# BERTRAND AND MANNHEIM CURVES IN THREE-DIMENSIONAL WALKER MANIFOLDS

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By
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

## BERTRAND AND MANNHEIM CURVES IN THREE-DIMENSIONAL WALKER MANIFOLDS

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We review the basic concepts of space curves, including curvature and torsion. We examine special curves such as Mannheim and Bertrand in a three-dimensional Euclidean space. We define Walker manifolds which are pseudo-Riemannian manifolds with a parallel null distribution. Then we compute Christoffel symbols and Levi-Civita connection components for an arbitrary three-dimensional Walker manifold.

Finally, we derive the curvature and torsion of a regular curve on a three dimensional Walker manifold. Then, we investigate necessary and sufficient conditions for Mannheim curves in a strict three-dimensional Walker manifold. Moreover, we also prove necessary and sufficient conditions of Bertrand curves in a three-dimensional Walker manifold.

*Keywords:* Walker manifold, Mannheim curves, Frenet frame.

# ÖZET

# ÜÇ BOYUTLU WALKER MANİFOLDLARINDA BERTRAND VE MANNHEİM EĞRİLERİ

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Eğrilik ve burulma da dahil olmak üzere uzay eğrilerinin temel kavramlarını gözden geçiriyoruz. Mannheim ve Bertrand gibi özel eğrileri üç boyutlu Öklid uzayında inceliyoruz.

Yarı-Riemann manifoldları olan Walker manifoldlarını paralel sıfır dağılımına sahip manifoldlar olarak tanımlıyoruz. Daha sonra Christoffel sembollerini ve Levi-Civita bağlantı bileşenlerini keyfi üç boyutlu Walker manifoldları için hesaplıyoruz.

Son olarak, düzenli bir eğrinin eğriliğini ve burulmasını üç boyutlu Walker manifoldu üzerinde elde ediyoruz. Daha sonra üç boyutlu Walker manifoldları üzerinde Mannheim eğrileri için gerek ve yeter şartlarını araştırıyoruz. Ayrıca üç boyutlu Walker manifoldları üzerinde tanımlanan Bertrand eğrileri için de gerek ve yeter şartları kanıtlıyoruz.

*Anahtar sözcükler*: Walker manifoldu, Mannheim eğrileri, Frenet çerçevesi.

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I would like to dedicate this thesis to all my family, especially to my father who was the reason I fell in love with Mathematics, and the first person to believe in me as a young mathematician.

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# Chapter 1

# Introduction

It is established that the existence of a parallel line field on a Riemannian manifold leads to a local decomposition of the manifold into a direct product. This characteristic also applies to semi-Riemannian manifolds if the line field is non-degenerate, meaning it is generated by a non-null, locally defined vector field (see [**1**], [**2**]). But, the geometric implications of the existence of a parallel degenerate line field on a manifold are not well understood yet. A Walker structure refers to a parallel degenerate plane field on a manifold. Many key differences between Riemannian and semi-Riemannian geometries arise from the existence of *Walker structures*. Walker found the canonical form for a space with a parallel field of null planes in [**3**]. Moreover, Walker metrics are recognized as an effective tool for creating interesting indefinite metrics that display diverse geometric properties not given by any positive definite metrics, see[**4**]. Indefinite metrics are essential in several physical contexts, including classical cosmological models (general relativity) and string theory, among others. Walker manifolds naturally emerge in numerous physical contexts and serve as examples of extreme mathematical cases [**9**] . Therefore, investigating properties of special curves on a Walker manifold is highly insightful. In [**4**], Gningue, Ndiaye, and Nkunzimana examined the geometry of biharmonic curves in a strict Walker 3-manifold and derived explicit parametric equations for both biharmonic curves and time-like biharmonic curves. Additionally, they discussed the conditions under which a speed curve qualifies as a slant helix in a Walker manifold.

General theorems of curves and surfaces in Euclidean space (or more generally in Riemannian manifolds) have attracted many studies. These studies have significantly deepened our understanding of both local and global geometry of Riemannian manifolds. This will be the approach we follow to study both Bertrand and Mannheim curves. Bertrand curves trace back to Saint-Venant, and have many applications in differential geometry, physics, biology.[**15**]. In[**12**], Bertrand and Mannheim curves are studied in $\mathbb{E}^3$. Honda and Takahashi give existence conditions of Bertrand and Mannheim regular space curves. Moreover, they define Bertrand and Mannheim curves of framed curves and prove that they are dependent on the moving frame.

In this thesis, we begin by introducing the Riemannian manifold theory guided by [**6**]. After establishing the fundamental concepts like a semi-Riemannian manifold, defined as a smooth manifold $M$ endowed with a metric tensor $g$, we move to section 2.2 where we define the curvature $\kappa$ and torsion $\tau$ of a curve in the Euclidean space. In section 2.3, we present the notion of a Frenet frame $\{T, N, B\}$ which is an orthonormal basis for $\mathbb{E}^3$, and give the Frenet-Serret equations of regular curves. In section 2.4, we introduce a Bertrand curve in $\mathbb{E}^3$ as a space curve whose principal normal aligns with the principal normal of another curve. Moreover, we define Bertrand mates and give some properties related to their curvature $\kappa$ and $\tau$ torsion. As for section 2.5, we introduce Mannheim curves. A Mannheim curve is characterized by having its principal normal line identical to the binormal line of another curve. We provide a variety of characteristics of both Mannheim and Mannheim partner curves in the subsections 2.5.1 and 2.5.2. Furthermore, we introduce the concept of connections in a semi-Riemannian manifold in section 2.6. The Levi-Civita connection is a unique affine connection on the manifold $M$ and is compatible with the metric. Finally in section 2.7, we define the covariant derivative using the levi-civita connection and give a well-known formula to calculate it.

In chapter 3, three-dimensional Walker manifolds are introduced. They are pseudo-Riemannian manifolds with a parallel degenerate line field. The manifold is referred to as a strictly Walker manifold if this line field can be generated by a null vector ([**2**]), ([**1**]). In order to describe these Walker manifolds we use a defining function $f = f(x, y, z)$ and local coordinates $\{x, y, z\}$. The class of Walker manifolds is extensive, based on the defining function $f$. The function $f = f(y, z)$ characterizes a strict Walker manifold. We define the Walker metric in section 3.1, then give the Walker vector product and form an orthonormal basis in section 3.2.([**5**]). In section 3.3, we compute the connections and Christoffel symbols coefficients for a general three-dimensional Walker manifold and give the Frenet frame equations. (see ([**2**]), ([**9**])).

Finally in chapter 4, we obtain the curvature $\kappa$ and torsion $\tau$ for a regular curve on any arbitrary three-dimensional Walker manifold. We then find $\kappa$ and $\tau$ for the case of a strict Walker manifold and some other special cases, particularly, when one of the components of the curve $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ is constant. Furthermore, we examine Mannheim curves on a Walker manifold through studying Mannheim characterizing property for the spacial case when $\alpha_2(t) = \mathit{constant}$ and the Walker manifold is strict, that is, when $f = f(y, z)$. Moreover, we give the proof of Bertrand mates for two nondegenerate curves on any three-dimensional Walker manifold.

# Chapter 2

# Preliminaries

In this section, we give fundamental concepts and definitions from semi-Riemannian Geometry and Euclidean geometry. These concepts and results are commonly found in various books like [**6**] and [**11**]. The definition we give for curvature aligns with that of [**6**].

## 2.1 Manifold Theory

In this section, we present some fundamental concepts from general manifold theory. In addition, we give some notational conventions.

In this study, we assume any manifold $M$ is connected, Hausdorff, paracompact, and smooth. $Tp(M)$ represents the set of all tangent vectors to $M$ at $p \in M$. $T(M)$ denotes the set of all tangent vectors to $M$. $I$ is an open and connected interval in the set of real numbers.

A smooth section $X$ of $T(M)$ represents a vector field in $M$. $\mathfrak{X}(M)$ denotes the collection of all vector fields in $M$. Suppose $\alpha : I \to M$ is a smooth curve, then the smooth mapping $V : I \to T(M)$ such that $V(t) \in T_{\alpha(t)}(M)$ for all $t \in I$, where $t$ is a parameter parametrizing the curve $\alpha$, is called a vector field along $\alpha$. Denote by $\mathfrak{X}(\alpha)$ the set of all vector fields along $\alpha$.

Let $M$ be a smooth manifold. A *metric tensor* $g$ on $M$ is a symmetric and non-degenerate $(0,2)$ tensor field with a constant index. The signature of $g$, denoted by $(r,s)$, indicates the number of negative eigenvalues $(r)$ and positive eigenvalues $(s = n - r)$, where $n$ is the dimension of the manifold.

If a smooth manifold $M$ is equipped with a metric tensor $g$, then $(M,g)$ is called a Semi-Riemannian manifold. If $g$ has no negative values, i.e, has index zero, then $(M,g)$ is a Riemannian manifold.

## 2.2 Curvature and Torsion of Curves in Euclidean Space

In this section, we define the concepts of curvature and torsion for a curve in three dimensional Euclidean Space. The definitions introduced here are compatible with [**11**].

Let $I$ be an open and connected interval in the set of real numbers.

A curve $\alpha(t)$ is a smooth map from an open interval $I \subset \mathbb{R}$ to the Euclidean space $\mathbb{R}^n : \alpha(t) : I \to \mathbb{R}^n$.

A regular curve $\alpha(t) : I \to \mathbb{R}^n$ is a curve whose speed never vanishes, $|\alpha'(t)| \neq 0$ for any $t \in I$.

A regular curve $\alpha : I \to \mathbb{R}^n$ is said to be degenerate or lightlike if $\langle \alpha'(t), \alpha'(t) \rangle = 0$ where $\langle , \rangle$ denotes the Euclidean inner product.

We denote the velocity of the curve $\alpha$ by $v(t)$, and its acceleration by $a(t)$, i.e, $v(t) = \alpha'(t), \quad a(t) = \alpha''(t)$.

$t(t)$, $n(t)$, and $b(t)$ denote the unit tangent, the unit normal, and the unit binormal vectors of the curve $\alpha$, respectively.

The prime symbol $(')$ denotes the first derivative with respect to the parameter $t$.

For two vectors $u$ and $v \in \mathbb{R}^n$, $u^\perp$ denotes the orthogonal complement of $u$, i.e, $u = u^\parallel + u^\perp$ where $u^\parallel$ is the projection of $u$ in the direction of $v$.

**Definition 2.2.1.** *Let $\alpha : I \to \mathbb{R}^n$ be a regular curve. $\alpha$ has a **curvature function** that is given as: $\kappa : I \to [0, \infty)$*

$$\kappa(t) = \frac{|a^\perp(t)|}{|v(t)|^2}. \tag{2.1}$$

**Proposition 2.2.2.** *Let $\alpha : I \to \mathbb{R}^n$ be a regular space curve. Its **curvature** can be expressed as:*

$$\kappa(t) = \frac{|v(t) \times a(t)|}{|v(t)|^3}. \tag{2.2}$$

*for all $t \in I$.*

**Proposition 2.2.3.** *Let $\alpha : I \to \mathbb{R}^3$ be a regular space curve. Its **torsion** at $t$ is defined as:*

$$\tau(t) = \frac{-\langle b'(t), n(t)\rangle}{|v(t)|}. \tag{2.3}$$

*for $t \in I$ such that $\kappa(t) \neq 0$.*

## 2.3 Frenet Frame in 3 Dimensional Euclidean Space

In order to understand the geometry of a curve $\alpha$ in $\mathbb{E}^3$, we need to establish a basis and define how this basis changes at each point along the curve. To achieve this, we use the Frenet frame $\{T(s), N(s), B(s)\}$ which is an orthonormal basis for $\mathbb{E}^3$ for each $s$ where $B = T \times N$.

**Definition 2.3.1.** *Assume $\alpha(s) : I \to \mathbb{R}^3$ is a regular space curve. Let $s \in I$ such that $\kappa(s) \neq 0$.*

*The basis* $\{T(s), N(s), B(s)\}$ *of* $\mathbb{R}^3$ *is called the Frenet frame at* $s$*, and is defined as:*

$$T(s) = \frac{v(s)}{|v(s)|}, \quad N(s) = \frac{a^{\perp}(s)}{|a^{\perp}(s)|} = \frac{T'(s)}{|T'(s)|}, \quad B(s) = T(s) \times N(s). \tag{2.4}$$

*Where* $T(s), N(s)$ *and* $B(s)$ *are the unit tangent, the unit normal, and the unit binormal vectors at* $s$*, respectively.*

## 2.4 Bertrand Curves in $\mathbb{E}^3$

In this section, our goal is to define Bertrand curves and give some of their properties in the three dimensional Euclidean space. Throughout this section, we follow the definitions from the book [**11**] and the paper [**12**].

