ENTANGLEMENT AND INVARIANCE OF QUBIT-QUBIT, QUBIT-QUTRIT AND QUTRIT-QUTRIT QUANTUM STATES

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BETÜL KIZILKAYA

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IZMIR

We approve the thesis of **BETÜL KIZILKAYA**

Examining Committee Members:

Prof. Dr. Oktay PASHAEV
Department of Mathematics, İzmir Institute of Technology

Doç. Dr. Özgür ÇAKIR
Department of Physics, İzmir Institute of Technology

Prof. Dr. Göktuğ KARPAT
Department of Physics, İzmir University of Economics

11 July 2024

Prof. Dr. Oktay PASHAEV
Supervisor, Department of Mathematics
İzmir Institute of Technology

Prof. Dr. Başak AY SAĞLAM
Head of the Department of
Mathematics

Prof. Dr. Mehtap EANES
Dean of the Graduate School

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ABSTRACT

ENTANGLEMENT AND INVARIANCE OF QUBIT-QUBIT, QUBIT-QUTRIT AND QUTRIT-QUTRIT QUANTUM STATES

The present thesis is devoted to studies of entanglement properties of pure two-qubit, qubit-qutrit and two-qutrit states. Entanglement is essentially non-classical characteristics of composite quantum states and it plays a key point in quantum computation and quantum information theory. To characterize entanglement of the states we use the reduced density matrix approach, which relates entanglement of pure composite state with mixed reduced density matrix. Von Neumann entropy of the reduced density matrix and the linear entropy, as squared concurrence are used to quantify entanglement. By using the unitary one qubit and one qutrit gates we show invariance of entanglement under the transformations. This allows us to construct continuously parametrized set of the same level entangled states. We apply the results for calculation of purification for given mixed state and for finding maximally entangled minimum of average energy for two-qubit spin XYZ model in magnetic field.

ÖZET

KÜBİT-KÜBİT, KÜBİT-KÜTRİT VE KÜTRİT-KÜTRİT KUANTUM DURUMLARININ DOLANIKLIĞI VE DEĞİŞMEZLİĞİ

Mevcut tez, saf iki kübit, kübit-kütrit ve iki kütrit durumlarının dolanıklık özelliklerinin incelenmesine ayrılmıştır. Dolanıklık, esasen bileşik kuantum durumlarının klasik olmayan bir özelliğidir ve kuantum hesaplamasında ve kuantum bilgi teorisinde önemli bir rol oynar. Durumların dolanıklığını karakterize etmek için, saf bileşik durumun dolanıklığını karışık azaltılmış yoğunluklu matrisle ilişkilendiren azaltılmış yoğunluk matrisi yaklaşımını kullanırız. Azaltılmış yoğunluk matrisinin Von Neumann entropisi ve karesel eşzamanlılık olarak doğrusal entropi, dolanıklığı ölçmek için kullanılır. Üniter bir kübit ve bir kütrit kapılarını kullanarak, dönüşümler altında dolanıklığın değişmezliğini gösteririz. Bu, aynı seviyedeki dolanık durumların sürekli olarak parametrelendirilmiş kümesini oluşturmamızı sağlar. Sonuçları, verilen karışık durum için arıtmanın hesaplanması ve manyetik alanda iki kübit spin XYZ modeli için ortalama enerjinin maksimum dolaşık minimumunu bulmak için uyguluyoruz.

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CHAPTER 1

INTRODUCTION

The entanglement of quantum states as non-classical and non-local correlation has been introduced by E. Schrodinger in 1935 (Schrödinger 1935, 555–63) and discussed in (Schrödinger 1936, 446–52). He was motivated by Einstein-Podolsky-Rosen (EPR) paradox (Einstein, Podolsky, and Rosen 1935, 777–80).

Experimental test, which was performed with entangled pair of photons (Aspect, Grangier, and Roger 1981, 460–63), showing violation of Bell's inequalities, opened laboratory investigation of entanglement. Alain Aspect, John Clauser, and Anton Zeilinger received the 2022 Nobel Prize in Physics for their work on quantum entanglement of photons. The entanglement becomes a key point in quantum information theory, quantum computers, quantum networks, secure quantum cryptography. Quantum superdense coding enables the transmission of two bits of classical information by manipulating a single entangled qubit out of a pair (Benenti et al. 2018, 7). Another utilization of entanglement is the process of teleporting quantum states (Benenti et al. 2018, 7), which enables the transmission of the quantum state from one system to another, even if they are physically far from one other.

The typical ingredient of entanglement applications is the EPR pair, described by two qubit maximally entangled Bell states. Due to entanglement, it is not possible to assign individual states to both subsystems, but only mixture, determined by reduced density matrix. If composite system of Alice and Bob is associated with Bell state

$$|\psi\rangle = \frac{(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)}{\sqrt{2}},$$
$$CNOT|i_0\rangle|i_1\rangle = |i_0\rangle|i_0 + i_1\rangle,$$

mod 2, where $i_0, i_1 = 0, 1$. By Hadamard H gate and CNOT gate possible to generate all four, maximally entangled Bell states from computational states.

The qubit state as unit of quantum information can be realized as a state in the Hilbert space of a two-level quantum system, corresponding to a binary position system with a base of two. In more realistic physical hardware, the quantum information unit is associated with multi-level quantum system. The qutrit quantum state for three-level quantum system, the ququad quantum state for four-level quantum system and the qudit quantum state for generic d-level quantum system. This requires higher dimensional Hilbert space and allows a more wide range of applications.

The Von Neumann entropy describes the entanglement properties of pure two qubit states, characterized by the linear version of this entropy, and denoted by the real number $0 < C < 1$, which is referred to as the concurrence. The entanglement of real qubit states (the rebit states) can be calculated from the reduced density matrix (Wootters 1998, 2245–48). The geometrical interpretation of the concurrence as the doubled area of the parallelogram, which is determined by two one qubit vectors, is straightforward and is associated with the two qubit states (Parlakgörür and Pashaev 2019, 3).

Similar geometric ideas were applied to concurrence of a real two-qutrit states and a relation between maximally entangled two-qutrit states and Pythagoras' theorem for tetrahedron areas, as de Gua's theorem was established in (Pashaev, Oktay K. 2023, 93–104).

In this thesis we are going to study entanglement characteristics of qubit-qubit, qubit-qutrit and qutri-qutrit states and corresponding invariance properties. The main tool of study is the reduced density matrix and the concurrence. Invariance properties of concurrence for two-qubits, qubit-qutrit and two-qutrit states, which are subject of present thesis, are important, since they allow to count the set of the pure states with the same level of entanglement. This allows from one side, to describe the wide set of purification states, associated with some reduced density matrix, and equally entangled. From another side, it allows to describe physical characteristics of these states in connection with entanglement. In (Pashaev and Gürkan 2012, 13) the set of maximally entangled two-qubit coherent states was constructed. In the limiting case the states reduce to the Bell states and for average energy function of XYZ model Hamiltonian, demonstrate a two- and three-qubit phase space's finite-energy localized structure with local extreme points.

The structure of the thesis is as follows. In Chapter 2 we review basic quantum computations for one qubit states, one-qubit quantum gates and universality of one qubit computations. In Chapter 3 we introduce separable and entangled two-qubit states, linear dependence of characteristic states and corresponding determinant relation. Chapter 4 is devoted to description of entanglement by reduced density matrix. The mathematical optimization problem for description of maximal value of the concurrence is described and solved. In Chapter 5 we define entanglement of a pure state in terms of the von Neumann entropy and concurrence. Chapter 6 is dealing with invariance of concurrence and entanglement under one-qubit unitary gates. In Chapter 7 we describe entanglement of qubit-qutrit states and invariance under unitary, one-qubit gates and one-qutrit gates. Entanglement and concurrence invariance of two qutrit states is subject of Chapter 8. In Chapter 9 we illustrate our results by application for purification of reduced density matrix for mixed states. In Chapter 10 we apply the set of maximally entangled two qubit states to minimize energy average for XYZ spin model in magnetic field. In Conclusions we briefly discuss results of the present thesis.

CHAPTER 2

QUBIT QUANTUM STATES

2.1. Representations of One Qubit Quantum States

Bits are the basic elements of information in classical computing, denoted as either 0 or 1. They can only exist in one state at a time. A qubit, sometimes referred to as a quantum bit, is the fundamental unit of information in quantum computing.(Benenti et al. 2018, 98) The quantum bit is the quantum equivalent of the conventional binary bit and is realized via the use of a two-state device. There are two basis states for the qubit, $|0\rangle$ and $|1\rangle$ in Hilbert space.

According to quantum mechanics, any system of this kind may be expressed in a superposition of states. Generally, the state of a qubit is characterized by sum of two vector in $\mathbb{C}^2$,

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \tag{2.1}$$

corresponding to the computational basis in this space

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $|\psi\rangle$ and $|\varphi\rangle$ are two states and $|\varphi\rangle^\dagger$ is the hermitian one qubit state,

$$\begin{aligned} |\psi\rangle &= c_0 |0\rangle + c_1 |1\rangle, \\ |\varphi\rangle &= d_0 |0\rangle + d_1 |1\rangle, \\ |\varphi\rangle^\dagger &= \langle\varphi| = \bar{d}_0\langle 0| + \bar{d}_1\langle 1|, \\ \langle\varphi \mid \psi\rangle &= \bar{d}_0 c_0 + \bar{d}_1 c_1. \end{aligned}$$

The computational basis vectors are both normalized and orthogonal, making them orthonormal,

$$\langle 0 \mid 0\rangle = 1 = \langle 1 \mid 1\rangle, \langle 0 \mid 1\rangle = 0 = \langle 1 \mid 0\rangle.$$

The normalization condition allows us to understand the geometric interpretation of a qubit state, as shown below:

$$\langle \psi \mid \psi\rangle = |c_0|^2 + |c_1|^2 = 1, \tag{2.2}$$

giving

$$|c_0|^2 + |c_1|^2 = 1,$$

where coefficients $|c_0| = \alpha_0 + i\beta_0$ and $|c_1| = \alpha_1 + i\beta_1$ are complex numbers.It corresponds to the unit sphere by following demonstration. In real variables, equation

$$\alpha_0^2 + \alpha_1^2 + \beta_0^2 + \beta_1^2 = 1,$$

describes the unit sphere in four dimensional real space $\mathbf{S}^3 \in \mathbb{R}^4$, where $(\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{R}^4$. The unit sphere $\mathbf{S}^3$ is reduced to $\mathbf{S}^2 \in \mathbb{R}^3$, by using global phase identification. In polar representation of a complex numbers,

$$c_0 = r_0 e^{i\varphi_0} \quad \text{and} \quad c_1 = r_1 e^{i\varphi_1},$$

the state is written

$$|\psi\rangle = r_0 e^{i\varphi_0} |0\rangle + r_1 e^{i\varphi_1} |1\rangle$$

in terms of four real parameters $r_0, r_1, \varphi_0, \varphi_1$. A quantum state remains unchanged, when it is multiplied by any number of unit norm. $e^{-i\varphi_0}$,

$$e^{-i\varphi_0}|\psi\rangle = e^{-i\varphi_0} \cdot \left(r_0 e^{i\varphi_0}|0\rangle + r_1 e^{i\varphi_1}|1\rangle\right) = r_0|0\rangle + r_1 e^{i(\varphi_1-\varphi_0)}|1\rangle.$$

Then, the state is characterized by only one parameter $\varphi := \varphi_1 - \varphi_0$. Also, we have the unit circle constraint $r_0^2 + r_1^2 = 1$. By using the parametric representation of this circle $r_0 = \cos\frac{\theta}{2}$ and $r_1 = \sin\frac{\theta}{2}$, and substituting into the state, we obtain the equivalent repsentation of $|\psi\rangle$:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{j\varphi}\sin\frac{\theta}{2}|1\rangle.$$

Definition 2.1 A representation of a state that contains one qubit is called the Bloch sphere

$$|\psi\rangle = |\theta, \varphi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle, \tag{2.3}$$

where

$$0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi.$$

In equation (2.3) any point (θ, φ) in S^2, on the Bloch sphere corresponds to quantum state and vice versa. The unit vector on the sphere jumps to the north or south poles, when the generic qubit state is measured, and the associated qubit state collapses to the $|0\rangle$ or $|1\rangle$ state, with the relevant probabilities. The probabilities are entirely determined by the angle θ :

1. The probability of obtaining the state $|0\rangle$ is denoted as $p_0 = |\langle 0 \mid \psi\rangle|^2 = \cos^2\frac{\theta}{2}$
2. The probability of obtaining the state $|1\rangle$ is denoted by $p_1 = |\langle 1 \mid \psi\rangle|^2 = \sin^2\frac{\theta}{2}$.

An addition of these probabilities is one : $p_0 + p_1 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$.

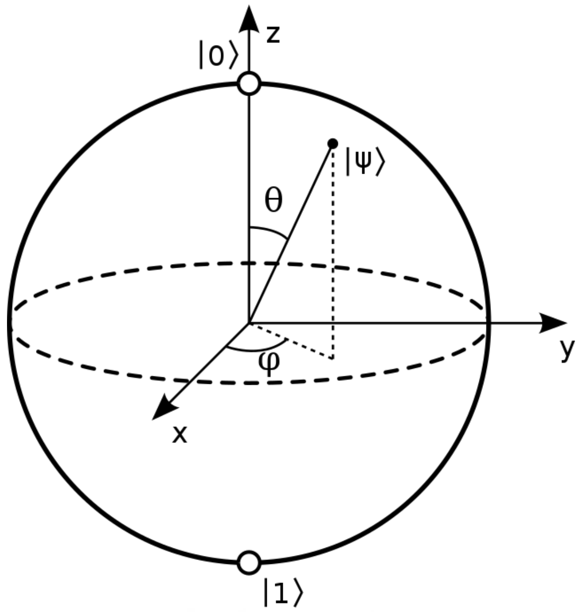

Figure 2.1. The Bloch Sphere

2.2. One Qubit Quantum Gates

2.2.1. Pauli Gates

The Pauli gates are important operations for quantum states(Benenti et al. 2018, 103).

Definition 2.2 The matrix representation of the Pauli gates X, Y, and Z are given with respect to the computational basis.

- The property of the X gate is that it rotates the Bloch Sphere as much as the π radians around the X axis and makes the values $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$.

$$X \equiv \sigma_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.4}$$

The X gate operates as

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle, \quad X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle,$$

or

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle.$$

- Similar to the X gate, Y gate rotates the Bloch Sphere. However, unlike the previous gate, this time the Y axis is rotated and makes the states $|0\rangle \rightarrow -i|1\rangle$ and $|1\rangle \rightarrow i|0\rangle$.

$$Y \equiv \sigma_2 \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

The Y gate operates as

$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|1\rangle, \quad Y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|0\rangle,$$

or

$$Y|0\rangle = i|1\rangle, \quad Y|1\rangle = -i|0\rangle.$$

- Similarly to the Pauli X and Y gates, rotation by Z gate is done on the Bloch Sphere. This time, the rotation is about the Z axis and makes the states $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow -|1\rangle$.

$$Z \equiv \sigma_3 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Z gate operates as

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \quad Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle,$$

or

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle.$$

Also, squared of Pauli gates gives us the identity matrix.

$$X^2 = Y^2 = Z^2 = I.$$

2.2.2. Hadamard Gate and Phase-Shift Gate

Definition 2.3 (Benenti et al. 2018, 103)The Hadamard gate transforms a qubit from the basis states $|0\rangle$ and $|1\rangle$ into a state that is equally composed of both states, creating a superposition. The Hadamard gate performs the following transformations:

$$|0\rangle \to \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and

$$|1\rangle \to \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

In this case, the matrix of the Hadamard gate is as follows:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The output of computational basis states, applied to the Hadamard gate is

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \equiv |+\rangle,$$
$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \equiv |-\rangle.$$

Thus, the computational basis states $\{|0\rangle, |1\rangle\}$ become transformed into the Hadamard basis states $\{|+\rangle, |-\rangle\}$.

Definition 2.4 (Benenti et al. 2018, 103) The phase-shift gate is

$$R_z(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}.$$

On computational basis, the Phase-Shift gate acts as

$$R_z(\theta)|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle,$$
$$R_z(\theta)|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\theta}|1\rangle,$$

where θ is any real number.

2.3. Unitary Transformation of One Qubit States

Definition 2.5 A unitary transformation is represented by a 2×2 unitary matrix

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$$

and

$$U^\dagger = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix},$$

so that

$$UU^\dagger = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} = (|a|^2 + |b|^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $|a|^2 + |b|^2 = 1$. Then $UU^\dagger = U^\dagger U = 1$.

Properties of unitary transformation :

1. $UU^\dagger = U^\dagger U = 1$

2. $detU = 1$

Here we apply this transformation to two one qubit states $|\psi\rangle$ and $|\varphi\rangle$.
Let

$$|\psi\rangle = c_0|0\rangle + b_0|1\rangle$$

and

$$|\varphi\rangle = b_0|0\rangle + b_1|1\rangle.$$

Then, calculating Hermitian inner product to $|\psi\rangle$ and $|\varphi\rangle$ we get,

$$\langle\varphi|\psi\rangle = \overline{b}_0 c_0 + \overline{b}_1 c_1.$$

By applying the unitary transformation $U|\psi\rangle = |\varphi\rangle$, due to $\langle\psi|\psi\rangle = |c_0|^2 + |c_1|^2 = 1$, we find

$$\langle\varphi|\varphi\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1,$$

where we have used unitary condition $U^\dagger U = UU^\dagger = I$.

2.4. Universal Gates and Universality of One Qubit Computation

It is possible to generate an arbitrary one qubit gate applying a sequence of the Hadamard and phase gates. The property is referred to as the universality of gates and calculations on one qubit (Benenti et al. 2018, 117). This demonstrates that using a set of universal gates, it is possible to convert any one qubit state to any other arbitrary qubit state.

Let $|\psi\rangle$ is the generic one qubit state. Firstly we apply Hadamard and phase-shift gates to basis state $|0\rangle$:

$$R_Z\left(\frac{\pi}{2}+\varphi\right)HR_Z(\theta)H|0\rangle \quad = \quad e^{i\frac{\theta}{2}}\left(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi}|1\rangle\right),$$

or

$$R_Z\left(\frac{\pi}{2}+\varphi\right)HR_Z(\theta)H|0\rangle = |\psi\rangle.$$

For two arbitrary qubits

$$\begin{aligned}|\psi_1\rangle &= \cos\frac{\theta_1}{2}|0\rangle + \sin\frac{\theta_1}{2}e^{i\varphi_1}|1\rangle,\\ |\psi_2\rangle &= \cos\frac{\theta_2}{2}|0\rangle + \sin\frac{\theta_1}{2}e^{i\varphi_2}|1\rangle,\end{aligned}$$

appyling this circuit

$$R_Z\left(\frac{\pi}{2}+\varphi_2\right)HR_Z\left(\theta_2-\theta_1\right)HR_Z\left(-\frac{\pi}{2}-\varphi_1\right)|\psi_1\rangle \quad = \quad e^{i\left(\frac{\theta_2}{2}-\frac{\theta_1}{2}\right)}\left(\cos\frac{\theta_2}{2}|0\rangle + \sin\frac{\theta_2}{2}e^{i\varphi_2}|1\rangle\right)$$

up to global phase we get relation

$$R_Z\left(\frac{\pi}{2}+\varphi_2\right)HR_Z\left(\theta_2-\theta_1\right)HR_Z\left(-\frac{\pi}{2}-\varphi_1\right)|\psi_1\rangle \quad = \quad |\psi_2\rangle.$$

This transformation enables the generation of arbitrary one qubit $|\psi_2\rangle$ from arbitrary qubit $|\psi_1\rangle$.

Furthermore, the Hadamard and phase gates can be utilized to represent X, Y, and Z gates as follows.

Proposition 2.1 The Pauli gates can be decomposed as

$$
\begin{aligned}
X &= HR_Z(\pi)H, \\
Y &= R_Z\left(\frac{\pi}{2}\right)HR_Z(\pi)HR_Z\left(-\frac{\pi}{2}\right), \\
Z &= R_Z(\pi).
\end{aligned}
$$

CHAPTER 3

SEPARABLE AND ENTANGLED TWO QUBIT STATES

In this chapter, we will examine what are the qubit-qubit states, the separability of these states by using the tensor product, and the linear dependence.

3.1. Two Qubit States

Definition 3.1 (Benenti et al. 2018, 111) The generic qubit-qubit state $|\psi\rangle$ is defined as

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle, \tag{3.1}$$

where normalization condition for state $|\psi\rangle$ is

$$\langle\psi|\psi\rangle = |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1. \tag{3.2}$$

3.2. Separable and Entangled States

Definition 3.2 (Benenti et al. 2018, 111) If $|\psi\rangle = |\psi_0\rangle \otimes |\psi_1\rangle = |\psi_0\rangle|\psi_1\rangle$ where $|\psi_0\rangle, |\psi_1\rangle$ are one qubit states, then $|\psi\rangle$ is called as separable two qubit state. If not, then the state is entangled.

Any generic two-qubit state can be represented by two one-qubit states, by using the left and the right decompositions.

1. The left decomposition is

$$\begin{aligned} |\psi\rangle &= |0\rangle \otimes \underbrace{(c_{00}|0\rangle + c_{01}|1\rangle)}_{|\varphi_0\rangle} + |1\rangle \otimes \underbrace{(c_{10}|0\rangle + c_{11}|1\rangle)}_{|\varphi_1\rangle} \\ &= |0\rangle \otimes |\varphi_0\rangle + |1\rangle \otimes |\varphi_1\rangle. \end{aligned} \tag{3.3}$$

where, $|\varphi_0\rangle = c_{00}|0\rangle + c_{01}|1\rangle$ and $|\varphi_1\rangle = c_{10}|0\rangle + c_{11}|1\rangle$.

2. The right decomposition is

$$\begin{aligned} |\psi\rangle &= \underbrace{(c_{00}|0\rangle + c_{10}|1\rangle)}_{|\psi_0\rangle} \otimes|0\rangle + \underbrace{(c_{01}|0\rangle + c_{11}|1\rangle)}_{|\psi_1\rangle} \otimes|1\rangle \\ &= |\psi_0\rangle \otimes |0\rangle + |\psi_1\rangle \otimes |1\rangle \end{aligned} \tag{3.4}$$

where, $|\psi_0\rangle = c_{00}|0\rangle + c_{10}|1\rangle$ and $|\psi_1\rangle = c_{01}|0\rangle + c_{11}|1\rangle$.

An arbitrary two-qubit state can be written in both forms (3.3) or (3.4).

Theorem 3.1 An arbitrary two qubit state $|\psi\rangle$ is separable if and only if, the one qubit states $|\varphi_0\rangle$ and $|\varphi_1\rangle$ or $|\psi_0\rangle$ and $|\psi_1\rangle$ are linearly dependent.

Proof

1. Let $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are linearly dependent,

$$|\varphi_0\rangle = \lambda|\varphi_1\rangle.$$

Then

$$\begin{aligned} |\psi\rangle &= |0\rangle\lambda|\varphi_1\rangle + |1\rangle\lambda|\varphi_1\rangle \\ &= (\lambda|0\rangle + |1\rangle)\,|\varphi_1\rangle \end{aligned}$$

and it is separable.

2. Let $|\psi\rangle$ is separable, and may be expressed as the following form

$$|\psi\rangle = (a_0|0\rangle + a_1|1\rangle)\,(b_0|0\rangle + b_1|1\rangle)\,.$$

Then

$$
\begin{aligned}
|\psi\rangle &= a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle \\
&= a_0|0\rangle\,(b_0|0 + b_1|1\rangle) + a_1|1\rangle\,(b_0|0\rangle + b_1|1\rangle)\,, \\
|\varphi_0\rangle &= a_0\,(b_0|0 + b_1|1\rangle)\,, \\
|\varphi_1\rangle &= a_1\,(b_0|0 + b_1|1\rangle)\,.
\end{aligned}
$$

So, the states $a_1|\varphi_0\rangle = a_0|\varphi_1\rangle$ are linearly dependent.

□

Corollary 3.1 If $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are linearly independent, then $|\psi\rangle$ is not separable. It is entangled.

3.3. Linear Dependence and Determinant

Lemma 3.1 Let

$$
\begin{aligned}
|\varphi_0\rangle &= c_{00}|0\rangle + c_{01}|1\rangle = \begin{pmatrix} c_{00} \\ c_{01} \end{pmatrix}, \\
|\varphi_1\rangle &= c_{10}|0\rangle + c_{11}|1\rangle = \begin{pmatrix} c_{10} \\ c_{11} \end{pmatrix}.
\end{aligned}
$$

States $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are linearly dependent if and only if

$$
\det\begin{pmatrix} c_{00} & c_{10} \\ c_{01} & c_{11} \end{pmatrix} = 0.
$$

Lemma 3.2 States $|\psi_0\rangle$ and $|\psi_1\rangle$

$$|\psi_0\rangle = c_{00}|0\rangle + c_{10}|1\rangle = \begin{pmatrix} c_{00} \\ c_{10} \end{pmatrix}$$

$$|\psi_1\rangle = c_{01}|0\rangle + c_{11}|1\rangle = \begin{pmatrix} c_{01} \\ c_{11} \end{pmatrix}$$

are linearly dependent if and only if

$$\det\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} = 0.$$

Corollary 3.2 States $|\varphi_0\rangle$ and $|\varphi_1\rangle$ ($|\psi_0\rangle$ and $|\psi_1\rangle$) are linearly dependent if and only if the real number

$$C = 2\left|det\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}\right| = 2\left\|\begin{matrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{matrix}\right\| = 0.$$

Corollary 3.3 If $C \neq 0$, then $|\varphi_0\rangle$ and $|\varphi_1\rangle$ ($|\psi_0\rangle$ and $|\psi_1\rangle$) are linearly independent and state $|\psi\rangle$ is entangled.

CHAPTER 4

ENTANGLEMENT OF PURE TWO QUBIT STATES

4.1. Density Matrix

It is possible to formulate quantum mechanics through state vectors, and more general, by density operator or density matrix concept.

Definition 4.1 (Benenti et al. 2018, 81) The density operator for two qubit state $|\psi\rangle$ from $H_A \otimes H_B$ Hilbert space is expressed as

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| = \sum_{i,j=0,1} c_{ij}|ij\rangle \sum_{i'j'} \overline{c_{i'j'}}\langle i'j'| \\ &= \sum_{i,j}\sum_{i'j'} c_{ij}\overline{c_{i'j'}}\ |ij\rangle\langle i'j'|. \end{aligned} \tag{4.1}$$

Properties of Density Matrix

1. $tr(\rho) = tr(|\psi\rangle\langle\psi|) = 1$
2. $tr(\rho^2) = 1$ means that the state ρ is pure.
3. $tr(\rho^2) < 1$ means that the state ρ is mixed.

4.2. Reduced Density Matrix

The reduced density matrix is a very useful, important formalism used for the identification and analysis of subsystems of composite quantum systems.
For subsystem A, the reduced density operator is written as

$$\rho_A = tr_B(\rho) = \sum_{k=0}^{1} {}_B\langle k \mid \rho \mid k\rangle_B$$

where $tr_B(\rho)$ is called the partial trace.
For generic two qubit state

$$\begin{aligned}
\rho_A &= tr_B(\rho) = \sum_{k=0}^{1} {}_B\langle k \mid \rho \mid k\rangle_B \\
&= \sum_{i,j}\sum_{i',j'} c_{ij}\overline{c_{i'j'}}\, |i\rangle_A\, {}_A\langle i'| \sum_k \underbrace{{}_B\langle k|j\rangle_B}_{\delta_{kj}} \underbrace{{}_B\langle j'|k\rangle_B}_{\delta_{j'k}} \\
&= \sum_{ij}\sum_{i'j'} c_{ij}\overline{c_{i'j'}}\, |i\rangle_A\, {}_A\langle i'| \underbrace{\sum_k \delta_{kj}\delta_{j'k}}_{\delta_{jj'}} \\
&= \sum_{ij}\sum_{i'j'} c_{ij}\overline{c_{i'j'}}\, |i\rangle_A\, {}_A\langle i'|\delta_{jj'} \\
&= \sum_{ij}\sum_{i'j} c_{ij}\overline{c_{i'j}}\, |i\rangle_A\, {}_A\langle i'|.
\end{aligned}$$

So, the reduced density matrix is written as

$$\rho_A = \sum_{j}\sum_{ii'} c_{ij}\overline{c_{i'j}}\, |i\rangle_A\, {}_A\langle i'| \tag{4.2}$$

or in explicit form

$$\rho_A = \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 & c_{00}\overline{c}_{01} + c_{01}\overline{c}_{11} \\ c_{10}\overline{c}_{00} + c_{11}\overline{c}_{01} & |c_{10}|^2 + |c_{11}|^2 \end{pmatrix}. \tag{4.3}$$

Properties of Reduced Density Matrix

1. $\rho_A^\dagger = \rho_A$
   This provides that all eigenvalues of ρ_A are real and eigenstates for distinct eigenvalues are orthogonal.

