REPUBLIC OF TÜRKİYE
YILDIZ TECHNICAL UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING

EXISTENCE, UNIQUENESS AND STABILITY RESULTS FOR SOME NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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DOCTOR OF PHILOSOPHY THESIS
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A thesis submitted by Faruk DEVELİ in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY** is approved by the committee on 02.08.2024 in Department of Mathematics, Program of Mathematics.

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Faruk DEVELİ

Signature

Dedicated to my lovely wife ESMA SULTAN

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Faruk DEVELİ

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LIST OF SYMBOLS

$\preceq$	Partial order relation
$\mathbb{R}_+$	Set of all nonnegative real numbers
$C(D, \mathbb{R})$	Set of all continuous functions $u : D \to \mathbb{R}$
$C^{1,2}(D, \mathbb{R})$	Set of functions $u(x,t) : D \to \mathbb{R}$ that are continuous along with their partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x \partial t}$

LIST OF ABBREVIATIONS

E&U	Existence and Uniqueness
iff	If and Only If
IVP	Initial Value Problem
FPTs	Fixed Point Theorems
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
RHS	Right-Hand Side
UH	Ulam-Hyers
UHR	Ulam-Hyers-Rassias
w.r.t	With Respect to

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ABSTRACT

Existence, Uniqueness and Stability Results For Some Nonlinear Hyperbolic Partial Differential Equations

Faruk DEVELİ

Department of Mathematics
Doctor of Philosophy Thesis

Supervisor: Prof. Dr. Canan ÇELİK KARAASLANLI

The main goal of this thesis is to contribute to the field of nonlinear analysis of partial differential equations (abbreviated as PDEs) by investigating the existence and uniqueness (E&U) of solutions using fixed point theory. More specifically, E&U results are obtained for several second-order nonlinear hyperbolic PDEs. Moreover, the stability of the corresponding nonlinear hyperbolic PDEs is also performed in the sense of Ulam-Hyers (UH) and Ulam-Hyers-Rassias (UHR). Finally, to support and illustrate the E&U and stability results obtained, several examples are provided.

Keywords: Existence of solution, fixed point theory, partial differential equation, Ulam-Hyers stability, Ulam-Hyers-Rassias stability.

YILDIZ TECHNICAL UNIVERSITY
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ÖZET

Bazı Doğrusal Olmayan Hiperbolik Kısmi Diferansiyel Denklemler İçin Varlık, Teklik ve Kararlılık Sonuçları

Faruk DEVELİ

Matematik Anabilim Dalı
Doktora Tezi

Danışman: Prof. Dr. Canan ÇELİK KARAASLANLI

Bu tezin temel amacı, sabit nokta teorisi yardımı ile çözümlerin varlık ve tekliğini (E&U) araştırarak kısmi diferansiyel denklemlerin (kısaca PDEs) doğrusal olmayan analizi alanına katkıda bulunmaktır. Daha spesifik olarak, ikinci dereceden doğrusal olmayan hiperbolik PDEs için E&U sonuçları elde edilmiştir. Ayrıca, ilgili doğrusal olmayan hiperbolik denklemlerin kararlılığı da Ulam-Hyers (UH) ve Ulam-Hyers-Rassias (UHR) anlamında gerçekleştirilmiştir. Son olarak, elde edilen E&U ve kararlılık sonuçlarını desteklemek ve örneklendirmek için birkaç örnek verilmiştir.

Anahtar Kelimeler: Çözümün varlığı, sabit nokta teorisi, kısmi diferansiyel denklemler, Ulam-Hyers kararlılık, Ulam-Hyers-Rassias kararlılık.

YILDIZ TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

Differential equations have been used as mathematical models to make a description of nature. That is why many of the general laws and phenomena in many fields such as physics, chemistry, biology and engineering are formulated. As can be seen in our life, reactions to a stimulus can often be delayed, even for a short time. The behavior of phenomena at a certain time depends on the past history/memory, in which case delays are observed in the mathematical descriptions (modelling) of the phenomena under consideration. Taking the recent Covid 19 pandemic as an example, the time delay in the models of this situation can be associated with the duration of the infections period. Such a class of differential equations containing delays in their formulations is called delay differential equations. Phenomena are also influenced by many parameters due to the behaviour of nature. For this reason, PDEs based on multiple independent variables are more suitable as compared to ordinary differential equations (ODEs) to describe phenomena in nature. In this manner, it allows us to analyze and shed mathematical light on the behaviour of dynamics for real-world problems. In order to have an insight into this behaviour, one of the most important analysis is the determination of whether the mathematical description has a solution. For instance, solving the corresponding differential equation serves to understand how phenomena will change over time. When the behavior of a phenomenon happens in only one way, the uniqueness of the solution allows us to make a single decision about this behavior, thus eliminating solutions that do not occur in reality. Sometimes it can be very difficult to determine the exact solution to a differential equation, even when it exists. For this challenge, algorithms are being developed to find approximate solutions, particularly in numerical analysis. To draw healthy inferences about the dynamics of an equation using these approximate solutions, it is natural to hope that they are close to the exact behaviour of the equation. The problem of whether there exists an exact solution close to a function that almost solves a given equation establishes the concept of stability for functional equations proposed by Ulam, which is also named after him.

Nonlinear effects can often be observed in phenomena such as the dynamics of the population in the interaction between predator and prey. In order to describe this phenomena in a more realistic way, nonlinear differential equations are used. They are thus quite common in different scientific fields and are an active area of research in mathematics. Their analysis is not a simple task. It often requires the use of special analytical techniques. As a powerful tool in nonlinear analysis, fixed point theory can be shown as an example. Fixed points hold significance as they denote states of equilibrium, stability, and serve as solutions to a range of problems. The theory of fixed points offers techniques and approaches for analysing the existence, properties and dynamics of these special points. For instance, this theory has been a mathematical material to John Nash's result in game theory, which earned him a Nobel Prize in economics, and has also played a significant role in investigating the E&U of solution for nonlinear differential equations. Moreover, just by looking at [1], one may say that the theory has interactions with many areas of mathematics from topology and analysis to algebra and geometry. Therefore, it would be appropriate to characterize this theory as interdisciplinary. As can be seen, this interdisciplinary theory has many applications in various fields of mathematics and other sciences. To illustrate the theory with an interesting problem, one can consider the question: "Is there always a pair of opposite points on the Earth's equator with the same temperature, where the temperature varies continuously?" At first sight it seems difficult to answer this question. However, one can answer this question in the affirmative with the help of the Borsuk-Ulam theorem. Because of the above mentioned, as Felix Browder said, "The theory of fixed points is one of the most powerful tools of modern mathematics".

This is followed by a review of the literature on some PDEs subjected to existence, uniqueness and stability analysis, which will be carried out throughout the thesis.

1.1 Literature Review

In the theory of differential equations, one of the main areas of research is whether there is a solution to the equation, and if so, its uniqueness. In this area, the contributions of Cauchy, Peano, Picard, Lindelöf and Lipschitz on the initial value problem (IVP)

$$u'(x) = f\big(x, u(x)\big), \quad u(x_0) = u_0 \tag{1.1}$$

have been pioneering studies in the literature. In [2, 3], Peano established that there is a solution to the equation assuming f is continuous, and extended this discovery to systems of ODEs by employing successive approximations. Peano's theorem offers a very easily controllable condition to verify the existence of a solution.

The fundamental theorem, however, merely guarantees existence and gives no information about its uniqueness. Another fundemantal theorem in ODEs is the so-called Picard-Lindelöf theorem, which requires a continuous function f to be Lipschitz with respect to (w.r.t.) u, but it guarantees both the E&U of solutions. Additionally, this theorem provides a method for approximating a solution. In the theory of ODEs, two significant theorems mentioned above have caught the interest of many authors, leading to the availability of many proofs today. Proof techniques can be categorized into two groups depending on "construction of a sequence of approximate solutions such as Tonelli sequence or the Euler-Cauchy polygons" and "fixed point theory". Among these two groups, fixed point theory provides elegant proofs for these classical theorems. For instance, the Peano and Picard-Lindelöf theorems are associated with the Schauder and Banach fixed point theorems (FPTs), respectively.

As an analogue of the ODE (1.1) in PDEs, the following example of hyperbolic type can be considered:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\Big(x, t, u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}\Big) \tag{1.2}$$

or

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\big(x, t, u(x,t)\big). \tag{1.3}$$

The boundary value problem so called "Darboux-problem", which is the equation (1.2) (or the equation (1.3)) together with the conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad u(0,t) = \psi(t) \quad \text{where} \quad \varphi(0) = \psi(0), \tag{1.4}$$

has been treated by many different methods such as method of successive approximation (today attributed to Picard), an analogue of the Euler-Cauchy polygon method and fixed point method. Darboux [4] and Kamke [5] obtained a unique solution to the equation (1.2)-(1.4) by applying the method of successive approximation under the conditions that the countinuous function $f(x,t,u,p,q)$, $p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial t}$, is bounded and satisfies a Lipschitz condition w.r.t. p, q and to u. Unlike these papers, Hartman and Winter [6] have shown that the Lipschitz condition in the argument u can be omitted to guarantee the existence of solutions, though not in the uniqueness result, see also [7–9]. Also, Lungu and Rus [10] recently proved the E&U result for this equation under the Lipschitz condition w.r.t. three variables as mentioned above by using the Banach FPT converting the differential equation into the corresponding integral system.

In the simple scenario that $f(x,t,u,p,q)$ is independent of the p and q (that is, the other equation (1.3)), Montel [11] proved that there is at least one solution to this equation with the conditions (1.4), but it is usually not unique, as shown by simple examples in [7] and [6]. His proof was an adaptation of the standard proof of Peano's existence theorem in the theory of ODEs. In addition, by considering this equation in Banach space, the existence of a solution is investigated in [12] and [13] by the use of the FPT of Sadovskii.

More specifically, the two types of Darboux problem (1.2) and (1.3) mentioned above has been also considered in functional PDEs. Czlapinski has mainly dealt with these problems for such types of PDEs in his studies [14–17] and discussed the existence of a solution. Rus has considered the type (1.3) of Darboux problem in a general framework and presented an E&U result in [18]. Other aspects like periodicity, upper and lower solutions for these two types of equations mentioned above have been considered by many authors besides the E&U of solutions [19–22]. For further details about these equations, see also [23] and to the references given in these papers. Unlike these types of equations, Rzepecki [24] has considered the right-hand side (RHS) function f as follows:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\left(x, t, u(x,t), \frac{\partial^2 u(x,t)}{\partial x \partial t}\right) \tag{1.5}$$

with the same boundary conditions as in (1.4) and has examined the existence of solution in the first coordinate plane $\mathbb{R}_+ \times \mathbb{R}_+$. However, the question of whether there is a solution to the equation (1.5)-(1.4) or not, has not been the subject of many papers. Moreover, a fractional counterpart of the aforementioned equation has recently derived by utilizing fractional derivatives, which has been currently examined, and the existence of solution has been discussed in [25], where the existence theorem has been proven via fixed point theory.

In addition to E&U problems, another subject discussed in the thesis is stability analysis and the background on the concept of stability is mentioned below. At a mathematical colloquium organised by the University of Wisconsin in 1940, Ulam gave an extended talk. During this talk he dealt with various important open questions [26]. One of the questions he posed concerned the stability of homomorphisms: Consider two groups: $(E_1, \circ)$ and $(E_2, \star)$, where the second is equipped with a metric $\rho(\cdot, \cdot)$. For a given $\epsilon > 0$, is it feasible to find an $\delta > 0$ such that if any function $h : E_1 \to E_2$ fulfilling the following relation

$$\rho(h(x \circ \overline{x}), h(x) \star h(\overline{x})) < \delta, \quad x, \overline{x} \in E_1,$$

then there is a homomorphism $g : E_1 \to E_2$ satisfying $\rho(h(x), g(x)) < \epsilon$ for

$x \in E_1$? The functional equation for homomorphisms is said to be stable if there is a real homomorphism nearby when a mapping is almost homomorphic. That is, the stability means that there is an exact solution near each approximate solution of the equation under consideration. An answer to Ulam's question was provided a year later by Hyers in [27]: Let B_1 and B_2 denote real Banach spaces and $\epsilon > 0$. If a function $h : B_1 \to B_2$ satisfies

$$\left\| h(x + \overline{x}) - \big(h(x) + h(\overline{x})\big) \right\| \leq \epsilon, \quad x, \overline{x} \in B_1, \tag{1.6}$$

then there is a unique additive function $g : B_1 \to B_2$ which fulfills

$$\left\| h(x) - g(x) \right\| \leq \epsilon, \quad x \in B_1.$$

With the help of Hyers to Ulam's question, this concept of stability was later recognized in the literature as UH stability. Moreover, Rassias [28] enhanced the result of Hyers by considering a function dependent on x and $\overline{x}$ instead of ϵ in (1.6), which is referred to as UHR stability in the literature. Furthermore, Obloza was the first author to study this type of stability in the context of linear differential equations [29]. Later, Alsina and Ger [30] demonstrated that for any differentiable function $\vartheta : I \to \mathbb{R}$ fulfilling $\left|\vartheta'(x) - \vartheta(x)\right| \leq \epsilon$ for $\epsilon > 0$ and all $x \in I$ (an open interval of reals), the differential equation $u'(x) = u(x)$ has a solution represented as u in a way that ensures $\left|u(x) - \vartheta(x)\right| \leq 3\epsilon$ for all $x \in I$. Jung [31] employed the fixed point method to establish the stability in the sense of both UH and UHR of the equation

$$u'(x) = f\big(x, u(x)\big).$$

This study extended the previous results to the nonlinear case. The concept of this stability was also discussed in many topics such as ODEs and PDEs [32–37].

1.2 Objective of the Thesis

In this thesis, the general structure of a second-order PDE for $u(x,t)$ is considered as follows:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} = g\Big(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\Big).$$

If the equation above is hyperbolic, a suitable transformation can reduce it to the canonical form below:

$$\frac{\partial^2 u}{\partial \eta \partial \gamma} = f\Big(\eta, \gamma, u, \frac{\partial u}{\partial \eta}, \frac{\partial u}{\partial \gamma}\Big).$$

Since there is no general method to solve nonlinear equations as compared to linear equations, these equations have to be treated as a separate problem. For that reason, this thesis aims to investigate the E&U of solutions as well as the stability of hyperbolic equations in canonical form with several functions f on the RHS. These equations are determined by considering the above equation and its various forms as the general second-order hyperbolic PDEs are reduced to this type of equation.

1.3 Thesis Outline

This thesis consists of five chapters and in the current chapter (Chapter 1) the importance of the analyses (existence, uniqueness and stability) that are discussed in this thesis is briefly emphasized. The objective of the thesis is then presented through a literature review.

In Chapter 2, there is an introduction where the basic concepts of PDEs are given. Then it is followed by the main theorems of fixed point theory used as a tool throughout thesis and the concept of stability handled in this thesis is also presented.

There are four sections in Chapter 3 where the E&U of solutions for various types of hyperbolic PDEs are analyzed by applying fixed point theory. Inspired by Burton's method called "progressive contractions", his methodology is extended to two-dimensional regions unlike the other related studies in Section 3.1. It is also applied to the hyperbolic PDEs with two delays:

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x \partial t} &= f\big(x,t,u(x,t),u(x-\alpha,t-\beta)\big) && (x,t)\in D \\ u(x,t) &= \phi(x,t) && (x,t)\in \tilde{D} \end{aligned}$$

with

$$u(x,0)=\varphi(x) \quad \text{and} \quad u(0,t)=\psi(t)$$

where $D=[0,a]\times[0,b]$, $f\in C(D\times\mathbb{R}^2,\mathbb{R})$, $\tilde{D}=[-\alpha,a]\times[-\beta,b]\backslash(0,a]\times(0,b]$, $\phi\in C(\tilde{D},\mathbb{R})$, $\varphi(x)$ and $\psi(t)$ are continuously differentiable mappings with $\varphi(x)=\phi(x,0)$, $\psi(t)=\phi(0,t)$ for the intervals $[0,a]$ and $[0,b]$, respectively. Later on, Banach's FPT is utilized to derive an E&U result for these equations. This method takes advantage of the sufficiency of the Lipschitz condition of the function f w.r.t. the third variable only, ignoring the Lipschitz condition w.r.t. the fourth variable. In Section 3.2, the same theorem is also utilized to establish the E&U of solutions for the hyperbolic functional PDEs:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\Big(x,t,u\big(g(x,t),h(x,t)\big)\Big), \quad (x,t)\in D$$

with

$$\begin{cases} u(x,0) = \varphi(x) \\ u(0,t) = \psi(t) \end{cases} \quad \text{such that} \quad \varphi(0) = \psi(0),$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}, \mathbb{R})$, $g \in C(D, [0,a])$, $h \in C(D, [0,b])$, $\varphi(x)$ and $\psi(t)$ belong to the space of continuously differentiable mappings defined on $[0,a]$ and $[0,b]$, respectively. After that, based on the unique solutions discovered in the bounded domains, the relevant result is then extended to an unbounded domain. Section 3.3 is devoted to investigating the E&U of solutions to the implicit hyperbolic PDEs:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\left(x, t, u(x,t), \frac{\partial^2 u(x,t)}{\partial x \partial t}\right), \quad (x,t) \in D$$

with

$$\begin{cases} u(x,0) = \varphi(x) \\ u(0,t) = \psi(t) \end{cases} \quad \text{such that} \quad \varphi(0) = \psi(0),$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}, \mathbb{R})$, $\varphi(x)$ and $\psi(t)$ belong to the space of continuously differentiable mappings defined on $[0,a]$ and $[0,b]$, respectively. Under appropriate conditions, the existence result is proved based on Schauder's FPT, and the uniqueness result is demonstrated using the Wendorff lemma. In Section 3.4, following the idea of utilizing Banach's FPT to construct well-defined mappings, a novel proof of the existence theorem originally proposed by Hartman and Winter for the following PDEs is constructed based on fixed point theory.

