum_{k=0}^{n} HO_k = \frac{1}{6}(n+1)(12j+(10j+4)n+(2j+2)n^2)$.

(d) $\sum_{k=0}^{n} Hp_k = \frac{1}{6}(n+1)(6j+9jn+(3j+3)n^2)$.

As a special case of the Theorem 3.7 (b), below, we outline the following corollary.

Corollary 3.8

(a) $\sum_{k=0}^{n} HT_{2k} = \frac{1}{6}(n+1)(6j+(5+11j)n+(4+4j)n^2)$.

(b) $\sum_{k=0}^{n} HH_{2k} = (3j+3)(n+1)$.

(c) $\sum_{k=0}^{n} HO_{2k} = \frac{1}{6}(n+1)(12j+(10+22j)n+(8+8j)n^2)$.

(d) $\sum_{k=0}^{n} Hp_{2k} = \frac{1}{6}(n+1)(6j+(3+21j)n+(12+12j)n^2)$.

As a special case of the Theorem 3.7 (c), the following corollary follows.

Corollary 3.9

(a) $\sum_{k=0}^{n} HT_{2k+1} = \frac{1}{6}(n+1)((6+18j)+(11+17j)n+(4+4j)n^2)$.

(b) $\sum_{k=0}^{n} HH_{2k+1} = (3j+3)(n+1)$.

(c) $\sum_{k=0}^{n} HO_{2k+1} = \frac{1}{6}(n+1)((12+36j)+(22+34j)n+(8+8j)n^2)$.

(d) $\sum_{k=0}^{n} Hp_{2k+1} = \frac{1}{6}(n+1)((6+30j)+(21+39j)n+(12+12j)n^2)$.

Now, we give the formula that yield the summation formulas of the generalized Guglielmo numbers with negative subscripts.

Theorem 3.8 *For $n \geq 1$, hyperbolic generalized Guglielmo numbers have the following formulas:*

(a) $\sum_{k=0}^{n} HW_{-k} = \frac{1}{6}(n+1)((2n+jn^2-jn+n^2)W_2+(6j-7n-2jn^2-jn-2n^2)$
$W_1+(5n+jn^2+2jn+n^2+6)W_0)$.

(b) $\sum_{k=0}^{n} HW_{-2k} = \frac{1}{6}(n+1)((5n+4jn^2-jn+4n^2)W_2+(6j-16n-8jn^2-4jn-8n^2)W_1+(11n+4jn^2+5jn+4n^2+6)W_0)$.

(c) $\sum_{k=0}^{n} HW_{-2k+1} = \frac{1}{6}(n+1)((6j-n+4jn^2-7jn+4n^2)W_2+(-4n-8jn^2+8jn-8n^2+6)W_1+(5n+4jn^2-jn+4n^2)W_0)$.

Proof.

(a) Note that using (3.1), we get

$$\sum_{k=0}^{n} HW_{-k} = \sum_{k=0}^{n} W_{-k} + j\sum_{k=0}^{n} W_{-k+1}$$

and using Proposition 1.4, the proof can be easily carried out..

(b) Note that using (3.1), we get

$$\sum_{k=0}^{n} HW_{-2k} = \sum_{k=0}^{n} W_{-2k} + j\sum_{k=0}^{n} W_{-2k+1}$$

and using Proposition 1.5, the proof can be easily established..

(c) Note that using (3.1), we get

$$\sum_{k=0}^{n} HW_{-2k+1} = \sum_{k=0}^{n} W_{-2k+1} + j\sum_{k=0}^{n} W_{-2k+2}$$

and using Proposition (1.5), the proof can be easily shown. □

As a special case of the Theorem 3.8 (a), we deduce the following corollary.

Corollary 3.10

(a) $\sum_{k=0}^{n} HT_{-k} = \frac{1}{6}(n+1)(6j+(-1-4j)n+(1+j)n^2)$.

(b) $\sum_{k=0}^{n} HH_{-k} = (3j+3)(n+1)$.

(c) $\sum_{k=0}^{n} HO_{-k} = \frac{1}{6}(n+1)(12j+(-2-8j)n+(2+2j)n^2)$.

(d) $\sum_{k=0}^{n} Hp_{-k} = \frac{1}{2}(n+1)(2j+(1-2j)n+(1+j)n^2)$.

As a special case of the Theorem 3.8 (b), we derive the following corollary.

Corollary 3.11

(a) $\sum_{k=0}^{n} HT_{-2k} = \frac{1}{6}\left(n+1\right)\left(6j+(-1-7j)n+(4+4j)n^{2}\right).$

(b) $\sum_{k=0}^{n} HH_{-2k} = \left(3j+3\right)\left(n+1\right).$

(c) $\sum_{k=0}^{n} HO_{-2k} = \frac{1}{3}\left(n+1\right)\left(6j+(-1-7j)n+(4+4j)n^{2}\right).$

(d) $\sum_{k=0}^{n} Hp_{-2k} = \frac{1}{6}\left(n+1\right)\left((6j)+(9-9j)n+(12+12j)n^{2}\right).$

As a special case of the Theorem 3.8 (c), we establish the following corollary.

Corollary 3.12

(a) $\sum_{k=0}^{n} HT_{-2k+1} = \frac{1}{6}\left(n+1\right)\left((6+18j)+(-7-13j)n+(4+4j)n^{2}\right).$

(b) $\sum_{k=0}^{n} HH_{-2k+1} = \left(3j+3\right)\left(n+1\right).$

(c) $\sum_{k=0}^{n} HO_{-2k+1} = \frac{1}{3}\left(n+1\right)\left((6+18j)+(-7-13j)n+(4+4j)n^{2}\right).$

(d) $\sum_{k=0}^{n} Hp_{-2k+1} = \frac{1}{6}\left(n+1\right)\left((6+30j)+(-9-27j)n+(12+12j)n^{2}\right).$

We will now provide a different theorem that allows us to calculate the finite sum of Gaussian numbers.

Theorem 3.9 *For every integer n, hyperbolic generalized Guglielmo numbers have the following formula:*

$$\sum_{k=0}^{n} HW_{n} = (A_{1}\widehat{\alpha}+\widehat{\beta}(A_{2}+A_{3}))(n+1)+(\widehat{\alpha}A_{2}+2\widehat{\beta}A_{3})\frac{n(n+1)}{2}+\widehat{\alpha}A_{3}\frac{n(n+1)(2n+1)}{6}.$$

Proof. The proof can be done easily by using identity (3.6).

Next we can get the following corollary by using Theorem 3.9.

(a) $\sum_{k=0}^{n} HT_{n} = \frac{1}{2}(\beta(n+1)+(\alpha+2\beta)\frac{n(n+1)}{2}+\alpha\frac{n(n+1)(2n+1)}{6}).$

(b) $\sum_{k=0}^{n} HH_{n} = 3\widehat{\alpha}(n+1).$

(c) $\sum_{k=0}^{n} HO_{n} = \beta(n+1)+(\alpha+2\beta)\frac{n(n+1)}{2}+\alpha\frac{n(n+1)(2n+1)}{6}.$

(d) $\sum_{k=0}^{n} Hp_{n} = \frac{1}{2}(2\beta(n+1)+(6\beta-\alpha)\,\frac{n(n+1)}{2}+3\alpha\frac{n(n+1)(2n+1)}{6}).$

3.5 MATRICES RELATED WITH HYPERBOLIC GENERALIZED GUGLIELMO NUMBERS

In this section, first we give the following Lemma and define matrices formulation related with Hyperbolic Generalized Guglielmo Numbers.

Lemma 3.10 *For all integers n the following identity is true*

$$\begin{pmatrix} HW_{n+2} \\ HW_{n+1} \\ HW_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}.$$

Proof. First, we consider $n \geq 0$. Lemma 3.10 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity holds.

$$\begin{pmatrix} HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

For $n = k + 1$, we get

$$\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} HW_2 \\ HW_1 \\ HW_0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} HW_{k+2} \\ HW_{k+1} \\ HW_k \end{pmatrix}$$

$$= \begin{pmatrix} 3HW_{k+2} - 3HW_{k+1} + HW_k \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} HW_{k+3} \\ HW_{k+2} \\ HW_{k+1} \end{pmatrix}.$$

If we get $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Theorem 3.11 *If we define the matrices N_{HW} and E_{HW} as follow*

$$N_{HW} = \begin{pmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{pmatrix},$$

$$E_{HW} = \begin{pmatrix} HW_{n+2} & HW_{n+1} & HW_n \\ HW_{n+1} & HW_n & HW_{n-1} \\ HW_n & HW_{n-1} & HW_{n-2} \end{pmatrix},$$

then the following identity is true:

$$A^n N_{HW} = E_{HW}.$$

Proof. Using (1.11), we can derive

$$A^nN_{HW} = \begin{pmatrix} T_{n+1} & -3T_n+T_{n-1} & T_n \\ T_n & -3T_{n-1}+T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2}+T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} HW_2 & HW_1 & HW_0 \\ HW_1 & HW_0 & HW_{-1} \\ HW_0 & HW_{-1} & HW_{-2} \end{pmatrix},$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$\begin{aligned}
a_{11} &= GHW_2T_{n+1} + HW_1(T_{n-1} - 3T_n) + HW_0T_n, \\
a_{12} &= HW_1T_{n+1} + HW_0(T_{n-1} - 3T_n) + HW_{-1}T_n, \\
a_{13} &= HW_0T_{n+1} + HW_{-1}(T_{n-1} - 3T_n) + HW_{-2}T_n, \\
a_{21} &= HW_2T_n + HW_1(T_{n-2} - 3T_{n-1}) + HW_0T_{n-1}, \\
a_{22} &= HW_1T_n + HW_0(T_{n-2} - 3T_{n-1}) + HW_{-1}T_{n-1}, \\
a_{23} &= HW_0T_n + HW_{-1}(T_{n-2} - 3T_{n-1}) + HW_{-2}T_{n-1}, \\
a_{31} &= HW_2T_{n-1} + HW_1(T_{n-3} - 3T_{n-2}) + HW_0T_{n-2}, \\
a_{32} &= HW_1T_{n-1} + HW_0(T_{n-3} - 3T_{n-2}) + HW_{-1}T_{n-2}, \\
a_{33} &= HW_0T_{n-1} + HW_{-1}(T_{n-3} - 3T_{n-2}) + HW_{-2}T_{n-2}.
\end{aligned}$$

Using the Theorem (3.6) the proof is done. □

From Theorem 3.11, we can write the following corollary.

Corollary 3.13

(a) *Let the matrices N_{HT} and E_{HT} be defined as following*

$$N_{HT} = \begin{pmatrix} HT_2 & HT_1 & HT_0 \\ HT_1 & HT_0 & HT_{-1} \\ HT_0 & HT_{-1} & HT_{-2} \end{pmatrix},$$

$$E_{HT} = \begin{pmatrix} HT_{n+2} & HT_{n+1} & HT_n \\ HT_{n+1} & HT_n & HT_{n-1} \\ HT_n & HT_{n-1} & HT_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, N_{HT}, E_{HT},

$$A^nN_{HT} = E_{HT},$$

(b) *Conside that the matrices N_{HO} and E_{HO} are defined as following*

$$N_{HO} = \begin{pmatrix} HO_2 & HO_1 & HO_0 \\ HO_1 & HO_0 & HO_{-1} \\ HO_0 & HO_{-1} & HO_{-2} \end{pmatrix},$$

$$E_{HO} = \begin{pmatrix} HO_{n+2} & HO_{n+1} & HO_n \\ HO_{n+1} & HO_n & HO_{n-1} \\ HO_n & HO_{n-1} & HO_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, N_{HO}, E_{HO},

$$A^n N_{HO} = E_{HO}.$$

(c) *The matrices N_{HH} and E_{HH} are defined as following*

$$N_{HH} = \begin{pmatrix} HH_2 & HH_1 & HH_0 \\ HH_1 & HH_0 & HH_{-1} \\ HH_0 & HH_{-1} & HH_{-2} \end{pmatrix},$$

$$E_{HH} = \begin{pmatrix} HH_{n+2} & HH_{n+1} & HH_n \\ HH_{n+1} & HH_n & HH_{n-1} \\ HH_n & HH_{n-1} & HH_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, N_{HH}, E_{HH},

$$A^n N_{HH} = E_{HH}.$$

(d) *We assume that the matrices N_{Hp} and E_{Hp} are defined as following*

$$N_{Hp} = \begin{pmatrix} Hp_2 & Hp_1 & Hp_0 \\ Hp_1 & Hp_0 & Hp_{-1} \\ Hp_0 & Hp_{-1} & Hp_{-2} \end{pmatrix},$$

$$E_{Hp} = \begin{pmatrix} Hp_{n+2} & Hp_{n+1} & Hp_n \\ Hp_{n+1} & Hp_n & Hp_{n-1} \\ Hp_n & Hp_{n-1} & Hp_{n-2} \end{pmatrix},$$

such that he following identity is true for A^n, N_{Hp}, E_{Hp},

$$A^n N_{Hp} = E_{Hp}.$$

CHAPTER 4

DUAL GENERALIZED GUGLIELMO NUMBERS

In this chapter, we define dual generalized Guglielmo numbers then we present reccurance relation, generating functions and Binet formulas.

4.1 DEFINITION AND PROPERTIES

We will now explore dual generalized Guglielmo numbers on $\mathbb{H}_{\mathbb{D}}$.The nth generalized dual Guglielmo numbers, with DW_0, DW_1, DW_2 being the initial conditions, are defined as follows

$$DW_n = W_n + \varepsilon W_{n+1}. \tag{4.1}$$

Furthermore (4.1) can be described to negative subscripts by defining,

$$DW_{-n} = W_{-n} + \varepsilon W_{-n+1} \tag{4.2}$$

So the identity (4.1) holds for all integers n.

Now we define some special cases of dual generalized Guglielmo numbers. The nth dual triangular numbers, the nth dual triangular-Lucas numbers, the nth dual oblong numbers and the nth dual pentegonal numbers, respectively, can be seen below.

