DEVELOPING ALGEBRAIC THINKING OF FIFTH GRADERS: AN INTERACTION WITH UNITS COORDINATION

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ABSTRACT

DEVELOPING ALGEBRAIC THINKING OF FIFTH GRADERS: AN INTERACTION WITH UNITS COORDINATION

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This study examined how fifth-grade students' progress in algebraic thinking interacted with their levels of units coordination. Four students from two different units coordination levels attended pre-assessment interviews, six teaching episodes, and post-assessment interviews. The findings demonstrated some patterns in students' progressions depending on both inter- and intra-level differences in units coordination. For example, the students with higher level of units coordination learned to write symbolic representations with fewer prompts and more quickly, used different forms of functional thinking interchangeably, and demonstrated structural thinking explicitly. The students with lower levels of units coordination relied on recursive thinking in every new context, and could not generalize the functional relationships in the form of $y = ax + b$ without help. However, all students achieved most of the learning goals despite the differences in their learning path. This is promising for introducing algebraic thinking practices in the early middle school years. The prominent differences in their learning path showed the need to consider the differences in the nature of problems and the level of units coordination.

Keywords: Algebraic Thinking, Units Coordination, Functional Thinking

ÖZ

BEŞİNCİ SINIF ÖĞRENCİLERİNİN CEBİRSEL DÜŞÜNMELERİNİN GELİŞTİRİLMESİ: BİRİM KOORDİNASYON İLE İLİŞKİSİ

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Bu çalışma, beşinci sınıf öğrencilerinin cebirsel düşünmedeki ilerlemelerinin birim koordinasyon düzeyleri ile etkileşimini incelemiştir. İki farklı birim koordinasyon düzeyinden dört öğrenci ön değerlendirme görüşmelerine, altı tane öğretim deneyine ve son değerlendirme görüşmelerine katılmıştır. Bulgular, birim koordinasyonda hem seviyeler arası hem de seviye içi farklılıklara bağlı olarak öğrencilerin ilerlemelerinde de farklılıklar olduğunu göstermiştir. Örneğin, birim koordinasyon düzeyi daha yüksek olan öğrenciler daha az yönergeyle ve daha hızlı biçimde sembolik ifadeler yazmayı öğrenmiş, fonksiyonel düşünmenin farklı biçimlerini kullanmış ve yapısal düşünme göstermiştir. Birim koordinasyon düzeyi daha düşük olan diğer öğrenciler ise yinelemeli düşünmeye eğilim göstermiş ve $y = ax + b$ şeklindeki fonksiyonel ilişkileri yardımsız genelleyememişlerdir. Öğrenme sırasındaki farklılıklarla beraber tüm öğrenciler öğrenme hedeflerinin çoğuna ulaşmıştır. Bu durum, ortaokulun ilk yıllarında cebirsel düşünme uygulamaları için umut vericidir. Bu süreçteki öne çıkan farklılıklar, problem çeşidinin ve birim koordinasyon seviyesindeki farklılıkların dikkate alınması gerektiğini göstermiştir.

Anahtar Kelimeler: Birim Koordinasyon, Cebirsel Düşünme, Fonksiyonel Düşünme

To my parents and my brothers

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LIST OF ABBREVIATIONS

ABBREVIATIONS

MC1: The First Multiplicative Concept

MC2: The Second Multiplicative Concept

MC3: The Third Multiplicative Concept

HLT: Hypothetical Learning Trajectory

CHAPTER 1

INTRODUCTION

The origins of algebra and algebraic thinking can be traced back to antiquity when people worked on word problems and presented solutions using verbal expressions including numbers and stories but not algebraic notations (Ponte & Guimaraes, 2014; Sfard, 1995). For example, the emergence of the function concept in antiquity is seen in the practices of counting, identifying the connections and relations between variables, determining a correspondence between a number of objects, and the notion of dependence between quantities in solving the problems of the social and physical world (Ponte, 1992). Algebra, as a "generalized computational process" has existed in different forms such as rhetorical (i.e., including all verbal expressions) and geometric (e.g., generating particular formulas and expressions through the areas of geometric shapes) until today's symbolic and abstract form (Sfard, 1995, p.18). Therefore, as a system of thinking, a form of reasoning and proof, and a language of generalizations (Usiskin, 1988; Stephens et al., 2021), it has always existed in people's lives for centuries in making decisions in various contexts and solving problems (Usiskin, 1995).

Considering the implicit or explicit presence of algebra and algebraic thinking in people's lives throughout history, algebra deserves great attention as a way of thinking, the language of relationships, the language for solving problems, and the language of generalizations (Stacey & MacGregor, 1997; Usiskin, 1995). The study of algebra provides students with the opportunity to develop their capacity for structured abstract thinking and materials for logical reasoning (Stacey & MacGregor, 1997). Therefore, in the early 1900s, the study of this subject was a significant component of the mathematics curriculum in secondary schools, with the objective of equipping students with the requisite skills to pursue further studies in calculus at the university level (Ponte & Guimaraes, 2014). Subsequently, mathematics educators proposed that it should be integrated with other mathematics subjects, such as arithmetic and geometry, rather than being isolated in the curriculum. However, in the 1980s, there emerged a divergence of opinions regarding the teaching of algebra (Baker, 2013). The discussion was about the challenges students face in learning algebra, the practical applications of the subject, and whether it should be a compulsory course (Baker, 2013; Chazan, 1996; MacGregor, 2004).

Students' difficulties in learning algebra and performance in various algebraic tasks were reported in the studies of mathematics educators such as a lack of understanding of algebraic structures and a deficiency in conceptual knowledge of fundamental algebra concepts (e.g., Bush & Karp, 2013; Ersoy & Erbaş, 2005; Kaput & Blanton, 2001; Kieran, 1992; Stacey & Macgregor, 1997). These difficulties highlighted its way of teaching and how it is presented in mathematics curricula (Chazan, 1996). The roots of difficulties were attributed to teaching of algebraic concepts without sufficient attention to conceptual understanding and to the superficial use of symbolic language without connections between the contexts rather than the nature of the subject (Carraher et al., 2008; Kaput, 2008; MacGregor, 2004; Sfard, 1995). The use of symbolic representations as rote learning without an understanding of the meanings inherent in the algebraic expressions can lead to an incomplete understanding of algebra and difficulties in advanced mathematics in subsequent years (Brizuela & Earnest, 2008; Carraher et al., 2008).

Radford (2000) indicated that the historical development of algebra as a product of a community of practice can have some implications for teaching it to today's mathematics educators. Introducing the main concepts through contextual situations in a problem case, operating with these mathematical entities, and then abstracting them as mathematical objects would be more meaningful in the process of learning algebra (Sfard, 1995). For example, Sfard (1992) indicated that the ancient studies with function were implicit and relied only on operations and computations rather than having an object characteristic. Throughout history, it has developed through many extensions such as having an analytic structure with algebraic expressions and graphic representations. In parallel, Sfard (1995) asserted that reification is an important process in constructing abstract mathematical concepts through algebraic processes, meaning "turning computational operations into permanent objects-like entities" (p. 16).

In response to the aforementioned discussions about the superficial presentation of algebraic concepts and the difficulties of students in learning algebra, researchers adopted an early algebra perspective in teaching algebra (e.g., Blanton et al., 2019; Blanton, Brizuela, et al., 2015; Carraher et al., 2008). For example, Blanton et al. (2017) asserted that the reason younger students or adolescents do not have a robust understanding of the variable concept is not because of their "lack of ability" (p. 199), rather because of the lack of opportunities given in the classroom to mathematize the variables in contextual problems. These opportunities can also allow what Sfard (1995) supported in abstracting mathematical concepts through appropriate and sufficient processes for reification. This approach in teaching algebra refers to an explicit and gradual presentation of main algebraic processes and concepts into the mathematics curriculum starting from the early elementary years (Blanton & Kaput, 2005; Carraher et al., 2008). Therefore, some common questions appeared in the literature such as whether there is a particular/appropriate time for learning algebra, and whether students should master arithmetic before learning algebra (Levin & Walkoe, 2022). These questions and interpretations about students' difficulty in learning and doing algebra raise the issue of how algebra can be introduced in the early years of schools, which is elaborated briefly in the next section.

1.1 Introducing Algebra in the Elementary Years

There are several suggestions for teaching and learning algebra with more understanding, such as beginning in early years, interconnecting algebra with other mathematics subjects, presenting various algebraic thinking forms, and considering the students' current capacities and abilities (Kaput, 1999). Therefore, there is increasing attention on early algebra studies in which algebraic reasoning, and some practices are implemented in early elementary years to see how students at an early age can practice algebra and achieve those processes (e.g., Blanton et al., 2019; Blanton, Brizuela, et al., 2015; Carraher et al., 2008). For example, Blanton et al. (2011) emphasized significant concepts and constructs as five big ideas in early algebra teaching which are generalized arithmetic, variable understanding, understanding of equivalence, quantitative reasoning and functional thinking.

The notion of "early algebra" appeared in many studies examining the teaching and learning of algebraic concepts such as generalization, functional thinking, and variable understanding in elementary years of schooling, even in kindergarten ages (e.g., Blanton & Kaput, 2005; Carraher et al., 2008). Researchers asserted that giving algebraic reasoning in early elementary years invests in the students' sophisticated and conceptual understanding of further mathematics learning (e.g., Blanton & Kaput, 2005; Carraher et al., 2008; Kaput, 1999; Radford, 2014). For example, Carraher et al. (2008) described a learning process where some key mathematical processes are applied such as using indeterminate quantities in contextual problems, interpreting the data in function tables, creating conjectures, generalizing, and representing the relationships in different formats.

Researchers observed that students in elementary years could identify problem quantities, generalize the relationships between variables, and represent these relationships using symbolic notations and equations (e.g., Blanton et al., 2019; Blanton, Stephens, et al., 2015; Brizuela et al., 2015; Carraher et al., 2006). For example, Brizuela et al. (2015) demonstrated that first graders could use algebraic notations to represent the relationships between covarying quantities by possessing different understandings of variables. They indicated that given opportunities through meaningful contexts and employing useful methods such as presenting the information in a function table and asking for generating a rule allowed students to recognize and work with indeterminate quantities, which is a key condition of algebraic thinking (Radford, 2014). Similarly, Blanton, Stephens, et al. (2015) found that third graders who took a one-year early algebra intervention focusing on generalized arithmetic, variable understanding, understanding of equal signs, and functional thinking demonstrated an improvement in algebraic thinking. They started to think relationally about the equal sign, use variable notations to represent unknown quantities, and generalize functional relationships between covarying quantities. In addition, Blanton et al. (2019) found that students in a longitudinal intervention showed improvement in various algebraic practices, such as generalizing arithmetic properties with symbols and expressing functional thinking in various forms, from third to fifth grade. They also noted that students were better at using symbols to represent the arithmetic properties than at writing equations for functional relationships. This note on the difference in the students' performances in various algebraic tasks was later discussed by Zwanch (2022a) on the role of certain cognitive factors, which is also mentioned in the next heading.

These findings of early algebra studies are very important in terms of dealing with students' difficulties in learning algebra, improving their algebraic thinking, and observing the roots of important algebraic subjects of advanced mathematics in young students' thinking processes. Despite the promising findings of early algebra studies, some researchers remarked on the constraints of young students' mental operations with known and unknown quantities in their various mathematical performances such as multiplicative reasoning, generalization, and algebraic reasoning (e.g., Hackenberg, 2013; Olive & Caglayan, 2008). Investigating reasoning with quantities to understand algebraic thinking (e.g., Olive & Caglayan, 2008; Smith & Thompson, 2008) puts further question marks in investigating young students' practices of algebraic thinking and reasoning. For example, there appeared questions such as how the students' understanding of quantities and their relationships affect their writing equations and generalizations of multiplicative relationships (e.g., Hackenberg & Lee, 2015; Olive & Caglayan, 2008; Steffe & Izsak, 2002).

1.2 Students' Mental Structures about Quantities

Analyzing the problems through the quantities and the relationships between the quantities refers to quantitative reasoning (Ellis, 2007; Thompson, 1990). Steffe and Izsak (2002) described algebraic reasoning through reasoning on unknown or known quantities. It involves the generalizations of the relationships between quantities (Blanton & Kaput, 2011). Therefore, researchers who take a quantitative reasoning perspective in investigating the teaching and learning algebra indicated that understanding quantities and relations between them is an essential factor for developing algebraic reasoning (e.g., Ellis, 2011; Fuji & Stephens, 2008; Hackenberg, 2013; Hackenberg et al., 2021; Olive & Caglayan, 2008; Smith & Thompson, 2008). For example, interpreting the relationships between quantities multiplicatively signifies the complexity of this reasoning process. This, in turn, allows students to recognize and generalize the multiplicative relationships in different forms such as verbal or symbolic (Hackenberg & Lee, 2015; Zwanch, 2022a).

Many researchers (e.g., Ellis, 2011; Stephens, Ellis, et al., 2017) pointed out that quantitative reasoning can be enhanced by providing problem situations dealing with various real-life quantities. However, it is remarked that real-life situations do not guarantee the students' recognition and operation with quantities and meaningful generalizations by themselves; the students' conceptions and the way of mental operations with quantities should be taken into account as well (Stephens, Ellis, et al., 2017). Therefore, students' conceptions of numbers and how they operate with quantities by using their current mental structures, as units coordinating activity, gained importance in investigating students' various mathematics performances such as algebraic reasoning, multiplicative reasoning and fraction understanding (e.g., Hackenberg, 2013; Olive & Caglayan, 2008; Steffe, 1992).

The quantitative complexity in students' work with quantities is described through their construction and coordination of units (Ulrich, 2015). For example, calculating the number of muffins in several rows each involving the same number of muffins requires students to iterate units (each muffin) and composite units (the number of muffins in each row) into another composite unit (the number of rows) (Hackenberg, 2010). The level of units students assimilate determines their levels of units coordination (i.e., MC1, MC2, and MC3). For instance, assimilating one level of units refers to students' calculating the total number of muffins through activity by taking one-level of units (one muffin) as given, such as counting by ones and signifying each count of a row of muffins. This is called stage 1 or the first multiplicative concept (MC1) in terms of the levels of units coordination (Hackenberg & Tillema, 2009; Hackenberg & Sevinc, 2024). Assimilating two-levels of units refers to students' recognition of this multiplying structure before activity (MC2). Therefore, they know that they need to find, for example, five (the number of rows) fours (the number of muffins in each row). When students assimilated three levels of units that means they can flexibly operate between the different units such as the number of rows, the number of muffins in each row and the total number of muffins as given structures it refers to the third multiplicative concept (MC3). Steffe (1992) indicated that analyzing multiplicative situations requires students to coordinate at least two composite units by distributing one composite unit over the other.

By integrating the framework of students' levels of units coordination and their performance in various algebraic tasks, researchers observed that students' transformation and coordination of units is an influential factor in algebraic reasoning in terms of writing equations and using letters for unknown quantities and generalizations (e.g., Hackenberg, 2013; Hackenberg & Lee, 2015; Zwanch, 2022a, 2022b). They suggest that students must possess a certain unit coordination level in order to generalize and represent the relationship between quantities using symbols. For example, researchers asserted that students who coordinate two-levels of units in activity would not perceive quantitative unknowns while the students at the upper levels (i.e., MC2 or MC3) can operate with quantitative unknowns such as partitioning and iterating (Hackenberg, 2013; Hackenberg et al., 2021).

Students at different levels of units coordination demonstrated qualitatively different ways of writing equations and generalizations of relationships between unknown quantities (Hackenberg & Lee, 2015; Zwanch, 2022a). For example, in a multiplicative relationship between two lengths, the longer one is five times the shorter one ($y = 5x$), students' assimilation of three-level of units helps them to analyze this situation in a more sophisticated way (Hackenberg & Lee, 2015). Representing this equation in symbols requires constructing the unknown *y* as a unit of five units of *x* as a three-level units structure. This, in turn, could allow students to generate other equations if they can reflect on this three-level of units structure. For example, taking *y* as a composite unit and constructing *x* by dividing *y* by the unit of fives (i.e., $y/5 = x$) represent students' sophistication of units coordination by internalizing three-levels of units.

Considering both promising findings of early algebra studies (e.g., Blanton et al., 2019; Carraher et al., 2006) and remarkable notes about the interaction between units coordination and algebraic reasoning, I aimed to incorporate those findings and investigate this interaction in a different context. In general, this study aimed to investigate fifth-grade students' progress in algebraic thinking with the potential interaction with their units coordination. The next section presents the purpose of the study in more detail.

1.3 Purpose and Research Questions

The point of departure in this study is the crucial role of algebra and algebraic thinking in mathematics and real life. It embedded many important thinking processes such as analytical thinking and structural thinking (Radford, 2014; Kieran, 2022), and useful practices such as generalizations, justifications, and the use of a new language in expressing the generalities (Ellis, 2007; Kaput, 2008; Usiskin, 1995). These are important components in solving problems, evaluating real-life situations and decision-making (Usiskin, 1995). In addition, notable findings from early algebra studies provided a new perspective, such as the gradual settling of algebraic processes, which allows young students to orient themselves to key processes such as functional thinking, generalization, and symbolic notation (Blanton & Kaput, 2005; Carraher et al., 2008).

Despite the promising findings of early algebra studies about improving young students' algebraic thinking and reasoning, researchers focusing on quantitative reasoning perspective and studying units coordination raised new questions on the way to the goal of this study. They asserted that certain cognitive factors of students such as constructing and operating with units are influential in students' algebraic thinking and reasoning (e.g., Hackenberg, 2013; Zwanch, 2022a). For example, Hackenberg (2013) indicated that students need to possess certain mental structures in terms of units coordination to achieve algebraic tasks such as writing equations. Accordingly, it may be difficult for elementary and early middle school students (i.e., fifth grade) to achieve algebraic reasoning, given the estimations and findings about students' levels of units coordination (e.g., Acar & Sevinc, 2021; Clark & Kamii, 1996; Kosko, 2019; Steffe, 2024). Researchers estimated that more than 20 percent of fifth graders could not demonstrate valid multiplicative thinking (Clark & Kamii, 1996; Kosko, 2019) such as operating at the MC1 level or pre-multiplicative stage (Acar & Sevinç, 2021; Steffe, 2024). This could be a significant constraint for administering an early algebra learning approach to young students.

Departing from this contradiction between early algebra studies and units coordination perspective, the goal of this study emerged. It was aimed to investigate students' progress in algebraic thinking in interaction with their units coordination. To achieve this goal, there was a need to design a detailed learning process including the specified learning goals, related learning tasks, and conjectures about students' learning process, which refers to a Hypothetical Learning Trajectory (Simon, 1995). In this process, I aimed to study with fifth graders by hypothesizing that I would encounter students at different levels of multiplicative concepts, starting from MC1 (Acar & Sevinç, 2021; Steffe, 2024). The fifth-grade level, as opposed to lower grades, can provide students who have attained at least an MC1 level, which represents the initial stage of multiplicative concepts in terms of units coordination.

At this stage, students can complete multiplicative tasks (e.g., Hackenberg, 2013; Hackenberg et al., 2021), which can be encountered in algebraic tasks. In addition, the fifth graders do not receive formal algebra, which could mitigate the adverse consequences of misaligned or erroneous preconceptions during the learning process.

For generating the domain-specific perspective in the hypothetical learning trajectory (HLT) and designing the learning tasks, I aimed to follow the researchers in the early algebra studies (e.g., Blanton, 2008; Blanton & Kaput, 2008; Carraher et al., 2008). They supported allowing students to study in meaningful contexts, to think about the relationships between quantities, and to make generalizations for helping them understand the key ideas and to move smoothly with a new symbolic language to represent and express the relationships (e.g., Blanton, Stephens, et al., 2015; Carraher et al., 2008; Stephens, Fonger, et al., 2017). These generated the overall characteristics of the framework of the HLT.

Conclusively, the main goal of this study was to investigate fifth-grade students' progress in algebraic thinking in interaction with their units coordination levels. I aimed to observe this progress and interaction during a learning sequence specified after actualizing the HLT that targeted the students' generalizations and symbolic representations of the relationships between unknown quantities or variables. Specifically, this study aimed to answer the following research questions:

1. What is the initial state of fifth-grade students' units coordination and algebraic thinking?
2. How can the units coordination levels of fifth-grade students interact with their progress in algebraic thinking during a learning sequence that focuses on the generalization of the relationships between unknown quantities and between variables?

2.1. How can the units coordination levels of fifth-grade students interact with their progress in algebraic thinking regarding *the relationships between unknown quantities*?

2.2. How can the units coordination levels of fifth-grade students interact with their progress in algebraic thinking regarding *the functional relationships between variables*?

The process from the initial motivation of this study to the formulation of the research questions described until this part is presented in Figure 1.1.

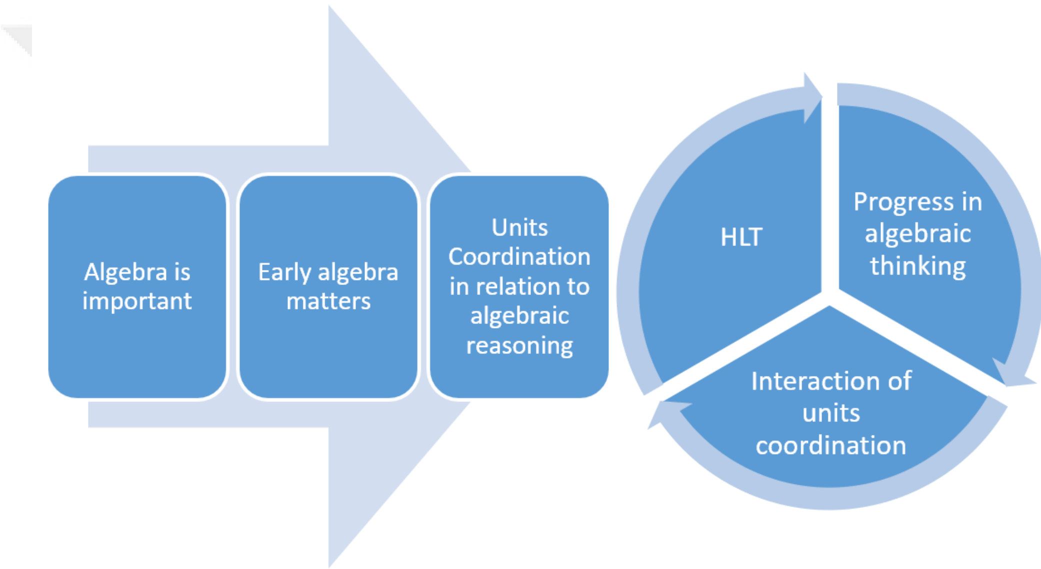


Figure 1.1 The overall process in formulation of the research questions

1.4 The Significance of the Study

There are many studies investigating the interaction between units coordination and performance on algebraic tasks in various aspects (e.g., Hackenberg, 2013; Hackenberg & Lee, 2015; Zwanch, 2022a, 2022b). This study differed from them in several aspects such as the characteristics of students, the design of the study, and the assessment procedure in terms of task characteristics.

First of all, this study involved fifth graders as participants whereas other studies had students at upper levels, ranging from 6^{th} to 10^{th} grade (e.g., Hackenberg et al., 2021; Hackenberg & Lee, 2015; Zwanch, 2022a, 2022b). Including younger students in the investigation of the interaction between algebraic thinking and units coordination was a challenge considering that a considerable amount of fifth graders may not be good at multiplicative thinking (Clark & Kamii, 1996; Kosko, 2019; Steffe, 2024) which is a significant element in both units coordination and algebraic reasoning tasks. However, the inclusion of students who had not yet demonstrated significant proficiency in these subjects was also intended. This was because I aimed to observe their progress in algebraic thinking through the HLT. This study also aimed to include MC1 students who were regarded as incapable of achieving some algebraic processes, such as perceiving quantitative unknowns (Hackenberg et al., 2021) and symbolic generalizations (Zwanch, 2022a). Given that other studies worked with students at upper levels of units coordination such as MC2 and MC3 (e.g., Hackenberg et al., 2017; Hackenberg et al., 2021), except Zwanch (2022a) who included MC1 students as well but at sixth and upper-grade levels, I conjectured that including MC1 students in this study would provide significant findings regarding how further these students could go in algebraic thinking. Conclusively, working with fifth graders and also MC1 students represents a departure from previous studies and offers a unique advantage in conducting this design study aligned with the goal of the study.