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\left(x, t, u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}\right), \quad (x,t) \in D$$

with

$$\begin{cases} u(x,0) = \varphi(x) \\ u(0,t) = \psi(t) \end{cases} \quad \text{such that} \quad \varphi(0) = \psi(0),$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}^3, \mathbb{R})$, $\varphi(x)$ and $\psi(t)$ belong to the space of continuously differentiable mappings defined on $[0,a]$ and $[0,b]$, respectively.

Chapter 4 is divided into three sections, each of which investigates the stability of corresponding PDEs discussed in the first three sections of Chapter 3, utilizing some tools from Picard operator theory, the Wendorff lemma, and weighted norms. Finally, the thesis ends up with a conclusion (Chapter 5).

2
FUNDAMENTAL CONCEPTS

In this chapter, some basic concepts of PDEs will be mentioned and the concepts, definitions and theorems that are used throughout the analyses performed in this thesis will be given.

2.1 Some Basic Concepts of PDEs

A PDE concerning a function $u(x,t,\cdots)$ establishes a relationship between u and its partial derivatives such as $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial t^2}, \cdots$, and can be expressed as:

$$F(x,t,\cdots,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x \partial t},\frac{\partial^2 u}{\partial t^2},\cdots)=0 \tag{2.1}$$

where F is a specified function, $x,t,\cdots$ represent independent variables, and $u(x,t,\cdots)$ is referred to as the dependent variable. The highest derivative that appears in equation (2.1) determines the order of this equation. The equation described above in this configuration is referred to as implicit PDE. If the highest order partial derivatives in this equation are isolated on one side, this equation is called explicit.

A PDE is called as **linear** if the coefficients of the unknown function u and all its derivatives are solely dependent on the independent variables, otherwise it is said to be **nonlinear**. Moreover, the family of nonlinear PDEs are divided into three categories as follows:

- A PDE of order k is called semi-linear if the coefficients of the k order partial derivatives of the unknown function depend solely on the independent variables.

- A PDE of order k is called quasi-linear if the coefficients of the k order partial derivatives of the unknown function depend on the independent variables and/or on partial derivatives of the unknown function of order at most $k-1$ (including the unknown function itself)

- If a (nonlinear) PDE is not quasi-linear, then it is classified as fully nonlinear.

It is clear from the above definitions that a semi-linear PDE is also a quasi-linear PDE. The above categorization of PDEs into linear, semilinear, quasilinear, and fully nonlinear represents a rough hierarchy of complexity in terms of studying and solving these equations. Indeed, the mathematical theory of linear PDEs is now well understood. On the other hand, less is known about semi-linear and quasi-linear PDEs, and even less about fully nonlinear PDEs.

Let us now consider the general structure of a second-order PDE with the unknown function $u(x,t)$ specified as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} = g\Big(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\Big). \tag{2.2}$$

The other classification of the aforementioned PDE depends on the sign of the following quantity called the discriminant of this equation, which is computed as

$$\Delta(x,t) := B^2(x,t) - 4A(x,t)C(x,t).$$

If $\Delta > 0$, the equation is said to be **hyperbolic**; if $\Delta = 0$, it is **parabolic**; and if $\Delta < 0$, it is **elliptic**. This classification is mathematically based on the potential for reducing the given equation to canonical form through coordinate transformation.

To illustrate the significance of the discriminant and thus the classification of the PDE (2.2), we demonstrate how to reduce this equation to its canonical form by transforming the variables (x,t) into the new ones (η,γ)

$$\eta = \eta(x,t) \quad \text{and} \quad \gamma = \gamma(x,t) \tag{2.3}$$

where both η and γ are twice continuously differentiable. Additionally, the Jacobian of this transformation given by

$$J = \frac{\partial(\eta,\gamma)}{\partial(x,t)} = \begin{vmatrix} \eta_x & \eta_t \\ \gamma_x & \gamma_t \end{vmatrix}$$

is non-zero in the region under consideration. Using the chain rule, we compute the terms of the equation (2.2) in these new variables

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial x}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \gamma}\frac{\partial \gamma}{\partial t}\end{aligned}$$

and then

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \eta^2}\Big(\frac{\partial \eta}{\partial x}\Big)^2 + 2\frac{\partial^2 u}{\partial \eta \partial \gamma}\frac{\partial \eta}{\partial x}\frac{\partial \gamma}{\partial x} + \frac{\partial^2 u}{\partial \gamma^2}\Big(\frac{\partial \gamma}{\partial x}\Big)^2\\ &\quad + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial u}{\partial \gamma}\frac{\partial^2 \gamma}{\partial x^2}\\ \frac{\partial^2 u}{\partial x \partial t} &= \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t} + \frac{\partial^2 u}{\partial \eta \partial \gamma}\Big(\frac{\partial \eta}{\partial x}\frac{\partial \gamma}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial \gamma}{\partial x}\Big)\\ &\quad + \frac{\partial^2 u}{\partial \gamma^2}\frac{\partial \gamma}{\partial x}\frac{\partial \gamma}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial t} + \frac{\partial u}{\partial \gamma}\frac{\partial^2 \gamma}{\partial x \partial t}\\ \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial \eta^2}\Big(\frac{\partial \eta}{\partial t}\Big)^2 + 2\frac{\partial^2 u}{\partial \eta \partial \gamma}\frac{\partial \eta}{\partial t}\frac{\partial \gamma}{\partial t} + \frac{\partial^2 u}{\partial \gamma^2}\Big(\frac{\partial \gamma}{\partial t}\Big)^2\\ &\quad + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial t^2} + \frac{\partial u}{\partial \gamma}\frac{\partial^2 \gamma}{\partial t^2}.\end{aligned}$$

Substitution of the above statements into equation (2.2) gives

$$\mathcal{A}^*\frac{\partial^2 u}{\partial \eta^2} + \mathcal{B}^*\frac{\partial^2 u}{\partial \eta \partial \gamma} + \mathcal{C}^*\frac{\partial^2 u}{\partial \gamma^2} = g\Big(\eta, \gamma, u, \frac{\partial u}{\partial \eta}, \frac{\partial u}{\partial \gamma}\Big) \tag{2.4}$$

where

$$\begin{aligned}\mathcal{A}^* &= A\Big(\frac{\partial \eta}{\partial x}\Big)^2 + B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t} + C\Big(\frac{\partial \eta}{\partial t}\Big)^2\\ \mathcal{B}^* &= 2A\frac{\partial \eta}{\partial x}\frac{\partial \gamma}{\partial x} + B\Big(\frac{\partial \eta}{\partial x}\frac{\partial \gamma}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial \gamma}{\partial x}\Big) + 2C\frac{\partial \eta}{\partial t}\frac{\partial \gamma}{\partial t}\\ \mathcal{C}^* &= A\Big(\frac{\partial \gamma}{\partial x}\Big)^2 + B\frac{\partial \gamma}{\partial x}\frac{\partial \gamma}{\partial t} + C\Big(\frac{\partial \gamma}{\partial t}\Big)^2\end{aligned}$$

Hence,

$$(\mathcal{B}^*)^2 - 4\mathcal{A}^*\mathcal{C}^* = \Big(\frac{\partial \eta}{\partial x}\frac{\partial \gamma}{\partial t} - \frac{\partial \eta}{\partial t}\frac{\partial \gamma}{\partial x}\Big)^2\Big(B^2 - 4AC\Big).$$

From this relation, it follows that the discriminants of the equation (2.2) and the transformed equation (2.4) have the same sign, and hence the type of equation will not change under the transformation (2.3).

Now let us consider the case of the equation (2.2) being hyperbolic, that is $B^2 - 4AC > 0$. This implies that the equation $A\lambda^2 + B\lambda + C = 0$ has two distinct real roots, say λ_1 and λ_2. Choosing η and γ in a manner that

$$\frac{\partial \eta}{\partial x} = \lambda_1 \frac{\partial \eta}{\partial t} \quad \text{and} \quad \frac{\partial \gamma}{\partial x} = \lambda_2 \frac{\partial \gamma}{\partial t}, \tag{2.5}$$

it can be seen as follows that the coefficients of $\frac{\partial^2 u}{\partial \eta^2}$ and $\frac{\partial^2 u}{\partial \gamma^2}$ will be zero:

$$\mathcal{A}^* = A\Big(\frac{\partial \eta}{\partial x}\Big)^2 + B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t} + C\Big(\frac{\partial \eta}{\partial t}\Big)^2 = \Big(A\lambda_1^2 + B\lambda_1 + C\Big)\Big(\frac{\partial \eta}{\partial t}\Big)^2 = 0,$$
$$\mathcal{C}^* = A\Big(\frac{\partial \gamma}{\partial x}\Big)^2 + B\frac{\partial \gamma}{\partial x}\frac{\partial \gamma}{\partial t} + C\Big(\frac{\partial \gamma}{\partial t}\Big)^2 = \Big(A\lambda_2^2 + B\lambda_2 + C\Big)\Big(\frac{\partial \gamma}{\partial t}\Big)^2 = 0.$$

To determine the existence of η and γ satisfying the equations (2.5), we consider the characteristic curves of these equations given by

$$\frac{dt}{dx} = -\lambda_1 \quad \text{and} \quad \frac{dt}{dx} = -\lambda_2.$$

The solution curves of the above equations, known as the Lagrange's auxiliary equation, are as follows:

$$t + \lambda_1 x = k_1 \quad \text{and} \quad t + \lambda_2 x = k_2$$

where k_1, k_2 are integration constants. With the following choices for $\eta(x,t)$ and $\gamma(x,t)$

$$\eta = t + \lambda_1 x \quad \text{and} \quad \gamma = t + \lambda_2 x,$$

one can obtain that

$$\frac{\partial^2 u}{\partial \eta \partial \gamma} = G\Big(\eta, \gamma, u, \frac{\partial u}{\partial \eta}, \frac{\partial u}{\partial \gamma}\Big), \quad G = \frac{1}{\mathcal{B}^*} g$$

which is another hyperbolic equation called "canonical form" of the original general hyperbolic equation (2.2).

Example 2.1. Let us examine the equation below:

$$\frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{on} \quad \Omega = \{(x,t) \in \mathbb{R}^2 \mid x \neq 0\}.$$

Comparing the given equation with (2.2), one can get $A = 1, B = 0$ and $C = -x^2$. Then the discriminant of this equation is $\Delta = B^2 - 4AC = 4x^2 > 0$ and so the equation is hyperbolic. The solutions of the characteristic equation $A\lambda^2 + B\lambda + C = 0$ are $\lambda_1 = x$ and $\lambda_2 = -x$. The corresponding characteristic curves are

$$\frac{dt}{dx} = \pm x$$

whose two solutions are $t + \frac{x^2}{2} = k_1$ and $t - \frac{x^2}{2} = k_2$. Taking the transformations η and γ as follows:

$$\eta = t + \frac{x^2}{2} \quad \text{and} \quad \gamma = t - \frac{x^2}{2},$$

we get

$$\begin{aligned}
\frac{\partial u}{\partial x} =& \frac{\partial u}{\partial \eta}x - \frac{\partial u}{\partial \gamma}x \\
\frac{\partial^2 u}{\partial x^2} =& x\left(\frac{\partial^2 u}{\partial \eta^2}x - \frac{\partial^2 u}{\partial \eta \partial \gamma}x\right) + \frac{\partial u}{\partial \eta} - x\left(\frac{\partial^2 u}{\partial \gamma \partial \eta}x - \frac{\partial^2 u}{\partial \gamma^2}x\right) - \frac{\partial u}{\partial \gamma} \\
=& x^2\left(\frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \eta \partial \gamma} + \frac{\partial^2 u}{\partial \gamma^2}\right) + \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \gamma}
\end{aligned}$$

and similarly

$$\begin{aligned}
\frac{\partial u}{\partial t} =& \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \gamma} \\
\frac{\partial^2 u}{\partial t^2} =& \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \gamma^2} + 2\frac{\partial^2 u}{\partial \eta \partial \gamma}.
\end{aligned}$$

Substituting these expression in the given equation, we have

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} - x^2\frac{\partial^2 u}{\partial t^2} =& x^2\left(\frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \eta \partial \gamma} + \frac{\partial^2 u}{\partial \gamma^2} - \frac{\partial^2 u}{\partial \eta^2} - \frac{\partial^2 u}{\partial \gamma^2} - 2\frac{\partial^2 u}{\partial \eta \partial \gamma}\right) \\
&+ \frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \gamma} = 0.
\end{aligned}$$

This yields that

$$\frac{\partial^2 u}{\partial \eta \partial \gamma} = \frac{1}{4x^2}\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \gamma}\right) = \frac{1}{4(\gamma - \eta)}\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \gamma}\right)$$

which is the canonical form of the corresponding original equation.

2.2 Fixed Point Theory

The theory of fixed points is a mathematical discipline concerned with the E&U of solutions to equations in the type $Tx = x$ (x represents the fixed point of T). Although this equation may seem simple, the fixed point has a profound impact as it represents the solution, stability, and equilibrium point of many problems. Since various problems in different fields can be reduced to the simple fixed point equation mentioned above, having such a special point enables us to make assertions about these problems. For instance, one can consider the IVP given below:

$$u'(x) = f\big(x, u(x)\big), \quad u(x_0) = u_0.$$

If f is continuous, then this equation can be converted into an integral equation as follows:

$$u(x) = \underbrace{u_0 + \int_{x_0}^{x} f\big(s, u(s)\big)ds.}_{:=Tu(x)}$$

Hence, the fixed point of the operator T represents the solution to the given problem. Now, let us initially give the concept of Lipschitz mappings to introduce Banach's FPT, which exemplifies mathematical beauty in terms of the simplicity and elegance of its proof and its wide range of applications.

Definition 2.1. Consider a metric space (E, ρ). A mapping $T : E \to E$ is referred to as Lipschitz if there exists a constant $L > 0$ such that

$$\rho(Tx, T\overline{x}) \leq L\rho(x, \overline{x}) \quad \text{for any} \quad x, \overline{x} \in E.$$

The smallest value of L satisfying this inequality is known as the Lipschitz constant of T. Then this mapping specifically called a contraction mapping when $L < 1$.

While the continuity of a Lipschitz mapping is evident, the opposite is not usually true. This fact can be demonstrated with the following example.

Example 2.2. Consider the mapping on $\mathbb{R}$ given below

$$Tx = \begin{cases} x \sin \frac{\pi}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

Obviously, this mapping is continuous on $\mathbb{R}$. However it is not a Lipschitz mapping. If it were not so, there would be $L > 0$ ensuring that

$$\big|Tx - T\overline{x}\big| \leq L\big|x - \overline{x}\big|$$

for any $x, \overline{x} \in \mathbb{R}$. Taking $x_n = \frac{1}{\frac{1}{2}+2n}$ and $\overline{x}_n = \frac{1}{2n}$, we get

$$\big|Tx_n - T\overline{x}_n\big| \leq L\big|x_n - \overline{x}_n\big|,$$

leading to the following

$$2 \leq \frac{4n+1}{4n(2n+\frac{1}{2})}.$$

This causes the contradiction $2 \leq 0$ as n goes to infinity.

The theorem stated below, proved by Banach in 1922, is one of the best-known FPTs.