The nth generalized dual triangular numbers DT_n, with DT_0, DT_1, DT_2 being the initial conditions, are given by the following expressions

$$DT_n = T_n + \varepsilon T_{n+1}$$

where

$$DT_0 = T_0 + \varepsilon T_1, DT_1 = T_1 + \varepsilon T_2, DT_2 = T_2 + \varepsilon T_3.$$

The nth generalized dual triangular-Lucas numbers DH_n, with DH_0, DH_1, DH_2 being the initial conditions, are expressed as follows

$DH_n = \ H_n + j\ H_{n+1}$

where

$DH_0 = \ H_0 + \varepsilon\ H_1, DH_1 = \ H_1 + \varepsilon\ H_2, DH_2 = \ H_2 + \varepsilon\ H_3.$

The nth generalized dual triangular numbers DO_n, with DO_0, DO_1, DO_2 being the initial conditions, are described by the following definitions

$DO_n = O_n + \varepsilon O_{n+1}$

where

$DO_0 = O_0 + \varepsilon O_1, DO_1 = O_1 + \varepsilon O_2, DO_2 = O_2 + \varepsilon O_3.$

The nth generalized dual triangular numbers Dp_n, with Dp_0, Dp_1, Dp_2 being the initial conditions, are defined as follows

$Dp_n = p_n + \varepsilon p_{n+1}$

where

$Dp_0 = p_0 + \varepsilon p_1, Dp_1 = p_1 + \varepsilon p_2, Dp_2 = p_2 + \varepsilon p_3.$

For dual triangular numbers, taking $W_n = T_n$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, we have

$DT_0 = 3\varepsilon, DT_1 = 1 + 6\varepsilon, DT_2 = 3 + 10\varepsilon,$

for dual triangular-Lucas numbers, taking $W_n = H_n$, $H_0 = 3$, $H_1 = 3$, $H_2 = 3$, we derive

$DH_0 = 3 + 3\varepsilon, DH_1 = 3 + 3\varepsilon, DH_2 = 3 + 3\varepsilon,$

for dual oblong numbers,taking $W_n = O_n$, $O_0 = 0$, $O_1 = 2$, $O_2 = 6$, we obtain

$DO_0 = 6\varepsilon, DO_1 = 2 + 12\varepsilon, DO_2 = 6 + 20\varepsilon,$

and for dual pentegonal numbers, taking $W_n = p_n$, $p_0 = 0$, $p_1 = 1$, $p_2 = 5$, we deduce

$Dp_0 = 5\varepsilon, Dp_1 = 1 + 12\varepsilon, Dp_2 = 5 + 22\varepsilon,$

Thus, by using (4.1), we can establish the following identity for non-negative integers n,

$$DW_n = 3DW_{n-1} - 3DW_{n-2} + DW_{n-3}. \tag{4.3}$$

Hence the sequence $\{DW_n\}_{n\geq 0}$ can be given as

$$DW_{-n} = 3DW_{-(n-1)} - 3DW_{-(n-2)} + DW_{-(n-3)},$$

for $n\in\{1, 2, 3....\}$ by using (4.2). Accordingly, recurrence (4.3) is true for all integer n. In the Table 4.1, We provide the initial dual generalized Guglielmo numbers with both positive and negative subscripts.

Table 4.1. Some dual generalized Guglielmo numbers

n	DW_n	DW_{-n}
0	DW_0	DW_0
1	DW_1	$3DW_0 - 3DW_1 + DW_2$
2	DW_2	$6DW_0 - 8DW_1 + 3DW_2$
3	$DW_0 - 3DW_1 + 3DW_2$	$10DW_0 - 15DW_1 + 6DW_2$
4	$3DW_0 - 8DW_1 + 6DW_2$	$15DW_0 - 24DW_1 + 10DW_2$
5	$6DW_0 - 15DW_1 + 10DW_2$	$21DW_0 - 35DW_1 + 15DW_2$
6	$10DW_0 - 24DW_1 + 15DW_2$	$28DW_0 - 48DW_1 + 21DW_2$

remember that

$$\begin{aligned} DW_0 &= W_0 + \varepsilon W_1, \\ DW_1 &= W_1 + \varepsilon W_2, \\ DW_2 &= W_2 + \varepsilon W_3. \end{aligned}$$

Some dual triangular numbers, dual triangular-Lucas numbers, dual oblong numbers, and dual pentagonal numbers with positive or negative subscripts are presented tables which is given below .

Table 4.2 Dual triangular numbers

n	DT_n	DT_{-n}
0	ε	
1	$1 + 3\varepsilon$	0
2	$3 + 6\varepsilon$	1
3	$6 + 10\varepsilon$	$3 + \varepsilon$
4	$10 + 15\varepsilon$	$6 + 3\varepsilon$
5	$15 + 21\varepsilon$	$10 + 6\varepsilon$

Table 4.3. Dual triangular-Lucas numbers

n	DH_n	DH_{-n}
0	$3 + 3\varepsilon$	
1	$3 + 3\varepsilon$	$3 + 3\varepsilon$
2	$3 + 3\varepsilon$	$3 + 3\varepsilon$
3	$3 + 3\varepsilon$	$3 + 3\varepsilon$
4	$3 + 3\varepsilon$	$3 + 3\varepsilon$
5	$3 + 3\varepsilon$	$3 + 3\varepsilon$

Table 4.4 Dual oblong numbers

n	DO_n	DO_{-n}
0	2ε	
1	$2+6\varepsilon$	
2	$6+12\varepsilon$	2
3	$12+20\varepsilon$	$6+2\varepsilon$
4	$20+30\varepsilon$	$12+6\varepsilon$
5	$30+42\varepsilon$	$20+12\varepsilon$

Table 4.5. Dual pentegonal numbers

n	Dp_n	Dp_{-n}
0	ε	
1	$1+5\varepsilon$	2
2	$5+12\varepsilon$	$7+2\varepsilon$
3	$12+22\varepsilon$	$15+7\varepsilon$
4	$22+35\varepsilon$	$26+15\varepsilon$
5	$35+51\varepsilon$	$40+26\varepsilon$

Now, we will establish Binet's formula for the dual generalized Guglielmo numbers, and for the remainder of the study, we will utilize the following notations:

$$\widetilde{\alpha} = 1+\varepsilon, \tag{4.4}$$

$$\widetilde{\beta} = \varepsilon. \tag{4.5}$$

Note that the following identities are true.

$$\begin{aligned}
\widetilde{\alpha}^2 &= 1+2\varepsilon, \\
\widetilde{\beta}^2 &= 0, \\
\widetilde{\alpha}\widetilde{\beta} &= \widetilde{\beta}.
\end{aligned}$$

4.1.1 The Binet's Formula For The Dual Generalized Guglielmo Numbers

Next theorem gives us the Binet's Formula of the dual generalized Guglielmo numbers.

Theorem 4.1 *(Binet's Formula) For any integer n, the nth dual generalized Guglielmo number can be expressed as follows*

$$DW_n = (\widetilde{\alpha}A_1 + \widetilde{\beta}(A_2+A_3)) + (\widetilde{a}A_2 + 2\widetilde{\beta}A_3)n + \widetilde{a}A_3n^2 \tag{4.6}$$

where $\widetilde{\alpha}$, $\widetilde{\beta}$ are given as (4.4)-(4.5).

Proof. Using Binet's formula given below

$$W_n = A_1 + A_2n + A_3n^2$$

of the generalized Guglielmo numbers, A_1, A_2, A_2 are given (1.3), (1.4), (1.5) we get

$$\begin{aligned}
DW_n &= W_n + \varepsilon W_{n+1} \\
&= A_1 + A_2n + A_3n^2 + (A_1 + A_2(n+1) + A_3(n+1)^2)\varepsilon \\
&= (\widetilde{\alpha}A_1 + \widetilde{\beta}(A_2+A_3)) + (\widetilde{a}A_2 + 2\widetilde{\beta}A_3)n + \widetilde{a}A_3n^2.
\end{aligned}$$

This proves (4.6). □

As special cases, for any integer n, the Binet's Formula of nth dual triangual number is expressed as

$$DT_n = \frac{1}{2}(\widetilde{\beta} + (\widetilde{\alpha} + 2\widetilde{\beta})n + \widetilde{\alpha}n^2), \tag{4.7}$$

the Binet's Formula of nth dual triangular-Lucas number is defined as

$$\widehat{H}_n = 3\widetilde{\alpha}, \tag{4.8}$$

the Binet's Formula of nth dual oblong number is stated as

$$DO_n = 2\widetilde{\beta} + \left(\widetilde{\alpha} + 2\widetilde{\beta}\right) n + \widetilde{\alpha}n^2, \tag{4.9}$$

and the Binet's Formula of nth dual pentegonal number given as

$$Dp_n = \frac{1}{2}(2\widetilde{\beta} + \left(6\widetilde{\beta} - \widetilde{\alpha}\right) n + 3\widetilde{\alpha}n^2). \tag{4.10}$$

4.1.2 The Generating Function of Dual Generalized Guglielmo Numbers

Next, we will introduce the generating function of the dual generalized Guglielmo numbers.

Theorem 4.2 *The generating function for the dual generalized Guglielmo numbers is*

$$f_{DW_n}(x) = \frac{DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + 3DW_0)x^2}{(1 - 3x + 3x^2 - x^3)}. \tag{4.11}$$

Proof. Let the generating function of the dual generalized Guglielmo numbers is given below

$$f_{DW_n}(x) = \sum_{n=0}^{\infty} DW_n x^n.$$

Following that, by utilizing the definition of the dual generalized Guglielmo numbers, and substracting $xg(x)$ and $x^2g(x)$ from $g(x)$, we get
$(1 - 3x + 3x^2 - x^3)f_{GDW_n}(x) = (1 - x)^3 \sum_{n=0}^{\infty} DW_n x^n$.

Hence

$$\begin{aligned}
(1-x)^3\sum_{n=0}^{\infty}DW_nx^n &= \sum_{n=0}^{\infty}DW_nx^n-3x\sum_{n=0}^{\infty}DW_nx^n+3x^2\sum_{n=0}^{\infty}DW_nx^n-x^3\sum_{n=0}^{\infty}DW_nx^n\\
&= \sum_{n=0}^{\infty}DW_nx^n-3\sum_{n=0}^{\infty}DW_nx^{n+1}+3\sum_{n=0}^{\infty}DW_nx^{n+2}-\sum_{n=0}^{\infty}DW_nx^{n+3}\\
&= \sum_{n=0}^{\infty}DW_nx^n-3\sum_{n=1}^{\infty}DW_{n-1}x^n+3\sum_{n=2}^{\infty}DW_{n-2}x^n-\sum_{n=3}^{\infty}DW_{n-3}x^n\\
&= (DW_0+DW_1x+DW_2x^2)-3(DWx+DW_1x^2)+3GW_0x^2\\
&\quad +\sum_{n=3}^{\infty}(DW_n-3DW_{n-1}+3DW_{n-2}-DW_{n-3})x^n\\
&= DW_0+DW_1x+DW_2x^2-3DW_0x-3DW_1x^2+3DW_0x^2\\
&= DW_0+(DW_1-3DW_0)x+(DW_2-3DW_1+3DW_0)x^2.
\end{aligned}$$

Note that we use the recurrence relation $DW_n = 3DW_{n-1} - 3DW_{n-2} + DW_{n-3}$. We rearrange equation which is given above then we obtain (4.11). □

As specific cases, the generating functions of the dual triangular, triangular-Lucas, oblong and dual pentegonal numbers are given by

$$\begin{aligned}
f_{\widehat{T}_n}(x) &= \frac{(j+3\varepsilon+6j\varepsilon)+(1-8j\varepsilon-3\varepsilon)\,x+(\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)},\\
f_{\widehat{H}_n}(x) &= \frac{(3+3j+3\varepsilon+3j\varepsilon)+(-6-6j-6\varepsilon-6j\varepsilon)\,x+(3+3j+3\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)},\\
f_{\widehat{O}_n}(x) &= \frac{(2j+6\varepsilon+12j\varepsilon)+(2-16j\varepsilon-6\varepsilon)\,x+(2\varepsilon+6j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)},\\
f_{\widehat{p}_n}(x) &= \frac{(j+5\varepsilon+12j\varepsilon)+(1+2j-3\varepsilon-14j\varepsilon)\,x+(2+\varepsilon+5j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)},
\end{aligned}$$

respectively. □

4.2 DRIVING BINET'S FORMULA FROM GENERATING FUNCTION OF DUAL GENERALIZED GUGLIELMO NUMBERS

Next, we will explore the Binet formula for the dual generalized Guglielmo numbers $\{DW_n\}$ by utilizing generating function $f_{DW_n}(x)$.

Theorem 4.3 *(Binet formula of dual generalized Guglielmo numbers)*

$$DW_n = (\widetilde{\alpha}A_1+\widetilde{\beta}(A_2+A_3))+(\widetilde{a}A_2+2\widetilde{\beta}A_3)n+\widetilde{a}A_3n^2. \tag{4.12}$$

Proof. We write

$$\sum_{n=0}^{\infty} DW_n x^n = \frac{DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + 3DW_0)x^2}{(1 - 3x + 3x^2 - x^3)} \quad (4.13)$$

$$= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3} \quad (4.14)$$

$$= \frac{d_1(1-x)^2 + d_2(1-x) + d_3}{(1-x)^3}. \quad (4.15)$$

then, we get

$$DW_0 + (DW_1 - 3DW_0)x + (DW_2 - 3DW_1 + 3DW_0)x^2 = (d_1 + d_2 + d_3) + (-2d_1 - d_2)x + d_1 x^2.$$

If we equalize the coefficients of the same degree terms of x in the above equation, we get

$$\begin{aligned} DW_0 &= d_1 + d_2 + d_3, \quad (4.16) \\ DW_1 - 3DW_0 &= -2d_1 - d_2, \\ DW_2 - 3DW_1 + 3DW_0 &= d_1. \end{aligned}$$

If we solve system of equation (4.16) we obtain

$$\begin{aligned} d_1 &= 3DW_0 - 3DW_1 + DW_2, \\ d_2 &= 5DW_1 - 3DW_0 - 2DW_2, \\ d_3 &= DW_0 - 2DW_1 + DW_2. \end{aligned}$$

Thus (4.14) can be given as

$$\begin{aligned} \sum_{n=0}^{\infty} DW_n x^n &= d_1 \sum_{n=0}^{\infty} x^n + d_2 \sum_{n=0}^{\infty} (n+1)x^n + d_3 \sum_{n=0}^{\infty} \frac{n^2+3n+2}{2} x^n \\ &= \sum_{n=0}^{\infty} (d_1 + d_2(n+1) + d_3 \frac{n^2+3n+2}{2}) x^n \\ &= \sum_{n=0}^{\infty} (DW_0 + \frac{1}{2}(-DW_2 + 4DW_1 - 3DW_0)n + \frac{1}{2}(DW_2 - 2DW_1 + DW_0)n^2) x^n. \end{aligned}$$

Hence, we get

$$DW_n = \widetilde{A}_1 + \widetilde{A}_2 n + \widetilde{A}_3 n^2$$

where

$$\begin{aligned} \widetilde{A}_1 &= DW, \\ \widetilde{A}_2 &= \frac{1}{2}(-DW_2 + 4DW_1 - 3DW_0), \\ \widetilde{A}_3 &= \frac{1}{2}(DW_2 - 2DW_1 + DW_0). \end{aligned}$$

Note that the following equalities given below are true:

$$\begin{aligned}
\widetilde{A}_1 &= DW_0 \\
&= W_0 + \varepsilon W_1 \\
&= (1+\varepsilon)W_0 + \varepsilon(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) + (\varepsilon)(\frac{1}{2}(W_2 - 2W_1 + W_0)) \\
&= \widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3.
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\widetilde{A}_2 &= \frac{1}{2}(-DW_2 + 4DW_1 - 3DW_0) \\
&= \frac{1}{2}((-3W_0 + 4W_1 - W_2) + \varepsilon(-W_0 + W_2) \\
&= (1+\varepsilon)(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) + \varepsilon((W_2 - 2W_1 + W_0)) \\
&= (\widehat{a}A_2 + 2\widehat{\beta}A_3).
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\widetilde{A}_3 &= \frac{1}{2}(DW_2 - 2DW_1 + DW_0) \\
&= \frac{1}{2}((W_2 - 2W_1 + W_0) + \varepsilon(W_2 - 2W_1 + W_0) \\
&= \widetilde{a}A_3.
\end{aligned} \tag{4.19}$$

Using (4.17), (4.18) and (4.19) , we get

$DW_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{a}A_3n^2.$ □

4.3 SOME IDENTITIES FOR DUAL GENERALIZED GUGLIELMO NUMBERS

We will now introduce some specific identities for the dual generalized Guglielmo sequence $\{DW_n\}$. The next theorem gives the Simpson's formula for the dual generalized Guglielmo numbers.