Let $n(t)$ denote the principal normal vector of $\alpha$ at $t$.
$\kappa(t)$ and $\tau(t)$ represent the curvature and torsion of a curve $\alpha$ at $t$, respectively.

**Definition 2.4.1.** *Let* $\alpha$ *and* $\overline{\alpha} : I \to \mathbb{R}^3$ *be two non-degenerate curves.*
*If there exists a smooth function* $\lambda : I \to \mathbb{R}$ *such that* $\overline{\alpha}(t) = \alpha(t) + \lambda(t)n(t)$ *and* $n(t) = \pm\overline{n}(t)$ *for all* $t \in I$*, then* $\alpha$ *and* $\overline{\alpha}$ *are called Bertrand mates.*

*We call* $\alpha : I \to \mathbb{R}^3$ *a Bertrand curve if there exists another curve* $\overline{\alpha} : I \to \mathbb{R}^3$ *such that* $\alpha$ *and* $\overline{\alpha}$ *are Bertrand mates.*

Assume that $\alpha$ and $\overline{\alpha}$ are Bertrand mates, thus they have the same principal normal line for each point. Notice that, we can assume that $n(t) = \overline{n}(t)$ if we consider $-\lambda$ instead of $\lambda$.

**Example 2.4.2.** *Let* $\alpha(t)$ *be the following general helix:*

$$\alpha(t) = \left(2\sqrt{1+t} - 2, \frac{2}{3}(2\sqrt{1+t} - 2)^{\frac{3}{2}}, (\sqrt{1+t} - 1)^2\right)$$

*It has the following Frenet frame:*

$$t = \left( \frac{1}{\sqrt{1+t}}, \frac{\sqrt{2\sqrt{1+t}-2}}{\sqrt{1+t}}, \frac{\sqrt{1+t}-1}{\sqrt{1+t}} \right)$$

$$n = \left( -\frac{\sqrt{2\sqrt{1+t}-2}}{\sqrt{1+t}}, \frac{2-\sqrt{1+t}}{\sqrt{1+t}}, \frac{\sqrt{2\sqrt{1+t}-2}}{\sqrt{1+t}} \right)$$

$$b = \left( \frac{\sqrt{1+t}-1}{\sqrt{1+t}}, -\frac{\sqrt{2\sqrt{1+t}-2}}{\sqrt{1+t}}, \frac{1}{\sqrt{1+t}} \right)$$

*Now, consider the following curve:*

$$\overline{\alpha}(t) = \left( -2+t+2\sqrt{1+t}, \frac{4\sqrt{2}}{3}(\sqrt{1+t}-1)^{\frac{3}{2}}, 2+2t-2\sqrt{1+t} \right)$$

*The frenet frame for $\overline{\alpha}(t)$ is :*

$$\bar{t} = \left( \frac{1+\sqrt{1+t}}{\sqrt{5+5t}}, \frac{\sqrt{2\sqrt{1+t}-2}}{\sqrt{5+5t}}, \frac{2\sqrt{1+t}-1}{\sqrt{5+5t}} \right)$$

$$\overline{n} = \left( -\frac{\sqrt{2\sqrt{1+t}-2}}{\sqrt{1+t}}, \frac{2-\sqrt{1+t}}{\sqrt{1+t}}, \frac{\sqrt{2\sqrt{1+t}-2}}{\sqrt{1+t}} \right)$$

$$\bar{b} = \left( \frac{2\sqrt{1+t}-3}{\sqrt{5+5t}}, -\frac{3\sqrt{2\sqrt{1+t}-2}}{\sqrt{5+5t}}, \frac{3-\sqrt{1+t}}{\sqrt{5+5t}} \right)$$

*since $n = \overline{n}$, we say $\alpha$ and $\overline{\alpha}$ are Bertrand mates.*

More details about the example above can be found in [**17**].

**Lemma 2.4.3.** *Let $\alpha : I \to \mathbb{R}^3$ be an arc-length parameterized non-degenerate curve. If $\alpha$ and $\overline{\alpha}$ are Bertrand mates, then $\lambda$ is a nonzero constant where $\lambda$ is as mentioned in definition 2.4.1*

**Theorem 2.4.4.** *Let $\alpha : I \to \mathbb{R}^3$ be an arc-length parameterized nondegenerate curve. We assume $\tau(s) \neq 0$ for all $s \in I$ and $\delta$ is a nonzero constant. Then $\alpha$ and $\overline{\alpha}$ are Bertrand mates with $\overline{\alpha}(s) = \alpha(s) + \delta n(s)$ if and only if there exists a constant $\beta$ such that $\delta\kappa(s) + \beta\tau(s) = 1$ and $\beta\kappa(s) - \delta\tau(s) \neq 0$ for all $s \in I$.*

The proof of theorem 2.4.4 can be found in the paper [**12**].

**Proposition 2.4.5.** *Let* $\alpha$ *and* $\overline{\alpha} : I \to \mathbb{R}^3$ *be two different nondegenerate curves. With the same assumptions in theorem 2.4.4, we assume that* $\alpha$ *and* $\overline{\alpha}$ *are Bertrand mates with* $\overline{\alpha}(s) = \alpha(s) + \delta n(s)$ *and* $\delta\kappa(s) + \beta\tau(s) = 1$ *for all* $s \in I$*, where* $\beta$ *is a constant. Then the curvature* $\overline{\kappa}$ *and the torsion* $\overline{\tau}$ *of* $\overline{\alpha}$ *are as follows:*

$$\overline{\kappa}(s) = \frac{|\beta\kappa(s) - \delta\tau(s)|}{(\delta^2+\beta^2)|\tau(s)|}, \qquad \overline{\tau}(s) = \frac{1}{(\delta^2+\beta^2)\tau(s)}. \tag{2.5}$$

**Corollary 2.4.6.** $\tau(s)\overline{\tau}(s)$ *is a positive constant.*

Furthermore, $\delta\kappa(s) + \beta\tau(s) = 1$ and $\beta\kappa(s) - \delta\tau(s) = 0$ for all $s \in I$ if and only if $\kappa(s) = \delta/(\delta^2+\beta^2)$ and $\tau(s) = \beta/(\delta^2+\beta^2)$. Thus, $\alpha$ is a helix up to congruence. So, $\alpha(s)$ is given as follows:

$$\alpha(s) = \big(\delta\cos\frac{s}{\sqrt{\delta^2+\beta^2}}, \delta\sin\frac{s}{\sqrt{\delta^2+\beta^2}}, \frac{\beta s}{\sqrt{\delta^2+\beta^2}}\big).$$

After a straightforward computation, we find

$$\mathbf{n}(s) = (-\cos(\frac{s}{\sqrt{\delta^2+\beta^2}})), -\sin(\frac{s}{\sqrt{\delta^2+\beta^2}}, 0)$$

Therefore,

$$\overline{\alpha}(s) = \alpha(s) + \lambda\mathbf{n}(s) = \big((\delta-\lambda)\cos\frac{s}{\sqrt{\delta^2+\beta^2}}, (\delta-\lambda)\sin\frac{s}{\sqrt{\delta^2+\beta^2}}, \frac{\beta s}{\sqrt{\delta^2+\beta^2}}\big),$$

where $\lambda$ is a constant.

Assume $\lambda = \delta$, then $\overline{\alpha}(s) = (0, 0, \beta s/\sqrt{\delta^2+\beta^2})$. So, $\overline{\alpha}(s)$ is degenerate, i.e, $\overline{\kappa}(s) = 0$ for all $s \in I$.

Now, if $\lambda \neq \delta$, then $\overline{\mathbf{n}}(s) = sgn(\delta-\lambda)\big(-\cos\frac{s}{\sqrt{\delta^2+\beta^2}}, -\sin\frac{s}{\sqrt{\delta^2+\beta^2}}, 0\big)$, where $sgn(\delta-\lambda) = 1$ if $\delta > \lambda$ and $sgn(\delta-\lambda) = -1$ if $\delta < \lambda$.

Thus, $\overline{\alpha}(s)$ is nondegenerate and $\alpha$ and $\overline{\alpha}$ are Bertrand mates.

## 2.5 Mannheim Curves in $\mathbb{E}^3$

The results presented in this section are taken from [**13**].

### 2.5.1 Mannheim Curves

In this section, we introduce Mannheim curves and provide a few of their characteristics in the three dimensional Euclidean space. We stick to the definitions provided in [**12**].

**Definition 2.5.1.** *If the principal normal line of a space curve $\alpha$ is the same as the binormal line of another curve $\gamma$ for every point, then $\alpha$ is called a Mannheim curve. $\alpha$ and $\gamma$ are called Mannheim partners.*

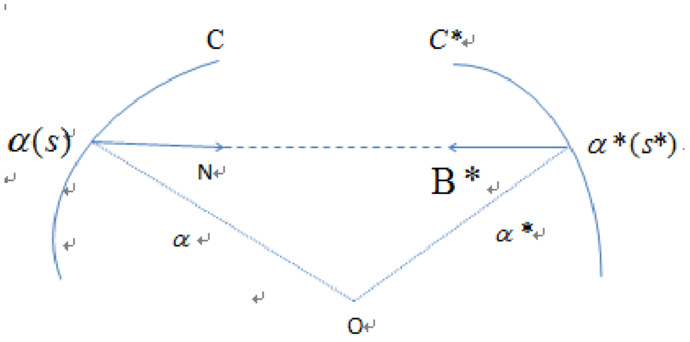

Figure 2.1: $\alpha(s)$ and $\alpha^*(s^*)$ are Mannheim partners in $\mathbb{E}^3$

(G.S.Atalay, Surfaces Family with a Common Mannheim Geodesic Curve, 2018, p.157).

**Theorem 2.5.2.** *The curvature $\kappa$ and torsion $\tau$ of a Mannheim curve in $\mathbb{E}^3$ satisfy the following equation:*

$$\kappa = \lambda(\kappa^2 + \tau^2) \tag{2.6}$$

$\lambda$ is a nonzero constant. [**14**]

**Theorem 2.5.3.** *Let $\alpha$ be a Mannheim curve.*
*If $\alpha$ is a general helix, then $\alpha$ is a circular helix. The curvature and torsion of $\alpha$ are then obtained in the following way:*

$$k = \frac{1}{\lambda(1+c^2)},$$

$$\tau = \frac{c}{\lambda(1+c^2)}.$$

*where $c = \frac{\tau}{\kappa}$ and $\lambda$ is the same as in theorem 2.5.2.*

*Proof.* Since $\alpha$ is a general helix, we can write

$$\frac{\tau}{\kappa} = c \tag{2.7}$$

where $c$ is a constant.
$\alpha$ is a Mannheim curve, so its curvature $\kappa$ satisfies the following:

$$\kappa = \lambda(\kappa^2 + \tau^2)$$

Substituting back in (2.7), we obtain the following expressions for the curvature $\kappa$ and torsion $\tau$:

$$k = \frac{1}{\lambda(1+c^2)},$$

$$\tau = \frac{c}{\lambda(1+c^2)}.$$

□

**Theorem 2.5.4.** *A rectifying Mannheim curve $\alpha(s)$ has the following curvature and torsion:*

$$k = \frac{1}{\lambda(1+(as+b)^2)}, \tag{2.8}$$

$$\tau = \frac{as+b}{\lambda(1+(as+b)^2)} \tag{2.9}$$

*where $\lambda$ is the same as in theorem 2.5.2 while $a$ and $b$ are constants such that $\frac{\tau}{\kappa} = as+b$.*

**Definition 2.5.5.** *A Salkowski curve is a space curve characterized by a constant curvature $\kappa$ and a non-constant torsion $\tau$.*

**Theorem 2.5.6.** *If $\alpha$ is a Mannheim curve, then $\alpha$ is not Salkowski.*

**Definition 2.5.7.** *An Anti-Salkowski curve is a space curve characterized by a constant torsion $\tau$ and a non-constant curvature $k$.*

**Theorem 2.5.8.** *If $\alpha$ is a Mannheim curve, then $\alpha$ is not anti-Salkowski.*

### 2.5.2 Mannheim Partner Curves

**Theorem 2.5.9.** *A Mannheim partner curve $\alpha(s_1)$ in $\mathbb{E}^3$ is a space curve whose curvature $k_1$ and torsion $\tau_1$ satisfy the following equation*

$$\dot{\tau_1} = \frac{d\tau_1}{ds_1} = \frac{k_1}{\lambda}(1+\lambda^2{\tau_1}^2). \tag{2.10}$$