Another distinguishing feature of this study is its methodology involving the conceptual framework, the design, and data collection. Firstly, this study aimed to examine students' progress in algebraic thinking throughout the HLT by interpreting it with their uınits coordination levels. The main goal of the HLT was to develop students' algebraic thinking in terms of generalizing the relationships between unknown quantities and generalizing functional relationships between variables. Therefore, this study mainly used a design-based approach to develop an HLT and adopted an early algebra perspective in teaching and designing the tasks. On the other hand, the majority of studies investigating students' diverse algebraic performances in relation to their units coordination levels employed a clinical interview approach (e.g., Hackenberg, 2013; Hackenberg & Lee, 2015; Zwanch, 2022a). Some researchers who used a design experiment approach (e.g., Hackenberg et al., 2017; Hackenberg et al., 2021) mainly focused on the students' fractional knowledge and expression of relationships between unknown lengths measured by different non-standard units. Furthermore, the aforementioned studies presented problems including fractional units and coefficients (e.g., Hackenberg et al., 2017; Hackenberg & Lee, 2015; Olive & Cağlayan, 2008). In contrast, all problems in this study included whole number quantities by focusing on their algebraic reasoning involving functional thinking, analytical thinking, and generalization.

To summarize, this study differs from other studies in terms of methodology, although there is some overlap in theoretical frameworks. Therefore, it gains importance through its design, contexts, and instructional tasks which are diverged from other studies by bringing together different algebraic reasoning processes such as the generalizing relationships between unknown quantities and generalizing functional relationships. In this way, the findings of this study would contribute to the existing body of knowledge on the teaching of algebra in the early middle school years and on the coordination of units to plan further instructions and mathematics curricula that integrate algebra more effectively in elementary or early middle school years.

1.5 Definitions of Important Terms

This section defined important terms that were used frequently throughout the dissertation text. The definitions or explanations of the terms were based on how I understood each concept grounded on the descriptions of various researchers or mathematics educators and the meaning I used throughout the text. Although there are many important mathematical concepts embedded in this study, that need to be clearly described, this section touches on only several of them that are important to understand the research objective from the outset. Other important concepts and terms were explained in Chapter 2 based on the conceptual and theoretical perspectives.

Algebraic thinking is one of the key concepts of this study. It refers to a thinking process involving working with indeterminate quantities in an analytic way which can be reflected in different ways such as verbal, symbols, or figures (Radford, 2014). For example, considering that ‘changing the order of any two numbers does not change their sum’ refers to algebraic thinking by taking indeterminate quantities (i.e., any two numbers), operating with them as abstract objects, and deducing a property for the addition operation in a structured way. This verbal statement of thought can be denoted in symbols like $a + b = b + a$.

Algebraic reasoning is described as “an activity of generalizing mathematical ideas” (Blanton & Kaput, 2011, p. 6) and it involves the processes of generalization, symbolical representation of these generalizations, and operating in this system of symbols (Kaput, 2008). In early algebra literature, this term is used interchangeably with algebraic thinking (Kieran, 2011). However, Kieran (2011) defended the use of a broader term, algebraic thinking, against the risk of a narrow interpretation of algebraic reasoning from a classical mathematical reasoning perspective. Therefore, I use this term to refer to any generalization action in problem situations.

Functional relationship refers to a covarying relationship between quantities in which the change in one quantity is expressed in terms of the change in the other quantity. For example, in a basic early algebra context, expressing the number of legs of dogs as ‘it increases by four when the number of dogs increases by one’ or ‘it is four times the number of dogs’ refers to expressing a functional relationship. Moreover, this way of thinking including recognition of functional relationships refers to *functional thinking* (Confrey & Smith, 1994; Smith, 2008) which is one dimension of algebraic thinking (Kieran, 2022).

Units coordination refers to the mental operations that construct the units in different levels and the relationships between units and coordinate them in different problem situations (Hackenberg & Sevinç, 2024; Steffe, 2001). Units coordination demonstrates the "quantitative complexity" of constructing and coordinating units and composite units (Ulrich, 2015, p. 3). For example, the number *five* as a unit consisting of five iterable units of ones refers to constructing composite units and coordinating two-levels of units. This composite unit is abstracted from a counting activity and separated from a sequence. Therefore, two different sequences such as 1-2-3-4-5 and 16-17-18-19-20 represent the same composite unit, five.

A Hypothetical Learning Trajectory (HLT) refers to teachers' or researchers' prediction of a learning path in a particular context including the learning goals, hypothesis about the students' learning and learning tasks (Simon, 1995). It is hypothetical because a teacher cannot know exactly what might happen during the learning process. He/she can only hypothesize what might happen based on his/her theoretical understanding, current knowledge of the students, findings from related literature, and previous experiences.

Learning sequence refers to which path students got from one point to the current one in a designed process (Bakker, 2018). I used this terminology in several places in this study to indicate the process or steps the students take during and after the HLT is actualized. After implementing the learning tasks in the HLT, the outcome was an actual learning process rather than a hypothetical one. Therefore, I used the term learning sequence in some parts of the study to emphasize that it is an actual path anymore.

Progress in algebraic thinking with interaction with units coordination refers to the development of students' way of algebraic thinking, which is potentially influenced by their levels of units coordination. It concerns the interpretation of students' development in algebraic thinking by relating the patterns in their progress to their units coordination.

CHAPTER 2

LITERATURE REVIEW

In this study, the purpose was to investigate the interaction between the students' units coordination levels and their progress in algebraic thinking through a hypothetical learning trajectory that centered on different algebraic reasoning tasks including generalization, functional thinking, and symbolic representation. As the investigation of this study involves many mathematical concepts and processes, this chapter presents the theoretical and conceptual framework that guided our understanding and the investigation.

Algebraic thinking and unit coordination are the two main concepts in this study. Within this framework, this chapter presents the nature of algebra and unit coordination in two different sections, respectively. Each section provides a detailed explanation of the relevant concept, accompanied by clear definitions and descriptions of the related terms. It also presents an overview of the related literature, which serves to demonstrate relevant findings and to guide this research in terms of methodology and interpretations of findings.

2.1 The Nature of Algebraic Thinking

The origins of algebra and algebraic thinking go back to ancient times when a minority of people gradually developed a symbolic system as a problem-solving strategy, which evolved into the use of symbols from verbally expressing the relationships in problems (Sfard, 1995). For example, in ancient times, the idea of function as a key concept in algebra appeared in verbal expressions of dependence without using symbols (Youschkevitch, 1976). By the end of the nineteenth century, the number of texts on algebra increased (Katz, 1997). These texts involved various definitions of algebra priorly emphasizing equation solving, general procedure of operations, unknown and known quantities, and notations. As algebra developed throughout history as a branch of mathematics, and as the number of people working on it increased, it gained a multifaceted nature (Katz, 1997).

Researchers interpret algebra as an important tool for various mathematical activities such as expressing generality and relationships, constructing equivalent expressions, and problem-solving (e.g., Stacey & MacGregor, 1997; MacGregor, 2004; Usiskin, 1995). Most of these activities require certain ways of thinking and reasoning such as analytical thinking, abstraction, deduction, and structural thinking (Kieran, 1989; Radford, 2014; Usiskin, 1995). Based on various ways of thinking and reasoning that are embedded in algebra, the following paragraphs identify and clarify the dimensions and conditions of algebraic thinking and the key processes and aspects of algebraic reasoning.

Researchers make a characterization of algebraic thinking to differentiate it from arithmetic thinking and to specify it in various mathematical processes (e.g., Kieran, 1989; Radford, 2014). Radford (2014) characterizes algebraic thinking through its analytic nature in which the operations are made with indeterminate quantities as if they were known quantities and rules or formulas are deduced as a consequence of this thinking/operation process. Hence, Radford (2014) put forward three conditions to specify algebraic thinking: indeterminacy, denotation, and analyticity.

The condition of indeterminacy refers to dealing with quantities that are not known and can take various numerical values (i.e., indeterminate) in a problem situation. These indeterminate quantities can be denoted in various ways such as symbols, natural language, and gestures, which refer to the denotation condition. Finally, the analyticity condition refers to the operations on these indeterminate quantities using a symbolic system. The condition of analyticity is important because it represents that algebraic thinking is an effective tool for deductive reasoning, which results in an identity or formula after operating with unknown quantities (MacGregor, 2004).

This deductive way of reasoning refers to establishing mathematical formulas or truths through the use of current mathematical rules and truths (Usiskin, 1995).

Furthermore, Kieran (2022) indicated three main dimensions of algebraic thinking: analytic thinking, structural thinking, and functional thinking. Similar to Radford (2014), Kieran (2022) remarked that the analytic manner of algebraic thinking is essential to differentiate it from arithmetic thinking. This form of thinking is described as holding unknowns and operating with them like they were known quantities. This is exemplified through some practices such as equation solving, transformation of equalities, and generalization of arithmetic properties. Radford (2014) asserted that analyticity is the way of algebraic thinking in which the formulas are formed through analytical deduction rather than guessing or trying, hence it is why François Viète, the French mathematician in the sixteenth century, called algebra an analytic art.

The dimension of structural thinking points out the ability to be aware of the relations and structures in mathematical notions and express and elaborate on these structures (Kieran, 2022). Structural thinking is considered an essential characterization of algebraic thinking by different researchers (e.g., Blanton & Kaput, 2005; Blanton, Stephens et al., 2015). It requires treating mathematical notions or expressions as abstract objects rather than operational processes (Sfard, 1991). For example, defining rational numbers as "pairs of integers (a member of a specially defined set of pairs)" rather than a result of "division of integers" represents a structural way of definition rather than an operational way (Sfard, 1991, p.5). Similarly, explaining the equality $12 + 15 = 15 + 12$ by remarking that reversing the numbers does not change the addition (i.e., reflecting the commutative property) represents a kind of structural thinking (Blanton, Stephens, et al., 2015). In an equation-solving task, considering the algebraic expression "$x + 5$" a mathematical object in an equation "$3(x + 5) = 36$" (p. 56) reflects an awareness of the structure and allows the students to conduct more meaningful ways for solving the equation rather than applying the procedures learned by rote such as multiplying 3 with $(x + 5)$ (p.56).

The last dimension of algebraic thinking is functional thinking (Kieran, 2022). Functional thinking can be explained through the concept of function and functional relationship.

The idea of function first appeared in verbal expressions and definitions of dependence rather than in symbolic expressions in ancient times (Youschkevitch, 1976). Different forms of functions existed in ancient times, such as "implying a correspondence between a set of given objects and a sequence of counting numbers," "four elementary arithmetical operations, which are functions of two variables," and "tables of reciprocals" (Ponte, 1992, p. 3). Freudenthal (1983) described the emergence of the function concept through the connections and relations between variables and the notion of dependence between the quantities in the social and physical world. Thereby, functions are described as a "world of relationships, world of processes and world of rules, patterns and laws" (Sierpinska, 1992, p.31).

Functional thinking is the recognition of the relationships among covarying quantities and the representation of this functional relationship in various forms including words, graphs, and symbols (Smith, 2008). During a functional thinking practice, there are multiple steps to carry out such as identifying covarying quantities, recording the values of the quantities reciprocally (i.e., making a table), determining patterns in the record, and representing this functional relationship by coordinating the patterns. Therefore, functional thinking refers to the thinking processes in interpreting functional relationships between covarying quantities in mathematical problems or real-life situations.

In defining a functional relationship, two common approaches are described as covariational and correspondence (Confrey & Smith, 1994). In the covariational approach, the functional relationship is defined through the rate of change by interpreting a corresponding increase in one variable with the increase/decrease in the other variable. In other words, a functional relationship is described by focusing on how each quantity varies from one step to the following (i.e., from x_1 to x_2 and from y_1 to y_2) and determining the functional relationship by coordinating this transition. In a correspondence approach, the relation is defined between "corresponding pairs of variables" (Smith, 2008, p.147) such as "the number of eyes equals 2 times the number of people" (i.e., $y = 2x$). Hence, in the correspondence approach, an invented rule or formula determines the value of one quantity (e.g., y) with respect to the other quantity.

In short, the practices of algebra incorporated important thinking processes. An important process starts with identifying and operating with indeterminate quantities. Additionally, there are abstracting the structures constructed through these operations and denoting or expressing the outcomes of the thinking process in various forms. In addition to characterizing algebraic thinking through its dimensions, understanding the nature of algebra requires describing the main aspects of algebraic reasoning processes. This is the topic of the next section.

2.1.1 Algebraic reasoning

Kaput (2008) remarked on the challenge of describing algebra because of its multifaceted nature evolved throughout history and different cultures. The integration of different activities of algebra such as the generalization of relationships and the use of indeterminate quantities provides a broader and general description of algebraic reasoning (e.g., Carraher & Schliemann, 2014; Kaput, 2008). In many research, algebraic reasoning is used interchangeably with algebraic thinking. However, Kieran (2011) remarked that algebraic reasoning does not involve as many thinking processes as algebraic thinking embodies.

Kaput (2008) differentiates algebra and algebraic reasoning by describing algebra as a body of knowledge by its structure in cultural contexts while defining algebraic reasoning as a "human activity" (p. 9), depending on thought processes. Kaput's (2008) description of algebraic reasoning includes two core aspects: one is "systematically symbolizing generalizations of regularities" and the other is "syntactically guided reasoning and actions on generalization expressed in conventional symbol systems” (p.11). He also describes several content strands that include, to a considerable extent, these main practices in algebra such as generalized arithmetic incorporating the structures of operations and the relations with quantitative reasoning, functions with the relationships between covarying quantities, and modeling applications of mathematics (see Figure 2.1).

The Two Core Aspects

(A) Algebra as systematically symbolizing generalizations of regularities and constraints.
(B) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems.

Core Aspects A & B Are Embodied in Three Strands

1. Algebra as the study of structures and systems astracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and in quantitative reasoning.
2. Algebra as the study of functions, relations, and joint variation.
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics.

Figure 2.1. Kaput’s (2008) framework describing algebraic reasoning (p.11)

Kaput (2008) puts two notions, generalization and symbolic representation, at the center of defining algebraic reasoning. For example, through generalizations, the students can identify and represent the relationships between the numbers and operations such as the commutative property of addition (i.e., $a + b = b + a$). This allows the students to see the mathematical structures and to make abstractions (Blanton et al., 2011; Dienes, 1963). Furthermore, the study of dependence between covarying quantities requires a generalization of the relationships and representing them in a symbolic system (Kaput, 2008). Similarly, the relations and structures recognized and generalized from different problem situations can be represented in this symbolic system through mathematical modeling languages, which is another aspect of algebraic reasoning.

As seen, generalization is a central activity in algebra comprising generalized arithmetic, functional thinking, and modeling (e.g., Mason, 1996; Lee, 1996).

Therefore, the following section describes the meaning of generalization in both mathematics and algebra to complement the nature of algebra in the conceptual domain.

2.1.2 Generalization

Generalization is emphasized as a central component of mathematics (Mason, 1996) while it has a kind of scientific characteristic that is not specific to mathematics (Radford, 1996). Dienes (1963) defined generalization as a “class extension” (p. 120) where the class is formed by bringing similar events or elements through an abstraction process. The generalization process is summarized in a dual nature: seeing the general in particular cases and applying this to others and finding the general, that is not known, from particular cases (Krutetskii, 1976; Mason, 1996).

Dumitrascu (2017) mentioned three inevitable mental processes in generalization performance: analysis, synthesis, and abstraction (as cited from Rubinshtein, 1994). In addition, two types of generalization are identified as empirical and theoretical generalization in terms of whether analysis and abstraction are used in the process. In empirical generalization, the main activity is to compare the cases according to their external similarities and differences, while in theoretical generalization the relations are internalized through analysis and abstraction (Davydov, 1990; Dörfler, 1991). Thus, abstraction is an essential process in generalization. Radford (1996) points out that generalization has "a logical aspect" (p. 108) that depends on the problem solver's way of thinking about the relations and objects of the problem. Therefore, it needs "an additional (logical) element in the classroom" (p. 109): validity, in other words, justification. Therefore, the students’ justification of the generalizations such as generic formulas or general statements would provide evidence for the abstraction and analysis processes, which represent a theoretical generalization (Lannin, 2005; Radford, 1996)

Radford (2010) put forward “a typology of forms of algebraic thinking” (p. 1) based on the students’ generalizations. In the first and basic form, factual algebraic thinking, (although they are not hierarchical) the students perceive the regularities in the patterns partially such as focusing on the recursive relationship between figures. Hence, they may not practically reach the bigger items in the figural patterns. We can observe the students' in-action-formulas through their gestures and verbal expressions implicitly such as finger movements and expressions like "every time there will be one more in the air" (p.5). The second form is contextual algebraic thinking in which the students can describe the generic formula verbally in a particular context. In this form, indeterminacy becomes explicit as different than factual algebraic thinking because the students form a general figure rather than focusing on specific cases. Lastly, standard algebraic thinking requires the generated formula to represent the students' experiences with the relationship between quantities. Hence, the formula can be represented in both symbols and narrative statements. If the symbolic representations are formed through an analytic and deductive way rather than trial-error calculations it can be regarded as standard algebraic thinking. Therefore, Radford (2010) remarked that the formula or symbolic representations of generalizations do not guarantee the analytic way in students' algebraic thinking.

In conclusion, the nature of algebra includes many thinking and reasoning processes such as analysis, generalization, representation, and justification. Our conceptual understanding of algebra involves the dimensions of algebraic thinking such as analytical and structural thinking that Radford (2014) defined, and the key processes and aspects of algebraic reasoning described by Kaput (2008) such as generalization and symbolic representations. Concerning this, we considered the main processes in developing the students' algebraic reasoning would be the generalization of relationships or patterns and symbolic representations of these generalized relationships, which requires various algebraic thinking processes such as analytical thinking, structural thinking, and functional thinking. Figure 2.2 illustrates the conceptual understanding of algebra and the key processes in terms of thinking and reasoning surrounding this understanding in the current study.

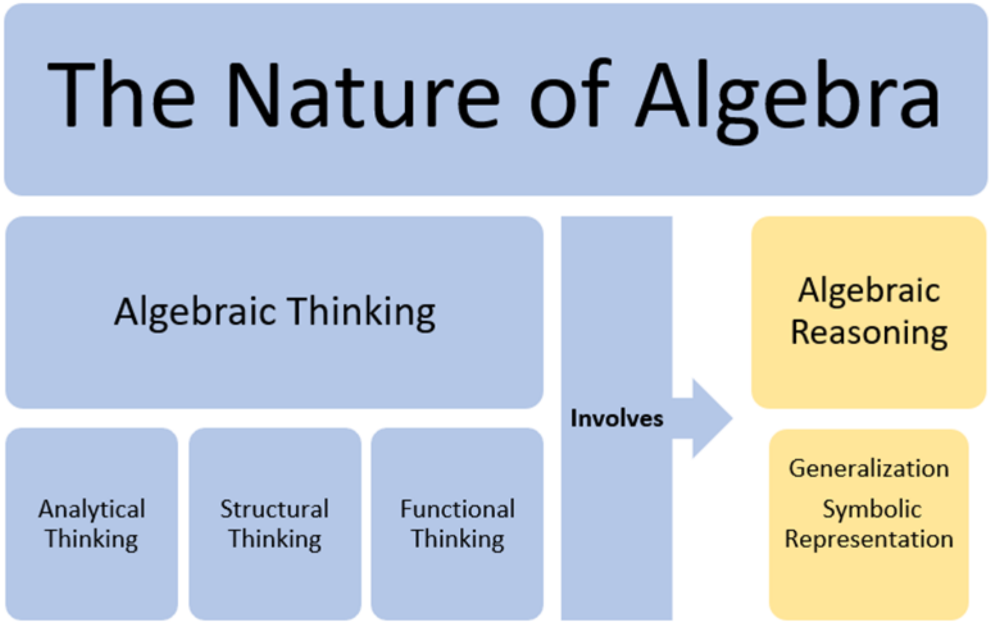


Figure 2.2 The conceptual understanding of the nature of algebra

Kaput (2008) needed to bring forward his description of algebra and algebraic reasoning to clarify the scope of early algebra which also means incorporating some mathematical processes that allow the students to be ready for advanced algebraic subjects (Carraher et al., 2008). In teaching and learning algebra, it is important to clarify important algebraic thinking and reasoning processes to determine the scope of school algebra. This would also be useful for designing the learning sequence to develop the students' algebraic reasoning.

2.1.3 Algebra in mathematics education

At the beginning of the twentieth century, the teaching and learning of algebra as a subject, including first and second-degree equations, proportions, algebraic expressions, trigonometry, etc., gained importance in secondary schools (Ponte & Guimaraes, 2014). In general, the community perceived the study of algebra as a gateway to college because it was compulsory in high schools (Kaput, 1999; Ponte & Guimaraes, 2014). In the 1980s, the slogan "Algebra for All" appeared in the

United States, emphasizing that all students should learn algebra (Chazan, 1996). On the other hand, many mathematics educators and curriculum makers have started to discuss the difficulties the students have in learning algebra and the practical applications of the subject as the level of necessity (Baker, 2013; Chazan, 1996; MacGregor, 2004).

Assessments of students' algebra performances and their understanding of algebraic structures showed that they lack conceptual knowledge of fundamental algebra concepts (e.g., Bush & Karp, 2013; Ersoy & Erbaş, 2005; Kaput & Blanton, 2001; Kieran, 1992). Researchers asserted that for many years, superficial teaching of algebra without sense-making and without going beyond the limited symbolic manipulation causes students to have difficulty studying algebra in further years and have a failure in mathematics (e.g., Blanton et al., 2017; Carraher et al., 2008; Kaput, 1999, 2008; MacGregor, 2004).

Concerning the discussions about the difficulties students have in learning algebra, Chazan (1996) defended focusing on the teaching of algebra and its place in the curriculum rather than discussing the subject's difficulty in improving the students' learning. Carraher and Schliemann (2007) asserted that many topics taught earlier are essential to prepare the grounds for later mathematics subjects. Presenting algebra like dropping out of the sky may make the students learning difficult because of a lack of connections and ground (Kieran, 1992). Hence, the notion of early algebra appeared as an essential body of research in algebra learning approaches.

Carraher et al. (2008) warned that early algebra should not be understood only as algebra early. It does not mean presenting the symbolic language of algebra in the early years. Emphasizing and directly manipulating algebra's notation system in the earlier years may cause the students to have a superficial understanding of the subject matter (Usiskin, 1988). Early algebra is distinguished from algebra presented in high schools and colleges through its gradual settlement in elementary mathematics topics (Carraher et al., 2008). Hence, when searching today's research about school algebra, many views emphasize the effective incorporation of algebra subject through the algebraic reasoning dimension in school mathematics for students at early ages (e.g., Blanton et al., 2019; Blanton, Brizuela, et al., 2015; Carraher et al., 2008). Furthermore, Blanton et al. (2011) described five big ideas that summarize all fundamental concepts and processes in teaching and learning algebra from starting the elementary years: generalized arithmetic, understanding equal sign as representing the equivalence of two quantities, variable understanding, quantitative reasoning for generalizations, and functional thinking as a gateway to algebra.

2.1.3.1 Generalized arithmetic

Researchers (e.g., Carpenter et al., 2005; Carraher & Schliemann, 2007) defend that arithmetic and algebra should not stay as distinct domains in mathematics education. Some procedures and concepts in arithmetic can be implicitly or explicitly connected to algebraic ideas; hence, educators should bridge those ideas in teaching (Carraher & Schliemann, 2007). There is a common view about the inclusion of algebra in elementary mathematics through generalized arithmetic using relations among sets of numbers, symbols, and properties (Blanton et al., 2011; Carpenter et al., 2003; Carpenter et al., 2005; Carraher et al., 2006; Chimoni et al., 2018; Kieran, 1992; Knuth et al., 2008).

Researchers identify generalized arithmetic as one of the representatives of algebraic thinking considering the idea of generalization from the relations and properties in arithmetic operations (Blanton & Kaput, 2005; Carpenter et al., 2005; Usiskin, 1988). Carpenter et al. (2005) indicated that teaching arithmetic through the relationships between operations and quantities, instead of focusing on getting a solution from an operation, would be an effective way to improve students' relational thinking and thus develop a foundation for algebra. For example, presenting addition and subtraction as inverse operations, and constructing and expressing the arithmetic properties (e.g., commutative property) through generalizations are regarded as key ideas for integrating algebraic reasoning in arithmetic (Blanton et al., 2011). Integration of arithmetic properties in algebraic reasoning activities develops elementary students’ representations of generalizations of these relationships (Strachota et al., 2018).