Theorem 2.3 (Banach's FPT)**.** *Consider (E, ρ) as a complete metric space, and let $T : E \to E$ be a contraction mapping. Then T possesses a unique fixed point in E. Additionally, the sequence $\{x_n\}$ given by*

$$x_{n+1} = Tx_n, \quad n = 1, 2, \cdots,$$

converge to the unique fixed point.

In the previous theorem, which is referred to the contraction mapping principle, Banach ensures the existence of unique fixed point for the mapping provided that it is a contraction and also gives the method of how to find the point. This method known as Picard iteration or successive approximations in the literature is roughly illustrated as follows:

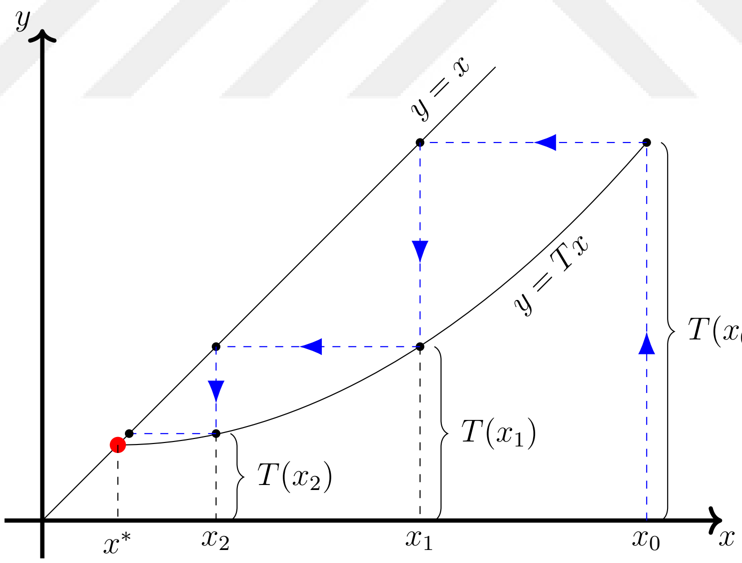

Figure 2.1 The convergence of Picard iteration to the fixed point x^* of T

Unlike Banach's FPT, Brouwer put forward a FPT known by his name, where he takes continuity as the only condition for the mapping. Before stating this theorem, let us give a familiar result from calculus:

Every continuous function $f : [a, b] \to [a, b]$ has a fixed point.

This simple case can be considered as the one dimensional version of Brouwer's FPT. The important theorem given below finds extensive applications in nonlinear analysis as well as many other fields of mathematics.

Theorem 2.4 (Brouwer's FPT-Version 1)**.** *Let T be a continuous mapping from a closed unit ball $\mathcal{B}$ in $\mathbb{R}^n$ into itself. Then T possesses a fixed point.*

An alternative variant of Brouwer's FPT is as follows:

Theorem 2.5 (Brouwer's FPT-Version 2)**.** *Suppose S is a convex, compact and nonempty subset of $\mathbb{R}^n$, and let $T : S \to S$ be a continuous mapping. Then T possesses a fixed point.*

Brouwer's FPT is only applicable in finite dimensinal spaces, not infinite ones. The following example explains this situation.

Example 2.6. Take into account the Banach space c_0 of sequences converging to 0, equipped with the norm

$$\|x\| = \sup_{n\in\mathbb{N}} |x_n|, \quad \text{where} \quad x = (x_1, x_2, \cdots).$$

Let $\mathcal{B}$ represent a closed unit ball in c_0, and define the operator $T : \mathcal{B} \to \mathcal{B}$ as follows:

$$Tx = (1, x_1, x_2, \cdots).$$

The continuity of T is evident as the equality $\|Tx - T\overline{x}\| = \|x - \overline{x}\|$ holds for every $x, \overline{x} \in \mathcal{B}$. However, T does not possess a fixed point in $\mathcal{B}$ because $Tx = x$ implies that $x_1 = x_2 = \cdots = 1$ and hence $x \notin \mathcal{B} \subset c_0$.

One of the generalizations of Brouwer's FPT was given by Schauder, who extended this theorem to infinite dimensional spaces as follows;

Theorem 2.7 (Schauder's FPT-Version 1)**.** *Suppose S is a convex, compact and nonempty subset of a Banach space E, and let $T : S \to S$ be a continuous mapping. Then T possesses a fixed point.*

Under some more assumptions, a continuous function may have a fixed point even in the absence of a mapping from a compact convex set into itself. In the following, it is stated an alternative version of Schauder's FPT that is more suitable for applications since compact sets are harder to find in an infinite dimensional space.

Theorem 2.8 (Schauder's FPT-Version 2). *Suppose S is a convex, closed and nonempty subset of a Banach space E. If T is a continuous mapping of S into itself such that $T(S)$ is relatively compact (that is, its closer is compact), then T possesses a fixed point.*

The Ascoli-Arzela theorem below offers criteria for the relative compactness required for Schauder's FPT.

Theorem 2.9 (Ascoli-Arzela Theorem). *Let E denote a compact metric space endowed with ρ. A subset S of $C(E, \mathbb{R}^n)$ is relatively compact if and only if (shortly, iff) it meets the following criteria:*

i) S is bounded, which means that $\|u(x)\| \leq M$ for some constant $M > 0$.

ii) S is equicontinuous, which means that for any $\epsilon > 0$, there is a $\delta > 0$ such that for all $u \in S$,

$$\|u(x) - u(\overline{x})\| \leq \epsilon$$

holds for all $x, \overline{x} \in E$ provided that $\rho(x, \overline{x}) < \delta$.

2.3 Stability Theory

This section presents the basic concepts of stability in the sense of Ulam-Hyers (UH) and Ulam-Hyers-Rassias (UHR), and the necessary tools to achieve them. These stability definitions will be given for the specific ODE of the general first order:

$$u'(x) = f\big(x, u(x)\big), \tag{2.6}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 2.2. [32] The equation (2.6) is called UH stable when there is a constant $\mathcal{C} > 0$ such that for every $\epsilon > 0$, the following property holds: Given any continuously differentiable function ϑ satisfying

$$\big|\vartheta'(x) - f\big(x, \vartheta(x)\big)\big| \leq \epsilon,$$

there exists a solution u to the equation (2.6) fulfilling $\big|\vartheta(x) - u(x)\big| \leq \mathcal{C}\epsilon$ for all $x \in [a, b]$.

Definition 2.3. [32] Let ϕ be a non negative function. The equation (2.6) is called UHR stable w.r.t. ϕ when there is a constant $\mathcal{C} > 0$ such that the following property holds: Given any continuously differentiable function ϑ satisfying

$$\left|\vartheta'(x) - f\big(x, \vartheta(x)\big)\right| \le \phi(x),$$

there exists a solution u to the equation (2.6) fulfilling $\left|\vartheta(x) - u(x)\right| \le \mathcal{C}\phi(x)$ for all $x \in [a, b]$.

These definitions will be modified for some nonlinear hyperbolic PDEs that are studied in this thesis. Now we state the concept of the Picard operator and the abstract Gronwall lemma from Picard operator theory.

Definition 2.4. [38, 39] Consider an operator $\mathcal{P} : E \to E$ on a metric space (E, ρ). If there exists a $z^* \in E$ such that

i) $F_{\mathcal{P}} = \{z^*\}$, where $F_{\mathcal{P}} = \{z \in E : \mathcal{P}(z) = z\}$ denotes the fixed point set of $\mathcal{P}$,

ii) The sequence $(\mathcal{P}^n(z_0))_{n\in\mathbb{N}}$ converges to z^* for all $z_0 \in E$,

then $\mathcal{P}$ is called Picard operator.

And the triplet $(E, \rho, \preceq)$ is called an ordered metric space if (E, ρ) forms a metric space and $\preceq$ represents a partial order relation on X.

Lemma 2.1. *[38, 39] Consider an increasing Picard operator $\mathcal{P} : E \to E$ with $F_{\mathcal{P}} = \{z^*\}$, and let $(E, \rho, \preceq)$ be an ordered metric space. For $z \in E$, if $z \preceq \mathcal{P}(z)$, then $z \preceq z^*$; whereas if $z \succeq \mathcal{P}(z)$, then $z \succeq z^*$.*

Next, the Wendorff lemma is stated, which is the extended form of the Gronwall lemma.

Lemma 2.2. *[40, 41] Let $z, h, k \in C([0, a] \times [0, b], \mathbb{R}_+)$, and let $h(x, t)$ be non-decreasing w.r.t. x and t. Suppose*

$$z(x, t) \le h(x, t) + \int_0^x \int_0^t k(r, s)z(r, s)dsdr, \quad x \in [0, a], \quad t \in [0, b].$$

Then the inequality stated below holds:

$$z(x, t) \le h(x, t) \exp\left(\int_0^x \int_0^t k(r, s)dsdr\right).$$

3
EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR HYPERBOLIC PDEs

In this chapter, the E&U results are performed for several nonlinear hyperbolic PDEs with or without time delay using fixed point theory which is the main objective of the thesis. After obtaining these E&U results, in the next chapter, we also investigate the stability in the sense of UH and UHR for these nonlinear hyperbolic PDEs.

3.1 Nonlinear Hyperbolic PDEs with Two Delays

In this section, we examine the following class of hyperbolic PDEs involving finite time delays α and β,

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f(x,t,u(x,t),u(x-\alpha,t-\beta)) \qquad (x,t) \in D \tag{3.1}$$

$$u(x,t) = \phi(x,t) \qquad (x,t) \in \tilde{D} \tag{3.2}$$

with

$$u(x,0) = \varphi(x) \quad \text{and} \quad u(0,t) = \psi(t) \tag{3.3}$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}^2, \mathbb{R})$, $\tilde{D} = [-\alpha,a] \times [-\beta,b] \backslash (0,a] \times (0,b]$, $\phi \in C(\tilde{D}, \mathbb{R})$, $\varphi(x)$ and $\psi(t)$ are continuously differentiable mappings with $\varphi(x) = \phi(x,0)$, $\psi(t) = \phi(0,t)$ for the intervals $[0,a]$ and $[0,b]$, respectively.

This section is dedicated to the investigation of the E&U of solutions to this class of PDEs. To prove the main result, we use a technique so called "progressive contractions" which is introduced by Burton, and is carried out in one dimension in [42, 43]. Recently, they applied this technique to a form of integral equations with delay and it allowed Burton and Purnaras to get rid of the function with delay in [44]. Inspired by Burton's method, we extend the progressive contraction technique to two-dimensional regions unlike the other related studies and then apply it to our problem. More specifically, we create nested rectangular regions by dividing the intervals $[0, a]$ on the x-axis and $[0, b]$ on the other properly. Using these partitions, we demonstrate that a unique solution exists in the first rectangular region. After that, we extend this region into an upper rectangular region and find a unique solution in this upper region by considering the solution function we found in the previous step as initial function. Continuing this process until we reach the whole domain, we can obtain a unique solution for the equation (3.1)-(3.3).

Before stating the main result, we recall Bielecki's norm [1] which is used to obtain a solution defined on the whole rectangle $\Omega := [-\alpha, a] \times [-\beta, b]$. The Bielecki norm $\|\cdot\|_B$ is defined by $\|u\|_B = \max_{(x,t)\in\Omega} e^{-\theta(x+t)} |u(x,t)|$ on $C(\Omega, \mathbb{R})$. Note that the maximum norm on $C(\Omega, \mathbb{R})$ is equivalent to the Bielecki norm, as indicated by the inequality:

$$e^{-\theta(a+b)} \max_{(x,t)\in\Omega} |u(x,t)| \leq \|u\|_B \leq e^{\theta(\alpha+\beta)} \max_{(x,t)\in\Omega} |u(x,t)| .$$

Consequently, it follows straightforwardly that $(C(\Omega, \mathbb{R}), \|\cdot\|_B)$ constitutes a Banach space. Let us now turn our attention to the proof of the E&U of solutions to our problem (3.1)-(3.3). This is stated in the following theorem. As mentioned above, we mainly apply the progressive contraction technique into two dimensional region.

Definition 3.1. A function $u \in C^{1,2}(D, \mathbb{R}) \cap C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ is called to be a solution of the proposed equation (3.1)-(3.3) if it satisfies the equations (3.1) and (3.2) on D, as well as (3.3) on $\tilde{D}$, where $C^{1,2}(D, \mathbb{R})$ denotes the set of functions $u(x,t) : D \to \mathbb{R}$ that are continuous along with their partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x \partial t}$.

Theorem 3.1. *Suppose the following conditions hold:*

i) The function $f : D \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

ii) There exists $L_f > 0$ which ensures that

$$\left| f(x,t,u,w) - f(x,t,\overline{u},w) \right| \leq L_f \left| u - \overline{u} \right| \tag{3.4}$$

for any $u, \overline{u}, w \in \mathbb{R}$ and $(x,t) \in D$.

Then the problem (3.1)-(3.3) has a unique solution on $[-\alpha, a] \times [-\beta, b]$.

[1] This norm was first used by Bielecki in [45]

Proof. We begin by transforming our problem (3.1)-(3.3) into a fixed point problem. For this purpose, we introduce the operator N as follows:

$$N : C([-\alpha, a] \times [-\beta, b], \mathbb{R}) \to C([-\alpha, a] \times [-\beta, b], \mathbb{R})$$

which is defined as:

$$Nu(x,t) = \begin{cases} \phi(x,t) & (x,t) \in \tilde{D} \\ \mu(x,t) + \int_0^x \int_0^t f\big(r, s, u(r,s), u(r-\alpha, s-\beta)\big) ds dr & (x,t) \in D, \end{cases}$$

where $\mu(x,t) = \varphi(x) + \psi(t) - \varphi(0)$. The intervals $[0,a]$ and $[0,b]$ are now appropriately divided on the x and t axes respectively. Let $0 < S < \alpha$ and $0 < T < \beta$, where $nS = a$ and $nT = b$. Observe that the following argument validates the existence of such n:

$$\exists N_S \in \mathbb{N} \quad \frac{a}{N_S} < \alpha \Longrightarrow \frac{a}{n} \leq \frac{a}{N_S} < \alpha \Longrightarrow S = \frac{a}{n}$$

and

$$\exists N_T \in \mathbb{N} \quad \frac{b}{N_T} < \beta \Longrightarrow \frac{b}{n} \leq \frac{b}{N_T} < \beta \Longrightarrow T = \frac{b}{n},$$

where $n = \max\{N_S, N_T\}$. The intervals are divided into the following partitions:

$$0 = S_0 < S_1 < \cdots < S_n = a, \quad S_i - S_{i-1} = S$$

and

$$0 = T_0 < T_1 < \cdots < T_n = b, \quad T_i - T_{i-1} = T.$$

To keep things simple, we shall utilize the following notations:

$$\begin{cases} \tilde{D}_i := [-\alpha, S_i] \times [-\beta, T_i] \backslash (0, S_i] \times (0, T_i] \\ D_i := [0, S_i] \times [0, T_i] \\ u_{(\alpha,\beta)}(x,t) := u(x-\alpha, t-\beta), \quad (x,t) \in D. \end{cases}$$

Now, our observation deduced from the above partition is the following fact

$$(x,t) \in D_{i+1} \Longrightarrow (x-\alpha, t-\beta) \in D_i \cup \tilde{D}_i. \tag{3.5}$$

We notice that if $(x,t) \in D_{i+1}$

$$x - \alpha \leq S_{i+1} - \alpha < S_{i+1} - S = S_i$$

and

$$t-\beta \le T_{i+1}-\beta < T_{i+1}-T = T_i,$$

that is $(x-\alpha, t-\beta) \in D_i \cup \tilde{D}_i$.

By using these partitions in the manner described below, we will show that there exists only one solution.

Step 1:

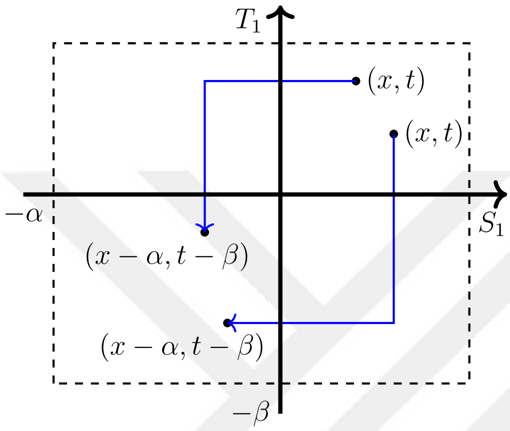

Figure 3.1 The first rectangular partition of progressive contraction

Let $\left(M_{(S_1,T_1)}, \|\cdot\|_1\right)$ be complete normed space of continuous functions

$$u : [-\alpha, S_1] \times [-\beta, T_1] \to \mathbb{R}$$

with the Bielecki norm

$$\|u\|_1 = \max_{[-\alpha,S_1]\times[-\beta,T_1]} e^{-\theta(x+t)} \left|u(x,t)\right|$$

and we take $u(x,t) = \phi(x,t)$ for $(x,t) \in \tilde{D}_1$.