*where $s_1$ is a parameter parameterizing $\alpha$.*

**Theorem 2.5.10.** *The curvature and torsion of a Mannheim partner curve $\alpha_1$ which is a general helix, are as follows:*

$$k_1 = \frac{e^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}} \tag{2.11}$$

$$\tau_1 = \frac{c e^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}} \tag{2.12}$$

*$c$ and $d$ are real constants.*

*Proof.* Let $\alpha_1$ be a general helix. Then, the the ratio of the curvature $\kappa_1$ and torsion $\tau_1$ is constant:

$$\frac{\tau_1}{\kappa_1} = c$$

so,

$$\tau_1 = c\kappa_1$$

since $\alpha_1$ is a Mannheim partner curve, it satisfies equation (2.10).
That is

$$\dot{\tau_1} = \frac{d\tau_1}{ds_1} = \frac{k_1}{\lambda}(1 + \lambda^2{\tau_1}^2)$$

Replacing $\tau_1$ with $c\kappa_1$ in the expression above, we obtain the following Bernoulli differential equation:

$$\dot{\kappa_1} = \frac{\kappa_1}{c\lambda} + \lambda c\kappa_1^3$$

We use the transformation $z = \kappa_1^{-2}$ to get the next linear equation:

$$\dot{z} + \frac{2z}{\lambda c} = -2\lambda c$$

solving this linear equation, we obtain:

$$k_1 = \frac{e^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}}$$

multiplying $\kappa_1$ with a constant, we obtain

$$\tau_1 = \frac{ce^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}}$$

□

**Theorem 2.5.11.** *Let $\alpha_1$ be a Mannheim partner curve.*
*If $\alpha_1$ is a rectifying curve, then curvature and torsion are given as:*

$$k_1 = \frac{1}{\sqrt{-\lambda^2(as_1 + b)^2 + \frac{d}{(as_1+b)^{(\frac{2}{\lambda a}-2)}}}} \tag{2.13}$$

$$\tau_1 = \frac{(as_1+b)}{\sqrt{-\lambda^2(as_1+b)^2 + \frac{d}{(as_1+b)^{(\frac{2}{\lambda a}-2)}}}} \tag{2.14}$$

*d is a real constant*

**Theorem 2.5.12.** *Let $\alpha_1$ be a Mannheim partner curve. If $\alpha_1$ is a Salkowski curve, then the curvature and torsion are given as follows:*

$$\begin{aligned} k_1 &= c, \\ \tau_1 &= \frac{\tan(cs_1 + \frac{d}{\lambda})}{\lambda}. \end{aligned} \tag{2.15}$$

*c and d are real constants*

**Theorem 2.5.13.** *Let $\alpha_1$ be a Mannheim partner curve, then $\alpha_1$ is not anti-Salkowski curve.*

### 2.5.3 Examples

Consider the following curve in $\mathbb{E}^3$:

$$\gamma(t) = \begin{pmatrix} \frac{-3}{\sqrt{2}}\cos(\sqrt{2}t)\sin t + 2\cos t\sin(\sqrt{2}t), \\ \frac{-3}{\sqrt{2}}\sin(\sqrt{2}t)\sin t - 2\cos t\cos(\sqrt{2}t), \\ \frac{1}{\sqrt{2}}\sin t \end{pmatrix}$$

with the Frenet frame:

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos(\sqrt{2}t)\cos t + \sin(\sqrt{2}t)\sin t \\ \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\cos t - \cos(\sqrt{2}t)\sin t \\ \frac{1}{\sqrt{2}}\cos(t) \end{pmatrix},$$

$$N = \left(\frac{\cos(\sqrt{2}t)}{\sqrt{2}}, \frac{\sin(\sqrt{2}t)}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos(\sqrt{2}t)\sin t - \sin(\sqrt{2}t)\cos t, \\ \frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\sin t + \cos(\sqrt{2}t)\cos t, \\ \frac{1}{\sqrt{2}}\sin(t) \end{pmatrix},$$

We can find another curve $\overline{\gamma}(t)$ :

$$\begin{pmatrix} -\sqrt{2}\cos(\sqrt{2}t)\sin t + 2\sin(\sqrt{2}t)\cos t, \\ -2\cos(\sqrt{2}t)\cos t - \sqrt{2}\sin(\sqrt{2}t)\sin t, \\ \sqrt{2}\sin(t), \end{pmatrix},$$

with the following Frenet frame:

$$\overline{T} = \left(\tfrac{\cos(\sqrt{2}t)}{\sqrt{2}}, \tfrac{\sin(\sqrt{2}t)}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$

$$\overline{N} = \left(-\sin(\sqrt{2}t), \cos(\sqrt{2}t), 0,\right)$$

$$\overline{B} = \left(-\tfrac{\cos(\sqrt{2}t)}{\sqrt{2}}, -\tfrac{\sin(\sqrt{2}t)}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$

such that

$$N = -\overline{B}$$

Therefore, $\gamma(t)$ is a Mannheim curve whose Mannheim partner curve is $\overline{\gamma}$ [**18**].

## 2.6 Levi Civita Connection and The Covariant Derivative

**Definition 2.6.1.** *Assume $M$ is a differentiable manifold.*
*We call a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ an* affine connection *in $M$ if:*

*(i)* $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$;

*(ii)* $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;

*(iii)* $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

*for any $X, Y, Z \in \mathfrak{X}(M)$ and where $f, g \in C^\infty(M, \mathbb{R})$.*

*The* ***covariant derivative*** *of $Y$ along $X$ is the vector field denoted by $\nabla_X Y$.*

**Theorem 2.6.2.** *Let $(M, g)$ be a semi-Riemannian Manifold and $g$ is the metric tensor on $M$.*
*The associated Levi-Civita connection $\nabla = \nabla^g$ is defined as a unique symmetric connection satisfying the following:*

*(i) $\nabla$ is torsion-free:* $\nabla_X Y - \nabla_Y X = [X, Y]$.

*(ii) $\nabla$ is Riemannian:* $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle$ *(Metric compatibility).*

*for all $X, Y, Z \in \mathfrak{X}(M)$.*

*$\nabla$ satisfies Koszul formula:*

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle.$$

**Definition 2.6.3.** *The* Christoffel symbols *are real-valued functions* $\Gamma^k_{ij}$, *on a neighborhood* $\mathbb{U}$ *in a semi-Riemannian manifold* $M$, *such that:*

$$\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{ij}\partial_k. \tag{2.16}$$

*where* $1 \leqslant i, j \leqslant n$ *and* $\nabla$ *is the Levi-Civita connection of* $M$.

**Proposition 2.6.4.** *Based on the definition above, we can give the* Christoffel symbols *as follows:*

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{2.17}$$

where $g^{ij} = (g_{ij})^{-1}$ are the entries of the canonical form of the metric tensor $g$. [**19**]

Examples:

$$\Gamma^1_{33} = \frac{1}{2}\Bigg[g^{11}(\frac{\partial g_{31}}{\partial z} + \frac{\partial g_{31}}{\partial z} - \frac{\partial g_{33}}{\partial x}) + 0 + g^{13}(\frac{\partial g_{33}}{\partial z} + \frac{\partial g_{33}}{\partial z} - \frac{\partial g_{33}}{\partial z})\Bigg]$$

$$\Gamma^1_{33} = \frac{1}{2}\Bigg[-f(\frac{\partial 1}{\partial z} + \frac{\partial 1}{\partial z} - \frac{\partial f}{\partial x}) + 0 + 1(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial z} - \frac{\partial f}{\partial z})\Bigg]$$

$$\Gamma^1_{33} = \frac{1}{2}(ff_x + f_z).$$

$$\Gamma^3_{12} = \frac{1}{2}\Bigg[g^{31}(\frac{\partial g_{21}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{12}}{\partial x}) + 0 + 0\Bigg] = 0$$

# Chapter 3

# Three Dimensional Walker Manifolds

In this chapter, we give the definition of Walker manifolds and strict Walker manifolds and give their curvature and torsion.

Let $M$ be a pseudo-Riemannian manifold with signature (m,n). Assume the tangent bundle can be written as $TM = V_1 \bigoplus V_2$ where the smooth subbundles $V_1$ and $V_2$ are called *distributions*. This will then define two complementary projections $\pi_1$ and $\pi_2$ of $TM$ onto $V_1$ and $V_2$. If $V_1$ is parallel, i.e, $\nabla \pi_1 = 0$, and the metric restricted to $V_1$ vanishes identically, then $V_1$ is said to be a *null parallel distribution*.
If a manifold admits a null parallel distribution, then it is called a Walker manifold, and a strict Walker manifold if the distribution is spanned by a null vector field [**2**],[**9**].

## 3.1 The Walker Metric

**Definition 3.1.1.** *A Walker manifold is a pseudo-Riemannian manifold $M$ that admits a parallel null distribution $D$.*

*A three-dimensional pseudo-Riemannian manifold $M$, which admits a null parallel distribution, has a canonical form given by the metric tensor expressed as*

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix},$$

*with the inverse*

$$g_f^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

where $(x, y, z)$ are the local coordinates, $f(x, y, z)$ is defined as a smooth function on the manifold $M$, and $\epsilon = \pm 1$. Note that we will consider $\epsilon = 1$ in this work.

**Remark 3.1.2.** *The Walker metric of a three-dimensional manifold $(M, g_f)$ with coordinates $(x, y, z)$ can also be given in the following way:*

$$g_f^\epsilon = dx \cdot dz + \epsilon dy^2 + f(x, y, z)dz^2. \tag{3.2}$$

We assume that $M$ is a three dimensional Walker manifold throughout this study.

## 3.2 The Cross Product

**Definition 3.2.1.** *Let $u$ and $v$ be vectors in $M$.*
*The cross-product with respect to the Walker metric $g_f^\epsilon$ is given by:*

$$g_f^\epsilon(u \times_f v, w) = det(u, v, w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \tag{3.3}$$

*for any arbitrary vector $w$ in $M$.*

**Proposition 3.2.2.** *The cross-product can be written explicitly as:*

$$u \times_f v = \left( \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) \vec{i} - \epsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{k}. \tag{3.4}$$

It is now possible to form an orthonormal frame on the manifold $M$ with respect to $g_f$. Let $u = (0, 1, 0)$ and $v = (0, 0, \frac{1}{\sqrt{|f|}})$. Then

$$u \times v = (-\sqrt{f}, 0, \frac{1}{\sqrt{f}}).$$

We have $|u| = |v| = |u \times v| = 1$. Here $f$ was assumed to be positive $f > 0$, since the case of $f < 0$ can be studied similarly. We finally have an orthonormal frame

$$e_1 = -\sqrt{f}\partial_x + \frac{1}{\sqrt{f}}\partial_z, \quad e_2 = \partial_y, \quad e_3 = \frac{1}{\sqrt{f}}\partial_z. \tag{3.5}$$

Furthermore,

$$\langle e_1, e_1 \rangle = -1, \quad \langle e_2, e_2 \rangle = 1, \quad \langle e_3, e_3 \rangle = 1.$$

$\{e_1, e_2, e_3\}$ constitutes an orthonormal basis for $T(M)$.

## 3.3 Connections and Curvature

We start with calculating the Christoffel symbols and the Levi-civita connection. Using the (2.16) formula, we can compute the Christoffel symbols and find the Levi-Civita connection.