2. $tr(\rho_A) = 1 \iff |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1.$

We'll show that the state $|\psi\rangle$ is separable or entangled state in the following way. First, we calculate

$$
\begin{aligned}
tr\rho_A^2 &= \left(|c_{00}|^2 + |c_{01}|^2\right)^2 + \left(|c_{10}|^2 + |c_{11}|^2\right)^2 \\
&+ 2\left(c_{00}\overline{c_{10}} + c_{01}\overline{c_{11}}\right)\left(\overline{c}_{00}c_{10} + \overline{c}_{01}c_{11}\right),
\end{aligned}
$$

and squared normalization condition (3.1),

$$
\begin{aligned}
1 &= |c_{00}|^4 + |c_{01}|^4 + |c_{10}|^4 + |c_{11}|^4 \\
&+ 2|c_{00}|^2|c_{01}|^2 + 2|c_{00}|^2|c_{10}|^2 + 2|c_{00}|^2|c_{11}|^2 \\
&+ 2|c_{01}|^2|c_{10}|^2 + 2|c_{01}|^2|c_{11}|^2 + 2|c_{10}|^2|c_{11}|^2.
\end{aligned}
$$

Taking difference

$$
tr\rho_A^2 - 1 = 2\left(c_{00}c_{11}\overline{c_{10}c_{01}} + c_{01}c_{10}\overline{c_{00}c_{11}} - c_{00}\overline{c_{00}}c_{11}\overline{c_{11}} - c_{01}\overline{c_{01}}c_{10}\overline{c_{10}}\right)
$$

we get

$$
1 - tr\rho_A^2 = 2\left\|\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}\right\|^2. \tag{4.4}
$$

Definition 4.2 The real number C is called the concurrence

$$
C = 2\left\|\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}\right\|. \tag{4.5}
$$

Then, the relation (4.4) becomes

$$
1 = tr\rho_A^2 + \frac{1}{2}C^2.
$$

This relation between concurrence and reduced density matrix provides criterium for entanglement of pure state $|\psi\rangle$:

1. $C = 0$ if and only if $tr\rho_A^2 = 1$ and reduced state is pure state.

2. $C \neq 0$ if and only if $tr\rho_A^2 < 1$ and reduced state is mixed state.

4.3. Maximal Value of Concurrence

Proposition 4.1 The level of concurrence, denoted as C,is bounded from above by maximum value

$$\max C = 1.$$

Proof By definition

$$C = 2|c_{00}c_{11} - c_{01}c_{10}|.$$

Let $c_{00} = |c_{00}|e^{i\alpha_{00}}$, $c_{01} = |c_{01}|e^{i\alpha_{01}}$, $c_{10} = |c_{10}|e^{i\alpha_{10}}$, $c_{11} = |c_{11}|e^{i\alpha_{11}}$, then

$$C = 2||c_{00}||c_{11}| - |c_{01}||c_{10}|e^{i\alpha}|,$$

where $\alpha = (\alpha_{01} + \alpha_{10} - \alpha_{00} - \alpha_{11})$. It gives

$$C^2 = 4\left(|c_{00}|^2|c_{11}|^2 + 2|c_{01}|^2|c_{10}|^2 - 2|c_{00}||c_{11}||c_{01}||c_{10}|\cos\alpha\right).$$

Since $|\cos\alpha| \leq 1$, the concurrence is bounded

$$\begin{aligned} C^2 &\leq 4\left(|c_{00}|^2|c_{11}|^2 + 2|c_{01}|^2|c_{10}|^2 + 2|c_{00}||c_{11}||c_{01}||c_{10}|\right) \\ &= 4\left(|c_{00}||c_{11}| + |c_{01}||c_{10}|\right). \end{aligned}$$

To evaluate this inequality we formulate following optimization problem.

4.3.1. Optimization Problem

We define an optimization problem to find the maximal value of concurrence. Then, we have to solve this problem with constraints, which becomes the Lagrange multiplier problem.

Let denote $|c_{00}| \equiv r_{00}$, $|c_{01}| \equiv r_{01}$, $|c_{10}| \equiv r_{10}$ and $|c_{11}| \equiv r_{11}$. The maximal value problem takes the form

$$F\left(r_{00}, r_{01}, r_{10}, r_{11}, \lambda\right) = 4\left(r_{00}r_{11} + r_{01}r_{10}\right)^2 + \lambda\left(\sum_{i,j=0,1} |r_{ij}|^2 - 1\right), \tag{4.6}$$

where $r_{00}, r_{11}, r_{01}, r_{10}$ are positive real numbers and λ is Lagrange multiplier.

Critical points of this function are determined by equations

$$\frac{\partial F}{\partial r_{ij}} = 0, \qquad i, j = 0, 1$$

and

$$\frac{\partial F}{\partial \lambda} = 0.$$

The last one gives constraint.

$$\sum_{i,j=0,1} |r_{ij}|^2 = 1.$$

For critical points we have the system,

$$\begin{aligned}
4\left(r_{00}r_{11} - r_{01}r_{10}\right) r_{11} &= \lambda r_{00}, \\
4\left(r_{00}r_{11} - r_{01}r_{10}\right) r_{00} &= \lambda r_{11}, \\
4\left(r_{00}r_{11} - r_{01}r_{10}\right) r_{10} &= \lambda r_{01}, \\
4\left(r_{00}r_{11} - r_{01}r_{10}\right) r_{01} &= \lambda r_{10}.
\end{aligned}$$

This gives relations

$$r_{00} = r_{11}, \qquad r_{01} = r_{10}$$

$$\lambda = -\left(r_{00}^2 + r_{01}^2\right),$$

and our function now becomes

$$F = 4\left(r_{00}^2 + r_{01}^2\right)^2 + \lambda\left(2\sum(r_{00}^2 + r_{01}^2) - 1\right),$$

with constraint

$$2(r_{00}^2 + r_{01}^2) = 1.$$

By parametrizing $r_{00} = \frac{1}{\sqrt{2}}\cos\theta$ and $r_{01} = \frac{1}{\sqrt{2}}\sin\theta$, we have equation of the circle $r_{00}^2 + r_{01}^2 = \frac{1}{2}$. In this parametrization,

$$F = 4\left(r_{00}^2 + r_{01}^2\right)^2 = 4\frac{1}{4} = 1.$$

This implies that $C^2 \leq 1$ and max $C = 1$. □

Corollary 4.1 The concurrence

$$C = 2\left\|\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}\right\|$$

is bounded real function,

$$0 \leq C \leq 1.$$

Thus, at the boundaries of C,the value $C = 1$ represents maximally entangled state and $C = 0$ represents separable states.
We may demonstrate this situation using an example.

Example 4.1 For the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right),$$

we have $C = 1$ and this state is the maximally entangled state.

CHAPTER 5

QUANTUM ENTROPY AND ENTANGLEMENT

In information theory, entropy is considered to be a measure of the amount of information lost along the way during the exchange between a recipient and the source of information.

5.1. Von Neumann and Shannon Entropy

Von Neumann's entropy is the basis of the theory of information and allows the calculation of quantum information contained in a system (Benenti et al. 2018, 252), (Wootters 1998, 2245).

Definition 5.1 (Benenti et al. 2018, 252) If we have a reduced density matrix, we may calculate its Von Neumann entropy to find the entanglement by using the following formula:

$$E_A = -tr\left(\rho_A \log_2 \rho_A\right).$$

It is connected with the concurrence. To find this relation with consider eigenvalue problem for ρ_A.

Definition 5.2 (Benenti et al. 2018, 248) The Shannon entropy of random variable X taking values $x_1, x_2, \ldots, x_n$ with probabilities $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$ is

$$E = -\sum_i (\lambda_i \log_2 \lambda_i).$$

These two entropies are related.
Let

$$\rho_A = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \overline{\rho_{01}} & \rho_{11} \end{pmatrix}.$$

Now, we can write the characteristic equation for ρ_A as follows:

$$\det|\rho_A - \lambda I| = 0$$

or in explicit form

$$\begin{vmatrix} \rho_{00} - \lambda & \rho_{01} \\ \overline{\rho_{10}} & \rho_{11} - \lambda \end{vmatrix} = 0.$$

Then, the characteristic equation becomes

$$\lambda^2 - \lambda \underbrace{(\rho_{00} + \rho_{11})}_{tr\rho_A} + \underbrace{\rho_{00}\rho_{11} - |\rho_{01}|^2}_{\det\rho_A} = 0$$

or

$$\lambda^2 - (tr\rho_A)\,\lambda + \det\rho_A = 0.$$

Since, for reduced pure state $tr\rho_A = 1$ we have

$$\lambda^2 - \lambda + \det\rho_A = 0.$$

Then

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \det\rho_A}$$

and also

$$
\begin{aligned}
\det \rho_A &= \rho_{00}\rho_{11} - |\rho_{01}|^2 \\
&= \left(|c_{00}|^2 + |c_{01}|^2\right)\left(|c_{10}|^2 + |c_{11}|^2\right) \\
&= \left(c_{00}c_{11} - c_{01}c_{10}\right)\left(\overline{c_{00}c_{11} - c_{01}c_{10}}\right)
\end{aligned}
$$

or in explicit form

$$
\det \rho_A = \left\| \begin{matrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{matrix} \right\|^2 .
$$

Then we have lemma.

Lemma 5.1

$$
\det \rho_A = \left\| \begin{matrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{matrix} \right\|^2 . \tag{5.1}
$$

Corollary 5.1 Due to Definition 4.2

$$
\left\| \begin{matrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{matrix} \right\|^2 = \frac{1}{4}C^2
$$

and

$$
\det \rho_A = \frac{1}{4}C^2
$$

Proposition 5.1 Eigenvalues of ρ_A are

$$\lambda_1 = \frac{1+\sqrt{1-C^2}}{2},$$
$$\lambda_2 = \frac{1-\sqrt{1-C^2}}{2}.$$

If $C = 0$ (separable states) then, $\lambda_1 = 1, \lambda_2 = 0$.
If $C = 1$ (maximally entangled states) then, $\lambda_1 = \frac{1}{2} = \lambda_2$. So we proved the following

Proposition 5.2 Von Neumann entropy of generic two qubit state (3.1) is

$$E_A = -\frac{1}{2}\left[(1+\sqrt{1-C^2})\log_2\frac{1+\sqrt{1-C^2}}{2} + (1-\sqrt{1-C^2})\log_2\frac{1-\sqrt{1-C^2}}{2}\right]$$

where C is the concurrence (4.5). For separable states, since $C = 0$, we have $E_A = 0$. Also, for maximally entangled states for $C = 1$ and we have $E_A = 1$.

Here are some examples.

Example 5.1 All of the Bell states

$$|\beta\rangle_\pm = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \qquad |\alpha\rangle_\pm = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}},$$

have $C = 1$, which means they are maximally entangled states. We can see this from the concurrence formula in the following determinant form:

$$2\begin{vmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = 1.$$

Then the entropy is $E_A = 1$.

Example 5.2 Let

$$|\epsilon\rangle = \frac{|00\rangle + \epsilon|01\rangle + |11\rangle}{\sqrt{2+\epsilon^2}},$$

where $\langle\epsilon|\epsilon\rangle = 1$. The concurrence

$$C = 2\begin{vmatrix} \frac{1}{\sqrt{2+\epsilon^2}} & \frac{\epsilon}{\sqrt{2+\epsilon^2}} \\ 0 & \frac{1}{\sqrt{2+\epsilon^2}} \end{vmatrix} = \frac{2}{2+\epsilon^2}$$

and the entropy is

$$E_A = -\frac{1}{2}\left(\left(1+\sqrt{1-C^2}\right)\log_2\left(1+\sqrt{1-C^2}\right)+\left(1-\sqrt{1-C^2}\right)\log_2\left(1-\sqrt{1-C^2}\right)\right).$$

CHAPTER 6

INVARIANCE OF ENTANGLEMENT UNDER UNITARY ONE-QUBIT TRANSFORMATIONS

The unitary transformation, performed by unitary operator, is a linear transformation that preserves the Hermitian inner product.

6.1. Unitary One Qubit Transformation

Let's consider a unitary operator U that acts on the one-qubit $|\psi\rangle$ vector. The tranformed vector shown as $|\tilde{\psi}\rangle$ is given by

$$U|\psi\rangle = |\tilde{\psi}\rangle,$$

where $UU^\dagger = I$ and it confirms that the Hermitian inner product is preserved under a unitary transformation $\langle\tilde{\psi}|\tilde{\psi}\rangle = \langle\psi|\psi\rangle$.
The operation of the unitary transformation on ρ_A is as follows:

$$U|\psi\rangle\langle\psi|U^\dagger = U\rho_A U^\dagger = \tilde{\rho}_A = |\tilde{\psi}\rangle\langle\tilde{\psi}|.$$

Since every Hermitian matrix may be transformed into a diagonal matrix using a unitary transformation:

$$\tilde{\rho}_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

exists such U that

$$\tilde{\rho}_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = U\rho_A U^\dagger.$$

Then

1. $tr\tilde{\rho}_A = tr\rho_A = 1$,

2. $\det\tilde{\rho}_A = \det\rho_A = \left\| \begin{matrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{matrix} \right\|^2$.

From first two properties, characteristic equations for both matrices are the same

$$\lambda^2 - tr\rho_A\lambda + \det\rho_A = 0,$$
$$\lambda^2 - tr\tilde{\rho}_A\lambda + \det\tilde{\rho}_A = 0,$$

which implies that ρ_A and $\tilde{\rho}_A$ should have the same eigenvalues.

Let

$$|\psi_i\rangle \equiv U|\varphi_i\rangle$$

and

$$\underbrace{\left(U\rho_A U^\dagger\right)}_{\tilde{\rho}_A}|\psi_i\rangle = \lambda_i|\psi_i\rangle.$$

Then we have

$$\tilde{\rho}_A|\psi_i\rangle = \lambda_i|\psi_i\rangle.$$

Lemma 6.1 The reduced density matrices ρ_A and $\tilde{\rho}_A$ have the same eigenvalues.

Proof It follows from characteristic equation in determinant form

$$\det|\rho_A - \lambda I| = 0$$
$$\Rightarrow \det|U^\dagger\tilde{\rho}_A U - \lambda U^\dagger U| = 0.$$

Then, if we take a common parenthesis

$$\Rightarrow \det|U^\dagger(\tilde{\rho}_A - \lambda I)\,U| = 0$$
$$\Rightarrow \det U^\dagger \det|\tilde{\rho}_A - \lambda I| \det U = 0$$
$$\Rightarrow \underbrace{\det U^\dagger \det U}_{\det U^\dagger \det U = \det I = 1} \det|\tilde{\rho}_A - \lambda I| = 0.$$

Thus, we have the same eigenvalues

$$\Rightarrow \det|\tilde{\rho}_A - \lambda I| = 0.$$

□

Corollary 6.1 The concurrence C is invariant under unitary transformation U: $\det\rho_A = \frac{C^2}{2}$ and $\det\tilde{\rho}_A = \frac{\tilde{C}^2}{2}$, and we get

$$C = \tilde{C}.$$

Proof This follows from identification $\det\rho_A = \frac{C^2}{2}, \det\tilde{\rho}_A = \frac{C^2}{2}$. □

6.2. One Qubit Unitary Gate Acting on Two Qubit States

Let

$$\begin{aligned} |\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \\ &= |0\rangle \otimes (c_{00}|0\rangle + c_{01}|1\rangle) + |1\rangle \otimes (c_{10}|0\rangle + c_{11}|1\rangle) \end{aligned}$$

decomposed as

$$|\psi\rangle = |0\rangle \otimes |\psi_0\rangle + |1\rangle \otimes |\psi_1\rangle.$$

Acting by unitary (2×2) matrix on one qubit states $|\psi_0\rangle$ and $|\psi_1\rangle$, we have

$$U|\psi_0\rangle = |\tilde{\psi}_0\rangle \quad and \quad U|\psi_1\rangle = |\tilde{\psi}_1\rangle.$$

This transformation preserves Hermitian inner products

$$\langle\psi_i|\psi_j\rangle = \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle.$$

Here $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ is the unitary matrix, and as a result we have transformed two qubit state as follows:

$$|\tilde{\psi}\rangle = |0\rangle \otimes |\tilde{\psi}_0\rangle + |1\rangle \otimes |\tilde{\psi}_1\rangle$$

6.3. Unitary One Qubit Gates and Concurrence

Let

$$\begin{aligned} |\psi\rangle &= |0\rangle_A|\psi_0\rangle_B + |1\rangle_A|\psi_1\rangle_B \\ &= |\varphi_0\rangle_A|0\rangle_B + |\varphi_1\rangle_A|1\rangle_B. \end{aligned}$$

Normalization condition implies

$$\langle\psi|\psi\rangle = 1 = \langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle = \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle.$$

The density matrix for this states

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= (|\varphi_0\rangle_A|0\rangle_B + |\varphi_1\rangle_A|1\rangle_B)\,({}_A\langle\varphi_0|_B\langle 0| +_A \langle\varphi_1|_B\langle 1|)\end{aligned}$$

or

$$\begin{aligned}\rho &= |0\rangle_A\, {}_A\langle 0|\,(|\psi_0\rangle_B\, {}_B\langle\psi_0|) + |0\rangle_A\, {}_A\langle 1|\,(|\psi_0\rangle_B\, {}_B\langle\psi_1|) \\ &+ |1\rangle_A\, {}_A\langle 0|\,(|\psi_1\rangle_B\, {}_B\langle\psi_0|) + |1\rangle_A\, {}_A\langle 1|\,(|\psi_1\rangle_B\, {}_B\langle\psi_1|)\,.\end{aligned}$$

The corresponding reduced density matrices are

$$\begin{aligned}\rho_A &= tr_B\rho = |\varphi_0\rangle_A\, {}_A\langle\varphi_0| + |\varphi_1\rangle_A\, {}_A\langle\varphi_1|, \\ \rho_B &= tr_A\rho = |\psi_0\rangle_B\, {}_B\langle\psi_0| + |\psi_1\rangle_B\, {}_B\langle\psi_1|.\end{aligned}$$

Then, we have

$$\begin{aligned}tr\rho_A &= {}_A\langle 0|\varphi_0\rangle\langle\varphi_0|0\rangle_A +_A \langle 1|\varphi_0\rangle\langle\varphi_0|1\rangle_A +_A \langle 0|\varphi_1\rangle\langle\varphi_1|0\rangle_A +_A \langle 1|\varphi_1\rangle\langle\varphi_1|1\rangle_A \\ &= \langle\varphi_0|0\rangle\langle 0|\varphi_0\rangle + \langle\varphi_0|1\rangle\langle 1|\varphi_0\rangle + \langle\varphi_1|0\rangle\langle 0|\varphi_1\rangle + \langle\varphi_1|1\rangle\langle 1|\varphi_1\rangle \\ &= \langle\varphi_0|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{I}|\varphi_0\rangle + \langle\varphi_1|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{I}|\varphi_1\rangle,\end{aligned}$$

and due to normalization

$$tr\rho_A = \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle = 1.$$

Similar way

$$
\begin{aligned}
tr\rho_B &= {}_B\langle 0|\psi_0\rangle\langle\psi_0|0\rangle_B +_B \langle 1|\psi_0\rangle\langle\psi_0|1\rangle_B +_B \langle 0|\psi_1\rangle\langle\psi_1|0\rangle_B +_B \langle 1|\psi_1\rangle\langle\psi_1|1\rangle_B \\
&= \langle\psi_0|0\rangle\langle 0|\psi_0\rangle + \langle\psi_0|1\rangle\langle 1|\psi_0\rangle + \langle\psi_1|0\rangle\langle 0|\psi_1\rangle + \langle\psi_1|1\rangle\langle 1|\psi_1\rangle \\
&= \langle\psi_0|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{I}|\psi_0\rangle + \langle\psi_1|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{I}|\psi_1\rangle,
\end{aligned}
$$

and

$$
tr\rho_B = \langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle = 1
$$

To find the determinant, we represent ρ_B in matrix form

$$
\begin{aligned}
\rho_B &= (c_{00}|0\rangle c_{01}|1\rangle)(\bar{c}_{00}|0\rangle\bar{c}_{01}|1\rangle) + (c_{10}|0\rangle c_{11}|1\rangle)(\bar{c}_{10}|0\rangle\bar{c}_{11}|1\rangle) \\
&= |0\rangle\langle 0|\left(|c_{00}|^2 + |c_{10}|^2\right) + |0\rangle\langle 1|(c_{00}\bar{c}_{01} + c_{10}\bar{c}_{11}) \\
&+ |1\rangle\langle 0|(c_{01}\bar{c}_{00} + c_{11}\bar{c}_{10}) + |1\rangle\langle 1|\left(|c_{01}|^2 + |c_{11}|^2\right).
\end{aligned}
$$

Then we have following matrix form

$$
\rho_B = \begin{pmatrix} |c_{00}|^2 + |c_{10}|^2 & c_{00}\bar{c}_{01} + c_{10}\bar{c}_{11} \\ c_{01}\bar{c}_{00} + c_{11}\bar{c}_{10} & |c_{01}|^2 + |c_{11}|^2 \end{pmatrix} = \begin{pmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_1|\psi_0\rangle \\ \langle\psi_0|\psi_1\rangle & \langle\psi_1|\psi_1\rangle \end{pmatrix}
$$

and by using relations

$$
\begin{aligned}
\langle\psi_0|\psi_0\rangle &= |c_{00}|^2 + |c_{10}|^2 \\
\langle\psi_1|\psi_0\rangle &= c_{00}\bar{c}_{01} + c_{10}\bar{c}_{11} \\
\langle\psi_0|\psi_1\rangle &= c_{01}\bar{c}_{00} + c_{11}\bar{c}_{10} \\
\langle\psi_1|\psi_1\rangle &= |c_{01}|^2 + |c_{11}|^2,
\end{aligned}
$$

we get

$$\det \rho_B \quad = \quad \langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle - |\langle\psi_0|\psi_1\rangle|^2.$$

By using the Cauchy-Schwarz Inequality,

$$\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle \geq |\langle\psi_0|\psi_1\rangle|^2$$

we conclude that, $\det \rho_B \geq 0$. The same result we have for matrix $\rho_A \geq 0$.
So we proved the Lemma.

Lemma 6.2 $\det \rho_A \geq 0, \det \rho_B \geq 0$.

Proposition 6.1

$$C^2 \quad = \quad 4 \begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_1|\psi_0\rangle \\ \langle\psi_0|\psi_1\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix} = 4 \left\| \begin{matrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{matrix} \right\|^2$$

From Corollary 5.1, $C^2 = 2 \det \rho_A$. This is invariant under transposition $|\psi_i\rangle \leftrightarrow |\varphi_i\rangle$.

$$C^2 = 2 \det \rho_A = 2 \det \rho_B.$$

6.4. Action of Unitary Transformation on Two Qubit States

Let U is 2×2 unitary transformation, which satisfies $UU^\dagger = I$. When transformation U is applied to $|\psi_0\rangle$ and $|\psi_1\rangle$, we have

$$U|\psi_0\rangle = |\tilde{\psi}_0\rangle$$

and

$$U|\psi_1\rangle = |\tilde{\psi}_1\rangle.$$

The Hermitian inner product is invariant under unitary transformation

$$\langle\psi_i|\psi_j\rangle = \langle\psi_i|U^\dagger U|\psi_j\rangle = \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle.$$

Then

$$\tilde{C}^2 = C^2$$

so that, we have

$$\tilde{C} = C.$$

Let

$$\begin{aligned} |\tilde{\psi}\rangle &= |0\rangle|\tilde{\psi}_0\rangle + |1\rangle|\tilde{\psi}_1\rangle \\ &= |0\rangle \otimes U|\psi_0\rangle + |1\rangle \otimes U|\psi_1\rangle \\ &= (I \otimes U)\underbrace{(|0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle)}_{|\psi\rangle} \end{aligned}$$

then

$$|\tilde{\psi}\rangle = (I \otimes U)\,|\psi\rangle.$$

Under this unitary transformation, $\tilde{C} = C$, where in explicit form

$$I \otimes U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes U = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix}.$$

Then we proved following result.

Proposition 6.2 The concurrence $\tilde{C} = C$, is invariant under unitary transformation

$$|\tilde{\psi}\rangle = (I \otimes U)\,|\psi\rangle.$$

Theorem 6.1 The concurrences of two two-qubit states, connected by unitary transformation $U = U_1 \otimes U_2$.
Thus,

$$|\tilde{\psi}\rangle = (U_1 \otimes U_2)\,|\psi\rangle$$

are equal.

Proof Let transformed density matrix is

$$\begin{aligned}
\tilde{\rho} &= |\tilde{\psi}\rangle\langle\tilde{\psi}| \\
&= (U_1 \otimes U_2)\,|\psi\rangle\langle\psi|\,(U_1 \otimes U_2)^\dagger \\
&= (U_1 \otimes U_2)\,(|0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle)\,(\langle 0|\langle\psi_0| + \langle 1|\langle\psi_1|)\,(U_1 \otimes U_2)^\dagger \\
\tilde{\rho} &= \left(U_1|0\rangle\langle 0|U_1^\dagger\right)\left(U_2|\psi_0\rangle\langle\psi_0|U_2^\dagger\right) \\
&+ \left(U_1|1\rangle\langle 1|U_1^\dagger\right)\left(U_2|\psi_1\rangle\langle\psi_1|U_2^\dagger\right) \\
&+ \left(U_1|0\rangle\langle 1|U_1^\dagger\right)\left(U_2|\psi_0\rangle\langle\psi_1|U_2^\dagger\right) \\
&+ \left(U_1|1\rangle\langle 0|U_1^\dagger\right)\left(U_2|\psi_1\rangle\langle\psi_0|U_2^\dagger\right).
\end{aligned}$$

Since $U_2|\psi_0\rangle = |\tilde{\psi}_0\rangle$ and $U_2|\psi_1\rangle = |\tilde{\psi}_1\rangle$, we have

$$\begin{aligned}\tilde{\rho} &= \left(U_1|0\rangle\langle 0|U_1^\dagger\right)\left(|\tilde{\psi}_0\rangle\langle_0\tilde{\psi}|\right)\\ &+ \left(U_1|1\rangle\langle 1|U_1^\dagger\right)\left(|\tilde{\psi}_1\rangle\langle_1\tilde{\psi}|\right)\\ &+ \left(U_1|0\rangle\langle 1|U_1^\dagger\right)\left(|\tilde{\psi}_0\rangle\langle_1\tilde{\psi}|\right)\\ &+ \left(U_1|1\rangle\langle 0|U_1^\dagger\right)\left(|\tilde{\psi}_1\rangle\langle_0\tilde{\psi}|\right).\end{aligned}$$

Then we check

$$\begin{aligned}tr_A\tilde{\rho} &= |\tilde{\psi}_0\rangle\langle_0\tilde{\psi}|\left(\underbrace{\langle 0|U_1|0\rangle}_{a_1}\underbrace{\langle 0|U_1^\dagger|0\rangle}_{\bar{a}_1}+\underbrace{\langle 1|U_1|0\rangle}_{-\bar{b}_1}\underbrace{\langle 0|U_1^\dagger|1\rangle}_{-b_1}\right)\\ &+ |\tilde{\psi}_1\rangle\langle_1\tilde{\psi}|\left(\underbrace{\langle 0|U_1|1\rangle}_{b_1}\underbrace{\langle 1|U_1^\dagger|0\rangle}_{\bar{b}_1}+\underbrace{\langle 1|U_1|1\rangle}_{\bar{a}_1}\underbrace{\langle 1|U_1^\dagger|1\rangle}_{a_1}\right)\\ &+ |\tilde{\psi}_0\rangle\langle_1\tilde{\psi}|\left(\underbrace{\langle 0|U_1|0\rangle}_{a_1}\underbrace{\langle 1|U_1^\dagger|0\rangle}_{\bar{b}_1}+\underbrace{\langle 1|U_1|0\rangle}_{-\bar{b}_1}\underbrace{\langle 1|U_1^\dagger|1\rangle}_{a_1}\right)\\ &+ |\tilde{\psi}_1\rangle\langle_0\tilde{\psi}|\left(\underbrace{\langle 0|U_1|1\rangle}_{b_1}\underbrace{\langle 0|U_1^\dagger|0\rangle}_{\bar{a}_1}+\underbrace{\langle 1|U_1|1\rangle}_{\bar{a}_1}\underbrace{\langle 0|U_1^\dagger|1\rangle}_{-b_1}\right)\end{aligned}$$

so that, we can write

$$\begin{aligned}tr_A\tilde{\rho} &= |\tilde{\psi}_0\rangle\langle_0\tilde{\psi}|\left(\underbrace{|a_1|^2+|b_1|^2}_{1}\right)\\ &+ |\tilde{\psi}_1\rangle\langle_1\tilde{\psi}|\left(\underbrace{|a_1|^2+|b_1|^2}_{1}\right)\\ &+ |\tilde{\psi}_0\rangle\langle_1\tilde{\psi}|\left(\underbrace{a_1\bar{b}_1-\bar{b}_1a_1}_{0}\right)\\ &+ |\tilde{\psi}_1\rangle\langle_0\tilde{\psi}|\left(\underbrace{b_1\bar{a}_1-\bar{a}_1b_1}_{0}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned} tr_A\tilde{\rho} &= \tilde{\rho}_B = |\tilde{\psi}_0\rangle\langle\tilde{\psi}_0| + |\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| \\ &= U_2\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|\right)U_2^{\dagger}, \end{aligned}$$

and we get

$$\tilde{\rho}_B = U_2\rho_B U_2^{\dagger}.$$

Here, we have

$$\begin{aligned} tr\tilde{\rho}_B &= tr\rho_B, \\ det\tilde{\rho}_B &= det\rho_B, \\ \tilde{C} &= C, \\ U &= U_1 \otimes U_2. \end{aligned}$$

□

Concurrence is the same $\tilde{C} = C$, when

$$|\tilde{\psi}\rangle = U_1 \otimes U_2|\psi\rangle.$$

Corollary 6.2 The maximally entangled states with $C = 1$, are related by transformation $|\tilde{\psi}\rangle = U_1 \otimes U_2|\psi\rangle$.