Define a mapping $N_1 : M_{(S_1,T_1)} \to M_{(S_1,T_1)}$

$$N_1u(x,t) = \begin{cases} \phi(x,t) & (x,t) \in \tilde{D}_1 \\ \mu(x,t) + \int_0^x \int_0^t f\left(r,s,u(r,s),u_{(\alpha,\beta)}(r,s)\right) dsdr & (x,t) \in D_1. \end{cases}$$

For $u, \overline{u} \in M_{(S_1,T_1)}$, $N_1u = N_1\overline{u}$ when $(x,t) \in \tilde{D}_1$, then we take $(x,t) \in D_1$. Hence,

$$\begin{aligned} \left\|N_1u - N_1\overline{u}\right\|_1 \le \max_{(x,t)\in D_1} e^{-\theta(x+t)} \int_0^x \int_0^t \Big| & f\left(r,s,u(r,s),u_{(\alpha,\beta)}(r,s)\right) \\ & - f\left(r,s,\overline{u}(r,s),\overline{u}_{(\alpha,\beta)}(r,s)\right)\Big| dsdr. \end{aligned}$$

By (3.5) and the definition of $M_{(S_1,T_1)}$, we get $u_{(\alpha,\beta)} = \overline{u}_{(\alpha,\beta)}$. And thus, it is enough to put the Lipschitz condition on the function f w.r.t. the third variable. Hence,

$$\begin{aligned}
\left\|N_1 u - N_1 \overline{u}\right\|_1 &\leq \max_{(x,t)\in D_1} e^{-\theta(x+t)} \int_0^x \int_0^t L_f \left|u(r,s) - \overline{u}(r,s)\right| ds dr \\
&\leq L_f \left\|u - \overline{u}\right\|_1 \max_{(x,t)\in D_1} e^{-\theta(x+t)} \int_0^x \int_0^t e^{\theta(r+s)} ds dr \\
&\leq \frac{L_f}{\theta^2} \left\|u - \overline{u}\right\|_1.
\end{aligned}$$

By taking $\theta > 0$ (through the other steps below) such that $\frac{L_f}{\theta^2} < 1$ in $\left\|.\right\|_1$ norm above, we obtain that N_1 is a contraction mapping. According to Banach's FPT, there exists a unique fixed point $\phi_1 \in M_{(S_1,T_1)}$ such that ϕ_1 satisfies the problem (3.1)-(3.3) on $[-\alpha, S_1] \times [-\beta, T_1]$.

Step 2: In this step, we extend the interval of Step 1 into $[-\alpha, S_2] \times [-\beta, T_2]$.

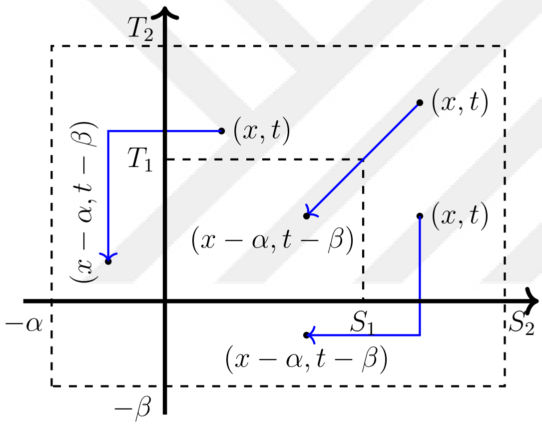

Figure 3.2 The second rectangular partition of progressive contraction

Let $\left(M_{(S_2,T_2)}, \left\|.\right\|_2\right)$ be complete normed space of continuous functions

$$u : [-\alpha, S_2] \times [-\beta, T_2] \to \mathbb{R}$$

equipped with the norm

$$\left\|u\right\|_2 = \max_{[-\alpha,S_2]\times[-\beta,T_2]} e^{-\theta(x+t)} \left|u(x,t)\right|$$

and we take

$$u(x,t) = \begin{cases} \phi(x,t) & (x,t) \in \tilde{D_2} \\ \phi_1(x,t) & (x,t) \in D_1. \end{cases}$$

Similarly, let us define a mapping $N_2 : M_{(S_2,T_2)} \to M_{(S_2,T_2)}$

$$N_2 u(x,t) = \begin{cases} \phi(x,t) & (x,t) \in \tilde{D_2} \\ \phi_1(x,t) & (x,t) \in D_1 \\ \mu(x,t) + \int_0^x \int_0^t f\big(r, s, u(r,s), u_{(\alpha,\beta)}(r,s)\big) ds dr & (x,t) \in D_2 \backslash D_1. \end{cases}$$

For $u, \overline{u} \in M_{(S_2,T_2)}$, $N_2 u = N_2 \overline{u}$ whenever $(x,t) \in D_1 \cup \tilde{D_2}$, then we consider

$(x,t) \in D_2 \backslash D_1$. Hence,

$$\left\|N_2 u - N_2 \overline{u}\right\|_2 \leq \max_{(x,t)\in D_2\backslash D_1} e^{-\theta(x+t)} \int_0^x \int_0^t \Big| f\big(r,s,u(r,s),u_{(\alpha,\beta)}(r,s)\big) - f\big(r,s,\overline{u}(r,s),\overline{u}_{(\alpha,\beta)}(r,s)\big)\Big| dsdr$$

$\Big($ by (3.5) and the definition of $M_{(S_2,T_2)}\Big)$

$$\begin{aligned}
&\leq \max_{(x,t)\in D_2\backslash D_1} e^{-\theta(x+t)} \int_0^x \int_0^t L_f \big|u(r,s) - \overline{u}(r,s)\big| dsdr \\
&\leq L_f \left\|u - \overline{u}\right\|_2 \max_{(x,t)\in D_2\backslash D_1} e^{-\theta(x+t)} \int_0^x \int_0^t e^{\theta(r+s)} dsdr \\
&\leq \frac{L_f}{\theta^2} \left\|u - \overline{u}\right\|_2.
\end{aligned}$$

Therefore N_2 is a contraction mapping and by Banach's FPT there exists a unique fixed point ϕ_2 in $M_{(S_2,T_2)}$, which serves as a solution to the equation (3.1)-(3.3) on $[-\alpha, S_2] \times [-\beta, T_2]$.

Step 3: Continuing this process up to n^{th} Step, we can obtain a unique continuous solution ϕ_n for the equation (3.1)-(3.3) on two dimensional region $[-\alpha, S_n] \times [-\beta, T_n] = [-\alpha, a] \times [-\beta, b]$.

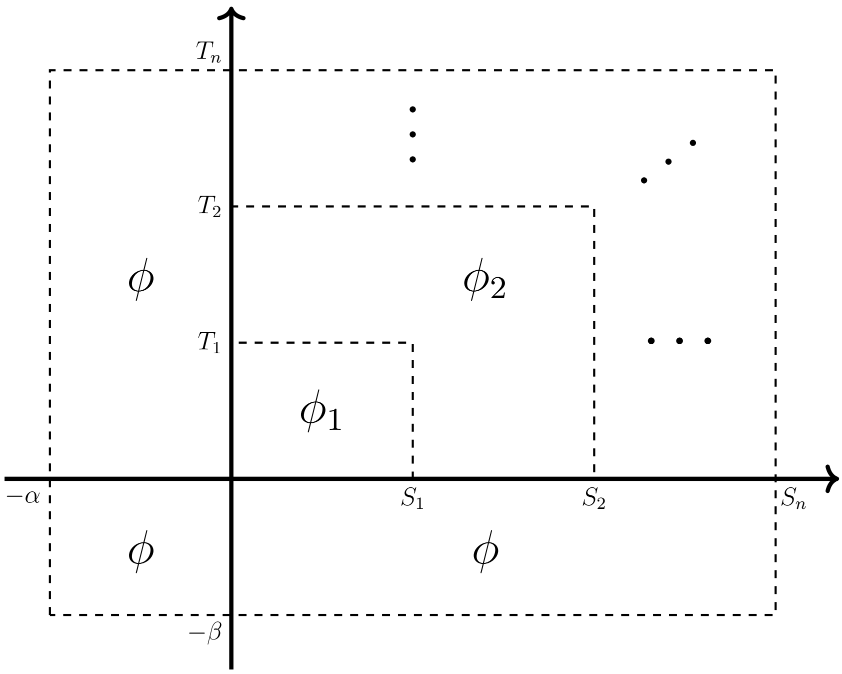

Figure 3.3 All rectangular partitions of progressive contraction

■

3.2 Nonlinear Hyperbolic Functional PDEs

In this section, we focus on the hyperbolic functional PDEs given by

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\Big(x, t, u\big(g(x,t), h(x,t)\big)\Big), \quad (x,t) \in D \tag{3.6}$$

with

$$\begin{cases} u(x,0) = \varphi(x) \\ u(0,t) = \psi(t) \end{cases} \qquad \text{such that} \quad \varphi(0) = \psi(0), \tag{3.7}$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}, \mathbb{R})$, $g \in C(D, [0,a])$, $h \in C(D, [0,b])$, $\varphi(x)$ and $\psi(t)$ belong to the space of continuously differentiable mappings defined on $[0,a]$ and $[0,b]$, respectively.

The current section is dedicated to establish the E&U of solutions to the proposed equation (3.6)-(3.7) on a bounded domain utilizing the Bielecki norm, and based on these solutions obtained on bounded domains, the finding is extended to an unbounded domain. Below, we present the main results regarding the E&U of solutions for the equation (3.6)-(3.7) in both bounded and unbounded domains.

By a solution to the equation (3.6)-(3.7) we refer to a function $u \in C^{1,2}(D, \mathbb{R})$ satisfying the equation (3.6) and the conditions (3.7).

Theorem 3.2. *Assume the following conditions are satisfied:*

(C1) $f \in C(D \times \mathbb{R}, \mathbb{R})$, $g \in C(D, [0,a])$, $h \in C(D, [0,b])$ *with* $g(x,t) \leq x$ *and* $h(x,t) \leq t$.

(C2) There exists $L > 0$ *which ensures that*

$$\big|f(x,t,u) - f(x,t,\overline{u})\big| \leq L\big|u - \overline{u}\big|$$

for any $u, \overline{u} \in \mathbb{R}$ *and* $(x,t) \in D$.

Then the equation (3.6)-(3.7) has a unique solution in $C(D, \mathbb{R})$.

Proof. Under the condition (C1), the proposed equation (3.6)-(3.7) is equivalent to the integral equation

$$u(x,t) = \mu(x,t) + \int_0^x \int_0^t f\Big(r, s, u\big(g(r,s), h(r,s)\big)\Big) ds dr, \tag{3.8}$$

where $\mu(x,t) = \varphi(x) + \psi(t) - \varphi(0)$. Converting this equation to a fixed point problem, we aim to find the fixed point of the mapping given below

$$F : C(D, \mathbb{R}) \to C(D, \mathbb{R})$$

defined by

$$Fu = \text{the RHS of the equation (3.8).}$$

We now demonstrate that F is contraction w.r.t. the Bielecki norm given by

$$\|u\|_B = \max_{(x,t)\in D} |u(x,t)| e^{-\theta(x+t)} \quad \text{where} \quad \theta > 0. \tag{3.9}$$

For any $u, \overline{u} \in C(D, \mathbb{R})$, we have

$$\begin{aligned} |Fu(x,t) - F\overline{u}(x,t)| \leq & \int_0^x \int_0^t \Big| f\Big(r, s, u\big(g(r,s), h(r,s)\big)\Big) \\ & - f\Big(r, s, \overline{u}\big(g(r,s), h(r,s)\big)\Big)\Big| dsdr \\ \leq & L \int_0^x \int_0^t e^{\theta(r+s)} \Big(\Big| u\big(g(r,s), h(r,s)\big) \\ & - \overline{u}\big(g(r,s), h(r,s)\big)\Big| e^{-\theta(r+s)}\Big) dsdr \\ \leq & L\|u - \overline{u}\|_B \int_0^x \int_0^t e^{\theta(r+s)} dsdr \leq \frac{L}{\theta^2} \|u - \overline{u}\|_B e^{\theta(x+t)}, \end{aligned}$$

which implies that

$$\|Fu - F\overline{u}\|_B \leq \frac{L}{\theta^2} \|u - \overline{u}\|_B.$$

By choosing $\theta > 0$ sufficiently large such that $\theta^2 > L$, we obtain that F is a contraction mapping, so the equation (3.6)-(3.7) has only one solution in $C(D, \mathbb{R})$ by Banach's FPT. ■

Now, we will show that the result proved above holds for unbounded domain. That is, Theorem 3.2 can be also proved if D is replaced by $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$ as shown below.

Theorem 3.3. *Assume the following conditions are satisfied:*

(C3) $f \in C(\mathbb{R}_+^2 \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}_+, \mathbb{R}_+)$, $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $g(x,t) \leq x$ *and* $h(x,t) \leq t$.

(C4) There exists $\mathbb{L} \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ *which ensures that*

$$|f(x,t,u) - f(x,t,\overline{u})| \leq \mathbb{L}(x,t)|u - \overline{u}|$$

for all $u, \overline{u} \in \mathbb{R}$ and $(x,t) \in \mathbb{R}_+^2$.

Then the equation (3.6)-(3.7) has a unique solution in $C(\mathbb{R}_+^2, \mathbb{R})$.

Proof. According to Theorem 3.2, for any $n \in \mathbb{N}$, there exists a unique continuous mapping $u_n : D_n \to \mathbb{R}$ such that

$$u_n(x,t) = \mu(x,t) + \int_0^x \int_0^t f\Big(r, s, u_n\big(g(r,s), h(r,s)\big)\Big) dsdr, \tag{3.10}$$

where $D_n = [0,n] \times [0,n]$, since the continuous function $\mathbb{L}$ is bounded on this compact domain. If $(x,t) \in D_n$, the following equality can be easily seen from the uniqueness of u_n

$$u_n(x,t) = u_{n+i}(x,t) \quad \text{for each} \quad i = 1, 2, 3, \cdots . \tag{3.11}$$

For any $(x,t) \in \mathbb{R}_+^2$, let us define $n(x,t) \in \mathbb{N}$ as

$$n(x,t) = \min\{n \in \mathbb{N} \mid (x,t) \in D_n\}.$$

Additionally, we introduce a mapping $u : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$u(x,t) = u_{n(x,t)}(x,t). \tag{3.12}$$

To demonstrate the continuity of u described above, we choose $n_1 = n(x_1, t_1)$ for an arbitrary $(x_1, t_1) \in \mathbb{R}_+^2$. Then (x_1, t_1) belongs to the interior of D_{n_1+1}. Thus, there exists an $\epsilon > 0$ such that $u(x,t) = u_{n_1+1}(x,t)$ for all $(x,t) \in B_\epsilon(x_1, t_1)$. Since u_{n_1+1} is continuous at (x_1, t_1), the mapping u is also continuous at this arbitrary point. Now we show that the mapping u satisfies the equation (3.8). For any $(x,t) \in \mathbb{R}_+^2$, there is an integer $n(x,t)$ such that $(x,t) \in D_{n(x,t)}$. It follows from (3.10) and (3.12) that

$$\begin{aligned} u(x,t) &= u_{n(x,t)}(x,t) \\ &= \mu(x,t) + \int_0^x \int_0^t f\Big(r, s, u_{n(x,t)}\big(g(r,s), h(r,s)\big)\Big) dsdr \\ &= \mu(x,t) + \int_0^x \int_0^t f\Big(r, s, u\big(g(r,s), h(r,s)\big)\Big) dsdr. \end{aligned}$$

where the last equality is obtained since $n\big(g(r,s), h(r,s)\big) \leq n(x,t)$ for any $(r,s) \in D_{n(x,t)}$ implies

$$u_{n(x,t)}\big(g(r,s), h(r,s)\big) = u_{n(g(r,s),h(r,s))}\big(g(r,s), h(r,s)\big) = u\big(g(r,s), h(r,s)\big)$$

by using (3.11) and (3.12). To prove the uniqueness, we suppose that ϑ is a continuous mapping which also satisfies (3.8). For an arbitrary $(x,t) \in \mathbb{R}^2_+$, since the restrictions $u\mid_{D_{n(x,t)}}$ and $\vartheta\mid_{D_{n(x,t)}}$ both satisfy (3.8) for all $(x,t) \in D_{n(x,t)}$, the uniqueness of $u_{n(x,t)} = u\mid_{D_{n(x,t)}}$ implies that

$$u(x,t) = u\mid_{D_{n(x,t)}}(x,t) = \vartheta\mid_{D_{n(x,t)}}(x,t) = \vartheta(x,t).$$

This completes the proof. ■

3.3 Nonlinear Implicit Hyperbolic PDEs

In this section, unlike the explicit hyperbolic problems as in Section 3.1 and 3.2, we deal with the existence of solutions to the following implicit hyperbolic PDEs:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\left(x, t, u(x,t), \frac{\partial^2 u(x,t)}{\partial x \partial t}\right), \quad (x,t) \in D \tag{3.13}$$

with

$$\begin{cases} u(x,0) = \varphi(x) \\ u(0,t) = \psi(t) \end{cases} \quad \text{such that} \quad \varphi(0) = \psi(0), \tag{3.14}$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}, \mathbb{R})$, $\varphi(x)$ and $\psi(t)$ belong to the space of continuously differentiable mappings defined on $[0,a]$ and $[0,b]$, respectively.