The emphasis on the generalization process between arithmetic and algebra also reflects Davydov's perspective (Schmittau, 2011), in which learning takes place from the abstract to the concrete: first, the general relationships are given; then concrete/arithmetic examples are solved (Davydov, 1990). In this approach, the students are taught to generalize relationships and operations, for example, through comparison activities (i.e., $A > B$) and conjecturing about part-whole relationships by using algebraic symbols rather than discrete numbers (Sutherland, 2004). In such a study, it was seen that the children at the end of the third grade could solve problems about proportional reasoning and rate, time, and distance (Schmittau, 2011).

2.1.3.2 Understanding of equal signs

One of five big ideas emphasized by Blanton et al. (2011) is the understanding of equal signs that represent the equivalence of two quantities written in an equation. Using notations or demonstrating generalizations in different ways is a key aspect of algebra and algebraic reasoning (Kaput, 2008). The symbolic system in algebra enables students to interact within a mathematical system (Strachota et al., 2018). One of the important notations in this system is the equal sign. Researchers emphasized the importance of understanding equality and equal sign as a building block for algebra (e.g., Carpenter et al., 2005; Chimoni et al., 2018; Kieran, 1992; Knuth et al., 2008).

An equal sign represents the sameness and a relationship of equivalence in mathematical expressions (Baroody & Ginsburg, 1983; Behr et al., 1980). However, the students can interpret the function and meaning of equal signs in different ways such as an operator symbol, sameness of two sides, “answer is coming” symbol, and equivalence (Behr et al., 1980; Baroody & Ginsburg, 1983; Knuth et al., 2006).

Therefore, researchers indicated two main understandings of equal sign which are relational and operational.

Relational understanding refers to the accurate understanding of equal signs as a comparison symbol that includes sameness and equivalence meanings (Behr et al., 1980; Baroody & Ginsburg, 1983; Matthews et al., 2012). It allows the students to understand the equivalent expressions in different forms relationally and accurately. For example, they can find some unfamiliar equations such as $3 = 3$ and $3 + 4 = 5 + 2$ meaningful by focusing on the equivalence of both sides. On the other hand, operational understanding refers to interpreting an equal sign as an operator or "do something signal" (Behr et al., 1980, p.15). In this situation, unfamiliar or nonstandard forms of equations such as operations on the right side (e.g., $7 = 3 + 4$) and operations on both sides (e.g., $3 + 5 = 10 - 2$) can not make sense for the students who have this understanding (Matthews et al., 2012).

In addition to differentiating the students' understanding of equal signs as relational and operational, researchers also specified some extra levels of understanding included in relational or operational levels of understanding (e.g., Matthews et al., 2012; Rittle-Johnson et al., 2011). They generated a construct map expressing the understanding of equal signs into four levels: "Rigid Operational, Flexible Operational, Basic Relational, and Comparative Relational" (Matthews et al., 2012, p. 320). The indicators and explanations of each level of understanding are represented in Table 2.1.

In this construct map, the highest level of understanding entails a sophisticated comparison and transformation between the sides of the equal sign such as a compensatory strategy and applying the same operation on both sides (Matthews et al., 2012). This way of reasoning represents the students' recognition and interpretation of the relations between numbers and equivalent expressions.

Table 2.1 The Construct Map: The Levels of Understanding of the Equal Sign

The Level of Understanding	Descriptions – Perceptions of equation structures
Rigid Operational	Operations-equals-answer structure ($a + b = c$) Operational definition of the equal sign
Flexible Operational	Non-standard equation structures ($c = a + b$; $a = a$) Operational definition of the equal sign
Basic Relational	Operations on both sides in equation structures ($a + b = c + d$) The relational definition of the equal sign makes sense
Comparative Relational	Equation solving involving a compensatory strategy (i.e., comparing both sides of the equation and applying transformations) e.g. $3 + 7 = 4 + 6$; 4 is one more than 3; 6 is one less than 7 to maintain the equivalence. Construct a relational definition of equal sign

Note: Matthews et al.'s (2012) construct map was summarized

At the basic relational level, the operational view of the equal sign may reappear occasionally. However, the students could solve equations involving operations on both sides, as different than the operational views of the equal sign. Two levels of operational views of the equal sign (i.e., rigid and basic) differ from each other in terms of the kinds of equation structures that the students can solve and interpret. According to Matthews et al. (2012) standard forms of equation structures, the operations on the left side and the solution on the right side, are the only forms that the students at the rigid operational level of understanding can interpret and work accurately. On the other hand, the students at the basic relational level can understand the nonstandard (unfamiliar) equation structures such as operations on the right side (i.e., $c = a + b$) or equations without operation on either side (i.e., $a = a$).

Students in different grade levels can still show an operational understanding of the equal sign (Baroody & Ginsburg, 1983; Behr et al., 1980; Blanton et al., 2018; Carpenter et al., 2003; McNeil & Alibali, 2005; Rittle-Johnson & Alibali, 1999).

Fyfe et al. (2018) reported that this construct map applies to middle school students, and it provides a link between the students' understanding of equivalence and their algebraic reasoning. The students who perceive the meaning of equal sign only as "the answer is coming" rather than "the equivalence of two sides" cannot perform well in equation solving (Alibali et al., 2007; Carpenter et al., 2003; Knuth et al., 2006). Therefore, the relational understanding of the equal sign and the meaning of the equivalence of two quantities gains importance in developing the students' algebraic reasoning and performance in algebraic tasks (Blanton et al., 2011).

2.1.3.3 Understanding the multifaceted nature of variables

Sfard (1995) indicated that French mathematician Viete's description of algebra as "science of species" or "types of things" (p.24) may be a point of departure for the emergence of the variable concept. Furthermore, the concept of variable has existed implicitly in ancient mathematicians' studies about the quantities although it is not named a "variable". Ely and Adams (2012) indicated that two important motives are effective in the development of the variable concept: one is using letters standing for a range of indeterminate values in addition to determinate unknown values; the second is practicing covariational reasoning in which one quantity changes as dependent to another quantity.

Alphanumeric symbols or notations are used for several purposes such as describing some laws by formulas, making generalizations, and substituting some solutions in mathematical problems (Schoenfeld & Arcavi, 1988). Usiskin (1988) focused on the meaning of the variable and its multifaceted nature to make sense of the symbols and letters used in algebraic equations and formulas. She interpreted different aspects of the variable concept concerning how it functions in various situations. For example, a variable can appear as an unknown in an equation or show a formula of area (e.g., $A = L.W$); it can represent an operational rule such as multiplicative identity (i.e., $1 = n.(1/n)$) or manipulate an argument in a function. Hence, the letters used in all these situations are regarded as variables by Usiskin (1988). Therefore, the variable concept has different images and meanings depending on where it is used (Schoenfeld & Arcavi, 1998; Usiskin, 1988).

Schoenfeld and Arcavi (1988) indicated many meanings of the term variable in the literature showing the multiplicity in its meaning. For example, there appeared such meanings: a symbol as a placeholder, a changing quantity, and a letter representing an indeterminate value. Hence, they called the letters used for these purposes as variables. Ely and Adams (2012) differentiated the meanings by incorporating the other researchers' definitions (e.g., Küchemann, 1978; Philipp, 1992; Sfard, 1995) into three ways: an unknown, variable, and placeholder. The word, unknown, is used for determinate quantities because it represents a specific number (or several numbers) in an equation. Therefore, a letter used as an unknown in an equation would be determined when the necessary information is available. Both variable and placeholder meanings refer to indeterminate quantities (Ely & Adams, 2012). The word, variable, means a varying quantity (Philipp, 1992). It refers to a set of values that a letter or variable would represent in a specific mathematical context (Ely & Adams, 2012). Therefore, a letter that is used as a variable is an indeterminate value and can represent any number among a set of values depending on what value other related indeterminate quantities represent. The function of a placeholder appears in the coefficients and parameters in the equations in which specific numbers will take the place of these letters according to the context. For example, in a linear function equation, $y = kx$, the letter *k* represents the parameter of the function and the slope of the line. For a given context, it will take a specific value, hence it is a placeholder (Ely & Adams, 2012).

Although the concept of variable has multiple meanings, the research showed that the students may not possess all those uses (e.g., Alvarez et al., 2015; Küchemann, 1981). Alvarez et al. (2015) observed that secondary school students in Spain and Mexico had difficulty in differentiating the multiple uses of variables, using them in problem situations, and interpreting their meanings. Similarly, MacGregor and Stacey (1997) found that middle school students interpreted algebraic letters in different ways such as unknowns, representing abbreviated words (i.e., h for height), generalized numbers, and objects. Knuth et al. (2005) observed that most middle school students understand variables as “representing more than one value” (p. 274) while in lower grades there is a weak understanding of variables through some misinterpretations.

Blanton et al. (2018) indicated that understanding the concept of variables is a crucial factor affecting the students’ representation of algebraic quantities in mathematical situations. Carraher and Schliemann (2007) pointed out that the unknowns in the missing value problems and equations should be interpreted as variables as early as elementary years so that the students can start to think about the concept of variables and variation as a significant concept within algebraic thinking. For example, Carraher et al. (2008) showed how third graders could start to use notations for variables in solving contextual problems from starting using pictures towards using symbols and letters together with operations when they discussed and imagined the problem situation in meaningful contexts. The researchers think that early possession of variable notation permits the students to grasp and represent easefully the relationships in problems (Blanton, Stephens, et al., 2015).

2.1.3.4 Quantitative reasoning

Quantitative reasoning starts with the recognition of the quantities in problem situations and quantifying them by giving numerical values to their qualities (Thompson, 1990). The examination of the relationship between quantities and operating with them using the relationships refers to quantitative reasoning. For example, comparing the lengths of two pencils, interpreting the additive relationships between the lengths of three ropes, or analysis of the multiplicative relationships between the lengths of two ropes are some examples of practices of quantitative reasoning (Blanton et al., 2011). Similarly, interpreting and generalizing the relationship between three quantities, *times, speed,* and *distance*, as ‘the amount of distance is the multiplication of time and speed’ corresponds to quantitative reasoning (Smith & Thompson, 2008). This provides the generality that is in algebraic reasoning. Therefore, providing the practices of quantitative reasoning in elementary and middle school years would help develop algebraic reasoning as well.

Understanding quantities and how they relate to each other provides significant grounds for developing algebraic reasoning (Blanton et al., 2011; Smith & Thompson, 2008). For example, Olive and Caglayan (2008) observed that eighth graders' making sense of the units of quantities (e.g., the value of coins) in word problems enabled them to write accurate equations in problem-solving. Further, how students approach and interpret the quantities and the relationship between them determines the complexity of their algebraic reasoning (Confrey & Smith, 1994). Ellis (2011) observed that middle school students who had a robust understanding of the problem quantities and the relationships between them, such as the length, width, and area, demonstrated various forms of functional thinking such as covariation and correspondence.

2.1.3.5 Functional thinking

Algebra is a study of patterns and functions (Blanton et al., 2011; Kaput, 2008) and a language of generalization in which the patterns between mathematical ideas and objects are realized, described, and extended (Usiskin, 1995). The researchers think that early pattern activities are useful for the students to understand the relationship between quantities, make conjectures, and generalize relationships as algebraic reasoning practices (e.g., Moss & Beatty, 2006; Zazkis & Liljedahl, 2002). Pattern activities initially present a meaningful context that is appropriate for young students to proceed to the generalization and abstraction earlier (Blanton & Kaput, 2004; Moss & Beatty, 2006; Zazkis & Liljedahl, 2002).

Growing pattern activities have the role of providing a path for a transition towards relational and functional thinking between independent and dependent variables (Blanton, Brizuela, et al., 2015). Studies showed that functional thinking besides growing patterns practices could be taught in the early elementary mathematics curriculums such as in the first and second grades (e.g., Blanton & Kaput, 2004;

Moss & McNab, 2011; Warren & Cooper, 2005) and even in kindergartens (e.g., Blanton & Kaput, 2004). Therefore, functional thinking, including "generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior", is the fifth big idea identified by Blanton et al. (2011, p.13).

In the early algebra studies, researchers examined the students' ways of generalization of functional thinking (e.g., Blanton et al., 2011; Blanton, Brizuela, et al., 2015; Smith, 2008; Stephens, Fonger, et al., 2017). An identification of the relationship in the problem differs according to the students' strategies and sophistication of the generalization. The basic strategies are generally followed by a recursive approach in which the students focus on a single variation and the difference between consecutive terms. Recursive thinking is observed in the early years of elementary school as a precursor to functional thinking (Blanton, Brizuela, et al., 2015). More advanced approaches appear as identified covariational relationships in which the students integrate both variables to construct a rule and express it explicitly with words or symbols. In the correspondence approach, students identify and express covariation using a generic rule. In functional thinking, the ultimate goal is to achieve covariation and correspondence thinking as the higher level of thinking processes (Blanton, Brizuela, et al., 2015).

In short, researchers regard the integration of the function concept in early mathematics teaching such as in elementary and middle schools through functional thinking activities for developing algebraic thinking (e.g., Blanton, Brizuela, et al., 2015; Usiskin, 1988). Because in this study, the participants are in the age range of 10-11 (i.e., fifth grade) and the targeted tasks included the generalizations of functional relationships and representing them using symbols, the studies about early functional thinking practices and their frameworks about the modes of functional thinking and ways of generalizations will guide the conceptual understanding in this study.

2.1.4 Recent findings about the generalization of functional thinking

In the context of early algebra, which I use for introducing algebraic processes before formal algebra instruction, proceeds from verbal to symbolic system and is regarded as a critical issue (Blanton & Kaput, 2011). As stated before, generalization, as a significant component of algebraic thinking, enables students to understand the relationships and to use symbolic notations to represent them even in elementary years (Blanton & Kaput, 2011). The transformations of students' mathematical expressions from understanding the meaning and relationships in contextual problems to representing and generalizing them in notations highlight the effectiveness of contextual situations involving patterns and functions (e.g., Blanton et al., 2019; Pang & Sunwoo, 2022; Stephens et al., 2021). Hence, researchers suggested representing functional thinking in elementary years through the problems in which the contextualized quantities change over time and while using graphs and tables (e.g., Kaput, 1999; Stephens et al., 2021).

Functional thinking is regarded as a gateway to algebra because it involves handling many key algebraic concepts and processes such as "generalizing relationships between covarying quantities; representing and justifying these relationships in multiple ways using natural language, variable notation, tables, and graphs; and reasoning fluently with these generalized representations in order to understand and predict functional behavior" (Blanton, Brizuela, et. al, 2015, p.512). For example, Blanton and Kaput (2005) described a problem situation in which students practice algebraic reasoning in earlier grades through functional thinking: the handshake problem. As students try to understand the relationship between the number of people and the total number of handshakes, they can express the relationship in different ways, such as words, drawings, and symbols. While they are dealing with this problem, they can practice algebraic reasoning through generalizations, understanding, and representing the functional relationship between the number of people (independent variable) and the number of handshakes (dependent variable).

Researchers used the function context in their studies to examine the students' ways of generalizations of functional relationships in different problems (e.g., Blanton, Brizuela, et al., 2015; Pinto & Canadas, 2021; Pittalis et al., 2020; Ramirez et al., 2020), and report their progress in algebraic reasoning through the intervention focused on functional relationships (e.g., Ayala-Altamirano et al., 2022; Blanton, Brizuela, et al., 2015; Blanton et al., 2019). Early algebra interventions used the generalization of functional relationships through different contextual situations involving the relationship between the number of people and the number of ears and the relationship between the amount of money in the piggy bank and the time elapsed (e.g., Blanton, Brizuela, et al., 2015; Blanton et al., 2019; Stephens, Fonger, et al., 2017).

2.1.4.1 Students' modes of functional thinking in early algebra

Studies that aim to develop the students' functional thinking in elementary grades demonstrated that students' functional thinking practices ranged between multiple levels starting from recursive thinking towards sophisticated use of both covariational and correspondence thinking (e.g., Blanton, Brizuela, et al., 2015; Stephens, Fonger, et al., 2017). Although researchers reported common levels or strategies of functional thinking, different levels or categories of functional thinking also appeared in some of them that show the sophistication of students' functional thinking.

Blanton, Brizuela, et al. (2015) described the first graders' functional thinking levels in an increasing sophistication from recursive to functional, with their particular sub-levels. Students in early grades can identify a pattern by focusing on the change in only one column or row in a function table, that is a recursive approach (Blanton, Brizuela, et al. 2015; Blanton et al., 2011). The researchers identified the sophistication of the students' thinking according to applied generality in the cases and accomplishment in the representation of the generality. For example, describing a functional relationship through specific cases is regarded as a functional-particular level of thinking while conceptualizing the relationship from a group of cases and stating it without specifying the relationship between two different variables is regarded as primitive functional-general. As the students increase the generality by focusing on all values and expressing the relationship by integrating both variables, the levels in functional thinking become more sophisticated up to condensed functional-general.

Similarly, Stephens, Fonger, et al. (2017) described the students' (grades 3-5) functional thinking levels into three main headings apart from the lowest level "no evidence of functional thinking" (p. 153): variational thinking, covariational thinking and correspondence thinking. Within variational thinking, there are two different levels of recursive thinking according to whether they articulate the relationship on particular cases or express the general relationship on all values of one variable. When the students describe the covariational relationship by expressing coordinated variables (e.g., every time you add a desk, you add two people) they are regarded as covariational thinkers. On the other hand, inventing a function rule refers to correspondence thinking. However, there are five more sub-levels within correspondence thinking with respect to how general the students described the rule or how much they could integrate both variables in the formula.

Pittalis et al. (2020) conducted interviews and specific measures with students (grades 3-5) to identify their functional thinking modes. They grouped the students' thinking modes into three categories. In the first category, the students represented a recursive thinking mode in all the functional thinking tasks. Their performance corresponds to the factual algebraic thinkers defined by Radford (2010) where they had difficulty in finding the larger values in the pattern situation. In the second category, three modes of thinking were observed separately or in combination for some students. Some students used both the recursive thinking mode and found the larger values in the tasks by applying the function rule that they constructed contextually. The researcher observed emergent-covariational and correspondence modes of thinking in this category with "pre-symbolic contextual strategies" (p. 658). Lastly, category 3 represented the students who could apply covariational and correspondence modes of thinking flexibly by using symbolic representations and providing explanations of their solutions. Therefore, their performance corresponds to the standard algebraic thinkers defined by Radford (2010) which demonstrates generalized abstraction.

Pinto and Canadas (2021) examined the third and fifth-grade students' generalizations of functional thinking in different forms of contextual problems such as $y = 3x$, $y = x + 5$, and $y = 2x + 6$. They found that only half of the third graders could generalize the functional relationships, verbally or using numerical representations (e.g., $20 + 2 = 22$). In addition, they observed that only two third graders demonstrated a covariational approach in their verbal generalizations and the others used a correspondence approach while none of them could use a symbolic representation. On the other hand, most of the fifth graders could generalize the functional relationship in a correspondence approach in which they could verbally state the general rule for $y = 2x + 6$ or use symbolic representation. For example, a student said: "Multiplying the number of white tiles by 2 and adding 6 gives you the result" (p. 128).

In summary, in the elementary years, when students did not receive formal algebra instruction, they represented a range of modes of functional thinking through intervention or without intervention, as reported in the studies above. Researchers observed that students in these grade levels could use recursive, correspondence, and covariational approaches in functional thinking. The following section presents the findings about the progress of students in functional thinking in the studies including intervention.

2.1.4.2 Students' progress in generalizing functional thinking through interventions

Blanton, Brizuela, et al. (2015) demonstrated how elementary school children can develop algebraic reasoning and functional thinking by working on covarying quantities in contextual problems and generalizing and representing the relationships between variables with literal symbols, tables, and graphs. Multiple representations such as drawings, tables, charts and graphs, and age-appropriate tasks in rich contexts helped the students to make sense of the variables and functional relationships and to make generalizations as initial steps into the algebraic concepts (e.g., Blanton et al., 2019; Blanton, Stephens, et al., 2015; Carraher et al., 2008).

In a longitudinal intervention program, the researchers (Blanton et al., 2019) implemented an instructional sequence for students from grades 3 to 5; that is, to foster algebraic thinking through generalizing, justifying, and reasoning with mathematical structures and functional relationships. Blanton et al. (2019) observed that at the end of each grade, the performance of students in the experimental group showed significant progress in algebraic reasoning compared to the control group, such as expressing the functional relationship in words, using notations (e.g., $y = 2x$) and interpreting mathematical structures (e.g., $a + b = c$ and $b + a = c$). In addition, they reported that all of the students, both experimental and control groups, struggled to use notations in representing the functional relationships more than representing the arithmetic properties in symbols. Moreover, in the same intervention, Stephens, Fonger, et al. (2017) found that third-grade students who could not represent and generalize the relationships before the treatment started to reason sophistically about the relationships between quantities. They could identify and describe the functional relationships through correspondence and a covariational approach after 18 hours of treatment while they used a recursive-patterning approach at the beginning.

Blanton, Stephens, et al. (2015) reported the progress of third graders in the experimental group during the one year of intervention in representing functional relationships. About one-fifth of the students in the experimental group expressed the functional relationships using the covariational strategy in words such as “each table you add adds two people” (p. 67) after the intervention. In addition, a small proportion of students (16%) could express the function rule using symbols while a smaller proportion of students (8%) could express this rule in words such as "number of tables times two plus two equals number of people" (p. 67). In addition to observing a progression in students' functional thinking as compared to the control group, a surprising finding was the students' more flexible use of symbols instead of using their own words to express the function rule. Performing better in using notations in expressing the functional relationships after the intervention is also observed by other researchers for students in grades 4 (e.g., Blanton et al., 2019) and 5 (e.g., Akın & Isler-Baykal, 2024; Blanton et al., 2019).

In another small-scale teaching experiment, Pinto and Canadas (2021) examined the third and fifth-grade students' generalizations of functional thinking in different forms of contextual problems such as $y = 3x$, $y = x + 5$, and $y = 2x + 6$. They found that only half of the third graders could generalize the functional relationships verbally or they could use numerical representations for expressing the generalization (e.g., $20 + 2 = 22$). Only two of the third graders demonstrated a covariational approach in their verbal generalizations and the others used a correspondence approach while none of them could use a symbolic representation. On the other hand, most of the fifth graders could generalize a functional relationship in a correspondence approach in which they could verbally state the general rule for $y = 2x + 6$ or use symbolic representation. For example, a student could say "multiplying the number of white tiles by 2 and adding 6 gives you the result" (p. 128). This study highlights the potential difference in the progress in algebraic thinking between various grade levels of students.

In another study (Ayala-Altamirano et al., 2022), researchers reported similar findings to Pinto and Canadas (2021) in terms of using symbolic notations in functional thinking. Ayala-Altamirano and her colleagues (2022) examined the fourth-grade students' generalizations of functional relationships in a teaching experiment targeting the development of students' algebraic thinking. They observed that a small proportion of students could generalize functional relationships using natural language after the intervention. However, students did not show an improvement in using letters or notations to represent the generalizations or interpreting the questions. Researchers observed that students had the most difficulty in the questions including letters and they refused to use letters in generalizations.

The above studies would be promising in terms of younger students' ability to generalize functional relationships and their progress through the advanced modes of functional thinking. However, researchers also reported some difficulties and differences that students showed in using letters and notations in generalizing functional relationships (e.g., Akın & Isler-Baykal, 2024; Ayala-Altamirano et al., 2022; Pinto & Canadas, 2021). Therefore, the literature on teaching and learning early algebra needs further research and investigation to elaborate and expand our knowledge about students' progress in algebraic thinking and their needs for this improvement in terms of the characteristics of students' mental processes, the teaching sequence, and the tasks used in interventions.

2.2 The Framework of Units Coordination

Constructing and conceptualizing whole numbers starting from the counting activities is described through the process of constructing arithmetical units (Steffe, 1992; von Glasersfeld, 1981). For example, to conceptualize the number *five* as a unit for use in different mathematical processes, children need to experience various sensory and mental actions. The ability to perceive numbers as arithmetical units depends upon reflective abstraction, a process whereby the mind can conceptualize the number without being constrained by the limits of sensory input and operate with it as a single entity through attentional processes (von Glasersfeld, 1981).

Von Glasersfeld (1981) explained the unitizing operation whereby numbers are constructed as discrete entities through the binding or sequencing of disparate sensory elements. This bounding or sequencing process results in a meaningful and associative whole. In this process, distinguishing various items or grouping them may not rely only on their perceptual characteristics, it is rather a cognitive framing that constructs a cognitive entity or object. This outcome, the constructed individual entity, refers to the term, unit, (Hackenberg & Sevinç, 2024; Ulrich, 2015), which will be further processed experientially and abstractly to construct new units and composite units (von Glasersfeld, 1981).