Theorem 4.4 *(Simpson's formula for dual generalized Guglielmo numbers) For all integers n we have,*

$$\begin{vmatrix} DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+1} & DW_n & DW_{n-1} \\ DW_n & DW_{n-1} & DW_{n-2} \end{vmatrix} = \begin{vmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{vmatrix}. \tag{4.20}$$

Proof. First we assume that $n \geqslant 0$. For the proof, we use the mathematical induction on n. For $n = 0$ identity (4.20) is true. Now we take (4.20) is true for $n = k$. Thus,we get the following identity

$$\begin{vmatrix} DW_{k+2} & DW_{k+1} & DW_k \\ DW_{k+1} & DW_k & DW_{k-1} \\ DW_k & DW_{k-1} & DW_{k-2} \end{vmatrix} = \begin{vmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{vmatrix}.$$

For $n = k+1$, we have

$$\begin{vmatrix} DW_{k+3} & DW_{k+2} & DW_{k+1} \\ DW_{k+2} & DW_{k+1} & DW_k \\ DW_{k+1} & DW_k & DW_{k-1} \end{vmatrix} = \begin{vmatrix} 3DW_{k+2} - 3DW_{k+1} + DW_k & DW_{k+2} & DW_{k+1} \\ 3DW_{k+1} - 3DW_k + DW_{k-1} & DW_{k+1} & DW_k \\ 3DW_k - 3DW_{k-1} + DW_{k-2} & DW_k & DW_{k-1} \end{vmatrix}$$

$$= 3\begin{vmatrix} DW_{k+2} & DW_{k+2} & DW_{k+1} \\ DW_{k+1} & DW_{k+1} & DW_k \\ DW_k & DW_k & DW_{k-1} \end{vmatrix}$$

$$-3\begin{vmatrix} DW_{k+1} & DW_{k+2} & DW_{k+1} \\ DW_k & DW_{k+1} & DW_k \\ DW_{k-1} & DW_k & DW_{k-1} \end{vmatrix} + \begin{vmatrix} DW_k & DW_{k+2} & DW_{k+1} \\ DW_{k-1} & DW_{k+1} & DW_k \\ DW_{k-2} & DW_k & DW_{k-1} \end{vmatrix}$$

$$= \begin{vmatrix} DW_{k+2} & DW_{k+1} & DW_k \\ DW_{k+1} & DW_k & DW_{k-1} \\ DW_k & DW_{k-1} & DW_{k-2} \end{vmatrix}$$

Note that if we consider $n < 0$ the proof can be conducted similarly. Thus, the proof is concluded. □

From Theorem 4.20, we get following corollary.

Corollary 4.1

(a) $$\begin{vmatrix} DT_{n+2} & DT_{n+1} & DT_n \\ DT_{n+1} & DT_n & DT_{n-1} \\ DT_n & DT_{n-1} & DT_{n-2} \end{vmatrix} = -(3\varepsilon + 1).$$

(b) $$\begin{vmatrix} DT_{n+2} & DT_{n+1} & DT_n \\ DT_{n+1} & DT_n & DT_{n-1} \\ DT_n & DT_{n-1} & DT_{n-2} \end{vmatrix} = 0.$$

(c) $\begin{vmatrix} DO_{n+2} & DO_{n+1} & DO_n \\ DO_{n+1} & DO_n & DO_{n-1} \\ DO_n & DO_{n-1} & DO_{n-2} \end{vmatrix} = -8(3\varepsilon+1).$

(d) $\begin{vmatrix} Dp_{n+2} & Dp_{n+1} & Dp_n \\ Dp_{n+1} & Dp_n & Dp_{n-1} \\ Dp_n & Dp_{n-1} & Dp_{n-2} \end{vmatrix} = -27(3\varepsilon+1).$

Now, we define Catalan's identity of dual generalized Guglielmo numbers.

Theorem 4.5 *(Catalan's identity) The following identity is true considering all integers n and m*

$$\begin{aligned} DW_{n+m}DW_{n-m} - DW_n^2 &= m^2(A_3^2\left(2\widetilde{\beta} + \widetilde{a}^2m^2 - 2\widetilde{a}^2n^2 - 4n\widetilde{\beta}\right) \\ &\quad -2A_2A_3\left(\widetilde{\beta} + \widetilde{a}^2n\right) - \widetilde{a}^2\left(A_2^2 - 2A_1A_3\right)). \end{aligned} \tag{4.21}$$

Proof. The proof can be done easily using identity (4.12). □

As special cases of the above theorem, we give Catalan's identity of dual triangular, Lucas-triangular, Oblong, pentagonal numbers.

We present Catalan's identity of dual triangular numbers.

Corollary 4.2 *(Catalan's identity for the dual triangular numbers) The following identity is true considering all integers n and m*

$$DT_{n+m}DT_{n-m} - DT_n^2 = -m^2(-\frac{1}{4}\widetilde{a}^2\left(-2n + m^2 - 2n^2 - 1\right) + \widetilde{\beta}n).$$

Proof. If we get $DW_n = DT_n$ in Theorem 4.5, we acquire the desired outcome. □

We give Catalan's identity of dual triangular-Lucas numbers.

Corollary 4.3 *(Catalan's identity for the dual Lucas-triangular numbers) For all integers n and m, the following identity holds:*

$$DH_{n+m}DH_{n-m} - DH_n^2 = 0.$$

Proof. If we get $DW_n = DH_n$ in Theorem 4.5, we achieve the desired result. □

We give Catalan's identity of dual oblong numbers.

Corollary 4.4 *(Catalan's identity for the dual oblong numbers) The following identity is true considering all integers n and m:*

$$DO_{n+m}DO_{n-m} - DO_n^2 = -m^2\left(-\widetilde{a}^2(-2n + m^2 - 2n^2 - 1) + 4\widetilde{\beta}n\right).$$

Proof. If we get $DW_n = DO_n$ in Theorem 4.5, we reach the goal we aimed for. □

We give Catalan's identity of dual pentagonal numbers.

Corollary 4.5 *(Catalan's identity for the dual penteganal numbers) The following identity is true considering all integers n and m:*

$$Dp_{n+m}Dp_{n-m} - Dp_n^2 = \frac{1}{4}m^2(\widetilde{a}^2\left(6n + 9m^2 - 18n^2 - 1\right) - 12\widetilde{\beta}\,(3n-2)).$$

Proof. If we get $DW_n = Dp_n$ in Theorem 4.5, we obtain the result required. □

By setting $m = 1$ in Catalan's identity, we have Cassini's identity for the dual generalized Guglielmo numbers. Thus, we present the following corollary.

Corollary 4.6 *(Cassini's identity for the dual generalized Guglielmo numbers) For all integers n, the following identities holds.*

(a) $DT_{n+1}DT_{n-1} - DT_n^2 = \frac{1}{4}\widetilde{a}^2\left(-2n - 2n^2\right) - \widetilde{\beta}n.$

(b) $DH_{n+1}DH_{n-1} - DH_n^2 = 0.$

(c) $DO_{n+1}DO_{n-1} - DO_n^2 = \widetilde{a}^2(-2n - 2n^2) - 4\widetilde{\beta}n.$

(d) $Dp_{n+1}Dp_{n-1} - Dp_n^2 = \frac{1}{4}\widetilde{a}^2 6n - 18n^2 + 8 - 3\widetilde{\beta}\,(3n-2)).$

Theorem 4.6 *We assume that n and m are integers, T_n is triangular numbers, the following identity is true:*

$$DW_{m+n} = T_{m-1}DW_{n+2} + (T_{m-3} - 3T_{m-2})DW_{n+1} + T_{m-2}DW_n. \tag{4.22}$$

Proof. The Theorem 4.6 can be proved by mathematical induction on m. First we take $n, m \geqslant 0$. If $m = 0$ we get

$$DW_n = T_{-1}DW_{n+2} + (T_{-3} - 3T_{-2})DW_{n+1} + T_{-2}DW_n$$

which is true by seeing that $T_{-1}=0, T_{-2}=1, T_{-3}=3$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
DW_{(k+1)+n} &= 3DW_{n+k}-3DW_{n+k-1}+DW_{n+k-2} \\
&= 3(T_{k-1}DW_{n+2}+(T_{k-3}-3T_{k-2})DW_{n+1}+T_{k-2}DW_{n}) \\
&\quad -3(T_{k-2}DW_{n+2}+(T_{k-4}-3T_{k-3})DW_{n+1}+T_{k-3}DW_{n}) \\
&\quad +(T_{k-3}DW_{n+2}+(T_{k-5}-3T_{k-4})DW_{n+1}+T_{k-4}DW_{n}) \\
&= (3T_{k-1}-3T_{k-2}+T_{k-3})DW_{n+2}+((3T_{k-3}-3T_{k-4}+T_{k-5}) \\
&\quad -3(3T_{k-2}-3T_{k-3}+T_{k-4}))DW_{n+1}+(3T_{k-2}-3T_{k-3}+T_{k-4})DW_{n} \\
&= T_{k}DW_{n+2}+(T_{k-2}-3T_{k-1})DW_{n+1}+T_{k-1}DW_{n} \\
&= T_{(k+1)-1}DW_{n+2}+(T_{(k+1)-3}-3T_{(k+1)-2})DW_{n+1}+T_{(k+1)-2}DW_{n}.
\end{aligned}$$

The other cases on n, m the proof can be done easily. Consequently, by mathematical induction on m, this proves Theorem 4.6. □

4.4 LINEAR SUM FORMULAS FOR DUAL GENERALIZED GUGLIELMO NUMBERS

In this section, we give the summation formulas of the generalized Guglielmo numbers with positive and negatif subscripts.

Now, we will introduce the formulas that allow us to find the sum of dual generalized Guglielmo numbers.

Theorem 4.7 *For $n \geq 0$, dual generalized Guglielmo numbers have the following formulas:*

(a) $\sum_{k=0}^{n} DW_{k} = \frac{1}{6}(n+1)((-n+\varepsilon n^{2}+2\varepsilon n+n^{2})W_{2}+(6\varepsilon+5n-2\varepsilon n^{2}-\varepsilon n-2n^{2})$ $W_{1}+(-4n+\varepsilon n^{2}-\varepsilon n+n^{2}+6)W_{0})$.

(b) $\sum_{k=0}^{n} DW_{2k} = \frac{1}{6}(n+1)((-n+4\varepsilon n^{2}+5\varepsilon n+4n^{2})W_{2}+(6\varepsilon+8n-8\varepsilon n^{2}-4\varepsilon n-8n^{2})W_{1}$ $+(-7n+4\varepsilon n^{2}-\varepsilon n+4n^{2}+6)W_{0})$.

(c) $\sum_{k=0}^{n} DW_{2k+1} = \frac{1}{6}(n+1)((6\varepsilon+5n+4\varepsilon n^{2}+11\varepsilon n+4n^{2})W_{2}+(6-8\varepsilon n^{2}-16\varepsilon n-$ $8n^{2}-4n)W_{1}+(-n+4\varepsilon n^{2}+5\varepsilon n+4n^{2})W_{0})$.

Proof.

(a) Note that using (4.1), we get

$$\sum_{k=0}^{n} DW_k = \sum_{k=0}^{n} W_k + \varepsilon \sum_{k=0}^{n} W_{k+1}$$

and using Proposition 1.2, the proof can be easily accomplished

(b) Note that using (4.1), we get

$$\sum_{k=0}^{n} DW_{2k} = \sum_{k=0}^{n} W_{2k} + \varepsilon \sum_{k=0}^{n} W_{2k+1}$$

and using Proposition 1.3, the proof can be easily carried out.

(c) Note that using (4.1), we get

$$\sum_{k=0}^{n} DW_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + \varepsilon \sum_{k=0}^{n} W_{2k+2}$$

and using Proposition 1.3, the proof can be easily demonstrated. □
As a special case of the Theorem 4.7 (a), we outline the following corollary.

Corollary 4.7

(a) $\sum_{k=0}^{n} DT_k = \frac{1}{6}(n+1)(6\varepsilon + (5\varepsilon+2)n + (\varepsilon+1)n^2)$.

(b) $\sum_{k=0}^{n} DH_k = (3\varepsilon+3)(n+1)$.

(c) $\sum_{k=0}^{n} DO_k = \frac{1}{6}(n+1)(12\varepsilon + (10\varepsilon+4)n + (2\varepsilon+2)n^2)$.

(d) $\sum_{k=0}^{n} Dp_k = \frac{1}{6}(n+1)(6\varepsilon + 9\varepsilon n + (3\varepsilon+3)n^2)$.

As a special case of the Theorem 4.7 (b), the following corollary follows.

Corollary 4.8

(a) $\sum_{k=0}^{n} DT_{2k} = \frac{1}{6}(n+1)(6\varepsilon + (5+11\varepsilon)n + (4+4\varepsilon)n^2)$.

(b) $\sum_{k=0}^{n} DH_{2k} = (3\varepsilon+3)(n+1)$.

(c) $\sum_{k=0}^{n} DO_{2k} = \frac{1}{6}(n+1)(12\varepsilon + (10+22\varepsilon)n + (8+8\varepsilon)n^2)$.

(d) $\sum_{k=0}^{n} Dp_{2k} = \frac{1}{6}(n+1)(6\varepsilon + (3+21\varepsilon)n + (12+12\varepsilon)n^2)$.

As a special case of the Theorem 4.7 (c), we present following corollary.

Corollary 4.9

(a) $\sum_{k=0}^{n} DT_{2k+1} = \frac{1}{6}(n+1)((6+18\varepsilon) + (11+17\varepsilon)n + (4+4\varepsilon)n^2)$.

(b) $\sum_{k=0}^{n} DH_{2k+1} = (3\varepsilon+3)(n+1)$.

(c) $\sum_{k=0}^{n} DO_{2k+1} = \frac{1}{6}(n+1)((12+36\varepsilon) + (22+34\varepsilon)n + (8+8\varepsilon)n^2)$.

(d) $\sum_{k=0}^{n} Dp_{2k+1} = \frac{1}{6}(n+1)((6+30\varepsilon) + (21+39\varepsilon)n + (12+12\varepsilon)n^2)$.

Now, we present the formula that yield the summation formulas of the generalized Guglielmo numbers with negative subscripts.

Theorem 4.8 *For $n \geq 1$, dual generalized Guglielmo numbers have the following formulas:*

(a) $\sum_{k=0}^{n} DW_{-k} = \frac{1}{6}(n+1)((2n+\varepsilon n^2 - \varepsilon n + n^2)W_2 + (6\varepsilon - 7n - 2\varepsilon n^2 - \varepsilon n - 2n^2)W_1 + (5n + \varepsilon n^2 + 2\varepsilon n + n^2 + 6)W_0)$.

(b) $\sum_{k=0}^{n} DW_{-2k} = \frac{1}{6}(n+1)((5n+4\varepsilon n^2 - \varepsilon n + 4n^2)W_2 + (6\varepsilon - 16n - 8\varepsilon n^2 - 4\varepsilon n - 8n^2)W_1 + (11n + 4\varepsilon n^2 + 5\varepsilon n + 4n^2 + 6)W_0)$.

(c) $\sum_{k=0}^{n} DW_{-2k+1} = \frac{1}{6}(n+1)((6\varepsilon - n + 4\varepsilon n^2 - 7\varepsilon n + 4n^2)W_2 + (-4n - 8\varepsilon n^2 + 8\varepsilon n - 8n^2 + 6)W_1 + (5n + 4\varepsilon n^2 - \varepsilon n + 4n^2)W_0)$.