Let us express that what we mean by a solution of the equation (3.13)-(3.14). A function $u \in C^{1,2}(D, \mathbb{R})$ is defined as a solution of this equation if it satisfies the equation (3.13) and the conditions (3.14) on D.

Before stating our existence result, let us provide the following lemma, which converts the proposed equation (3.13)-(3.14) into a fixed point problem.

Lemma 3.1. *Suppose $f : D \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. The function u is a solution of the equation (3.13)-(3.14) iff it satisfies*

$$u(x,t) = \mu(x,t) + \int_0^x \int_0^t h(r,s) ds dr$$

where $\mu(x,t) = \varphi(x) + \psi(t) - \psi(0)$ and $h \in C(D, \mathbb{R})$ holds the functional equation $h(x,t) = f\big(x, t, u(x,t), h(x,t)\big)$.

Theorem 3.4. *Suppose the following conditions hold:*

(H1) The function $f : D \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(H2) There is a positive constant $L < 1$ which ensures that

$$\big|f(x,t,u,z_1) - f(x,t,u,z_2)\big| \leq L\big|z_1 - z_2\big|$$

for $u, z_i \in \mathbb{R}$ $(i = 1, 2)$ and $(x,t) \in D$.

(H3) There exist $p, q, d \in C(D, \mathbb{R}_+)$ with $d^* = \sup_{(x,t)\in D} d(x,t) < 1$ which ensure that

$$\big|f(x,t,u,z)\big| \leq p(x,t) + q(x,t)\big|u\big| + d(x,t)\big|z\big|$$

for all $(x,t) \in D$ and $u, z \in \mathbb{R}$.

Then the equation (3.13)-(3.14) has at least one solution.

Proof. Let $p^* = \sup_{(x,t)\in D} p(x,t)$ and $q^* = \sup_{(x,t)\in D} q(x,t)$. Define the bounded, closed and convex subset of $\Omega := C(D, \mathbb{R})$ as follows:

$$S_R = \{u \in \Omega : \big|u(x,t)\big| \leq Re^{\theta(x+t)}\}$$

where $\theta > 0$ is chosen sufficiently large such that $q^*/(1 - d^*) < \theta^2$. Consider an operator $\mathcal{P}$ on S_R into Ω given by

$$\mathcal{P}u(x,t) := \mu(x,t) + \int_0^x \int_0^t h(r,s)dsdr$$

where $h \in \Omega$ satisfies the equation:

$$h(x,t) = f\big(x,t,u(x,t),h(x,t)\big). \tag{3.15}$$

To demonstrate the well-definedness of the operator $\mathcal{P}$, there must exist an h satisfying the equation (3.15) and it must be unique for each $u \in S_R$. Utilizing assumption (H2), this can be achieved through the application of Banach's FPT to the following operator:

$$h(\cdot,\cdot) \to f\big(\cdot,\cdot,u(\cdot,\cdot),h(\cdot,\cdot)\big).$$

Lemma 3.1 indicates that the fixed point of the operator $\mathcal{P}$ corresponds to the solution of the given equation (3.13)-(3.14). The existence of a fixed point to this operator is discussed below using Schauder's FPT.

Step I: The operator $\mathcal{P}$ maps S_R into itself with the value of R to be determined below.

Let $u \in S_R$ be arbitrary element. Then,

$$\left|\mathcal{P}u(x,t)\right| \leq \left|\mu(x,t)\right| + \int_0^x \int_0^t \left|h(r,s)\right| dsdr.$$

By (H3),

$$\begin{aligned}\left|h(x,t)\right| &= f\big(x,t,u(x,t),h(x,t)\big)\\ &\leq p(x,t) + q(x,t)\left|u(x,t)\right| + d(x,t)\left|h(x,t)\right|\\ &\leq p^* + q^* R e^{\theta(x+t)} + d^*\left|h(x,t)\right|,\end{aligned}$$

which yields

$$\left|h(x,t)\right| \leq \frac{p^* + q^* R e^{\theta(x+t)}}{1-d^*}. \tag{3.16}$$

Then,

$$\begin{aligned}\left|\mathcal{P}u(x,t)\right| \leq &\left|\mu(x,t)\right| + \frac{p^*}{(1-d^*)} \int_0^x \int_0^t dsdr\\ &+ \frac{q^* R}{(1-d^*)} \int_0^x \int_0^t e^{\theta(r+s)} dsdr.\end{aligned}$$

One can obtain the following

$$\begin{aligned}\left|\mathcal{P}u(x,t)\right| &\leq \underbrace{\sup_{(x,t)\in D}\left|\mu(x,t)\right| + \frac{p^* ab}{(1-d^*)}}_{:=\Delta} + \underbrace{\frac{q^*}{(1-d^*)\theta^2}}_{:=\Upsilon} R e^{\theta(x+t)}\\ &= \Delta + \Upsilon R e^{\theta(x+t)} \leq R e^{\theta(x+t)},\end{aligned}$$

here the last inequality holds by the choice of $R > 0$ satisfying the following inequality

$$\frac{\Delta}{1-\Upsilon} \leq R.$$

Then we get $\mathcal{P} : S_R \to S_R$.

Step II: $\mathcal{P}$ is continuous operator on S_R.

Consider a sequence u_n in S_R that converges to u within S_R. Then we obtain

$$\left|\mathcal{P}u_n(x,t) - \mathcal{P}u(x,t)\right| \leq \int_0^x \int_0^t \left|h_n(r,s) - h(r,s)\right| dsdr$$

where $h_n, h \in \Omega$ satisfy

$$h_n(x,t) = f\big(x,t,u_n(x,t),h_n(x,t)\big) \quad \text{and} \quad h(x,t) = f\big(x,t,u(x,t),h(x,t)\big).$$

By (H2),

$$\begin{aligned}\big|h_n(x,t)-h(x,t)\big| \leq& \big|f\big(x,t,u_n(x,t),h_n(x,t)\big)-f\big(x,t,u_n(x,t),h(x,t)\big)\big|\\ &+\big|f\big(x,t,u_n(x,t),h(x,t)\big)-f\big(x,t,u(x,t),h(x,t)\big)\big|\\ \leq& \big|f\big(x,t,u_n(x,t),h(x,t)\big)-f\big(x,t,u(x,t),h(x,t)\big)\big|\\ &+L\big|h_n(x,t)-h(x,t)\big|\end{aligned}$$

then

$$\big\|h_n-h\big\|_\infty \leq \frac{1}{1-L}\big\|f\big(\cdot,\cdot,u_n(\cdot,\cdot),h(\cdot,\cdot)\big)-f\big(\cdot,\cdot,u(\cdot,\cdot),h(\cdot,\cdot)\big)\big\|_\infty.$$

As n goes to infinity, we get $h_n \to h$ since f is continuous. Hence

$$\big\|\mathcal{P}u_n-\mathcal{P}u\big\|_\infty \leq ab\big\|h_n-h\big\|_\infty \to 0 \quad \text{as} \quad n\to\infty$$

that is $\mathcal{P}$ is continuous operator.

Step III: $\mathcal{P}(S_R)$ is uniformly bounded.

This is evident from the fact that $\mathcal{P}(S_R)\subseteq S_R$ and S_R is bounded.

Step IV: $\mathcal{P}(S_R)$ is equicontinuous.

Without loss of generality, let $(x_1,t_1),(x_2,t_2)\in D$ be such that $x_1<x_2$ and $t_1<t_2$,

$$\begin{aligned}&\big|\mathcal{P}u(x_1,t_1)-\mathcal{P}u(x_2,t_2)\big|\\ &\quad\leq\big|\mu(x_1,t_1)-\mu(x_2,t_2)\big|+\left|\int_0^{x_1}\int_0^{t_1}h(r,s)dsdr-\int_0^{x_2}\int_0^{t_2}h(r,s)dsdr\right|\\ &\quad\leq\big|\varphi(x_1)-\varphi(x_2)\big|+\big|\psi(t_1)-\psi(t_2)\big|+\int_0^{x_1}\int_{t_1}^{t_2}\big|h(r,s)\big|dsdr\\ &\qquad+\int_{x_1}^{x_2}\int_0^{t_2}\big|h(r,s)\big|dsdr\end{aligned}$$

where $h\in\Omega$ holds the following from (3.16)

$$\big|h(x,t)\big|\leq M:=\frac{p^*+q^*Re^{\theta(a+b)}}{1-d^*}.$$

Therefore, we get

$$\begin{aligned}\big|\mathcal{P}u(x_1,t_1)-\mathcal{P}u(x_2,t_2)\big| \leq & \big|\varphi(x_1)-\varphi(x_2)\big| + \big|\psi(t_1)-\psi(t_2)\big| \\ & + M\Big(a\big|t_1-t_2\big| + b\big|x_1-x_2\big|\Big).\end{aligned}$$

Since φ and ψ are uniformly continuous on the compact domains $[0,a]$ and $[0,b]$ respectively, we can conclude that $\mathcal{P}(S_R)$ is an equicontinuous set.

By virtue of Steps I to IV together with the Ascoli-Arzela theorem, $\mathcal{P}(S_R)$ is relatively compact. As a consequence of Schauder's FPT, the operator $\mathcal{P}$ possesses a fixed point in S_R that is the solution of the equation (3.13)-(3.14). This complete the proof. ■

Let us now apply the Wendorff lemma to obtain the uniqueness of solution.

Theorem 3.5. *Suppose that (H1) and the following condition is satisfied:*

(H4) There exist constants $K>0$ and $0<L<1$ which ensure that

$$\big|f(x,t,u_1,z_1)-f(x,t,u_2,z_2)\big| \leq K\big|u_1-u_2\big| + L\big|z_1-z_2\big|$$

for all $u_i, z_i \in \mathbb{R}$ $(i=1,2)$ and $(x,t)\in D$.

Then the equation (3.13)-(3.14) has only one solution.

Proof. By Theorem 3.4, we proved that the proposed equation (3.13)-(3.14) has a solution u. Let w be another solution of this equation. Then, we have

$$\big|u(x,t)-w(x,t)\big| \leq \int_0^x\int_0^t \big|h_u(r,s)-h_w(r,s)\big|dsdr$$

where $h_u, h_w \in C(D,\mathbb{R})$ such that

$$h_u(x,t) = f\big(x,t,u(x,t),h_u(x,t)\big) \quad \text{and} \quad h_w(x,t) = f\big(x,t,w(x,t),h_w(x,t)\big).$$

By (H4), we get

$$\begin{aligned}\big|h_u(x,t)-h_w(x,t)\big| &= \big|f\big(x,t,u(x,t),h_u(x,t)\big) - f\big(x,t,w(x,t),h_w(x,t)\big)\big| \\ &\leq K\big|u(x,t)-w(x,t)\big| + L\big|h_u(x,t)-h_w(x,t)\big|\end{aligned}$$

which implies

$$\big|h_u(x,t) - h_w(x,t)\big| \leq \frac{K}{1-L}\big|u(x,t) - w(x,t)\big|.$$

Hence we obtain that

$$\big|u(x,t) - w(x,t)\big| \leq \frac{K}{1-L}\int_0^x \int_0^t \big|u(r,s) - w(r,s)\big| dsdr.$$

Thanks to the Wendorff lemma, we find $\big|u(x,t) - w(x,t)\big| = 0$ for all $(x,t) \in D$, which yields that the solution of the equation is unique. ■

3.4 Nonlinear Hyperbolic PDEs Involving First Order Derivatives

In this section, we focus on the following hyperbolic PDEs with a RHS function f involving the first order derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\Big(x, t, u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}\Big), \quad (x,t) \in D \tag{3.17}$$

with

$$\begin{cases} u(x,0) = \varphi(x) \\ u(0,t) = \psi(t) \end{cases} \quad \text{such that} \quad \varphi(0) = \psi(0), \tag{3.18}$$

where $D = [0,a] \times [0,b]$, $f \in C(D \times \mathbb{R}^3, \mathbb{R})$, $\varphi(x)$ and $\psi(t)$ belong to the space of continuously differentiable mappings defined on $[0,a]$ and $[0,b]$, respectively.

In the current part, we propose a novel proof of the theorem given by Hartman and Winter, which is related to the existence of solution for the PDEs (3.17)-(3.18). Our proof is based on fixed point theory. In the proof of this theorem, we employ two FPTs which are Banach's and Schauder's theorems. To explain briefly method applied in the proof, we use the Banach FPT to construct well-defined mappings and we apply Schauder's FPT to prove the existence result. This kind of use of Banach's FPT appears in some papers, especially in FPTs involving the sum and the product of two operators (see [46, 47]). By motivating these papers, we approach the proposed PDEs via fixed point theory.

Before we construct a new proof for the following theorem proved by Hartman and Wintner, let us first state what the solution means: A function $u \in C^{1,2}(D, \mathbb{R})$ that satisfies the equation (3.17) and the conditions (3.18) is called a solution of the equation (3.17)-(3.18).

Theorem 3.6. *Suppose the following conditions hold:*

i) $f \in C(D \times \mathbb{R}^3)$ *and* f *is bounded in absolute value, that is, there exists non-negative constant* M *which ensures that*

$$\left|f(x,t,u,p,q)\right| \leq M$$

for all $u,p,q \in \mathbb{R}$ *and* $(x,t) \in D$*.*

ii) f *satisfies the lipschitz condition in two arguments, that is, there is a constant* $L > 0$ *which ensures that*

$$\left|f(x,t,u,p_1,q_1) - f(x,t,u,p_2,q_2)\right| \leq L\left(\left|p_1 - p_2\right| + \left|q_1 - q_2\right|\right)$$

for $u, p_i, q_i \in \mathbb{R}$ $(i = 1,2)$ *and* $(x,t) \in D$*.*

Then the equation (3.17)-(3.18) has a solution.