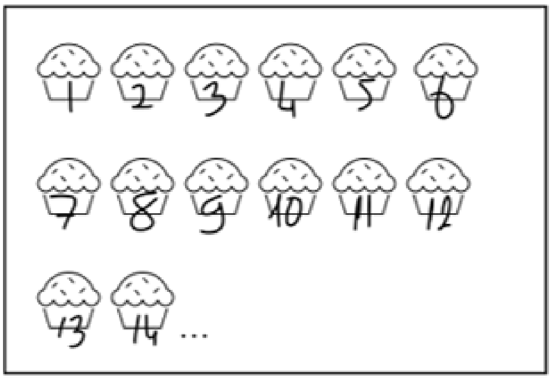
In short, each number is framed as an arithmetic unit through the process of reflective abstraction in which the bounded conceptual structures are released from sensory-motor experience (von Glasersfeld, 1981). For example, the number *five* can be perceived as a unit comprised of five equal units of ones through grouping or chunking without further need for sensory materials (e.g., fingers). Hence, this new unit can be used in counting by fives. By extension, von Glasersfeld (1981) indicated that each number is different from one another through their abstract characteristics.

Researchers explained how students conceptualize numbers and operations in different contexts starting from the counting schemes (e.g., von Glasersfeld, 1981; Steffe, 2001). At first, it is expected that the students need sensory-motor materials to keep track of counting items in an experiential situation. After an experience with this activity, they can re-process the counting activity through figural materials (e.g., fingers) without a direct sensory operation with materials. They gradually construct the units in a more complex activity as detached from the figural characteristics. The goal of this process is to facilitate the assimilation of students' complex activities and to facilitate their advancement to a higher level of proficiency and understanding in quantitative operations.

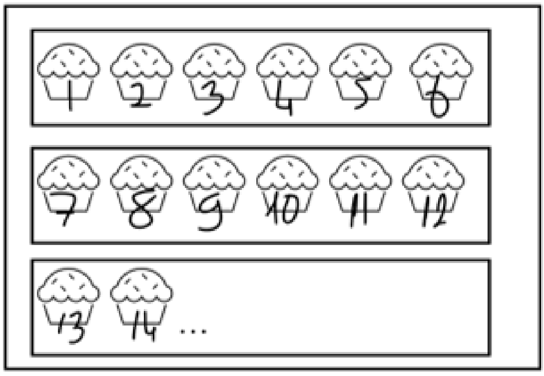
In sum, the theorization of units coordination deals with the "quantitative complexity" (p.3) in children's work with units and their construction of relationships between units (Ulrich, 2015). Accordingly, units coordination refers to the mental operations that describe people's construction of units, and operations with various levels of units (Hackenberg & Sevinç, 2024; Steffe, 2001; Ulrich, 2015). Therefore, units coordination is related to various mathematical situations such as developing the number sense, counting, operations with numbers, and multiplicative reasoning (Glasersfeld, 1981; Steffe, 1992; Ulrich, 2015, 2016a). The students' construction of the number concept and their operations with quantities are described in hierarchical levels or developmental stages considering their construction of and operations with units (Ulrich, 2015).

Glasersfeld (1981) explained the students' progression in units coordination through the types of units that they deal with during counting activities or arithmetic operations such as using figural materials (e.g., fingers), constructing arithmetic units, and using composite units respectively. He indicated that as long as a child needs sensory materials in counting, the number five, for example, does not refer to a number unit that is comprised of five units of one. This is characterized as the pre-numerical stage. Reflective abstraction is required to construct a higher level of the conceptual structure of numbers and interiorization of counting occurs.

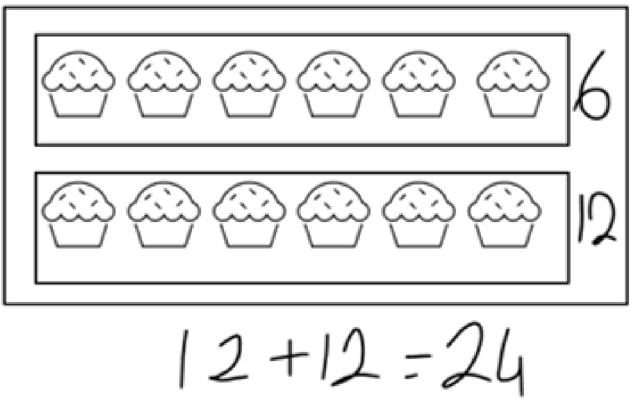
To exemplify the units coordination ability in a problem situation, we can think about counting or calculating the number of muffins in rows such that there are four rows and each row involves six muffins (Hackenberg, 2010). Counting the number of muffins one by one in a figure or by drawings deals with coordinating one-levels of units in activity (see Figure 2.3a). On the other hand, in a more complex operation, coordinating two levels of units in activity through interiorized one-level of units can be reflected by the behavior of stopping after each six-count to represent one row. They could reach the result of 24 but they will need to carry out the same activity when the problem is again asked (see Figure 2.3b). Through the interiorization of two levels of units, that is following a higher level of units coordination, a student can reach the result of 24 without needing a figurative material and she/he is already aware of the insertion of each six muffins into the rows. For example, if she knows two sixes are 12, she could add two 12s and get 24 (Figure 2.3c). Lastly, the interiorization of three levels of units means being aware that 24 is four units of six and 24 units of ones. This is given to students without any activity and the students can flexibly transform the different levels of units such as adding other identical rows to the current ones as any number of units including six units of ones.



a) A student's counting at the pre-multiplicative level



b) A student's coordination two-levels of units in activity



c) A student's assimilation of two-levels of units

Figure 2.3 Students' mental operations in different levels of units coordination

Several frameworks explained the students' construction of number sequences (Steffe, 2010) and multiplicative concepts (Hackenberg & Tillema, 2009) in terms of units coordination. Ulrich (2015, 2016a), on the other hand, preferred to call this hierarchy by levels, differentiating the number of levels of units that students deal with when operating with numbers and problem situations. These frameworks will be explained in the following sections.

2.2.1 The number sequences

Based on the model of von Glasersfeld (1981) for unitizing and constructing numerical units, the students' construction of different "numerical counting schemes" is characterized by the structure of number sequences (Steffe, 2010, p.27). A number sequence is explained as "a sequence of abstract unit items that contain records of counting acts" (Steffe, 2001, p. 267). Four distinct constructions of number sequences are the initial number sequence (INS), the tacitly nested number sequence (TNS), the explicitly nested number sequence (ENS), and the generalized number sequence (GNS).

Students start to generate initial number sequences (INS) after constructing numerical composites which is "the sequence of abstract unit items" (Steffe, 1992, p. 266) by uttering a number word (von Glasersfeld, 1981). They operate with one level of units. A key indicator of this level is the ability to count on (Steffe, 1992, 2024). Students with the INS need no further perceptual materials to count, like in the pre-numerical stage, but they need to use figurative materials like fingers to keep track of counting after starting a number. A number word is regenerated as a place in a sequence rather than a collection of ones that is another unit composed of ones. Therefore, INS students do not conceive of a number as nested in a bigger number (Wilkins et al., 2021). For example, INS students can count by threes from a number, say eight, like 9,10,11; 12,13,14; 15,16,17;…. but these threes are not distinct units, rather they are still counted units of ones in activity (Steffe, 1992). During counting by the numerical composites, they would mostly lose track of the number of trios. This process by which students construct arithmetical units refers to the operations with the first level of units in activity (Ulrich, 2015).

One of the constraints of the INS stage is that the students do not conceive of a number being embedded within a larger number (Wilkins et al., 2021). Therefore, students with INS need to count on the previous number in order to understand which number is bigger by focusing on which comes first. Another constraint of the INS is that it does not allow the students to conclude multiplicative tasks which include coordinating two units, such as the number of rows including a specific number of blocks (Steffe, 1992). Because the students had difficulty in differentiating the multiple units and keeping track of the counting through composite units, Steffe (1992) called this stage “pre-multiplicative” (p. 304).

Constructing the TNS refers to “an enactive units-coordination” or “units coordination in activity” (Steffe, 2024, p. 33). Students with TNS can coordinate two number sequences and construct composite units in activity (Steffe, 1992). In other words, while the students can interiorize arithmetical units of ones by the INS, students with TNS can construct units bigger than one (Ulrich, 2015), which some researchers called stage 1 of units coordination (e.g., Hackenberg & Sevinc, 2024).

Students with TNS use the result of a counting activity recursively in further counting operations (Steffe, 1992). For example, the students with TNS can calculate that there are four threes in 12 by counting by ones, and concurrently keeping track of how many times they counted by threes (Norton et al., 2015). Here, we can observe a double counting activity; one is counting by ones, and the other is counting the number of threes, which is an identifying indicator of TNS (Steffe, 1992). Students with TNS can conceive a number, say five, both five units of ones and a single unit comprising five individual units in activity (Ulrich, 2015). Hence, they can nest numbers in other numbers without needing any counting act such as seeing five as nested within seven (Wilkins et al., 2021).

In the TNS stage, the students’ conception of composite units is tacit because they construct composite units in activity implicitly and there is no explicit reflection on composite units as ready structures (Ulrich, 2015; Zwanch & Wilkins, 2021). In addition, the composite units constructed during activities can decay after the activity is ended. This also indicates that their multiplicative activity does not represent explicit multiplicative reasoning. Moreover, Ulrich and Wilkins (2017) differentiated the students who operate early in the TNS stage (eTNS) and the students who have advanced mental operations in the TNS stage (aTNS). The aTNS is a transition stage between the TNS and the ENS in which the students can reflect on composite units and can be "tacitly aware of embedded units."

Explicitly nested number sequence (ENS) refers to a two-level structure in which the students conceive numbers as collections of units of ones, construct and count with composite units (Norton et al., 2015; Ulrich, 2016a). Hackenberg and Sevinç (2024) called this level, stage 2 of units coordination. Students with ENS can compare two numbers by being aware that the smaller one is a subset of the bigger one and the difference between them is the amount of remainder, as another unit (Ulrich, 2016a). This demonstrates their explicit reflection on the additive relationship between numbers and the nestedness of number sequences. In a multiplicative situation, ENS students can develop an understanding of a composite unit, say 12, as consisting of four units of threes and also 12 units of ones that are identical and equivalent (Hackenberg & Sevinç, 2024). Because they intentionally construct and work with a composite unit of composite units (i.e., 12 is a composite unit of ones consisting of three composite units of fours), this reflects their coordination of three levels of units in activity as well.

An important difference of the ENS from the lower levels of numerical reasoning is that students with ENS can disembed units from the composite units (Steffe & Olive, 2010). Disembedding operation refers to envisioning a unit as pulled out of a composite unit (Steffe & Olive, 2010). For example, in finding how many fives there are in 45, "the ENS students can disembed a sequence of 5 from the sequence of 45, iterate it 8 more times to get a sequence of nine 5s and then re-embed their result into the original sequence of 45 to equate 45 singleton units with 9 composite units of 5". (Ulrich, 2016a, p. 37). In short, the assimilation of numbers as composite units and disembedding operations are key indicators of the ENS.

The students who constructed a GNS can reflect on a three-level structure through composite units in which they come up with a composite unit of 12 as constructed by iterating 4 times threes, which are other composite units formed by iterated ones (Hackenberg & Sevinç, 2024). This is called stage 3 of units coordination. At this stage, composite units are now iterable (Ulrich, 2016a). While students with the ENS can iterate composite units in activity, this becomes assimilatory for the GNS students (Ulrich, 2016a). Students with the GNS can monitor the iterations of composite units as being assimilated in multiplicative comparison tasks. At this stage, the students exhibit more complex strategies in additive and multiplicative reasoning. For example, it is expected that a GNS student can make use of the commutative property by himself without a need to be taught.

In short, the students' construction of numbers as arithmetic units through reflective abstraction, their further construction of composite units, and operations with these units are described in terms of the levels of number sequences (von Glasersfled, 1981; Steffe, 1992; 2024; Ulrich, 2015, 2016a). According to Steffe's (2024) estimations, half of the students starting the first grade will construct INS or TNS and only one-tenth of the first graders will be able to construct the ENS. At the middle school level, he estimated that 40% of fifth graders will have constructed the ENS and only a quarter of fifth graders will have constructed assimilated three levels of units. He considered these estimates to be of significant value in evaluating the students' mathematical activities within the curriculum.

2.2.2 Students' multiplicative concepts

The students' units coordination is also defined in the context of whole number multiplicative concepts (Hackenberg & Tillema, 2009). The level of multiplicative concepts that students interiorize through multiplying schemes describes their units coordination in the context of multiplicative relationships (Hackenberg, 2013). The students at the pre-multiplicative level cannot complete multiplicative tasks which include coordinating two units such as the number of rows including a specific number of blocks in each row (Steffe, 1992; Ulrich, 2015). The students start to deal with multiplicative concepts when they construct composite units in activity (i.e., dealing with two levels of units) such as representing 12 as three fours (Steffe, 1992). This level of unit coordination refers to the first multiplicative concept (i.e., MC1) (Hackenberg & Tillema, 2009).

The first multiplicative concept (i.e., MC1) refers to the level of TNS in the number sequences framework (Ulrich, 2015). Therefore, the students operating with the MC1 coordinate two levels of units in activity, like TNS students. For example, in computing the number of cakes in four rows, each containing six cakes, a student with MC1 can find that there are a total of 24 cakes by iterating the composite units, six cakes, four times into another composite unit, the number of rows (Hackenberg, 2013). However, this operation and its result will not be maintained for another operation. The students have to enact the same process of units coordination in a new problem situation because the two-levels of units they constructed are not assimilated.

The second multiplicative concept (MC2) refers to "the interiorization of two levels of units" (Hackenberg & Tillema, 2009, p.3). Therefore, it refers to the ENS in the number sequences framework (Ulrich, 2016a). One difference between MC2 and MC1 is the ability to use the result of a unit coordinating activity (e.g., dissembeding a part from a composite unit and using that part) in further operations (Hackenberg & Tillema, 2009). In addition, MC2 allows students to coordinate three levels of units in activity in which the students make a number, like 12, as a unit containing four units of three units of units of ones. On the other hand, they cannot use this structure in further operations as a ready material, rather this structure is curtailed in the further operations by staying as two levels of units structure at the end.

Lastly, the third multiplicative concept (MC3) refers to "the interiorization of three levels of units (Hackenberg & Tillema, 2009, p. 4). Since the three levels of units are available before any multiplicative operation, the students with MC3 can coordinate more than three levels of units in activity (Ulrich, 2016a). This refers to the GNS in the number sequences framework. For example, in a similar cake and rows problem, when five more rows are added, each containing six cakes, MC3 students could flexibly operate between the different levels of units such as the initial 4 rows of six cakes and the additional five rows of six cakes and the total 9 rows of six cakes, which are all available before activity (Hackenberg & Tillema, 2009).

In this study, I employed the terminology of multiplicative concepts in explaining the students' mental operations in terms of units coordination. Although these two frameworks (i.e., multiplicative concepts and number sequences) used different terminologies in defining the levels of units coordination, both have the same indicators and the levels of this framework correspond to the stages in the number sequences framework starting from the TNS (Ulrich, 2015, 2016a). Since the algebraic tasks in this study involved multiplicative relationships through recursive thinking, functional thinking, and quantitative reasoning, the terminology of whole number multiplicative concepts described by Hackenberg and Tillema (2009) would be more appropriate to describe the students' units coordination. The indicators for each corresponding level of units coordination are represented in Table 2.2, which was adapted from Ulrich (2016a). The next section will present the findings of related literature in terms of the interaction between units coordination and algebraic thinking and reasoning.

Table 2.2. The Indicators of Units Coordination Levels in Different Frameworks

Number Sequences	**Multiplicative Concepts**	**Operations**
None	None	Construction of one level of units in activity. 4 chairs + 3 chairs = 1, 2, 3, 4, 5, 6, 7 chairs
INS	None	Reflecting on one level of units A number as a counting sequence is unitized. 4 chairs + 3 chairs = 4 - 5, 6, 7 chairs
TNS	MC1	Construction of two levels of units (composite units) in activity.

Table 2.2 (Continued)

TNS	MC1	Additive comparison tasks (how much more) Coordination of two levels of units in activity Three times one unit of 4 = 4 + 4 + 4 = 12 (a composite unit of 12) 4 is a composite unit containing 4 equal units of one in any sequence (1-4 or 16-20)
ENS	MC2	Reflecting on two levels of units Coordination of three levels of units in activity Disembedding operation Understanding of interchangeable identical units 4 is four iterations of one Multiplicative reasoning
GNS	MC3	Reflecting on three levels of units Coordinating four or more levels of units Iterations of composite units Understanding of interchangeable identical composite units

2.2.3 The literature on units coordination

Researchers interpreted the students' performances in various mathematical subjects such as multiplicative reasoning, fraction knowledge, and algebraic reasoning through their units coordination levels and current mental structures of operations and numbers (e.g., Hackenberg, 2013; Hackenberg et al., 2021; Olive & Caglayan, 2008; Ulrich & Wilkins, 2017). They indicated that teaching mathematics should be enhanced regarding students' multiplicative concepts and coordination of units in unknown and known quantities. For example, Ulrich and Wilkins (2017) pointed out that being able to think multiplicatively - a major objective during middle school years - is "an awareness of a multiplicative relationship between two quantities" (p. 18) more than solving problems including multiplication. Therefore, the researchers claimed that unit coordination is a prerequisite and an associative factor for being successful in various mathematical tasks including multiplicative relationships such as generalizing the multiplicative relationships between unknown quantities, solving equations, and operating with fractions (e.g., Hackenberg, 2013; Olive & Caglayan, 2008; Wilkins et al., 2021; Zwanch, 2022a).

In the algebraic reasoning context, understanding and operating with quantities and investigating the relationships between them takes great attention by the researchers taking a quantity-based approach in teaching algebra (e.g., Chazan, 2000; Fuji & Stephens, 2008; Olive & Caglayan, 2008). Algebraic reasoning is described through reasoning with unknown or known quantities (Steffe & Izsak, 2002), and how students approach and interpret the quantities and their relationship determines the complexity of their algebraic reasoning (Smith & Thompson, 2008). For example, Olive and Caglayan (2008) observed that students who comprehended the units of quantities (e.g., the value of coins) in word problems, subsequently were able to write precise equations to solve the problems. Similarly, recognizing the covarying quantities in the functional thinking practices in early algebra is linked to students' understanding of problem quantities (Ellis, 2011).

The main goal of this study is to investigate students' progress in algebraic thinking with a potential interaction with their units coordination. Therefore, this section presents the related findings of the studies on this subject considering the link between students' performance in various algebraic tasks and their mental operations in terms of units coordination. Algebraic thinking and reasoning involve many interrelated elements, including understanding variables, generalization, symbolic representation, and solving equations. Researchers have focused on different aspects of algebraic thinking and reasoning while examining the impact or relationship of students' unit coordination structures. Researchers asserted that students' construction and coordination of units and their specific mental operations related to their number sequences (e.g., disembedding, iteration of composite units) can determine their performance in working with unknown quantities, writing equations and generalizations (e.g., Hackenberg & Lee, 2015; Olive & Çağlayan, 2008; Zwanch, 2022a). The following sections present these findings in more detail.

2.2.3.1 Students' interpretation of quantitative unknowns

An understanding of and ability to operate with variables or unknown quantities is a significant component of algebraic thinking and reasoning (Blanton et al., 2018; Kieran, 2022; Usiskin, 1988). Researchers reported that students' understanding of unknown quantities and how they interpret their use in algebraic expressions was affected by their levels of units coordination (e.g., Hackenberg & Lee, 2016; Hackenberg et al., 2017; Hackenberg et al., 2021). They indicated that the use and interpretation of unknown quantities were different in the performances of MC3 and MC2 students.

Hackenberg et al. (2021) implemented iterative design experiments on 13 MC2 and MC3 students (6-9 grades) to examine their understanding of unknowns by using problems including two unknowns with a known multiplicative relationship. They observed that MC3 students could construct a meaning of quantitative unknowns such as one unknown unit consisting of a certain number of smaller units. The MC3 students revised their equations and interpretations during the discussions from representing the relationship between measurement units to representing the relationship between the values of the unknown heights or lengths. On the other hand, most of the MC2 students represented the relationships between measurement units (e.g., the number of straw or pen lengths) rather than between the values of unknown measurements (e.g., the height of a house which is measured by the length of a pen). In addition, Hackenberg et al. (2021) observed that MC2 students were inclined to use knowns instead of working with indeterminate quantities. They generated two separate two-levels of unit structures. The literal symbols represented the measurement units instead of quantitative unknowns which are measured by an indeterminate unit, each consisting of a smaller number of units. Some of the MC2 students demonstrated a similar performance to what MC3 students did.

Similarly, Hackenberg et al. (2017) observed that the MC2 students from seventh and eighth grades had difficulty in representing the multiplicative relationship between two unknowns in figures and equations. The researchers indicated that the main reason for the MC2 students' struggle was their need to simplify the involved unit coordination so that they could work with two levels of units as given because they cannot operate further with three levels of units. Hence, the MC2 students represented three common approaches to simplify the unit coordination such as thinking of the multiplicative relationship as approximate, giving numerical examples for the unknowns, and conceiving the multiplicative relationship during making drawings and through teacher prompts.

Hackenberg and Lee (2016) found that the use of symbolic representations by the MC2 students was not algebraic while MC3 students algebraically use notation by writing equivalent forms of the same relationships. For example, MC2 students could use fraction multipliers to express the relationship "What is the three-fifths of a candy bar length represented by a?" by only writing "$3/5a$". However, MC3 students could use different forms of expressions through explanations and relating to other representations such as "$3/5a$" and "$a/5 . 3$". In addition, in expressing the three-fifths of the sum of three unknown quantities MC2 students could not write an appropriate algebraic expression (e.g., $3/5\ c\ b\ a$) while some MC3 students could express it using different forms again such as " $(a + b + c) \times 3/5$" and "$(3/5a + 3/5b + 3/5c)$" that the researchers attributed it to their distributive partitioning scheme. Therefore, they explained these findings as a relationship between the students' units coordination in terms of fractional knowledge and their algebraic reasoning.

A similar finding was also presented by Hackenberg and Sevinç (2022) who investigated the relationship between MC3 students' (7^{th} and 8^{th} graders) rational number knowledge and their reciprocal reasoning in writing equations using unknowns. They indicated that using unknowns in problems helped students' reciprocal reasoning in writing equations and vice versa. The MC3 students who constructed iterative unit fraction schemes interpreted the unknowns in equations in a more sophisticated way by using reciprocal reasoning and fractional multipliers. Moreover, Hackenberg and Sevinç (2021) observed that an advanced MC2 student, Milo, could show some evidence of reciprocal reasoning in some problems as well such as writing $S \times 3/7 = L$ and $L \times 0.42 = S$.

In general, it is asserted that the students at the first stage in multiplicative concepts (MC1) would not perceive quantitative unknowns while the students at the upper levels (MC2 and MC3) can imagine the partitioning of quantitative unknowns with unknown units (Hackenberg, 2013; Hackenberg et al., 2021). In addition, using numerical examples for the unknowns is seen as a common approach in MC2 students (e.g., Hackenberg & Lee, 2015; Hackenberg et al., 2017).

2.2.3.2 Writing equations using symbols

Another aspect of algebraic performance examined by the researchers in a relationship with units coordination is writing and solving equations in problems involving unknowns (Hackenberg, 2013; Hackenberg & Lee, 2015; Olive & Çağlayan, 2008). Researchers observed that students having different mental structures about the units and quantities demonstrated qualitatively different ways of writing equations (e.g., Hackenberg & Lee, 2015; Hackenberg & Sevinc, 2022). For example, Olive and Çağlayan (2008) asserted that constructing equations for solving word problems involving three different quantities with one unknown required the assimilation of three levels of units (MC3), which was accomplished by some of the eighth graders in their study. Additionally, they observed that some students who had assimilated two levels of units experienced difficulties in writing a complete equation representing the monetary values on each side consistently by using only one unknown. They indicated that the MC2 students' lack of reflection on three levels of units prevents them from representing the relationship between three quantitative units (i.e., the number of coins, the monetary value obtained from each coin, and the total monetary value). The researchers suggested that "unit coordination and unit conservation are cognitive prerequisites for constructing an equation when reasoning quantitatively about a situation" (Olive & Çağlayan, 2008, p. 32). On the other hand, in a recent study, Zwanch (2022b) observed that MC2 students could generate a system of linear equations through the disembedding and assimilation of two levels of units. Additionally, she indicated that MC1 students could not represent these equations because of the lack of disembedding operation.