Proof.

(a) Note that using (4.1), we get

$$\sum_{k=0}^{n} DW_{-k} = \sum_{k=0}^{n} W_{-k} + \varepsilon \sum_{k=0}^{n} W_{-k+1}$$

and using Proposition 1.4, the proof can be easily executed.

(b) Note that using (4.1), we get

$$\sum_{k=0}^{n} DW_{-2k} = \sum_{k=0}^{n} W_{-2k} + \varepsilon \sum_{k=0}^{n} W_{-2k+1}$$

and using Proposition 1.5, the proof can be easily carried out.

(c) Note that using (4.1), we get

$$\sum_{k=0}^{n} DW_{-2k+1} = \sum_{k=0}^{n} W_{-2k+1} + \varepsilon \sum_{k=0}^{n} W_{-2k+2}$$

and using Proposition 1.5, the proof can be done easily. □

As a special case of the Theorem 4.8 (a), we get the following corollary.

Corollary 4.10

(a) $\sum_{k=0}^{n} DT_{-k} = \frac{1}{6}\left(n+1\right)\left(6\varepsilon + (-1-4\varepsilon)n + (1+\varepsilon)n^2\right).$

(b) $\sum_{k=0}^{n} DH_{-k} = \left(3\varepsilon + 3\right)\left(n+1\right).$

(c) $\sum_{k=0}^{n} DO_{-k} = \frac{1}{6}\left(n+1\right)\left(12\varepsilon + (-2-8\varepsilon)n + (2+2\varepsilon)n^2\right).$

(d) $\sum_{k=0}^{n} Dp_{-k} = \frac{1}{2}\left(n+1\right)\left(2\varepsilon + (1-2\varepsilon)n + (1+\varepsilon)n^2\right).$

As a special case of the Theorem 4.8 (b), we have the following corollary.

Corollary 4.11

(a) $\sum_{k=0}^{n} DT_{-2k} = \frac{1}{6}\left(n+1\right)\left(6\varepsilon + (-1-7\varepsilon)n + (4+4\varepsilon)n^2\right).$

(b) $\sum_{k=0}^{n} DH_{-2k} = \left(3\varepsilon + 3\right)\left(n+1\right).$

(c) $\sum_{k=0}^{n} DO_{-2k} = \frac{1}{3}\left(n+1\right)\left(6\varepsilon + (-1-7\varepsilon)n + (4+4\varepsilon)n^2\right).$

(d) $\sum_{k=0}^{n} Dp_{-2k} = \frac{1}{6}\left(n+1\right)\left((6\varepsilon) + (9-9\varepsilon)n + (12+12\varepsilon)n^2\right).$

As a special case of the Theorem 4.8 (c), we derive the following corollary.

Corollary 4.12

(a) $\sum_{k=0}^{n} DT_{-2k+1} = \frac{1}{6}(n+1)((6+18\varepsilon)+(-7-13\varepsilon)n+(4+4\varepsilon)n^2)$.

(b) $\sum_{k=0}^{n} DH_{-2k+1} = (3\varepsilon+3)(n+1)$.

(c) $\sum_{k=0}^{n} DO_{-2k+1} = \frac{1}{3}(n+1)((6+18\varepsilon)+(-7-13\varepsilon)n+(4+4\varepsilon)n^2)$.

(d) $\sum_{k=0}^{n} Dp_{-2k+1} = \frac{1}{6}(n+1)((6+30\varepsilon)+(-9-27\varepsilon)n+(12+12\varepsilon)n^2)$.

We will now provide a different theorem that allows us to calculate the finite sum of dual generalized Gaussian numbers.

Theorem 4.9 *Let x, y, m are integers. the following sum formula are true:*

$$\begin{aligned}\sum_{k=0}^{m} DW_{xk+y} &= (\widetilde{\alpha}A_1+\widetilde{\beta}(A_2+A_3))(m+1)+(\widetilde{\alpha}A_2+2\widetilde{\beta}A_3)\frac{(m+1)}{2}(xm+2y)\\ &\quad +\widetilde{\alpha}A_3\frac{(m+1)}{2}(x^2\frac{m(2m+1)}{3}+2xym+2y^2)\end{aligned}$$

Proof. For the proof we use Binet's formula of dual generalized guglielmo numbers and we write

$$\begin{aligned}\sum_{k=0}^{m} DW_{xk+y} &= \sum_{k=0}^{m}(\widetilde{\alpha}A_1+\widetilde{\beta}(A_2+A_3))+(\widetilde{\alpha}A_2+2\widetilde{\beta}A_3)\sum_{k=0}^{m}(xk+y)+\widetilde{\alpha}A_3\sum_{k=0}^{m}(xk+y)^2\\ &= (\widetilde{\alpha}A_1+\widetilde{\beta}(A_2+A_3))(m+1)+(\widetilde{\alpha}A_2+2\widetilde{\beta}A_3)\frac{(m+1)}{2}(xm+2y)\\ &\quad +\widetilde{\alpha}A_3\frac{(m+1)}{2}(x^2\frac{m(2m+1)}{3}+2xym+2y^2).\end{aligned}$$

Thus, the proof has been completed. □

From the Theorem 4.9 we can write the following corollary.

Corollary 4.13

(a) $\sum_{k=0}^{m} DT_{xk+y} = \widetilde{\beta}(m+1)+(\frac{1}{2}\widetilde{\alpha}+\widetilde{\beta})\frac{(m+1)}{2}(xm+2y)+\widetilde{\alpha}\frac{(m+1)}{4}(x^2\frac{m(2m+1)}{3}+2xym+2y^2)$.

(b) $\sum_{k=0}^{m} DH_{xk+y} = 3\widetilde{\alpha}(m+1)$.

(c) $\sum_{k=0}^{m} DO_{xk+y} = 2\widetilde{\beta}(m+1)+(2\widetilde{\alpha}+2\widetilde{\beta})\frac{(m+1)}{2}(xm+2y)+\widetilde{\alpha}\frac{(m+1)}{2}(x^2\frac{m(2m+1)}{3}+2xym+2y^2)$.

(d) $\sum_{k=0}^{m} Dp_{xk+y} = \widetilde{\beta}(m+1)+(-\frac{1}{2}\widetilde{\alpha}+3\widetilde{\beta})\frac{(m+1)}{2}(xm+2y)+3\widetilde{\alpha}\frac{(m+1)}{4}(x^2\frac{m(2m+1)}{3}+2xym+2y^2)$.

4.5 MATRICES RELATED WITH DUAL GENERALIZED GUGLIELMO NUMBERS

In this section, we give matrix formulations related to dual generalized Guglielmo Numbers. Thus, we give following lemma.

Lemma 4.10 *For all integers n the following identity is true*

$$\begin{pmatrix} DW_{n+2} \\ DW_{n+1} \\ DW_n \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

Proof. First, we get $n \geq 0$. Lemma 4.10 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}.$$

For $n = k+1$, we get

$$\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} DW_2 \\ DW_1 \\ DW_0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} DW_{k+2} \\ DW_{k+1} \\ DW_k \end{pmatrix}$$

$$= \begin{pmatrix} 3DW_{k+2} - 3DW_{k+1} + DW_k \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} DW_{k+3} \\ DW_{k+2} \\ DW_{k+1} \end{pmatrix}.$$

If we get $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Theorem 4.11 *If we define the matrices N_{DW} and E_{DW} as follow*

$$N_{DW} = \begin{pmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{pmatrix},$$

$$E_{DW} = \begin{pmatrix} DW_{n+2} & DW_{n+1} & DW_n \\ DW_{n+1} & DW_n & DW_{n-1} \\ DW_n & DW_{n-1} & DW_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{DW} = E_{DW}.$$

Proof. Using (1.11), we obtain the following identities

$$A^n N_{DW} = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} DW_2 & DW_1 & DW_0 \\ DW_1 & DW_0 & DW_{-1} \\ DW_0 & DW_{-1} & DW_{-2} \end{pmatrix},$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{11} = DW_2T_{n+1} + DW_1(T_{n-1} - 3T_n) + DW_0T_n,$$
$$a_{12} = DW_1T_{n+1} + DW_0(T_{n-1} - 3T_n) + DW_{-1}T_n,$$
$$a_{13} = DW_0T_{n+1} + DW_{-1}(T_{n-1} - 3T_n) + DW_{-2}T_n,$$
$$a_{21} = DW_2T_n + DW_1(T_{n-2} - 3T_{n-1}) + DW_0T_{n-1},$$
$$a_{22} = DW_1T_n + DW_0(T_{n-2} - 3T_{n-1}) + DW_{-1}T_{n-1},$$
$$a_{23} = DW_0T_n + DW_{-1}(T_{n-2} - 3T_{n-1}) + DW_{-2}T_{n-1},$$
$$a_{31} = DW_2T_{n-1} + DW_1(T_{n-3} - 3T_{n-2}) + DW_0T_{n-2},$$
$$a_{32} = DW_1T_{n-1} + DW_0(T_{n-3} - 3T_{n-2}) + DW_{-1}T_{n-2},$$
$$a_{33} = DW_0T_{n-1} + DW_{-1}(T_{n-3} - 3T_{n-2}) + DW_{-2}T_{n-2}.$$

Using the Theorem 4.6 the proof is done. □

From Theorem 4.11, we can write the following corollary.

Corollary 4.14

(a) *We assume that the matrices N_{DT} and E_{DT} are defined as following*

$$N_{DT} = \begin{pmatrix} DT_2 & DT_1 & DT_0 \\ DT_1 & DT_0 & DT_{-1} \\ DT_0 & DT_{-1} & DT_{-2} \end{pmatrix},$$

$$E_{D\widetilde{T}} = \begin{pmatrix} DT_{n+2} & DT_{n+1} & DT_n \\ DT_{n+1} & DT_n & DT_{n-1} \\ DT_n & DT_{n-1} & DT_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, N_{DT}, E_{DT}:

$$A^n N_{DT} = E_{DT},$$

(b) *We assume that the matrices N_{DH} and E_{DH} are defined as following*

$$N_{DH} = \begin{pmatrix} DH_2 & DH_1 & DH_0 \\ DH_1 & DH_0 & DH_{-1} \\ DH_0 & DH_{-1} & DH_{-2} \end{pmatrix},$$

$$E_{DH} = \begin{pmatrix} DH_{n+2} & DH_{n+1} & DH_n \\ DH_{n+1} & DH_n & DH_{n-1} \\ DH_n & DH_{n-1} & DH_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, N_{DH}, E_{DH}:

$$A^n N_{DH} = E_{DH}.$$

(c) *We assume that the matrices N_{DO} and E_{DO} are defined as following*

$$N_{DO} = \begin{pmatrix} DO_2 & DO_1 & DO_0 \\ DO_1 & DO_0 & DO_{-1} \\ DO_0 & DO_{-1} & DO_{-2} \end{pmatrix},$$

$$E_{DO} = \begin{pmatrix} DO_{n+2} & DO_{n+1} & DO_n \\ DO_{n+1} & DO_n & DO_{n-1} \\ DO_n & DO_{n-1} & DO_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, N_{DO}, E_{DO}:

$$A^n N_{DO} = E_{DO}.$$

(d) *We assume that the matrices N_{Dp} and E_{Dp} are defined as following*

$$N_{Dp} = \begin{pmatrix} Dp_2 & Dp_1 & Dp_0 \\ Dp_1 & Dp_0 & Dp_{-1} \\ Dp_0 & Dp_{-1} & Dp_{-2} \end{pmatrix},$$

$$E_{Dp} = \begin{pmatrix} Dp_{n+2} & Dp_{n+1} & Dp_n \\ Dp_{n+1} & Dp_n & Dp_{n-1} \\ Dp_n & Dp_{n-1} & Dp_{n-2} \end{pmatrix}.$$

such that the following identity is true for A^n, N_{Dp}, E_{Dp}:

$$A^n N_{Dp} = E_{Dp}.$$

CHAPTER 5

DUAL HYPERBOLIC GENERALIZED GUGLIELMO NUMBERS

In this chapter, we define dual hyperbolic generalized Guglielmo numbers then we present generating functions and Binet formulas for them.

5.1 DEFINITION AND PROPERTIES

We now investigate dual hyperbolic generalized Guglielmo numbers over $\mathbb{H}_{\mathbb{D}}$. The nth dual hyperbolic generalized Guglielmo number is

$$\widehat{W}_n = W_n + jW_{n+1} + \varepsilon W_{n+2} + j\varepsilon W_{n+3}. \tag{5.1}$$

with the initial values $\widehat{W}_0, \widehat{W}_1, \widehat{W}_2$. (5.1) can be written to negative subscripts by defining,

$$\widehat{W}_{-n} = W_{-n} + jW_{-n+1} + \varepsilon W_{-n+2} + j\varepsilon W_{-n+3}. \tag{5.2}$$

So identity (5.1) holds for all integers n.

Now we define some special cases of dual hyperbolic generalized Guglielmo numbers. The nth dual hyperbolic triangular numbers, the nth dual hyperbolic triangular-Lucas numbers, the nth dual hyperbolic oblong numbers and the nth dual hyperbolic pentagonal numbers, respectively, are given as

The nth dual hyperbolic triangular numbers $\widehat{T}_n = T_n + jT_{n+1} + \varepsilon T_{n+2} + j\varepsilon T_{n+3}$, with the initial values as

$$\begin{aligned}
\widehat{T}_0 &= T_0 + jT_1 + \varepsilon T_2 + j\varepsilon T_3, \\
\widehat{T}_1 &= T_1 + jT_2 + \varepsilon T_3 + j\varepsilon T_4, \\
\widehat{T}_2 &= T_2 + jT_3 + \varepsilon T_4 + j\varepsilon T_5.
\end{aligned}$$

The nth dual hyperbolic triangular-Lucas numbers $\widehat{H}_n = H_n + jH_{n+1} + \varepsilon H_{n+2} + j\varepsilon H_{n+3}$ with the initial values as

$$
\begin{aligned}
\widehat{H}_0 &= H_0 + jH_1 + \varepsilon H_2 + j\varepsilon H_3, \\
\widehat{H}_1 &= H_1 + jH_2 + \varepsilon H_3 + j\varepsilon H_4, \\
\widehat{H}_2 &= H_2 + jH_3 + \varepsilon H_4 + j\varepsilon H_5.
\end{aligned}
$$

The nth dual hyperbolic oblong numbers $\widehat{O}_n = O_n + jO_{n+1} + \varepsilon O_{n+2} + j\varepsilon O_{n+3}$ with the initial values as

$$
\begin{aligned}
\widehat{O}_0 &= O_0 + jO_1 + \varepsilon O_2 + j\varepsilon O_3, \\
\widehat{O}_1 &= O_1 + jO_2 + \varepsilon O_3 + j\varepsilon O_4, \\
\widehat{O}_2 &= O_2 + jO_3 + \varepsilon O_4 + j\varepsilon O_5.
\end{aligned}
$$

The nth dual hyperbolic pentagonal numbers $\widehat{p}_n = p_n + jp_{n+1} + \varepsilon p_{n+2} + j\varepsilon p_{n+3}$ with the initial values as

$$
\begin{aligned}
\widehat{p}_0 &= p_0 + jp_1 + \varepsilon p_2 + j\varepsilon p_3, \\
\widehat{p}_1 &= p_1 + jp_2 + \varepsilon p_3 + j\varepsilon p_4, \\
\widehat{p}_2 &= p_2 + jp_3 + \varepsilon p_4 + j\varepsilon p_5.
\end{aligned}
$$