Proof. Let us transform the main equation into a corresponding system of integral equations by integrating w.r.t. the variables x and t

$$u(x,t) = \mu(x,t) + \int_0^x \int_0^t f\big(r,s,u(r,s),v(r,s),w(r,s)\big)dsdr$$

where $\mu(x,t) = \varphi(x) + \psi(t) - \varphi(0)$ and also the pair of v,w holds the below equations:

$$v(x,t) = \varphi'(x) + \int_0^t f\big(x,s,u(x,s),v(x,s),w(x,s)\big)ds$$
$$w(x,t) = \psi'(t) + \int_0^x f\big(r,t,u(r,t),v(r,t),w(r,t)\big)dr$$

or briefly

$$(u,v,w)(x,t) = \big(A_1(u,v,w), A_2(u,v,w), A_3(u,v,w)\big)(x,t). \tag{3.19}$$

The proposed equation can be converted into a fixed point problem for the following operator

$$\mathcal{A} : C(D,\mathbb{R})^3 \to C(D,\mathbb{R})^3,$$

which is defined by the RHS of the equation (3.19). Let us consider the product space $\big(C(D,\mathbb{R}) \times C(D,\mathbb{R}), \left\|(\cdot,\cdot)\right\|\big)$ such that it constitutes a Banach space equipped with the

$$\|(v,w)\| = \|v\|_B + \|w\|_B \quad \text{where} \quad \|v\|_B = \max_{(x,t)\in D} |v(x,t)| e^{-\theta(x+t)}.$$

By taking $u \in C(D,\mathbb{R})$ as a constant, we define a mapping from $C(D,\mathbb{R}) \times C(D,\mathbb{R})$ to itself as follows:

$$\mathcal{P}_u(v,w) := \big(A_2(u,v,w), A_3(u,v,w)\big).$$

For $(v,w), (\overline{v},\overline{w}) \in C(D,\mathbb{R}) \times C(D,\mathbb{R})$,

$$\begin{aligned}
&\big|A_2(u,v,w) - A_2(u,\overline{v},\overline{w})\big| \\
&\qquad\qquad \leq \int_0^t \Big| f\big(x,s,u(x,s),v(x,s),w(x,s)\big) \\
&\qquad\qquad\quad - f\big(x,s,u(x,s),\overline{v}(x,s),\overline{w}(x,s)\big)\Big| ds \\
&\qquad\qquad \leq L \int_0^t \Big(\big|v(x,s) - \overline{v}(x,s)\big| + \big|w(x,s) - \overline{w}(x,s)\big| \Big) ds \\
&\qquad\qquad \leq L \Big(\|v - \overline{v}\|_B + \|w - \overline{w}\|_B \Big) \int_0^t e^{\theta(x+s)} ds \\
&\qquad\qquad \leq \frac{L}{\theta} \big\|(v,w) - (\overline{v},\overline{w})\big\| e^{\theta(x+t)},
\end{aligned}$$

and consequently we get

$$\big\|A_2(u,v,w) - A_2(u,\overline{v},\overline{w})\big\|_B \leq \frac{L}{\theta} \big\|(v,w) - (\overline{v},\overline{w})\big\|. \tag{3.20}$$

Similarly,

$$\big\|A_3(u,v,w) - A_3(u,\overline{v},\overline{w})\big\|_B \leq \frac{L}{\theta} \big\|(v,w) - (\overline{v},\overline{w})\big\|. \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\big\|\mathcal{P}_u(v,w) - \mathcal{P}_u(\overline{v},\overline{w})\big\| \leq \frac{2L}{\theta} \big\|(v,w) - (\overline{v},\overline{w})\big\|.$$

If we choose $\theta > 0$ such that $\frac{2L}{\theta} < 1$, then $\mathcal{P}_u$ is a contraction mapping and so $\mathcal{P}_u$ has a unique fixed point in $C(D,\mathbb{R}) \times C(D,\mathbb{R})$. Now we can construct two well-defined mappings from $C(D,\mathbb{R})$ to $C(D,\mathbb{R})$ as follows;

$$\begin{aligned} \mathcal{F} : C(D,\mathbb{R}) &\to C(D,\mathbb{R}) \\ u &\to \mathcal{F}u \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{T} : C(D,\mathbb{R}) &\to C(D,\mathbb{R}) \\ u &\to \mathcal{T}u \end{aligned}$$

where $(\mathcal{F}u, \mathcal{T}u)$ is the unique fixed point of $\mathcal{P}_u$ and this pair satisfy

$$\begin{aligned}\mathcal{F}u(x,t) &= \varphi'(x) + \int_0^t f\big(x,s,u(x,s),\mathcal{F}u(x,s),\mathcal{T}u(x,s)\big)ds\\ \mathcal{T}u(x,t) &= \psi'(t) + \int_0^x f\big(r,t,u(r,t),\mathcal{F}u(r,t),\mathcal{T}u(r,t)\big)dr.\end{aligned}$$

Now we show that these mappings are continuous. Let $\|u_n - u\|_B \to 0$. Then we have

$$\begin{aligned}\big|\mathcal{F}u_n(x,t) - &\mathcal{F}u(x,t)\big|\\ &\le \int_0^t \Big|f\big(x,s,u_n(x,s),\mathcal{F}u_n(x,s),\mathcal{T}u_n(x,s)\big)\\ &\quad - f\big(x,s,u(x,s),\mathcal{F}u(x,s),\mathcal{T}u(x,s)\big)\Big|ds\\ &\le \int_0^t \Big|f\big(x,s,u_n(x,s),\mathcal{F}u_n(x,s),\mathcal{T}u_n(x,s)\big)\\ &\quad - f\big(x,s,u_n(x,s),\mathcal{F}u(x,s),\mathcal{T}u(x,s)\big)\Big|ds\\ &\quad + \int_0^t \Big|f\big(x,s,u_n(x,s),\mathcal{F}u(x,s),\mathcal{T}u(x,s)\big)\\ &\quad - f\big(x,s,u(x,s),\mathcal{F}u(x,s),\mathcal{T}u(x,s)\big)\Big|ds\\ &\le \frac{L}{\theta}\Big(\|\mathcal{F}u_n - \mathcal{F}u\|_B + \|\mathcal{T}u_n - \mathcal{T}u\|_B\Big)e^{\theta(x+t)}\\ &\quad + \frac{1}{\theta}\Big\|f\big(\cdot,\cdot,u_n(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)\\ &\quad - f\big(\cdot,\cdot,u(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)\Big\|_B e^{\theta(x+t)},\end{aligned}$$

consequently

$$\begin{aligned}\|\mathcal{F}u_n - \mathcal{F}u\|_B \le &\frac{L}{\theta}\Big(\|\mathcal{F}u_n - \mathcal{F}u\|_B + \|\mathcal{T}u_n - \mathcal{T}u\|_B\Big)\\ &+ \frac{1}{\theta}\Big\|f\big(\cdot,\cdot,u_n(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)\\ &- f\big(\cdot,\cdot,u(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)\Big\|_B.\end{aligned} \tag{3.22}$$

Similarly

$$\begin{aligned}\|\mathcal{T}u_n - \mathcal{T}u\|_B \le &\frac{L}{\theta}\Big(\|\mathcal{F}u_n - \mathcal{F}u\|_B + \|\mathcal{T}u_n - \mathcal{T}u\|_B\Big)\\ &+ \frac{1}{\theta}\Big\|f\big(\cdot,\cdot,u_n(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)\\ &- f\big(\cdot,\cdot,u(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)\Big\|_B.\end{aligned} \tag{3.23}$$

Adding up the two inequalities (3.22) and (3.23) above, we obtain the following inequality:

$$\left(1-\frac{2L}{\theta}\right)\Big(\big\|\mathcal{F}u_n-\mathcal{F}u\big\|_B+\big\|\mathcal{T}u_n-\mathcal{T}u\big\|_B\Big)$$
$$\leq\frac{2}{\theta}\Big\|f\big(\cdot,\cdot,u_n(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)$$
$$-f\big(\cdot,\cdot,u(\cdot,\cdot),\mathcal{F}u(\cdot,\cdot),\mathcal{T}u(\cdot,\cdot)\big)\Big\|_B.$$

Since f is a continuous mapping, when n goes to infinity, we have $\mathcal{F}u_n$ and $\mathcal{T}u_n$ converge to $\mathcal{F}u$ and $\mathcal{T}u$ respectively, that is $\mathcal{F}$ and $\mathcal{T}$ continuous mappings. Since $\|\cdot\|_B$ and the maximum norm $\|\cdot\|$ are equivalent, these mappings are also continuous w.r.t. the maximum norm. Then we consider a mapping

$$\Upsilon:\big(C(D,\mathbb{R}),\|\cdot\|\big)\to\big(C(D,\mathbb{R}),\|\cdot\|\big)$$

which is given as

$$\Upsilon u(x,t)=A_3(u,\mathcal{F}u,\mathcal{T}u)(x,t).$$

The continuity of Υ is deduced from the continuity of f and the mappings $\mathcal{F}$ and $\mathcal{T}$. For each $u\in C(D,\mathbb{R})$,

$$\big\|\Upsilon u\big\|\leq\big\|\mu\big\|+abM.$$

Let $(x_1,t_1),(x_2,t_2)\in D$, $x_1<x_2$, $t_1<t_2$

$$\big|\Upsilon u(x_1,t_1)-\Upsilon u(x_2,t_2)\big|\leq\big|\varphi(x_1)-\varphi(x_2)\big|+\big|\psi(t_1)-\psi(t_2)\big|$$
$$+M\Big(b\big|x_1-x_2\big|+a\big|t_1-t_2\big|\Big)$$

Hence $\Upsilon\big(C(D,\mathbb{R})\big)$ is a bounded and equicontinuous subset of $C(D,\mathbb{R})$. The Ascoli-Arzela theorem indicates that it is relatively compact. Then there exists an $u^*\in C(D,\mathbb{R})$ that equals to Υu^* according to Schauder's FPT. The triple of $(u^*,\mathcal{F}u^*,\mathcal{T}u^*)$ is solution of our problem (3.17)-(3.18). We notice that

$$u^*=\Upsilon u^*$$
$$(\mathcal{F}u^*,\mathcal{T}u^*)=\mathcal{P}_{u^*}(\mathcal{F}u^*,\mathcal{T}u^*),$$

or

$$u^*(x,t) = \mu(x,t) + \int_0^x \int_0^t f\big(r,s,u^*(r,s),\mathcal{F}u^*(r,s),\mathcal{T}u^*(r,s)\big)dsdr$$
$$\mathcal{F}u^*(x,t) = \varphi'(x) + \int_0^t f\big(x,s,u^*(x,s),\mathcal{F}u^*(x,s),\mathcal{T}u^*(x,s)\big)ds$$
$$\mathcal{T}u^*(x,t) = \psi'(t) + \int_0^x f\big(r,t,u^*(r,t),\mathcal{F}u^*(r,t),\mathcal{T}u^*(r,t)\big)dr.$$

Thus the proof is completed. ■

As an application of our result, we give an illustrative example as follows;

Example 3.7. Let us consider the given equation

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = \cos(u^2(x,t)) + \frac{e^{x+t}}{1+\left|\frac{\partial u(x,t)}{\partial x}\right|}, \quad (x,t) \in [0,5]\times[0,7],$$

with

$$\begin{cases} u(x,0) = 1+x, & x\in[0,5] \\ u(0,t) = \cos(t), & t\in[0,7]. \end{cases}$$

Set $f(x,t,u,p,q) = \cos(u^2) + \frac{e^{x+t}}{1+|p|}$. It is clear that f is bounded and satisfies

$$|f(x,t,u,p_1,q_1) - f(x,t,u,p_2,q_2)| \le e^2\Big(|p_1-p_2| + |q_1-q_2|\Big)$$

for all $u, p_i, q_i \in \mathbb{R}$ $(i=1,2)$ and $(x,t)\in[0,5]\times[0,7]$. Then there is at least one solution of the above problem by Theorem 3.6.

4
STABILITY RESULTS FOR NONLINEAR HYPERBOLIC PDEs

This chapter is devoted to the derivation of stability results for the nonlinear hyperbolic PDEs for which the E&U of solutions is discussed in the previous chapter. At this stage, the Bielecki norm, the Wendorff lemma, and the abstract Gronwall lemma from Picard operator theory serve as tools for obtaining stability findings.

4.1 Stability of Nonlinear Hyperbolic PDEs with Two Delays

In this section, a stability result in the sense of UH is obtained for the following hyperbolic PDEs:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f(x,t,u(x,t),u(x-\alpha,t-\beta)), \quad (x,t) \in D. \tag{4.1}$$

The E&U of solutions for this class of equations is analysed in Section 3.1. Before stating the relevant stability result, the definition of UH stability is given below:

Definition 4.1. The equation (4.1) is said to be UH stable when there exists a constant $\mathcal{C} > 0$ such that the following statement is true for any $\epsilon > 0$: If $\vartheta \in C^{1,2}(D,\mathbb{R}) \cap C([-\alpha,a] \times [-\beta,b],\mathbb{R})$ satisfies the inequality

$$\left| \frac{\partial^2 \vartheta(x,t)}{\partial x \partial t} - f\Big(x,t,\vartheta(x,t),\vartheta(x-\alpha,t-\beta)\Big) \right| \leq \epsilon, \quad (x,t) \in D \tag{4.2}$$

then there exists a solution u of the equation (4.1) fulfilling

$$\Big|\vartheta(x,t) - u(x,t)\Big| \leq \mathcal{C}\epsilon, \quad (x,t) \in [-\alpha,a] \times [-\beta,b].$$

Theorem 4.1. *Suppose the following conditions hold:*

i) $f \in C(D \times \mathbb{R}^2,\mathbb{R})$.

ii) There exist $L_1, L_2 > 0$ which ensure that

$$\left|f(x,t,u_1,v_1) - f(x,t,u_2,v_2)\right| \leq L_1\left|u_1 - u_2\right| + L_2\left|v_1 - v_2\right|,$$

for $(x,t) \in D$ and $u_i, v_i \in \mathbb{R}$, $i = 1, 2$.

Then, the equation (4.1) is UH stable.

Proof. Let $\vartheta \in C^{1,2}(D,\mathbb{R}) \cap C([-\alpha, a] \times [-\beta, b], \mathbb{R})$ fulfils the inequality (4.2), that is

$$\left|\frac{\partial^2 \vartheta(x,t)}{\partial x \partial t} - f\Big(x, t, \vartheta(x,t), \vartheta(x-\alpha, t-\beta)\Big)\right| \leq \epsilon, \quad (x,t) \in D.$$

By Theorem 3.1, we indicate u as a unique solution to the proposed equation:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\Big(x, t, u(x,t), u(x-\alpha, t-\beta)\Big), \quad (x,t) \in D$$

with $u(x,t) = \vartheta(x,t)$ on $\tilde{D}$. Equivalently,

$$\begin{aligned} u(x,t) =& \vartheta(x,0) + \vartheta(0,t) - \vartheta(0,0) \\ &+ \int_0^x \int_0^t f\Big(r, s, u(r,s), u(r-\alpha, s-\beta)\Big) ds dr \end{aligned}$$

for $(x,t) \in D$. It is also evident that $\left|\vartheta(x,t) - u(x,t)\right| = 0$ for $(x,t) \in \tilde{D}$. Since ϑ holds the inequality (4.2), then there is a function $k \in C(D,\mathbb{R})$ such that

$$\left|k(x,t)\right| \leq \epsilon \quad \text{and} \quad \frac{\partial^2 \vartheta(x,t)}{\partial x \partial t} = f\Big(x, t, \vartheta(x,t), \vartheta(x-\alpha, t-\beta)\Big) + k(x,t).$$

Clearly, the following inequality can be derived for $(x,t) \in D$

$$\begin{aligned} \Big|\vartheta(x,t) - &\vartheta(x,0) - \vartheta(0,t) + \vartheta(0,0) \\ &- \int_0^x \int_0^t f\Big(r, s, \vartheta(r,s), \vartheta(r-\alpha, s-\beta)\Big) ds dr\Big| \leq \epsilon x t. \end{aligned}$$

Furthermore, we obtain the following based on condition ii)

$$\begin{aligned}
&\big|\vartheta(x,t)-u(x,t)\big| \qquad (4.3)\\
\leq&\Big|\vartheta(x,t)-\vartheta(x,0)-\vartheta(0,t)+\vartheta(0,0)\\
&-\int_0^x\int_0^t f\Big(r,s,\vartheta(r,s),\vartheta(r-\alpha,s-\beta)\Big)dsdr\Big|\\
&+\int_0^x\int_0^t \Big|f\Big(r,s,\vartheta(r,s),\vartheta(r-\alpha,s-\beta)\Big)\\
&-f\Big(r,s,u(r,s),u(r-\alpha,s-\beta)\Big)\Big|dsdr\\
\leq&\epsilon xt+\int_0^x\int_0^t L_1\big|\vartheta(r,s)-u(r,s)\big|\\
&+L_2\big|\vartheta(r-\alpha,s-\beta)-u(r-\alpha,s-\beta)\big|dsdr.
\end{aligned}$$

For $\omega\in C([-\alpha,a]\times[-\beta,b],\mathbb{R}_+)$, we define

$$A: C([-\alpha,a]\times[-\beta,b],\mathbb{R}_+)\to C([-\alpha,a]\times[-\beta,b],\mathbb{R}_+)$$

by

$$A\omega(x,t)=\begin{cases}\epsilon xt+\int_0^x\int_0^t L_1\omega(r,s)+L_2\omega(r-\alpha,s-\beta)dsdr & (x,t)\in D\\ 0 & (x,t)\in\tilde{D}\end{cases}$$

To establish that A is a Picard operator, we demonstrate that it is a contraction mapping on $C([-\alpha,a]\times[-\beta,b],\mathbb{R}_+)$ equipped with the Bielecki norm:

$$\big\|\omega\big\|_B=\max_{(x,t)\in[-\alpha,a]\times[-\beta,b]} e^{-\theta(x+t)}\big|\omega(x,t)\big|.$$

For $\omega,\overline{\omega}\in C([-\alpha,a]\times[-\beta,b],\mathbb{R}_+)$,

$$\begin{aligned}
\big\|A\omega-A\overline{\omega}\big\|_B\leq&\max_{(x,t)\in D} e^{-\theta(x+t)}\int_0^x\int_0^t L_1\big|\omega(r,s)-\overline{\omega}(r,s)\big|\\
&+L_2\big|\omega(r-\alpha,s-\beta)-\overline{\omega}(r-\alpha,s-\beta)\big|dsdr\\
\leq&(L_1+L_2)\big\|\omega-\overline{\omega}\big\|_B\max_{(x,t)\in D} e^{-\theta(x+t)}\int_0^x\int_0^t e^{\theta(r+s)}dsdr\\
\leq&\frac{(L_1+L_2)}{\theta^2}\big\|\omega-\overline{\omega}\big\|_B.
\end{aligned}$$

If $\theta>0$ is chosen large enough so that $(L_1+L_2)<\theta^2$, A is a contraction w.r.t. the Bielecki norm on $C([-\alpha,a]\times[-\beta,b],\mathbb{R}_+)$. Therefore, A is a Picard operator and the below equality is valid by means of the Banach FPT.

$$\omega^*(x,t) = \epsilon xt + \int_0^x \int_0^t L_1\omega^*(r,s) + L_2\omega^*(r-\alpha, s-\beta)dsdr, \quad (x,t) \in D.$$

Additionally, we observe that $\omega^*(r-\alpha, s-\beta) \leq \omega^*(r,s)$ due to the solution ω^* is increasing. Obviously, then

$$\omega^*(x,t) \leq \epsilon xt + (L_1+L_2)\int_0^x \int_0^t \omega^*(r,s)dsdr. \tag{4.4}$$

Now applying the Wendorff lemma to the inequality (4.4), we get

$$\omega^*(x,t) \leq \epsilon abe^{(L_1+L_2)ab},$$

for all $(x,t) \in [-\alpha, a] \times [-\beta, b]$. Particularly, $\omega \leq A\omega$ if we select $\omega = \left|\vartheta - u\right|$ in (4.3). This means that as a result of A being an increasing Picard operator, the inequality $\omega \leq \omega^*$ is satisfied by the abstract Gronwall lemma. Consequently, we get

$$\left|\vartheta(x,t) - u(x,t)\right| \leq \mathcal{C}\epsilon, \quad \text{where} \quad \mathcal{C} = abe^{(L_1+L_2)ab},$$

for all $(x,t) \in [-\alpha, a] \times [-\beta, b]$. Thus we proved that our delayed hyperbolic partial differential equation (4.1) is UH stable.