6.4.1. Maximally Entangled Two Qubit States

As we have seen $U_1 \otimes U_2$ transformation preserves the concurrence and entanglement. If we apply it to Bell state ($C = 1$),

$$(U_1 \otimes U_2) \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} (U_1|0\rangle U_2|0\rangle + U_1|1\rangle U_2|1\rangle).$$

By using $U_1 = \begin{pmatrix} a_1 & b_1 \\ -\bar{b}_1 & \bar{a}_1 \end{pmatrix}$ and $U_2 = \begin{pmatrix} a_2 & b_2 \\ -\bar{b}_2 & \bar{a}_2 \end{pmatrix}$, we find the maximally entangled ($C = 1$) state.

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}[(a_1 a_2 + b_1 b_2)|00\rangle + \left(\bar{a}_1 \bar{a}_2 + \bar{b}_1 \bar{b}_2\right)|11\rangle \\ &+ \left(\bar{a}_2 b_1 - a_1 \bar{b}_2\right)|01\rangle - \left(a_2 \bar{b}_1 - \bar{a}_1 b_2\right)|10\rangle]. \end{aligned}$$

This state can be rewritten as

$$|\psi\rangle = c_{00}|00\rangle + c_{11}|11\rangle + c_{01}|01\rangle + c_{10}|10\rangle,$$

where the numbers

$$c_{00} = \frac{a_1 a_2 + b_1 b_2}{\sqrt{2}}, \quad c_{11} = \bar{c}_{00}, \quad c_{01} = \frac{\bar{a}_2 b_1 - a_1 \bar{b}_2}{\sqrt{2}}, \quad \text{and} \quad c_{10} = -\bar{c}_{01}.$$

Then we have the following Proposition.

Proposition 6.3 Normalized two qubit state in the form

$$|\psi\rangle = c_{00}|00\rangle + \bar{c}_{00}|11\rangle + c_{01}|01\rangle - \bar{c}_{01}|10\rangle \tag{6.1}$$

where $\langle\psi|\psi\rangle = 1$, is maximally entangled state, $C = 1$.

Proof Normalization condition implies

$$|c_{00}|^2 + |\bar{c}_{00}|^2 + |c_{01}|^2 + |-\bar{c}_{01}|^2 = 1,$$

or

$$|c_{00}|^2 + |c_{01}|^2 = \frac{1}{2}. \tag{6.2}$$

From another side concurrence for state (6.1) is

$$C = 2\det\begin{vmatrix} c_{00} & c_{01} \\ -\bar{c}_{01} & \bar{c}_{00} \end{vmatrix} = 2\left(|c_{00}|^2 + |c_{01}|^2\right),$$

and due to (6.2)

$$C = 1.$$

□

Definition 6.1 Matrix

$$\hat{C} = \sqrt{2}\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}$$

we call the concurrence matrix.

Proposition 6.4 The concurrence is equal

$$C = 2\left| det \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} \right| = \left|\det \hat{C}\right|$$

Corollary 6.3 For maximally entangled state (6.1) the concurrence matrix is the unitary matrix in $SU(2)$.

Proof Indeed for (6.1) matrix $\hat{C}$ is

$$\hat{C} = \begin{pmatrix} \sqrt{2}c_{00} & \sqrt{2}c_{01} \\ -\sqrt{2}\bar{c}_{01} & \sqrt{2}\bar{c}_{00} \end{pmatrix} \equiv \underbrace{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}}_{U} \in SU(2)$$

$$\left|\det \hat{C}\right| = |a|^2 + |b|^2 = 1 = |\det U|$$

because $\det \hat{C} = 2\left(|c_{00}|^2 + |c_{01}|^2\right)$ and it is equal 1 due to normalization.

And due to these

$$\hat{C}\hat{C}^\dagger = \hat{C}^\dagger\hat{C} = I$$

unitary condition.

Comments 1: For two qubit state (6.1) the normalization condition $|c_{00}|^2 + |c_{01}|^2 = \frac{1}{2}$ leads to maximal value of entanglement

$$C = 2\left(|c_{00}|^2 + |c_{01}|^2\right) = 1.$$

Comments 2: Form of state (6.2) implies that $|a|^2 + |b|^2 = 1$ for all unitary transformation $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, produces maximally entangled state $C = 1$ of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(a|00\rangle + \bar{a}|11\rangle + b|01\rangle - \bar{b}|10\rangle\right).$$

Example 6.1 Let

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Then we can apply $(I \otimes U)$ to state $|\psi\rangle$, we have

$$\begin{aligned} I \otimes U|\psi\rangle &= \frac{|0\rangle}{\sqrt{2}}U|0\rangle + \frac{|1\rangle}{\sqrt{2}}U|1\rangle \\ &= \frac{|0\rangle}{\sqrt{2}}\left(a|0\rangle - \bar{b}|1\rangle\right) + \frac{|1\rangle}{\sqrt{2}}(b|0\rangle + \bar{a}|1\rangle) \end{aligned}$$

and we get

$$
\begin{aligned}
|\tilde{\psi}\rangle &= \frac{a}{\sqrt{2}}|00\rangle + \frac{\bar{a}}{\sqrt{2}}|11\rangle + \frac{-\bar{b}}{\sqrt{2}}|01\rangle + \frac{b}{\sqrt{2}}|10\rangle \\
&= \frac{a|00\rangle + \bar{a}|11\rangle}{\sqrt{2}} + \frac{-\bar{b}|01\rangle + b|10\rangle}{\sqrt{2}}.
\end{aligned}
$$

Then, normalization condition gives us

$$
|a|^2 + |b|^2 = 1 \rightarrow a = \cos\frac{\theta}{2}e^{i\varphi_a} \quad b = \sin\tfrac{\theta}{2}e^{i\varphi_b}
$$

and

$$
|\tilde{\psi}\rangle = \cos\frac{\theta}{2}\frac{e^{i\varphi_a}|00\rangle + e^{-i\varphi_a}|11\rangle}{\sqrt{2}} + \sin\frac{\theta}{2}\frac{-e^{-i\varphi_b}|01\rangle + e^{i\varphi_b}|10\rangle}{\sqrt{2}},
$$

up to phase

$$
|\tilde{\psi}\rangle = \cos\frac{\theta}{2}\frac{|00\rangle + e^{-i2\varphi_a}|11\rangle}{\sqrt{2}} + \sin\frac{\theta}{2}\frac{-e^{-i\varphi_b}|01\rangle + e^{i\varphi_b}|10\rangle}{\sqrt{2}},
$$

$-2\varphi_a = \lambda + \mu \pm \pi \quad \lambda = -\varphi_b \pm \pi$

$-2\varphi_a = \pm\pi \qquad \mu = \varphi_b \pm \pi$

$-2\varphi_a = 3\pi, \pi \quad \lambda + \mu = 0, 2\pi.$

Then

$$
|\psi\rangle = \cos\frac{\theta}{2}\left(\frac{|00\rangle - e^{i(\varphi_{01}-\varphi_{00}+\varphi_{10}-\varphi_{00})}|11\rangle}{\sqrt{2}}\right) + \sin\frac{\theta}{2}\left(\frac{e^{i(\varphi_{01}-\varphi_{00})}|01\rangle + e^{i(\varphi_{10}-\varphi_{00})}|10\rangle}{\sqrt{2}}\right),
$$

where $\varphi_{01} - \varphi_{00} \equiv \lambda, \varphi_{10} - \varphi_{00} \equiv \mu$, is the form of maximally entangled two qubit state

$$
|\psi\rangle = \cos\frac{\theta}{2}\frac{|00\rangle - e^{i(\lambda+\mu)}|11\rangle}{\sqrt{2}} + \sin\frac{\theta}{2}\frac{e^{i\lambda}|01\rangle + e^{i\mu}|10\rangle}{\sqrt{2}}. \tag{6.3}
$$

The concurrence for this state is

$$\begin{aligned} C &= 2\begin{vmatrix} \frac{\cos\frac{\theta}{2}}{\sqrt{2}} & \frac{\sin\frac{\theta}{2}}{\sqrt{2}}e^{i\lambda} \\ \frac{\sin\frac{\theta}{2}}{\sqrt{2}}e^{i\mu} & -\frac{\cos\frac{\theta}{2}}{\sqrt{2}}e^{i(\lambda+\mu)} \end{vmatrix} \\ &= \left|-\cos^2\frac{\theta}{2}e^{i(\lambda+\mu)} - \sin^2\frac{\theta}{2}e^{i(\lambda+\mu)}\right| \\ &= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1. \end{aligned}$$

If $\theta = 0$, then

$$|\psi\rangle = \frac{|00\rangle - e^{i(\lambda+\mu)}|11\rangle}{\sqrt{2}}.$$

If $\theta = \pi$, then

$$|\psi\rangle = \frac{e^{i\lambda}|01\rangle + e^{i\mu}|10\rangle}{\sqrt{2}} = e^{i\lambda}\frac{|01\rangle + e^{i(\mu-\lambda)}|10\rangle}{\sqrt{2}}.$$

For state (6.3) we have generalized Bell states.

$$\cos\frac{\theta}{2}\frac{e^{-i\frac{\lambda+\mu}{2}}|00\rangle - e^{i\frac{\lambda+\mu}{2}}|11\rangle}{\sqrt{2}} + \sin\frac{\theta}{2}\frac{e^{-i\lambda-i\frac{\lambda+\mu}{2}}|01\rangle + e^{i\mu-i\frac{\lambda+\mu}{2}}|10\rangle}{\sqrt{2}}. \tag{6.4}$$

Let $a = ie^{-i\frac{\lambda+\mu}{2}}\cos\frac{\theta}{2}$ and $\bar{a} = -ie^{i\frac{\lambda+\mu}{2}}\cos\frac{\theta}{2}$, $\bar{b} = -i\sin\frac{\theta}{2}e^{i\frac{\lambda-\mu}{2}}$ and $b = i\sin\frac{\theta}{2}e^{-i\frac{\lambda-\mu}{2}}$ so that

$$\begin{aligned} i|\psi\rangle &= \frac{i\cos\frac{\theta}{2}e^{-i\frac{\lambda+\mu}{2}}|00\rangle - ie^{i\frac{\lambda+\mu}{2}}|11\rangle}{\sqrt{2}} + \frac{i\sin\frac{\theta}{2}e^{i\frac{\lambda-\mu}{2}}|01\rangle + ie^{-i\frac{\lambda-\mu}{2}}|10\rangle}{\sqrt{2}} \\ &= \frac{a|00\rangle + \bar{a}|11\rangle}{\sqrt{2}} + \frac{-\bar{b}|01\rangle + b|10\rangle}{\sqrt{2}}. \end{aligned}$$

Theorem 6.2 All maximally entangled state are of the form

$$|\psi\rangle = \cos\frac{\theta}{2}\frac{|00\rangle - e^{i(\lambda+\mu)}|11\rangle}{\sqrt{2}} + \sin\frac{\theta}{2}\frac{e^{i\lambda}|01\rangle + e^{i\mu}|10\rangle}{\sqrt{2}}$$

or up to global phase,

$$|\psi\rangle = \frac{a|00\rangle + \bar{a}|11\rangle}{\sqrt{2}} + \frac{-\bar{b}|01\rangle + b|10\rangle}{\sqrt{2}},$$

and it can be represented as

$$|\psi\rangle = (I \otimes U)\,\frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

One more form of this state is

$$|\psi\rangle = \cos\frac{\theta}{2}\frac{|00\rangle + |11\rangle}{\sqrt{2}} + \sin\frac{\theta}{2}\frac{e^{-i\mu}|01\rangle - e^{i\mu}|10\rangle}{\sqrt{2}}.$$

In following examples we calculate action of $H \otimes H$ gate on Bell states.

Example 6.2

$$\begin{aligned}
(H \otimes H)\,\frac{|00\rangle + |11\rangle}{\sqrt{2}} &= \frac{H|0\rangle \otimes H|0\rangle + H|1\rangle \otimes H|1\rangle}{\sqrt{2}} \\
&= \frac{|h_+\rangle \otimes |h_+\rangle + |h_-\rangle \otimes |h_-\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}1\\-1\end{pmatrix}\right] \\
&= \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}
\end{aligned}$$

Example 6.3

$$\begin{aligned}(H\otimes H)\frac{|00\rangle-|11\rangle}{\sqrt{2}} &= \frac{H|0\rangle\otimes H|0\rangle-H|1\rangle\otimes H|1\rangle}{\sqrt{2}}\\ &= \frac{|h_+\rangle\otimes|h_+\rangle-|h_-\rangle\otimes|h_-\rangle}{\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\begin{pmatrix}1\\-1\end{pmatrix}\right]\\ &= \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}=\frac{|01\rangle+|10\rangle}{\sqrt{2}}\end{aligned}$$

Example 6.4

$$\begin{aligned}(H\otimes H)\frac{|01\rangle+|10\rangle}{\sqrt{2}} &= \frac{H|0\rangle\otimes H|1\rangle+H|1\rangle\otimes H|0\rangle}{\sqrt{2}}\\ &= \frac{|h_+\rangle\otimes|h_-\rangle+|h_-\rangle\otimes|h_+\rangle}{\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\right]\\ &= \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}+\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right]=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\frac{|00\rangle-|11\rangle}{\sqrt{2}}\end{aligned}$$

Example 6.5

$$
\begin{aligned}
(H\otimes H)\,\frac{|01\rangle-|10\rangle}{\sqrt{2}} &= \frac{H|0\rangle\otimes H|1\rangle - H|1\rangle\otimes H|0\rangle}{\sqrt{2}}\\
&= \frac{|h_+\rangle\otimes|h_-\rangle - |h_-\rangle\otimes|h_+\rangle}{\sqrt{2}}\\
&= \frac{1}{\sqrt{2}}\frac{1}{2}\left[\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\-1\end{pmatrix}-\begin{pmatrix}1\\-1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\right]\\
&= \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}-\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} = -\frac{|01\rangle-|10\rangle}{\sqrt{2}}
\end{aligned}
$$

Due to above examples we have proposition.

Proposition 6.5

$$
\begin{aligned}
H\otimes H\frac{|00\rangle+|11\rangle}{\sqrt{2}} &= \frac{|00\rangle+|11\rangle}{\sqrt{2}}\\
H\otimes H\frac{|00\rangle-|11\rangle}{\sqrt{2}} &= \frac{|01\rangle+|10\rangle}{\sqrt{2}}\\
H\otimes H\frac{|01\rangle-|10\rangle}{\sqrt{2}} &= -\frac{|01\rangle-|10\rangle}{\sqrt{2}}\\
H\otimes H\frac{|01\rangle+|10\rangle}{\sqrt{2}} &= \frac{|00\rangle-|11\rangle}{\sqrt{2}}
\end{aligned}
$$

Example 6.6

$$\frac{|00\rangle + e^{i\varphi}|11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\e^{i\varphi}\end{pmatrix}$$

$$\begin{aligned}(H\otimes H)\frac{|00\rangle + e^{i\varphi}|11\rangle}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix}1+e^{i\varphi}\\1-e^{i\varphi}\\1-e^{i\varphi}\\1+e^{i\varphi}\end{pmatrix} = e^{i\frac{\varphi}{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\frac{\varphi}{2}\\-i\sin\frac{\varphi}{2}\\-i\sin\frac{\varphi}{2}\\\cos\frac{\varphi}{2}\end{pmatrix}\\ &= e^{i\frac{\varphi}{2}}\left[\cos\frac{\varphi}{2}\frac{|00\rangle+|11\rangle}{\sqrt{2}} - i\sin\frac{\varphi}{2}\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right];\\ C &= 2\begin{vmatrix}\frac{\cos\frac{\varphi}{2}}{\sqrt{2}} & -i\frac{\sin\frac{\varphi}{2}}{\sqrt{2}}\\ -i\frac{\sin\frac{\varphi}{2}}{\sqrt{2}} & \frac{\cos\frac{\varphi}{2}}{\sqrt{2}}\end{vmatrix}\\ &= 2\frac{1}{2}\left(\cos^2\frac{\varphi}{2} + \sin^{\frac{\varphi}{2}}\right) = 1\end{aligned}$$

Example 6.7

$$\begin{aligned}|\psi\rangle &= \frac{|00\rangle + |11\rangle + \epsilon|01\rangle}{\sqrt{2+\epsilon^2}}\\ C &= 2\begin{vmatrix}\frac{1}{\sqrt{2+\epsilon^2}} & \frac{\epsilon}{\sqrt{2+\epsilon^2}}\\ 0 & \frac{1}{\sqrt{2+\epsilon^2}}\end{vmatrix} = \frac{2}{2+\epsilon^2} = \frac{1}{1+\frac{\epsilon^2}{2}}\\ (H\otimes H)|\psi\rangle &= \frac{1}{2}\frac{1}{\sqrt{2+\epsilon^2}}\begin{pmatrix}2+\epsilon\\-\epsilon\\\epsilon\\2-\epsilon\end{pmatrix}\\ &= \frac{1}{\sqrt{2+\epsilon^2}}\left(\left(1+\frac{\epsilon}{2}\right)|00\rangle - \frac{\epsilon}{2}|01\rangle + \frac{\epsilon}{2}|10\rangle + \left(1-\frac{\epsilon}{2}\right)|11\rangle\right)\\ \tilde{C} &= 2\frac{1}{2+\epsilon^2}\begin{vmatrix}1+\frac{\epsilon}{2} & -\frac{\epsilon}{2}\\ \frac{\epsilon}{2} & 1-\frac{\epsilon}{2}\end{vmatrix} = \frac{2}{2+\epsilon^2}\end{aligned}$$

$$C = \tilde{C}$$

Example 6.8 Let arbitrary two qubit state $|\psi\rangle$ is

$$\begin{aligned} |\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \\ &= |0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle, \end{aligned}$$

where $|\psi_0\rangle = c_{00}|0\rangle + c_{01}|1\rangle$ and $|\psi_1\rangle = c_{10}|0\rangle + c_{11}|1\rangle$. Then

$$\begin{aligned} |\tilde{\psi}\rangle &= H \otimes H|\psi\rangle \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes H|\psi_0\rangle + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes H|\psi_1\rangle \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}c_{00}\\c_{01}\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}c_{10}\\c_{11}\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}c_{00}+c_{01}\\c_{00}-c_{01}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}c_{10}+c_{11}\\c_{10}-c_{11}\end{pmatrix} \\ &= \frac{1}{2}[(c_{00}+c_{01}+c_{10}+c_{11})\,|00\rangle + (c_{00}-c_{01}+c_{10}-c_{11})\,|01\rangle \\ &+ (c_{00}+c_{01}-c_{10}-c_{11})\,|10\rangle + (c_{00}-c_{01}-c_{10}+c_{11})\,|11\rangle]. \end{aligned}$$

Then concurrence $\tilde{C}$ written as follows:

$$\begin{aligned} \tilde{C} &= 2\begin{vmatrix} \frac{c_{00}+c_{01}+c_{10}+c_{11}}{2} & \frac{c_{00}-c_{01}+c_{10}-c_{11}}{2} \\ \frac{c_{00}+c_{01}-c_{10}-c_{11}}{2} & \frac{c_{00}-c_{01}-c_{10}+c_{11}}{2} \end{vmatrix} = \frac{1}{2}\left|4|c_{00}||c_{11}| - 4|c_{01}||c_{11}|\right| \\ &= 2\left|c_{00}c_{11} - c_{01}c_{10}\right| \end{aligned}$$

Thus, we get

$$C = \tilde{C}.$$

CHAPTER 7

QUBIT-QUTRIT QUANTUM STATES

7.1. Qutrit Quantum States

Definition 7.1 Every vector in $\mathbf{C}^3$ can be expressed as a superposition of three vectors,

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle + c_2|2\rangle = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}, \tag{7.1}$$

where $|0\rangle, |1\rangle$ and $|2\rangle$ are computational basis states in matrix forms,

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and normalization condition is

$$|c_0|^2 + |c_1|^2 + |c_2|^2 = 1.$$

7.2. One Qubit and One Qutrit States

Definition 7.2 The generic qubit-qutrit state $|\psi\rangle$ is defined as

$$\begin{aligned} |\psi\rangle &= c_{00}|0\rangle|0\rangle + c_{01}|0\rangle|1\rangle + c_{02}|0\rangle|2\rangle \\ &+ c_{10}|1\rangle|0\rangle + c_{11}|1\rangle|1\rangle + c_{12}|1\rangle|2\rangle, \end{aligned} \tag{7.2}$$

where $\langle\psi|\psi\rangle = 1 \Rightarrow |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 + |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 = 1.$

7.3. Separable and Entangled Qubit-Qutrit States

An arbitrary generic qubit-qutrit state is separable if

$$|\psi\rangle = |\varphi\rangle_2 \otimes |\chi\rangle_3,$$

where $|\varphi\rangle_2$ and $|\chi\rangle_3$ are one qubit and one qutrit states, respectively.
To find separability criterium we use the left and right decomposition of states (7.2).
1. The left decomposition is

$$\begin{aligned} |\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{02}|02\rangle + c_{10}|10\rangle + c_{11}|11\rangle + c_{12}|12\rangle \\ &= |0\rangle \underbrace{(c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle)}_{|\varphi_0\rangle} + |1\rangle \underbrace{(c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle)}_{|\varphi_1\rangle} \end{aligned}$$

Then, we get

$$|\psi\rangle = |0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle. \tag{7.3}$$

where the pair of one qubit states $|\varphi_0\rangle = c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle$ and $|\varphi_1\rangle = c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle$.

Corollary 7.1 An arbitrary qubit-qutrit state $|\psi\rangle$ is separable if and only if states $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are linearly dependent.

Proof

1. Let $|\varphi_0\rangle$ and $|\varphi_1\rangle$ states are linearly dependent,

$$|\varphi_0\rangle = \lambda|\varphi_1\rangle.$$

Then

$$\begin{aligned} |\psi\rangle &= |0\rangle \otimes \lambda|\varphi_1\rangle + |1\rangle \otimes |\varphi_1\rangle \\ &= \underbrace{(\lambda|0\rangle + |1\rangle)}_{qubit} \otimes \underbrace{|\varphi_1\rangle}_{qutrit} \end{aligned}$$

and it is separable.

2. Let $|\psi\rangle = |qubit\rangle \otimes |qutrit\rangle$ is separable.
   Then

$$\begin{aligned} |\psi\rangle &= (c_0|0\rangle + c_1|1\rangle)\,|\varphi\rangle \\ &= c_0|0\rangle|\varphi\rangle + c_1|1\rangle|\varphi\rangle \\ |\psi\rangle &= |0\rangle \underbrace{(c_0|\varphi\rangle)}_{|\varphi_0\rangle} + |1\rangle \underbrace{(c_1|\varphi\rangle)}_{|\varphi_1\rangle} \end{aligned}$$

   If we define state $|\varphi_0\rangle$ as follows

$$|\varphi_0\rangle = \frac{c_0}{c_1}|\varphi_1\rangle,$$

   then we can say states $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are linearly dependent.

□

Corollary 7.2 If $|\varphi_0\rangle$ and $|\varphi_1\rangle$ are linearly independent, then the state is not separable. It is entangled.

Now, we have examined another side of decomposition.
2. The right decomposition is

$$\begin{aligned} |\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{02}|02\rangle + c_{10}|10\rangle + c_{11}|11\rangle + c_{12}|12\rangle \\ &= \underbrace{(c_{00}|0\rangle + c_{10}|1\rangle)}_{|\psi_0\rangle}|0\rangle + \underbrace{(c_{01}|0\rangle + c_{11}|1\rangle)}_{|\psi_1\rangle}|1\rangle + \underbrace{(c_{02}|0\rangle + c_{12}|1\rangle)}_{|\psi_2\rangle}|2\rangle. \end{aligned}$$

Then, we get

$$|\psi\rangle = |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle, \tag{7.4}$$

where three one qubit states are defined as $|\psi_0\rangle = (c_{00}|0\rangle + c_{10}|1\rangle)$, $|\psi_1\rangle = (c_{01}|0\rangle + c_{11}|1\rangle)$ and $|\psi_2\rangle = (c_{02}|0\rangle + c_{12}|1\rangle)$.

Corollary 7.3 An arbitrary qubit-qutrit state $|\psi\rangle$ is separable if and only if states $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly dependent and related as $|\psi_0\rangle = \lambda_1|\psi_1\rangle = \lambda_2|\psi_2\rangle$.

Proof

1. Let $|\psi\rangle = |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle$ and $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly dependent,

$$|\psi_1\rangle = \lambda_1|\psi_0\rangle,$$

$$|\psi_2\rangle = \lambda_2|\psi_0\rangle.$$

   Then,

$$\begin{aligned} |\psi\rangle &= |\psi_0\rangle|0\rangle + \lambda_1|\psi_0\rangle|1\rangle + \lambda_2|\psi_0\rangle|2\rangle \\ &= |\psi_0\rangle\left(|0\rangle + \lambda_1|1\rangle + \lambda_2|2\rangle\right) \end{aligned}$$

   and it is separable.

2. Let $|\psi\rangle = |qubit\rangle \otimes |qutrit\rangle$ is separable.
   Then

$$\begin{aligned} |\psi\rangle &= |\chi\rangle \otimes (c_0|0\rangle + c_1|1\rangle + c_2|2\rangle) \\ &= \underbrace{c_0|\chi\rangle}_{|\psi_0\rangle}|0\rangle + \underbrace{c_1|\chi\rangle}_{|\psi_1\rangle}|1\rangle + \underbrace{c_2|\chi\rangle}_{|\psi_2\rangle}|2\rangle \\ |\psi\rangle &= |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle \end{aligned}$$

   and $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly dependent, and $|\psi_0\rangle = \lambda_1|\psi_1\rangle$, $|\psi_0\rangle = \lambda_2|\psi_2\rangle$.

□

Corollary 7.4 If $|\psi_0\rangle, |\psi_1\rangle$ and $|\varphi_2\rangle$ are linearly independent, then it is not separable. It is entangled.

An arbitrary qubit-qutrit states can be written in both forms (7.3) or (7.4).