For dual hyperbolic triangular numbers (taking $W_n = T_n$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$) we obtain

$$
\begin{aligned}
\widehat{T}_0 &= j + 3\varepsilon + 6j\varepsilon, \\
\widehat{T}_1 &= 1 + 3j + 6\varepsilon + 10j\varepsilon, \\
\widehat{T}_2 &= 3 + 6j + 10\varepsilon + 15j\varepsilon.
\end{aligned}
$$

For dual hyperbolic triangular-Lucas numbers (taking $W_n = H_n$, $H_0 = 3$, $H_1 = 3$, $H_2 = 3$) we obtain

$$
\begin{aligned}
\widehat{H}_0 &= 3 + 3j + 3\varepsilon + 3j\varepsilon, \\
\widehat{H}_1 &= 3 + 3j + 3\varepsilon + 3j\varepsilon, \\
\widehat{H}_2 &= 3 + 3j + 3\varepsilon + 3j\varepsilon.
\end{aligned}
$$

For dual hyperbolic oblong numbers (taking $W_n = O_n$, $O_0 = 0$, $O_1 = 2$, $O_2 = 6$) we obtain

$$\begin{aligned}
\widehat{O}_0 &= 2j + 6\varepsilon + 12j\varepsilon, \\
\widehat{O}_1 &= 2 + 6j + 12\varepsilon + 20j\varepsilon, \\
\widehat{O}_2 &= 6 + 12j + 20\varepsilon + 30j\varepsilon.
\end{aligned}$$

For dual hyperbolic pentagonal numbers (taking $W_n = p_n$, $p_0 = 0$, $p_1 = 1$, $p_2 = 5$) we obtain

$$\begin{aligned}
\widehat{p}_0 &= j + 5\varepsilon + 12j\varepsilon, \\
\widehat{p}_1 &= 1 + 5j + 12\varepsilon + 22j\varepsilon, \\
\widehat{p}_2 &= 5 + 12j + 22\varepsilon + 35j\varepsilon.
\end{aligned}$$

So, using (5.1) we can write the following identity for non negative integers n,

$$\widehat{W}_n = 3\widehat{W}_{n-1} - 3\widehat{W}_{n-2} + \widehat{W}_{n-3}. \tag{5.3}$$

Hence the sequence $\{\widehat{W}_n\}_{n\geq 0}$ can be given as

$$\widehat{W}_{-n} = 3\widehat{W}_{-(n-1)} - 3\widehat{W}_{-(n-2)} + \widehat{W}_{-(n-3)},$$

for $n = 1, 2, 3, ...$ by using (5.2). As a result, recurrence (5.3) holds for all integer n.

In the Table 5.1, we present the first few dual hyperbolic generalized Guglielmo numbers with positive subscript and negative subscript.

Table 5.1. A few dual hyperbolic generalized Guglielmo numbers

n	$\widehat{W}_n$	$\widehat{W}_{-n}$
0	$\widehat{W}_0$	$\widehat{W}_0$
1	$\widehat{W}_1$	$3\widehat{W}_0 - 3\widehat{W}_1 + \widehat{W}_2$
2	$\widehat{W}_2$	$6\widehat{W}_0 - 8\widehat{W}_1 + 3\widehat{W}_2$
3	$\widehat{W}_0 - 3\widehat{W}_1 + 3\widehat{W}_2$	$10\widehat{W}_0 - 15\widehat{W}_1 + 6\widehat{W}_2$
4	$3\widehat{W}_0 - 8\widehat{W}_1 + 6\widehat{W}_2$	$15\widehat{W}_0 - 24\widehat{W}_1 + 10\widehat{W}_2$
5	$6\widehat{W}_0 - 15\widehat{W}_1 + 10\widehat{W}_2$	$21\widehat{W}_0 - 35\widehat{W}_1 + 15\widehat{W}_2$
6	$10\widehat{W}_0 - 24\widehat{W}_1 + 15\widehat{W}_2$	$28\widehat{W}_0 - 48\widehat{W}_1 + 21\widehat{W}_2$

Remember that

$$\begin{aligned}
\widehat{W}_0 &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon W_3, \\
\widehat{W}_1 &= W_1 + jW_2 + \varepsilon W_3 + j\varepsilon W_4, \\
\widehat{W}_2 &= W_2 + jW_3 + \varepsilon W_4 + j\varepsilon W_5.
\end{aligned}$$

A few dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers and dual hyperbolic pentagonal numbers with positive subscript and negative subscript are given in the following Table 5.2, Table 5.3, Table 5.4 and Table 5.5.

While writing the tables given below, we give some negative and positive values for n.

By using $\widehat{T}_n = T_n + jT_{n+1} + \varepsilon T_{n+2} + j\varepsilon T_{n+3}$,

where

$T_0 = 0$, $T_1 = 1$, $T_2 = 3$.

We get a few dual hyperbolic triangular numbers in the the following tables.

Table 5.2. Dual hyperbolic triangular numbers

n	$\widehat{T}_n$	$\widehat{T}_{-n}$
0	$j + 3\varepsilon + 6j\varepsilon$	
1	$1 + 3j + 6\varepsilon + 10j\varepsilon$	$\varepsilon + 3j\varepsilon$
2	$3 + 6j + 10\varepsilon + 15j\varepsilon$	$1 + j\varepsilon$
3	$6 + 10j + 15\varepsilon + 21j\varepsilon$	$3 + j$
4	$10 + 15j + 21\varepsilon + 28j\varepsilon$	$6 + 3j + \varepsilon$
5	$15 + 21j + 28\varepsilon + 36j\varepsilon$	$10 + 6j + 3\varepsilon + j\varepsilon$

By using $\widehat{H}_n = H_n + jH_{n+1} + \varepsilon H_{n+2} + j\varepsilon H_{n+3}$,

where

$H_0 = 3$, $H_1 = 3$, $H_2 = 3$.

We get a few dual hyperbolic triangular-Lucas numbers in the the following tables.

Table 5.3. Dual hyperbolic triangular-Lucas numbers

n	$\widehat{H}_n$	$\widehat{H}_{-n}$
0	$3 + 3j + 3\varepsilon + 3j\varepsilon$	
1	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
2	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
3	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
4	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$
5	$3 + 3j + 3\varepsilon + 3j\varepsilon$	$3 + 3j + 3\varepsilon + 3j\varepsilon$

By using $\widehat{O}_n = O_n + jO_{n+1} + \varepsilon O_{n+2} + j\varepsilon O_{n+3}$,

where

$O_0 = 0$, $O_1 = 2$, $O_2 = 6$.

We get a few dual hyperbolic oblong numbers in the the following tables.

Table 5.4. Dual hyperbolic oblong numbers

n	$\widehat{O}_n$	$\widehat{O}_{-n}$
0	$2j+6\varepsilon+12j\varepsilon$	
1	$2+6j+12\varepsilon+20j\varepsilon$	$2\varepsilon+6j\varepsilon$
2	$6+12j+20\varepsilon+30j\varepsilon$	$2+2\varepsilon j$
3	$12+20j+30\varepsilon+42j\varepsilon$	$6+2j$
4	$20+30j+42\varepsilon+56j\varepsilon$	$12+6j+2\varepsilon$
5	$30+42j+56\varepsilon+72j\varepsilon$	$20+12j+6\varepsilon+2j\varepsilon$

By using $\widehat{p}_n = p_n + jp_{n+1} + \varepsilon p_{n+2} + j\varepsilon p_{n+3}$,

where

$p_0 = 0$, $p_1 = 1$, $p_2 = 5$.

We get a few dual hyperbolic pentegonal numbers in the the following tables.

Table 5.5. Dual hyperbolic pentegonal numbers

n	$\widehat{p}_n$	$\widehat{p}_{-n}$
0	$j+5\varepsilon+12j\varepsilon$	
1	$1+5j+12\varepsilon+22j\varepsilon$	$2+\varepsilon+5j\varepsilon$
2	$5+12j+22\varepsilon+35j\varepsilon$	$7+2j+j\varepsilon$
3	$12+22j+35\varepsilon+51j\varepsilon$	$15+7j+2\varepsilon$
4	$22+35j+51\varepsilon+70j\varepsilon$	$26+15j+7\varepsilon+2j\varepsilon$
5	$35+51j+70\varepsilon+92j\varepsilon$	$40+26j+15\varepsilon+7j\varepsilon$

Now, we will settle Binet's formula for the dual hyperbolic generalized Guglielmo numbers and in the rest of the study, we use the following notations:

$$\widehat{\alpha} = 1+j+\varepsilon+j\varepsilon, \tag{5.4}$$

$$\widehat{\beta} = j+2\varepsilon+3j\varepsilon, \tag{5.5}$$

$$\widehat{\gamma} = j+4\varepsilon+9j\varepsilon. \tag{5.6}$$

It is important to note that we have the following identities:

$$
\begin{aligned}
\widehat{\alpha}^2 &= 2+2j+4\varepsilon+4j\varepsilon,\\
\widehat{\beta}^2 &= 1+6\varepsilon+4j\varepsilon,\\
\widehat{\gamma}^2 &= 1+18\varepsilon+8j\varepsilon,\\
\widehat{\alpha}\widehat{\beta} &= 1+j+6\varepsilon+6j\varepsilon,\\
\widehat{\alpha}\widehat{\gamma} &= 1+j+14\varepsilon++14j\varepsilon,\\
\widehat{\beta}\widehat{\gamma} &= 1+6j\varepsilon+12\varepsilon.
\end{aligned}
$$

5.1.1 The Binet's Formula For The Dual Hyperbolic Generalized Guglielmo Numbers

Next theorem gives us the Binet's Formula of the dual hyperbolic generalized Guglielmo numbers.

Theorem 5.1 *(Binet's Formula) For any integer n, the nth dual hyperbolic generalized Guglielmo number is*

$$\widehat{W}_n = (\widehat{\alpha}A_1+\widehat{\beta}A_2+\widehat{\gamma}A_3)+(\widehat{a}A_2+2\widehat{\beta}A_3)n+\widehat{a}A_3n^2 \tag{5.7}$$

*where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ are given in (5.4)-(**??**).*

Proof. Using Binet's formula given below

$$W_n = A_1+A_2n+A_3n^2$$

of the generalized Guglielmo numbers, A_1, A_2, A_2 are given (1.3), (1.4), (1.5), we get

$$
\begin{aligned}
\widehat{W}_n &= W_n+jW_{n+1}+\varepsilon W_{n+2}+j\varepsilon W_{n+3}\\
&= A_1+A_2n+A_3n^2+(A_1+A_2(n+1)+A_3(n+1)^2)j+(A_1+A_2(n+2)+A_3(n+2)^2)\varepsilon\\
&\quad +(A_1+A_2(n+3)+A_3(n+3)^2)j\varepsilon\\
&= (\widehat{\alpha}A_1+\widehat{\beta}A_2+\widehat{\gamma}A_3)+(\widehat{a}A_2+2\widehat{\beta}A_3)n+\widehat{\alpha}A_3n^2.
\end{aligned}
$$

This proves (5.7). □

As special cases, for any integer n, the Binet's Formula of nth dual hyperbolic triangular number is given below

$$\widehat{T}_n = \frac{1}{2}((\beta+\gamma)+(\alpha+2\beta)n+\alpha n^2), \tag{5.8}$$

the Binet's Formula of nth dual hyperbolic triangular-Lucas number can be seen below

$$\widehat{H}_n = 3\alpha, \tag{5.9}$$

the Binet's Formula of nth dual hyperbolic oblong number can be presented below

$$\widehat{O}_n = (\beta+\gamma) + (\alpha+2\beta)\, n + \alpha n^2, \tag{5.10}$$

and the Binet's Formula of nth dual hyperbolic pentegonal number is shown below

$$\widehat{p}_n = \frac{1}{2}((-\beta+3\gamma) + (6\beta-\alpha)n + 3\alpha n^2). \tag{5.11}$$

5.1.2 The Generating Function of Dual Hyperbolic Generalized Guglielmo Numbers

Next, we present generating function of the dual hyperbolic generalized Guglielmo numbers.

Theorem 5.2 *The generating function for the dual hyperbolic generalized Guglielmo numbers is*

$$f_{\widehat{W}_n}(x) = \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2}{(1-3x+3x^2-x^3)}. \tag{5.12}$$

Proof. Let

$$f_{\widehat{W}_n}(x) = \sum_{n=0}^{\infty} \widehat{W}_n x^n$$

be generating function of the dual hyperbolic generalized Guglielmo numbers. Then, using the definition of the dual hyperbolic generalized Guglielmo numbers, and substracting $xf_{\widehat{W}_n}(x)$, $x^2 f_{\widehat{W}_n}(x)$ and $x^3 f_{\widehat{W}_n}(x)$ from $f_{\widehat{W}_n}(x)$, we get

$$\begin{aligned}
(1-3x+3x^2-x^3) f_{G\widehat{W}_n}(x) &= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 3x\sum_{n=0}^{\infty} \widehat{W}_n x^n + 3x^2 \sum_{n=0}^{\infty} \widehat{W}_n x^n - x^3 \sum_{n=0}^{\infty} \widehat{W}_n x^n \\
&= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 3\sum_{n=0}^{\infty} \widehat{W}_n x^{n+1} + 3\sum_{n=0}^{\infty} \widehat{W}_n x^{n+2} - \sum_{n=0}^{\infty} \widehat{W}_n x^{n+3} \\
&= \sum_{n=0}^{\infty} \widehat{W}_n x^n - 3\sum_{n=1}^{\infty} \widehat{W}_{n-1} x^n + 3\sum_{n=2}^{\infty} \widehat{W}_{n-2} x^n - \sum_{n=3}^{\infty} \widehat{W}_{n-3} x^n \\
&= (\widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2) - 3(\widehat{W}_0 x + \widehat{W}_1 x^2) + 3GW_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (\widehat{W}_n - 3\widehat{W}_{n-1} + 3\widehat{W}_{n-2} - \widehat{W}_{n-3}) x^n \\
&= \widehat{W}_0 + \widehat{W}_1 x + \widehat{W}_2 x^2 - 3\widehat{W}_0 x - 3\widehat{W}_1 x^2 + 3\widehat{W}_0 x^2 \\
&= \widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2.
\end{aligned}$$

Note that we use the recurrence relation $\widehat{W}_n = 3\widehat{W}_{n-1} - 3\widehat{W}_{n-2} + \widehat{W}_{n-3}$. Rearranging above equation, we get (5.12). □

As special cases, the generating functions of the dual hyperbolic triangular, triangular-Lucas, oblong and dual pentegonal numbers are given by

$$\begin{aligned} f_{\widehat{T}_n}(x) &= \frac{(j+3\varepsilon+6j\varepsilon)+(1-8j\varepsilon-3\varepsilon)\,x+(\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\ f_{\widehat{H}_n}(x) &= \frac{(3+3j+3\varepsilon+3j\varepsilon)+(-6-6j-6\varepsilon-6j\varepsilon)\,x+(3+3j+3\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\ f_{\widehat{O}_n}(x) &= \frac{(2j+6\varepsilon+12j\varepsilon)+(2-16j\varepsilon-6\varepsilon)\,x+(2\varepsilon+6j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\ f_{\widehat{p}_n}(x) &= \frac{(j+5\varepsilon+12j\varepsilon)+(1+2j-3\varepsilon-14j\varepsilon)\,x+(2+\varepsilon+5j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \end{aligned}$$

respectively. □

5.2 OBTAINING BINET'S FORMULA FROM GENERATING FUNCTION OF DUAL HYPERBOLIC GUGLIELMO NUMBERS

We next investigate Binet formula of dual hyperbolic generalized Guglielmo number $\{\widehat{W}_n\}$ by using generating function $f_{\widehat{W}_n}(x)$.