■

4.2 Stability of Nonlinear Hyperbolic Functional PDEs

This part examines the stability in the sense of UH and UHR for the following equations, whose E&U results are already proven in Section 3.2, on both bounded and unbounded domains

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\Big(x,t,u\big(g(x,t),h(x,t)\big)\Big), \quad (x,t) \in D. \tag{4.5}$$

Let us now give the stability definitions for the proposed equation.

Definition 4.2. If for $\vartheta \in C^{1,2}(D,\mathbb{R})$ satisfying the inequality

$$\left|\frac{\partial^2 \vartheta(x,t)}{\partial x \partial t} - f\Big(x,t,\vartheta\big(g(x,t),h(x,t)\big)\Big)\right| \leq \Phi(x,t), \tag{4.6}$$

there exists a solution $u \in C^{1,2}(D,\mathbb{R})$ of the equation (4.5) and a positive number $\mathcal{C}$ with

$$\left|\vartheta(x,t) - u(x,t)\right| \leq \mathcal{C}\Phi(x,t), \quad (x,t) \in D,$$

then we say that the equation (4.5) is UHR stable w.r.t. $\Phi \in C(D, \mathbb{R}_+)$.

Especially if Definition 4.2 is provided for each positive constant instead of Φ in above inequalities, we say that the equation (4.5) is UH stable.

Remark 4.1. A function $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfies the inequality (4.6) iff there is a function $k \in C(D, \mathbb{R})$ which satisfies

$$\big|k(x,t)\big| \leq \Phi(x,t) \quad \text{and} \quad \frac{\partial^2 \vartheta(x,t)}{\partial x \partial t} = f\Big(x, t, \vartheta\big(g(x,t), h(x,t)\big)\Big) + k(x,t).$$

Remark 4.2. If $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfies the inequality (4.6), it also satisfies the integral inequality:

$$\Bigg|\vartheta(x,t) - \vartheta(x,0) - \vartheta(0,t) + \vartheta(0,0) - \int_0^x \int_0^t f\Big(r, s, \vartheta\big(g(r,s), h(r,s)\big)\Big) ds dr\Bigg| \leq \int_0^x \int_0^t \Phi(r,s) ds dr.$$

Note that if we replace D by $\mathbb{R}_+^2$, analogously we have the aforementioned definitions and remarks.

4.2.1 Stability results on bounded domain

In this subsection, we present two stability results for the equation (4.5) on bounded domain. First, we prove a UH stability result in Theorem 4.2 by using the effectiveness of Bielecki norm. In Theorem 4.3, we prove a UHR stability result inspired by Otrocol and Ilea's paper [34].

Theorem 4.2. *The equation (4.5) is UH stable under the conditions (C1) and (C2) in Theorem 3.2.*

Proof. Let $\epsilon > 0$ and $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfy the following inequality

$$\left|\frac{\partial^2 \vartheta(x,t)}{\partial x \partial t} - f\Big(x, t, \vartheta\big(g(x,t), h(x,t)\big)\Big)\right| \leq \epsilon.$$

Based on Theorem 3.2, a unique solution (referred to as u) can be found for the equation (4.5) with the following conditions:

$$\begin{cases} u(x,0) = \vartheta(x,0), & x \in [0,a] \\ u(0,t) = \vartheta(0,t), & t \in [0,b] \end{cases} \tag{4.7}$$

Hence, the solution $u(x,t)$ satisfies

$$u(x,t) = \vartheta(x,0) + \vartheta(0,t) - \vartheta(0,0) + \int_0^x \int_0^t f\Big(r, s, u\big(g(r,s), h(r,s)\big)\Big) dsdr.$$

From Remark 4.2, we derive the inequality

$$\begin{aligned}
&\big|\vartheta(x,t) - u(x,t)\big| \\
&\quad \leq \Bigg|\vartheta(x,t) - \vartheta(x,0) - \vartheta(0,t) + \vartheta(0,0) \\
&\qquad - \int_0^x \int_0^t f\Big(r, s, \vartheta\big(g(r,s), h(r,s)\big)\Big) dsdr\Bigg| \\
&\qquad + \int_0^x \int_0^t \Big|f\Big(r, s, \vartheta\big(g(r,s), h(r,s)\big)\Big) - f\Big(r, s, u\big(g(r,s), h(r,s)\big)\Big)\Big| dsdr \\
&\quad \leq \epsilon xt + L \int_0^x \int_0^t \Big|\vartheta\big(g(r,s), h(r,s)\big) - u\big(g(r,s), h(r,s)\big)\Big| dsdr \\
&\quad \leq \epsilon ab + L\big\|\vartheta - u\big\|_B \int_0^x \int_0^t e^{\theta(r+s)} dsdr \\
&\quad \leq \epsilon ab + \frac{L}{\theta^2}\big\|\vartheta - u\big\|_B e^{\theta(x+t)},
\end{aligned}$$

which implies that

$$\Big(1 - \frac{L}{\theta^2}\Big)\big\|\vartheta - u\big\|_B \leq \epsilon ab.$$

By taking $\theta > 0$ sufficiently large so that $\theta^2 > L$, then

$$\big|\vartheta(x,t) - u(x,t)\big| e^{-\theta(x+t)} \leq \big\|\vartheta - u\big\|_B \leq \frac{\epsilon ab}{1 - L/\theta^2}.$$

Consequently, we get

$$\big|\vartheta(x,t) - u(x,t)\big| \leq \mathcal{C}\epsilon, \quad \mathcal{C} := \frac{abe^{\theta(a+b)}}{1 - L/\theta^2}.$$

for all $(x,t) \in D$. Thus the equation (4.5) is UH stable. ■

Theorem 4.3. *If the conditions (C1) and (C2) in Theorem 3.2 and the following condition hold:*

(C5) There exists $\lambda > 0$ which ensures that

$$\int_0^x \int_0^t \Phi(r,s) dsdr \leq \lambda \Phi(x,t), \quad (x,t) \in D.$$

Then the equation (4.5) is UHR stable w.r.t. $\Phi \in C(D, \mathbb{R}_+)$.

Proof. Let $\vartheta \in C^{1,2}(D,\mathbb{R})$ satisfy the inequality (4.6) and let u represent a solution for the equation (4.5)-(4.7). From Remark 4.2, we derive the inequality

$$\begin{aligned}
&\big|\vartheta(x,t) - u(x,t)\big| \\
&\quad \leq \bigg|\vartheta(x,t) - \vartheta(x,0) - \vartheta(0,t) + \vartheta(0,0) \\
&\qquad - \int_0^x \int_0^t f\Big(r,s,\vartheta\big(g(r,s),h(r,s)\big)\Big) dsdr \bigg| \\
&\qquad + \int_0^x \int_0^t \Big| f\Big(r,s,\vartheta\big(g(r,s),h(r,s)\big)\Big) - f\Big(r,s,u\big(g(r,s),h(r,s)\big)\Big)\Big| dsdr \\
&\quad \leq \int_0^x \int_0^t \Phi(r,s) dsdr + L \int_0^x \int_0^t \Big|\vartheta\big(g(r,s),h(r,s)\big) \\
&\qquad - u\big(g(r,s),h(r,s)\big)\Big| dsdr.
\end{aligned}$$

For $\omega \in C(D,\mathbb{R}_+)$, we define $\mathcal{P}: C(D,\mathbb{R}_+) \to C(D,\mathbb{R}_+)$

$$\mathcal{P}(\omega)(x,t) = \int_0^x \int_0^t \Phi(r,s) dsdr + L \int_0^x \int_0^t \omega\big(g(r,s),h(r,s)\big) dsdr.$$

Considering the Bielecki norm defined in (3.9), the following inequality can be obtained as in Theorem 3.2 for any $\omega, \overline{\omega}$ in $C(D,\mathbb{R}_+)$

$$\big\|\mathcal{P}\omega - \mathcal{P}\overline{\omega}\big\|_B \leq \delta \big\|\omega - \overline{\omega}\big\|_B \quad \text{where} \quad \delta = \frac{L}{\theta^2}.$$

Taking $\theta > 0$ sufficiently large such that $\delta < 1$, we get that $\mathcal{P}$ is a Picard operator ($F_{\mathcal{P}} = \{\omega^*\}$). Then the following equality holds by Banach's FPT

$$\omega^*(x,t) = \int_0^x \int_0^t \Phi(r,s) dsdr + L \int_0^x \int_0^t \omega^*\big(g(r,s),h(r,s)\big) dsdr.$$

Since ω^* is increasing, one get $\omega^*\big(g(r,s),h(r,s)\big) \leq \omega^*(r,s)$ due to $g(r,s) \leq r$ and $h(r,s) \leq s$, which yields that

$$\omega^*(x,t) \leq \int_0^x \int_0^t \Phi(r,s) dsdr + L \int_0^x \int_0^t \omega^*(r,s) dsdr.$$

Applying the Wendorff lemma to the above inequality, we obtain from the condition (C5) that

$$\omega^*(x,t) \leq e^{xtL} \int_0^x \int_0^t \Phi(r,s) dsdr \leq \lambda e^{xtL} \Phi(x,t), \quad (x,t) \in D.$$

Specifically, it is possible to have $\omega \leq \mathcal{P}\omega$ choosing $\omega = \big|\vartheta - u\big|$. Since $\mathcal{P}$ is an increasing Picard operator, the inequality $\omega \leq \omega^*$ is obtained by the abstact

Gronwall lemma. As a result, we get

$$\big|\vartheta(x,t)-u(x,t)\big| \le \mathcal{C}\Phi(x,t), \quad \mathcal{C}:=\lambda e^{abL}.$$

Hence the equation (4.5) is UHR stable w.r.t. $\Phi \in C(D,\mathbb{R}_+)$. ■

4.2.2 Stability result on unbounded domain

Theorem 4.4. *Suppose that the conditions (C3)-(C4) in Theorem 3.3 and the following condition hold:*

(C6) There exists $\lambda > 0$ which ensures that

$$\int_0^x \int_0^t \Phi(r,s)dsdr \le \lambda\Phi(x,t), \quad (x,t)\in\mathbb{R}_+^2.$$

If the double integration of $\mathbb{L}$ in (C4) is finite, the equation (4.5) is UHR stable w.r.t. $\Phi \in C(\mathbb{R}_+^2,\mathbb{R}_+)$.

Proof. Let $\vartheta \in C^{1,2}(\mathbb{R}_+^2,\mathbb{R})$ satisfy the inequality (4.6). One can obtain the unique solution (denoted by u) of the following equation from Theorem 3.3

$$\frac{\partial^2 u(x,t)}{\partial x\partial t} = f\Big(x,t,u\big(g(x,t),h(x,t)\big)\Big), \quad (x,t)\in\mathbb{R}_+^2,$$

with

$$\begin{cases} u(x,0)=\vartheta(x,0) \\ u(0,t)=\vartheta(0,t). \end{cases}$$

For arbitrary $(x,t)\in\mathbb{R}_+^2$, there is an $n\in\mathbb{N}$ such that $(x,t)\in D_n$. By considering the restrictions on the domain D_n of functions ϑ and u, it can be deduced that

$$\big|\vartheta(x,t)-u(x,t)\big| = \Big|\vartheta\mid_{D_n}(x,t) - u\mid_{D_n}(x,t)\Big|$$

(in view of Theorem 4.3)

$$\begin{aligned} &\le \lambda\Phi(x,t)\exp\Big(\int_0^x\int_0^t \mathbb{L}(r,s)dsdr\Big) \\ &= \mathcal{C}\Phi(x,t), \end{aligned}$$

where $\mathcal{C} = \lambda\exp\Big(\int_0^\infty\int_0^\infty \mathbb{L}(r,s)dsdr\Big)$. Hence the equation (4.5) is UHR stable w.r.t. $\Phi\in C(\mathbb{R}_+^2,\mathbb{R}_+)$. ■

To support our theoretical findings, let us now give some examples.

Example 4.5. Consider the following equation

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = \frac{\sin xt}{1 + \left|u(xt, t^2)\right|}, \quad (x,t) \in [0,6] \times [0,1] \tag{4.8}$$

with

$$\begin{cases} u(x,0) = e^x, & x \in [0,6] \\ u(0,t) = \cos t, & t \in [0,1]. \end{cases} \tag{4.9}$$

In Theorem 3.2, we set

$$f\big(x, t, u\big(g(x,t), h(x,t)\big)\big) = \frac{\sin xt}{1 + \left|u\big(g(x,t), h(x,t)\big)\right|}, \quad (x,t) \in [0,6] \times [0,1]$$

where $g(x,t) = xt$ and $h(x,t) = t^2$. For each $u, \overline{u} \in \mathbb{R}$ and $(x,t) \in [0,6] \times [0,1]$, it is obvious that

$$\left|f(x,t,u) - f(x,t,\overline{u})\right| \le \left|u - \overline{u}\right|.$$

Then, by Theorem 3.2, the equation (4.8)-(4.9) has a unique solution on $[0,6] \times [0,1]$. Moreover, applying Theorem 4.2, we obtain that the equation (4.8) is stable in the sense of UH. Taking $k(x,t) = xt$ in the condition (C5) of Theorem 4.3,

$$\int_0^x \int_0^t k(r,s)dsdr \le \frac{3}{2}xt := \lambda k(x,t).$$

So the condition (C5) is satisfied with $\lambda = \frac{3}{2}$. Hence this equation is also UHR stable w.r.t. k.

Example 4.6. Consider the following equation

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = \frac{x \cos t + u(x, te^{-x})}{e^{x+t}}, \quad (x,t) \in \mathbb{R}_+^2 \tag{4.10}$$

with

$$\begin{cases} u(x,0) = x^2, & x \in \mathbb{R}_+ \\ u(0,t) = t, & t \in \mathbb{R}_+. \end{cases} \tag{4.11}$$

In Theorem 3.3, let us take

$$f\big(x, t, u\big(g(x,t), h(x,t)\big)\big) = \frac{x \cos t + u\big(g(x,t), h(x,t)\big)}{e^{x+t}}, \quad (x,t) \in \mathbb{R}_+^2$$

with $g(x,t) = x$ and $h(x,t) = te^{-x}$. For each $u, \overline{u} \in \mathbb{R}$ and $(x,t) \in \mathbb{R}_+^2$, we have

$$\left|f(x,t,u) - f(x,t,\overline{u})\right| \le \mathbb{L}(x,t)\left|u - \overline{u}\right| \quad \text{where} \quad \mathbb{L}(x,t) = e^{-(x+t)}.$$

Then, it follows from Theorem 3.3 that the equation (4.10)-(4.11) has a unique solution on $\mathbb{R}^2_+$. Also if we denote $k(x,t) = e^{x+t}$ in the condition (C6) of Theorem 4.4, we get

$$\int_0^x \int_0^t k(r,s)dsdr \leq e^{x+t} := \lambda k(x,t).$$

Therefore the condition (C6) is satisfied with $\lambda = 1$. Hence the equation (4.10) is UHR stable w.r.t. k.