**Proposition 3.3.1.** *Using the formula given in 2.6.4,*

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right),$$

*the Christoffel symbols can be calculated, and the nonzero Christoffel symbols of a Walker metric $g_f$ are found as:*

$$\Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2}f_x,$$

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2}f_y,$$

$$\Gamma^1_{33} = \frac{1}{2}(f_z + ff_x),$$

$$\Gamma^2_{33} = \frac{1}{2}f_y,$$

$$\Gamma^3_{33} = -\frac{1}{2}f_x.$$

*where $f_x$, $f_y$, and $f_z$ are, respectively, the first derivative of $f$ w.r.t $x$, $y$, and $z$.*

*Proof.* We can calculate the Christoffel symbols, using (2.16), in the following manner:

$$\Gamma^1_{23} = \frac{1}{2}\sum_{l=1}^{n} g^{1l}\left(\frac{\partial g_{3l}}{\partial x^2} + \frac{\partial g_{2l}}{\partial x^3} - \frac{\partial g_{23}}{\partial x^l}\right)$$

$$\Gamma^1_{23} = \frac{1}{2}\left(g^{11}(\frac{\partial g_{31}}{\partial y}+\frac{\partial g_{21}}{\partial z}-\frac{\partial g_{23}}{\partial x})+g^{12}(\frac{\partial g_{32}}{\partial y}+\frac{\partial g_{22}}{\partial z}-\frac{\partial g_{23}}{\partial y})+g^{13}(\frac{\partial g_{33}}{\partial y}+\frac{\partial g_{23}}{\partial z}-\frac{\partial g_{23}}{\partial z})\right)$$

$$\Gamma^1_{23} = \frac{1}{2} f_y,$$

$$\Gamma^2_{33} = \frac{1}{2}\sum_{l=1}^{n} g^{2l}\left(\frac{\partial g_{3l}}{\partial x^3} + \frac{\partial g_{3l}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^l}\right)$$

$$\Gamma^2_{33} = \frac{1}{2}\left(g^{21}(\frac{\partial g_{31}}{\partial z}+\frac{\partial g_{31}}{\partial z}-\frac{\partial g_{33}}{\partial x})+g^{22}(\frac{\partial g_{32}}{\partial z}+\frac{\partial g_{32}}{\partial z}-\frac{\partial g_{33}}{\partial y})+g^{23}(\frac{\partial g_{33}}{\partial z}+\frac{\partial g_{33}}{\partial z}-\frac{\partial g_{33}}{\partial z})\right)$$

$$\Gamma^2_{33} = -\frac{\epsilon}{2} f_y.$$

In the same way, we can find all the other components $\Gamma^i_{jk}$ of the Christoffel symbols as mentioned above. □

After finding the Christoffel symbols, we proceed to calculating the components of the Levi-Civita connection.

**Theorem 3.3.2.** *If $\nabla$ is the Levi-Civita connection of an arbitrary metric $g_f$, then its nonzero components are:*

$$\nabla_{\partial_x}\partial_z = \frac{1}{2} f_x \partial_x,$$

$$\nabla_{\partial_y}\partial_z = \frac{1}{2} f_y \partial_x,$$

$$\nabla_{\partial_z}\partial_z = \frac{1}{2}(f f_x + f_z)\partial_x + \frac{1}{2} f_y \partial_y - \frac{1}{2} f_x \partial_z.$$

*Proof.* Using (2.16) and 2.6.4, we can directly find all the components of the Levi-Civita connection:

$$\nabla_{\partial_x}\partial_y = \sum_k \Gamma^k_{12}\partial_k,$$

$$\nabla_{\partial_x}\partial_y = [\frac{\partial w^1}{\partial x} + \Gamma^1_{12} w^2]\partial_x + [\frac{\partial w^2}{\partial x} + \Gamma^2_{12} w^2]\partial_y + [\frac{\partial w^3}{\partial x} + \Gamma^3_{12} w^2]\partial_z,$$

$$\implies \nabla_{\partial_x}\partial_y = 0.$$

$$\nabla_{\partial_z}\partial_z = \big[\frac{\partial w^1}{\partial z} + \Gamma^1_{33} w^3\big]\partial_x + \big[\frac{\partial w^2}{\partial z} + \Gamma^2_{33} w^3\big]\partial_y + \big[\frac{\partial w^3}{\partial z} + \Gamma^3_{33} w^3\big]\partial_z,$$

$$\implies \nabla_{\partial_z}\partial_z = (\frac{1}{2} f_x f + \frac{1}{2} f_z)\partial_x - \frac{\epsilon}{2} f_y \partial_y - \frac{1}{2} f_x \partial_z,$$

$$\implies \nabla_{\partial_z}\partial_z = \frac{1}{2}\Bigg[(f f_x + f_z)\partial_x - \epsilon f_y \partial_y - f_x \partial_z\Bigg].$$

All the other components $\nabla_{\partial_i}\partial j$ of the Levi-Civita can be found in a similar way. □

### 3.3.1 Frenet Frame Equations

In this section, we find the Frenet frame equations for a curve on a three dimensional Walker manifold. Let $U$ be a vector field. Then there exist smooth functions $a_1, a_2$, and $a_3$ such that:

$$U = a_1 T(s) + a_2 N(s) + a_3 B(s), \tag{3.6}$$

where $a_1, a_2$, and $a_3$ are functions of $s$ and $(T, N, B)$ is the Frenet frame of the curve $\alpha$. Recall that

$$\begin{aligned} T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N &= N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B &= B_1 e_1 + B_2 e_2 + B_3 e_3. \end{aligned}$$

Differentiating (3.6) with respect to $s$, we find:

$$\begin{aligned} \nabla_T U &= \nabla_T (a_1 T + a_2 N + a_3 B) \\ &= \nabla_T (a_1 T) + \nabla_T (a_2 N) + \nabla_T (a_3 B) \\ &= a_1' T(s) + a_1 \nabla_T T + a_2' N + a_2 \nabla_T N + a_3' B + a_3 \nabla_T B. \end{aligned}$$

Recall that the covariant derivative with respect to a curve is calculated using the following formula:

$$\nabla_{\alpha'} z' = \sum_k \left( \frac{d(z')^k}{dt} + \sum_{ij} \Gamma^k_{ij} \frac{d(x^i \circ \alpha')}{dt} (z')^j \right) \partial_k \tag{3.7}$$

Therefore,

$$\begin{aligned}
\nabla_T(a_1 T) &= \sum_k \left( \frac{d(a_1 T)^k}{dt} + \sum_{ij} \Gamma^k_{ij} \frac{d(x^i \circ \alpha')}{dt} (a_1 T)^j \right) \partial_k \\
\nabla_T(a_2 N) &= \sum_k \left( \frac{d(a_2 N)^k}{dt} + \sum_{ij} \Gamma^k_{ij} \frac{d(x^i \circ \alpha')}{dt} (a_2 N)^j \right) \partial_k \\
\nabla_T(a_3 B) &= \sum_k \left( \frac{d(a_3 B)^k}{dt} + \sum_{ij} \Gamma^k_{ij} \frac{d(x^i \circ \alpha')}{dt} (a_3 B)^j \right) \partial_k
\end{aligned}$$

After calculating the above covariant derivatives, we substitute them back into $\nabla_T U$:

$$\begin{aligned}
\nabla_T U = {} & a_1' T(s) + a_2' N(s) + a_3' B(s) + a_1 \Big[ (T_1' + \frac{1}{2} f_x (\alpha_1'' T_3 + \alpha_3'' T_1 + f \alpha_3'' T_3) \\
& + \frac{1}{2} f_y (\alpha_2'' T_3 + \alpha_3'' T_2) + \frac{1}{2} f_z \alpha_3'' T_3) \partial_x + (T_2' - \frac{1}{2} f_y \alpha_3'' T_3) \partial_y + (T_3' - \frac{1}{2} f_x \alpha_3'' T_3) \partial_z \Big] \\
& + a_2 \Big[ (N_1' + \frac{1}{2} f_x (\alpha_1'' N_3 + \alpha_3'' N_1 + f \alpha_3'' N_3) + \frac{1}{2} f_y (\alpha_2'' N_3 + \alpha_3'' N_2) + \frac{1}{2} f_z \alpha_3'' N_3) \partial_x \\
& + (N_2' - \frac{1}{2} f_y \alpha_3'' N_3) \partial_y + (N_3' - \frac{1}{2} f_x \alpha_3'' N_3) \partial_z \Big] + a_3 \Big[ (B_1' + \frac{1}{2} f_x (\alpha_1'' B_3 + \alpha_3'' B_1 \\
& + f \alpha_3'' B_3) + \frac{1}{2} f_y (\alpha_2'' B_3 + \alpha_3'' B_2) + \frac{1}{2} f_z \alpha_3'' B_3) \partial_x + (B_2' - \frac{1}{2} f_y^2 \alpha_3'' B_3) \partial_y + (B_3' \\
& - \frac{1}{2} f_x \alpha_3'' B_3) \partial_z ].
\end{aligned}$$

Therefore, the Frenet frame equations are found as:

$$\begin{aligned}
\nabla_T T = & \Big( T_1' + \frac{1}{2} f_x (\alpha_1'' T_3 + \alpha_3'' T_1 + f \alpha_3'' T_3) + \frac{1}{2} f_y (\alpha_2'' T_3 + \alpha_3'' T_2) + \frac{1}{2} f_z \alpha_3'' T_3 \Big) \partial_x \\
& + \Big( T_2' - \frac{1}{2} f_y \alpha_3'' T_3 \Big) \partial_y + \Big( T_3' - \frac{1}{2} f_x \alpha_3'' T_3 \Big) \partial_z.
\end{aligned}$$

$$\begin{aligned}\nabla_T N =&\Big(N_1' + \frac{1}{2}f_x(\alpha_1''N_3 + \alpha_3''N_1 + f\alpha_3''N_3) + \frac{1}{2}f_y(\alpha_2''N_3 + \alpha_3''N_2) + \frac{1}{2}f_z\alpha_3''N_3\Big)\partial_x\\ &+ \Big(N_2' - \frac{1}{2}f_y\alpha_3''N_3\Big)\partial_y + \Big(N_3' - \frac{1}{2}f_x\alpha_3''N_3\Big)\partial_z.\end{aligned}$$

$$\begin{aligned}\nabla_T B =&\Big(B_1' + \frac{1}{2}f_x(\alpha_1''B_3 + \alpha_3''B_1 + f\alpha_3''B_3) + \frac{1}{2}f_y(\alpha_2''B_3 + \alpha_3''B_2) + \frac{1}{2}f_z\alpha_3''B_3\Big)\partial_x\\ &+ \Big(B_2' - \frac{1}{2}f_y^2\alpha_3''B_3\Big)\partial_y + \Big(B_3' - \frac{1}{2}f_x\alpha_3''B_3\Big)\partial_z.\end{aligned}$$

**Theorem 3.3.3.** *Consider an arc-length parametrized curve* $\alpha(s) : I \in \mathbb{R} \to (M, g_f^\epsilon)$.
*The trihedron* $\{T(s), N(s), B(s)\}$, *where* $T(s)$ *is the tangent,* $N(s)$ *is the principal normal, and* $B(s)$ *is the binormal vector, forms the Frenet frame of* $\alpha$. *These three vectors satisfy the Frenet formulas:*

$$\begin{aligned}\nabla_T T(s) =&\epsilon_2\kappa(s)N(s) && (3.8)\\ \nabla_T N(s) =&-\epsilon_1\kappa T(s) - \epsilon_3\tau B(s) && (3.9)\\ \nabla_T B(s) =&\epsilon_2\tau(s)N(s) && (3.10)\end{aligned}$$

where $\tau(s)$ and $\kappa(s)$ are, respectively, the torsion and the curvature of $\alpha$.
with $\epsilon_1 = g_f(T, T)$, $\epsilon_2 = g_f(N, N)$, and $\epsilon_3 = g_f(B, B)$. [**4**]

# Chapter 4

# Bertrand and Mannheim Curves on a Three Dimensional Walker Manifold

## 4.1 Curves in 3 Dimensional Walker Manifolds

In this chapter, we investigate Mannheim curves' properties in the three dimensional Walker manifold.

Recall that a space curve $\alpha : I \to \mathbb{R}^3$ is *regular* if and only if $\alpha'(t) \neq 0$ for any $t \in I$, and *non-degenerate* if $\langle \alpha'(t), \alpha'(t) \rangle \neq 0$ for all $t \in I$.

Throughout this work, we take $\alpha' = \alpha_1' \frac{\partial}{\partial x} + \alpha_2' \frac{\partial}{\partial y} + \alpha_3' \frac{\partial}{\partial z}$ and $\alpha'' = \nabla_{\alpha'} \alpha'$.