Hackenberg and Lee (2015) observed differences in written equations of MC3 and MC2 students for the multiplicative relationships between unknowns. For example, MC3 students could write accurate equations in a quick time by using both whole number and fraction coefficients and could write the reverse forms of initial equations (i.e., $x = (1/5)y$ for $y = 5x$). However, they remarked that most of the MC3 students were eighth and tenth graders who took a regular algebra course. On the other hand, MC2 students struggled to write equations representing the multiplicative relationships between unknowns. They needed more prompts from the teacher, and they could not write the equations by using reciprocal reasoning. Researchers also observed that MC2 students were inclined to give numerical examples to the unknowns.

In another study, Hackenberg (2013) selected six MC1 students from seventh and eighth grades to examine the relationship between their algebraic reasoning, in terms of equation writing and generalization, and fractional knowledge, in terms of some mental operations such as splitting, partitioning, disembedding, and iterating. She found that the majority of MC1 students could not write an equation representing the multiplicative relationship between two unknowns such as one unknown length is given as five times the other unknown length (the Cord Problem represented in Chapter 3). Only two students could write the equation through the teacher's coaching and after making some errors. Hackenberg (2013) indicated that the lack of disembedding operation, which is quite necessary for algebraic reasoning, caused some constraints for the MC1 students in writing equations for the multiplicative relationship.

Another example of the differing equation writing of students from different levels of units coordination is represented by Hackenberg and Sevinc (2022). They observed that all MC3 students demonstrated the ability to express the multiplicative relationships between two unknowns in a flexible manner, using whole number multipliers and reverse equations such as “$S \div 7 \times 3 = L$” and “$L \div 3 \times 7 = S$”. Moreover, they could also construct reciprocal reasoning by using fractional multipliers to express this relationship which is one unknown is three-seventh of another unknown thanks to the iterative fraction schemes. On the other hand, two MC3 students who could not construct iterative unit fraction schemes could only construct reversible and additive reasoning using whole number multipliers but not being able to relate it to the equations involving fraction multipliers. Therefore, the researchers indicated a connection between the students' units coordination in terms of fractional knowledge and their performance in writing algebraic expressions.

2.2.3.3 Generalizations

An important aspect of algebraic reasoning is generalization (Kaput, 2008) where the students recognize the relationships between quantities or variables and express these relationships in different ways. Similar to the writing equations, researchers also observed an impact of students’ units coordination levels on their generalization performances by analyzing their verbal and symbolic expressions representing the particular relationships or formulas (e.g., Hackenberg, 2013; Zwanch, 2022a).

Hackenberg (2013) studied with MC1 students to examine the interaction between their levels of units coordination and their generalizations and generating rules. She observed the generalization activities of MC1 students in their work on finding the number of small squares on the border of a bigger square, which is called the Border Problem. The generated rules by the students demonstrated their algebraic reasoning. However, Hackenberg (2013) indicated that their generalizations were not very sophisticated by involving structural thinking. In addition, the students had difficulty verbalizing the generated rules. This is attributed to the absence of disembedding operations in MC1 students. On the other hand, their iteration operations helped them generate some methods for finding the number of small squares in terms of generalization.

In another study, Zwanch (2022a) examined middle school students' generalization (grades 6 to 9) by relating it to the students' stages of number sequences. She noted that all of the ENS (MC2) students could represent the generalization using symbols, whereas nearly half of the aTNS (advanced MC1) students were able to use symbolic representations for generalizations. She found that TNS students relied on recursive patterns in problems to find certain steps in the patterns and they could not generate a symbolic generalization. On the other hand, aTNS students' performances showed differences according to the problem types. They could write symbolic representations for the generalization of pattern situations. In addition, she remarked that figural reasoning may be more accessible to aTNS students instead of numerical reasoning.

Relying on recursive thinking in patterns to reach the bigger steps is explained by the TNS students' construction of composite units in activity, and lack of reflection on composite units (Zwanch, 2022a). Hence, Zwanch (2022a) posited that this inadequacy leads the TNS students to fail to generalize relationships into patterns. Zwanch (2022) remarked that reflecting on composite units as a characteristic of aTNS and ENS students might allow the students to generalize the relationships verbally and symbolically. In addition, a lack of disembedding operation can create constraints in symbolic generalizations in pattern situations because the students cannot reflect on the relationships between unknowns, as also explained by Hackenberg (2013).

2.3 Summary

This section presented key terms and concepts about algebraic thinking and units coordination. In short, algebraic thinking involves many ways of thinking such as analytical, structural, and functional thinking (Radford, 2014; Kieran, 2022). The indicators of these ways of thinking can be seen in generalization activity which is a key practice of algebraic reasoning (Kaput, 2008). Symbolic expression of these generalizations is another part of algebraic reasoning. If these practices involve certain patterns of thinking, such as being analytical, abstracting ideas and relationships, and structurally interpreting results, they may represent developed algebraic thinking (Kieran, 2022). Writing a symbolic formula does not always demonstrate algebraic thinking (Radford, 2010). Sometimes students arrive at a formula by trial and error or by guessing, repeatedly changing some terms in the formula until they get the right result. Therefore, this is not "an analytical way of thinking about indeterminate quantities" which is "the main characteristic of algebraic thinking" (p.9). During a clinical interview or classroom experiments, students' algebraic thinking and reasoning can be assessed and improved through contextual tasks that require generalizations and interpretations of relationships between unknown quantities or variables, as seen in the aforementioned studies (e.g., Ayala-Altamirano & Molina, 2020; Blanton et al., 2019; Stephens, Fonger, et al., 2017).

Units coordination is expressed as mental operations involving certain structures that demonstrate how individuals identify and construct the units and the relationships between units. Individuals' mental structures about units and composite units demonstrate the complexity of their work with mathematical quantities. Therefore, it is asserted that individuals' units coordination is an influential factor in many mathematical performances (Hackenberg, 2013). For example, researchers observed that students with different levels of units coordination represent qualitatively different ways of writing symbolic mathematical expressions (e.g., Hackenberg & Lee, 2015). Moreover, students' generalization in verbal or symbolic interacts with their units coordination (Zwanch, 2022a). Therefore, it may be a novel case in algebra studies to interpret students' mental structures about units as an influential factor in developing algebraic thinking.

As seen in the literature, there is a growing interest in design-based research, which is concerned with how important cognitive processes impact students' algebraic thinking and reasoning during the learning process. Some researchers investigated the interaction between the students' units coordination and their algebra performance through design experiments in terms of different contexts such as students' rational number knowledge and equation writing (e.g., Hackenberg et al., 2017) and students' rational number knowledge and their representations of quantitative unknowns (e.g., Hackenberg et al., 2021). This study differently presented an HLT that was designed in accordance with the findings of early algebra studies and involved conjectures and outcomes about the students' learning processes with the aim of investigating the interaction between their units coordination and their progress in algebraic thinking.

CHAPTER 3

METHODS

The main goal of this study was to investigate the interaction between fifth-grade students' progress in algebraic thinking and their units coordination levels. To accomplish this, the current study answered the following research questions:

1. What is the initial state of fifth-grade students' units coordination and algebraic thinking?
2. How can the units coordination levels of fifth-grade students interact with their progress in algebraic thinking during a learning sequence that focuses on the generalization of the relationships between unknown quantities and between variables?
    2.1. How can the units coordination levels of fifth-grade students interact with their progress in algebraic thinking regarding *the relationships between unknown quantities*?
    2.2. How can the units coordination levels of fifth-grade students interact with their progress in algebraic thinking regarding *the functional relationships between variables*?

Departing from the aim of the study, this chapter presents the methodology including the design of the study, participants, data collection, data analysis, and trustworthiness.

3.1 Design of the Study

The starting point of this study was the controversial findings of two different lines of research: one is units coordination and the other is early algebra. On the one hand, researchers have asserted that algebraic reasoning, such as equation writing (e.g., Hackenberg, 2013, Zwanch, 2022b) and generalization (e.g., Zwanch, 2022a), necessitates specific mental operations that pertain to a specific level of unit coordination or the possession of some multiplicative concepts. On the other hand, researchers observed that even in the early elementary grades, when it is highly unlikely that students have developed the necessary multiplicative concepts (Clark & Kamii, 1996; Kosko, 2019), they could use algebraic notations and generalize the relationships between quantities through appropriate instructional designs (e.g., Blanton et al., 2019; Carraher et al., 2006). In this regard, the starting point of the study was to fill in some gaps and specify the points in explaining the interaction between students' algebraic reasoning and units coordination. Bringing together research on early algebra and unit coordination could validate or improve current perspectives on algebraic reasoning and student learning by specifying the issues involved.

In line with this, the study aimed to create a new perspective by filling in the empty parts of the claim such as: "A learning process including the aspects of […] would develop the students' algebraic thinking when they have characteristics of […]" or "Students who have […] would progress in algebraic thinking when the learning activities start with/involve […]". This required exploring the characteristics of a learning environment that supports the algebraic thinking of students who demonstrate distinct mental operations in terms of units coordination. Additionally, the study also aimed to provide a new perspective on the theory of the interaction between algebraic thinking and units coordination. To achieve the aims of the study, a theory-oriented, interventionist, and iterative approach was necessary, indicating design-based research (Cobb et al., 2003). Therefore, this was primarily a validation study, a form of design-based research, that aims to develop or validate theories through educational interventions based on certain principles (Plomp, 2013). Concerning this, the current study, adopting a design-based research approach, aimed to design intervention by employing the relevant theoretical frameworks and to provide new insights into these frameworks through the analysis of the intervention. The following paragraphs describe the main principles of design-based research and the adaptation of the current study to this approach, highlighting the rationale for the actions taken.

Developing theories by providing new perspectives is an important characteristic of design-based research (Cobb et al., 2003) which is one goal of this study. The second characteristic highlights its interventionist nature. Learning ecologies include many factors affecting the learning process such as mathematical tasks, student characteristics, and classroom discourse. This creates a complex “interacting system” (p. 9) and requires an engineering issue in a real context. This simultaneous interaction between developing a learning ecology and understanding the interrelationships between different factors through experimentation allows the researchers to understand and improve the characteristics of a new design.

Since one of the goals of this study was to explore the characteristics of a learning process aimed at the development of algebraic thinking, this could be achieved by designing a hypothetical learning trajectory as an intervention plan. A hypothetical learning trajectory (HLT) refers to the teachers’ conjectures and expectations about a learning path and how the learning might proceed under certain circumstances and for certain goals (Simon, 1995). Mathematics educators base their instructional design decisions on prior conjectures before implementation, which is why it is called hypothetical. Simon (1995) stated three key components of HLTs: *the learning goals*, *activities to be used in class*, and *the statements hypothetically talking about the learning process*. Therefore, it was aimed to specify the learning goals for developing the students’ algebraic thinking, preparing mathematical tasks parallel to the learning goals, and writing hypotheses about the student learning process by being informed of their characteristics and the learning goals (Simon & Tzur, 2004).

The development of an HLT and intervention is possible through the reflective and prospective nature of design-based research which is the third feature (Bakker, 2018; Cobb et al., 2003). During the prospective phase, mathematics educators formulate conjectures about the learning process through thought experiments (Gravemeijer & Cobb, 2013). The ideas are presented as hypothetical cases to be designed and developed. Therefore, researchers must generate conjectures about the specific learning environment as informed by the respective theories (Plomp, 2013). In the reflective section, mathematics educators compare the conjectures with actual learning by experimenting with the developed product (Gravemeijer & Cobb, 2013). Invalidated conjectures are adjusted, or new ones are developed and tested again.

Assessments of student learning inform all components of HLTs through an ongoing teacher decision-making process (Simon, 1995). These components also provide mathematics educators with information about the learning process through ongoing intervention and assessments. Therefore, interventions and designs are informed by and inform the respective theories (Plomp, 2013). This evolving cyclical process, which is another characteristic of design-based research continues to bring a developed and working product into practice (Cobb et al., 2003). This iterative nature allows mathematics educators to test and revise conjectures or generate new ones to test in the next cycle (Bakker, 2018).

In addition to the main characteristics, in each cycle of design-based research, there are three main phases: *preparation and design phase*, *conducting teaching experiments*, and *retrospective analysis* (Bakker, 2018). All phases of this study were completed to explore the characteristics of an HLT that develops students' algebraic thinking. Therefore, the following sections explain the details of the methodology adopted in this study including the selection of participants, the characteristics of the context, the design process, the instruments used in data collection, and data analysis.

3.2 Research Team, Participants, and the Context

In this design-based research, the research team consisted of two main researchers and other mathematics educators who were occasionally consulted. The first researcher, who is also the author of this dissertation, has ten years of experience as a mathematics teacher in a government school. The author's role also included implementing the teaching experiments for students whom she had not previously taught. Therefore, she will be referred to interchangeably as a teacher and a teacher-researcher throughout the text. The other member of the research team was an Associate Professor of Mathematics Education who specialized in units coordination. During the design process, the team consulted multiple times with other mathematics educators to ensure that the characteristics of the HLT were appropriate.

The sampling process of the study involved several steps: the selection of the grade, the school, and the students, respectively. The selection of the grade level was purposeful and theory-driven, which addressed the research aim (Creswell, 2009). In line with the purpose of the study, to contribute to the literature on early algebra and better examine the progress of algebraic thinking along the learning trajectory, the research team has decided to work with fifth graders who were new to middle school and had not yet received formal algebra instruction according to the Turkish Mathematics Curriculum (MoNE, 2018). According to the Turkish Mathematics Curriculum (MoNE, 2018), fifth graders, who are in their first year of middle school in Turkiye's education system. The strand of algebra is first introduced in sixth grade. According to the current mathematics curriculum, a student who has completed fourth grade is expected to know arithmetic operations (i.e., addition, subtraction, multiplication, and division), the meaning of equality, and how to present data in tables (MoNE, 2018). Considering the potential mathematical tasks to be included in the HLT, including contextual situations with functional relationships as presented in the literature for primary school students' algebraic thinking and reasoning (e.g., Blanton et al., 2011; Blanton et al., 2019), the research team objected to work with the students who can think multiplicatively, understand a mathematical expression including equal sign, and read and construct a table.

Another reason for choosing fifth-grade students, rather than those in lower grades, is to work with students who have achieved at least an MC1 level, which is the first stage of multiplicative concepts in terms of units coordination. MC1 level would enable students to accomplish multiplicative tasks (e.g., Hackenberg, 2013; Hackenberg et al., 2021), which are commonly used in tasks including functional relationships, which were initially aimed to give place in the learning trajectory. In a research surveying unit coordination of middle school students in Turkiye (Acar & Sevinc, 2021), more than half of the fifth graders were found at the MC1 level. Furthermore, researchers estimated that half of the fifth graders started to reason multiplicatively reflecting their construction of composite units (e.g., Kosko, 2019; Steffe, 2024). In sum, the units coordination level was selected to attain these research goals based on the literature and theory by using a purposeful sampling strategy (Creswell, 2009).

The school of the students was chosen through convenience sampling by considering the geographical accessibility, and the willingness of both the students and their parents to participate in the study (Miles et al., 2014). The researcher had an advantage in obtaining permission and reaching out to the school principal and parents since her workplace was the school in question. The school was a small middle school situated in a neighborhood with medium socio-economic status in İstanbul. It had one fifth-grade class consisting of 20 students.

Lastly, the students' selection among the fifth graders in this middle school was purposeful, considering various factors to obtain relevant data for research purposes (Creswell, 2009) such as their units coordination level, self-expression skills, and achievement levels. It was critical to have students who could easily share their thinking processes and provide significant data during interviews and teaching experiments as this study necessitates qualitative data collection and analysis. To obtain extensive data, it was also necessary to involve students who were not low in general academic skills, including reading comprehension, numeracy, and interpreting tables and figures. Following consultation with the fifth graders' mathematics teachers, the researcher requested a selection of students who had achieved high or moderate scores on mathematics exams and the teacher's evaluation in the classroom in terms of their self-expression skills. The mathematics teacher provided six students who scored between 60-100 on mathematics exams.

To investigate the fifth-grade students' progress in algebraic reasoning based on their units coordination, the study primarily targeted students with varying levels of units coordination. Specifically, the research team aimed to implement teaching experiments by pairing the students who shared similar cognitive operations and generating two different groups demonstrating distinct units coordination levels. Therefore, the selection of students was based on their units coordination levels during data collection. Four students were selected and grouped into two based on their units coordination, as indicated by the initial data analysis in the Findings section. It is important to note that the interview for units coordination was used to decide on the students, but it primarily demonstrated the students' mental operations in interpreting the main data of this study. The findings section presents a detailed analysis of the students' unit coordination levels; still it is important here to pinpoint the units coordination of participating students to distinguish their similarities and differences.

In conclusion, there were four fifth-grade students (two males and two females, 11-year-olds) each two having a different level of units coordination (see Table 3.1). The names of the students were pseudonyms.

Table 3.1 Study Participants

	Roy	Belle	Sara	Luke
Units Coordination Levels	MC2	MC2	MC1	MC1
General Academic Performance Score	95.1	87.5	79.7	81.2
Mathematics Score	97.8	95.6	67.9	73.4
Gender	Male	Female	Female	Male

3.3 Ethical Considerations

Before the data collection process, the researcher provided the research ethics committee approval taken from the Middle East Technical University (see Appendix

A), received permission from the school principal and the İstanbul provincial directorate for national education (see Appendix B) to collect data from students in the school and to implement teaching experiments. Then, the parents and students received information about the study. I, as the researcher-teacher, informed them that the issue of confidentiality would be addressed in this study and that they would not be harmed during the study. The students voluntarily participated in the study and their parents provided signed permission forms before the research began (Appendix C).

3.4 Design Process

This study consisted of three phases following a design-based research approach. The first phase involved preparing the HLT, which included how conjectures and learning goals were generated, the influence of local instruction theories, the role of the research team, and the initial assessments of participants as another influential factor. The second phase explained how teaching experiments were conducted including the role of researchers, a description of learning ecology, and the elements of theoretical perspective in teaching and learning. Lastly, the last phase included how the retrospective analysis of teaching experiments was conducted. Therefore, in this section, these three phases and the HLT as the end product were described.

3.4.1 Preliminary research phase: Before teaching experiments

The initial phase of design-based research begins with a literature review that provides insight into the specific educational problem and potential solutions (Nieven & Folmer, 2013). This review also helps generate the initial design principles for enhancing the situation. In this phase, the characteristics of the design are established through an ongoing literature review, the analysis of context, and the adoption of a conceptual and theoretical framework (Plomp, 2013). These characteristics, which were derived from various literature sources, generate the starting points of a design. The design is developed further through thought experiments on anticipated student performances and interactions (Gravemeijer, 2004). In this process, researchers must consider several aspects to ensure an effective design and analysis process. These aspects include the cognitive characteristics of the design, the type of interaction between students and teachers, and the students' access to the resources of the design (Collins et al., 2004). These considerations allow the researchers to develop a local instruction theory by expanding and adjusting current perspectives to the hypothetical learning process (Gravemeijer & Cobb, 2013).

In this study, after specifying the research problem and determining the grade level of students, an extensive exploration of literature and theoretical perspectives continued. This review provided a significant perspective on learning algebraic thinking and reasoning before formal algebra instruction (e.g., Blanton et al., 2011; Blanton et al., 2019; Carraher et al., 2006), students' mental operations in terms of units coordination (e.g., Hackenberg et al., 2021; Steffe, 1992) and theories for learning and teaching (Cobb et al., 1992; Simon, 1995; von Glasersfeld, 1995;). Domain-specific perspectives about teaching and learning algebra shaped the conjectures about what students can do and which algebraic tasks and processes can be effective in the development of students' algebraic thinking. Additionally, the previous research about units coordination provided insight into the students' mental operations and helped to generate new conjectures regarding the students' multiplicative concepts and potential performances in multiplicative tasks. Lastly, general instructional and epistemological theories such as the Realistic Mathematics Education, and socio-constructivism approach helped to determine task characteristics, the teacher's role, the type of questioning of the teacher during teaching experiments, and socio-mathematical norms. The following sections outline the principles and characteristics of the HLT design, including the theoretical and conceptual perspectives, characteristics of mathematical tasks, the teacher's role, and socio-mathematical norms.

3.4.1.1 Theoretical perspectives on algebraic thinking in the HLT

This section explains the interpretation of students' development of algebraic thinking through domain-specific perspectives at the cognitive level, as a part of local instruction theory. In this way, the goals of the HLT are generated by starting with a general goal and then becoming more specific. The primary objective of the entire learning sequence is to develop students' algebraic thinking. To achieve this, the first step was to examine how algebraic thinking is defined and approached in literature. We concentrated on the widely used descriptions of algebraic thinking (Kieran, 1989; Radford, 2014) and algebraic reasoning (Kaput, 1999, 2008) to identify the elements of learning tasks in the HLT. Descriptions of algebraic thinking and reasoning share common components and are similar in form. For example, Radford (2014) identified the dimensions of algebraic thinking as indeterminacy, denotation, and analyticity. Similarly, Kieran (2022) framed early algebraic thinking into three dimensions: analytical thinking, structural thinking, and functional thinking. Kieran (2022) describes analytical thinking as mental operations involving indeterminate quantities (indeterminacy) and denoting relationships in different ways (denotation), which aligns with Radford's (2014) dimensions.

Furthermore, Kaput (2008) presented two core aspects of algebraic reasoning that should be incorporated in teaching algebra: "a) Algebra as systematically symbolizing generalizations of regularities and constraints b) Algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems" (p. 11). He also indicated three strands embodying these core aspects: the study of structures in arithmetic and quantitative reasoning, the study of functions, and modeling applications. These forms also incorporate the dimensions of algebraic thinking that are specified by Kieran (2022) and Radford (2014) such as manipulation of formalism including denotation and analytical thinking, study of structures including structural thinking, and study of functions including functional thinking. Therefore, these components have become the study's starting point in terms of comprehension of the main variable of the design (see Table 3.2).

Table 3.2 Starting Point: The Components of Algebraic Thinking and Reasoning

	Algebraic Reasoning	Algebraic Thinking
Thinking processes	Generalization of Regularities	Analytical Thinking Functional thinking Structural Thinking
Observable behavior	Generalizations expressed verbally or by the conventional symbol system	The denotation/expression of the operations with indeterminate quantities

In this way, more specific learning goals emerged at the cognitive level through the interpretation of algebraic thinking and reasoning (see Table 3.3).

Table 3.3 The Learning Goals in the HLT Presented in the Order of Generation

Learning Goals	Components
Developing the students’ algebraic thinking Using indeterminate quantities in expressing the variables, relationships, and regularities	Analytical Thinking
Expressing the variables, relationships, and regularities using symbols	Analytical Thinking Denotation Indeterminacy Generalization
Expressing the variables, relationships, and regularities in different forms (e.g., writing multiple equations, constructing tables and drawings)	Structural Thinking
Identifying functional relationships	Functional Thinking Generalization
Expressing functional relationships using symbols	Functional Thinking Generalization Analytical Thinking

After generating the initial learning goals, I have decided to give place for mathematical activities including the aspects of generalization and symbolic manipulation of generalizations with the dimensions of algebraic thinking such as analytical, structural, and functional thinking. To determine the order of tasks and objectives for studying with students who have not received formal algebra instruction, I conducted a literature review on early algebra and its introduction in primary school years.

As a starting point, Blanton et al. (2011) gave us a clear and robust perspective on teaching algebra through their objectives, sample mathematical tasks, and big ideas. They described five big ideas that summarize fundamental concepts and processes in early algebra teaching and learning which are commonly adopted in lots of research about algebraic reasoning: a) Arithmetic context for algebraic thinking, b) Equivalence of two quantities, c) Variable understanding, d) Quantitative reasoning for generalizations, and d) Functional thinking as a gateway to algebra. In addition to highlighting the structures and relationships in an arithmetic context, they remarked on some concepts that are important in algebra such as variables and equivalence. Hence, they start with the comparison of unknown quantities using variables and comparison symbols. They also highlighted the processes of quantitative reasoning (e.g., Ellis, 2011; Smith & Thompson, 2008), generalization (Kaput, 2008; Kieran, 2007; MacGregor, 2004), and functional thinking (Carraher et al., 2008) like many researchers did. Therefore, I have decided to start the lessons with comparisons of unknown quantities so that the students could have a chance to think over unknown quantities and use comparison symbols (i.e., <, >, =) with unknowns as a new thinking form and language. In this way, it was aimed to give place for the meanings of comparison symbols, the meaning of equivalence, and comparing different quantities as in the framework in Blanton et al. (2011).