In Examples 4.5 and 4.6, we have illustrated the existence, uniqueness and stability results on both bounded and unbounded domains.

4.3 Stability of Nonlinear Implicit Hyperbolic PDEs

In addition to the E&U results obtained for the following types of equations in Section 3.3, the stability of such equations is investigated in this current section

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\left(x, t, u(x,t), \frac{\partial^2 u(x,t)}{\partial x \partial t}\right), \quad (x,t) \in D. \tag{4.12}$$

The stability concepts for the problem under consideration are as follows:

Definition 4.3. If for $\vartheta \in C^{1,2}(D, \mathbb{R})$ satisfying the inequality

$$\left| \frac{\partial^2 \vartheta(x,t)}{\partial x \partial t} - f\left(x, t, \vartheta(x,t), \frac{\partial^2 \vartheta(x,t)}{\partial x \partial t}\right) \right| \leq \Phi(x,t), \tag{4.13}$$

there exists a solution $u \in C^{1,2}(D, \mathbb{R})$ of the equation (4.12) and a positive number $\mathcal{C}$ with

$$\left|\vartheta(x,t) - u(x,t)\right| \leq \mathcal{C}\Phi(x,t), \quad (x,t) \in D,$$

then we say that the equation (4.12) is UHR stable w.r.t. $\Phi \in C(D, \mathbb{R}_+)$.

Especially if Definition 4.3 is provided for each positive constant instead of Φ in above inequalities, we say that the equation (4.12) is UH stable.

Theorem 4.7. *Suppose the following condition holds:*

(H5) There exists $\Delta_\Phi > 0$ which ensures that

$$\int_0^x \int_0^t \Phi(r,s)dsdr \leq \Delta_\Phi \Phi(x,t) \quad (x,t) \in D.$$

Under the assumptions (H1) and (H4) in Theorem 3.5, the equation (4.12) is UHR stable w.r.t. $\Phi \in C(D, \mathbb{R}_+)$.

Proof. Let $\vartheta \in C^{1,2}(D,\mathbb{R})$ be a solution of the inequality (4.13). According to Theorem 3.5, there exists a unique solution (denoted by u) for the equation (4.12) with the following conditions:

$$\begin{cases} u(x,0) = \vartheta(x,0) \\ u(0,t) = \vartheta(0,t). \end{cases}$$

Then, we derive from Lemma 3.1 that

$$u(x,t) = \vartheta(x,0) + \vartheta(0,t) - \vartheta(0,0) + \int_0^x \int_0^t g_u(r,s)dsdr$$

where $g_u \in C(D,\mathbb{R})$ satisfies the functional equation

$$g_u(x,t) = f\big(x,t,u(x,t),g_u(x,t)\big). \tag{4.14}$$

Since $\vartheta \in C^{1,2}(D,\mathbb{R})$ holds the inequality (4.13), there is a function $k \in C(D,\mathbb{R})$ such that

$$\frac{\partial^2\vartheta(x,t)}{\partial x\partial t} = f\left(x,t,\vartheta(x,t),\frac{\partial^2\vartheta(x,t)}{\partial x\partial t}\right) + k(x,t) \quad \text{where} \quad \big|k(x,t)\big| \le \Phi(x,t).$$

Again in the light of Lemma 3.1, we can express the given ϑ as follows:

$$\vartheta(x,t) = \vartheta(x,0) + \vartheta(0,t) - \vartheta(0,0) + \int_0^x \int_0^t g_\vartheta(r,s)dsdr$$

where $g_\vartheta \in C(D,\mathbb{R})$ satisfies the functional equation

$$g_\vartheta(x,t) = f\big(x,t,\vartheta(x,t),g_\vartheta(x,t)\big) + k(x,t). \tag{4.15}$$

Then we have

$$\big|\vartheta(x,t) - u(x,t)\big| \le \int_0^x \int_0^t \big|g_\vartheta(r,s) - g_u(r,s)\big|dsdr$$

where g_u and g_ϑ are as stated in (4.14) and (4.15). By (H4), it is evident that

$$\begin{aligned}\big|g_\vartheta(x,t) - g_u(x,t)\big| \le& \big|k(x,t)\big| + \big|f\big(x,t,\vartheta(x,t),g_\vartheta(x,t)\big) \\ &- f\big(x,t,u(x,t),g_u(x,t)\big)\big| \\ \le& \Phi(x,t) + K\big|\vartheta(x,t) - u(x,t)\big| + L\big|g_\vartheta(x,t) - g_u(x,t)\big|\end{aligned}$$

which yields that

$$\left|g_{\vartheta}(x,t)-g_{u}(x,t)\right| \leq \frac{\Phi(x,t)}{1-L}+\frac{K}{1-L}\left|\vartheta(x,t)-u(x,t)\right|.$$

Hence, we get that

$$\left|\vartheta(x,t)-u(x,t)\right| \leq \int_0^x\int_0^t \frac{\Phi(r,s)}{1-L}dsdr+\frac{K}{1-L}\int_0^x\int_0^t \left|\vartheta(r,s)-u(r,s)\right|dsdr.$$

The application of the Wendorff lemma to the above inequality yields that

$$\begin{aligned}\left|\vartheta(x,t)-u(x,t)\right| &\leq \exp\left(\frac{abK}{1-L}\right)\int_0^x\int_0^t \frac{\Phi(r,s)}{1-L}dsdr\\ &\leq \exp\left(\frac{abK}{1-L}\right)\frac{\Delta_{\Phi}\Phi(x,t)}{1-L} \quad \text{by the hypothesis (H5).}\end{aligned}$$

Consequently, the following inequality is satisfied

$$\left|\vartheta(x,t)-u(x,t)\right| \leq \left[\exp\left(\frac{abK}{1-L}\right)\frac{\Delta_{\Phi}}{1-L}\right]\Phi(x,t) := \mathcal{C}_{\Phi}\Phi(x,t).$$

Thus the equation (4.12) is UHR stable w.r.t. $\Phi \in C(D,\mathbb{R}_{+})$. ■

Theorem 4.8. *Under the assumptions (H1) and (H4) of Theorem 3.5, the equation (4.12) is UH stable.*

Proof. Let $\vartheta \in C^{1,2}(D,\mathbb{R})$ satisfy the inequality:

$$\left|\frac{\partial^2\vartheta(x,t)}{\partial x\partial t}-f\left(x,t,\vartheta(x,t),\frac{\partial^2\vartheta(x,t)}{\partial x\partial t}\right)\right| \leq \epsilon, \quad \epsilon>0. \tag{4.16}$$

And we denote by $u \in C^{1,2}(D,\mathbb{R})$ the unique solution to the equation (4.12) under the conditions:

$$\begin{cases} u(x,0)=\vartheta(x,0)\\ u(0,t)=\vartheta(0,t).\end{cases}$$

In the same way as the proof of Theorem 4.7, we can easily observe that the following is valid

$$\left|\vartheta(x,t)-u(x,t)\right| \leq \frac{\epsilon ab}{1-L}+\frac{K}{1-L}\int_0^x\int_0^t \left|\vartheta(r,s)-u(r,s)\right|dsdr.$$

By considering the Bielecki norm given as follows:

$$\left\|z\right\|_B=\max_{(x,t)\in D}\left|z(x,t)\right|e^{-\theta(x+t)} \quad \text{where} \quad \theta>0,$$

we get

$$\begin{aligned}\left|\vartheta(x,t)-u(x,t)\right| \leq & \frac{\epsilon ab}{1-L}+\frac{K}{1-L}\int_0^x\int_0^t e^{\theta(r+s)} \\ & \times\left(e^{-\theta(r+s)}\left|\vartheta(r,s)-u(r,s)\right|\right)dsdr \\ \leq & \frac{\epsilon ab}{1-L}+\frac{K}{(1-L)\theta^2}\left\|\vartheta-u\right\|_B e^{\theta(x+t)}.\end{aligned}$$

Then we have

$$(1-\Lambda)\left\|\vartheta-u\right\|_B \leq \frac{\epsilon ab}{1-L} \quad \text{where} \quad \Lambda := \frac{K}{(1-L)\theta^2}.$$

Taking $\theta > 0$ large enough so that $\Lambda < 1$, we get

$$\left|\vartheta(x,t)-u(x,t)\right|e^{-\theta(x+t)} \leq \left\|\vartheta-u\right\|_B \leq \frac{\epsilon ab}{(1-\Lambda)(1-L)}.$$

It follows that

$$\left|\vartheta(x,t)-u(x,t)\right| \leq \mathcal{C}\epsilon, \quad \mathcal{C} := \frac{abe^{\theta(a+b)}}{(1-\Lambda)(1-L)}$$

for all $(x,t) \in D$. Thus, the equation (4.12) is UH stable. ■

We illustrate our theoretical results in this section with two examples to support our findings.

Example 4.9. Let us consider the given equation

$$\frac{\partial^2 u(x,t)}{\partial x\partial t} = \frac{e^t\left(x+\left|u(x,t)\right|\right)+\frac{\partial^2 u(x,t)}{\partial x\partial t}}{7+\left|u(x,t)\right|}, \quad (x,t) \in [0,3]\times[0,3] \tag{4.17}$$

with

$$u(x,0)=0 \quad \text{and} \quad u(0,t)=t, \quad x,t \in [0,3]. \tag{4.18}$$

Set

$$f(x,t,u,z) = \frac{e^t\left(x+\left|u\right|\right)+z}{7+\left|u\right|}, \quad x,t \in [0,3] \quad \text{and} \quad u,z \in \mathbb{R}.$$

It is clear that the following is provided

$$\left|f(x,t,u,z)-f(x,t,u,\overline{z})\right| \leq \frac{1}{7}\left|z-\overline{z}\right|$$

and

$$\left|f(x,t,u,z)\right| \leq xe^t + e^t|u| + \frac{1}{7}|z|$$

for all $u, z, \overline{z} \in \mathbb{R}$ and $(x,t) \in [0,3] \times [0,3]$. As a result, the equation (4.17)-(4.18) possesses at least one solution because all of the requirements of Theorem 3.4 are fulfilled.

Example 4.10. Let us consider another equation below

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = \frac{5}{1+\left|u(x,t)\right|} + \frac{1}{2}\sin\left(\frac{\partial^2 u(x,t)}{\partial x \partial t}\right), \quad (x,t) \in [0,1] \times [0,8] \tag{4.19}$$

with

$$u(x,0) = 1, \quad x \in [0,1] \quad \text{and} \quad u(0,t) = e^t, \quad t \in [0,8]. \tag{4.20}$$

Let

$$f(x,t,u,z) = \frac{5}{1+|u|} + \frac{\sin z}{2}, \quad (x,t) \in [0,1] \times [0,8] \quad \text{and} \quad u, z \in \mathbb{R}.$$

For each $(x,t) \in [0,1] \times [0,8]$ and $u, \overline{u}, z, \overline{z} \in \mathbb{R}$, we get

$$\left|f(x,t,u,z) - f(x,t,\overline{u},\overline{z})\right| \leq 5|u - \overline{u}| + \frac{1}{2}|z - \overline{z}|.$$

Consequently, Theorem 3.5 demonstrates that there is only one solution to this problem (4.19)-(4.20) and also the equation (4.19) is UH stable by Theorem 4.8.

5
CONCLUSION

In this thesis, we present E&U and stability results for nonlinear second-order hyperbolic PDEs in canonical form:

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} = f\Big(x, t, u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}\Big) \tag{5.1}$$

which are obtained by the reduction of general second-order hyperbolic equations under appropriate transformations. Due to the lack of a general method for solving nonlinear equations, we consider these types of equations with several functions f individually and tackle each of them with different approaches. We investigate the E&U of solutions for these hyperbolic PDEs in Chapter 3 on the basis of fixed point theory.

More precisely, in Section 3.1, the E&U of solutions for nonlinear hyperbolic PDEs with delays is obtained based on the Banach FPT by considering the RHS of the equation (5.1) as $f\big(x, t, u(x,t), u(x-\alpha, t-\beta)\big)$. Here we extend the method of Burton called "progressive contractions" to two dimensions in contrast to previous studies conducted in one dimension, and apply it to our problem. We use the Bielecki norm to apply Banach's FPT due to the increasing contractivity constants at each step of this proof. Applying Burton's method to our problem in PDEs give us the advantage that the Lipschitz condition on the function f is sufficient only w.r.t. the third variable. Otherwise, it would be necessary to impose a Lipschitz condition on the fourth variable as well.

In Section 3.2, we also apply the Banach FPT to the nonlinear hyperbolic functional PDEs by taking the RHS of the equation (5.1) as $f\big(x, t, u\big(g(x,t), h(x,t)\big)\big)$ to establish the E&U of solutions. Studying in the space equipped with the Bielecki norm, we first derive the existence of a unique solution in the bounded domains based on this theorem. Afterwards, we extend our finding to the unbounded domain based on the unique solutions discovered for the bounded domains.

In Section 3.3, we consider $f\big(x,t,u(x,t),\frac{\partial^2 u(x,t)}{\partial x \partial t}\big)$ to investigate the results concerning the existence of solutions to implicit PDEs and also its uniqueness. Here we provide suitable criteria to guarantee the existence of solutions to our problem. After that we give the uniqueness result using the Wendorff lemma. To emphasise the significance of our approach in the proof of our existence result, we reapply the same technique to the following fractional counterpart of the problem:

$$\overline{\mathcal{D}}^{\rho} u(x,t) = f\big(x,t,u(x,t),\overline{\mathcal{D}}^{\rho} u(x,t)\big).$$

Then the advantage of our result in [48] is outlined below by comparing it with the result of a highly cited paper [25] in the literature:

- While the existence of the solution in Theorem 5.3 of [25] depends on the following condition:

$$d^* + \frac{q^* a^{\rho_1} b^{\rho_2}}{\Gamma(1+\rho_1)\Gamma(1+\rho_2)} < 1$$

  where $\rho = (\rho_1, \rho_2) \in (0,1)^2$ is the order of fractional derivative and Γ is the gamma function, our result requires a weaker condition $d^* < 1$ to establish the existence result.

- Instead of the Lipschitz condition (H4) imposed on f (for more information, see also [49, 50]), we utilize the more general condition (H2) in our result.

We believe that the method applied in the existence result will inspire similar equations in the literature; for instance, see [51].

In Section 3.4, we present a new proof based on the fixed point theory of the existence result proposed by Hartman and Winter [6], which differs from the standard approach in the literature by considering f as indicated in (5.1). Following the idea of using the Banach's FPT to construct well-defined mappings, we provide our proof by meeting the requirements of Schauder's FPT.

In Chapter 4, we deal with stability analyses in the sense of UH and UHR of the first three equations that have been investigated in the previous chapter on the existence of a solution. Together with the obtained existence results, we derive stability results for these equations using Picard operator theory, the Wendorff lemma, and the Bielecki norm which are the main tools for us to perform our analyses.

As a conclusion, by choosing appropriate methods for some nonlinear hyperbolic PDEs, we obtain the E&U and stability results under weaker conditions, more general hypotheses, or with different approaches/techniques compared to similar studies in the literature. In addition, we believe that the approaches applied in their proofs rather than the results themselves will be a source of inspiration for future studies, and we think that this thesis will contribute to the literature in these respects.

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PUBLICATIONS FROM THE THESIS

Papers

1. C. Çelik, F. Develi, "Existence and Hyers–Ulam stability of solutions for a delayed hyperbolic partial differential equation," *Periodica Mathematica Hungarica*, Vol. 84, no. 2, pp. 211-220, 2022.

2. F. Develi, C. Çelik, "A new proof of Hartman and Winter's theorem," *Journal of Fixed Point Theory and Applications*, Vol. 25, no. 1, pp. 1-6, 2023.

3. C. Çelik, F. Develi, "On the stability analysis of Darboux problem on both bounded and unbounded domains," *Applications of Mathematics*, Vol 69, no. 1, pp. 139-150, 2024.

4. C. Çelik, F. Develi, "Sufficient conditions to the existence of solutions and stability for implicit partial hyperbolic differential equations", Under Review

Conference Papers

1. C. Çelik, F. Develi, "Existence and Hyers–Ulam stability of solutions for a delayed hyperbolic partial differential equation", *International Conference on Applied Mathematics in Engineering (ICAME)*, Balikesir, Turkey, pp. 81, 1-3 September, 2021.