For a general parameter $t$, we have the following:

$$\mathbf{T}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \qquad \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t), \qquad \mathbf{B}(t) = \frac{\alpha'(t) \times \alpha''(t)}{|\alpha'(t) \times \alpha''(t)|}. \tag{4.1}$$

The **curvature** $\kappa$ and the **torsion** $\tau$ of a regular nondegenerate space curve are given as:

$$\kappa(t) = \frac{|\alpha'(t) \times_f \alpha''(t)|}{|\alpha'(t)|^3}, \qquad \tau(t) = \frac{det(\alpha'(t), \alpha''(t), \alpha'''(t))}{|\alpha'(t) \times_f \alpha''(t)|^2}. \tag{4.2}$$

### 4.1.1 Calculation of $\kappa$

Let $\alpha = \alpha(t) = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z}$, be a regular nondegenarate curve in a 3-dimensional Walker manifold $(M, g_f)$.
Then: $\alpha'' = \nabla_T T$
Applying the properties of the covariant derivative, we obtain:

$$\begin{aligned} \alpha'' &= \nabla_T(\alpha'_1 \frac{\partial}{\partial x} + \alpha'_2 \frac{\partial}{\partial y} + \alpha'_3 \frac{\partial}{\partial z}) \\ &= \nabla_T(\alpha'_1 \frac{\partial}{\partial x}) + \nabla_T(\alpha'_2 \frac{\partial}{\partial y}) + \nabla_T(\alpha'_3 \frac{\partial}{\partial z}) \\ &= \alpha'_1 \nabla_{\frac{\partial}{\partial x}}(\alpha'_1 \frac{\partial}{\partial x}) + \alpha'_2 \nabla_{\frac{\partial}{\partial y}}(\alpha'_1 \frac{\partial}{\partial x}) + \cdots + \alpha'_3 \nabla_{\frac{\partial}{\partial z}}(\alpha'_3 \frac{\partial}{\partial z}) \end{aligned}$$

On the other hand, we know that $\alpha''$ components w.r.t $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ can be calculated using the following expression:

$$\frac{d^2(x^i \circ \alpha)}{dt^2} + \sum_{m,n} \Gamma^i_{mn}(\alpha) \frac{d(x^m \circ \alpha)}{dt} \frac{d(x^n \circ \alpha)}{dt} = 0 \tag{4.3}$$

for $1 \leq i \leq 3$, and $1 \leq m, n \leq 3$. [**6**]

hence,

$$\alpha'' = \mathbb{A} \frac{\partial}{\partial x} + \mathbb{B} \frac{\partial}{\partial y} + \mathbb{C} \frac{\partial}{\partial z}$$

Where:

$$\mathbb{A} = \frac{d^2(x \circ \alpha)}{dt^2} + \sum_{i,j} \Gamma^1_{ij}(\alpha) \frac{d(x^i \circ \alpha)}{dt} \frac{d(x^j \circ \alpha)}{dt},$$

$$\mathbb{B} = \frac{d^2(y \circ \alpha)}{dt^2} + \sum_{i,j} \Gamma^2_{ij}(\alpha) \frac{d(x^i \circ \alpha)}{dt} \frac{d(x^j \circ \alpha)}{dt},$$

$$\mathbb{C} = \frac{d^2(z \circ \alpha)}{dt^2} + \sum_{i,j} \Gamma^3_{ij}(\alpha) \frac{d(x^i \circ \alpha)}{dt} \frac{d(x^j \circ \alpha)}{dt},$$

In these calculations, we take $\epsilon = 1$. We obtain the following:

$$\mathbb{A} = \frac{d^2\alpha_1}{dt^2} + f_x \frac{d\alpha_1}{dt} \frac{d\alpha_3}{dt} + f_y \frac{d\alpha_2}{dt} \frac{d\alpha_3}{dt} + \frac{1}{2} f f_x \left( \frac{d\alpha_3}{dt} \right)^2 + \frac{1}{2} f_z \left( \frac{d\alpha_3}{dt} \right)^2,$$

$$\mathbb{B} = \frac{d^2\alpha_2}{dt^2} - \frac{1}{2} f_y \left( \frac{d\alpha_3}{dt} \right)^2,$$

.

$$\mathbb{C} = \frac{d^2\alpha_3}{dt^2} - \frac{1}{2} f_x \left( \frac{d\alpha_3}{dt} \right)^2,$$

Next, we need to find $\alpha' \times_f \alpha''$:

$$\alpha' \times_f \alpha'' =$$

$$\Bigg(\alpha'_1\alpha''_2 - \tfrac{1}{2} f_y \alpha'_1 (\alpha'_3)^2 - \alpha'_2\alpha''_1 - f_x \alpha'_1\alpha'_2\alpha'_3 - f_y (\alpha'_2)^2 \alpha'_3 - \tfrac{1}{2}(f f_x + f_z)\alpha'_2(\alpha'_3)^2 - f(\alpha'_2\alpha''_3 -$$

$$\tfrac{1}{2} f_x \alpha'_2 (\alpha'_3)^2 - \alpha'_3\alpha''_2 + \tfrac{1}{2} f_y (\alpha'_3)^3)\Bigg) \tfrac{\partial}{\partial x} + \Bigg( - \alpha'_1\alpha''_3 + \tfrac{1}{2} f_x (\alpha'_3)^2 \alpha'_1 + \alpha'_3\alpha''_1 + f_x \alpha'_1 (\alpha'_3)^2 +$$

$$f_y \alpha'_2 (\alpha'_3)^2 + \tfrac{1}{2}(f f_x + f_z)(\alpha'_3)^3 \Bigg) \tfrac{\partial}{\partial y} + \Bigg( \alpha'_2\alpha''_3 - \tfrac{1}{2} f_x \alpha'_2 (\alpha'_3)^2 - \alpha'_3\alpha''_2 + \tfrac{1}{2} f_y (\alpha'_3)^3 \Bigg) \tfrac{\partial}{\partial z}.$$

$$|\alpha' \times_f \alpha''| = \sqrt{g_f^{\epsilon}(\alpha' \times_f \alpha'', \alpha' \times_f \alpha'')}$$

We need to find $|\alpha'(t)|^3$:

We know that $\alpha' = \alpha'_1 \frac{\partial}{\partial x} + \alpha'_2 \frac{\partial}{\partial y} + \alpha'_3 \frac{\partial}{\partial z}$,
and since $|\alpha'(t)| = \sqrt{g_f^{\epsilon}(\alpha', \alpha')}$,
we have $|\alpha'(t)| = \sqrt{\alpha'_1 \cdot \alpha'_3 + (\alpha'_2)^2 + f(\alpha'_3)^2}$
so,

$$|\alpha'(t)|^3 = \Bigg(\alpha'_1 \cdot \alpha'_3 + (\alpha'_2)^2 + f(\alpha'_3)^2\Bigg)^{\frac{3}{2}}.$$

$$\kappa(t) = \tfrac{|\alpha'(t) \times_f \alpha''(t)|}{|\alpha'(t)|^3}$$

$$\begin{aligned}\kappa(t) = \Bigg(& \alpha'_1[\alpha'_2\alpha''_2\alpha''_3 - (\alpha''_2)^2\alpha'_3 + \alpha'_1(\alpha''_3)^2 - 2\alpha''_1\alpha'_3\alpha''_3] + \alpha''_1[\alpha'_2\alpha''_2\alpha'_3 \\ & -(\alpha'_2)^2\alpha''_3 + \alpha''_1(\alpha'_3)^2] + \alpha'_1\alpha'_2[\tfrac{1}{2}f_x\alpha''_2(\alpha'_3)^2 - \tfrac{5}{2}f_y(\alpha'_3)^2\alpha''_3 + \tfrac{11}{4}f_xf_y(\alpha'_3)^4 + \tfrac{1}{2}f_x^2\alpha'_2(\alpha'_3)^3 - \\ & f_x\alpha'_2\alpha'_3\alpha''_3] + \alpha'_1\alpha'_3[f_y\alpha''_2(\alpha'_3)^2 - \tfrac{1}{4}f_y^2(\alpha'_3)^4 - 3f_x\alpha'_1\alpha'_3\alpha''_3 - (ff_x + f_z)(\alpha'_3)^2\alpha''_3 + \\ & \tfrac{9}{4}f_x^2(\alpha'_3)^3\alpha'_1 + 3f_x\alpha''_1(\alpha'_3)^2 + \tfrac{3}{2}f_x(ff_x + f_z)(\alpha'_3)^4] + \alpha'_2\alpha'_3[\tfrac{1}{2}f_x\alpha''_1\alpha'_2\alpha'_3 + \tfrac{3}{2}f_y\alpha''_1(\alpha'_3)^2 - \\ & f_y(\alpha'_2)^2\alpha''_3 + \tfrac{1}{2}f_xf_y(\alpha'_2)^2(\alpha'_3)^2 + f_y\alpha'_2\alpha'_3\alpha''_2 + \tfrac{1}{2}f_y^2\alpha'_2(\alpha'_3)^3 + \tfrac{1}{4}f_x(ff_x + f_z)\alpha'_2(\alpha'_3)^3 + \\ & \tfrac{3}{4}f_y(ff_x + f_z)(\alpha'_3)^4 - \tfrac{1}{2}ff_xf_y(\alpha'_3)^4 + ff_y(\alpha'_3)^2\alpha''_3 + \tfrac{1}{2}(ff_x + f_z)\alpha'_3(\alpha''_2\alpha'_3 - \alpha'_2\alpha''_3)] + \\ & (\alpha'_3)^4[ff_y(\tfrac{1}{2}f_y(\alpha'_3)^2 - \alpha''_2) + (ff_x + f_z)(\alpha''_1 + \tfrac{1}{4}(\alpha'_3)^2(ff_x + f_z))]\Bigg)^{\frac{1}{2}} \div \Bigg(\alpha'_1 \cdot \alpha'_3 + \\ & (\alpha'_2)^2 + f(\alpha'_3)^2\Bigg)^{\frac{3}{2}}\end{aligned}$$

### 4.1.2 Calculation of $\tau$

The aim of this section is to find the general formula of the torsion $\tau$ of a curve on a three dimensional Walker Manifold. The first step is to calculate $\alpha'''(s)$.

**Remark 4.1.1.** *In the specific case where* $z = \alpha'$, *the derivative* $z' = \alpha''$ *represents the acceleration of the curve* $\alpha$.

*For a vector field $z$ on $\alpha$, it might be tempting to write $z' = \nabla_{\alpha'} z$ and therefore $\alpha'' = \nabla_{\alpha'}\alpha'$. Although $z$ and $\alpha'$ are not vector fields on $M$, these equations can be valid, but only at points $\alpha(t)$ where $\alpha'(t) \neq 0$ [**6**].*

**Theorem 4.1.2.** *The covariant derivative with respect to a curve is given by the following formula:*

$$z' = \sum_k \left\{ \frac{d(z)^k}{dt} + \sum_{i,j} \Gamma^k_{i,j} \frac{d(x^i \circ \alpha)}{dt} (z)^j \right\} \partial_k \tag{4.4}$$

Hence,

$$\alpha'''(s) = \nabla_{\alpha'}\alpha'' = (\alpha'')' = (z')' = \sum_k \left\{ \frac{d(z')^k}{dt} + \sum_{i,j} \Gamma^k_{i,j} \frac{d(x^i \circ \alpha')}{dt} (z')^j \right\} \partial_k \tag{4.5}$$

We know that
$z' = \alpha'' = [\alpha_1''' + f_x(\alpha_1''\alpha_3' + \alpha_1'\alpha_3'') + f_y(\alpha_2''\alpha_3' + \alpha_2'\alpha_3'') + (ff_x + f_z)\alpha_3'\alpha_3'' + \frac{1}{2} f_x\alpha_1''(\alpha_3'' - \frac{1}{2} f_x(\alpha_3')^2) + \frac{1}{2} f_y\alpha_2''(\alpha_3'' - \frac{1}{2} f_x(\alpha_3')^2 + \frac{1}{2} f_x\alpha_3''(\alpha_1'' + f_x\alpha_1'\alpha_3' + f_y\alpha_2'\alpha_3' + \frac{1}{2}(ff_x + f_z)(\alpha_3')^2 + \frac{1}{2}(ff_x + f_z)\alpha_3''(\alpha_3'' - \frac{1}{2} f_x(\alpha_3')^2)]\frac{\partial}{\partial_x} + [\alpha_2''' - f_y\alpha_3'\alpha_3'' - \frac{1}{2} f_y\alpha_3''(\alpha_3'' - \frac{1}{2} f_x(\alpha_3')^2)]\frac{\partial}{\partial_y} + [\alpha_3''' - f_x\alpha_3'\alpha_3'' - \frac{1}{2} f_x\alpha_3''(\alpha_3'' - \frac{1}{2} f_x(\alpha_3')^2)]\frac{\partial}{\partial_z}$.