Theorem 5.3 *(Binet formula of dual hyperbolic generalized Guglielmo numbers)*

$$\widehat{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{\alpha}A_2 + 2\widehat{\beta}A_3)n + \widehat{\alpha}A_3n^2 \tag{5.13}$$

Proof. We write

$$\begin{aligned} \sum_{n=0}^{\infty}\widehat{W}_n x^n &= \frac{\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2}{(1-3x+3x^2-x^3)} \\ &= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3}. \end{aligned} \tag{5.14}$$

So that

$$\begin{aligned} \sum_{n=0}^{\infty}\widehat{W}_n x^n &= \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3} \\ &= \frac{d_1(1-x)^2 + d_2(1-x) + d_3}{(1-x)^3}. \end{aligned}$$

Then, the following equality can be derived

$$\widehat{W}_0 + (\widehat{W}_1 - 3\widehat{W}_0)x + (\widehat{W}_2 - 3\widehat{W}_1 + 3\widehat{W}_0)x^2 = (d_1 + d_2 + d_3) + (-2d_1 - d_2)x + d_1x^2.$$

If we equalize the coefficients of the same degree terms of x in the above equation, we get

$$\begin{aligned} \widehat{W_0} &= d_1 + d_2 + d_3, \\ \widehat{W_1} - 3\widehat{W_0} &= -2d_1 - d_2, \\ \widehat{W_2} - 3\widehat{W_1} + 3\widehat{W_0} &= d_1. \end{aligned} \tag{5.15}$$

If we solve system of equations (5.15), we obtain

$$\begin{aligned} d_1 &= 3\widehat{W_0} - 3\widehat{W_1} + \widehat{W_2}, \\ d_2 &= 5\widehat{W_1} - 3\widehat{W_0} - 2\widehat{W_2}, \\ d_3 &= \widehat{W_0} - 2\widehat{W_1} + \widehat{W_2}. \end{aligned}$$

Thus (5.14) can be given as

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{W}_n x^n &= d_1 \sum_{n=0}^{\infty} x^n + d_2 \sum_{n=0}^{\infty} (n+1)x^n + d_3 \sum_{n=0}^{\infty} \frac{n^2+3n+2}{2} x^n, \\ &= \sum_{n=0}^{\infty} (d_1 + d_2(n+1) + d_3 \frac{n^2+3n+2}{2})x^n, \\ &= \sum_{n=0}^{\infty} (\widehat{W_0} + \frac{1}{2}(-\widehat{W_2} + 4\widehat{W_1} - 3\widehat{W_0})n + \frac{1}{2}(\widehat{W_2} - 2\widehat{W_1} + \widehat{W_0})n^2)x^n. \end{aligned}$$

Hence, we get

$$\widehat{W}_n = \widehat{A_1} + \widehat{A_2} n + \widehat{A_3} n^2$$

where

$$\begin{aligned} \widehat{A_1} &= \widehat{W_0}, \\ \widehat{A_2} &= \frac{1}{2}(-\widehat{W_2} + 4\widehat{W_1} - 3\widehat{W_0}), \\ \widehat{A_3} &= \frac{1}{2}(\widehat{W_2} - 2\widehat{W_1} + \widehat{W_0}). \end{aligned}$$

Note that the following equalities given below are true:

$$\begin{aligned} \widehat{A_1} &= \widehat{W_0} \\ &= W_0 + jW_1 + \varepsilon W_2 + j\varepsilon(W_0 - 3W_1 + 3W_2) \\ &= (1 + j + \varepsilon + j\varepsilon)W_0 + (j + 2\varepsilon + 3j\varepsilon)(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) \\ &\quad + (j + 4\varepsilon + 9j\varepsilon)(\frac{1}{2}(W_2 - 2W_1 + W_0)) \\ &= \widehat{\alpha} A_1 + \widehat{\beta} A_2 + \widehat{\gamma} A_3, \end{aligned} \tag{5.16}$$

$$
\begin{aligned}
\widehat{A_2} &= \frac{1}{2}(-\widehat{W_2}+4\widehat{W_1}-3\widehat{W_0}) && (5.17)\\
&= \frac{1}{2}((-3W_0+4W_1-W_2)+j(\frac{1}{2}(-W_0+W_2))\\
&\quad +\varepsilon(W_0-4W_1+3W_2)+j\varepsilon(3W_0-8W_1+5W_2))\\
&= (1+j+\varepsilon+j\varepsilon)(\frac{1}{2}(-W_2+4W_1-3W_0))\\
&\quad +2(j+2\varepsilon+3j\varepsilon)(\frac{1}{2}(W_2-2W_1+W_0))\\
&= (\widehat{a}A_2+2\widehat{\beta}A_3),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{A_3} &= \frac{1}{2}(\widehat{W_2}-2\widehat{W_1}+\widehat{W_0}) && (5.18)\\
&= \frac{1}{2}((W_2-2W_1+W_0)+j(W_2-2W_1+W_0)\\
&\quad +\varepsilon(W_2-2W_1+W_0)+j\varepsilon(W_2-2W_1+W_0)\\
&= \widehat{a}A_3.
\end{aligned}
$$

Using (5.16), (5.17) and (5.18) , we get

$\widehat{W_n}=(\widehat{\alpha}A_1+\widehat{\beta}A_2+\widehat{\gamma}A_3)+(\widehat{a}A_2+2\widehat{\beta}A_3)n+\widehat{a}A_3n^2$. □

5.3 SOME IDENTETIES FOR DUAL HYPERBOLIC GUGLIELMO NUMBERS

We now present some special identities for the dual hyperbolic generalized Guglielmo sequence $\{\widehat{W_n}\}$. The following theorem gives the Simpson's formula for the dual hyperbolic generalized Guglielmo numbers.

Theorem 5.4 *(Simpson's formula for dual hyperbolic generalized Guglielmo numbers) For all integers n we have,*

$$
\begin{vmatrix}
\widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_{n}\\
\widehat{W}_{n+1} & \widehat{W}_{n} & \widehat{W}_{n-1}\\
\widehat{W}_{n} & \widehat{W}_{n-1} & \widehat{W}_{n-2}
\end{vmatrix}
=
\begin{vmatrix}
\widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0}\\
\widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1}\\
\widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2}
\end{vmatrix}. \qquad (5.19)
$$

Proof. First we assume that $n \geqslant 0$. For the proof we use mathematical induction. For $n=0$ identity (5.19) is true. Now we suppose that (5.19) is true for $n=k$. We prove the (5.19) is satisfied for $n=k+1$. Thus,we write the identity given below,

$$
\begin{vmatrix}
\widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_{k}\\
\widehat{W}_{k+1} & \widehat{W}_{k} & \widehat{W}_{k-1}\\
\widehat{W}_{k} & \widehat{W}_{k-1} & \widehat{W}_{k-2}
\end{vmatrix}
=
\begin{vmatrix}
\widehat{W}_{2} & \widehat{W}_{1} & \widehat{W}_{0}\\
\widehat{W}_{1} & \widehat{W}_{0} & \widehat{W}_{-1}\\
\widehat{W}_{0} & \widehat{W}_{-1} & \widehat{W}_{-2}
\end{vmatrix}.
$$

For $n = k + 1$, we get

$$\begin{vmatrix} \widehat{W}_{k+3} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k+1} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix} = \begin{vmatrix} 3\widehat{W}_{k+2} - 3\widehat{W}_{k+1} + \widehat{W}_{k} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ 3\widehat{W}_{k+1} - 3\widehat{W}_{k} + \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ 3\widehat{W}_{k} - 3\widehat{W}_{k-1} + \widehat{W}_{k-2} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix}$$
$$= 3\begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k+1} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix} - 3\begin{vmatrix} \widehat{W}_{k+1} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k-1} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix}$$
$$+ \begin{vmatrix} \widehat{W}_{k} & \widehat{W}_{k+2} & \widehat{W}_{k+1} \\ \widehat{W}_{k-1} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k-2} & \widehat{W}_{k} & \widehat{W}_{k-1} \end{vmatrix}$$
$$= \begin{vmatrix} \widehat{W}_{k+2} & \widehat{W}_{k+1} & \widehat{W}_{k} \\ \widehat{W}_{k+1} & \widehat{W}_{k} & \widehat{W}_{k-1} \\ \widehat{W}_{k} & \widehat{W}_{k-1} & \widehat{W}_{k-2} \end{vmatrix}.$$

Note that if we take $n < 0$ the proof can be done similarly. Thus, the proof is completed.

□

From Theorem 5.19, we get following corollary.

Corollary 5.1

(a) $$\begin{vmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_{n} \\ \widehat{T}_{n+1} & \widehat{T}_{n} & \widehat{T}_{n-1} \\ \widehat{T}_{n} & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{vmatrix} = -4(3\varepsilon + 1)(j + 1).$$

(b) $$\begin{vmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_{n} \\ \widehat{H}_{n+1} & \widehat{H}_{n} & \widehat{H}_{n-1} \\ \widehat{H}_{n} & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{vmatrix} = 0.$$

(c) $$\begin{vmatrix} \widehat{O}_{n+2} & \widehat{O}_{n+1} & \widehat{O}_{n} \\ \widehat{O}_{n+1} & \widehat{O}_{n} & \widehat{O}_{n-1} \\ \widehat{O}_{n} & \widehat{O}_{n-1} & \widehat{O}_{n-2} \end{vmatrix} = -32(3\varepsilon + 1)(j + 1).$$

(d) $$\begin{vmatrix} \widehat{p}_{n+2} & \widehat{p}_{n+1} & \widehat{p}_{n} \\ \widehat{p}_{n+1} & \widehat{p}_{n} & \widehat{p}_{n-1} \\ \widehat{p}_{n} & \widehat{p}_{n-1} & \widehat{p}_{n-2} \end{vmatrix} = -108(3\varepsilon + 1)(j + 1).$$

Now, we define Catalan's identity of dual hyperbolic generalized Guglielmo numbers.

Theorem 5.5 *(Catalan's identity) For all integers n and m, the following identity holds:*

$$\begin{aligned}\widehat{W}_{n+m}\widehat{W}_{n-m}-\widehat{W}_n^2 &= m^2(A_3^2\left(2\widehat{a}\widehat{\gamma}-4\widehat{\beta}^2+\widehat{a}^2m^2-2\widehat{a}^2n^2-4\widehat{a}n\beta\right) \\ &\quad -2\widehat{a}A_2A_3\left(\widehat{\beta}+\widehat{a}n\right)-\widehat{a}\left(\widehat{a}A_2^2-2\widehat{\alpha}A_1A_3\right)).\end{aligned} \tag{5.20}$$

Proof. Using the Binet Formula given below

$$\widehat{W}_n=(\widehat{\alpha}A_1+\widehat{\beta}A_2+\widehat{\gamma}A_3)+(\widehat{a}A_2+2\widehat{\beta}A_3)n+\widehat{a}A_3n^2.$$

The proof is completed. □

As special cases of the above theorem, we give Catalan's identity of dual hyperbolic triangular, Lucas-triangular, Oblong, pentagonal numbers.

We present Catalan's identity of dual hyperbolic triangular numbers.

Corollary 5.2 *(Catalan's identity for the dual hyperbolic triangular numbers) For all integers n and m, the following identity holds:*

$$\widehat{T}_{n+m}\widehat{T}_{n-m}-\widehat{T}_n^2=\frac{1}{4}m^2((-\widehat{a}^2-2\widehat{a}\widehat{\beta}+2\widehat{a}\widehat{\gamma}-4\widehat{\beta}^2)-2\widehat{a}n\left(\widehat{a}+2\widehat{\beta}\right)+\widehat{a}^2\left(m^2-2n^2\right)).$$

Proof. Taking $\widehat{W}_n=\widehat{T}_n$ in Theorem 5.5, we achieve the desired result. □

We give Catalan's identity of dual hyperbolic triangular-Lucas numbers.

Corollary 5.3 *(Catalan's identity for the dual hyperbolic Lucas-triangular numbers) For all integers n and m, the following identity holds:*

$$\widehat{H}_{n+m}\widehat{H}_{n-m}-\widehat{H}_n^2=0.$$

Proof. Taking $\widehat{W}_n=\widehat{H}_n$ in Theorem 5.5, we achieve the required result. □

We give Catalan's identity of dual hyperbolic oblong numbers.

Corollary 5.4 *(Catalan's identity for the dual hyperbolic oblong numbers) For all integers n and m, the following identity holds:*

$$\widehat{O}_{n+m}\widehat{O}_{n-m}-\widehat{O}_n^2=m^2((-\widehat{a}^2-2\widehat{a}\widehat{\beta}+2\widehat{a}\widehat{\gamma}-4\widehat{\beta}^2)-2\widehat{a}n\left(\widehat{a}+2\widehat{\beta}\right)+\widehat{a}^2\left(m^2-2n^2\right)).$$

Proof. Taking $\widehat{W}_n=\widehat{O}_n$ in Theorem 5.5, we have the result that aimed for. □

We give Catalan's identity of dual hyperbolic pentagonal numbers.

Corollary 5.5 *(Catalan's identity for the dual hyperbolic pentagonal numbers) For all integers* n *and* m, *the following identity holds:*

$$\widehat{p}_{n+m}\widehat{p}_{n-m}-\widehat{p}_{n}^{2}=\frac{1}{4}m^{2}((-\widehat{a}^{2}+6\widehat{a}\widehat{\beta}+18\widehat{a}\widehat{\gamma}-36\widehat{\beta}^{2})+6\widehat{a}n\left(\widehat{a}-6\widehat{\beta}\right)+9\widehat{a}^{2}\left(m^{2}-2n^{2}\right)).$$

Proof. Taking $\widehat{W}_n=\widehat{p}_n$ in Theorem 5.5, we obtain the required result. □

Note that for $m=1$ in Catalan's identity, we get the Cassini's identity for the dual hyperbolic generalized Guglielmo numbers.hence,we present corollary given below.

Corollary 5.6 *(Cassini's identity for the dual hyperbolic generalized Guglielmo numbers) For all integers* n, *the following identities holds:*

(a) $\widehat{T}_{n+1}\widehat{T}_{n-1}-\widehat{T}_{n}^{2}=-\frac{1}{2}((\widehat{a}\widehat{\beta}-\widehat{a}\widehat{\gamma}+2\widehat{\beta}^{2})+n(\widehat{a}^{2}+2\widehat{a}\widehat{\beta})+\widehat{a}^{2}n^{2})$.

(b) $\widehat{H}_{n+1}\widehat{H}_{n-1}-\widehat{H}_{n}^{2}=0$.

(c) $\widehat{O}_{n+1}\widehat{O}_{n-1}-\widehat{O}_{n}^{2}=-2((\widehat{a}\widehat{\beta}-\widehat{a}\widehat{\gamma}+2\widehat{\beta}^{2})+n(\widehat{a}^{2}+2\widehat{a}\widehat{\beta})+\widehat{a}^{2}n^{2})$.