7.4. Qubit-Qutrit Entanglement

Let $|\psi\rangle$ is the generic qubit-qutrit state,

$$\begin{aligned} |\psi\rangle &= c_{00}|0\rangle|0\rangle + c_{01}|0\rangle|1\rangle + c_{02}|0\rangle|2\rangle + c_{10}|1\rangle|0\rangle + c_{11}|1\rangle|1\rangle + c_{12}|1\rangle|2\rangle \\ &= (c_{00}|0\rangle + c_{10}|1\rangle)\,|0\rangle + (c_{01}|0\rangle + c_{11}|1\rangle)\,|1\rangle + (c_{02}|0\rangle + c_{12}|1\rangle)\,|2\rangle \\ |\psi\rangle &= |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle, \end{aligned}$$

where $|\psi_0\rangle = c_{00}|0\rangle + c_{10}|1\rangle, |\psi_1\rangle = c_{01}|0\rangle + c_{11}|1\rangle$ and $|\psi_2\rangle = c_{02}|0\rangle + c_{12}|1\rangle$.

7.5. Density Matrix

For qubit-qutrit state $|\psi\rangle$, the density operator can be written as

$$\begin{aligned} \rho &= |\psi\rangle\langle\psi| \\ &= (|\psi_0\rangle|0\rangle_B + |\psi_1\rangle|1\rangle_B + |\psi_2\rangle|2\rangle_B)\,(\langle\psi_0|_B\langle 0| + \langle\psi_1|_B\langle 1| + \langle\psi_2|_B\langle 2|) \\ &= |\psi_0\rangle\langle\psi_0|\,(|0\rangle_B\,{}_B\langle 0|) + |\psi_0\rangle\langle\psi_1|\,(|0\rangle_B\,{}_B\langle 1|) + |\psi_0\rangle\langle\psi_2|\,(|0\rangle_B\,{}_B\langle 2|) \\ &+ |\psi_1\rangle\langle\psi_0|\,(|1\rangle_B\,{}_B\langle 0|) + |\psi_1\rangle\langle\psi_1|\,(|1\rangle_B\,{}_B\langle 1|) + |\psi_1\rangle\langle\psi_2|\,(|1\rangle_B\,{}_B\langle 2|) \\ &+ |\psi_2\rangle\langle\psi_0|\,(|2\rangle_B\,{}_B\langle 0|) + |\psi_2\rangle\langle\psi_1|\,(|2\rangle_B\,{}_B\langle 1|) + |\psi_2\rangle\langle\psi_2|\,(|2\rangle_B\,{}_B\langle 2|)\,. \end{aligned}$$

7.6. Reduced Density Matrix

For generic qubit-qutrit states, we can express the reduced density operator for subsystem A in the following form:

$$
\begin{aligned}
\rho_A &= tr_B\rho = \langle 0|\rho|0\rangle + \langle 1|\rho|1\rangle + \langle 2|\rho|2\rangle \\
&= |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|,
\end{aligned}
$$

and trace of this density matrix is

$$
\begin{aligned}
tr\rho_A &= tr\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right) \\
&= \langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle = 1. \qquad (7.5)
\end{aligned}
$$

The squared reduced density matrix is written as

$$
\begin{aligned}
\rho_A^2 &= \left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right)\left(|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right) \\
&= |\psi_0\rangle\langle\psi_0|\langle\psi_0|\psi_0\rangle + |\psi_0\rangle\langle\psi_1|\langle\psi_0|\psi_1\rangle + |\psi_0\rangle\langle\psi_2|\langle\psi_0|\psi_2\rangle \\
&+ |\psi_1\rangle\langle\psi_0|\langle\psi_1|\psi_0\rangle + |\psi_1\rangle\langle\psi_1|\langle\psi_1|\psi_1\rangle + |\psi_1\rangle\langle\psi_2|\langle\psi_1|\psi_2\rangle \\
&+ |\psi_2\rangle\langle\psi_0|\langle\psi_2|\psi_0\rangle + |\psi_2\rangle\langle\psi_1|\langle\psi_2|\psi_1\rangle + |\psi_2\rangle\langle\psi_2|\langle\psi_2|\psi_2\rangle
\end{aligned}
$$

or in explicit matrix form

$$
\rho_A^2 = \begin{pmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle & \langle\psi_0|\psi_2\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle & \langle\psi_0|\psi_2\rangle \\ \langle\psi_2|\psi_0\rangle & \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle \end{pmatrix},
$$

where $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$ are basis.

Then, we calculate trace of squared reduced density matrix (See Appendix A)

$$\begin{aligned} tr\rho_A^2 &= |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_0|\psi_1\rangle|^2 + |\langle\psi_0|\psi_2\rangle|^2 \\ &+ |\langle\psi_1|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_0|\psi_2\rangle|^2 \\ &+ |\langle\psi_2|\psi_0\rangle|^2 + |\langle\psi_2|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2, \end{aligned}$$

and squared normalization condition (7.5)

$$\begin{aligned} 1 = |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2 + 2\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle \\ +2\langle\psi_0|\psi_0\rangle\langle\psi_2|\psi_2\rangle + 2\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle. \end{aligned}$$

Taking difference

$$\begin{aligned} 1 - tr\rho_A^2 &= 2\left(\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle - |\langle\psi_0|\psi_1\rangle|^2\right) \\ &+ 2\left(\langle\psi_0|\psi_0\rangle\langle\psi_2|\psi_2\rangle - |\langle\psi_0|\psi_2\rangle|^2\right) \\ &+ 2\left(\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle - |\langle\psi_1|\psi_2\rangle|^2\right) \end{aligned}$$

or

$$1 - tr\rho_A^2 = 2\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_2\rangle \\ \langle\psi_2|\psi_0\rangle & \langle\psi_2|\psi_2\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle \end{vmatrix},$$

we get (see Appendix A)

$$1 - tr\rho_A^2 = 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2,$$

where the partial concurrences are defined as

$$C_{01} = 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}, C_{02} = 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix} \quad \text{and} \quad C_{12} = 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}. \tag{7.6}$$

As a result we have

$$1 - tr\rho_A^2 = \frac{1}{2}\left(C_{01}^2 + C_{02}^2 + C_{12}^2\right) \equiv \frac{1}{2}C^2,$$

giving the relation

$$tr\rho_A^2 + \frac{1}{2}C^2 = 1. \tag{7.7}$$

Therefore, the total concurrence C satisfies

$$C^2 = C_{01}^2 + C_{02}^2 + C_{12}^2,$$

or

$$C = \sqrt{C_{01}^2 + C_{02}^2 + C_{12}^2}.$$

This way we get following proposition.

Proposition 7.1 For qubit-qutrit state (7.2), the total concurrence C is equal

$$C = |\vec{C}| = 2\sqrt{|C_{01}|^2 + |C_{02}|^2 + |C_{12}|^2},$$

where the concurrence vector

$$\vec{C} = (C_{01}, C_{02}, C_{12}).$$

In a similar way we decompose

$$|\psi\rangle \quad = \quad |0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle$$

in terms of two qutrit states $|\varphi_0\rangle = c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle$ and $|\varphi_1\rangle = c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle$. Then

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| = (|0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle)\,(\langle 0|\langle\varphi_0| + \langle 1|\langle\varphi_1|)\\ &= |0\rangle\langle 0|(|\varphi_0\rangle\langle\varphi_0|) + |0\rangle\langle 1|(|\varphi_0\rangle\langle\varphi_1|)\\ &+ |1\rangle\langle 0|(|\varphi_1\rangle\langle\varphi_0|) + |1\rangle\langle 1|(|\varphi_1\rangle\langle\varphi_1|)\end{aligned}$$

or in explicit form

$$\rho = \begin{pmatrix} |\varphi_0\rangle\langle\varphi_0| & |\varphi_0\rangle\langle\varphi_1| \\ |\varphi_1\rangle\langle\varphi_0| & |\varphi_1\rangle\langle\varphi_1| \end{pmatrix}.$$

We can demonstrate also that, for subsystem B, the reduced density matrix is

$$\rho_B = tr_A\rho = |\varphi_0\rangle\langle\varphi_0| + |\varphi_1\rangle\langle\varphi_1|,$$

and trace of this matrix is

$$tr\rho_B = 1 \Rightarrow \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle = 1.$$

Then, the squared reduced density matrix is

$$\begin{aligned}\rho_B^2 &= |\varphi_0\rangle\langle\varphi_0|\langle\varphi_0|\varphi_0\rangle + |\varphi_1\rangle\langle\varphi_1|\langle\varphi_1|\varphi_1\rangle\\ &+ |\varphi_0\rangle\langle\varphi_1|\langle\varphi_1|\varphi_0\rangle + |\varphi_1\rangle\langle\varphi_0|\langle\varphi_0|\varphi_1\rangle\end{aligned}$$

and the trace of this matrix is equal

$$tr\rho_B^2 = |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_1|\varphi_0\rangle|^2 + |\langle\varphi_0|\varphi_1\rangle|^2.$$

The squared normalization condition gives

$$1 = |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle.$$

Then, by taking difference

$$\begin{aligned} 1 - tr\rho_B^2 &= 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle - 2|\langle\varphi_0|\varphi_1\rangle|^2 \\ &= 2\begin{vmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle \end{vmatrix} \end{aligned}$$

we get (See Appendix B)

$$1 - tr\rho_B^2 = 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2,$$

where we define the same concurrence vector (7.6)

$$C_{01} = 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}, C_{02} = 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix} \quad \text{and} \quad C_{12} = 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}.$$

So, we have

$$1 - tr\rho_B^2 = \frac{1}{2}\left(C_{01}^2 + C_{02}^2 + C_{12}^2\right) = \frac{1}{2}C^2.$$

Thus, this gives the same relation

$$tr\rho_B^2 + \frac{1}{2}C^2 = 1. \qquad (7.8)$$

Comparing equations (7.7) and (7.8) we get identity

$$\begin{aligned} tr\rho_B^2 &= |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + 2|\langle\varphi_0|\varphi_1\rangle|^2 \\ &= |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2 \\ &+ 2|\langle\psi_0|\psi_1\rangle|^2 + 2|\langle\psi_0|\psi_2\rangle|^2 + 2|\langle\psi_1|\psi_2\rangle|^2 \\ &= tr\rho_A^2 \\ \Rightarrow tr\rho_A^2 &= tr\rho_B^2. \end{aligned}$$

7.7. Maximal Value of Concurrence

Proposition 7.2 For generic state

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{02}|02\rangle + c_{10}|10\rangle + c_{11}|11\rangle + c_{12}|12\rangle,$$

$$\max C = 1.$$

Proof

$$\begin{aligned} |\psi\rangle &= |0\rangle_A \underbrace{(c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle)}_{|\varphi_0\rangle} + |1\rangle_A \underbrace{(c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle)}_{|\varphi_1\rangle} \\ &= \underbrace{(c_{00}|0\rangle + c_{10}|1\rangle)}_{|\psi_0\rangle}|0\rangle + \underbrace{(c_{01}|0\rangle + c_{11}|1\rangle)}_{|\psi_1\rangle}|1\rangle + \underbrace{(c_{02}|0\rangle + c_{12}|1\rangle)}_{|\psi_2\rangle}|2\rangle, \end{aligned}$$

and

$$\begin{aligned} \rho &= (|0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle)(\langle 0|\langle\varphi_0| + \langle 1|\langle\varphi_1|) \\ &= (|\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle)(\langle\psi_0|\langle 0| + \langle\psi_1|\langle 1| + \langle\psi_2|\langle 2|), \end{aligned}$$

and the reduced density matrices are

$$\begin{aligned}
\rho_A &= tr_B\rho = |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|, \\
\rho_B &= tr_A\rho = |\varphi_0\rangle\langle\varphi_0| + |\varphi_1\rangle\langle\varphi_1|.
\end{aligned}$$

Then,

$$\begin{aligned}
\rho_B &= (c_{00}|0\rangle + c_{10}|1\rangle)(\bar{c}_{00}\langle 0| + \bar{c}_{10}\langle 1|) \\
&+ (c_{01}|0\rangle + c_{11}|1\rangle)(\bar{c}_{01}\langle 0| + \bar{c}_{11}\langle 1|) \\
&+ (c_{02}|0\rangle + c_{12}|1\rangle)(\bar{c}_{02}\langle 0| + \bar{c}_{12}\langle 1|),
\end{aligned}$$

or in explicit form

$$\rho_B = \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12} \\ c_{10}\bar{c}_{00} + c_{11}\bar{c}_{10} + c_{12}\bar{c}_{02} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 \end{pmatrix}.$$

Due to $\rho_B = \rho_B^\dagger$, exists unitary transformation which can diagonalize ρ_B as

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where λ_1, λ_2 are real numbers.
To find λ_1, λ_2 we should solve characteristic equation.

$$\begin{aligned}
&|\rho_B - \lambda I| = 0 \\
\Rightarrow &\begin{vmatrix} \rho_{00} - \lambda & \rho_{01} \\ \rho_{10} & \rho_{11} - \lambda \end{vmatrix} = 0 \\
\Rightarrow &\lambda^2 - \lambda tr\rho_B + det\rho_B = 0.
\end{aligned}$$

Since $tr\rho_B = 1$ we have

$$\lambda^2 - \lambda + det\rho_B = 0,$$

so, we get

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - det\rho_A}.$$

Since, $\rho_B = \rho_B^\dagger \Rightarrow \lambda_1, \lambda_2$ are real, and as follows

$$det\rho_B \leq \frac{1}{4}.$$

The transformation implies,

$$\rho_B \rightarrow \tilde{\rho}_B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$
$$\rho_B^2 \rightarrow \tilde{\rho}_B^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix},$$
$$tr\rho_B^2 = tr\tilde{\rho}_B^2 = \lambda_1^2 + \lambda_2^2.$$

and

1. $tr\rho_B = 1 \Rightarrow \lambda_1 + \lambda_2 = 1$
2. $det\rho_B \equiv \frac{1}{4}C^2 \Rightarrow C^2 = 4det\rho_B$
3. $det\rho_B = det\tilde{\rho}_B = \lambda_1\lambda_2$

So, $C^2 = 4\lambda_1\lambda_2$ and

$$\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 = (\lambda_1 + \lambda_2)^2 = 1$$
$$\Rightarrow tr\rho_B^2 + \frac{C^2}{2} = 1.$$

Now we can show that max $C = 1$. We have two relations

$$\lambda_1 + \lambda_2 = 1,$$
$$C^2 = 4\lambda_1\lambda_2,$$

and we need to find critical points of function

$$F(\lambda_1, \lambda_2, \lambda) = 4\lambda_1\lambda_2 - \lambda(\lambda_1 + \lambda_2 - 1).$$

Taking partial derivatives

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow \lambda_1 + \lambda_2 = 1,$$
$$\frac{\partial F}{\partial \lambda_1} = 4\lambda_2 - \lambda = 0 \Rightarrow \lambda_2 = \frac{\lambda}{4},$$
$$\frac{\partial F}{\partial \lambda_2} = 4\lambda_1 - \lambda = 0 \Rightarrow \lambda_1 = \frac{\lambda}{4},$$

this gives $\lambda_1 = \lambda_2$, and then

$$\lambda_1 + \lambda_2 = 1 \Rightarrow 2\lambda_1 = 1 \Rightarrow \lambda_1 = \lambda_2 = \frac{1}{2}.$$

Finally, we get

$$max\, C^2 = 4\frac{1}{2}\frac{1}{2} = 1.$$

□

Example 7.1 Let

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

is pure Bell state. Then $c_{00} = c_{11} = \frac{1}{\sqrt{2}}$, and as follows $C = 1$.

Example 7.2 Let

$$|\psi\rangle = \frac{|00\rangle + |12\rangle}{\sqrt{2}}.$$

Then the concurrence is

$$C^2 = 4\left(\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + \begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + \begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2\right).$$

Since, $c_{00} = c_{12} = \frac{1}{\sqrt{2}}$ and all others are 0, then, $C = 1$.

7.8. Entanglement Invariance Under Unitary Transformations

Here we are going to show invariance of entanglement under unitary transformation of special form.

Proposition 7.3 The concurrence $\tilde{C} = C$, is invariant under unitary transformation

$$|\tilde{\psi}\rangle = (I_A \otimes U_B)\,|\psi\rangle,$$

where $U_B \in SU(3)$ is arbitrary one qubit unitary gate.

Proof

$$(I_A \otimes U_B)\,|\psi\rangle = |0\rangle \underbrace{(U_B|\varphi_0\rangle)}_{|\tilde{\varphi}_0\rangle} + |1\rangle \underbrace{(U_B|\varphi_1\rangle)}_{|\tilde{\varphi}_1\rangle}$$

Since

$$\langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle = \langle\varphi_0|\varphi_0\rangle \quad \langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle = \langle\varphi_1|\varphi_1\rangle$$
$$\langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle = \langle\varphi_0|\varphi_1\rangle \quad \langle\tilde{\varphi}_1|\tilde{\varphi}_0\rangle = \langle\varphi_1|\varphi_0\rangle$$

Then, $C = \tilde{C}$. □

Proposition 7.4 The concurrence $\tilde{C} = C$, is invariant under unitary transformation

$$|\tilde{\psi}\rangle = (U_A \otimes I_B)\,|\psi\rangle,$$

where $U_A \in SU(2)$ is arbitrary one qubit unitary gate.

Proof

$$(U_A \otimes I_B)\,|\psi\rangle \;=\; (U_A|\psi_0\rangle)\,|0\rangle + (U_A|\psi_1\rangle)\,|1\rangle + (U_A|\psi_2\rangle)\,|2\rangle$$

so that

$$|\tilde{\psi}\rangle \;=\; |\tilde{\psi}_0\rangle|0\rangle + |\tilde{\psi}_1\rangle|1\rangle + |\tilde{\psi}_2\rangle|2\rangle.$$

Due to

$$\langle\tilde{\psi}_i|\tilde{\psi}_j\rangle \;=\; \langle\psi_i|U_A^{\dagger}U_A|\psi_j\rangle = \langle\psi_i|\psi_j\rangle.$$

Then,

$$\tilde{C} = C.$$

□

Proposition 7.5 The concurrence $\tilde{C} = C$, is invariant under unitary transformation

$$|\tilde{\psi}\rangle = (U_A \otimes U_B)\,|\psi\rangle.$$

where $U_A \in SU(2)$, $U_B \in SU(3)$.

Proof Let $|\psi\rangle$ is qubit-qutrit state and $(U_A \otimes U_B)$ acts on this state, we have

$$(U_A \otimes U_B)\,|\psi\rangle \;=\; (U_A|0\rangle)\,(U_B|\varphi_0\rangle) + (U_A|1\rangle)\,(U_B|\varphi_1\rangle)\,,$$

where $U_A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $U_A|0\rangle = a|0\rangle - \bar{b}|1\rangle$ and $U_A|1\rangle = b|0\rangle + \bar{a}|1\rangle$.

Then,

$$\begin{aligned} |\tilde{\psi}\rangle &= \left(a|0\rangle - \bar{b}|1\rangle\right)(U_B|\varphi_0\rangle) + (b|0\rangle + \bar{a}|1\rangle)\,(U_B|\varphi_1\rangle) \\ &= |0\rangle \underbrace{(aU_B|\varphi_0\rangle + bU_B|\varphi_1\rangle)}_{|\varphi_0'\rangle} + |1\rangle \underbrace{\left(-\bar{b}U_B|\varphi_0\rangle + \bar{a}U_B|\varphi_1\rangle\right)}_{|\varphi_1'\rangle} \end{aligned}$$

where $|\varphi_0'\rangle = aU_B|\varphi_0\rangle + bU_1|\varphi_1\rangle$ and $|\varphi_1'\rangle = -\bar{b}U_B|\varphi_0\rangle + \bar{a}U_B|\varphi_1\rangle$.

We have the following inner products

$$\begin{aligned} \langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle &= \left(\langle\varphi_0|U_B^\dagger\bar{a} + \langle\varphi_1|U_B^\dagger\bar{b}\right)(aU_B|\varphi_0\rangle + bU_1|\varphi_1\rangle) \\ &= |a|^2\langle\varphi_0|\varphi_0\rangle + |b|^2\langle\varphi_1|\varphi_1\rangle + \bar{a}b\langle\varphi_0|\varphi_1\rangle + a\bar{b}\langle\varphi_1|\varphi_0\rangle \\ \langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle &= |b|^2\langle\varphi_0|\varphi_0\rangle + |a|^2\langle\varphi_1|\varphi_1\rangle - a\bar{b}\langle\varphi_1|\varphi_0\rangle - \bar{a}b\langle\varphi_0|\varphi_1\rangle \\ \langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle &= -\bar{a}\bar{b}\langle\varphi_0|\varphi_0\rangle + \bar{a}\bar{b}\langle\varphi_1|\varphi_1\rangle - \bar{b}^2\langle\varphi_1|\varphi_0\rangle + \bar{a}^2\langle\varphi_0|\varphi_1\rangle \\ \langle\tilde{\varphi}_1|\tilde{\varphi}_0\rangle &= -ab\langle\varphi_0|\varphi_0\rangle + ab\langle\varphi_1|\varphi_1\rangle - b^2\langle\varphi_0|\varphi_1\rangle + a^2\langle\varphi_1|\varphi_0\rangle. \end{aligned}$$

We write $(U_A \otimes U_B)\,|\psi\rangle$ in matrix form

$$|\tilde{\psi}\rangle = \begin{pmatrix} \langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle \\ \langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle \\ \langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle \\ \langle\tilde{\varphi}_1|\tilde{\varphi}_0\rangle \end{pmatrix} \;=\; \begin{pmatrix} |a|^2 & \bar{a}b & a\bar{b} & |b|^2 \\ -\bar{a}\bar{b} & \bar{a}^2 & -\bar{b}^2 & \bar{a}\bar{b} \\ -ab & -b^2 & a^2 & ab \\ |b|^2 & -\bar{a}b & -a\bar{b} & |a|^2 \end{pmatrix} \begin{pmatrix} \langle\varphi_0|\varphi_0\rangle \\ \langle\varphi_1|\varphi_1\rangle \\ \langle\varphi_0|\varphi_1\rangle \\ \langle\varphi_1|\varphi_0\rangle \end{pmatrix}$$

Then

$$|\tilde{\psi}\rangle = \begin{pmatrix} \langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle & \langle\tilde{\varphi}_1|\tilde{\varphi}_0\rangle \\ \langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle & \langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle \end{pmatrix} = U_A G^T U_A^\dagger = U_A \begin{pmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_1|\varphi_0\rangle \\ \langle\varphi_0|\varphi_1\rangle & \langle\varphi_1|\varphi_1\rangle \end{pmatrix} U_A^\dagger = \tilde{G}^T$$

where, G is Gram matrix.

$$G \equiv \begin{pmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_1|\varphi_0\rangle \\ \langle\varphi_0|\varphi_1\rangle & \langle\varphi_1|\varphi_1\rangle \end{pmatrix};$$

and

$$\tilde{G} = \begin{pmatrix} \langle\tilde{\varphi}_0|\tilde{\varphi}_0\rangle & \langle\tilde{\varphi}_1|\tilde{\varphi}_0\rangle \\ \langle\tilde{\varphi}_0|\tilde{\varphi}_1\rangle & \langle\tilde{\varphi}_1|\tilde{\varphi}_1\rangle \end{pmatrix}.$$

Then, transformation $U_A \otimes U_B$ generates

$$\tilde{G}^T = U_A G^T U_A^\dagger \Leftrightarrow \tilde{G} = (U_A)^T G (U_A^\dagger)^T.$$

Since

$$det\ \tilde{G}^T = det\ \tilde{G}$$

and

$$det\ G^T = det\ G$$
$$\Rightarrow det\ \tilde{G} = det\ G.$$

Thus, we have $\tilde{C} = C$. □

CHAPTER 8

TWO QUTRIT QUANTUM STATES

8.1. Two Qutrit States

Definition 8.1 Let $|\psi\rangle$ is the generic two-qutrit state in the following form

$$\begin{aligned} |\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{02}|02\rangle \\ &+ c_{10}|10\rangle + c_{11}|11\rangle + c_{12}|12\rangle \\ &+ c_{20}|20\rangle + c_{21}|21\rangle + c_{22}|22\rangle, \end{aligned} \tag{8.1}$$

where normalization condition for state $|\psi\rangle$ is

$$\langle\psi|\psi\rangle = |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 + |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 + |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 = 1. \tag{8.2}$$

8.2. Separable and Entangled States

Definition 8.2 If $|\psi\rangle = |\varphi\rangle \otimes |\chi\rangle$, where $|\varphi\rangle$ and $|\chi\rangle$ are the one qutrit states, then $|\psi\rangle$ is separable. If not, then the state is entangled. An arbitrary generic two qutrit state (8.1) can be represented by three one qutrit states. For these representations we use the left and the right decompositions.

1. The left decomposition is

$$\begin{aligned}|\psi\rangle &= |0\rangle \underbrace{(c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle)}_{|\varphi_0\rangle} \\ &+ |1\rangle \underbrace{(c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle)}_{|\varphi_1\rangle} \\ &+ |2\rangle \underbrace{(c_{20}|0\rangle + c_{21}|1\rangle + c_{22}|2\rangle)}_{|\varphi_2\rangle}\end{aligned}$$

and we can write

$$|\psi\rangle = |0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle + |2\rangle|\varphi_2\rangle, \tag{8.3}$$

where,

$$|\varphi_0\rangle = c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle,$$

$$|\varphi_1\rangle = c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle,$$

$$|\varphi_2\rangle = c_{20}|0\rangle + c_{21}|1\rangle + c_{22}|2\rangle.$$

2. The right decomposition is

$$\begin{aligned}|\psi\rangle &= \underbrace{(c_{00}|0\rangle + c_{10}|1\rangle + c_{20}|2\rangle)}_{|\psi_0\rangle} |0\rangle \\ &+ \underbrace{(c_{01}|0\rangle + c_{11}|1\rangle + c_{21}|2\rangle)}_{|\psi_1\rangle} |1\rangle \\ &+ \underbrace{(c_{02}|0\rangle + c_{12}|1\rangle + c_{22}|2\rangle)}_{|\psi_2\rangle} |2\rangle\end{aligned}$$

and we can write

$$|\psi\rangle = |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle, \tag{8.4}$$

where,

$$|\psi_0\rangle = c_{00}|0\rangle + c_{10}|1\rangle + c_{20}|2\rangle,$$

$$|\psi_1\rangle = c_{01}|0\rangle + c_{11}|1\rangle + c_{21}|2\rangle,$$

$$|\psi_2\rangle = c_{02}|0\rangle + c_{12}|1\rangle + c_{22}|2\rangle.$$

An arbitrary two qutrit states can be written in both forms (8.3) or (8.4).

Theorem 8.1 An arbitrary two qutrit state $|\psi\rangle$ is separable if and only if one qutrit states $|\varphi_0\rangle$, $|\varphi_1\rangle$ and $|\varphi_2\rangle$, or $|\psi_0\rangle$,$|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly dependent.

Proof

1. Let $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\varphi_2\rangle$ are linearly dependent,

$$\begin{aligned} |\varphi_1\rangle &= \lambda_1|\varphi_0\rangle \\ |\varphi_2\rangle &= \lambda_2|\varphi_0\rangle. \end{aligned}$$

   Then,

$$\begin{aligned} |\psi\rangle &= |0\rangle|\varphi_0\rangle + \lambda_1|1\rangle|\varphi_0\rangle + \lambda_2|2\rangle|\varphi_0\rangle \\ &= (|0\rangle + \lambda_1|1\rangle + \lambda_2|2\rangle)\,|\varphi_0\rangle \end{aligned}$$

   and it is separable.

2. Let $|\psi\rangle$ is separable, and can be written as

$$|\psi\rangle = (a_0|0\rangle + a_1|1\rangle + a_2|2\rangle)\,(b_0|0\rangle + b_1|1\rangle + b_2|2\rangle)$$

Then

$$
\begin{aligned}
|\psi\rangle &= c_{00}|00\rangle + c_{01}|01\rangle + c_{02}|02\rangle + c_{10}|10\rangle + c_{11}|11\rangle \\
&+ c_{12}|12\rangle + c_{20}|20\rangle + c_{21}|21\rangle + c_{22}|22\rangle \\
&= a_0b_0|00\rangle + a_0b_1|01\rangle + a_0b_2|02\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle \\
&+ a_1b_2|12\rangle + a_2b_0|20\rangle + a_2b_1|21\rangle + a_2b_2|22\rangle \\
&= a_0|0\rangle\,(b_0|0\rangle + b_1|1\rangle + b_2|2\rangle) \\
&+ a_1|1\rangle\,(b_0|0\rangle + b_1|1\rangle + b_2|2\rangle) \\
&+ a_2|2\rangle\,(b_0|0\rangle + b_1|1\rangle + b_2|2\rangle) \\
|\varphi_0\rangle &= a_0|0\rangle\,(b_0|0\rangle + b_1|1\rangle + b_2|2\rangle) = c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle \\
|\varphi_1\rangle &= a_1|1\rangle\,(b_0|0\rangle + b_1|1\rangle + b_2|2\rangle) = c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle \\
|\varphi_2\rangle &= a_2|2\rangle\,(b_0|0\rangle + b_1|1\rangle + b_2|2\rangle) = c_{20}|0\rangle + c_{21}|1\rangle + c_{22}|2\rangle
\end{aligned}
$$

So, the states $\begin{array}{l} |\varphi_0\rangle = a_0|\chi\rangle \\ |\varphi_1\rangle = a_1|\chi\rangle \\ |\varphi_2\rangle = a_2|\chi\rangle \end{array} \Rightarrow \begin{array}{l} |\varphi_1\rangle = \lambda_1 \mid \varphi_0\rangle \\ |\varphi_2\rangle = \lambda_2\,|\varphi_0\rangle \end{array}$ are linearly dependent.