Similarly,
$\alpha'''(t) = [\alpha_1''' + f_x\alpha_1''\alpha_3' + f_x\alpha_1'\alpha_3'' + f_y\alpha_2''\alpha_3' + f_y\alpha_2'\alpha_3'' + ff_x\alpha_3'\alpha_3'' + f_z\alpha_3'\alpha_3'' + \frac{1}{2} f_x\alpha_1''\alpha_3'' - \frac{1}{4} f_x^2\alpha_1''(\alpha_3')^2 + \frac{1}{2} f_y\alpha_2''\alpha_3'' - \frac{1}{4} f_y f_x\alpha_2''(\alpha_3')^2 + \frac{1}{2} f_x\alpha_1''\alpha_3'' + \frac{1}{2} f_x^2\alpha_1'\alpha_3'\alpha_3'' + \frac{1}{2} f_x f_y\alpha_2'\alpha_3'\alpha_3'' + \frac{1}{4} ff_x^2(\alpha_3')^2\alpha_3'' + \frac{1}{4} f_x f_z(\alpha_3')^2\alpha_3'' + \frac{1}{4} ff_x^2(\alpha_3'')^3 - \frac{1}{8} ff_x^3(\alpha_3')^2(\alpha_3'')^2 + \frac{1}{4} f_x f_z(\alpha_3'')^3 - \frac{1}{8} f_x^2 f_z(\alpha_3')^2(\alpha_3'')^5]\frac{\partial}{\partial_x} + [\alpha_2''' - f_y\alpha_3'\alpha_3'' - \frac{1}{2} f_y(\alpha_3'')^2 + \frac{1}{4} f_x f_y(\alpha_3')^2\alpha_3'']\frac{\partial}{\partial_y} + [\alpha_3''' - f_x\alpha_3'\alpha_3'' - \frac{1}{2} f_x(\alpha_3'')^2 + \frac{1}{4} f_x^2(\alpha_3')^2\alpha_3'']\frac{\partial}{\partial_z}$.

The next step is to find $g_f^\epsilon(\alpha' \times_f \alpha'', \alpha''')$.

$g_f^\epsilon(\alpha' \times_f \alpha'', \alpha''') = \alpha_3'''[\alpha_1'\alpha_2'' - \alpha_1''\alpha_2' + ff_y(\alpha_3')^3 - 2f\alpha_2''\alpha_3'] + \alpha_2'''[\alpha_1''\alpha_3' - \alpha_1'\alpha_3'' + \frac{(\alpha_3')^3}{2}(ff_x + f_z)] - f_x\alpha_1'\alpha_2''\alpha_3''[\alpha_3' + \frac{\alpha_3''}{2} + \frac{(\alpha_3')^2}{4} f_x] + \alpha_1'\alpha_3'[f_x^2\alpha_2'(\alpha_3'\alpha_3'' + \frac{1}{2}(\alpha_3'')^2 - \frac{1}{4} f_x(\alpha_3')^2\alpha_3'') + \alpha_3''(f_y\alpha_3'' - f_x f_y(\alpha_3')^2 + \frac{1}{4} f - x^2 f_y(\alpha_3')^3) + \frac{3}{2} f_x\alpha_2'''\alpha_3' - f_x\alpha_2'\alpha_3''' - \frac{1}{2} f_y\alpha_3'\alpha_3'''] + \alpha_2'\alpha_3'[f_y\alpha_2'''\alpha_3' - f_y\alpha_2'\alpha_3''' + f_x f_y\alpha_2'\alpha_3'\alpha_3''(1 - \frac{\alpha_3'}{4} f_x) - ff_x\alpha_3'\alpha_3''' + \alpha_3''(\frac{1}{2} f_x f_y\alpha_2'\alpha_3'' + \frac{3}{4} ff_x^2\alpha_3'\alpha_3'' + ff_x^2(\alpha_3')^2 - \frac{3}{8} ff_x^3(\alpha_3')^3 + f_x f_z\alpha_3'(\frac{\alpha_3''}{4} + \frac{\alpha_3'}{2} - \frac{(\alpha_3')^2}{8} f_x) - f_y^2\alpha_3'(\alpha_3' + \frac{\alpha_3''}{2}) + f_x(\alpha_3')^2(\frac{1}{2} ff_x + \frac{\alpha_3'}{4} f_y^2)) - \frac{1}{2}\alpha_3'\alpha_3'''(f_z + ff_x)] + \alpha_2'\alpha_3''[\frac{1}{2} f_x\alpha_1''(\alpha_3'' -$

$\frac{1}{2}f_x(\alpha_3')^2)] + \alpha_2''\alpha_3''[2ff_x(\alpha_3')^2 + ff_x\alpha_3'\alpha_3'' - \frac{1}{2}ff_x^2(\alpha_3')^3] + \alpha_3'\alpha_3''[\frac{(\alpha_3')^2}{2}ff_xf_y(\frac{(\alpha_3')^2}{4}f_x - \frac{\alpha_3''}{2} - \alpha_3')] + \alpha_3''[\frac{\alpha_3'}{2}\alpha_1''f_y(\frac{(\alpha_3')^2}{2}f_x - \alpha_3'') - \frac{(\alpha_3')^3}{2}f_yf_z(\alpha_3' + \frac{\alpha_3''}{2}) + f_xf_y(\alpha_3')^3(-f\alpha_3' + \frac{(\alpha_3')^2}{4}ff_x - \frac{\alpha_3''}{2}f + \frac{(\alpha_3')^2}{8}f_z) + \frac{\alpha_1'}{2}f_y(\alpha_3'')^2 - f_y\alpha_1''(\alpha_3')^2]$

Hence, the torsion $\tau$ of a curve on a three dimensional Walker manifold is given as follows:

$$\tau(s) = \frac{A(s)}{B(s)},$$

where

$$\begin{aligned}
A(s) \;=\;& \alpha_3'''[\alpha_1'\alpha_2'' - \alpha_1''\alpha_2' + ff_y(\alpha_3')^3 - 2f\alpha_2''\alpha_3'] + \alpha_2'''[\alpha_1''\alpha_3' - \alpha_1'\alpha_3'' + \frac{(\alpha_3')^3}{2}(ff_x \\
+\;& f_z)] - f_x\alpha_1'\alpha_2''\alpha_3''[\alpha_3' + \frac{\alpha_3''}{2} + \frac{(\alpha_3')^2}{4}f_x] + \alpha_1'\alpha_3'[f_x^2\alpha_2'(\alpha_3'\alpha_3'' + \frac{1}{2}(\alpha_3'')^2 \\
-\;& \frac{1}{4}f_x(\alpha_3')^2\alpha_3'') + \alpha_3''(f_y\alpha_3'' - f_xf_y(\alpha_3')^2 + \frac{1}{4}f_x^2f_y(\alpha_3')^3) + \frac{3}{2}f_x\alpha_2'''\alpha_3' - f_x\alpha_2'\alpha_3''' \\
-\;& \frac{1}{2}f_y\alpha_3'\alpha_3'''] + \alpha_2'\alpha_3'[f_y\alpha_2'''\alpha_3' - f_y\alpha_2'\alpha_3''' + f_xf_y\alpha_2'\alpha_3'\alpha_3''(1 - \frac{\alpha_3'}{4}f_x) - ff_x\alpha_3'\alpha_3''' \\
+\;& \alpha_3''(\frac{1}{2}f_xf_y\alpha_2'\alpha_3'' + \frac{3}{4}ff_x^2\alpha_3'\alpha_3'' + ff_x^2(\alpha_3')^2 - \frac{3}{8}ff_x^3(\alpha_3')^3 + f_xf_z\alpha_3'(\frac{\alpha_3''}{4} + \frac{\alpha_3'}{2} \\
-\;& \frac{(\alpha_3')^2}{8}f_x) - f_y^2\alpha_3'(\alpha_3' + \frac{\alpha_3''}{2}) + f_x(\alpha_3')^2(\frac{1}{2}ff_x + \frac{\alpha_3'}{4}f_y^2)) - \frac{1}{2}\alpha_3'\alpha_3'''(f_z + ff_x)] \\
+\;& \alpha_2'\alpha_3''[\frac{1}{2}f_x\alpha_1''(\alpha_3'' - \frac{1}{2}f_x(\alpha_3')^2)] + \alpha_2''\alpha_3''[2ff_x(\alpha_3')^2 + ff_x\alpha_3'\alpha_3'' - \frac{1}{2}ff_x^2(\alpha_3')^3] \\
+\;& \alpha_3'\alpha_3''[\frac{(\alpha_3')^2}{2}ff_xf_y(\frac{(\alpha_3')^2}{4}f_x - \frac{\alpha_3''}{2} - \alpha_3')] + \alpha_3''[\frac{\alpha_3'}{2}\alpha_1''f_y(\frac{(\alpha_3')^2}{2}f_x - \alpha_3'') \\
-\;& \frac{(\alpha_3')^3}{2}f_yf_z(\alpha_3' + \frac{\alpha_3''}{2}) + f_xf_y(\alpha_3')^3(-f\alpha_3' + \frac{(\alpha_3')^2}{4}ff_x - \frac{\alpha_3''}{2}f + \frac{(\alpha_3')^2}{8}f_z) \\
+\;& \frac{\alpha_1'}{2}f_y(\alpha_3'')^2 - f_y\alpha_1''(\alpha_3')^2].
\end{aligned}$$

and

$$
\begin{aligned}
B(s) &= \alpha_1'[\alpha_2'\alpha_2''\alpha_3'' - (\alpha_2'')^2\alpha_3' + \alpha_1'(\alpha_3'')^2 - 2\alpha_1''\alpha_3'\alpha_3''] + \alpha_1''[\alpha_2'\alpha_2''\alpha_3' - (\alpha_2')^2\alpha_3'' \\
&+ \alpha_1''(\alpha_3')^2] + \alpha_1'\alpha_2'[\frac{1}{2}f_x\alpha_2''(\alpha_3')^2 - \frac{5}{2}f_y(\alpha_3')^2\alpha_3'' + \frac{11}{4}f_xf_y(\alpha_3')^4 + \frac{1}{2}f_x^2\alpha_2'(\alpha_3')^3 \\
&- f_x\alpha_2'\alpha_3'\alpha_3''] + \alpha_1'\alpha_3'[f_y\alpha_2''(\alpha_3')^2 - \frac{1}{4}f_y^2(\alpha_3')^4 - 3f_x\alpha_1'\alpha_3'\alpha_3'' - (ff_x + f_z)(\alpha_3')^2\alpha_3'' \\
&+ \frac{9}{4}f_x^2(\alpha_3')^3\alpha_1' + 3f_x\alpha_1''(\alpha_3')^2 + \frac{3}{2}f_x(ff_x + f_z)(\alpha_3')^4] + \alpha_2'\alpha_3'[\frac{1}{2}f_x\alpha_1''\alpha_2'\alpha_3' \\
&+ \frac{3}{2}f_y\alpha_1''(\alpha_3')^2 - f_y(\alpha_2')^2\alpha_3'' + \frac{1}{2}f_xf_y(\alpha_2')^2(\alpha_3')^2 + f_y\alpha_2'\alpha_3'\alpha_2'' + \frac{1}{2}f_y^2\alpha_2'(\alpha_3')^3 \\
&+ \frac{1}{4}f_x(ff_x + f_z)\alpha_2'(\alpha_3')^3 + \frac{3}{4}f_y(ff_x + f_z)(\alpha_3')^4 - \frac{1}{2}ff_xf_y(\alpha_3')^4 + ff_y(\alpha_3')^2\alpha_3'' \\
&+ \frac{1}{2}(ff_x + f_z)\alpha_3'(\alpha_2''\alpha_3' - \alpha_2'\alpha_3'')] + (\alpha_3')^4[ff_y(\frac{1}{2}f_y(\alpha_3')^2 - \alpha_2'') + (ff_x + f_z)(\alpha_1'' \\
&+ \frac{1}{4}(\alpha_3')^2(ff_x + f_z))].
\end{aligned}
$$