(d) $\widehat{p}_{n+1}\widehat{p}_{n-1}-\widehat{p}_{n}^{2}=-\frac{1}{2}((-4\widehat{a}^{2}-3\widehat{a}\widehat{\beta}-9\widehat{a}\widehat{\gamma}+18\widehat{\beta}^{2})+3n(6\widehat{a}\widehat{\beta}-\widehat{a}^{2})+9\widehat{a}^{2}n^{2})$.

Theorem 5.6 *Suppose that* n *and* m *be positive integers,* T_n *is triangular numbers, the following identity is true:*

$$\widehat{W}_{m+n}=T_{m-1}\widehat{W}_{n+2}+(T_{m-3}-3T_{m-2})\widehat{W}_{n+1}+T_{m-2}\widehat{W}_{n}. \tag{5.21}$$

Proof. The Theorem 5.6 can be proved by mathematical induction on m. If $m=0$ we get

$$\widehat{W}_{n}=T_{-1}\widehat{W}_{n+2}+(T_{-3}-3T_{-2})\widehat{W}_{n+1}+T_{-2}\widehat{W}_{n}$$

which is true by seeing that $T_{-1}=0, T_{-2}=1, T_{-3}=3$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\widehat{W}_{(k+1)+n} &= 3\widehat{W}_{n+k}-3\widehat{W}_{n+k-1}+\widehat{W}_{n+k-2} \\
&= 3(T_{k-1}\widehat{W}_{n+2}+(T_{k-3}-3T_{k-2})\widehat{W}_{n+1}+T_{k-2}\widehat{W}_{n}) \\
&\quad -3(T_{k-2}\widehat{W}_{n+2}+(T_{k-4}-3T_{k-3})\widehat{W}_{n+1}+T_{k-3}\widehat{W}_{n}) \\
&\quad +(T_{k-3}\widehat{W}_{n+2}+(T_{k-5}-3T_{k-4})\widehat{W}_{n+1}+T_{k-4}\widehat{W}_{n}) \\
&= (3T_{k-1}-3T_{k-2}+T_{k-3})\widehat{W}_{n+2}+((3T_{k-3}-3T_{k-4}+T_{k-5}) \\
&\quad -3(3T_{k-2}-3T_{k-3}+T_{k-4}))\widehat{W}_{n+1}+(3T_{k-2}-3T_{k-3}+T_{k-4})\widehat{W}_{n} \\
&= T_{k}\widehat{W}_{n+2}+(T_{k-2}-3T_{k-1})\widehat{W}_{n+1}+T_{k-1}\widehat{W}_{n} \\
&= T_{(k+1)-1}\widehat{W}_{n+2}+(T_{(k+1)-3}-3T_{(k+1)-2})\widehat{W}_{n+1}+T_{(k+1)-2}\widehat{W}_{n}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves Theorem 5.6. □

5.4 LINEAR SUM FORMULAS FOR DUAL HYPERBOLIC GUGLIELMO NUMBERS

In this section, we give the summation formulas of the dual hyperbolic generalized Guglielmo numbers with positive and negatif subscripts.

Next, we present the formulas which give the summation of the dual hyperbolic generalized Guglielmo numbers.

Theorem 5.7 *For $n \geq 0$, dual hyperbolic generalized Guglielmo numbers have the following formulas:*

(a) $\sum_{k=0}^{n}\widehat{W}_{k}=\frac{1}{6}(n+1)((-n+6\varepsilon+18j\varepsilon+5n\varepsilon+jn^{2}+n^{2}\varepsilon+2jn+n^{2}+8jn\varepsilon+jn^{2}\varepsilon)$ $W_{2}+(6j+5n-18j\varepsilon-7n\varepsilon-2jn^{2}-2n^{2}\varepsilon-jn-2n^{2}-13jn\varepsilon-2jn^{2}\varepsilon)W_{1}+(-4n+$ $6j\varepsilon+2n\varepsilon+jn^{2}+n^{2}\varepsilon-jn+n^{2}+5jn\varepsilon+jn^{2}\varepsilon+6)W_{0})$.

(b) $\sum_{k=0}^{n}\widehat{W}_{2k}=\frac{1}{6}(n+1)((-n+6\varepsilon+18j\varepsilon+11n\varepsilon+4jn^{2}+4n^{2}\varepsilon+5jn+4n^{2}+17jn\varepsilon+4jn^{2}\varepsilon)$ $W_{2}+(6j+8n-18j\varepsilon-16n\varepsilon-8jn^{2}-8n^{2}\varepsilon-4jn-8n^{2}-28jn\varepsilon-8jn^{2}\varepsilon)W_{1}$ $+(-7n+6j\varepsilon+5n\varepsilon+4jn^{2}+4n^{2}\varepsilon-jn+4n^{2}+11jn\varepsilon+4jn^{2}\varepsilon+6)W_{0})$.

(c) $\sum_{k=0}^{n}\widehat{W}_{2k+1}=\frac{1}{6}(n+1)((6j+5n+18\varepsilon+36j\varepsilon+17n\varepsilon+4jn^{2}+4n^{2}\varepsilon+11jn+4n^{2}+$ $23jn\varepsilon+4jn^{2}\varepsilon)W_{2}+(6-18\varepsilon-48j\varepsilon-28n\varepsilon-8jn^{2}-8n^{2}\varepsilon-16jn-8n^{2}-40jn\varepsilon-$ $8jn^{2}\varepsilon-4n)W_{1}+(-n+6\varepsilon+18j\varepsilon+11n\varepsilon+4jn^{2}+4n^{2}\varepsilon+5jn+4n^{2}+17jn\varepsilon$

$+4jn^2\varepsilon)W_0)$.

Proof.

(a) Note that using (5.1), we get

$$\sum_{k=0}^{n}\widehat{W}_k=\sum_{k=0}^{n}W_k+j\sum_{k=0}^{n}W_{k+1}+\varepsilon\sum_{k=0}^{n}W_{k+2}+j\varepsilon\sum_{k=0}^{n}W_{k+3}$$

and using Proposition 1.2, the proof can be easily handled.

(b) Note that using (5.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{2k}=\sum_{k=0}^{n}W_{2k}+j\sum_{k=0}^{n}W_{2k+1}+\varepsilon\sum_{k=0}^{n}W_{2k+2}+j\varepsilon\sum_{k=0}^{n}W_{2k+3}$$

and using Proposition 1.3, the proof can be easily resolved.

(c) Note that using (5.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{2k+1}=\sum_{k=0}^{n}W_{2k+1}+j\sum_{k=0}^{n}W_{2k+2}+\varepsilon\sum_{k=0}^{n}W_{2k+3}+j\varepsilon\sum_{k=0}^{n}W_{2k+4}$$

and using Proposition 1.3, the proof can be done easily. □

As a special case of the Theorem 5.7 (a), the following corollary hold.

Corollary 5.7

(a) $\sum_{k=0}^{n}\widehat{T}_k=\frac{1}{6}(n+1)((6j+18\varepsilon+36j\varepsilon)+(5j+8\varepsilon+11j\varepsilon+2)n+(j+\varepsilon+j\varepsilon+1)n^2)$.

(b) $\sum_{k=0}^{n}\widehat{H}_k=(3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c) $\sum_{k=0}^{n}\widehat{O}_k=\frac{1}{6}(n+1)((12j+36\varepsilon+72j\varepsilon)+(10j+16\varepsilon+22j\varepsilon+4)n+(2j+2\varepsilon+2j\varepsilon+2)n^2)$.

(d) $\sum_{k=0}^{n}\widehat{p}_k=\frac{1}{6}(n+1)((6j+30\varepsilon+72j\varepsilon)+(18\varepsilon+9j+27j\varepsilon)n+(3j+3\varepsilon+3+3j\varepsilon)n^2)$.

As a special case of the Theorem 5.7 (b), we present following corollary.

Corollary 5.8

(a) $\sum_{k=0}^{n}\widehat{T}_{2k}=\frac{1}{6}(n+1)((6j+18\varepsilon+36j\varepsilon)+(5+17\varepsilon+11j+23j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2)$.

(b) $\sum_{k=0}^{n}\widehat{H}_{2k}=(3j+3\varepsilon+3j\varepsilon+3)\,(n+1)\,.$

(c) $\sum_{k=0}^{n}\widehat{O}_{2k}=\frac{1}{6}\,(n+1)\,((12j+36\varepsilon+72j\varepsilon)+(10+22j+34\varepsilon+46j\varepsilon)n+(8+8j+8\varepsilon+8j\varepsilon)n^2).$

(d) $\sum_{k=0}^{n}\widehat{p}_{2k}=\frac{1}{6}\,(n+1)\,((6j+30\varepsilon+72j\varepsilon)+(3+21j+39\varepsilon+57j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^2).$

As a special case of the Theorem 5.7 (c), we have following corollary.

Corollary 5.9

(a) $\sum_{k=0}^{n}\widehat{T}_{2k+1}=\frac{1}{6}\,(n+1)\,((6+18j+36\varepsilon+60j\varepsilon)+(11+17j+23\varepsilon+29j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^2).$

(b) $\sum_{k=0}^{n}\widehat{H}_{2k+1}=(3j+3\varepsilon+3j\varepsilon+3)\,(n+1)\,.$

(c) $\sum_{k=0}^{n}\widehat{O}_{2k+1}=\frac{1}{6}\,(n+1)\,((12+36j+72\varepsilon+120j\varepsilon)+(22+46\varepsilon+34j+58j\varepsilon)n+(8+8j+8\varepsilon+8j\varepsilon)n^2).$

(d) $\sum_{k=0}^{n}\widehat{p}_{2k+1}=\frac{1}{6}\,(n+1)\,((6+30j+72\varepsilon+132j\varepsilon)+(21+39j+57\varepsilon+75j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^2).$

Now, we present the formula which give the summation formulas of the generalized Guglielmo numbers with negative subscripts.

Theorem 5.8 *For $n\geq 0$, dual hyperbolic generalized Guglielmo numbers have the following formulas:*

(a) $\sum_{k=0}^{n}\widehat{W}_{-k}=\frac{1}{6}\,(n+1)\,((2n+6\varepsilon+18j\varepsilon-4n\varepsilon+jn^2+n^2\varepsilon-jn+n^2-7jn\varepsilon+jn^2\varepsilon)W_2+(6j-7n-18j\varepsilon+5n\varepsilon-2jn^2-2n^2\varepsilon-jn-2n^2+11jn\varepsilon-2jn^2\varepsilon)W_1+(5n+6j\varepsilon-n\varepsilon+jn^2+n^2\varepsilon+2jn+n^2-4jn\varepsilon+jn^2\varepsilon+6)W_0).$

(b) $\sum_{k=0}^{n}\widehat{W}_{-2k}=\frac{1}{6}\,(n+1)\,((5n+6\varepsilon+18j\varepsilon-7n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2-13jn\varepsilon+4jn^2\varepsilon)W_2+(6j-16n-18j\varepsilon+8n\varepsilon-8jn^2-8n^2\varepsilon-4jn-8n^2+20jn\varepsilon-8jn^2\varepsilon)W_1+(11n+6j\varepsilon-n\varepsilon+4jn^2+4n^2\varepsilon+5jn+4n^2-7jn\varepsilon+4jn^2\varepsilon+6)W_0).$

(c) $\sum_{k=0}^{n}\widehat{W}_{-2k+1}=\frac{1}{6}\left(n+1\right)((6j-n+18\varepsilon+36j\varepsilon-13n\varepsilon+4jn^2+4n^2\varepsilon-7jn+4n^2-19jn\varepsilon+4jn^2\varepsilon)W_2+(20n\varepsilon-18\varepsilon-48j\varepsilon-4n-8jn^2-8n^2\varepsilon+8jn-8n^2+32jn\varepsilon-8jn^2\varepsilon+6)W_1+(5n+6\varepsilon+18j\varepsilon-7n\varepsilon+4jn^2+4n^2\varepsilon-jn+4n^2-13jn\varepsilon+4jn^2\varepsilon)W_0)$.

Proof.

(a) Note that using (5.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-k}=\sum_{k=0}^{n}W_{-k}+j\sum_{k=0}^{n}W_{-k+1}+\varepsilon\sum_{k=0}^{n}W_{-k+2}+j\varepsilon\sum_{k=0}^{n}W_{-k+3}$$

and using Proposition 1.4, the proof can be easily handled.

(b) Note that using (5.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k}=\sum_{k=0}^{n}W_{-2k}+j\sum_{k=0}^{n}W_{-2k+1}+\varepsilon\sum_{k=0}^{n}W_{-2k+2}+j\varepsilon\sum_{k=0}^{n}W_{-2k+3}$$

and using Proposition (1.5) the proof easily can be done .

(c) Note that using (5.1), we get

$$\sum_{k=0}^{n}\widehat{W}_{-2k+1}=\sum_{k=0}^{n}W_{-2k+1}+j\sum_{k=0}^{n}W_{-2k+2}+\varepsilon\sum_{k=0}^{n}W_{-2k+3}+j\varepsilon\sum_{k=0}^{n}W_{-2k+4}$$

and using Proposition 1.5, the proof can be easily completed. □

Next, we give different sum formulas of the dual hyperbolic generalized Guglielmo numbers.

As a special case of the Theorem 5.8 (a), we have the following corollary.

Corollary 5.10

(a) $\sum_{k=0}^{n}\widehat{T}_{-k}=\frac{1}{6}\left(n+1\right)((6j+18\varepsilon+36j\varepsilon)+(-1-4j-7\varepsilon-10j\varepsilon)n+(1+j+\varepsilon+j\varepsilon)n^2)$.

(b) $\sum_{k=0}^{n}\widehat{H}_{-k}=\left(3j+3\varepsilon+3j\varepsilon+3\right)\left(n+1\right).$

(c) $\sum_{k=0}^{n}\widehat{O}_{-k}=\frac{1}{6}\left(n+1\right)((12j+36\varepsilon+72j\varepsilon)+(-2-8j-14\varepsilon-20j\varepsilon)n+(2+2j+2\varepsilon+2j\varepsilon)n^2)$.

(d) $\sum_{k=0}^{n}\widehat{p}_{-k}=\frac{1}{2}\left(n+1\right)((2j+10\varepsilon+24j\varepsilon)+(1-2j-5\varepsilon-8j\varepsilon)n+(1+j+\varepsilon+j\varepsilon)n^2)$.

As a special case of the Theorem 5.8 (b), we obtain the following corollary.

Corollary 5.11

(a) $\sum_{k=0}^{n}\widehat{T}_{-2k}=\frac{1}{6}(n+1)((6j+18\varepsilon+36j\varepsilon)+(-1-7j-13\varepsilon-19j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^{2})$.

(b) $\sum_{k=0}^{n}\widehat{H}_{-2k}=(3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c) $\sum_{k=0}^{n}\widehat{O}_{-2k}=\frac{1}{3}(n+1)((6j+18\varepsilon+36j\varepsilon)+(-1-7j-13\varepsilon-19j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^{2})$.

(d) $\sum_{k=0}^{n}\widehat{p}_{-2k}=\frac{1}{6}(n+1)((6j+30\varepsilon+72j\varepsilon)+(9-9j-27\varepsilon-45j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^{2})$.

As a special case of the Theorem 5.8 (c), we present the following corollary.

Corollary 5.12

(a) $\sum_{k=0}^{n}\widehat{T}_{-2k+1}=\frac{1}{6}(n+1)((6+18j+36\varepsilon+60j\varepsilon)+(-7-13j-19\varepsilon-25j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^{2})$.