□

8.3. Entanglement of Pure Two Qutrit State

8.3.1. Density Matrix

Definition 8.3 For two qutrit state $|\psi\rangle$, the density matrix is given by

$$
\begin{aligned}
\rho &= |\psi\rangle\langle\psi| = \sum_{i,j=0}^{2} c_{ij}|ij\rangle \sum_{i',j'=0}^{2} \bar{c}_{i'j'}\langle i'j'| \\
&= \sum_{i,j=0}^{2}\sum_{i',j'=0}^{2} c_{ij}\bar{c}_{i'j'}|ij\rangle\langle i'j'|, \qquad (8.5)
\end{aligned}
$$

or by (8.5) in terms of one qutrit states

$$\begin{aligned}
\rho &= (|0\rangle_A|\varphi_0\rangle_B + |1\rangle_A|\varphi_1\rangle_B + |2\rangle_A|\varphi_2\rangle_B)\,({}_A\langle 0|_B\langle\varphi_0| +_A \langle 1|_B\langle\varphi_1| +_A \langle 2|_B\langle\varphi_2|) \\
&= (|0\rangle_{A\,A}\langle 0|)\,(|\varphi_0\rangle_{B\,B}\langle\varphi_0|) + (|0\rangle_{A\,A}\langle 1|)\,(|\varphi_0\rangle_{B\,B}\langle\varphi_1|) + (|0\rangle_{A\,A}\langle 2|)\,(|\varphi_0\rangle_{B\,B}\langle\varphi_2|) \\
&+ (|1\rangle_{A\,A}\langle 0|)\,(|\varphi_1\rangle_{B\,B}\langle\varphi_0|) + (|1\rangle_{A\,A}\langle 1|)\,(|\varphi_1\rangle_{B\,B}\langle\varphi_1|) + (|1\rangle_{A\,A}\langle 2|)\,(|\varphi_1\rangle_{B\,B}\langle\varphi_2|) \\
&+ (|2\rangle_{A\,A}\langle 0|)\,(|\varphi_2\rangle_{B\,B}\langle\varphi_0|) + (|2\rangle_{A\,A}\langle 1|)\,(|\varphi_2\rangle_{B\,B}\langle\varphi_1|) + (|2\rangle_{A\,A}\langle 2|)\,(|\varphi_2\rangle_{B\,B}\langle\varphi_2|)\,.
\end{aligned}$$

In explicit matrix form in Alice computational basis $\{|i\rangle_A\}$, where $i = 0, 1, 2$

$$\rho_B = \begin{pmatrix} |\varphi_0\rangle\langle\varphi_0| & |\varphi_0\rangle\langle\varphi_1| & |\varphi_0\rangle\langle\varphi_2| \\ |\varphi_1\rangle\langle\varphi_0| & |\varphi_1\rangle\langle\varphi_1| & |\varphi_1\rangle\langle\varphi_2| \\ |\varphi_2\rangle\langle\varphi_0| & |\varphi_2\rangle\langle\varphi_1| & |\varphi_2\rangle\langle\varphi_2| \end{pmatrix} \tag{8.6}$$

8.3.2. Reduced Density Matrix

1. **Proposition 8.1** For subsystem B, we can write the reduced density matrix as

$$\rho_B = tr_A(\rho) = |\varphi_0\rangle_{B\,B}\langle\varphi_0| + |\varphi_1\rangle_{B\,B}\langle\varphi_1| + |\varphi_2\rangle_{B\,B}\langle\varphi_2|,$$

where $tr_A(\rho)$ is called the partial trace, and

$$tr_B\rho_B = \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle + \langle\varphi_2|\varphi_2\rangle = 1.$$

Proof Due to the reduced density matrix is following form

$$\rho_B = |\varphi_0\rangle\langle\varphi_0| + |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|.$$

Trace of ρ_B is

$$
\begin{aligned}
tr\rho_B &= {}_B\langle 0 \mid \rho_B \mid 0\rangle_B +_B \langle 1 \mid \rho_B \mid 1\rangle_B +_B \langle 2 \mid \rho_B \mid 2\rangle_B \\
&= {}_B\langle 0|\varphi_0\rangle\langle\varphi_0|0\rangle_B +_B \langle 1|\varphi_0\rangle\langle\varphi_0|1\rangle_B +_B \langle 2|\varphi_0\rangle\langle\varphi_0|2\rangle_B \\
&+ {}_B\langle 0|\varphi_1\rangle\langle\varphi_1|0\rangle_B +_B \langle 1|\varphi_1\rangle\langle\varphi_1|1\rangle_B +_B \langle 2|\varphi_1\rangle\langle\varphi_1|2\rangle_B \\
&+ {}_B\langle 0|\varphi_2\rangle\langle\varphi_2|0\rangle_B +_B \langle 1|\varphi_2\rangle\langle\varphi_2|1\rangle_B +_B \langle 2|\varphi_2\rangle\langle\varphi_2|2\rangle_B \\
&= \langle\varphi_0|0\rangle_B\, {}_B\langle 0|\varphi_0\rangle + \langle\varphi_0|1\rangle_B\, {}_B\langle 1|\varphi_0\rangle + \langle\varphi_0|2\rangle_B\, {}_B\langle 2|\varphi_0\rangle \\
&+ \langle\varphi_1|0\rangle_B\, {}_B\langle 0|\varphi_1\rangle + \langle\varphi_1|1\rangle_B\, {}_B\langle 1|\varphi_1\rangle + \langle\varphi_1|2\rangle_B\, {}_B\langle 2|\varphi_1\rangle \\
&+ \langle\varphi_2|0\rangle_B\, {}_B\langle 0|\varphi_2\rangle + \langle\varphi_2|1\rangle_B\, {}_B\langle 1|\varphi_2\rangle + \langle\varphi_2|2\rangle_B\, {}_B\langle 2|\varphi_2\rangle \\
&= \langle\varphi_0|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I}|\varphi_0\rangle \\
&+ \langle\varphi_1|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I}|\varphi_1\rangle \\
&+ \langle\varphi_2|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I}|\varphi_2\rangle \\
&= \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle + \langle\varphi_2|\varphi_2\rangle
\end{aligned}
$$

and it gives

$$
tr\rho_B = 1 \Rightarrow \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle + \langle\varphi_2|\varphi_2\rangle = 1.
$$

□

Then, the squared reduced density matrix is

$$
\begin{aligned}
\rho_B^2 &= \sum_{i,j=0}^{2}\langle\varphi_i \mid \varphi_j\rangle|\varphi_i\rangle\langle\varphi_j| \\
&= |\varphi_0\rangle\langle\varphi_0|\langle\varphi_0|\varphi_0\rangle + |\varphi_0\rangle\langle\varphi_1|\langle\varphi_1|\varphi_0\rangle + |\varphi_0\rangle\langle\varphi_2|\langle\varphi_2|\varphi_0\rangle \\
&+ |\varphi_1\rangle\langle\varphi_0|\langle\varphi_0|\varphi_1\rangle + |\varphi_1\rangle\langle\varphi_1|\langle\varphi_1|\varphi_1\rangle + |\varphi_1\rangle\langle\varphi_2|\langle\varphi_2|\varphi_1\rangle \\
&+ |\varphi_2\rangle\langle\varphi_0|\langle\varphi_0|\varphi_2\rangle + |\varphi_2\rangle\langle\varphi_1|\langle\varphi_1|\varphi_2\rangle + |\varphi_2\rangle\langle\varphi_2|\langle\varphi_2|\varphi_2\rangle.
\end{aligned}
$$

The trace of this matrix is equal

$$\begin{aligned} tr\rho_B^2 &= |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\ &+ \langle\varphi_0|\varphi_1\rangle^2 + \langle\varphi_0|\varphi_2\rangle^2 + \langle\varphi_1|\varphi_0\rangle^2 \\ &+ \langle\varphi_1|\varphi_2\rangle^2 + \langle\varphi_2|\varphi_0\rangle^2 + \langle\varphi_2|\varphi_1\rangle^2 \\ &= |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\ &+ 2|\langle\varphi_0|\varphi_1\rangle|^2 + 2|\langle\varphi_0|\varphi_2\rangle|^2 + 2|\langle\varphi_1|\varphi_2\rangle|^2, \end{aligned}$$

and the squared normalization condition gives

$$\begin{aligned} 1 &= (\langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle + \langle\varphi_2|\varphi_2\rangle)^2 \\ &= |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\ &+ 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle + 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_2|\varphi_2\rangle + 2\langle\varphi_1|\varphi_1\rangle\langle\varphi_2|\varphi_2\rangle. \end{aligned}$$

Then, by taking difference

$$\begin{aligned} 1 - tr\rho_B^2 &= (|\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\ &+ 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle + 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_2|\varphi_2\rangle + 2\langle\varphi_1|\varphi_1\rangle\langle\varphi_2|\varphi_2\rangle) \\ &- |\langle\varphi_0|\varphi_0\rangle|^2 - |\langle\varphi_1|\varphi_1\rangle|^2 - |\langle\varphi_2|\varphi_2\rangle|^2 \\ &- 2|\langle\varphi_0|\varphi_1\rangle|^2 - 2|\langle\varphi_0|\varphi_2\rangle|^2 - 2|\langle\varphi_1|\varphi_2\rangle|^2 \\ &= 2\begin{vmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_2\rangle \\ \langle\varphi_2|\varphi_0\rangle & \langle\varphi_2|\varphi_2\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle \\ \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_2\rangle \end{vmatrix}. \end{aligned}$$

We have following relation

$$\begin{aligned} 1 &- tr\rho_B^2 \equiv \frac{1}{2}C_B^2 \\ \Rightarrow 1 &= tr\rho_B^2 + \frac{C_B^2}{2} \\ \Rightarrow C_B^2 &= 2(1 - tr\rho_B^2) \\ C_B^2 &= 2\sum_{i,j=0}^{2}\begin{vmatrix} \langle\varphi_i|\varphi_i\rangle & \langle\varphi_i|\varphi_j\rangle \\ \langle\varphi_j|\varphi_i\rangle & \langle\varphi_j|\varphi_j\rangle \end{vmatrix}. \end{aligned}$$

When $i = j$, the diagonal terms in the sum cancel each other. For of-diagonal terms with $i < j$, finally we get

$$C_B^2 = 4 \sum_{0=i<j}^{2} \begin{vmatrix} \langle \varphi_i | \varphi_i \rangle & \langle \varphi_i | \varphi_j \rangle \\ \langle \varphi_j | \varphi_i \rangle & \langle \varphi_j | \varphi_j \rangle \end{vmatrix},$$

where $|\varphi_i\rangle = \sum_{j=0}^{2} c_{ij} |j\rangle$. For generic two-qutrit states, we can express

$$\begin{aligned}
\rho_B &= tr_A(\rho) = \sum_{k=0}^{2} {}_A\langle k \mid \rho \mid k \rangle_A \\
&= \sum_{i,j} \sum_{i',j'} c_{ij} \overline{c_{i'j'}} |i\rangle_B \, {}_B\langle i'| \sum_k \underbrace{{}_A\langle k|j\rangle_A}_{\delta_{kj}} \underbrace{{}_A\langle j'|k\rangle_A}_{\delta_{j'k}} \\
&= \sum_{ij} \sum_{i'j'} c_{ij} \overline{c_{i'j'}} |i\rangle_B \, {}_B\langle i'| \underbrace{\sum_k \delta_{kj} \delta_{j'k}}_{\delta_{jj'}} \\
&= \sum_{ij} \sum_{i'j'} c_{ij} \overline{c_{i'j'}} |i\rangle_B \, {}_B\langle i'| \delta_{jj'} \\
&= \sum_{ij} \sum_{i'j} c_{ij} \overline{c_{i'j}} |i\rangle_B \, {}_B\langle i'|.
\end{aligned}$$

The Hermitian inner product

$$\langle \varphi_i \mid \varphi_k \rangle = \sum_{j'} \overline{c_{ij'}} \langle j' | \underbrace{\sum_j c_{kj} |j\rangle}_{\delta_{jj'}} = \sum_{jj'} \delta_{jj'} \overline{c_{ij'}} c_{kj}$$

or

$$\begin{aligned}
&= \sum_j \overline{c_{ij}} c_{kj} = \sum_j (c^\dagger)_{ji} c_{kj} \\
\langle \varphi_i \mid \varphi_k \rangle &= \sum_j c_{kj} (c^\dagger)_{ji} = (\hat{C}\hat{C}^\dagger)_{ki},
\end{aligned}$$

where matrix $(\hat{C})_{kj} = c_{kj}$, and

$$\hat{C} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix} \tag{8.7}$$

and

$$\hat{C}^{\dagger} = \begin{pmatrix} \overline{c_{00}} & \overline{c_{10}} & \overline{c_{20}} \\ \overline{c_{01}} & \overline{c_{11}} & \overline{c_{21}} \\ \overline{c_{02}} & \overline{c_{12}} & \overline{c_{22}} \end{pmatrix}.$$

We define the Gram matrix,

$$(G)_{ij} = \langle\varphi_i|\varphi_j\rangle,$$

so that

$$G^T = \hat{C}\hat{C}^{\dagger}.$$

Then we have 2×2 minors of Gram matrix

$$M_{ij} = \begin{vmatrix} \langle\varphi_i|\varphi_i\rangle & \langle\varphi_i|\varphi_j\rangle \\ \langle\varphi_j|\varphi_i\rangle & \langle\varphi_j|\varphi_j\rangle \end{vmatrix} = \begin{vmatrix} (\hat{C}\hat{C}^{\dagger})_{ii} & (\hat{C}\hat{C}^{\dagger})_{ij} \\ (\hat{C}\hat{C}^{\dagger})_{ji} & (\hat{C}\hat{C}^{\dagger})_{jj} \end{vmatrix}. \tag{8.8}$$

By explicit calculations we have

$$G^T = \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & c_{00}\overline{c_{10}} + c_{01}\overline{c_{11}} + c_{02}\overline{c_{12}} & c_{00}\overline{c_{20}} + c_{01}\overline{c_{21}} + c_{02}\overline{c_{22}} \\ c_{10}\overline{c_{00}} + c_{11}\overline{c_{01}} + c_{12}\overline{c_{02}} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 & c_{10}\overline{c_{20}} + c_{11}\overline{c_{21}} + c_{12}\overline{c_{22}} \\ c_{20}\overline{c_{00}} + c_{21}\overline{c_{01}} + c_{22}\overline{c_{02}} & c_{20}\overline{c_{10}} + c_{21}\overline{c_{11}} + c_{22}\overline{c_{12}} & |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \end{pmatrix}$$

Then,

$$
\begin{aligned}
M_{01} &= \begin{vmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & c_{00}\overline{c_{10}} + c_{01}\overline{c_{11}} + c_{02}\overline{c_{12}} \\ c_{10}\overline{c_{00}} + c_{11}\overline{c_{01}} + c_{12}\overline{c_{02}} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 \end{vmatrix} \\
&= \left(|c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2\right)\left(|c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2\right) \\
&- (c_{00}\overline{c_{10}} + c_{01}\overline{c_{11}} + c_{02}\overline{c_{12}})(c_{10}\overline{c_{00}} + c_{11}\overline{c_{01}} + c_{12}\overline{c_{02}}) \\
&= 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2, \\
M_{02} &= \begin{vmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & c_{00}\overline{c_{20}} + c_{01}\overline{c_{21}} + c_{02}\overline{c_{22}} \\ c_{20}\overline{c_{00}} + c_{21}\overline{c_{01}} + c_{22}\overline{c_{02}} & |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \end{vmatrix} \\
&= \left(|c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2\right)\left(|c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2\right) \\
&- (c_{00}\overline{c_{20}} + c_{01}\overline{c_{21}} + c_{02}\overline{c_{22}})(c_{20}\overline{c_{00}} + c_{21}\overline{c_{01}} + c_{22}\overline{c_{02}}) \\
&= 2\begin{vmatrix} c_{00} & c_{01} \\ c_{20} & c_{21} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{20} & c_{22} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{21} & c_{22} \end{vmatrix}^2
\end{aligned}
$$

and

$$
\begin{aligned}
M_{12} &= \begin{vmatrix} |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 & c_{10}\overline{c_{20}} + c_{11}\overline{c_{21}} + c_{12}\overline{c_{22}} \\ c_{20}\overline{c_{10}} + c_{21}\overline{c_{11}} + c_{22}\overline{c_{12}} & |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \end{vmatrix} \\
&= \left(|c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2\right)\left(|c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2\right) \\
&- (c_{10}\overline{c_{20}} + c_{11}\overline{c_{21}} + c_{12}\overline{c_{22}})(c_{20}\overline{c_{10}} + c_{21}\overline{c_{11}} + c_{22}\overline{c_{12}}) \\
&= 2\begin{vmatrix} c_{10} & c_{11} \\ c_{20} & c_{21} \end{vmatrix}^2 + 2\begin{vmatrix} c_{10} & c_{12} \\ c_{20} & c_{22} \end{vmatrix}^2 + 2\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^2.
\end{aligned}
$$

Thus, we have

$$1 - tr\rho_B^2 = \frac{1}{2}(M_{01}^2 + M_{02}^2 + M_{12}^2) = \frac{1}{2}C_B^2,$$

due to relation

$$tr\rho_B^2 + \frac{1}{2}C_B^2 = 1. \tag{8.9}$$

Therefore, the total concurrence C_B is equal

$$C_B^2 = M_{01}^2 + M_{02}^2 + M_{12}^2,$$

or

$$C_B = \sqrt{M_{01}^2 + M_{02}^2 + M_{12}^2}.$$

□

2. In a similar way we decompose

$$|\psi\rangle = |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle$$

in terms of three qutrit states $|\psi_0\rangle = c_{00}|0\rangle+c_{10}|1\rangle+c_{20}|2\rangle$, $|\psi_1\rangle = c_{01}|0\rangle+c_{11}|1\rangle+c_{21}|2\rangle$ and $|\psi_2\rangle = c_{02}|0\rangle + c_{12}|1\rangle + c_{22}|2\rangle$.
Then,

$$\begin{aligned}
\rho &= (|\psi_0\rangle_A|0\rangle_B + |\psi_1\rangle_A|1\rangle_B + |\psi_2\rangle_A|2\rangle_B)\,({}_A\langle\psi_0|_B\langle 0| +_A \langle\psi_1|_B\langle 1| +_A \langle\psi_2|_B\langle 2|) \\
&= (|\psi_0\rangle_A\,{}_A\langle\psi_0|)\,(|0\rangle_B\,{}_B\langle 0|) + (|\psi_0\rangle_A\,{}_A\langle\psi_1|)\,(|0\rangle_B\,{}_B\langle 1|) + (|\psi_0\rangle_A\,{}_A\langle\psi_2|)\,(|0\rangle_B\,{}_B\langle 2|) \\
&+ (|\psi_1\rangle_A\,{}_A\langle\psi_0|)\,(|1\rangle_B\,{}_B\langle 0|) + (|\psi_1\rangle_A\,{}_A\langle\psi_1|)\,(|1\rangle_B\,{}_B\langle 1|) + (|\psi_1\rangle_A\,{}_A\langle\psi_2|)\,(|1\rangle_B\,{}_B\langle 2|) \\
&+ (|\psi_2\rangle_A\,{}_A\langle\psi_0|)\,(|2\rangle_B\,{}_B\langle 0|) + (|\psi_2\rangle_A\,{}_A\langle\psi_1|)\,(|2\rangle_B\,{}_B\langle 1|) + (|\psi_2\rangle_A\,{}_A\langle\psi_2|)\,(|2\rangle_B\,{}_B\langle 2|)
\end{aligned}$$

or in matrix form for B-basis

$$\rho = \begin{pmatrix} |\psi_0\rangle\langle\psi_0| & |\psi_0\rangle\langle\psi_1| & |\psi_0\rangle\langle\psi_2| \\ |\psi_1\rangle\langle\psi_0| & |\psi_1\rangle\langle\psi_1| & |\psi_1\rangle\langle\psi_2| \\ |\psi_2\rangle\langle\psi_0| & |\psi_2\rangle\langle\psi_1| & |\psi_2\rangle\langle\psi_2| \end{pmatrix}. \qquad (8.10)$$

For subsystem A, the reduced density operator gives us

$$\rho_A = tr_B(\rho) = |\psi_0\rangle_A\,{}_A\langle\psi_0| + |\psi_1\rangle_A\,{}_A\langle\psi_1| + |\psi_2\rangle_A\,{}_A\langle\psi_2|,$$

where $tr_B(\rho)$ is called the partial trace, and

$$tr_A\rho_A = \langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle = 1.$$

Then, the squared reduced density matrix is

$$\begin{aligned}
\rho_A^2 &= \sum_{i,j=0}^{2} \langle \psi_i \mid \psi_j \rangle |\psi_i\rangle\langle\psi_j| \\
&= |\psi_0\rangle\langle\psi_0|\langle\psi_0|\psi_0\rangle + |\psi_0\rangle\langle\psi_1|\langle\psi_1|\psi_0\rangle + |\psi_0\rangle\langle\psi_2|\langle\psi_2|\psi_0\rangle \\
&+ |\psi_1\rangle\langle\psi_0|\langle\psi_0|\psi_1\rangle + |\psi_1\rangle\langle\psi_1|\langle\psi_1|\psi_1\rangle + |\psi_1\rangle\langle\psi_2|\langle\psi_2|\psi_1\rangle \\
&+ |\psi_2\rangle\langle\psi_0|\langle\psi_2|\psi_0\rangle + |\psi_2\rangle\langle\psi_1|\langle\psi_1|\psi_2\rangle + |\psi_2\rangle\langle\psi_2|\langle\psi_2|\psi_2\rangle
\end{aligned}$$

and the trace of this matrix is equal

$$\begin{aligned}
tr_A\rho_A^2 &= |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2 \\
&+ \langle\psi_0|\psi_1\rangle^2 + \langle\psi_0|\psi_2\rangle^2 + \langle\psi_1|\psi_0\rangle \\
&+ \langle\psi_1|\psi_2\rangle + \langle\psi_2|\psi_0\rangle + \langle\psi_2|\psi_1\rangle \\
&= |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2 \\
&+ 2|\langle\psi_0|\psi_1\rangle|^2 + 2|\langle\psi_0|\psi_2\rangle|^2 + 2|\langle\psi_1|\psi_2\rangle|^2.
\end{aligned}$$

The squared normalization condition gives

$$\begin{aligned}
1 &= (\langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle)^2 \\
&= |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2 \\
&+ 2\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle + 2\langle\psi_0|\psi_0\rangle\langle\psi_2|\psi_2\rangle + 2\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle.
\end{aligned}$$

Then, by taking difference

$$\begin{aligned}
1 - tr_A\rho_A^2 &= |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2 \\
&+ 2\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle + 2\langle\psi_0|\psi_0\rangle\langle\psi_2|\psi_2\rangle + 2\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle \\
&- |\langle\psi_0|\psi_0\rangle|^2 - |\langle\psi_1|\psi_1\rangle|^2 - |\langle\psi_2|\psi_2\rangle|^2 \\
&- 2|\langle\psi_0|\psi_1\rangle|^2 - 2|\langle\psi_0|\psi_2\rangle|^2 - 2|\langle\psi_1|\psi_2\rangle|^2 \\
&= 2\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_2\rangle \\ \langle\psi_2|\psi_0\rangle & \langle\psi_2|\psi_2\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle \end{vmatrix},
\end{aligned}$$

we have following relation

$$
\begin{aligned}
1 &- tr_A\rho_A^2 \equiv \frac{1}{2}C_A^2 \\
\Rightarrow 1 &= tr_A\rho_A^2 + \frac{C_A^2}{2} \\
\Rightarrow C_A^2 &= 2\left(1 - tr_A\rho_A^2\right) \\
C_A^2 &= 2\sum_{i,j=0}^{2}\begin{vmatrix}\langle\psi_i|\psi_i\rangle & \langle\psi_i|\psi_j\rangle \\ \langle\psi_j|\psi_i\rangle & \langle\psi_j|\psi_j\rangle\end{vmatrix}.
\end{aligned}
$$

The diagonal terms with $i = j$ in the sum cancel each other.For of-diagonal terms with $i < j$, we get

$$
C_A^2 = 4\sum_{0=i<j}^{2}\begin{vmatrix}\langle\psi_i|\psi_i\rangle & \langle\psi_i|\psi_j\rangle \\ \langle\psi_j|\psi_i\rangle & \langle\psi_j|\psi_j\rangle\end{vmatrix},
$$

where $|\psi_i\rangle = \sum_{i=0}^{2} c_{ji}|j\rangle$. The Hermitian inner product is

$$
\begin{aligned}
\langle\psi_i \mid \psi_k\rangle &= \sum_{j'}\langle j'|\overline{c_{j'i}}\underbrace{\sum_j c_{jk}|j\rangle}_{\delta_{jj'}} = \sum_{jj'}\delta_{jj'}\overline{c_{j'i}}c_{jk} \\
&= \sum_j \overline{c_{ji}}c_{jk} = \sum_j\left(C^\dagger\right)_{ij}C_{jk}
\end{aligned}
$$

or

$$
\langle\psi_i \mid \psi_k\rangle = \left(\hat{C}^\dagger\hat{C}\right)_{ik},
$$

where matrix $(\hat{C})_{jk} = c_{jk}$, and

$$
\hat{C} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix} \tag{8.11}
$$

and

$$\hat{C}^{\dagger} = \begin{pmatrix} \overline{c_{00}} & \overline{c_{10}} & \overline{c_{20}} \\ \overline{c_{01}} & \overline{c_{11}} & \overline{c_{21}} \\ \overline{c_{02}} & \overline{c_{12}} & \overline{c_{22}} \end{pmatrix}.$$

We define the Gram matrix,

$$(G)_{ij} = \langle\psi_i|\psi_j\rangle,$$

so that

$$G^T = \hat{C}^{\dagger}\hat{C}.$$

Then we have 2x2 minors of Gram matrix

$$M_{ij} \begin{vmatrix} \langle\psi_i|\psi_i\rangle & \langle\psi_i|\psi_j\rangle \\ \langle\psi_j|\psi_i\rangle & \langle\psi_j|\psi_j\rangle \end{vmatrix} = \begin{vmatrix} (\hat{C}^{\dagger}\hat{C})_{ii} & (\hat{C}^{\dagger}\hat{C})_{ij} \\ (\hat{C}^{\dagger}\hat{C})_{ji} & (\hat{C}^{\dagger}\hat{C})_{jj} \end{vmatrix},$$

By explicit calculations we have

$$G^T = \begin{pmatrix} |c_{00}|^2 + |c_{10}|^2 + |c_{20}|^2 & \overline{c_{00}}c_{01} + \overline{c_{10}}c_{11} + \overline{c_{20}}c_{21} & \overline{c_{00}}c_{02} + \overline{c_{10}}c_{12} + \overline{c_{20}}c_{22} \\ \overline{c_{01}}c_{00} + \overline{c_{11}}c_{10} + \overline{c_{21}}c_{20} & |c_{01}|^2 + |c_{11}|^2 + |c_{21}|^2 & \overline{c_{01}}c_{02} + \overline{c_{11}}c_{12} + \overline{c_{21}}c_{22} \\ \overline{c_{02}}c_{00} + \overline{c_{12}}c_{10} + \overline{c_{22}}c_{20} & \overline{c_{02}}c_{01} + \overline{c_{12}}c_{11} + \overline{c_{22}}c_{21} & |c_{02}|^2 + |c_{12}|^2 + |c_{22}|^2 \end{pmatrix}$$