Furthermore, Carraher et al. (2008) emphasized a learning process critical for early algebra within a function context. In this learning, some key mathematical processes are applied such as using indeterminate quantities in contextual problems, interpreting the data in function tables, creating conjectures, making generalizations, and representing the relationships with symbols. Functional thinking is regarded as a gateway to algebra because it involves handling many key algebraic concepts and processes such as “generalizing relationships between covarying quantities; representing and justifying these relationships in multiple ways using natural language, variable notation, tables, and graphs; and reasoning fluently with these generalized representations in order to understand and predict functional behavior” (Blanton, Brizuela et. al, 2015, p.512). Kaput (1999) remarked on the importance of functions in many ways of mathematical thinking through the ideas of causality, covariation, and rate of change as conceptual roots of algebraic reasoning. Hence, he suggests ways of representing functional thinking in elementary years through the problems in which the contextualized quantities change over time and uses graphs and tables (Kaput, 1999). Multiple representations such as drawings, tables, charts and graphs, and age-appropriate tasks in rich contexts can help the students to make sense of the variables and the functional relationships and to make generalizations as the initial steps into the algebraic concepts (Carraher et al., 2008; Blanton, Stephens, et al., 2015). From this point of view, functional thinking was prioritized with rich contextual problems in the HLT because it entails many key algebraic processes such as variable understanding, generalizing the relationship between quantities, and using multiple representations such as tables and equations to express the relationships.

In conclusion, the design approach in this study followed Kaput's (2008) description of algebraic reasoning and Kieran's (2022) and Radford's (2014) formulation of algebraic thinking as the main theoretical framework (see Table 3.2). Thus, the design of the HLT focused on mathematical processes such as functional thinking and quantitative reasoning, as well as symbolic generalizations. In addition, the ideas for the algebra learning process supported by Carraher et al. (2008) such as using indeterminate quantities in contextual problems, interpreting the data in function tables, creating conjectures, making generalizations, and representing the relationships with symbols and five big ideas presented by Blanton et al. (2011) shaped the initial version of the HLT. In sum, the HLT was based on a framework with four essential mathematical processes: quantitative reasoning, multiplicative reasoning, generalization and functional relationship, and three key objectives: use of variables, writing equations, and understanding functions as shown in Figure 3.1.

Mathematical Processes or Components in the HLT	**Intended Learner Outcomes**
• Quantitative Reasoning • Multiplicative Reasoning • Generalization • Functional Relationships	• Use of variables • Writing equations • Functional thinking

Figure 3.1. The framework of the HLT

3.4.1.2 Theoretical worldview about teaching and learning in the HLT

In the design and ordering of the learning activities in the HLT, a multifaceted approach to teaching and learning mathematics was employed. The aim was to follow a comprehensive approach that draws on the strengths of different theories and perspectives. Therefore, the HLT integrated various worldviews in the development of mathematics education by selectively incorporating certain aspects of each perspective while dismissing others, as suggested by Simon (2009). With respect to this, we, as a research team, adopted principles from the Realistic Mathematics Education (RME) theory, incorporated the worldviews of researchers in the field of early algebra, and included the views of the emergent perspective of Cobb and Yackel (1996).

The RME distinguishes between two types of mathematization: horizontal mathematization and vertical mathematization as crucial processes (van den Heuvel-Panhuizen, 2000). Horizontal mathematization involves transforming concepts from their real-world representation into mathematical symbols, while vertical mathematization involves transforming concepts within the mathematical symbolic system. In the context of developing algebraic thinking, we aimed to incorporate real-life representations of concepts, such as comparison situations and equivalence, and transform them into mathematical expressions. For instance, the objective was to have students mathematize the significant elements in contextual situations, such as variables and relationships, through a process of horizontal mathematization. This entailed identifying them in realistic situations and expressing the phenomena in mathematical ways. In the process of vertical mathematization, the objective was to have students work within mathematical expressions including verbal expressions of generalizations, the use of tables for functional relationships, and the use of alphanumeric symbols in equations.

Initially, the learning activities in the HLT were informed by the reality principle of the RME. The reality principle involves incorporating real-life contexts into mathematical abstractions, allowing students to imagine problem situations (van den Heuvel-Panhuizen, 2000). This approach aims to make the problem situations more tangible for students. It is important to ensure that students can imagine the mathematical situation, as this makes it more realistic for them. Using rich contextual situations to develop understanding and make sense for children is also a suggestion of researchers in the field of early algebra (e.g., Blanton et al., 2011; Carraher et al., 2008; Kaput, 1999).

Another fundamental principle that has been incorporated into the design of the HLT is the level principle. This principle facilitates the transition from informal to more formal mathematical discourse (van den Heuvel-Panhuizen, 2000). This also corresponds to the abstraction process in the constructivist theory of mathematical knowledge (von Glasersfeld, 1996). At the informal level, students' learning can be supported by using models and objects and by allowing them to move between different models (van den Heuvel-Panhuizen, 2000). In the early parts of the HLT, the aim was to use figures from real-life contexts for comparisons of unknown quantities before representing the relationships and concepts more formally and abstractly. Additionally, the objective was to utilize tables of values for each variable, develop a verbal generalization for the relationship between variables, and then express this generalization through symbols. This approach would facilitate the gradual increase in the level of formalization by providing multiple representations of generalizations, as suggested by Kaput (1991).

The final principle embraced by the HLT was the intertwinement principle of the RME, which places a strong emphasis on the connection between the various content areas of mathematics (van den Heuvel-Panhuizen, 2000). This principle also corresponds to the nature of algebra and algebraic thinking. For example, understanding the relationship between operations and the properties of operations is an aspect of algebra through generalization and structural thinking (Carpenter et al., 2005; Usiskin, 1988). In this regard, the aim was to guide students to express the relationships using symbols in different ways so that they could use the relationships between operations and arithmetic properties. Therefore, the HLT included many aspects of mathematics through the nature of algebra such as the relationships between operations, patterns, and multiplicative reasoning.

3.4.1.3 Hypothetical learning trajectory (HLT)

Based on the theoretical views about teaching mathematics and algebra, we, as the research team, created an HLT including six episodes (see Appendix D). In light of the learning goals (see Table 3.3), we specified learning outcomes for each episode and adapted and designed instructional activities. Each episode included contextual tasks for student interpretation and progression. The tasks were adapted from various studies that investigated and described students' early algebra instruction (e.g., Blanton, Brizuela, et al. 2015; Blanton et al., 2011; Carraher et al., 2006). The tasks, their descriptions, the mathematical ideas, and key cognitive operations are presented in the following paragraphs.

Before introducing contextual situations involving functional relationships, the goal was for the student to gain experience in writing mathematical sentences in various contexts using variables and symbols. We thought that fifth graders may lack experience in writing mathematical sentences to represent relationships beyond solving given arithmetic operations and missing value problems. Therefore, the first and second episodes aimed to support students in writing mathematical equations using variable notations for unknown quantities.

Episode 1. Episode 1 started with the principle that algebraic thinking requires recognizing unknown quantities/variables, comparing them, and using symbols. The comparison tasks were regarded as important to reflect how many ways two quantities can relate to each other under the big idea, *using quantitative reasoning to generalize relationships* (Blanton et al., 2011). Therefore, in the first episode, the learning outcomes were to compare unknown quantities represented in figures and to express the comparison in different ways, such as using verbal expressions, hypothetical values, and symbols.

For this aim, three tasks asked the students to interpret different ways of comparing two quantities and express the unknown quantities and the comparison of them using variable notation and symbols. The first two tasks in Episode 1 ask students to compare two unknown quantities, such as two pencils of different unknown lengths and two pencils of the same length. These activities were adapted from Blanton et al. (2011). In the third task, students continue to compare unknown quantities in different contexts, such as un/balanced scales, and jars of sugars (see Figure 3.2). By giving place for both equal and unequal situations it was also aimed to remark on the meaning of equal sign as a comparison symbol.

Figure 3.2. Sample figures from the comparison activities in Episode 1

In all the activities, scaffolding questions were added assisting students to move to the next steps and to comprehend the main idea. For example, to help students understand the concept of variables and make generalizations, questions were added that asked them to estimate the lengths of pencils and record their guesses in a table. Furthermore, they were asked to use literal symbols to represent the lengths or weights of the objects in each activity to help them become more comfortable working with literal symbols. Table 3.4 represents the key components of Episode 1 which is the first part of the HLT.

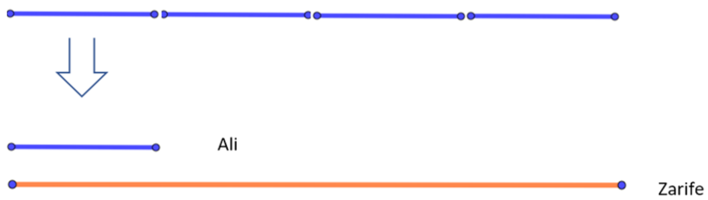
Table 3.4 The Structure of Episode 1 in the HLT

EPISODE 1: Comparison of unknown quantities using equality and inequality
Learning Outcomes
• Express the comparison of unknown quantities verbally (e.g., it is longer/heavier/older than the other) • Attain hypothetical values for unknown quantities by using tables. • Assing letters/symbols to represent an unknown quantity. • Use letters/symbols to represent the comparison between unknown quantities using equality and inequality. • Understand the relational meaning of the equal sign.
Tasks and Their Structures
Task 1: Expressing the multiplicative relationship between two unknowns by using symbolic expressions **Task 2:** Expressing the additive relationship between three unknowns by using symbolic expressions • Including contextual models and scenarios • Including a comparison of two or more unknown quantities • Allowing using tables of hypothetical values for the unknowns • Including quantitative reasoning through the comparison of different quantities • Generalizing from hypothetical values to symbols
Conjectures
a) MC1 and MC2 students would compare the unknown quantities and express them verbally at the beginning of tasks. b) MC1 and MC2 students would attain values for each unknown instead of using literal symbols

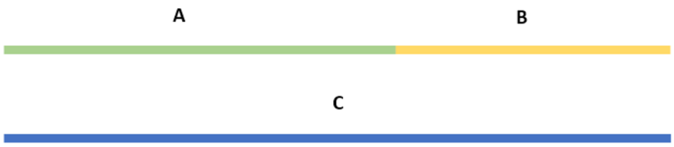
Table 3.4 (Continued)
c) MC1 students would not understand how they represent the relationship using symbols. d) Roy or both MC2 students would use the assigned letters to represent the comparison with symbols towards the end of the episode. e) MC1 students would continue to assign values to unknown quantities instead of using symbols. f) MC1 and MC2 students would have difficulty in representing the comparison between three unknowns on un/balanced scales
Instructional Moves Aligning with the Conjectures
• **Conjectures b – c – e** -Assign multiple values for each unknown on a table and discuss the generalized comparison -Discuss the comparison symbols in expressing the numerical situations in mathematical language (e.g., =, <, >) -Direct the student to use letters for unknowns by saying "Let the length of yellow pencil '*a*' and the length of orange pencil '*b*'." • **Conjecture f** -Use a table to assign values to three unknowns on an un/balanced scale and discuss how to represent two unknown weights on one side in comparison to the other on the other side.

Episode 2. Episode 2 involved the principle that algebraic reasoning involves recognizing the multiplicative and additive relationships between unknown quantities and representing them using symbols. The goal of the second episode was to support students in identifying the multiplicative and additive relationships between unknowns represented in models, representing the relationships by attaining numerical values and expressing the relationships using symbols such as equal signs and letters. In the previous episode, I conjectured that students would learn how to compare quantities and represent relationships using mathematical equations. In this lesson, they encounter multiple contexts that involve both multiplicative and additive relationships.

To achieve the goals, two activities were designed that incorporate model representations, aligning with the reality principle of the RME. The first activity involved a multiplicative relationship, while the second involved an additive relationship (see Figure 3.3). The second activity involving an additive relationship was adapted from Blanton et al. (2011, p. 44). It is introduced for developing students' quantitative reasoning and generalization skills as one dimension of algebraic reasoning. Blanton, Brizuela, et al. (2015) indicated that the student can represent a function such as $y = mx$ easier as compared to a functional relationship such as $y = x + b$. Considering that students would express a multiplicative relationship such as $y = 4x$ easier than expressing a relationship such as $y = x + a$, the activity involving a multiplicative relationship took first place in this Episode.



Multiplicative and additive relationship between unknown quantities



Additive Relationship between unknown quantities

Figure 3.3. The models for multiplicative and additive relationships

Using variables by operating with them as mathematical objects and by representing functional relationships indicates a high level of understanding of variables (Blanton et al., 2017). Hence, this task serves for the students to develop a sophisticated understanding of variables which is an important step before using them in representing functional relationships. Carraher et al. (2006) indicated that "the children's initial intuitions about order, change, and equality first arise in additive situations" (p. 94). Hence, before introducing functional relationships in multiple contexts, these two activities would develop an understanding of order, equality, and symbolic notations. Furthermore, the order of steps in each activity corresponds to the level principle of the RME through the use of models, from models to numerical values and from numerical values to symbolic representations. Therefore, the aim was to highlight multiple ways of expression of a multiplicative and additive relationship in an increasing abstraction. Table 3.5 represents the key components of Episode 2 which is the second part of the HLT.

Table 3.5 The Structure of Episode 2

EPISODE 2: Comparison of unknown quantities with additive and multiplicative relationships using equality
Learning Outcomes
• Recognize multiplicative and additive relationships between unknown quantities through models. • Create different scenarios by attaining hypothetical values to unknown quantities having multiplicative or additive relationships by using tables. • Assing letters/symbols to represent an unknown quantity. • Express the multiplicative and additive relationship between unknown quantities verbally. • Express the multiplicative and additive relationship between unknown quantities using symbols. • Show the relational meaning of the equal sign.
Tasks and Their Structures
Task 1: Expressing the multiplicative relationship between two unknowns by using symbolic expressions. **Task 2:** Expressing the additive relationship between three unknowns by using symbolic expressions • Including contextual models and scenarios (Reality and level principle) • Including a comparison of two or more unknown quantities • Allowing using tables of hypothetical values for the unknowns • Including quantitative reasoning through the multiplicative and additive relationships (Intertwinement principle) • Generalizing from hypothetical values to symbols (Level principle)
Conjectures
a) MC1 and MC2 students would express the additive and multiplicative relationships verbally

Table 3.5 (Continued)
b) MC1 students use the letters to represent the lengths, but they would not write the equations c) MC1 students would assign values for the length of ropes such as 1 and 4 or 2 and 8. d) MC1 students would give numerical values to the unknowns and do operations, but they would not represent the additive and multiplicative relationship using symbolic expressions e) MC2 students would express the additive and multiplicative relationship verbally and symbolically by using letters, operations, and equality. f) MC2 students would write different algebraic expressions representing the same multiplicative relationship (e.g., 4 x a: it is the longest rope; 4 x r = longer rope; r + r + r + r = longer rope; 4 x r = s; s / 4 = r).
Instructional Moves Aligning with the Conjectures
• **Conjectures a – b – c – d:** -Ask them to use letters and describe the same thing by using symbols - Ask and discuss "Is there another way to represent the relationship (addition /division/ multiplication/subtraction)?" - Emphasize that we do not know the lengths. Ask and discuss the relationship between the assigned numbers. For example, ask: "Which operation can you do to find one?" • **Conjecture e – f:** -Ask and discuss "How differently can you demonstrate the same relationship?"

Episodes 3-6. Since Episode 3, the design of the HLT followed the principle that developing algebraic thinking through the dimension of functional thinking requires recognizing the variables in contextual problems, the functional relationship between variables, and representing the relationship by using symbols. Regarding this, Episodes 3 and 4 aimed to develop students' functional thinking through various contexts with functional relationships in the form of $y = ax$ while Episodes 5 and 6 included functional relationships in the form of $y = ax + b$. Throughout these episodes, students complete some tasks in each activity such as identifying problem variables, representing the possible scenarios in function tables, interpreting and generalizing the data in tables, and expressing the functional relationship between the variables.

Episode 3 involved two activities including contextual problems which were taken from Blanton et al. (2011) and Blanton, Brizuela, et al. (2015). The problems included simple and discrete variables. The first activity involved a functional relationship between two variables: one is the number of chairs, and the other is the number of legs (Blanton et al., 2011). To enrich the contextual situation and to strengthen the students' understanding there were similar tasks in the following sections of the episode (see Figure 3.4). The students are asked to find similar relationships between the number of dogs and legs; the number of people and their ears; and the number of people and their noses (Blanton, Brizuela, et al. 2015).

The Chair and Legs Problem: Suppose that you have some chairs, and each chair has four legs. How would you describe the relationship between the number of chairs and the corresponding number of chair legs? (Blanton et al., 2011, p. 9).

- **Dogs and Legs:** How can you express the relationship between the number of dogs and the total number of legs on the dogs? (Blanton, Brizuela, et al. 2015)
- **People and Ears:** How can you express the relationship between the number of people and the total number of ears on the people? (Blanton, Brizuela, et al. 2015).
- **People and Noses:** How can you express the relationship between the number of people and the total number of noses on the people? (Blanton, Brizuela, et al. 2015).

Figure 3.4. Sample Problems in Episode 3

It was assumed that students could identify relationships in different ways through these tasks. One possibility is by observing a pattern in only one column of the function table through recursive thinking (Blanton, Brizuela, et al. 2015; Blanton et al., 2011). For instance, students may focus on the pattern in the column of the number of legs, noting that it increases by four. However, the learning trajectory aims to develop functional thinking through covariational and correspondence approaches. It was assumed that this development would occur at the end of the episodes.

Episode 4 continued the learning process from Episode 3 by providing additional contexts for the same form of the functional relationship ($y = ax$). It began with the Saving Money Problem which prompts students to express the relationship between the number of weeks and the amount of money saved when saving the same amount each week. This problem is important as it includes different continuous variables such as time and amount of money. After solving the initial problem and representing the relationship, students were presented with additional situations in a table format, such as saving varying amounts of money each month. The second problem was the Pool Problem which introduced a different context involving the relationship between time and the amount of water in a pool. The steps for this problem were the same as the previous one, including identifying variables, constructing a table, and expressing the relationship verbally and symbolically. In the same context at the end of the activity, there was another situation with a new table of data, similar to the first activity. These generalization practices were important for developing their covariational and correspondence thinking as ways of functional thinking.

In the final activity of Episode 4, students were required to determine the relationship between the number of tables and the number of people seated based on a given seating arrangement in The Birthday Party 1 Problem (Blanton, Brizuela, et al. 2015). It was anticipated that students who have completed the previous tasks would be able to express the functional relationship and provide additional context. To enhance students' understanding, it may be helpful to generalize functional relationships across different contextual problems by presenting them sequentially and highlighting similarities (Blanton et al., 2011; Carraher et al., 2008). For example, one could construct a connection between saving money and filling the pool contexts through the independent and dependent variables and shared linearity concept. This approach can facilitate comprehension by drawing connections between seemingly disparate contexts. These connections can assist students in transitioning from context-related situations to more general and formal ways of functional thinking while moving from informal to formal mathematics under the level principle in the RME (van den Heuvel-Panhuizen, 2000). Table 3.6 represents the key components of Episodes 3-4 which is the third part of the HLT.

Table 3.6 The Structure of Episodes 3 and 4 in the HLT

EPISODES 3-4: Representing functional relationships between variables in the form of $y = ax$

Learning Outcomes

- Identify the problem variables.
- Construct a function table.
- Identify and generalize the functional relationship in the table of data.
- Understand and express the functional relationship between two variables through recursive, covariational, and correspondence approach
- Represent the functional relationship using equation and variable notation.
- Connection between multiple representations of functional relationship (verbal, table and symbolic)

Tasks and Their Structures

3-1) The Chair and Legs Problem: The relationship between the number of chairs and the number of legs ($y = 4x$)

3-2) The contexts like the first task: The number of dogs/people/ and the number of legs/ ears /noses (e.g., $y = 2x$ and $y = x$)

4-1) The Saving Money Problem: The relationship between time and the total amount of money ($y = 5x$)

4-2) The Pool Problem: The relationship between the amount of water in a pool and the elapsed time ($y = 2x$)

4-3) The Birthday Party 1 Problem: The relationship between the number of tables and the number of people who are seated ($y = 2x$)

- Including contextual problems (Reality principle)
- Relationship between discrete quantities (Episode 3)
- Relationship between continuous quantities (Episode 4)
- Using function tables to represent the data before generalization (Level principle)

Conjectures

a) MC2 students would calculate any corresponding value in function tables.

Table 3.6 (Continued)
b) MC1 students would not calculate the larger values in function tables because of recursive thinking.
c) MC2 students would indicate the functional relationship verbally by using indeterminate quantities and write equations by using symbols.
d) MC1 students would not indicate the functional relationship by using indeterminate quantities and letters.
e) MC1 students would have difficulty understanding the problem about the relationship between the number of tables and the number of people seated around the tables
Instructional Moves Aligning with the Conjectures
• **Conjecture a – c:**
-Ask them to use different strategies and explain the relationship by using different expressions and equations (Developing structural thinking)
• **Conjecture b – d:**
-Fill the table together on the board and ask about the relationship between two variables. Indicate the names of each variable in discussing each case. Let the students interpret the change in both variables simultaneously.
• **Conjecture e:**
-Ask them to draw models to represent each situation. Show one table, two tables, and three tables on the board respectively, and ask them to interpret the situation.

In Episodes 5 and 6, the aim was to present functional relationships in the form of $y = ax + b$ through contextual problems. Episode 5 included two problems with similar contexts to the problems in the previous episode. The first problem of Episode 5 sought the relationship between the number of people and the number of tables, which was also the focus of the last activity in Episode 4 (The Birthday Party 2 Problem). In this scenario, there is a constant number representing the number of individuals seated at the table ends. Likewise, in the second problem of Episode 5, the Bank Account Problem, the students must focus on saving money context as they did in the previous episode (see Figure 3.5). In this scenario, an initial sum of money is given, and it increases each month at a constant rate.

The Bank Account Problem (Episode 5): Ali has 30 liras in his bank account. Ali decides to deposit 15 liras into his account every month.

a) Fill in the table below according to the information above.

Elapsed Time (the number of months)	1	2	3	4	5	30
The amount of money saved in the account (Liras)						

b) Let's call the number of months (elapsed time) a. How would you express the total amount of money saved in the account?

c) Can you write an equation that shows the relationship between the total amount of money saved in the account and the elapsed time (number of months)?

The People and Hats Problem (Episode 6): Think about a hat with a height of 20 cm. How would you describe the relationship between a person's height without a hat and with a hat? (Carraher et al., 2006)

Figure 3.5. Sample problems from Episodes 5 and 6

Episode 6 continues with three different problems for the same form of functional relationships, $y = ax + b$. The first one, the People and Hats Problem, was about the relationship between a person's height without a hat and with a hat with 20 cm height, which was taken from the study of Carraher et al. (2006) (see Figure 3.5). The second problem, Credit Card, in Episode 6 pertains to a credit card reward of a constant amount granted for any spending ($y = x - 20$). It asked for the relationship between the initial amount of spending and the amount of debt. The relationships between the variables in the People and Hats and the Credit Card Problems are written in the form of $y = x \pm a$, which differs from the previous problems in Episode 5 in that it has a constant rate of change of one. The last problem in Episode 6 was the Sapling Problem, which involved a relationship between the height of a tree sapling and the elapsed time. The initial height of the sapling is 35 cm, and it grows 2 cm each day ($y = 35 + 2x$). This problem was incorporated into the lesson plan shortly after the completion of Episode 5, as the students had difficulty in determining the relationships between the variables in the problems of this form, including those with a coefficient that differs from one. The research members added the Sapling Problem into the plan to provide further practice with this specific functional relationship. The problems present novel contexts to allow students to practice what they have learned as well. Table 3.7 represents the key components of Episodes 5-6 which is the last part of the HLT.