(b) $\sum_{k=0}^{n}\widehat{H}_{-2k+1}=(3j+3\varepsilon+3j\varepsilon+3)(n+1)$.

(c) $\sum_{k=0}^{n}\widehat{O}_{-2k+1}=\frac{1}{3}(n+1)((6+18j+36\varepsilon+60j\varepsilon)+(-7-13j-19\varepsilon-25j\varepsilon)n+(4+4j+4\varepsilon+4j\varepsilon)n^{2})$.

(d) $\sum_{k=0}^{n}\widehat{p}_{-2k+1}=\frac{1}{6}(n+1)((6+30j+72\varepsilon+132j\varepsilon)+(-9-27j-45\varepsilon-63j\varepsilon)n+(12+12j+12\varepsilon+12j\varepsilon)n^{2})$.

5.5 MATRICES RELATED WITH DUAL HYPERBOLIC GUGLIELMO NUMBERS

In this section. we give some fundemental matrix formulations related with dual hyperbolic generalized Guglielmo Numbers. First, we give following lemma.

Lemma 5.9 *For all integers n the following identity is true*

$$\begin{pmatrix} \widehat{W}_{n+2} \\ \widehat{W}_{n+1} \\ \widehat{W}_{n} \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}.$$

Proof. First, we get $n \geq 0$. Lemma 5.9 can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{0} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix}.$$

For $n = k + 1$, we have

$$\begin{aligned}
\begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} &= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} \widehat{W}_{2} \\ \widehat{W}_{1} \\ \widehat{W}_{0} \end{pmatrix} \\
&= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \\ \widehat{W}_{k} \end{pmatrix} \\
&= \begin{pmatrix} 3\widehat{W}_{k+2} - 3\widehat{W}_{k+1} + \widehat{W}_{k} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} \widehat{W}_{k+3} \\ \widehat{W}_{k+2} \\ \widehat{W}_{k+1} \end{pmatrix}.
\end{aligned}$$

If we obtain $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Theorem 5.10 *If we define the matrices $N_{\widehat{W}}$ and $E_{\widehat{W}}$ as follow*

$$
\begin{aligned}
N_{\widehat{W}} &= \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\
E_{\widehat{W}} &= \begin{pmatrix} \widehat{W}_{n+2} & \widehat{W}_{n+1} & \widehat{W}_n \\ \widehat{W}_{n+1} & \widehat{W}_n & \widehat{W}_{n-1} \\ \widehat{W}_n & \widehat{W}_{n-1} & \widehat{W}_{n-2} \end{pmatrix}.
\end{aligned}
$$

then the following identity is true:

$$A^n N_{\widehat{W}} = E_{\widehat{W}}.$$

Proof. Using (1.11), we obtain the following identities

$$
\begin{aligned}
A^n N_{\widehat{W}} &= \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} \widehat{W}_2 & \widehat{W}_1 & \widehat{W}_0 \\ \widehat{W}_1 & \widehat{W}_0 & \widehat{W}_{-1} \\ \widehat{W}_0 & \widehat{W}_{-1} & \widehat{W}_{-2} \end{pmatrix}, \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} &= G\widehat{W}_2 T_{n+1} + \widehat{W}_1 (T_{n-1} - 3T_n) + \widehat{W}_0 T_n, \\
a_{12} &= \widehat{W}_1 T_{n+1} + \widehat{W}_0 (T_{n-1} - 3T_n) + \widehat{W}_{-1} T_n, \\
a_{13} &= \widehat{W}_0 T_{n+1} + \widehat{W}_{-1} (T_{n-1} - 3T_n) + \widehat{W}_{-2} T_n, \\
a_{21} &= \widehat{W}_2 T_n + \widehat{W}_1 (T_{n-2} - 3T_{n-1}) + \widehat{W}_0 T_{n-1}, \\
a_{22} &= \widehat{W}_1 T_n + \widehat{W}_0 (T_{n-2} - 3T_{n-1}) + \widehat{W}_{-1} T_{n-1}, \\
a_{23} &= \widehat{W}_0 T_n + \widehat{W}_{-1} (T_{n-2} - 3T_{n-1}) + \widehat{W}_{-2} T_{n-1}, \\
a_{31} &= \widehat{W}_2 T_{n-1} + \widehat{W}_1 (T_{n-3} - 3T_{n-2}) + \widehat{W}_0 T_{n-2}, \\
a_{32} &= \widehat{W}_1 T_{n-1} + \widehat{W}_0 (T_{n-3} - 3T_{n-2}) + \widehat{W}_{-1} T_{n-2}, \\
a_{33} &= \widehat{W}_0 T_{n-1} + \widehat{W}_{-1} (T_{n-3} - 3T_{n-2}) + \widehat{W}_{-2} T_{n-2}.
\end{aligned}
$$

Using the Theorem 5.6, the proof is done. □

From Theorem 5.10, we can write the following corollary.

Corollary 5.13

(a) *We assume that the matrices* $N_{\widehat{T}}$ *and* $E_{\widehat{T}}$ *are defined as following*

$$N_T = \begin{pmatrix} \widehat{T}_2 & \widehat{T}_1 & \widehat{T}_0 \\ \widehat{T}_1 & \widehat{T}_0 & \widehat{T}_{-1} \\ \widehat{T}_0 & \widehat{T}_{-1} & \widehat{T}_{-2} \end{pmatrix},$$

$$E_{\widehat{T}} = \begin{pmatrix} \widehat{T}_{n+2} & \widehat{T}_{n+1} & \widehat{T}_n \\ \widehat{T}_{n+1} & \widehat{T}_n & \widehat{T}_{n-1} \\ \widehat{T}_n & \widehat{T}_{n-1} & \widehat{T}_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, $N_{\widehat{T}}$, $E_{\widehat{T}}$*:*

$$A^n N_{\widehat{T}} = E_{\widehat{T}},$$

(b) *We assume that the matrices* $N_{\widehat{O}}$ *and* $E_{\widehat{O}}$ *are defined as following*

$$N_{\widehat{O}} = \begin{pmatrix} \widehat{O}_2 & \widehat{O}_1 & \widehat{O}_0 \\ \widehat{O}_1 & \widehat{O}_0 & \widehat{O}_{-1} \\ \widehat{O}_0 & \widehat{O}_{-1} & \widehat{O}_{-2} \end{pmatrix},$$

$$E_{\widehat{O}} = \begin{pmatrix} \widehat{O}_{n+2} & \widehat{O}_{n+1} & \widehat{O}_n \\ \widehat{O}_{n+1} & \widehat{O}_n & \widehat{O}_{n-1} \\ \widehat{O}_n & \widehat{O}_{n-1} & \widehat{O}_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, $N_{\widehat{O}}$, $E_{\widehat{O}}$*:*

$$A^n N_{\widehat{O}} = E_{\widehat{O}}.$$

(c) *We assume that the matrices* $N_{\widehat{H}}$ *and* $E_{\widehat{H}}$ *are defined as following*

$$N_{\widehat{H}} = \begin{pmatrix} \widehat{H}_2 & \widehat{H}_1 & \widehat{H}_0 \\ \widehat{H}_1 & \widehat{H}_0 & \widehat{H}_{-1} \\ \widehat{H}_0 & \widehat{H}_{-1} & \widehat{H}_{-2} \end{pmatrix},$$

$$E_{\widehat{H}} = \begin{pmatrix} \widehat{H}_{n+2} & \widehat{H}_{n+1} & \widehat{H}_n \\ \widehat{H}_{n+1} & \widehat{H}_n & \widehat{H}_{n-1} \\ \widehat{H}_n & \widehat{H}_{n-1} & \widehat{H}_{n-2} \end{pmatrix},$$

such that the following identity is true for A^n, $N_{\widehat{H}}$, $E_{\widehat{H}}$*:*

$$A^n N_{\widehat{H}} = E_{\widehat{H}}.$$

(d) *We assume that the matrices $N_{\widehat{p}}$ and $E_{\widehat{p}}$ are defined as following*

$$N_{\widehat{p}} = \begin{pmatrix} \widehat{p}_2 & \widehat{p}_1 & \widehat{p}_0 \\ \widehat{p}_1 & \widehat{p}_0 & \widehat{p}_{-1} \\ \widehat{p}_0 & \widehat{p}_{-1} & \widehat{p}_{-2} \end{pmatrix},$$

$$E_{\widehat{p}} = \begin{pmatrix} \widehat{p}_{n+2} & \widehat{p}_{n+1} & \widehat{p}_n \\ \widehat{p}_{n+1} & \widehat{p}_n & \widehat{p}_{n-1} \\ \widehat{p}_n & \widehat{p}_{n-1} & \widehat{p}_{n-2} \end{pmatrix}.$$

such that he following identity is true for A^n, $N_{\widehat{p}}$, $E_{\widehat{p}}$:

$$A^n N_{\widehat{p}} = E_{\widehat{p}}.$$

CHAPTER 6

CONCLUSION

In chaper 1, we provided some basic properties that are needed for the rest of the thesis. These include the third-order sequence, often denoted as the (r; s; t) sequence, defined by $a_n = ra_{n-1} + sa_{n-2} + ta_{n-3}$ where r, s and t are constants, and the initial values a_0, a_1 and a_2 are given. In subsection 1.1, we gave the recurrence relation of generalized Guglielmo numbers. In Theorem 1.1, we presented the Binet formula of generalized Guglielmo numbers. Table 1.1 showed a few generalized Guglielmo numbers with positive subscripts and negative subscripts. In identities (1.7)-(1.10), we presented the reccurrence relations of four special cases called triangular, Lucas-triangular, oblong and pentagonal sequences. Table 1.2 presents some values of the unique third-order Triangular and Triangular-Lucas, oblong and pentagonal numbers with positive and negative indices. In subsection 1.1.1, we showed some linear sum formulas about generalized Guglielmo numbers. In subsection 1.2, we presented some matrix formulations related to generalized Guglielmo sequence. In subsection 1.3 we presented some information about Gaussian numbers, hyperbolic numbers, dual numbers, dual hyperbolic numbers and other special numbers and some fundamental properties related to these numbers. Moreover, we gave a literature search on these numbers.

In chapter 2, we defined generalized Gaussian Guglielmo sequences and four special cases called Gaussian triangular numbers, Gaussian triangular-Lucas numbers, Gaussian oblong numbers and Gaussian pentagonal numbers. In Table 2.1 and Table 2.2, we presented the first few generalized Gaussian Guglielmo numbers with positive subscripts and negative subscripts. In Table 2.3, we showed some special cases of Gaussian generalized Guglielmo numbers with positive and negative subscripts. In Theorem 2.1, we defined the Binet's formula for the Gaussian generalized Guglielmo numbers. In Theorem 2.2, we defined the generating function of Gaussian generalized Guglielmo numbers. In Theorem 2.3, we presented some identities involving Gaussian triangular, Gaussian triangular-Lucas, Gaussian oblong, Gaussian pentagonal numbers. In Theorem 2.5, we detailed the generating functions for the even and odd-indexed generalized Guglielmo sequences. In Theorem 2.7, we presented Simpson's formula of generalized Gaussian Guglielmo numbers. In Theorem 2.6, we defined an identitiy related to Gaussian Guglielmo numbers and triangular numbers. In Theorem 2.8 and Theorem 2.9, we defined some sum formulas of generalized Gaussian Guglielmo numbers. In Theorem 5.10, defined some fundemental formulas related to matrix formulation of GW_n.

In chapter 3, we defined hyperbolic generalized Guglielmo numbers then we present generating functions and Binet formulas for them. In the Table 3.1, we presented the first few hyperbolic generalized Guglielmo numbers with positive subscripts and negative subscripts. In Table 3.2-Table 3.5, we presented a few hyperbolic triangular numbers, hyperbolic triangular-Lucas numbers, hyperbolic oblong numbers and hyperbolic pentagonal numbers with positive subscripts and negative subscripts. In Theorem 5.1, we defined the Binet's formula for the hyperbolic generalized Guglielmo numbers. In Theorem 3.2, we provided the generating function for the hyperbolic generalized Guglielmo numbers. In Theorem 3.3, we defined the Binet formula of hyperbolic generalized Guglielmo number $\{HW_n\}$ utilizing generating function $f_{HW_n}(x)$. In Theorem 3.4, we defined the Simpson's formula for the hyperbolic generalized Guglielmo numbers. In Theorem 3.5, we defined Catalan's identity of hyperbolic generalized Guglielmo numbers. In Theorem 3.6, we defined an identitiy related to hyperbolic generalized Guglielmo numbers and triangular numbers. In Theorem 3.7, Theorem 3.8 and Theorem 3.9, we defined some summation formulas of the hyperbolic generalized Guglielmo numbers. In Theorem 3.11, we defined matrices formulation related with Hyperbolic Generalized Guglielmo Numbers.

In chapter 4, we defined dual generalized Guglielmo numbers then we present recurrence relation, generating functions and Binet formulas. In the Table 4.1, we provided the initial dual generalized Guglielmo numbers with both positive and negative subscripts. In Table 4.2-Table 4.5, we presented some dual triangular numbers, dual triangular-Lucas numbers, dual oblong numbers, and dual pentagonal numbers with positive or negative subscript. In Theorem 4.1, we defined the Binet's Formula of the dual generalized Guglielmo numbers. In Theorem 4.2, we introduced the generating function of the dual generalized Guglielmo numbers. In Theorem 4.3, we explored the Binet formula for the dual generalized Guglielmo numbers $\{DW_n\}$ by utilizing generating function $f_{DW_n}(x)$. In Theorem 4.4, we defined the Simpson's formula for the dual generalized Guglielmo numbers. In Theorem 4.5, we defined Catalan's identity of dual generalized Guglielmo numbers. In Theorem 4.6, we defined an identitiy related to dual generalized Guglielmo numbers and triangular numbers. In Theorem 4.7, Theorem 4.8 and Theorem 4.9, we defined some summation formulas of the dual generalized Guglielmo numbers. In Theorem 4.11, we defined matrix formulations related to dual generalized Guglielmo Numbers.

In chapter 5, we define dual hyperbolic generalized Guglielmo numbers then we present generating functions and Binet formulas for them. In the Table 5.1, we presented the first few dual hyperbolic generalized Guglielmo numbers with positive subscripts and negative subscripts. In the Table 5.2-Table 5.5, we presented dual hyperbolic triangular numbers, dual hyperbolic triangular-Lucas numbers, dual hyperbolic oblong numbers and dual hyperbolic pentagonal numbers with positive subscripts and negative subscripts. In Theorem 5.1, we defined the Binet's Formula of the dual hyperbolic generalized Guglielmo numbers. In Theorem 5.2, we defined generating function of the dual hyperbolic generalized Guglielmo numbers. In Theorem 5.3, we investigated Binet formula of dual hyperbolic generalized Guglielmo number $\{\widehat{W}_n\}$ by using generating function $f_{\widehat{W}_n}(x)$. In Theorem 5.4, we defined the Simpson's formula for the dual hyperbolic generalized Guglielmo numbers. In Theorem 5.5, we defined Catalan's identity of dual hyperbolic generalized Guglielmo numbers. In Theorem 5.6, we defined an identitiy related to dual hyperbolic generalized Guglielmo numbers and triangular numbers. In Theorem 5.7 and Theorem 5.8, we defined some summation formulas of the dual hyperbolic generalized Guglielmo numbers. In Theorem 5.10, we defined some fundamental matrix formulations related with dual hyperbolic generalized Guglielmo Numbers.

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