Then

$$\begin{aligned} M_{01} &= \begin{vmatrix} |c_{00}|^2 + |c_{10}|^2 + |c_{20}|^2 & \overline{c_{00}}c_{01} + \overline{c_{10}}c_{11} + \overline{c_{20}}c_{21} \\ \overline{c_{01}}c_{00} + \overline{c_{11}}c_{10} + \overline{c_{21}}c_{20} & |c_{01}|^2 + |c_{11}|^2 + |c_{21}|^2 \end{vmatrix} \\ &= \left(|c_{00}|^2 + |c_{10}|^2 + |c_{20}|^2\right)\left(|c_{01}|^2 + |c_{11}|^2 + |c_{21}|^2\right) \\ &- \left(\overline{c_{00}}c_{01} + \overline{c_{10}}c_{11} + \overline{c_{20}}c_{21}\right)\left(\overline{c_{01}}c_{00} + \overline{c_{11}}c_{10} + \overline{c_{21}}c_{20}\right) \\ &= 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{01} \\ c_{20} & c_{21} \end{vmatrix}^2 + 2\begin{vmatrix} c_{10} & c_{11} \\ c_{20} & c_{21} \end{vmatrix}^2 \\ M_{02} &= \begin{vmatrix} |c_{00}|^2 + |c_{10}|^2 + |c_{20}|^2 & \overline{c_{00}}c_{02} + \overline{c_{10}}c_{12} + \overline{c_{20}}c_{22} \\ \overline{c_{02}}c_{00} + \overline{c_{12}}c_{10} + \overline{c_{22}}c_{20} & |c_{02}|^2 + |c_{12}|^2 + |c_{22}|^2 \end{vmatrix} \\ &= \left(|c_{00}|^2 + |c_{10}|^2 + |c_{20}|^2\right)\left(|c_{02}|^2 + |c_{12}|^2 + |c_{22}|^2\right) \\ &- \left(\overline{c_{00}}c_{02} + \overline{c_{10}}c_{12} + \overline{c_{20}}c_{22}\right)\left(\overline{c_{02}}c_{00} + \overline{c_{12}}c_{10} + \overline{c_{22}}c_{20}\right) \\ &= 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{20} & c_{22} \end{vmatrix}^2 + 2\begin{vmatrix} c_{10} & c_{12} \\ c_{20} & c_{22} \end{vmatrix}^2 \end{aligned}$$

and

$$\begin{aligned}
M_{12} &= \begin{vmatrix} |c_{01}|^2+|c_{11}|^2+|c_{21}|^2 & \overline{c_{01}}c_{02}+\overline{c_{11}}c_{12}+\overline{c_{21}}c_{22} \\ \overline{c_{02}}c_{01}+\overline{c_{12}}c_{11}+\overline{c_{22}}c_{21} & |c_{02}|^2+|c_{12}|^2+|c_{22}|^2 \end{vmatrix} \\
&= \left(|c_{01}|^2+|c_{11}|^2+|c_{21}|^2\right)\left(|c_{02}|^2+|c_{12}|^2+|c_{22}|^2\right) \\
&- \left(\overline{c_{01}}c_{02}+\overline{c_{11}}c_{12}+\overline{c_{21}}c_{22}\right)\left(\overline{c_{02}}c_{01}+\overline{c_{12}}c_{11}+\overline{c_{22}}c_{21}\right) \\
&= 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{21} & c_{22} \end{vmatrix}^2 + 2\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^2 .
\end{aligned}$$

Thus, we have

$$1 - tr\rho_A^2 = \frac{1}{2}(M_{01}^2 + M_{02}^2 + M_{12}^2) = \frac{1}{2}C_A^2,$$

due to relation

$$tr\rho_A^2 + \frac{1}{2}C_A^2 = 1. \tag{8.12}$$

Therefore the total concurrence C_A is equal

$$C_A^2 = M_{01}^2 + M_{02}^2 + M_{12}^2,$$

or

$$C_A = \sqrt{M_{01}^2 + M_{02}^2 + M_{12}^2}.$$

Comparing equations (8.9) and (8.12) we get identity

$$tr_B\rho_B^2 = tr_A\rho_A^2.$$

Thus,

$$C_A = C_B.$$

8.4. Maximal Value of Concurrence

Let generic two qutrit state is decomposed in two different ways

$$
\begin{aligned}
|\psi\rangle &= \underbrace{(c_{00}|0\rangle + c_{10}|1\rangle + c_{20}|2\rangle)}_{|\psi_0\rangle}|0\rangle \\
&+ \underbrace{(c_{01}|0\rangle + c_{11}|1\rangle + c_{21}|2\rangle)}_{|\psi_1\rangle}|1\rangle \\
&+ \underbrace{(c_{02}|0\rangle + c_{12}|1\rangle + c_{22}|2\rangle)}_{|\psi_2\rangle}|2\rangle \\
&= |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle \\
&= |0\rangle\underbrace{(c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle)}_{|\varphi_0\rangle} \\
&+ |1\rangle\underbrace{(c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle)}_{|\varphi_1\rangle} \\
&+ |2\rangle\underbrace{(c_{20}|0\rangle + c_{21}|1\rangle + c_{22}|2\rangle)}_{|\varphi_2\rangle} \\
&= |0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle + |2\rangle|\varphi_2\rangle.
\end{aligned}
$$

Then, corresponding density matrix is

$$
\begin{aligned}
\rho &= (|\psi_0\rangle_A|0\rangle_B + |\psi_1\rangle_A|1\rangle_B + |\psi_2\rangle_A|2\rangle_B)\,({}_A\langle\psi_0|_B\langle 0| +_A \langle\psi_1|_B\langle 1| +_A \langle\psi_2|_B\langle 2|) \\
&= (|0\rangle_A|\varphi_0\rangle_B + |1\rangle_A|\varphi_1\rangle_B + |2\rangle_A|\varphi_2\rangle_B)\,({}_A\langle 0|_B\langle\varphi_0| +_A \langle 1|_B\langle\varphi_1| +_A \langle 2|_B\langle\varphi_2|)\,.
\end{aligned}
$$

For the reduced density matrices we have

$$
\rho_A = tr_B(\rho) = |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|,
$$

and

$$
\rho_B = tr_A(\rho) = |\varphi_0\rangle\langle\varphi_0| + |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|.
$$

For the last one

$$
\begin{aligned}
\rho_B &= (c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle)(\overline{c}_{00}\langle 0| + \overline{c}_{01}\langle 1| + \overline{c}_{02}\langle 2|) \\
&+ (c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle)(\overline{c}_{10}\langle 0| + \overline{c}_{11}\langle 1| + \overline{c}_{12}\langle 2|) \\
&+ (c_{20}|0\rangle + c_{21}|1\rangle + c_{22}|2\rangle)(\overline{c}_{20}\langle 0| + \overline{c}_{21}\langle 1| + \overline{c}_{22}\langle 2|)
\end{aligned}
$$

in explicit form we get

$$
\rho_B = \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & c_{00}\overline{c}_{10} + c_{01}\overline{c}_{11} + c_{02}\overline{c}_{12} & c_{00}\overline{c_{20}} + c_{01}\overline{c_{21}} + c_{02}\overline{c_{22}} \\ \overline{c_{00}}c_{10} + \overline{c_{01}}c_{11} + \overline{c_{02}}c_{12} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 & \overline{c_{10}}c_{20} + \overline{c_{11}}c_{21} + \overline{c_{12}}c_{22} \\ \overline{c_{00}}c_{20} + \overline{c_{01}}c_{21} + \overline{c_{02}}c_{22} & \overline{c_{10}}c_{20} + \overline{c_{11}}c_{21} + \overline{c_{12}}c_{22} & |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \end{pmatrix}.
$$

Trace of this reduced density matrix is

$$
tr\rho_B = |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 + |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 + |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 = 1.
$$

Due to the self-adjointness of density matrix, exists an unitary transformation, which can diagonalize ρ_B as

$$
\tilde{\rho}_B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},
$$

and

$$
\tilde{\rho}_B^2 = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}, \tag{8.13}
$$

where λ_1, λ_2 and λ_3 are real numbers. Then,

$$\begin{aligned} tr\rho_B &= \lambda_1 + \lambda_2 + \lambda_3 = 1, \\ det\rho_B &= \lambda_1\lambda_2\lambda_3, \\ tr\rho_B^2 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \end{aligned}$$

and

$$1 - tr\rho_B^2 = 2\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right).$$

We define C_B concurrence according to (8.9),

$$\frac{C_B^2}{2} = 2\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)$$

or

$$\Rightarrow C_B^2 = 4\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right).$$

In order to determine the maximum value of concurrence, we establish the following optimization problem.

8.4.1. Optimization Problem

To find maximal value of concurrence, we must find a solution to the following optimization problem with constraint, which becomes the Lagrange multiplier problem. Find $\lambda_1, \lambda_2, \lambda_3$ satisfying equation

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

and giving maximal value for C_B^2

$$C_B^2 = 4\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right).$$

The variational function takes the form

$$F(\lambda_1, \lambda_2, \lambda_3, \lambda) \quad = \quad 4\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) + \lambda(\lambda_1 + \lambda_2 + \lambda_3 - 1),$$

where λ is Lagrange multiplier.

Critical points of this function are determined by equations

$$\begin{aligned}
\frac{\partial F}{\partial \lambda} &= \lambda_1 + \lambda_2 + \lambda_3 - 1 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 1, \\
\frac{\partial F}{\partial \lambda_1} &= 4(\lambda_2 + \lambda_3) + \lambda = 0, \\
\frac{\partial F}{\partial \lambda_2} &= 4(\lambda_1 + \lambda_3) + \lambda = 0, \\
\frac{\partial F}{\partial \lambda_3} &= 4(\lambda_1 + \lambda_2) + \lambda = 0.
\end{aligned}$$

This gives $\lambda_1 = \lambda_2 = \lambda_3$, and then

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \Rightarrow 3\lambda_1 = 1 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}.$$

Therefore, we have

$$\begin{aligned}
C_B^2 = 4\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) = 4\left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right) &= \frac{4}{3} \\
\Rightarrow C_B &= \frac{2}{\sqrt{3}}
\end{aligned}$$

Finally, we get following proposition.

Proposition 8.2 Maximal value of concurrence for two-qutrit states, defined by (8.9) is

$$max\, C = \frac{2}{\sqrt{3}}.$$

Example 8.1 Let

$$|\psi\rangle \;=\; \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}}$$

is pure state. Then $c_{00} = c_{11} = c_{22} = \frac{1}{\sqrt{3}}$ and as follows $C = \frac{2}{\sqrt{3}}$. This is maximally entangled two qutrit state.

8.5. De Gua's Theorem

A tetrahedron with all three face angles at a single vertex being right angles is called a trirectangular tetrahedron. When this occurs, the area formula becomes as between orthogonal vectors, every scalar products are zero.
Let areas of faces are

$$A_{\triangle 01} = \frac{1}{2}A_{01}, \quad A_{\triangle 02} = \frac{1}{2}A_{02}, \quad A_{\triangle 12} = \frac{1}{2}A_{12},$$

and the area square of the face opposite tot the origin is

$$A_{\triangle}^2 = \frac{1}{4}(r_0^2 r_1^2 + r_0^2 r_2^2 + r_1^2 r_2^2). \tag{8.14}$$

For area of corresponding parallelogram faces we have

$$A_{01}^2 = \left|\vec{r}_0 \times \vec{r}_1\right|^2 = r_0^2 r_1^2,$$
$$A_{02}^2 = \left|\vec{r}_0 \times \vec{r}_2\right|^2 = r_0^2 r_2^2,$$
$$A_{12}^2 = \left|\vec{r}_1 \times \vec{r}_2\right|^2 = r_1^2 r_2^2.$$

Therefore, the relation between parallelogram areas is

$$A^2 = A_{01}^2 + A_{02}^2 + A_{12}^2. \tag{8.15}$$

A tetrahedron can be represented by the three-dimensional analogue of Pythagoras' theorem, known as De Gua's theorem. The square of the face area , which is located opposite the right-angle corner the trirectangular tetrahedron is equal to the total of the squares of the areas of the other three faces:

$$A_{\triangle}^2 = A_{\triangle 01}^2 + A_{\triangle 02}^2 + A_{\triangle 12}^2.$$

Then, given a maximally mixed state, this statement comes from (8.14) and (8.15), since

$$A_{\triangle 01} = A_{01} = \frac{1}{2}|r_0|\,|r_1|\,, \quad A_{\triangle 02} = A_{02} = \frac{1}{2}|r_0|\,|r_2| \quad A_{\triangle 12} = A_{12} = \frac{1}{2}|r_1|\,|r_2|\,.$$

Hence, we have demonstrated a relationship between De Gua's theorem, which is the Pythagorean theorem extended to a tetrahedron, and the maximally entangled two-retrit state (Pashaev, Oktay K. 2023, 93–104).

8.6. Entanglement Invariance Under Unitary Transformations

Proposition 8.3 The concurrence $C' = C$, is invariant under unitary transformation

$$|\psi'\rangle = (U_A \otimes I_B)\,|\psi\rangle,$$

where $UU^{\dagger} = I$ and $U_A \in SU(3)$ is arbitrary one qutrit unitary gate.

Proof Let

$$|\psi\rangle = |\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle + |\psi_2\rangle|2\rangle$$

and

$$
\begin{aligned}
|\psi\rangle &= (U_A \otimes I_B) = U|\psi_0\rangle|0\rangle + U|\psi_1\rangle|1\rangle + U|\psi_2\rangle|2\rangle \\
&= |\psi'_0\rangle|0\rangle + |\psi'_1\rangle|1\rangle + |\psi'_2\rangle|2\rangle
\end{aligned}
$$

where $|\psi'_i\rangle = U|\psi_i\rangle$.
The concurrence C_A^2 is

$$
C_A^2 = 4\sum_{0=i<j}^{2} \begin{vmatrix} \langle\psi_i|\psi_i\rangle & \langle\psi_i|\psi_j\rangle \\ \langle\psi_j|\psi_i\rangle & \langle\psi_j|\psi_j\rangle \end{vmatrix}.
$$

Since

$$
\langle\psi_i|\psi_j\rangle = \langle\psi'_i|U_A^\dagger U_A|\psi'_j\rangle = \langle\psi'_i|\psi'_j\rangle
$$

we have

$$
C_A'^2 = 4\sum_{0=i<j}^{2} \begin{vmatrix} \langle\psi'_i|\psi'_i\rangle & \langle\psi'_i|\psi'_j\rangle \\ \langle\psi'_j|\psi'_i\rangle & \langle\psi'_j|\psi'_j\rangle \end{vmatrix} = C_A^2.
$$

Thus,

$$
C'_A = C_A
$$

□

Proposition 8.4 The concurrence $C' = C$, is invariant under unitary transformation

$$
|\psi'\rangle = (I_A \otimes U_B)\,|\psi\rangle,
$$

where $UU^\dagger = I$ and $U_A \in SU(3)$ is arbitrary one qutrit unitary gate.

Proof Let

$$|\psi\rangle = |0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle + |2\rangle|\varphi_2\rangle$$

and

$$\begin{aligned} |\psi'\rangle &= (I_A \otimes U_B)\,|\psi\rangle = |0\rangle U|\varphi_0\rangle + |1\rangle U|\varphi_1\rangle + |2\rangle U|\varphi_2\rangle \\ &= |0\rangle|\varphi_0'\rangle + |1\rangle|\varphi_1'\rangle + |2\rangle|\varphi_2'\rangle \end{aligned}$$

where $|\varphi_i'\rangle = U|\varphi_i\rangle$. The concurrence C_B^2 is

$$C_B^2 = 4\sum_{0=i<j}^{2} \begin{vmatrix} \langle\varphi_i|\varphi_i\rangle & \langle\varphi_i|\varphi_j\rangle \\ \langle\varphi_j|\varphi_i\rangle & \langle\varphi_j|\varphi_j\rangle \end{vmatrix}.$$

Due to

$$\langle\varphi_i|\varphi_j\rangle = \langle\varphi_i'|U^\dagger U|\varphi_j'\rangle = \langle\varphi_i'|\varphi_j'\rangle$$

we have

$$C_B'^2 = 4\sum_{0=i<j}^{2} \begin{vmatrix} \langle\varphi_i'|\varphi_i'\rangle & \langle\varphi_i'|\varphi_j'\rangle \\ \langle\varphi_j'|\varphi_i'\rangle & \langle\varphi_j'|\varphi_j'\rangle \end{vmatrix} = C_B^2.$$

Thus,

$$C_B = C_B'.$$

□

Proposition 8.5 The concurrence $C' = C$, is invariant under unitary transformation

$$|\psi'\rangle = (U_A \otimes U_B)\,|\psi\rangle,$$

where $UU^\dagger = I$ and $U_A, U_B \in SU(3)$ are arbitrary one qutrit unitary gates.

Proof Let

$$|\psi\rangle = |0\rangle_A|\varphi_0\rangle_B + |1\rangle_A|\varphi_1\rangle_B + |2\rangle_A|\varphi_2\rangle_B,$$

where

$$
\begin{aligned}
|\varphi_0\rangle &= c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle, \\
|\varphi_1\rangle &= c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle, \\
|\varphi_2\rangle &= c_{20}|0\rangle + c_{21}|1\rangle + c_{22}|2\rangle.
\end{aligned}
$$

Then, applying unitary transformation

$$
\begin{aligned}
|\psi'\rangle &= (U_A \otimes U_B)\,|\psi\rangle \\
&= (U_A|0\rangle_A)\,(U_B|\varphi_0\rangle_B) + (U_A|1\rangle_A)\,(U_B|\varphi_1\rangle_B) + (U_A|2\rangle_A)\,(U_B|\varphi_2\rangle_B)
\end{aligned}
$$

where

$$
\begin{aligned}
U_A|k\rangle_A &= \sum_{s=0}^{2} |s\rangle\langle s|U_A \sum_{l=0}^{2} |l\rangle_A\ {}_A\langle l|k\rangle_A \\
&= \sum_{s=0}^{2}\sum_{l=0}^{2} \langle s|U_A|l\rangle|s\rangle_A\ {}_A\langle l|k\rangle_A
\end{aligned}
$$

and $k = 0, 1, 2$ and $\sum_{k=0}^{2} |k\rangle\langle k| = I$, we have

$$
\begin{aligned}
|\psi'\rangle &= \sum_{k=0}^{2} U_A|k\rangle_A\,(U_B|\varphi_k\rangle_B) \\
&= \sum_{s=0}^{2}\sum_{l=0}^{2}\sum_{k=0}^{2}\ {}_A\langle s|U_A|l\rangle_A|s\rangle_A\,\underbrace{{}_A\langle l|k\rangle_A}_{\delta_{lk}} \otimes (U_B|\varphi_k\rangle_B) \\
&= \sum_{s=0}^{2}\sum_{k=0}^{2}\ {}_A\langle s|U_A|k\rangle_A|s\rangle_A \otimes (U_B|\varphi_k\rangle_B) \\
&= \sum_{s=0}^{2} |s\rangle_A \underbrace{\left(\sum_{k=0}^{2} \langle s|U_A|k\rangle\,(U_B|\varphi_k\rangle_B)\right)}_{|\varphi'_s\rangle_A}.
\end{aligned}
$$

Here

$$
\begin{aligned}
|\varphi'_s\rangle_A &= \sum_{k=0}^{2} \langle s|U_A|k\rangle U_B|\varphi_k\rangle_B \\
|\varphi'_t\rangle_A &= \sum_{l=0}^{2} \langle t|U_A|l\rangle U_B|\varphi_l\rangle_B \\
{}_A\langle\varphi'_t| &= \sum_{l=0}^{2} \overline{\langle t|U_A|l\rangle}\,{}_B\langle\varphi_l|U_B^\dagger
\end{aligned}
$$

and for the inner product we get

$$
{}_A\langle\varphi'_t|\varphi'_s\rangle_A = \sum_{l=0}^{2}\sum_{k=0}^{2} \underbrace{{}_B\langle\varphi_l|U_B^\dagger U_B|\varphi_k\rangle_B}_{{}_B\langle\varphi_l|\varphi_k\rangle_B} \langle s|U_A|k\rangle\overline{\langle t|U_A|l\rangle}.
$$

The Gram matrix G is defined as the inner product matrix

$$
G_{lk} \equiv \langle\varphi_l|\varphi_k\rangle.
$$

Then,

$$
\begin{aligned}
G'_{ts} &= \sum_{l=0}^{2}\sum_{k=0}^{2} \langle s|U_A|k\rangle\langle l|U_A^\dagger|t\rangle G_{lk} \\
&= \sum_{l=0}^{2}\sum_{k=0}^{2} \underbrace{\langle s|U_A|k\rangle}_{(U_A)_{sk}}\ \underbrace{G_{lk}}_{G^T_{kl}}\ \underbrace{\langle l|U_A^\dagger|t\rangle}_{(U_A^\dagger)_{lt}} \\
&= \sum_{l=0}^{2}\sum_{k=0}^{2} (U_A)_{sk}\, G^T_{kl} \left(U_A^\dagger\right)_{lt} \\
&= \left(U_A G^T U_A^\dagger\right)_{st}. \\
\Rightarrow G'^T &= U_A G^T U_A^\dagger.
\end{aligned}
$$

The concurrence

$$\begin{aligned} C^2 &= 2\sum_{i,j=0}^{2}\begin{vmatrix}\langle\varphi_i|\varphi_i\rangle & \langle\varphi_i|\varphi_j\rangle\\ \langle\varphi_j|\varphi_i\rangle & \langle\varphi_j|\varphi_j\rangle\end{vmatrix}\\ &= 2\sum_{i,j=0}^{2} det\begin{vmatrix}G_{ii} & G_{ij}\\ G_{ji} & G_{jj}\end{vmatrix}. \end{aligned}$$

The determinant of Gram matrix elements the (ij) minor is

$$\begin{aligned} G'_{ii}G'_{jj}-G'_{ij}G'_{ji} &= \left(U_AG^TU_A^\dagger\right)_{ii}\left(U_AG^TU_A^\dagger\right)_{jj}-\left(U_AG^TU_A^\dagger\right)_{ij}\left(U_AG^TU_A^\dagger\right)_{ji}\\ &= \sum_{k,l,k',l'=0}^{2}((U_A)_{ik}\,G^T_{kl'}\left(U_A^\dagger\right)_{l'i}(U_A)_{jk'}\,G^T_{k'l}\left(U_A^\dagger\right)_{lj}\\ &- (U_A)_{ik}\,G^T_{kl}\left(U_A^\dagger\right)_{lj}(U_A)_{jk'}\,G^T_{k'l'}\left(U_A^\dagger\right)_{l'i})\\ &= \sum_{k,l,k',l'=0}^{2}(U_A)_{ik}\,(U_A)_{jk'}\left[G^T_{kl'}G^T_{k'l}-G^T_{kl}G^T_{k'l'}\right]\left(U_A^\dagger\right)_{lj}\left(U_A^\dagger\right)_{l'i}. \end{aligned}$$

Then, we have

$$\begin{vmatrix}G'_{ii} & G'_{ij}\\ G'_{ji} & G'_{jj}\end{vmatrix}=\sum_{k,l,k',l'=0}^{2}(U_A)_{ik}(U_A)_{jk'}\begin{vmatrix}G^T_{kl'} & G^T_{kl}\\ G^T_{k'l'} & G^T_{k'l}\end{vmatrix}(U_A^\dagger)_{lj}(U_A^\dagger)_{l'i}.$$

Finally, the concurrence

$$\begin{aligned} C'^2 &= 2\sum_{i,j=0}^{2}\begin{vmatrix}G'_{ii} & G'_{ij}\\ G'_{ji} & G'_{jj}\end{vmatrix}\\ &= 2\sum_{i,j=0}^{2}\sum_{k,l,k',l'=0}^{2}\begin{vmatrix}G^T_{kl'} & G^T_{kl}\\ G^T_{k'l'} & G^T_{k'l}\end{vmatrix}(U_A)_{ik}(U_A)_{jk'}(U_A^\dagger)_{lj}(U_A^\dagger)_{l'i}\\ &= 2\sum_{i,j=0}^{2}\sum_{k,l,k',l'=0}^{2}\begin{vmatrix}G^T_{kl'} & G^T_{kl}\\ G^T_{k'l'} & G^T_{k'l}\end{vmatrix}(U_A^\dagger)_{l'i}(U_A)_{ik}(U_A^\dagger)_{lj}(U_A)_{jk'}\\ &= 2\sum_{i,j=0}^{2}\sum_{k,l,k',l'=0}^{2}\begin{vmatrix}G^T_{kl'} & G^T_{kl}\\ G^T_{k'l'} & G^T_{k'l}\end{vmatrix}\underbrace{(U_A^\dagger U_A)_{l'k}}_{\delta_{l'k}}\underbrace{(U_A^\dagger U_A)_{lk'}}_{\delta_{lk'}}. \end{aligned}$$

Therefore,

$$\begin{aligned} C'^2 &= 2\sum_{k,l,k',l'=0}^{2} \begin{vmatrix} G^T_{kl'} & G^T_{kl} \\ G^T_{k'l'} & G^T_{k'l} \end{vmatrix} \delta_{l'k}\delta_{lk'} \\ &= 2\sum_{k,l=0}^{2} \begin{vmatrix} G^T_{kl'} & G^T_{kl} \\ G^T_{k'l'} & G^T_{k'l} \end{vmatrix} \delta_{l'k}\delta_{lk'} = C^2. \end{aligned}$$

Thus, we have

$$C' = C.$$

□

CHAPTER 9

PURIFICATION AND ENTANGLEMENT

For given quantum system A, which is describes by density matrix ρ_A it is possible to introduce another system, the B-system(an ancillary system), so that the state of composite system $|\psi\rangle$ is a pure state, and

$$\rho_A = Tr_B|\psi\rangle\langle\psi|.$$

The procedure is called the purification (Benenti et al. 2018, 91). It allows relates a pure state $|\psi\rangle$ with density matrix ρ_A and it allows to work with pure states instead of density matrix.
Here we should notice, that procedure of purification is not unique, and could exist more than one ancillatory systems and pure states $|\psi_1\rangle, |\psi_2\rangle, \ldots$, leading to the same reduced density matrix ρ_A. Since $tr\rho_A^2$ determines level of mixture for state ρ_A in form of the concurrence C, the purified states $|\psi_1\rangle, |\psi_2\rangle, \ldots$ should have the same level of entanglement and equal concurrence. This leaded us to study invariance of concurrence for two qubit(qutrit) pure states in chapters 6, 7.8 and 8.6.
Combinet system A+B is described by generic pure state,

$$|\psi\rangle \quad = \quad \sum c_{ij}|i\rangle_A|j\rangle_B, \tag{9.1}$$

where $\{|i\rangle_A\}$ is basis set for Hilbert space H_A and $\{|j\rangle_B\}$ for the space H_B. The density matrix of the state is

$$\rho \quad = \quad \sum_{i,j}\sum_{i',j'} c_{ij}\bar{c}_{i'j'}|i\rangle_A|j\rangle_B\, {}_A\langle i'|_B\langle j'|. \tag{9.2}$$

Let given density matrix for state A is

$$\rho_A = \sum_{i,i'} (\rho_A)_{ii'} |i\rangle_A \, {}_A\langle i'|. \tag{9.3}$$

Then, the pure state $|\psi\rangle$ as purification of ρ_A satisfies

$$\begin{aligned}
\rho_A &= Tr_B|\psi\rangle\langle\psi| = \sum_k {}_B\langle k|\psi\rangle\langle\psi|k\rangle_B \\
&= \sum_k {}_B\langle k| \sum_{i,j} \sum_{i',j'} c_{ij}\bar{c}_{i'j'} |i\rangle_A \, {}_A\langle i'|_B\langle k|j\rangle_B \, {}_B\langle j'|\rangle_B \\
\rho_A &= \sum_{i,j} \sum_{i',j'} c_{ij}\bar{c}_{i'j'} |i\rangle_A \, {}_A\langle i'| \underbrace{{}_B\langle j'| \underbrace{\sum_k |k\rangle_B \, {}_B\langle k|}_{I} |j\rangle_B}_{\delta_{jj'}} \\
\rho_A &= \sum_{i,j} \sum_{i'} c_{ij}\bar{c}_{i'j} |i\rangle_A \, {}_A\langle i'|.
\end{aligned} \tag{9.4}$$

Comparison of (9.3) and (9.4) implies that

$$(\rho_A)_{ii'} = \sum_j c_{ij}\bar{c}_{i'j}. \tag{9.5}$$

This equation can be rewritten in matrix form

$$(\rho_A)_{ii'} = \sum_j c_{ij}(c^\dagger)_{ji'} \tag{9.6}$$

or

$$\rho_A = \hat{C}\hat{C}^\dagger \tag{9.7}$$

where $\hat{C}$ is matrix of coefficients (9.1). The matrix elements $(\rho_A)_{ij}$ are considered as given and the goal is to determine corresponding matrix $\hat{C}$, by solving quadratic equations (9.6) or (9.7). These equations always can be solved for sufficiently large system B.