Table 3.7 The Structure of Episodes 5 and 6 in the HLT

EPISODES 5-6: Representing functional relationships between variables in the form of $y = ax + b$
Learning Outcomes
• Identify the variables and the constant term in the problem • Construct a function table. • Identify and generalize the functional relationship in the table of data. • Understand and express the functional relationship between two variables through recursive, covariational, and correspondence approach • Represent the functional relationship using equation and variable notation. • Connection between multiple representations of functional relationship (verbal, table and symbolic)
Tasks and Their Structures
5-1) The Birthday Party 2 Problem: The relationship between the number of tables and the number of people who are seated ($y = 2x + 2$) **5-2) The Bank Account Problem:** The relationship between time and the total amount of money saved in the account ($y = 15x + 30$) **6-1) The People and Hats Problem:** The relationship between a person's height without a hat and with a hat ($y = x + 20$) **6-2) The Credit Card Problem:** The relationship between the initial amount expenditure and total debt amount ($y = x - 20$)

Table 3.7 (Continued)

6-3) The Sapling Problem: The relationship between the elapsed time (days) and the height of the sapling. ($y = 2x + 35$)

- Including contextual problems (Reality principle)
- Relationship between discrete quantities (Episode 5)
- Relationship between continuous quantities (Episodes 5-6)
- Using function tables to represent the data before generalization (Level principle)

Conjectures

a) MC2 students would calculate any corresponding value in the function tables and indicate the functional relationship verbally by using indeterminate quantities

b) MC1 students would not calculate the larger values in the function tables because of ignoring the constant value and they would not indicate the functional relationship by using indeterminate quantities and letters.

c) MC1 and MC2 students would have difficulty in writing the equations representing the functional relationship such as ignoring the constant value

d) MC1 students would have difficulty in writing the equations representing the functional relationship in the form of $y = x \pm a$ which is different from the previous problems

Instructional Moves Aligning with the Conjectures

- **Conjecture a**

-Ask them to explain the relationship by using different expressions and equations (Developing structural thinking)

- **Conjecture b**

- Fill in the table together on the board and ask the relationship between two variables. Let the students interpret the change in both variables simultaneously.

- **Conjecture c**

-Ask them to pay attention to how they fill in the table and what operation they did in calculating one variable by using the value of another variable.

- **Conjecture d**

-Discuss the meaning of problem variables, pay attention to the table of values, and highlight the covariation

We believed that relating the contexts in the problems to the previous episode would help students connect the tasks they have worked on. However, it is assumed that the initial amount as a constant in the problem would create a perturbation which is an opportunity to expand their understanding (von Glasersfeld, 1993) while writing equations. Therefore, the students were expected to identify the similarities and also differences between the contexts in both situations (i.e., $y = ax$ and $y = ax + b$) so that they could construct an adapted knowledge of writing equations for functional relationships. All parts of the HLT are placed in Appendix D.

3.4.2 Phase 2: Conducting teaching experiments

During phase 2 of this design study, as a teacher-researcher, I conducted six teaching episodes to achieve all the learning outcomes that were specified in the previous phase. Each lesson followed the same learning sequence, including individual student work, sharing responses and ideas, discussing the responses, and using the same steps together with the teacher on the board. I adapted this sequence based on theoretical principles such as didactic constructivism (von Glasersfeld, 2001), the emergent perspective of Cobb and Yackel (1996), and the principles of the RME (van den Heuvel-Panhuizen, 2000).

von Glasersfeld (2001) introduced several principles for teaching with radical constructivism, which he termed "didactic constructivism." In this perspective, the teachers are expected to facilitate students' thinking and encourage them to verbalize their thinking. He recommends that teachers utilize "neutral questions" (p. 171) to guide students' thinking when necessary. Rather than emphasizing the attainment of a correct response, it is essential to foster an appreciation of the construction process. The HLT incorporated teacher-initiated questions and prompts to stimulate students' thinking by considering potential student conceptions that can emerge during the learning process, as proposed by von Glasersfeld (2001).

The RME approach emphasizes the activity principle, which posits that students learn best by doing and participating (van den Heuvel-Panhuizen, 2000). This approach makes students active participants in the learning process. Therefore, each contextual problem included small tasks to guide students toward the final step. The activities in each episode started with an individual work in which a student reads the contextual problem and applies the procedures required in each step of the activity. For example, students begin by interpreting the problem variables. Then, they construct a table with possible values for each variable and identify the relationships as explained in the previous section. Finally, they describe the relationships using verbal and symbolic language. This time given to each student at the beginning provides opportunities for students to think about this “novel” way of mathematical situations and come up with their ideas for further discussion (von Glasersfeld, 2001).

In conducting teaching experiments, I worked with two students instead of one in each episode to enrich the interaction process during learning. Cobb and Yackel (1996) put forward the emergent perspective that incorporates the interaction aspect in sociocultural theories and the psychological aspect in the constructivist approach to learning and the mathematical way of knowing. In this perspective, the microculture in the classroom, the roles of the teacher and students, and the way of developing mathematical knowledge as taken as shared gain importance at both societal and individual levels. In this regard, our initial goal was for the students to thoroughly think about each problem situation, construct their own responses, and share them in the class. In this process, they would come up with diverse outputs from different mental processes and possible misinterpretations (von Glasersfeld, 2001). We aimed for the students to hear the responses of each other after they worked on the problem by themselves. Verbalization of their thinking processes and responses would let the students hear different responses and adapt their answers in this small social context (Cobb et al., 1993; von Glasersfeld, 2001).

Although I objected to group students with similar mental operations in terms of units coordination, I expected and observed that they could have still diverse mental operations within the same problems. The similarities in their mental operations would allow them to understand each other more easily. Furthermore, the differences between the mental operations would provide productive and useful interaction between the students through different strategies and more sophisticated responses. For example, one would scaffold the other in case one of them had struggled with the problem. In addition, one would learn a more sophisticated approach from her/his peer.

While the students actively worked on each task, the role of the teacher was guidance, which is another principle of the RME (van den Heuvel-Panhuizen, 2000). Although the students worked on the problems individually in the beginning, when they needed help, the teacher intervened and helped them understand the point by using additional questions. For example, if a student does not understand a problem, the teacher can paraphrase or explain it. If a student cannot achieve a step of the problem, such as identifying the multiplicative relationship between quantities, the teacher asks small questions scaffolding the student for the further steps. Some examples of teacher prompts and scaffolding questions are represented in Table 3.8 which were prepared in the design process of the HLT.

Table 3.8 Teacher Prompting Questions and Scaffolding for Students

Students' possible performances	Teacher responses and prompts
Episode 2: They may only give numerical values for the unknown quantities instead of symbolic representation.	Remember that we do not know the lengths. How do you attain the value of the length of the smaller and longer bar? What kind of relationship is there between each hypothetical pair of values? How to represent these varying values for each rope by using letters?
Episode 2: They may write an equation: 4 x a: it is the longest rope	Is there another way to represent this relationship?
Episode 3: They may not write the equation by using x and y for the relationship between the number of chairs and the number of legs	What do x and y represent? How did you calculate the number of legs, I mean y? (while working on the table) What did you do with the number of chairs, I mean x? (while working on the table). Now you can use these letters instead of numbers.

In this context, the teacher's reactions play a crucial role when a student presents an incorrect equation or flawed reasoning. Prompting questions are essential to allow the student to reflect on their response (von Glasersfeld, 1996). My goal was to guide and support students in identifying their own mistakes and developing a new way of thinking with the teacher's guidance.

To summarize, each episode focused on three central processes: the students' individual work on tasks, verbalizing their reasoning and explaining responses, and reviewing the steps together on the board (see Table 3.9). This process constituted the classroom discourse, the role of the teacher and students in learning.

Table 3.9 Overview of Processes in Activities in Each Episode

Main Processes	**Intermediate Processes**	**Theoretical concepts**
Individual work on tasks	Teacher guidance	Activity principle-RME Guidance principle-RME
Verbalization of thinking	Teacher guidance Peer interaction Comparing the responses	Activity principle-RME Guidance principle-RME Interaction aspect -Emergent perspective
Reviewing the responses and tasks	Peer interaction Student adaptation Constructing new material	Activity principle-RME Guidance principle-RME Interaction aspect -Emergent perspective

3.4.3 Phase 3: Retrospective analysis

In the course of a retrospective analysis, the data derived from each teaching episode, including students' written work, verbal statements, and the results of in-class assessments, were analyzed by comparing the conjectures. The analysis of each teaching episode informed the subsequent teaching process by providing new conjectures or revising previous conjectures. Furthermore, the retrospective analysis conducted after each teaching episode permitted the revision of specific elements of the subsequent episode, which had been constructed before starting teaching episodes. For example, Episode 6 involved two problems that included functional relationships between variables in the form of $y = ax + b$. However, after conducting Episode 5 and analyzing the entire data, I put an additional problem that is similar to those in Episode 5 for further practice due to the students' difficulties observed in those problems.

3.5 Data Collection

There were three phases in the data collection:

1) Interviews before the teaching experiments assessing the students' units coordination, understanding of equal sign and variables, and algebraic thinking.
2) Teaching experiments involving the students' written works and end-of-lesson assessments
3) Post-assessment interview after teaching experiments for evaluating the students' overall achievement in algebraic thinking.

All phases of data collection are represented in Figure 3.6

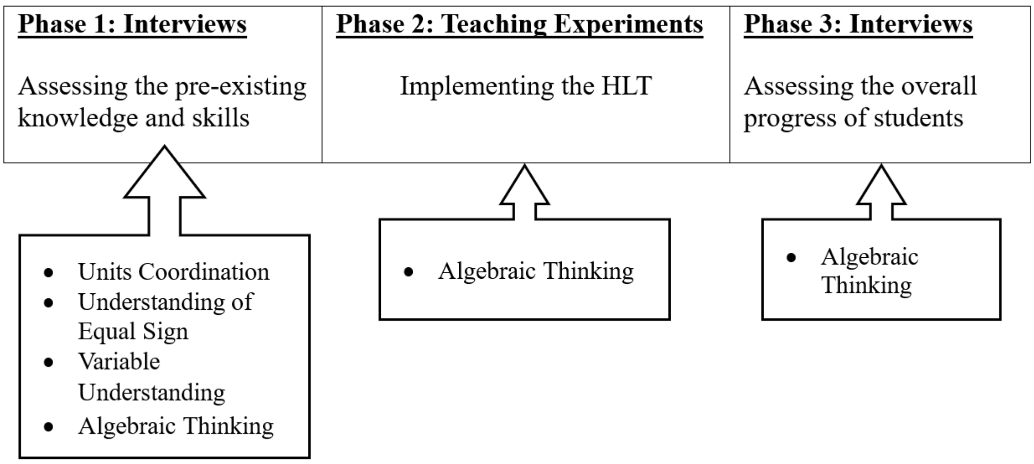


Figure 3.6. Three phases of the data collection process

Pre-assessment interviews were also used for documenting the prior knowledge level of the students which provides the means for developing the teaching (Cobb et al., 2003). The students' units coordination in terms of multiplicative concepts was determined for two purposes. The first purpose was to select students for the study and to group them according to their levels, demonstrating similar mental operations. The second purpose was to examine the effect of their mental operations on their performance in algebraic thinking. Therefore, their units coordination levels were determined before starting teaching experiments.

To design the HLTs and determine students' needs and preparedness for the subject matter, I evaluated their understanding of the equal sign and variables before teaching experiments. In this way, I aimed to make decisions on certain aspects of the HLTs which are related to using equal signs, variables, and alphanumeric symbols.

The interviews (i.e., the first phase) started almost three months before the teaching experiments and were spread out over time. There were four interviews for each student in total. Each student was interviewed separately, and each session was video recorded which took between 15 to 30 minutes. The problems were taken and adapted from different studies about units coordination (e.g., Hackenberg & Lee, 2015) and early algebra (e.g., Blanton, Brizuela et al., 2015), which are all explained in further sections.

The first interview was to determine the students' multiplicative concepts and select them to participate in the study. Each interview lasted between about 25 and 30 minutes, and all four students' units coordination interviews were completed in one month. Afterward, the Algebraic Thinking Interview was conducted with four students to determine their performance in determining their identification of the relationships between variables and between unknown quantities, generalization of the relationship, and representing the relationships using symbols. Each interview lasted approximately 25 minutes and the interviews of all four students were completed within 10 days. The third interview was conducted to assess the students' understanding of equal signs. Each interview lasted from 10 to 17 minutes. All four students' equal sign understanding interviews were completed within a week.

Finally, the fourth interview was to assess the students' variable understanding such as interpreting the literal symbols assigned to the problem quantities and identifying the indeterminate quantities in problems and operating with them. Each interview lasted from 10 to 19 minutes. All the variable understanding interviews were completed within one week.

During the second phase of the data collection, the teaching experiments were conducted in six episodes for each group over three weeks. Each group of students attended two teaching episodes per week after school with each episode lasting between 60 and 75 minutes. Each teaching episode was videotaped. Additionally, each student demonstrated his/her work on activity sheets and end-of-lesson assessment papers. One of the MC1 students, Sara, could not attend the last episode due to some special circumstances.

After completing the teaching experiments, I interviewed each student in the third phase of the data collection to evaluate their overall progress in algebraic thinking. I used the problems that I selected from the previous interviews. Each interview took around 25 minutes. One of the MC1 students, Sara, could not attend the general assessment as well. In the following sections, all constructs and how they were assessed are presented in detail.

3.5.1 Units coordination assessment

There were four problems in the first interview protocol that examined the students' multiplicative concepts in terms of units coordination. Researchers (e.g., Hackenberg & Lee, 2015; Ulrich & Wilkins, 2017) used these problems to assess the students' units coordination levels in previous studies. I have translated two problems (The Bar Problem and The Cupcake Problem) and adapted two problems (The Crate Problem and The Chairs-in-Rows Problem). I took an expert opinion in terms of comprehensibility.

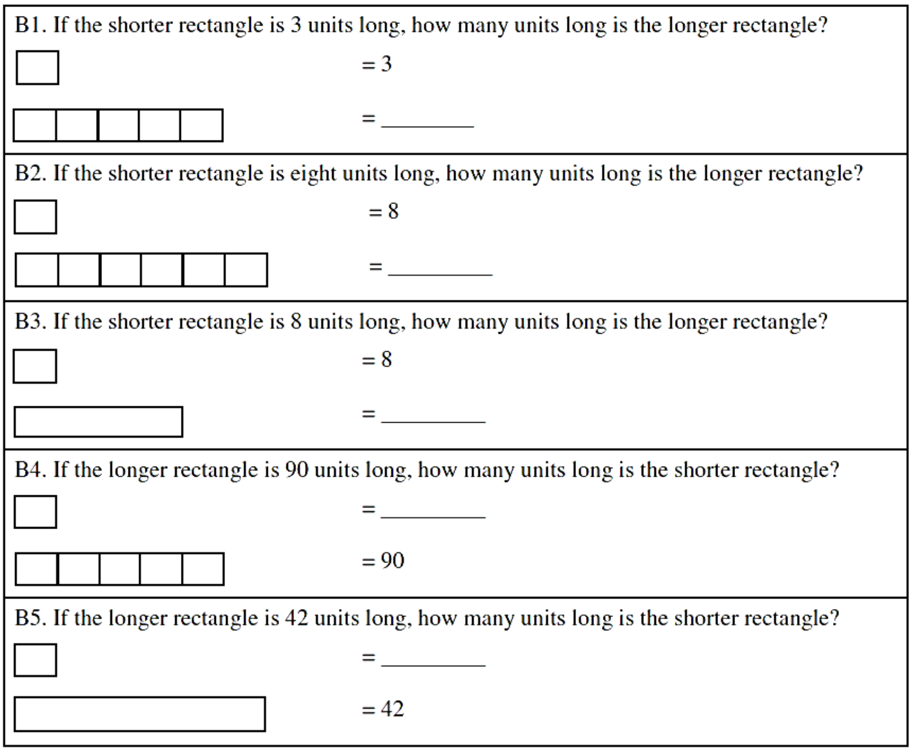
The first two problems were adapted from the studies assessing the units coordination of middle and high school students (e.g., Hackenberg, 2013; Hackenberg & Lee, 2015) (see Table 3.10)

Table 3.10 The Problems for Assessing the Units Coordination Levels of Students

The Crate Problem: There are 6 chocolates in a package and 8 packages of chocolate in a box. A crate contains 5 boxes. How can you find how many chocolates are in a crate? Can you draw a picture to show how you find it? (Hackenberg & Lee, 2015)

The Chairs-in-Rows Problem: There are 6 rows in a movie theater with 4 chairs in each row. 12 more chairs were brought to this hall. In the last case, how many rows can be made in total with 4 chairs in each row? In the last case, how many chairs are there in the hall? (Hackenberg & Lee, 2015)

The Bar Problem: (Ulrich & Wilkins, 2017, p. 9)


B1. If the shorter rectangle is 3 units long, how many units long is the longer rectangle?
= 3
= ________

B2. If the shorter rectangle is eight units long, how many units long is the longer rectangle?
= 8
= ________

B3. If the shorter rectangle is 8 units long, how many units long is the longer rectangle?
= 8
= ________

B4. If the longer rectangle is 90 units long, how many units long is the shorter rectangle?
= ________
= 90

B5. If the longer rectangle is 42 units long, how many units long is the shorter rectangle?
= ________
= 42

The Cupcake Problem: There are 3 rows of 6 cupcakes that are unboxed. If there are 9 rows of cupcakes in all, how many cupcakes are hidden in the box? (Ulrich & Wilkins, 2017, p. 14)

The Crate Problem involves embedded four levels of units (chocolates, packages, boxes, and crates) which helps in distinguishing the students who coordinated three levels of units (at least MC2). Similarly, in The Chairs-in-Rows Problem, there are different levels of units that the students can perceive at the same time such as the number of rows and the number of seats, or they can operate with just one of them at a time (Ulrich, 2015). The kind of student operations demonstrates the quantitative complexity the student is dealing with during the problem.

The other two problems were translated from the study of Ulrich and Wilkins (2017, p.9). The face validity of the translated problems was ensured through an expert opinion and piloting with a student. In the Bar Problem, there is a hierarchy of difficulty from the first (B1) to the last one (B5), and the researchers remarked that there was a significant association between the performance of each task and the stages of the units coordination (Ulrich & Wilkins, 2017). They found that the tasks helped distinguish the students at different levels of units coordination since they required operations such as constructing a composite unit in activity (B2), operating with assimilated composite units (B3 and B4), and constructing iterable units (B5). For example, the questions with unpartitioned bars (B3 and B5) were considered good at identifying students who assimilated with composite units (i.e., advanced MC1 and higher levels).

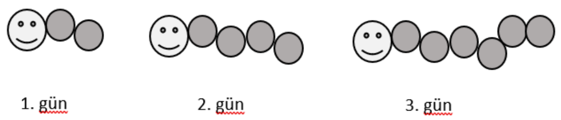
The Cupcake Problem involves "a composite of composite units" (p. 15) which increases the difficulty of the task for students without assimilated composite units (i.e., hidden or shown cupcakes, total cupcakes, and rows of cupcakes). Therefore, this task was regarded as hardly accessible for the students at lower levels. The researchers observed various student solutions including distinguishing mental operations of units coordination such as operating with the composite units, disembedding composite units, and using figurative materials.

3.5.2 Algebraic thinking assessment

In the interview assessing the student’s algebraic thinking, there were five problems adapted from different researchers (see Table 3.11).

Table 3.11 The Problems Used to Assess Students’ Algebraic Thinking

The Growing Caterpillar Problem: The following pictures show the change in body size of a caterpillar every day. Answer the questions accordingly. (An **e**xample question: Let T be the size of the caterpillar, and G the number of days passed. Can you write an equation that gives the length of the caterpillar?) (Blanton, 2008)



The Bouncing Ball Problem: The table below represents a recording of how high a ball rises after each hit the ground and from what height the ball is dropped. Can you write an equation showing the relationship between the height at which the ball is dropped and the amount of rise after it bounces off the ground? (Lucariello et al., 2014).

The height the ball is left (y)	80 cm	100 cm
The height after bouncing (x)	40 cm	50 cm

The Penny Bank Problem: Ali has 10 liras in his penny bank. Ali decides to save money by putting 3 liras in his penny bank every day. How many liras will Ali have in his penny bank after 8 days? How can you express the relationship between the number of days and the amount of money in the penny bank?

The Caterpillar and Leaf Problem: Ali is keeping 2 caterpillars at home. Each day brings 6 leaves to feed these 2 caterpillars. If Ali had 12 caterpillars, how many leaves would he have to bring each day to feed these caterpillars? Can you write a mathematical expression showing the relationship between the number of caterpillars and the number of leaves that need to be fed? (Blanton, Stephens, et al., 2015)

The Cord Problem: The charging cord of Sinan's phone is of some number of lengths. His charging cord is 5 times longer than Zeynep's charging cord. Could you draw a picture of this situation? Can you write an equation for this situation? (Hackenberg & Lee, 2015, p. 20)

The first two problems included pattern situations with a given tabular (The Bouncing Ball Problem) (Lucariello et al., 2014) or figural data (The Growing Caterpillar Problem) (Blanton, 2008). Hence, the aim was to assess how students identify the functional relationships between variables in the form of $y = ax$ and generalize and express them verbally or using symbols.

The third problem, the Penny Bank, similarly addressed the generalization and representation of a functional relationship between variables, but in the form of $y = ax + b$. Roy could not see this problem in his pre-assessment because this problem was not included in the initial form of the interview. Roy was the first student to be interviewed and it was not possible to interview Roy again to ask about this problem before teaching episodes.

The other two problems, The Caterpillar and Leaf Problem and The Cord Problem, put forward the quantitative reasoning aspect of algebra. The Caterpillar and Leaf Problem was adapted from Blanton, Stephens, et al. (2015) and included the multiplicative relationship between two known quantities and requires the students to generalize their solution by using indeterminate quantities as a dimension of algebraic thinking. Lastly, The Cord Problem was adapted from Hackenberg and Lee (2015). It assessed the students' representation of the multiplicative relationship between two unknown quantities by using symbolic language which requires the standard level of algebraic thinking.

3.5.3 Equal sign understanding assessment

For assessing the students' understanding of equal signs, a structured interview was constructed based on various sources about equal sign understanding (e.g., Behr et al., 1980; Fyfe et al., 2018; Hattikudur & Alibali, 2010; Matthews et al., 2012). The interview protocol involved five questions that were adapted from these studies. The questions involved describing comparison symbols, classifying symbols (e.g., 4, +, <, >, =), interpreting standard and non-standard forms of equations, and calculating missing values in equations. The questions addressed the reported misinterpretations and conceptualization of the equal sign concept.

The first two questions addressed the students' relational definition and conceptual understanding of equal sign by asking for the definition of comparison symbols and classifying various symbols including comparison and operation symbols and numbers (Hattikudur & Alibali, 2010; Matthews et al., 2012). In the third question (see Table 3.12) various equation structures were presented in typical (operation-answer structure) or atypical forms such as answer-operation and operation-operation structures (Baroody & Ginsburg, 1983) for the students to indicate whether they make sense or not (Behr et al., 1980; Matthews et al., 2012).

Table 3.12 Sample Problems in the Equal Sign Understanding Interview

Question 3. Look at the math expressions below, which ones make sense and which don't? Why do they make sense? a) 7 = 7 b) 7 = 2 + 5 c) 8 - 6 = 5 d) 5 + 3 = 5 – 3 e) 23 + 34 = 57 f) 47 + 52 = 48 + 51 g) 2 + 7 = 9 – 2 h) 4 + 6 + 3 = 10 + 3
Question 5. What are the numbers that should be in the spaces below? How did you find them? • 9 + 4 + 3 = 9 + __ • 3 + 5 + 7 = __ + 7 • 5 + 3 = ___ + 4

This assessed the students' understanding level of the equal sign and the level of reasoning in evaluating the equivalence of both sides (Fyfe et al., 2018). In the fourth and fifth questions, the students calculated the missing values in different number sentences. The aim was to identify the students' conceptions of equal signs, either relational or operational, their use of sophisticated solution strategies, and any potential misuse of equal signs. Questions 3 and 5 were represented in Table 3.12 as sample problems from the Equal Sign Understanding Interview.

3.5.4 Variable understanding assessment

For assessing the students' understanding of variables, a structured interview was constructed based on various sources about variable understanding (e.g., Ayala-Altamirano & Molina, 2020; Blanton et al., 2017; Lucariello et al., 2014). There were five questions to assess how the students identified the problem variables, how they viewed letters in contextual problems, and whether they had certain misinterpretations of letters in mathematical situations.

The first question asked about the meaning of the variable to identify if the students had heard it before and what they understood by the term. The second question was adapted from Blanton et al (2017) who studied the thinking of first-grade students about variables in a function context. This question and its sub-questions allow us to see whether the students mathematized the problem variables or how they mathematized variable quantities (Blanton et al., 2017). The use of letters in the question as prompts for the students (e.g., Let's call the number of cats D, what can it be and how can we express the number of total ears?) could allow for analyzing the students' interpretation of those letters assigned for the variable quantities whether they see as an object name, a constant value or indeterminate value (see Table 3.13).

The third and fourth questions were similar to the second question by including new contextual problems such as selling bagels and a constantly growing sapling. The problems included both qualitative questions with given variables but not values for certain quantities in each case. The aim was to incorporate more variable quantities in different roles, such as an unknown value (e.g., the price of a bagel and the initial length of a sapling) and a varying quantity (the length of a sapling in any day and the amount of money earned in a day) and observe how the students identify and interpret the variables in a mathematical problem situation. The third question (see Table 3.13) involved a functional relationship between two variables going through the origin (i.e., the number of bagels sold and earned money in a day: $y = ax$).