For the special case, when the Walker manifold is strict, i.e, $f_x = 0$, then the torsion is found as :

$$\tau(s) = \frac{C(s)}{D(s)},$$

where

$$
\begin{aligned}
C(s) &= \alpha_3'''[\alpha_1'\alpha_2'' - \alpha_1''\alpha_2' + ff_y(\alpha_3')^3 - 2f\alpha_2''\alpha_3'] + \alpha_2'''[\alpha_1''\alpha_3' - \alpha_1'\alpha_3'' + \frac{(\alpha_3')^3}{2}f_z] \\
&+ \alpha_1'\alpha_3'[\alpha_3''(f_y\alpha_3'') - \frac{1}{2}f_y\alpha_3'\alpha_3'''] + \alpha_2'\alpha_3'[f_y(\alpha_2'''\alpha_3' - \alpha_2'\alpha_3''') + \alpha_3''(-f_y^2\alpha_3'(\alpha_3' \\
&+ \frac{\alpha_3''}{2})) - \frac{\alpha_3'}{2}\alpha_3'''f_z] + \alpha_3''[\frac{-\alpha_3'}{2}\alpha_1''f_y\alpha_3'' - \frac{(\alpha_3')^4}{2}f_yf_z - \frac{(\alpha_3')^3}{4}\alpha_3''f_yf_z \\
&+ \frac{\alpha_1'}{2}f_y(\alpha_3'')^2 - f_y\alpha_1''(\alpha_3')^2].
\end{aligned}
$$

and

$$
\begin{aligned}
D(s) &= \alpha_1'[\alpha_2'\alpha_2''\alpha_3'' - (\alpha_2'')^2\alpha_3' + \alpha_1'(\alpha_3'')^2 - 2\alpha_1''\alpha_3'\alpha_3''] + \alpha_1''[\alpha_2'\alpha_2''\alpha_3' - (\alpha_2')^2\alpha_3'' \\
&+ \alpha_1''(\alpha_3')^2] + \alpha_1'\alpha_2'[-\frac{5}{2}f_y(\alpha_3')^2\alpha_3''] + \alpha_1'\alpha_3'[f_y\alpha_2''(\alpha_3')^2 - \frac{1}{4}f_y^2(\alpha_3')^4 - f_z(\alpha_3')^2\alpha_3''] \\
&+ \alpha_2'\alpha_3'[\frac{3}{2}f_y\alpha_1''(\alpha_3')^2 - f_y(\alpha_2')^2\alpha_3'' + f_y\alpha_2'\alpha_3'\alpha_2'' + \frac{1}{2}f_y^2\alpha_2'(\alpha_3')^3 + \frac{3}{4}f_yf_z(\alpha_3')^4 \\
&+ ff_y(\alpha_3')^2\alpha_3'' + \frac{1}{2}f_z\alpha_3'(\alpha_2''\alpha_3' - \alpha_2'\alpha_3'')] + (\alpha_3')^4[ff_y(\frac{1}{2}f_y(\alpha_3')^2 - \alpha_2'') + f_z\alpha_1'' \\
&+ \frac{1}{4}f_z^2(\alpha_3')^2].
\end{aligned}
$$

### 4.1.3 Special Cases

In this section, we investigate the $\kappa$ and $\tau$ for some special cases of a curve on a strict Walker manifold.

$\bullet$**Case 1**: $\alpha(s) = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1 = constant$ and $\alpha_2, \alpha_3 \neq constant$.

$$\tau = \Big[\alpha_3'''(ff_y(\alpha_3')^3 - 2f\alpha_2''\alpha_3') + \tfrac{1}{2}f_z\alpha_2'''(\alpha_3')^3 + \alpha_2'\alpha_3'(f_y\alpha_2'''\alpha_3' - f_y\alpha_2'\alpha_3''' + \alpha_3''(-f_y^2\alpha_3'(\alpha_3'+\tfrac{\alpha_3''}{2})) - \tfrac{1}{2}\alpha_3'\alpha_3'''f_z) + \alpha_3''(-\tfrac{(\alpha_3')^3}{2}f_yf_z(\alpha_3'+\tfrac{\alpha_3''}{2}))\Big] \div \Big[\alpha_2'\alpha_3'(-f_y(\alpha_2')^2\alpha_3'' + f_y\alpha_2'\alpha_3'\alpha_2'' + \tfrac{1}{2}f_y^2\alpha_2'(\alpha_3')^3 + \tfrac{3}{4}f_yf_z(\alpha_3')^4 + ff_y(\alpha_3')^2\alpha_3'' + \tfrac{1}{2}f_z\alpha_3'(\alpha_2''\alpha_3' - \alpha_2'\alpha_3'')) + (\alpha_3')^4(\tfrac{1}{2}ff_y^2(\alpha_3')^2 - ff_y\alpha_2'' + \tfrac{1}{4}(\alpha_3')^2f_z^2)\Big].$$

$$\kappa = \Big[\alpha_2'\alpha_3'(-f_y(\alpha_2')^2\alpha_3'' + f_y\alpha_2'\alpha_3'\alpha_2'' + \tfrac{1}{2}f_y^2\alpha_2'(\alpha_3')^3 + \tfrac{3}{4}f_yf_z(\alpha_3')^4 + ff_y(\alpha_3')^2\alpha_3'' + \tfrac{1}{2}f_z\alpha_3'(\alpha_2''\alpha_3' - \alpha_2'\alpha_3'')) + (\alpha_3')^4(\tfrac{1}{2}ff_y^2(\alpha_3')^2 - ff_y\alpha_2'' + \tfrac{1}{4}(\alpha_3')^2f_z^2)\Big]^{\frac{1}{2}} \div \Big[(\alpha_2')^2 + f(\alpha_3')^2\Big]^{\frac{3}{2}}.$$

$\bullet$**Case 2:** The second case we study is when $\alpha_2 = constant$ and $\alpha_1, \alpha_3 \neq constant$.

$$\tau = \frac{\Big[\alpha_3'''(ff_y(\alpha_3')^3) + \alpha_3''(-\frac{(\alpha_3')^3}{2}f_yf_z(\alpha_3' + \frac{\alpha_3''}{2}))\Big]}{\Big[(\alpha_3')^4(\frac{1}{2}ff_y^2(\alpha_3')^2 + \frac{1}{4}(\alpha_3')^2f_z^2)\Big]}$$

$$\kappa = \Big[\alpha_1'(\alpha_1'(\alpha_3'')^2 - 2\alpha_1''\alpha_3'\alpha_3'') + \alpha_1''(\alpha_1''(\alpha_3')^2) + \alpha_1'\alpha_3'(-\tfrac{1}{4}f_y^2(\alpha_3')^4 - f_z(\alpha_3')^2\alpha_3'') + (\alpha_3')^4(\tfrac{1}{2}ff_y^2(\alpha_3')^2 + \alpha_1''f_z + \tfrac{1}{4}(\alpha_3')^2f_z^2)\Big]^{\frac{1}{2}} \div [\alpha_1'\alpha_3' + f(\alpha_3')^2]^{\frac{3}{2}}.$$

$\bullet$**Case 3:** $\alpha(s) = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_3 = constant$ and $\alpha_1, \alpha_2 \neq constant$.
If $\alpha_3 = constant$, then $\alpha' \times_f \alpha'' = (\alpha_1'\alpha_2'' - \alpha_2'\alpha_1'')\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}$.
Thus,

$$g_f^{\epsilon}(\alpha' \times \alpha'') = (\alpha_1'\alpha_2'' - \alpha_2'\alpha_1'') \cdot 0 + \epsilon \cdot 0 + f \cdot 0 = 0$$

that is $|\alpha' \times_f \alpha''| = 0$. Therefore, this dot product is indefinite, and the vector $\alpha' \times \alpha''$ is light-like.
This case is unstudiable since there is no valid Frenet frame.

## 4.2 Mannheim Curves in Three Dimensional Walker Manifolds

In this section, we investigate Mannheim curves on a three dimensional Walker manifold. Recall that a Mannheim curve is a curve characterized by having its principal normal line identical to the binormal line of another curve.

**Definition 4.2.1.** *Let* $\gamma$ *and* $\overline{\gamma} : I \to \mathbb{R}^3$ *be two non-degenerate curves. If there exists a smooth function* $\lambda : I \to \mathbb{R}$ *such that* $\overline{\gamma}(t) = \gamma(t) + \lambda(t)N(t)$ *and* $N(t) = \pm\overline{B}(t)$ *for all* $t \in I$*, then* $\gamma$ *and* $\overline{\gamma}$ *are called Mannheim mates.*

$\gamma : I \to \mathbb{R}^3$ *is called a Mannheim curve if there exists another nondegenerate curve* $\overline{\gamma} : I \to \mathbb{R}^3$ *such that* $\gamma$ *and* $\overline{\gamma}$ *are Mannheim mates.*

Recall that a regular curve is Mannheim if and only if the ratio $\frac{\kappa}{\kappa^2+\tau^2} = constant$.

We calculate this ratio for a regular curve on a three dimensional Walker manifold:

We focus mainly on the special case when the second componenet function of the curve $\gamma$ is constant.

Let $\gamma(s) = (\alpha, \alpha_2, \alpha_3)$ where $\alpha_2 = constant$.

Then

$$\begin{aligned}
\frac{\kappa}{\kappa^2+\tau^2} &= \Big[(\alpha_3')^{\frac{9}{2}}(\frac{1}{2}ff_y^2 + \frac{1}{4}f_z^2)\big[(\alpha'\alpha_3'' - \alpha''\alpha_3')(\alpha'\alpha_3'' - \alpha''\alpha_3' - f_z(\alpha_3')^3) \\
&+ (\alpha_3')^6(\frac{1}{2}ff_y^2 + \frac{1}{4}f_z^2) - \frac{1}{4}f_y^2\alpha'(\alpha_3')^5\Big]^{\frac{1}{2}} \div \Big[(\alpha' + f\alpha_3')^{\frac{3}{2}}\big((\alpha_3' \\
&+ \frac{1}{2}\alpha_3'')(-\frac{1}{4}(\alpha_3'')^2 f_y^2 f_z^2(\alpha_3' + \frac{1}{2}\alpha_3'') - \alpha_3''\alpha_3''' ff_y^2 f_z) + (\alpha_3''')^2 f^2 f_y^2].
\end{aligned}$$

We proceed to study an example of a Mannheim curve.

Let $\gamma(s) = (\alpha, 2, sin(s))$, and $f(y,z) = e^y$.

The curve $\gamma(s)$ will be a Mannheim curve if and only if $\frac{\kappa}{\kappa^2+\tau^2} = constant$.

So, let

$$\frac{\kappa}{\kappa^2+\tau^2} = 1$$

After substituting the curvature and torsion expressions in the equation above, we obtain the following nonhomogeneous nonlinear second order differential equation for $\alpha$ :

$$\cos^2(s)(\alpha'')^2+2\cos(s)\sin(s)\alpha'\alpha''-4e^{2y}\cos^{-5}(s)(\alpha')^3+(\sin^2(s)-12e^{3y}\cos^{-4}(s))(\alpha')^2$$
$$-(\frac{1}{4}e^{2y}\cos^5(s)+12e^{4y}\cos^{-3}(s))\alpha'=-\frac{1}{2}e^{3y}\cos^6(s)+4e^{5y}\cos^{-2}(s)$$

**•Case 1**: If the following condition holds $8f^2=\cos^8(s)$, then the equation above becomes homogeneous. Solving for $\alpha$ numerically, we find the curve $\gamma(s)$ as shown in the following graphs:

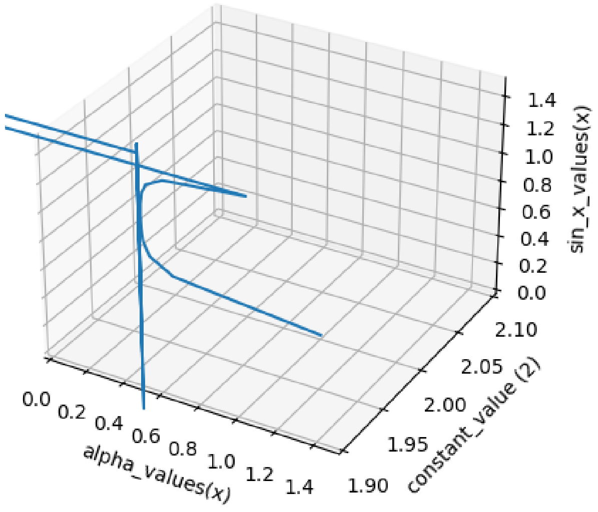

Figure 4.1: $\gamma(s)=(\alpha(s),2,sin(s))$

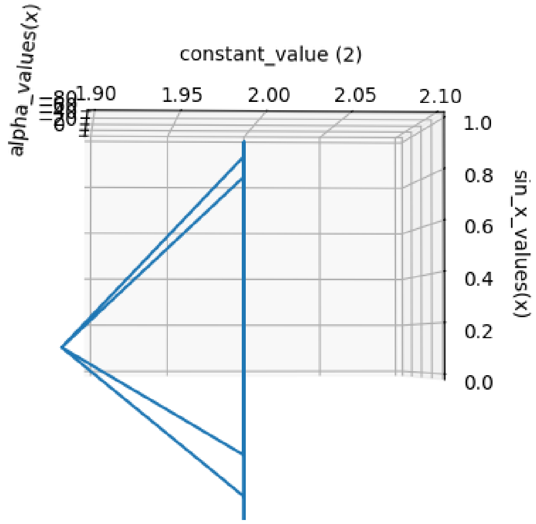

Figure 4.2: $\gamma(s) = (\alpha(s), 2, sin(s))$

•**Case 2**: The equation above is nonhomogeneous if $8f^2 \neq \cos^8(s)$. We solve for $\alpha$ numerically. The curve $\gamma(s)$ is plotted in the following graphs:

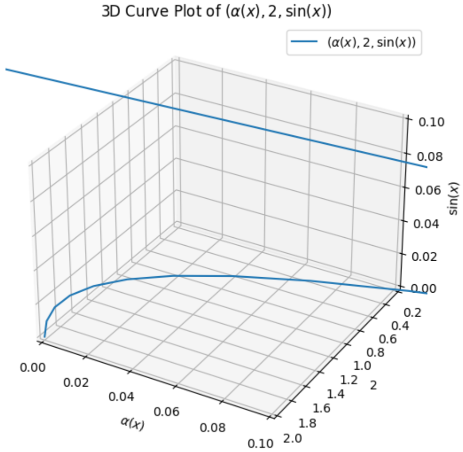

Figure 4.3: $\gamma(s) = (\alpha(s), 2, sin(s))$

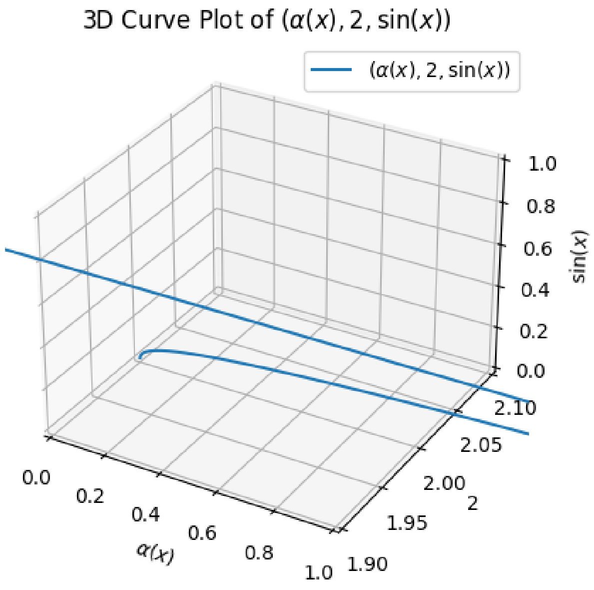

Figure 4.4: $\gamma(s) = (\alpha(s), 2, sin(s))$

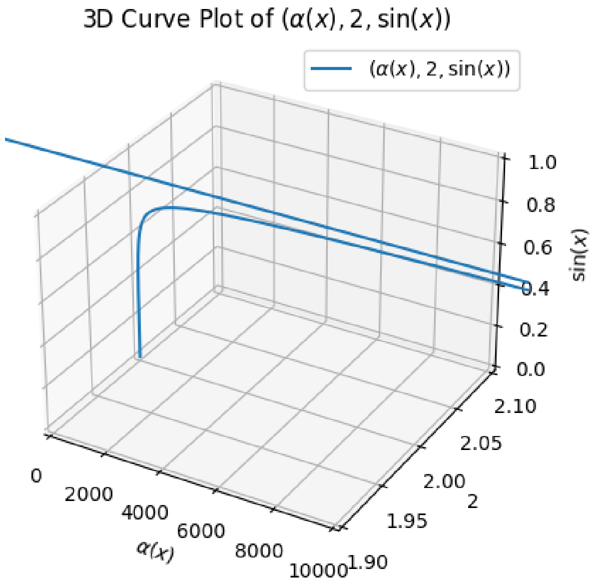

Figure 4.5: $\gamma(s) = (\alpha(s), 2, sin(s))$

## 4.3 Bertrand Curves in Three Dimensional Walker Manifolds

In this section, we investigate *Bertrand* curves on a three dimensional Walker manifold. Recall that a Bertrand curve is a space curve whose principal normal aligns with the principal normal of another curve.

In section 4.3, we have seen that two curves $\alpha$ and $\overline{\alpha}$ are Bertrand mates if and only if there exists a constant $\beta$ such that

$$\delta\kappa(s) + \beta\tau(s) = 1,$$

and

$$\beta\kappa(s) - \delta\tau(s) \neq 0.$$

To find this criteria for a regular curve $\alpha(s) = (\alpha_1, \alpha_2, \alpha_3)$ on a strict Walker manifold. We replace the $\kappa(s)$ and the $\tau(s)$ with the curvature and the torsion obtained for the special case when $\alpha_2 =$ constant.

For the Walker manifold case, we find that two curves $\alpha$ and $\overline{\alpha}$ are Bertrand mates if and only if there exists a constant $\beta$ such that

$$\delta\kappa(s) + \beta\tau(s) = 1,$$

and

$$\beta\kappa(s) + \delta\tau(s) \neq 0.$$

**Theorem 4.3.1.** *Let* $\alpha : I \to \mathbb{R}^3$ *be an arc-length parametrized nondegenerate curve. Assume* $\tau(s) \neq 0$ *for all* $s \in I$ *and* $\delta$ *is a nonzero constant. Then,* $\alpha$ *and* $\overline{\alpha}$ *are Betrand mates with* $\overline{\alpha}(s) = \alpha(s) + \delta\boldsymbol{N}(s)$ *if and only if there exists a constant* $\beta$ *such that* $\delta\kappa(s) + \beta\tau(s) = 1$ *and* $\beta\kappa(s) + \delta\tau(s) \neq 0$ *for all* $s \in I$.

*Proof.* ($\Longrightarrow$)
Assume that $\overline{\alpha}(s) = \alpha(s) + \delta\mathbf{N}(s)$ and $\mathbf{N}(s) = \overline{\mathbf{N}}(s)$.

Remark: Keep in mind that $s$ does not represent the arc-length parameter for $\overline{\alpha}$.

We differentiate $\overline{\alpha}(s) = \alpha(s) + \delta\mathbf{N}(s)$ w.r.t s :

$$\overline{\alpha}'(s) = \alpha'(s) + \delta\mathbf{N}'(s)$$

here $\mathbf{N}' = \nabla_T\mathbf{N}$

From the frenet-Serret equations for the Walker manifold, we have:

$$\begin{aligned} |\overline{\alpha}'(s)|\overline{T}(s) &= T(s) + \delta(-\epsilon_1\kappa(s)T(s) - \epsilon_3\tau B(s)) \\ &= T(s) - \delta\kappa(s)T(s) + \delta\tau(s)B(s) \\ &= (1 - \delta\kappa(s))T(s) + \delta\tau(s)B(s), \end{aligned}$$

where $\epsilon_1 = 1, \epsilon_3 = -1$.

From the assumption, we know that $\mathbf{N}(s) = \overline{\mathbf{N}}(s)$.

Therefore there exists a smooth function $\theta : I \to \mathbb{R}$ such that:

$$\begin{pmatrix} \overline{B}(s) \\ \overline{T}(s) \end{pmatrix} = \begin{pmatrix} \cos(\theta)(s) & -\sin(\theta)(s) \\ \sin(\theta)(s) & \cos(\theta)(s) \end{pmatrix} \begin{pmatrix} B(s) \\ T(s) \end{pmatrix}.$$

Then

$$|\overline{\alpha}'(s)|(\sin\theta(s)B(s) + \cos\theta(s)T(s)) = (1 - \delta\kappa(s))T(s) + \delta\tau(s)B(s),$$

hence,

$$|\overline{\alpha}'(s)|\sin\theta(s) = \delta\tau(s),$$

$$|\overline{\alpha}'(s)|\cos\theta = 1 - \delta\kappa(s),$$

which yields,

$$(1 - \delta\kappa)\sin\theta(s) - \delta\tau(s)\cos\theta(s) = 0. \tag{4.6}$$

By differentiating (taking the covariant derivative of both sides):

$$\overline{T}(s) = \sin\theta(s)B(s) + \cos\theta(s)T(s).$$

We have

$$\nabla_{\overline{T}}\overline{T} = \nabla_T(\sin\theta(s)B(s)) + \nabla_T(\cos\theta(s)T(s))$$

By the third property of the covariant derivative (Leibniz Rule):

$$\nabla_{\overline{T}}\overline{T} = (T \circ \sin\theta)B(s) + sin\theta\nabla_T B + (T \circ \cos\theta)T(s) + \cos\theta\nabla_T T,$$

hence,

$$\nabla_{\overline{T}}\overline{T} = \theta' \cos\theta(s)B(s) + \sin\theta\nabla_T B - \theta' \sin\theta(s)T(s) + \cos\theta\nabla_T T.$$

Using the frenet-Serret equations we substitute $\nabla_{\overline{T}}\overline{T}$, $\nabla_T B$, and $\nabla_T T$ in the equation above:

$$\overline{\kappa}(s)|\overline{\alpha}'(s)|\overline{N}(s) = \theta' \cos\theta(s)B(s) - \theta' \sin\theta(s)T(s) + \big(sin\theta\tau(s) + cos\theta\kappa(s)\big)N(s).$$

now, since $\overline{N}(s) = N(s)$, $\theta'$ must be zero. Therefore, $\theta$ is constant.

We found earlier that $|\overline{\alpha}'(s)|\sin\theta = \delta\tau(s)$, and from the assumption we know $\tau(s) \neq 0$. hence, $\sin\theta \neq 0$.
We go back to equation (4.4) and divide by $sin\theta$:

$$1 - \delta\kappa - \delta\tau\frac{\cos\theta}{\sin\theta} = 0,$$

$$\delta\kappa(s) + \delta\tau(s)\frac{\cos\theta}{\sin\theta} = 1.$$

If we choose $\beta$ as $\beta = \delta\frac{\cos\theta}{\sin\theta}$, then:

$$\delta\kappa(s) + \beta\tau(s) = 1,$$

for all $s \in I$

Furthermore, we found that,

$$\overline{\kappa}(s)|\overline{\alpha}'(s)| = \sin\theta\tau(s) + \cos\theta\kappa(s),$$

$$\overline{\kappa}(s)|\overline{\alpha}'(s)| = \frac{\sin\theta}{\delta}\big(\delta\tau(s) + \beta\kappa(s)\big),$$

since $\overline{\kappa}(s) \neq 0$, we have:

$$\delta\tau(s) + \beta\kappa(s) \neq 0.$$

($\Longleftarrow$) Conversely, assume we can find a constant $\beta$ such that:
$\delta\kappa(s) + \beta\tau(s) = 1$, $\delta\tau(s) + \beta\kappa(s) \neq 0$, and $\overline{\alpha}(s) = \alpha(s) + \delta N(s)$ for every $s \in I$.

We found earlier that $|\overline{\alpha}'(s)|\overline{T}(s) = (1 - \delta\kappa(s))T(s) + \delta\tau(s)B(s)$,
So,

$$\begin{aligned} |\overline{\alpha}'(s)|\overline{T}(s) =& \beta\tau(s)T(s) + \delta\tau(s)B(s) \\ =& \tau(s)(\beta T(s) + \delta B(s)). \end{aligned}$$

Taking the magnitude of both sides, we obtain:

$$|\overline{\alpha}'(s)| = \sqrt{\delta^2 + \beta^2}|\tau(s)|.$$

We have

$$\overline{T}(s) = sgn(\tau(s))\frac{1}{\sqrt{\delta^2 + \beta^2}}\big(\beta T(s) + \delta B(s)\big),$$

where $sgn(\tau(s)) = 1$ if $\tau(s) > 0$ and $sgn(\tau(s)) = -1$ if $\tau(s) < 0$.
By taking the covariant derivative of $\overline{T}(s)$, we find:

$$\begin{aligned}
|\overline{\alpha}'(s)|\overline{\kappa}(s)\overline{N}(s) =& sgn(\tau(s))\frac{1}{\sqrt{\delta^2+\beta^2}}\nabla_T(\beta T(s)+\delta B(s))\\
=& sgn(\tau(s))\frac{1}{\sqrt{\delta^2+\beta^2}}\big(\nabla_T(\beta T(s))+\nabla_T(\delta B(s))\big)\\
=& sgn(\tau(s))\frac{1}{\sqrt{\delta^2+\beta^2}}\big(\beta\nabla_T T+\delta\nabla_T B\big)\\
=& sgn(\tau(s))\frac{1}{\sqrt{\delta^2+\beta^2}}\big((\beta\kappa(s)+\delta\tau(s))N(s)\big)
\end{aligned}$$

from our assumption, we know $\beta\kappa(s)+\delta\tau(s)\neq 0$,
hence,

$$\overline{N}(s)=\pm N(s)$$

for every $s\in I$.
Thus, $\alpha$ and $\overline{\alpha}$ are **Bertrand mates.** □

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