Example 9.1 *(Benenti et al. 2018, 92)* Let for spin-$\frac{1}{2}$ system A the density matrix ρ_A is known. Then it is sufficient to add the second spin-$\frac{1}{2}$ system, for purification of ρ_A. Due to (9.6) we need to solve the system of equations

$$
\begin{aligned}
(\rho_A)_{00} &= |c_{00}|^2 + |c_{01}|^2 && (9.8)\\
(\rho_A)_{11} &= |c_{10}|^2 + |c_{11}|^2 && (9.9)\\
(\rho_A)_{01} &= (\bar{\rho}_A)_{10} = c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} && (9.10)
\end{aligned}
$$

Simplest solution is : $c_{01} = 0$. It implies

$$(\rho_A)_{00} = |c_{00}|^2,$$

$$(\rho_A)_{01} = c_{00}\bar{c}_{10},$$

$$(\rho_A)_{11} = |c_{10}|^2 + |c_{11}|^2.$$

$$
\begin{aligned}
\bar{c}_{10} &= \frac{(\rho_A)_{01}}{c_{00}} \to c_{10} = \frac{(\bar{\rho}_A)_{01}}{\bar{c}_{00}},\\
c_{00} &= \sqrt{(\rho_A)_{00}} \to c_{10} = \frac{(\bar{\rho}_A)_{01}}{\sqrt{(\rho_A)_{00}}}, && (9.11)\\
c_{11} &= \sqrt{(\rho_A)_{11} - \frac{|(\rho_A)_{01}|^2}{|c_{00}|^2}} = \sqrt{\frac{(\rho_A)_{00}(\rho_A)_{11} - |(\rho_A)_{01}|^2}{(\rho_A)_{00}}},\\
c_{11} &= \sqrt{\frac{det\rho_A}{(\rho_A)_{00}}} = \frac{C}{2}\frac{1}{\sqrt{(\rho_A)_{00}}}.
\end{aligned}
$$

Example 9.2 Let

$$\rho_A = \frac{1}{2}|0\rangle_A\, {}_A\langle 0| + \frac{1}{2}|1\rangle_A\, {}_A\langle 1| \qquad (9.12)$$

so that $(\rho_A)_{00} = \frac{1}{2}$, $(\rho_A)_{11} = \frac{1}{2}$, $(\rho_A)_{01} = 0$.

1. The first solution: By choosing $c_{10} = 0$, and $c_{00} = \frac{1}{\sqrt{2}}$, $c_{11} = \pm\frac{1}{\sqrt{2}}$. The purified state

is the first pair of Bell states

$$|\psi\rangle \;=\; \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B \pm |1\rangle_A|1\rangle_B\right).$$

2. Another solution: We choose $c_{00} = 0$

$$(\rho_A)_{00} = \frac{1}{2} = |c_{01}|^2,$$
$$(\rho_A)_{11} = \frac{1}{2} = |c_{10}|^2 + |c_{11}|^2,$$

and

$$c_{01}\bar{c}_{11} = 0 \Rightarrow c_{11} = 0 \Rightarrow |c_{11}|^2 = 0$$
$$\Rightarrow (\rho_A)_{11} = \frac{1}{2} = |c_{10}|^2$$
$$|c_{01}|^2 = \frac{1}{2} \quad |c_{10}|^2 = \frac{1}{2} \rightarrow c_{01} = c_{10} = \frac{\pm 1}{\sqrt{2}}.$$

The purified state is the second pair of Bell state

$$|\psi\rangle \;=\; \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B \pm |1\rangle_A|0\rangle_B\right).$$

It shows that the Bell states are purifications of state A and their are maximally entangled states. Now, if we apply unitary transformations to the states, preserving entanglement, we will get the set of purified states, corresponding to given state ρ_A.

9.1. Equations of Purification

9.1.1. Qubit-Qubit Case:

Let $\rho_A = CC^\dagger$, where

$$\hat{C} = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} \qquad \hat{C}^\dagger = \begin{pmatrix} \bar{c}_{00} & \bar{c}_{10} \\ \bar{c}_{01} & \bar{c}_{11} \end{pmatrix}$$

and the normalization condition is

$$|c_{00}|^2 + |c_{11}|^2 + |c_{01}|^2 + |c_{10}|^2 = 1.$$

Then,

$$\rho_A = \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 & c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} \\ c_{10}\bar{c}_{00} + c_{11}\bar{c}_{01} & |c_{11}|^2 + |c_{10}|^2 \end{pmatrix} = \begin{pmatrix} (\rho_A)_{00} & (\rho_A)_{01} \\ (\rho_A)_{10} & (\rho_A)_{11} \end{pmatrix},$$

and the equations are

$$\begin{aligned} |c_{00}|^2 + |c_{01}|^2 &= (\rho_A)_{00}, \\ |c_{11}|^2 + |c_{10}|^2 &= (\rho_A)_{11}, \\ c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} &= (\rho_A)_{01}. \end{aligned}$$

Particular solution in this case was discussed in above examples (9.1) and (9.2). Applications of unitary transformations from Chapter 6, to any solution of these equations, provide the family of purified states, with the same level of entanglement.

9.1.2. Qubit-Qutrit Case:

Let

$$\hat{C} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \end{pmatrix}, \qquad \hat{C}^\dagger = \begin{pmatrix} \bar{c}_{00} & \bar{c}_{10} \\ \bar{c}_{01} & \bar{c}_{11} \\ \bar{c}_{02} & \bar{c}_{12} \end{pmatrix}.$$

Then,

$$\hat{C}\hat{C}^\dagger = \rho_A = \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12} \\ c_{10}\bar{c}_{00} + c_{11}\bar{c}_{01} + c_{12}\bar{c}_{02} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 \end{pmatrix}$$

and the equations are

$$\begin{aligned} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 &= (\rho_A)_{00}, \\ |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 &= (\rho_A)_{11}, \\ c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12} &= (\rho_A)_{01}. \end{aligned}$$

Example 9.3 Let

$$\rho_A = \frac{1}{2}\left(|0\rangle_A \, {}_A\langle 0| + |1\rangle_A \, {}_A\langle 1|\right).$$

1. We choose

$$c_{00} = c_{11} = \frac{1}{\sqrt{2}},$$

   and all other terms are zero

$$c_{01} = c_{02} = c_{10} = c_{12} = 0.$$

The purified state is the Bell state

$$|\psi_1\rangle \quad = \quad \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

2. We choose

$$c_{00} = c_{12} = \frac{1}{\sqrt{2}},$$

and all other terms are zero

$$c_{01} = c_{02} = c_{10} = c_{11} = 0.$$

The purified state is the Bell state

$$|\psi_2\rangle \quad = \quad \frac{|00\rangle + |12\rangle}{\sqrt{2}}.$$

Both purified states $|\psi_1\rangle$ and $|\psi_2\rangle$ are max entangled qubit-qutrit states with $C = 1$. In this case unitary transformations from Section 7.8 can give the set of purified states.

9.1.3. Qutrit-Qutrit Case:

Let

$$\hat{C} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}, \qquad \hat{C}^\dagger = \begin{pmatrix} \bar{c}_{00} & \bar{c}_{10} & \bar{c}_{20} \\ \bar{c}_{01} & \bar{c}_{11} & \bar{c}_{21} \\ \bar{c}_{02} & \bar{c}_{12} & \bar{c}_{22} \end{pmatrix}.$$

Then,

$$\rho_A \quad = \quad \begin{pmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12} & c_{00}\bar{c}_{20} + c_{01}\bar{c}_{21} + c_{02}\bar{c}_{22} \\ c_{10}\bar{c}_{00} + c_{11}\bar{c}_{01} + c_{12}\bar{c}_{02} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 & c_{10}\bar{c}_{20} + c_{11}\bar{c}_{21} + c_{12}\bar{c}_{22} \\ c_{20}\bar{c}_{00} + c_{21}\bar{c}_{01} + c_{22}\bar{c}_{02} & c_{20}\bar{c}_{10} + c_{21}\bar{c}_{11} + c_{22}\bar{c}_{12} & |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \end{pmatrix}$$

and the equations are

$$\begin{aligned}
|c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 &= (\rho_A)_{00}, \\
|c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 &= (\rho_A)_{11}, \\
|c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 &= (\rho_A)_{22}, \\
c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12} &= (\rho_A)_{01}, \\
c_{00}\bar{c}_{20} + c_{01}\bar{c}_{21} + c_{02}\bar{c}_{22} &= (\rho_A)_{02}, \\
c_{10}\bar{c}_{20} + c_{11}\bar{c}_{21} + c_{12}\bar{c}_{22} &= (\rho_A)_{12}.
\end{aligned}$$

Example 9.4 Let

$$\rho_A = \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2|.$$

By choosing $c_{00} = c_{11} = c_{22} = \frac{1}{\sqrt{3}}$, and all other terms are 0. The purification state is

$$|\psi\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}}.$$

Then, the state is maximally entangled state with $C = \frac{2}{\sqrt{3}}$. Unitary transformations from Section 8.6 can generate the set of purified states for two qutrit case.

9.2. Diagonal Reduced Density Matrix

For reduced density matrix in diagonal form

$$\rho_A = \sum_i p_i |i\rangle_A\, {}_A\langle i|,$$

where $\sum_i p_i = 1$, for purification it is sufficient to have for B system the same state space as system A (Benenti et al. 2018, 93). The purification for this density matrix is given by state

$$|\psi\rangle = \sum_i \sqrt{p_i}|i\rangle_A|i\rangle_B.$$

This form of pure state coincides with Schmidt decomposition.

CHAPTER 10

ENERGY MINIMIZATION FOR MAXIMALLY ENTANGLED STATES

Here we are going to apply our maximally entangled two-qubit states to minimize the average energy in two interacting spins models. It was shown in Chapter 6, Theorem 6.2, that all maximally entangled two-qubit states have the form

$$|\psi\rangle = \frac{a|00\rangle + \bar{a}|11\rangle}{\sqrt{2}} + \frac{-\bar{b}|01\rangle + b|10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} a \\ -\bar{b} \\ b \\ \bar{a} \end{bmatrix} \tag{10.1}$$

where $|a|^2 + |b|^2 = 1$. Hamiltonian of XYZ model for two spins in homogeneous magnetic field B and non-homogeneous one b_0 is

$$H = \frac{1}{2}\left[J_x X \otimes X + J_y Y \otimes Y + J_z Z \otimes Z + B_+ Z \otimes I + B_- I \otimes Z\right],$$

where J_x, J_y, J_z are constants(exchange integrals) and $B_\pm \equiv B \pm b_0$. By calculating tensor products

$$X \otimes X = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, \quad Y \otimes Y = \begin{pmatrix} 0 & -iY \\ iY & 0 \end{pmatrix}, \quad Z \otimes Z = \begin{pmatrix} Z & 0 \\ 0 & -Z \end{pmatrix},$$
$$Z \otimes I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad I \otimes Z = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}$$

we have Hamiltonian

$$H = \frac{1}{2}\begin{bmatrix} J_z Z + B_+ I + B_- Z & J_x X - iJ_y Y \\ J_x X + iJ_y Y & -J_z Z - B_+ I + B_- Z \end{bmatrix}$$

or in matrix form

$$H = \frac{1}{2}\begin{bmatrix} J_z + B_+ + B_- & 0 & 0 & J_x - J_y \\ 0 & -J_z + B_+ - B_- & J_x + J_y & 0 \\ 0 & J_x + J_y & -J_z - B_+ + B_- & 0 \\ J_x - J_y & 0 & 0 & J_z - B_+ - B_- \end{bmatrix}. \quad (10.2)$$

We are going to find the average energy in state (10.1):

$$\langle\psi|H|\psi\rangle = E$$

as function of a & b. Applying

$$H|\psi\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix} (J_z + B_+ + B_-)a + J_-\bar{a} \\ (-J_z + B_+ - B_-)(-\bar{b}) + J_+b \\ J_+(-\bar{b}) + (-J_z - B_+ + B_-)b \\ J_-a + (J_z - B_+ - B_-)\bar{a} \end{bmatrix},$$

where $J_\pm \equiv J_x \pm J_-,\ B_+ + B_- = 2B,\ B_+ - B_- = 2b_0$, or

$$H|\psi\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix} (J_z + 2B)a + J_-\bar{a} \\ (-J_z + 2b_0)(-\bar{b}) + J_+b \\ -J_+\bar{b} + (-J_z - 2b_0)b \\ J_-a + (J_z - 2B)\bar{a} \end{bmatrix},$$

we get

$$\begin{aligned} \langle\psi|H|\psi\rangle &= \frac{1}{4}\{(J_z + 2B)|a|^2 + J_-\bar{a}^2 + (-J_z + 2b_0)|b|^2 - J_+b^2 \\ &- J_+\bar{b}^2 - (J_z + 2b_0)|b|^2 + J_-a^2 + (J_z - 2B)|a|^2\} \\ &= \frac{1}{4}\{(J_z + 2B + J_z22B)|a|^2 + (-J_z + 2b_0 - J_z - 2b_0)|b|^2 \\ &+ J_-(a^2 + \bar{a}^2) - J_+(B^2 + \bar{b}^2)\} \end{aligned}$$

or

$$\langle\psi|H|\psi\rangle \;=\; \frac{1}{4}\{J_z(|a|^2-|b|^2)+J_-(a^2+\bar{a}^2)-J_+(b^2+\bar{b}^2)\}.$$

Let $a=|a|e^{i\varphi_a}, \quad b=|b|e^{i\varphi_b}$ then,

$$\begin{aligned}
a^2+\bar{a}^2 &= |a|^2(e^{2\varphi_a}+e^{-2\varphi_a})=2|a|^2\cos 2\varphi_a,\\
b^2+\bar{b}^2 &= 2|b|^2\cos 2\varphi_b,\\
\langle\psi|H|\psi\rangle &= \frac{1}{4}\{J_z(|a|^2+|b|^2)+2J_-|a|^2\cos 2\varphi_a-2J_+|b|^2\cos 2\varphi_b.
\end{aligned}$$

Since $|a|^2+|b|^2=1$, we can choose $|a|=\cos\frac{\theta}{2}$ and $|b|=\sin\frac{\theta}{2}$, where $0\le\theta\le\pi$. Then,

$$|a|^2-|b|^2=\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}=\cos\theta,$$

and

$$\langle\psi|H|\psi\rangle \;=\; \frac{1}{4}\{J_z\cos\theta+2J_-\cos^2\frac{\theta}{2}\cos 2\varphi_a-2J_+\sin^2\frac{\theta}{2}\cos 2\varphi_b\}.$$

We have another form by using $1+\cos\theta=2\cos^2\frac{\theta}{2}$ and $1-\cos\theta=2\sin^2\frac{\theta}{2}$

$$\langle\psi|H|\psi\rangle \;=\; \frac{1}{4}\{J_z\cos\theta+J_-(1+\cos\theta)\cos 2\varphi_a-J_+(1-\cos\theta)\cos 2\varphi_b\}.$$

This gives the energy as function

$$E=\langle\psi|H|\psi\rangle=E(\varphi_a,\varphi_b,\theta),$$

of the maximally entangled states parameters

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\frac{\theta}{2}e^{i\varphi_a} \\ -\sin\frac{\theta}{2}e^{-i\varphi_b} \\ \sin\frac{\theta}{2}e^{i\varphi_b} \\ \cos\frac{\theta}{2}e^{-i\varphi_a} \end{pmatrix} = \left\{\cos\frac{\theta}{2}\frac{e^{i\varphi_a}|00\rangle + e^{-i\varphi_a}|11\rangle}{\sqrt{2}} + \sin\frac{\theta}{2}\frac{-e^{-i\varphi_b}|01\rangle + e^{i\varphi_b}|10\rangle}{\sqrt{2}}\right\},$$

where $0 \le \varphi_a \le 2\pi,\ 0 \le \varphi_b \le 2\pi,\ 0 \le \theta \le \pi$.

In Figures 10.1, 10.2 and 10.3 we plot the energy surface for maximally entangled states in special cases.

10.1. Particular Cases

1. XXX model $J_x = J_y = J_z \Rightarrow J_- = 0 \quad \& \quad J_+ = 2J_z$

$$E = \frac{1}{4}\{J_z\cos\theta - 2J_z(1-\cos\theta)\cos 2\varphi_b\} = E(\theta,\varphi_b)$$

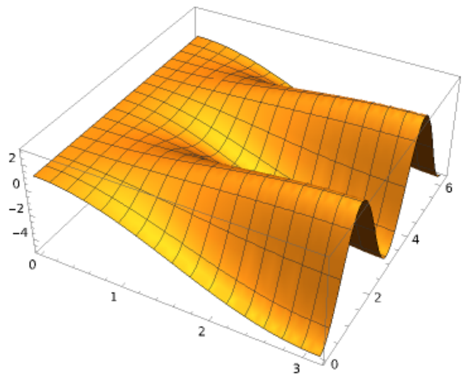

Figure 10.1. XXX Model

2. XXZ model $J_x = J_y \Rightarrow J_- = 0 \quad \& \quad J_+ = 2J_x$

$$E(\varphi_b, \theta) = \frac{1}{4}\{J_z \cos\theta - 2J_x(1 - cos\theta)\cos 2\varphi_b\}$$

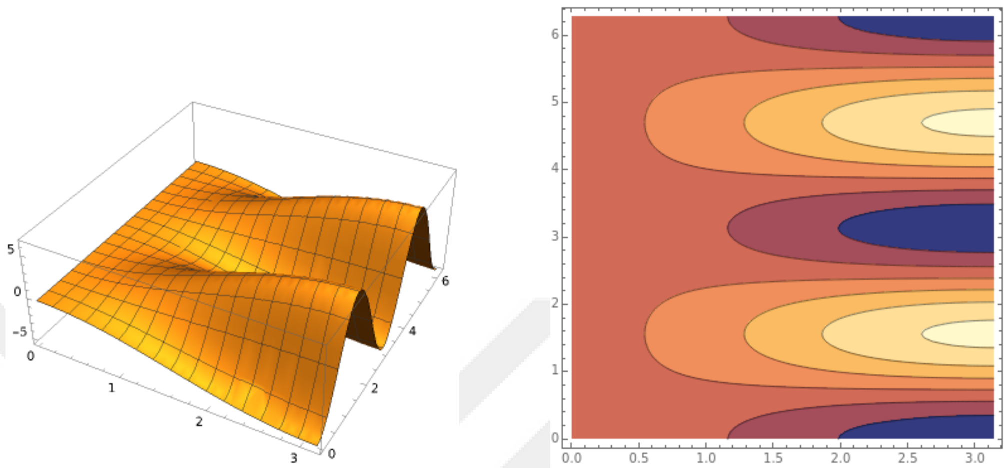

Figure 10.2. XXZ Model

3. XYZ model

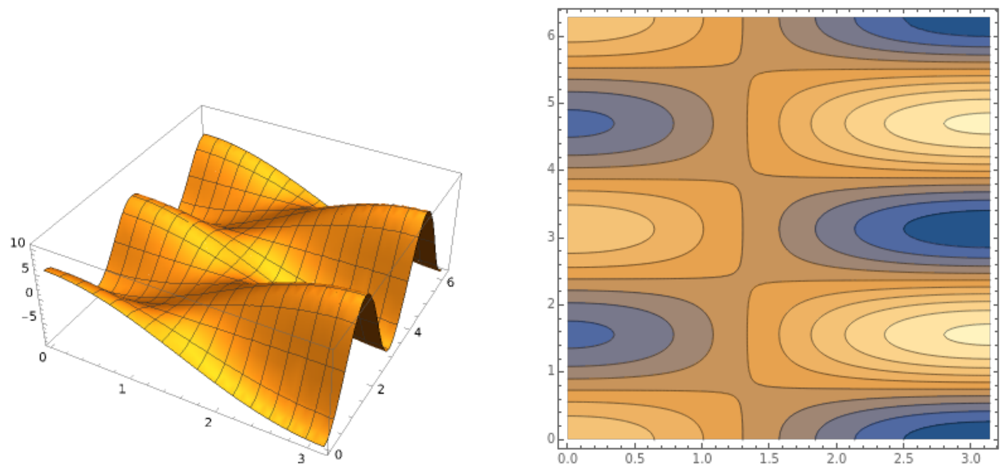

Figure 10.3. XYZ Model

CHAPTER 11

CONCLUSIONS

In present thesis we studied entanglement characterization for pure qubit-qubit, qubit-qutrit and qutrit-qutrit states and invariance properties of the entanglement. Entanglement quantification of the sates was performed by using reduced density matrix approach, the linear entropy and the von Neumann entropy. It was shown that the linear entropy plays the role of the concurrence square, and it is a simpler characteristic of entanglement, than the von Neumann entropy. In all three cases we studied unitary one-qubit or one-qutrit gates, and show invariance of entanglement under these transformations. This allowed us to describe the continuum parametrized set of the states with the same level of entanglement. The results were applied to construct the set of purification states from the given mixed state, described by density matrix. In addition, for two-qubit spin XYZ model in magnetic field we used the set of maximally entangled states as trial states for average value of the energy and found local minima occurring in the energy.

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APPENDIX A

LINEAR ENTROPY FOR ρ_A

$$
\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= |\psi_0\rangle\langle\psi_0|\,(|0\rangle_B\,{}_B\langle 0|) + |\psi_0\rangle\langle\psi_1|\,(|0\rangle_B\,{}_B\langle 1|) + |\psi_0\rangle\langle\psi_2|\,(|0\rangle_B\,{}_B\langle 2|) \\
&+ |\psi_1\rangle\langle\psi_0|\,(|1\rangle_B\,{}_B\langle 0|) + |\psi_1\rangle\langle\psi_1|\,(|1\rangle_B\,{}_B\langle 1|) + |\psi_1\rangle\langle\psi_2|\,(|1\rangle_B\,{}_B\langle 2|) \\
&+ |\psi_2\rangle\langle\psi_0|\,(|2\rangle_B\,{}_B\langle 0|) + |\psi_2\rangle\langle\psi_1|\,(|2\rangle_B\,{}_B\langle 1|) + |\psi_2\rangle\langle\psi_2|\,(|2\rangle_B\,{}_B\langle 2|) \\
\rho_A &= tr_B\rho = \langle 0|\rho|0\rangle + \langle 1|\rho|1\rangle + \langle 2|\rho|2\rangle \\
&= |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|
\end{aligned}
$$

and

$$
\begin{aligned}
tr\rho_A &= {}_A\langle 0|\rho_A|0\rangle_A +_A \langle 1|\rho_A|1\rangle_A \\
&= {}_A\langle 0|\psi_0\rangle\langle\psi_0|0\rangle_A +_A \langle 0|\psi_1\rangle\langle\psi_1|0\rangle_A +_A \langle 0|\psi_2\rangle\langle\psi_2|0\rangle_A \\
&+ {}_A\langle 1|\psi_0\rangle\langle\psi_0|1\rangle_A +_A \langle 1|\psi_1\rangle\langle\psi_1|1\rangle_A +_A \langle 1|\psi_2\rangle\langle\psi_2|1\rangle_A \\
&= \langle\psi_0|0\rangle_A\,{}_A\langle 0|\psi_0\rangle + \langle\psi_0|1\rangle_A\,{}_A\langle 1|\psi_0\rangle \\
&+ \langle\psi_1|0\rangle_A\,{}_A\langle 0|\psi_1\rangle + \langle\psi_1|1\rangle_A\,{}_A\langle 1|\psi_1\rangle \\
&+ \langle\psi_2|0\rangle_A\,{}_A\langle 0|\psi_2\rangle + \langle\psi_2|1\rangle_A\,{}_A\langle 1|\psi_2\rangle \\
&= \langle\psi_0|\underbrace{(|0\rangle_A\,{}_A\langle 0| + |1\rangle_A\,{}_A\langle 1|)}_{I}|\psi_0\rangle \\
&+ \langle\psi_1|\underbrace{(|0\rangle_A\,{}_A\langle 0| + |1\rangle_A\,{}_A\langle 1|)}_{I}|\psi_1\rangle \\
&+ \langle\psi_2|\underbrace{(|0\rangle_A\,{}_A\langle 0| + |1\rangle_A\,{}_A\langle 1|)}_{I}|\psi_2\rangle \\
tr\rho_A &= \langle\psi_0|\psi_0\rangle + \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle = 1.
\end{aligned}
$$

Then, the squared reduced matrix is

$$\begin{aligned}\rho_A^2 &= |\psi_0\rangle\langle\psi_0|\langle\psi_0|\psi_0\rangle + |\psi_0\rangle\langle\psi_1|\langle\psi_0|\psi_1\rangle + |\psi_0\rangle\langle\psi_2|\langle\psi_0|\psi_2\rangle \\ &+ |\psi_1\rangle\langle\psi_0|\langle\psi_1|\psi_0\rangle + |\psi_1\rangle\langle\psi_1|\langle\psi_1|\psi_1\rangle + |\psi_1\rangle\langle\psi_2|\langle\psi_1|\psi_2\rangle \\ &+ |\psi_2\rangle\langle\psi_0|\langle\psi_2|\psi_0\rangle + |\psi_2\rangle\langle\psi_1|\langle\psi_2|\psi_1\rangle + |\psi_2\rangle\langle\psi_2|\langle\psi_2|\psi_2\rangle.\end{aligned}$$

Then, we calculate trace of ρ_A^2

$$\begin{aligned}tr\rho_A^2 &= tr\left(|\psi_0\rangle\langle\psi_0|\right)\langle\psi_0|\psi_0\rangle + tr\left(|\psi_0\rangle\langle\psi_1|\right)\langle\psi_0|\psi_1\rangle + tr\left(|\psi_0\rangle\langle\psi_2|\right)\langle\psi_0|\psi_2\rangle \\ &+ tr\left(|\psi_1\rangle\langle\psi_0|\right)\langle\psi_1|\psi_0\rangle + tr\left(|\psi_1\rangle\langle\psi_1|\right)\langle\psi_1|\psi_1\rangle + tr\left(|\psi_1\rangle\langle\psi_2|\right)\langle\psi_1|\psi_2\rangle \\ &+ tr\left(|\psi_2\rangle\langle\psi_0|\right)\langle\psi_2|\psi_0\rangle + tr\left(|\psi_2\rangle\langle\psi_1|\right)\langle\psi_2|\psi_1\rangle + tr\left(|\psi_2\rangle\langle\psi_2|\right)\langle\psi_2|\psi_2\rangle \\ &= \langle\psi_1|\psi_0\rangle\langle\psi_1|\psi_0\rangle + \langle\psi_0|\psi_1\rangle\langle\psi_0|\psi_1\rangle + \langle\psi_0|\psi_2\rangle\langle\psi_0|\psi_2\rangle \\ &+ \langle\psi_1|\psi_0\rangle\langle\psi_1|\psi_0\rangle + \langle\psi_1|\psi_1\rangle\langle\psi_1|\psi_1\rangle + \psi_1|\psi_2\rangle\psi_1|\psi_2\rangle \\ &+ \langle\psi_2|\psi_0\rangle\langle\psi_2|\psi_0\rangle + \langle\psi_2|\psi_1\rangle\langle\psi_2|\psi_1\rangle + \langle\psi_2|\psi_2\rangle\langle\psi_2|\psi_2\rangle \\ tr\rho_A^2 &= |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_0|\psi_1\rangle|^2 + |\langle\psi_0|\psi_2\rangle|^2 \\ &+ |\langle\psi_1|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_0|\psi_2\rangle|^2 \\ &+ |\langle\psi_2|\psi_0\rangle|^2 + |\langle\psi_2|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2\end{aligned}$$

and squared normalization condition

$$\begin{aligned}1 = |\langle\psi_0|\psi_0\rangle|^2 + |\langle\psi_1|\psi_1\rangle|^2 + |\langle\psi_2|\psi_2\rangle|^2 + 2\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle \\ +2\langle\psi_0|\psi_0\rangle\langle\psi_2|\psi_2\rangle + 2\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle.\end{aligned}$$

Taking difference

$$\begin{aligned}1 - tr\rho_A^2 &= 2\left(\langle\psi_0|\psi_0\rangle\langle\psi_1|\psi_1\rangle - |\langle\psi_0|\psi_1\rangle|^2\right) \\ &+ 2\left(\langle\psi_0|\psi_0\rangle\langle\psi_2|\psi_2\rangle - |\langle\psi_0|\psi_2\rangle|^2\right) \\ &+ 2\left(\langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle - |\langle\psi_1|\psi_2\rangle|^2\right)\end{aligned}$$

or

$$1 - tr\rho_A^2 \quad = \quad 2\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_2\rangle \\ \langle\psi_2|\psi_0\rangle & \langle\psi_2|\psi_2\rangle \end{vmatrix} + 2\begin{vmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle \end{vmatrix}.$$

Let

$$\begin{aligned} \langle\psi_i|\psi_j\rangle &= (\langle 0|\bar{c}_{0i} + \langle 1|\bar{c}_{1i})\left(c_{0j}|0\rangle + c_{1j}|1\rangle\right) \\ &= \bar{c}_{0i}c_{0j} + \bar{c}_{1i}c_{1j} \end{aligned}$$

Then we checked

$$\begin{aligned} \begin{vmatrix} \langle\psi_i|\psi_i\rangle & \langle\psi_i|\psi_j\rangle \\ \langle\psi_j|\psi_i\rangle & \langle\psi_j|\psi_j\rangle \end{vmatrix} &= \langle\psi_i|\psi_i\rangle\langle\psi_j|\psi_j\rangle - \langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_i\rangle \\ &= \left(|c_{0i}|^2 + |c_{1i}|^2\right)\left(|c_{0j}|^2 + |c_{1j}|^2\right) - \left(\bar{c}_{0i}c_{0j} + \bar{c}_{1i}c_{1j}\right)\left(c_{0i}\bar{c}_{0j} + c_{1i}\bar{c}_{1j}\right) \\ &= |c_{0i}|^2|c_{0j}|^2 + |c_{0i}|^2|c_{1j}|^2 + |c_{1i}|^2|c_{0j}|^2 + |c_{1i}|^2|c_{1j}|^2 \\ &- |c_{0i}|^2|c_{0j}|^2 - |c_{1i}|^2|c_{1j}|^2 - \bar{c}_{0i}c_{0j}c_{1i}\bar{c}_{1j} - \bar{c}_{1i}c_{1j}c_{0i}\bar{c}_{0j} \\ &= c_{0i}\bar{c}_{0i}c_{1j}\bar{c}_{1j} - \bar{c}_{0i}c_{0j}c_{1i}\bar{c}_{1j} - c_{1i}\bar{c}_{1i}c_{0j}\bar{c}_{0j} - \bar{c}_{1i}c_{1j}c_{0i}\bar{c}_{0j} \\ &= \bar{c}_{0i}\bar{c}_{1j}\left(c_{0i}c_{1j} - c_{0j}c_{1i}\right) + \bar{c}_{1i}\bar{c}_{0j}\left(c_{0j}c_{1i} - c_{0i}c_{1j}\right) \\ &= \left(c_{0i}c_{1j} - c_{0j}c_{1i}\right)\left(\bar{c}_{0i}\bar{c}_{1j} - \bar{c}_{1i}\bar{c}_{0j}\right) \\ &= \begin{vmatrix} c_{0i} & c_{0j} \\ c_{1i} & c_{1j} \end{vmatrix}^2, \end{aligned}$$

and we get

$$\begin{aligned} \begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_1\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_1\rangle \end{vmatrix} &= \begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2, \\ \begin{vmatrix} \langle\psi_0|\psi_0\rangle & \langle\psi_0|\psi_2\rangle \\ \langle\psi_1|\psi_0\rangle & \langle\psi_1|\psi_2\rangle \end{vmatrix} &= \begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2, \\ \begin{vmatrix} \langle\psi_0|\psi_1\rangle & \langle\psi_0|\psi_2\rangle \\ \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle \end{vmatrix} &= \begin{vmatrix} c_{01} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2. \end{aligned}$$

Thus, we have

$$1 - tr\rho_A^2 = 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2 .$$

APPENDIX B

LINEAR ENTROPY FOR ρ_B

$$
\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= |0\rangle\langle 0|(|\varphi_0\rangle\langle\varphi_0|) + |0\rangle\langle 1|(|\varphi_0\rangle\langle\varphi_1|) \\
&+ |1\rangle\langle 0|(|\varphi_1\rangle\langle\varphi_0|) + |1\rangle\langle 1|(|\varphi_1\rangle\langle\varphi_1|) \\
\rho_B &= tr_A\rho =_B \langle 0|\rho|0\rangle_B +_B \langle 1|\rho|1\rangle_B \\
&= |\varphi_0\rangle\langle\varphi_0| + |\varphi_1\rangle\langle\varphi_1|,
\end{aligned}
$$

and trace of reduced density matrix is

$$
\begin{aligned}
tr\rho_B &= {}_B\langle 0|\rho_B|0\rangle_B +_B \langle 1|\rho_B|1\rangle_B +_B \langle 2|\rho_B|2\rangle_B \\
&= {}_B\langle 0|\varphi_0\rangle\langle\varphi_0|0\rangle_B +_B \langle 1|\varphi_0\rangle\langle\varphi_0|1\rangle_B +_B \langle 2|\varphi_0\rangle\langle\varphi_0|2\rangle_B \\
&= {}_B\langle 0|\varphi_1\rangle\langle\varphi_1|0\rangle_B +_B \langle 1|\varphi_1\rangle\langle\varphi_1|1\rangle_B +_B \langle 2|\varphi_1\rangle\langle\varphi_1|2\rangle_B \\
&= \langle\varphi_0|0\rangle_B\, {}_B\langle 0|\varphi_0\rangle + \langle\varphi_0|1\rangle_B\, {}_B\langle 1|\varphi_0\rangle + \langle\varphi_0|2\rangle_B\, {}_B\langle 2|\varphi_0\rangle \\
&+ \langle\varphi_1|0\rangle_B\, {}_B\langle 0|\varphi_1\rangle + \langle\varphi_1|1\rangle_B\, {}_B\langle 1|\varphi_1\rangle + \langle\varphi_1|2\rangle_B\, {}_B\langle 2|\varphi_1\rangle \\
&= \langle\varphi_0|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I}|\varphi_0\rangle + \langle\varphi_1|\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I}|\varphi_1\rangle \\
&= \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle \\
tr\rho_B &= 1 \Rightarrow \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle = 1.
\end{aligned}
$$

Then, the squared reduced density matrix is

$$
\begin{aligned}
\rho_B^2 &= |\varphi_0\rangle\langle\varphi_0|\langle\varphi_0|\varphi_0\rangle + |\varphi_1\rangle\langle\varphi_1|\langle\varphi_1|\varphi_1\rangle \\
&+ |\varphi_0\rangle\langle\varphi_1|\langle\varphi_1|\varphi_0\rangle + |\varphi_1\rangle\langle\varphi_0|\langle\varphi_0|\varphi_1\rangle
\end{aligned}
$$

and the trace of this matrix is equal

$$tr\rho_B^2 = |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_1|\varphi_0\rangle|^2 + |\langle\varphi_0|\varphi_1\rangle|^2.$$

The squared normalization condition gives

$$1 = |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle.$$

Then, by taking difference

$$\begin{aligned} 1 - tr\rho_B^2 &= 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle - 2|\langle\varphi_0|\varphi_1\rangle|^2 \\ &= 2\begin{vmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle \end{vmatrix}. \end{aligned}$$

Let

$$\begin{aligned} \langle\varphi_i|\varphi_j\rangle &= (\langle 0|\bar{c}_{0i} + \langle 1|\bar{c}_{1i})\left(c_{0j}|0\rangle + c_{1j}|1\rangle\right) \\ &= \bar{c}_{0i}c_{0j} + \bar{c}_{1i}c_{1j} \end{aligned}$$

Then we checked

$$\begin{aligned} \langle\varphi_0|\varphi_0\rangle &= |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2, \\ \langle\varphi_1|\varphi_1\rangle &= |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2, \\ \langle\varphi_0|\varphi_1\rangle &= \bar{c}_{00}c_{10} + \bar{c}_{01}c_{11} + \bar{c}_{02}c_{12}, \\ \langle\varphi_1|\varphi_0\rangle &= c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12}. \end{aligned}$$

Applying this conditions

$$
\begin{aligned}
1 - tr\rho_B^2 &= 2\begin{vmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle \end{vmatrix} \\
&= 2\begin{vmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & \bar{c}_{00}c_{10} + \bar{c}_{01}c_{11} + \bar{c}_{02}c_{12} \\ c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 \end{vmatrix} \\
&= 2\Big(\left(|c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2\right)\left(|c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2\right) \\
&- \left(\bar{c}_{00}c_{10} + \bar{c}_{01}c_{11} + \bar{c}_{02}c_{12}\right)\left(c_{00}\bar{c}_{10} + c_{01}\bar{c}_{11} + c_{02}\bar{c}_{12}\right)\Big) \\
&= 2\{|c_{00}|^2|c_{10}|^2 + |c_{00}|^2|c_{11}|^2 + |c_{00}|^2|c_{12}|^2 + |c_{01}|^2|c_{10}|^2 \\
&+ |c_{01}|^2|c_{11}|^2 + |c_{01}|^2|c_{12}|^2 + |c_{02}|^2|c_{10}|^2 + |c_{02}|^2|c_{11}|^2 \\
&+ |c_{02}|^2|c_{12}|^2 - |c_{00}|^2|c_{10}|^2 - |c_{01}|^2|c_{11}|^2 - |c_{02}|^2|c_{12}|^2 \\
&- \bar{c}_{00}c_{10}c_{01}\bar{c}_{11} - \bar{c}_{00}c_{10}c_{02}\bar{c}_{12} - \bar{c}_{01}c_{11}c_{00}\bar{c}_{10} \\
&- \bar{c}_{01}c_{11}c_{02}\bar{c}_{12} - \bar{c}_{02}c_{12}c_{00}\bar{c}_{10} - \bar{c}_{02}c_{12}c_{01}\bar{c}_{11}\} \\
&= \{\bar{c}_{00}c_{00}c_{11}\bar{c}_{11} + \bar{c}_{00}c_{00}c_{12}\bar{c}_{12} + \bar{c}_{01}c_{01}c_{10}\bar{c}_{10} \\
&+ \bar{c}_{01}c_{01}c_{12}\bar{c}_{12} + \bar{c}_{02}c_{02}c_{10}\bar{c}_{10} + \bar{c}_{02}c_{02}c_{11}\bar{c}_{11} \\
&- \bar{c}_{00}c_{10}c_{01}\bar{c}_{11} - \bar{c}_{00}c_{10}c_{02}\bar{c}_{12} - \bar{c}_{01}c_{11}c_{00}\bar{c}_{10} \\
&- \bar{c}_{01}c_{11}c_{02}\bar{c}_{12} - \bar{c}_{02}c_{12}c_{00}\bar{c}_{10} - \bar{c}_{02}c_{12}c_{01}\bar{c}_{11}\} \\
&= 2\{\bar{c}_{00}\bar{c}_{11}\left(c_{00}c_{11} - c_{10}c_{01}\right) + \bar{c}_{00}\bar{c}_{12}\left(c_{00}c_{12} - c_{10}c_{02}\right) \\
&+ \bar{c}_{01}\bar{c}_{10}\left(c_{01}c_{10} - c_{11}c_{00}\right) + \bar{c}_{01}\bar{c}_{12}\left(c_{01}c_{12} - c_{11}c_{02}\right) \\
&+ \bar{c}_{02}\bar{c}_{10}\left(c_{02}c_{10} - c_{12}c_{00}\right) + \bar{c}_{02}\bar{c}_{11}\left(c_{02}c_{11} - c_{12}c_{01}\right)\} \\
&= 2\{\left(\bar{c}_{00}\bar{c}_{11} - \bar{c}_{01}\bar{c}_{10}\right)\left(c_{00}c_{11} - c_{10}c_{01}\right) \\
&+ \left(\bar{c}_{00}\bar{c}_{12} - \bar{c}_{02}\bar{c}_{10}\right)\left(c_{00}c_{12} - c_{02}c_{10}\right) \\
&+ \left(\bar{c}_{01}\bar{c}_{12} - \bar{c}_{02}\bar{c}_{11}\right)\left(c_{01}c_{12} - c_{11}c_{02}\right)\}
\end{aligned}
$$

Thus, we have

$$
1 - tr\rho_B^2 = 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2 .
$$

APPENDIX C

ENTANGLEMENT OF PURE TWO QUTRIT STATE

Definition C.1 For two qutrit state $|\psi\rangle$, the density operator is

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| = \sum_{i,j=0}^{2} c_{ij}|ij\rangle \sum_{i',j'=0}^{2} \bar{c}_{i'j'}\langle i'j'| \\ &= \sum_{i,j=0}^{2}\sum_{i',j'=0}^{2} c_{ij}\bar{c}_{i'j'}|ij\rangle\langle i'j'|.\end{aligned}$$

The density operator for generic two qutrit state is

$$\begin{aligned}\rho &= (|0\rangle_A|\varphi_0\rangle_B + |1\rangle_A|\varphi_1\rangle_B + |2\rangle_A|\varphi_2\rangle_B)\,({}_A\langle 0|_B\langle\varphi_0| +_A \langle 1|_B\langle\varphi_1| +_A \langle 2|_B\langle\varphi_2|) \\ &= (|0\rangle_A\, {}_A\langle 0|)\,(|\varphi_0\rangle_B\, {}_B\langle\varphi_0|) + (|0\rangle_A\, {}_A\langle 1|)\,(|\varphi_0\rangle_B\, {}_B\langle\varphi_1|) + (|0\rangle_A\, {}_A\langle 2|)\,(|\varphi_0\rangle_B\, {}_B\langle\varphi_2|) \\ &+ (|1\rangle_A\, {}_A\langle 0|)\,(|\varphi_1\rangle_B\, {}_B\langle\varphi_0|) + (|1\rangle_A\, {}_A\langle 1|)\,(|\varphi_1\rangle_B\, {}_B\langle\varphi_1|) + (|1\rangle_A\, {}_A\langle 2|)\,(|\varphi_1\rangle_B\, {}_B\langle\varphi_2|) \\ &+ (|2\rangle_A\, {}_A\langle 0|)\,(|\varphi_2\rangle_B\, {}_B\langle\varphi_0|) + (|2\rangle_A\, {}_A\langle 1|)\,(|\varphi_2\rangle_B\, {}_B\langle\varphi_1|) + (|2\rangle_A\, {}_A\langle 2|)\,(|\varphi_2\rangle_B\, {}_B\langle\varphi_2|)\end{aligned}$$

or in explicit matrix form

$$\rho = \begin{pmatrix} |\varphi_0\rangle\langle\varphi_0| & |\varphi_0\rangle\langle\varphi_1| & |\varphi_0\rangle\langle\varphi_2| \\ |\varphi_1\rangle\langle\varphi_0| & |\varphi_1\rangle\langle\varphi_1| & |\varphi_1\rangle\langle\varphi_2| \\ |\varphi_2\rangle\langle\varphi_0| & |\varphi_2\rangle\langle\varphi_1| & |\varphi_2\rangle\langle\varphi_2| \end{pmatrix}$$

For subsystem B, the reduced density operator in the following form

$$\rho_B = tr_A(\rho) = |\varphi_0\rangle_B\, {}_B\langle\varphi_0| + |\varphi_1\rangle_B\, {}_B\langle\varphi_1| + |\varphi_2\rangle_B\, {}_B\langle\varphi_2|$$

where $tr_A(\rho)$ is called the partial trace.

For generic two qutrit state

$$
\begin{aligned}
\rho_B &= tr_A(\rho) = \sum_{k=0}^{2} {}_A\langle k \mid \rho \mid k\rangle_A \\
&= \sum_{i,j}\sum_{i',j'} c_{ij}\overline{c_{i'j'}}|i\rangle_B\, {}_B\langle i'| \sum_k \underbrace{{}_A\langle k|j\rangle_A}_{\delta_{kj}} \underbrace{{}_A\langle j'|k\rangle_A}_{\delta_{j'k}} \\
&= \sum_{ij}\sum_{i'j'} c_{ij}\overline{c_{i'j'}}|i\rangle_B\, {}_B\langle i'| \underbrace{\sum_k \delta_{kj}\delta_{j'k}}_{\delta_{jj'}} \\
&= \sum_{ij}\sum_{i'j'} c_{ij}\overline{c_{i'j'}}|i\rangle_B\, {}_B\langle i'|\delta_{jj'} \\
&= \sum_{ij}\sum_{i'j} c_{ij}\overline{c_{i'j}}|i\rangle_B\, {}_B\langle i'|
\end{aligned}
$$

So the reduced density matrix is expressed as

$$\rho_B = |\varphi_0\rangle\langle\varphi_0| + |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2|,$$

and trace of reduced density matrix is

$$
\begin{aligned}
tr\rho_B &= {}_B\langle 0 \mid \rho_B \mid 0\rangle_B +_B \langle 1 \mid \rho_B \mid 1\rangle_B +_B \langle 2 \mid \rho_B \mid 2\rangle_B \\
&= {}_B\langle 0|\varphi_0\rangle\langle\varphi_0|0\rangle_B +_B \langle 1|\varphi_0\rangle\langle\varphi_0|1\rangle_B +_B \langle 2|\varphi_0\rangle\langle\varphi_0|2\rangle_B \\
&+ {}_B\langle 0|\varphi_1\rangle\langle\varphi_1|0\rangle_B +_B \langle 1|\varphi_1\rangle\langle\varphi_1|1\rangle_B +_B \langle 2|\varphi_1\rangle\langle\varphi_1|2\rangle_B \\
&+ {}_B\langle 0|\varphi_2\rangle\langle\varphi_2|0\rangle_B +_B \langle 1|\varphi_2\rangle\langle\varphi_2|1\rangle_B +_B \langle 2|\varphi_2\rangle\langle\varphi_2|2\rangle_B \\
&= \langle\varphi_0|0\rangle_B\, {}_B\langle 0|\varphi_0\rangle + \langle\varphi_0|1\rangle_B\, {}_B\langle 1|\varphi_0\rangle + \langle\varphi_0|2\rangle_B\, {}_B\langle 2|\varphi_0\rangle \\
&+ \langle\varphi_1|0\rangle_B\, {}_B\langle 0|\varphi_1\rangle + \langle\varphi_1|1\rangle_B\, {}_B\langle 1|\varphi_1\rangle + \langle\varphi_1|2\rangle_B\, {}_B\langle 2|\varphi_1\rangle \\
&+ \langle\varphi_2|0\rangle_B\, {}_B\langle 0|\varphi_2\rangle + \langle\varphi_2|1\rangle_B\, {}_B\langle 1|\varphi_2\rangle + \langle\varphi_2|2\rangle_B\, {}_B\langle 2|\varphi_2\rangle \\
&= \langle\varphi_0| \underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I} |\varphi_0\rangle \\
&+ \langle\varphi_1| \underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I} |\varphi_1\rangle \\
&+ \langle\varphi_2| \underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|)}_{I} |\varphi_2\rangle \\
&= \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle + \langle\varphi_2|\varphi_2\rangle
\end{aligned}
$$

and it gives

$$tr\rho_B = 1 \Rightarrow \langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle + \langle\varphi_2|\varphi_2\rangle = 1.$$

Then, the squared reduced density matrix is

$$\begin{aligned}
\rho_B^2 &= \sum_{i,j=0}^{2}\langle\varphi_i \mid \varphi_j\rangle|\varphi_i\rangle\langle\varphi_j| \\
&= |\varphi_0\rangle\langle\varphi_0|\langle\varphi_0|\varphi_0\rangle + |\varphi_0\rangle\langle\varphi_1|\langle\varphi_1|\varphi_0\rangle + |\varphi_0\rangle\langle\varphi_2|\langle\varphi_2|\varphi_0\rangle \\
&+ |\varphi_1\rangle\langle\varphi_0|\langle\varphi_0|\varphi_1\rangle + |\varphi_1\rangle\langle\varphi_1|\langle\varphi_1|\varphi_1\rangle + |\varphi_1\rangle\langle\varphi_2|\langle\varphi_2|\varphi_1\rangle \\
&+ |\varphi_2\rangle\langle\varphi_0|\langle\varphi_0|\varphi_2\rangle + |\varphi_2\rangle\langle\varphi_1|\langle\varphi_1|\varphi_2\rangle + |\varphi_2\rangle\langle\varphi_2|\langle\varphi_2|\varphi_2\rangle
\end{aligned}$$

and the trace of this matrix is equal

$$\begin{aligned}
tr\rho_B^2 &= |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\
&+ \langle\varphi_0|\varphi_1\rangle^2 + \langle\varphi_0|\varphi_2\rangle^2 + \langle\varphi_1|\varphi_0\rangle^2 \\
&+ \langle\varphi_1|\varphi_2\rangle^2 + \langle\varphi_2|\varphi_0\rangle^2 + \langle\varphi_2|\varphi_1\rangle^2 \\
&= |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\
&+ 2|\langle\varphi_0|\varphi_1\rangle|^2 + 2|\langle\varphi_0|\varphi_2\rangle|^2 + 2|\langle\varphi_1|\varphi_2\rangle|^2.
\end{aligned}$$

The squared normalization condition gives

$$\begin{aligned}
1 &= (\langle\varphi_0|\varphi_0\rangle + \langle\varphi_1|\varphi_1\rangle + \langle\varphi_2|\varphi_2\rangle) \\
&= |\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\
&+ 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle + 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_2|\varphi_2\rangle + 2\langle\varphi_1|\varphi_1\rangle\langle\varphi_2|\varphi_2\rangle.
\end{aligned}$$

Then, by taking difference

$$\begin{aligned}
1 - tr\rho_B^2 &= (|\langle\varphi_0|\varphi_0\rangle|^2 + |\langle\varphi_1|\varphi_1\rangle|^2 + |\langle\varphi_2|\varphi_2\rangle|^2 \\
&+ 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_1|\varphi_1\rangle + 2\langle\varphi_0|\varphi_0\rangle\langle\varphi_2|\varphi_2\rangle + 2\langle\varphi_1|\varphi_1\rangle\langle\varphi_2|\varphi_2\rangle) \\
&- |\langle\varphi_0|\varphi_0\rangle|^2 - |\langle\varphi_1|\varphi_1\rangle|^2 - |\langle\varphi_2|\varphi_2\rangle|^2 \\
&- 2|\langle\varphi_0|\varphi_1\rangle|^2 - 2|\langle\varphi_0|\varphi_2\rangle|^2 - 2|\langle\varphi_1|\varphi_2\rangle|^2 \\
&= 2\begin{vmatrix}\langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle\end{vmatrix} + 2\begin{vmatrix}\langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_2\rangle \\ \langle\varphi_2|\varphi_0\rangle & \langle\varphi_2|\varphi_2\rangle\end{vmatrix} + 2\begin{vmatrix}\langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle \\ \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_2\rangle\end{vmatrix}
\end{aligned}$$

where the pair of one qubit states

$$\begin{aligned}
|\varphi_0\rangle &= c_{00}|0\rangle + c_{01}|1\rangle + c_{02}|2\rangle \\
|\varphi_1\rangle &= c_{10}|0\rangle + c_{11}|1\rangle + c_{12}|2\rangle \\
|\varphi_2\rangle &= c_{20}|0\rangle + c_{21}|1\rangle + c_{22}|2\rangle.
\end{aligned}$$

By calculating hermitian inner product we get,

$$\begin{aligned}
\langle\varphi_0|\varphi_0\rangle &= |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 \\
\langle\varphi_1|\varphi_1\rangle &= |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 \\
\langle\varphi_2|\varphi_2\rangle &= |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \\
\langle\varphi_0|\varphi_1\rangle &= \overline{c_{00}}c_{10} + \overline{c_{01}}c_{11} + \overline{c_{02}}c_{12} \\
\langle\varphi_0|\varphi_2\rangle &= \overline{c_{00}}c_{20} + \overline{c_{01}}c_{21} + \overline{c_{02}}c_{22} \\
\langle\varphi_1|\varphi_0\rangle &= \overline{c_{10}}c_{00} + \overline{c_{11}}c_{01} + \overline{c_{12}}c_{02} \\
\langle\varphi_1|\varphi_2\rangle &= \overline{c_{10}}c_{20} + \overline{c_{11}}c_{21} + \overline{c_{12}}c_{22} \\
\langle\varphi_2|\varphi_0\rangle &= \overline{c_{20}}c_{00} + \overline{c_{21}}c_{01} + \overline{c_{22}}c_{02} \\
\langle\varphi_2|\varphi_1\rangle &= \overline{c_{20}}c_{10} + \overline{c_{21}}c_{11} + \overline{c_{22}}c_{12}
\end{aligned}$$

and then,

$$
\begin{aligned}
1 - tr\rho_B^2 &= 2\begin{vmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & \overline{c_{00}}c_{10} + \overline{c_{01}}c_{11} + \overline{c_{02}}c_{12} \\ \overline{c_{10}}c_{00} + \overline{c_{11}}c_{01} + \overline{c_{12}}c_{02} & |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 \end{vmatrix} \\
&+ 2\begin{vmatrix} |c_{00}|^2 + |c_{01}|^2 + |c_{02}|^2 & \overline{c_{00}}c_{20} + \overline{c_{01}}c_{21} + \overline{c_{02}}c_{22} \\ \overline{c_{20}}c_{00} + \overline{c_{21}}c_{01} + \overline{c_{22}}c_{02} & |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \end{vmatrix} \\
&+ 2\begin{vmatrix} |c_{10}|^2 + |c_{11}|^2 + |c_{12}|^2 & \overline{c_{10}}c_{20} + \overline{c_{11}}c_{21} + \overline{c_{12}}c_{22} \\ \overline{c_{20}}c_{10} + \overline{c_{21}}c_{11} + \overline{c_{22}}c_{12} & |c_{20}|^2 + |c_{21}|^2 + |c_{22}|^2 \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
1 - tr\rho_B^2 &= \underbrace{2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix}^2}_{\begin{vmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_1\rangle \\ \langle\varphi_1|\varphi_0\rangle & \langle\varphi_1|\varphi_1\rangle \end{vmatrix}} \\
&+ \underbrace{2\begin{vmatrix} c_{00} & c_{01} \\ c_{20} & c_{21} \end{vmatrix}^2 + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{20} & c_{22} \end{vmatrix}^2 + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{21} & c_{22} \end{vmatrix}^2}_{\begin{vmatrix} \langle\varphi_0|\varphi_0\rangle & \langle\varphi_0|\varphi_2\rangle \\ \langle\varphi_2|\varphi_0\rangle & \langle\varphi_2|\varphi_2\rangle \end{vmatrix}} \\
&+ \underbrace{2\begin{vmatrix} c_{10} & c_{11} \\ c_{20} & c_{21} \end{vmatrix}^2 + 2\begin{vmatrix} c_{10} & c_{12} \\ c_{20} & c_{22} \end{vmatrix}^2 + 2\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^2}_{\begin{vmatrix} \langle\varphi_1|\varphi_1\rangle & \langle\varphi_1|\varphi_2\rangle \\ \langle\varphi_2|\varphi_1\rangle & \langle\varphi_2|\varphi_2\rangle \end{vmatrix}}
\end{aligned}
$$

where the partial concurrences are defined as

$$\begin{aligned}
C_{01} &= 2\begin{vmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{vmatrix} + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{10} & c_{12} \end{vmatrix} + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{11} & c_{12} \end{vmatrix} \\
C_{02} &= 2\begin{vmatrix} c_{00} & c_{01} \\ c_{20} & c_{21} \end{vmatrix} + 2\begin{vmatrix} c_{00} & c_{02} \\ c_{20} & c_{22} \end{vmatrix} + 2\begin{vmatrix} c_{01} & c_{02} \\ c_{21} & c_{22} \end{vmatrix} \\
C_{12} &= 2\begin{vmatrix} c_{10} & c_{11} \\ c_{20} & c_{21} \end{vmatrix} + 2\begin{vmatrix} c_{10} & c_{12} \\ c_{20} & c_{22} \end{vmatrix} + 2\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}.
\end{aligned}$$

As a result we have

$$1 - tr\rho_B^2 = \frac{1}{2}(C_{01}^2 + C_{02}^2 + C_{12}^2) = \frac{1}{2}C^2$$

and giving relation

$$tr\rho_B^2 + \frac{1}{2}C^2 = 1.$$