Table 3.13 Sample Questions from Variable Understanding Interview

Questions	Purposes
Q2. In a cat-only animal shelter, how many tails/ears/legs do 3/10/40 cats have? Why?	Identification of problem variables
a) If the number of cats in the shelter is unknown, how can we express the number of cats? Why?	Mathematizing variable quantities
b) Let's call the number of cats in the shelter D. How many can D be? How many tails/ears/legs are there? Why?	Interpretation of letters assigned to the problem variables
Q3. Ali decides to sell bagels every day on holiday. How can we calculate the money Ali will earn in one day? What do we need to know to make this calculation?	Identification of problem variables
a) What is the variable(s) in this question?	Mathematizing variable quantities
b) If the number of bagels sold in a day is S and one bagel is 5 liras, how many liras does Ali earn in a day?	Interpretation of letters assigned to the problem variables
c) If the number of bagels sold in a day is 50 and one bagel is T liras, how many liras does Ali earn in a day?	

The fourth problem involved another functional relationship between two variables going through the axis (i.e., the height of a sapling and the time passed after it is planted while it is growing the same amount each day: $y = ax + b$). In this way, the students' awareness of variables in distinct forms and contexts, and their interpretation of given letters to indeterminate quantities could be identified, which shows their level of understanding. Lastly, in the fifth question, an erroneous understanding of students was addressed by asking "What does L represent in the expression L + 4?". It reveals whether the students recognize the letters as labels or objects instead of variables (Lucariello et al., 2014).

3.5.5 Overall post-assessment

The students' overall assessments after completing the teaching experiments were carried out through critical interviews including six questions. The problems included distinct aspects of algebraic thinking such as focusing on functional thinking, being about the relationship between two unknown quantities, and being about understanding equal signs.

The first problem was The Caterpillar and Leaf Problem (Blanton, Stephens, et al., being2015) which was used in the Algebraic Thinking Interview before the teaching experiments (see Table 3.11). The aim was to observe the progress of the students in terms of analytical and structural thinking and symbolic representation of the relationship between two known quantities.

The second problem was The Cord Problem (Hackenberg & Lee, 2015) which was another problem from the first interview before teaching experiments. In terms of symbolic representation, the students had performed insufficiently in this task. Therefore, the aim was to observe how the students showed progress in interpreting the same problem and representing the multiplicative relationship between two unknown quantities.

Two problems (3rd and 4th) addressed functional thinking, each involving a different form of function. One was The Growing Caterpillar Problem (Blanton, 2008) which was also included in the Algebraic Thinking Interview before the teaching experiments (see Table 3.11). It involved figural data and addressed functional thinking in the form of $y = ax$. The fourth problem, the Penny Bank, (see Table 3.11) addressed functional thinking in the form of $y = ax + b$. The given context pertains to saving money, which is similar to the problems covered in Episodes 5 and 6. The problem's structure was similar to other functional thinking or pattern problems in an algebraic thinking context. It involved asking for the dependent variable when the independent variable has a larger value, using a function table, and representing the relationship between variables through symbols.

In the fifth question, there were two missing value operations. It aimed to assess whether certain students still struggled with the misinterpretation of the equal sign after the teaching experiments. This would indicate the effectiveness of the experiments in students' understanding of equal signs, which is a crucial concept in algebraic reasoning.

Finally, a question from the units coordination interview was used in the post-assessment interview, specifically The Crate Problem (see Table 3.10). During the initial interview, the students performed poorly on this problem. The aim was to observe any changes in the students' units coordination throughout the process. The questions in the post-assessment interview are presented in Appendix E by including a student's answers as sample data.

3.6 Data Analysis

There were two sources of data analysis in this research. The first source was the interviews conducted before and after the teaching experiments along with students' written work. Through these interviews, the students' algebraic thinking, units coordination levels, equal sign understanding, and variable understanding were evaluated. The second source of data analysis was teaching experiments including video records and written records of students' work. Each teaching episode in these experiments allowed for the evaluation of students' development of algebraic thinking through intervention. The following paragraphs describe the analysis of the interview records and teaching experiments, respectively.

Each interview session and teaching episode was videotaped and transcribed. The video recordings and student worksheets were used as the 'raw material' for data analysis, which did not include any direct interference from the researcher (Miles & Huberman, 1994, p. 46). By using these raw materials, I generated additional written sources of data for the analysis such as transcriptions, reflective memos, and field notes during and after the interviews and teachings. Because these "partially processed data" (p. 46) involved the researcher's selective attention and interpretation, this was the early part of data analysis (Miles & Huberman, 1994). For example, during the observation and transcription process, I highlighted some sentences, made comments for further analysis, replayed the video, or reviewed the students' worksheets.

After compiling all raw and processed data, the coding process began. The interviews were coded based on the corresponding variable. For example, the data coming from the unit coordination interviews were analyzed using the code sets generated from units coordination literature. The data from the interviews regarding algebraic thinking were coded based on the conception of algebraic thinking and reasoning described in Chapter 2. Finally, the teaching episodes were coded in terms of algebraic thinking. Although I observed the students' development and performance in algebraic thinking during the teaching experiments, I also somewhat evaluated their operations in terms of units coordination because the interaction between two constructs was crucial. The related coding sets and framework are presented in the following headings.

3.6.1 The students' units coordination levels

For coding the transcriptions of the units coordination interviews, I generated an initial set of codes that included specific mental operations, each of which referred to a specific level of multiplicative concept, based on the conceptual framework of units coordination (e.g., Hackenberg & Tillema, 2009; Steffe, 2002; Ulrich & Wilkins, 2017). Therefore, I used the mental operations of each level of multiplicative concepts identified by researchers in this literature as provisional codes, which means a starting list of codes generated from the literature before starting the coding process (Saldana, 2015). During the transcription and initial coding, some additional codes emerged as *process codes* by coding the students' mental and physical actions such as "drawing to check the answer" and "adding and subtracting different levels of units" (Saldana, 2015). These codes were grouped into categories according to which units coordination levels they primarily indicate. The main code list is presented in Table 3.14.

Table 3.14 Codes for Units Coordination Levels

Codes	**Categories**	
	MC2	**MC1**
Coordinating three levels of units in activity	X	
Interiorization of two levels of units	X	
Explicit reflection on the composite units	X	
Operate with composite units	X	
Disembedding	X	
Drawing to demonstrate after the teacher's request*	X	X
Equipartitioning	X	
Drawing to check the answer *	X	X
Coordinating two levels of units in activity		X
Need to draw in solving problems		X
Decaying composite units constructed during activity		X
Difficulty in keeping track of the multiple quantities		X
Iterative counting to construct composite unit*		X
Adding/Subtracting different levels of units *		X

*The codes generated during the coding process

Using the code list in Table 3.14, two researchers determined each student's level of multiplicative concept separately regarding the frequency of codes belonging to a certain level and the pattern in the students' mental operations. The researchers first analyzed each student's mental operations separately. In addition, the performance on each task was evaluated by comparing and contrasting the students. For example, firstly, the interview with Roy was coded and analyzed in terms of his mental operations related to a certain level of units coordination. Then his performance on the first task in the interview was compared with the performance of other students on the first task. This allowed for the analysis of each interview both across students and across tasks.

To enhance the inter-coder reliability, two researchers cross-checked their evaluations at the beginning of the analysis. The reliability percent was calculated for one student's results by using the reliability calculation mentioned in Miles and Huberman (1994). Therefore, the number of agreements was divided by the total number of decisions. The ratio was calculated as 75%. The decision from each coder is represented in Table 3.15.

Table 3.15 The Decisions of the Coders for Roy to Calculate the Intercoder Agreement

Tasks	**Coder 1**	**Coder 2**	**Final Decision**
Task 1	MC2	MC2 or Advanced MC1	Advanced MC1
Task 2	MC2	MC2	MC2
Task 3	MC2	Advanced MC1	MC2
Task 4	MC2	MC2	MC2

Due to the complexity of the mental operations and variations of those in different problems throughout the interview (Hackenberg & Sevinc, 2024), there were instances of inconsistencies in students' performances and their level of multiplicative concepts. The researchers discussed these instances to differentiate the students within the same level of units coordination. For example, being slow or fluent in some operations, needing a check after finishing the problem, and making trial and error were some differentiating factors of the students within the same level of multiplicative concepts. This intra-level differentiation was observed in several studies such as a seventh grader, Milo, who performed as an advanced MC2 in working with fractional relationships between unknowns (Hackenberg & Sevinc, 2022), and a sixth grader, Adam, who did advanced operations at the MC1 level (Ulrich, 2016b). Therefore, the researcher also reviewed those studies to compare and contrast the intra-level differentiation, which served as triangulation by theory (Lincoln & Guba, 1985).

3.6.2 The students' algebraic thinking

In teaching experiments and interviews, the students' algebraic thinking was analyzed by focusing on algebraic thinking processes and descriptions of algebraic reasoning specified by key researchers in the field such as Kaput (2008), Radford (2010; 2014) and Blanton (2008). A big part of the study required the students to identify and generalize relationships between variables or unknowns, and to express relationships verbally and symbolically, which is based on Kaput's (2008) description of algebraic reasoning. Therefore, the students' performances in algebra problems were analyzed by focusing on several units of analysis. These were a) students' identification of problem variables and relationships b) students' generalization of relationships in problems and c) the structure in students' expression of generalizations.

In analyzing their generalizations and expressions of generalizations, the dimensions of algebraic thinking such as analytical thinking, functional thinking, and structural thinking (Kieran, 2022; Radford, 2014) were provisional codes (Saldana, 2015) as indicators of students' algebraic thinking. However, these major codes included subcodes that were generated during the coding process. These subcodes described the students' thinking processes which were used for indicators of major codes or demonstration of a certain thinking level in a particular major code. For example, using indeterminate quantities in expressing a problem situation, such as the number of chair legs equal to four times the number of chairs, was used as a subcode of analytical thinking which is a dimension of algebraic thinking (Radford, 2014). On the other hand, giving numerical values to the unknown quantities to represent the relationship was an indicator of a lack of analyticity. Furthermore, writing different forms of operations in representing a relationship between variables demonstrated structural thinking. In another example, recursive thinking in a pattern situation was coded as a pre-level of functional thinking (Blanton et al., 2011). The identification of students' way of functional thinking was based on the categorization of covariational thinking and correspondence thinking (Smith, 2008). The sample codes and examples from student responses are represented in Table 3.16.

Table 3.16 Sample Codes in Analyzing Students' Performances in Algebra Problems

Provisional Code examples	**Subcode examples**	**Examples from student responses**
Analytical Thinking	a) Using indeterminate quantities b) Writing equations c) Expressing the problem variables d) Verbal generalization	(a) (d) "The number of chair legs equal to four times the number of chairs" (a) (b) $y = 4x$
Structural Thinking	a) Symbolic representation b) Writing equations in different forms c) Reversing the equations	(a) $y = 4x$ (b) (c) $y \div 4 = x$ and $y = 4x$
Functional Thinking	a) Recursive thinking b) Covariational thinking c) Correspondence thinking d) Finding the bigger items in a pattern problem	(a) It increases four by four (c)"If I multiply the number of months by five, I find the amount of money"

After categorizing all the codes and patterns in students' thinking, in terms of whether students construct in-action formulas to calculate a certain part of a problem (symbolic representation) and reach the bigger items in a pattern situation, students' algebraic thinking was categorized according to algebraic thinking forms such as factual, contextual and standard algebraic thinkers (Radford, 2010). For example, the students who could represent the generalizations in symbols and equations by showing many indicators of analytical thinking were determined as standard algebraic thinkers. On the other hand, the students who could only express the general relationship verbally, were determined as contextual algebraic thinkers. The students who were unable to reach any of these levels of generalization and who continued to engage in recursive thinking were identified as factual algebraic thinkers (see Table 3.17).

Table 3.17 Categories and Sample Code Patterns in Students' Algebraic Thinking

Categories in students' generalizations	**Generalization of relationships**
Factual Algebraic Thinking	Recursive thinking in pattern situations
Contextual Algebraic Thinking	Finding the bigger steps in a pattern situation
	Generalizing the relationship/rule between variables verbally
Standard Algebraic Thinking	Symbolic representation
	Explicitly using indeterminate quantities

In short, students' identification of the relationships and structures in problems was analyzed by focusing on how they think about and interpret the problem variables/quantities and their work on function tables. Students' generalizations were analyzed by focusing on their verbal statements, calculation of a random value of a variable in a problem, and symbolic and functional thinking forms. Lastly, students' symbolic representations were examined whether they were accurate equations and the explicit or implicit structures in written equations. All of these analyses revealed patterns and capacities in students' algebraic thinking and reasoning (see Table 3.18).

In addition to analyzing the students' thinking and reasoning in algebraic tasks, their progress was also analyzed in terms of the interaction with the teacher and peers in teaching episodes. Following the emergent perspective of social constructivism, in which the mathematical activities are seen as social events and the interactions and roles of individuals in this small social setting are important for knowledge development (Cobb & Yackel, 1996), I analyzed the students' progression in teaching experiments considering the roles and support of the teacher and peers. For example, the analysis distinguished between the performances of two students: one who completed a task with the support of only the task sequence and another who completed the same task with the support of the teacher's prompting questions or the work of a peer. Therefore, the analysis of students' progression in algebraic thinking involved some aspects of teaching such as interaction, the use of additional questions, and the timing of teacher interference.

Table 3.18 Analysis of Students' Performance on Algebra Problems

Focused Performances	**Categories of Analyses**	
	Substantial	**Weak**
Identification of relationships	Using table of values Recursive thinking Using indeterminate quantities Finding several values/examples for a variable	Giving numerical examples Erroneous identification
Generalization	Finding the bigger steps in a pattern situation Verbal statement Using indeterminate quantities Recursive / correspondence / covariational thinking	No generalization Could not find the bigger steps in a pattern situation
Expression of relationships	Symbolic representation Different forms of equations Different models/symbols	No symbolic representation Wrong equation/symbolic representation

3.6.3 The students' understanding of equal signs

In the analysis of students' understanding of equal signs, the aim was to determine whether they view equal signs as operators or signs indicating the equivalence of both sides, which is a relational view (Behr et al., 1980; Kieran, 1981). In the operational view, the students interpret equal signs as "do something signals" (Behr et al., 1980, p.15), and the signal for a coming *answer* (Kieran, 1981) while in the relational view, they see equal signs as comparison symbols showing an equivalence (Behr et al., 1980).

Matthews et al. (2012) described the students' responses according to their understanding of equal signs. They distinguished the students' understanding of equal signs into four categories: rigid operational, flexible operational, basic relational, and comparative relational. At the rigid operational level, the students accept the equations including the operations only on the left side while at the flexible operational level, atypical equation structures (i.e., $a = a$ or $a = b + c$) can make sense for the students. At the basic relational level, operations on both sides can work for the students. At the comparative relational level, the students can apply compensation strategy in solving missing value problems in equations as a more sophisticated understanding of equivalence. In this study, I evaluated each student's understanding of equal signs across the questions in his/her interview, and then his/her understanding was determined to be mainly operational or relational by focusing on the description and examples in the literature (e.g., Kieran, 1981; Matthews et al., 2012). Sample student answers and the corresponding codes are given in Table 3.19.

Table 3.19 Sample Student Answers and the Corresponding Codes

Example Student Answers	**Corresponding Codes**
Grouping the comparison symbols together "<, >, ="	Relational Understanding
Grouping the equal sign with the operation symbols "+, - , ="	Operational Understanding
Writing 8 into the blank in the equation $5 + 3 = \_\_ + 4$	Operational Understanding
Writing 4 into the blank in the equation $5 + 3 = \_\_ + 4$ by using compensation strategy	Relational Understanding

In the first two questions, the students' descriptions of comparison symbols (i.e., <, >, =) and their categorization with other symbols such as operation symbols and numbers provided codes indicating a relational or operational interpretation. For example, indicating that comparison symbols are used with operations or grouping an equal sign with operation symbols instead of comparison symbols were coded as operational understanding. On the other hand, grouping an equal sign with other comparison symbols was coded as relational understanding.

In addition, students' interpretations of various equation sentences were coded in terms of four levels of the framework in Matthews et al. (2012). For example, when an equation "$7 = 2 + 5$" was evaluated as not making sense or incorrect due to the order of the result and operation, it indicated a rigid operational level of understanding. If the students indicated that the equation, $7 = 7$ makes sense the response was coded as the flexible operational level of understanding. The students who accepted the equations including operations on both sides (Items h and f in question 3, in Table 3.12) were regarded at least at a basic relational level of understanding. Lastly, interpreting some equations or finding missing values using a compensation strategy indicated the comparative relational level of understanding (Matthews et al., 2012).

Questions 4 and 5 required the students to fill in the blanks in various equation sentences. The numbers the students wrote in the blanks were coded according to how they viewed equal signs. Their responses providing the equivalence of both sides demonstrated a relational view of equal signs. In addition, using a compensation strategy to find a missing value demonstrated a comparative relational level of understanding (Matthews et al., 2012).

3.6.4 The students' variable understanding

Student answers in this interview were coded in terms of how the students mathematized the problem variables by specifying them in problem situations, how they used them in operations, and how they interpreted the letters assigned for the variables. The use of letters assigned for certain variables in each problem as prompts, (e.g., Let's call the number of cats D, what can it be and how can we express the number of total ears?) allowed for analyzing the students' interpretation of those letters whether they see them as an object name, a constant value or indeterminate value.

Firstly, the students' identification of problem variables determined their understanding of variables in problem situations. For example, some students identified two variables in the sapling problem such as how much it grows each day and the number of days while some students identified only one variable: how much it grows each day. This represents some students are aware of the problem variables more than others which allows them to move on to mathematize the variables. In addition, it showed us those who could not recognize the problem variables need more support to improve their variable understanding in teaching episodes.

Students' interpretation of the letters assigned to the problem variables represented their understanding of the variables, whether they viewed them as objects, unknown or varying quantities. In this case, their variable understanding was analyzed according to the framework of Blanton et al (2017). They constructed a set of levels to identify the first graders' variable understanding in the context of functional relationships (see Table 3.20).

Table 3.20 The Levels of the Students Based on Their Variable Understanding

Levels	**Explanation**
Level 1	Pre-variable and Pre-symbolic: The students do not conceive the mathematical quantities as variables and they cannot use any symbolic representation
Level 2	Letters as labels or representing objects
Level 3	Letters as representing variables with fixed, deterministic values
Level 4	Letters as representing variables with fixed but arbitrarily chosen values
Level 5	Letters as representing variables that are varying unknowns
Level 6	Letters as representing variables as mathematical objects

There are several reasons for choosing this framework. First, it was shaped by a well-defined learning trajectory based on functional relationships. Second, it embodied and connected to previous frameworks about variable understanding (e.g., Küchemann, 1981). Third, because the students in this study were fifth graders who were not familiar with the variable construct, this extended framework was appropriate because it addressed levels from very basic to sophisticated.

For example, when students say that they are unable to calculate a particular variable by using letters in a problem situation, it showed that they understood letters as representing variables with fixed, deterministic values, Level 3 (Blanton et al., 2017). If they give numerical values to the letters assigned to the problem variables (e.g., If the height of a sapling that grows 2 cm every day is L) just before mathematizing the situation, it shows they see letters as representing variables with fixed but arbitrarily chosen values, Level 4 (Blanton et al., 2017). Lastly, a student who recognizes the problem variables as mathematical objects that s/he could operate on and create new expressions using these objects represents the highest level of variable understanding, Level 6.

Finally, I analyzed the students' use of variables in mathematical operations, including how they performed operations with letters and quantities assigned to the problem variables. For example, their operations with letters such as S × T (i.e., S for the number of bagels and T for the price of one bagel) were coded as an understanding of letters representing variables as mathematical objects (i.e., Level 6) (Blanton et al., 2017). On the other hand, the students who could not make operations with letters without giving numerical values were regarded at a lower level such as Level 4: representing variables with fixed but arbitrarily chosen values.

3.7 Trustworthiness

Design-based research follows the principles of qualitative research in terms of reliability and validity, which are key standards of research methodology (Bakker, 2018). Validity addresses the question of how accurately a method measures what it is intended to measure (Bakker, 2018; Lincoln & Guba, 1985). Internal validity is the degree to which the results of a study can be relied upon without bias, while external validity refers to the degree of generalizability of the results. On the other hand, reliability concerns how consistent the results would be in similar contexts and with similar participants.

In qualitative studies, internal validity is substituted by the term credibility (Lincoln & Guba, 1985). For ensuring credibility (i.e., internal validity) some common methods are using adequate and high-quality data, doing member checks, and data triangulation (Bakker, 2018; Lincoln & Guba, 1985). To ensure credibility, we used these methods in different phases of the study.

Firstly, the research members prepared the data collection tools, and teaching materials in line with the related literature and research. They were presented to different mathematics educators to ensure the construct validity. After four mathematics educators, including the teacher-researcher, agreed on conceptual understanding, consistency, and relevance to the purpose of the study, the data collection process began. We provided substantial data collection through observations of teaching episodes, video records of individual interviews and teaching sessions, written responses of the students, and reflective memos. This provides high-quality data by integrating multiple data sources and the credibility of the data analysis process. (Gravemeijer & Cobb, 2013). It allowed the teacher-researcher to check multiple sources during the examination such as watching the videos again, reading the reflective memos, and checking the students' written answers, as a kind of data triangulation (Lincoln & Guba, 1985). Furthermore, conducting end-of-lesson assessments after each teaching episode enhanced the credibility of the data as well (Nieveen & Folmer, 2013). In addition to using multiple sources of data, we used a multi-perspective approach in data analysis. For example, the research members conducted a joint examination of some of the transcribed data and the initial data analysis was shared with two mathematics educators. Moreover, the related literature provided information about the analysis of certain student responses. These methods enhanced the credibility of this study through triangulation.

In qualitative studies, external validity is replaced by the term transferability (Lincoln & Guba, 1985). In design research, it may be possible to transfer findings to other situations through analytical or theoretical generalization (Bakker, 2018). For this, a detailed explanation of the design in terms of “how, when, and why it works” in addition to “what works” is important to transfer the design to other situations in further steps (Cobb et al., 2003, p. 13). With respect to this, this study provides a detailed description of participants, and contexts to ensure transferability. Considering the students’ self-expression skills in the selection process was another factor for transferability to collect “thick descriptive data” (Guba, 1981, p. 86).

For the reliability and objectivity issues which are other criteria for research quality, multiple steps were taken to ensure that the findings are independent from the researcher bias. For example, multiple researchers have seen and interpreted a part of the data. We also used transcriptions of the recordings and an initial coding list in the data analysis. This allowed us to compare our codes and argumentations (Bakker, 2018).

CHAPTER 4

FINDINGS

This study aimed to investigate the interaction between fifth-grade students' progress in algebraic thinking and their units coordination levels by developing an HLT targeting the development of students' generalizations and symbolic representations of the relationships between unknown quantities or variables. This section presents the students' mental operations in terms of units coordination assessed before the teaching experiments and their initial performance in algebraic thinking, their progress throughout the teaching episodes, and their final level in algebraic thinking, organized under the three main headings. The first heading outlines the students' initial performances in pre-assessments involving units coordination, algebraic thinking, equal sign understanding, and variable understanding. The second heading, which is the main part of this section, presents the students' progress in algebraic thinking throughout the teaching episodes from the first to the sixth. Finally, the third heading presents an overall progress of the students including an evaluation of the difference between pre and post-assessment performances.

4.1 The Students' Initial Performances Before Teaching Experiments

This part first summarizes the students' mental operations according to the analyses of the Units Coordination Interview focusing on their multiplicative concepts. Second, it presents the students' evaluations in terms of algebraic thinking involving the understanding of some key terms in algebra such as the equal sign, and variable understanding.

4.1.1 Students' units coordinations in terms of multiplicative concepts

The students were divided into two groups according to their level of multiplicative concepts. Two students, Roy and Belle, demonstrated mostly the indicators of the MC2 level, such as performing the disembedding operation, operating with composite units, and coordinating three levels of units in activity. The other two students, Sara and Luke mostly demonstrated indicators of the MC1 level, such as using drawings to solve the problems, reducing composite units they constructed during the activity, and coordinating two levels of units in activity. Table 4.1 summarizes some indicators of multiplicative concepts that students demonstrated in different problems.

Table 4.1 Students' Mental Operations in terms of Units Coordination

	MC2		**MC1**	
	Roy	**Belle**	**Sara**	**Luke**
Coordinating three-levels of units in activity	1,2,3,4	1,2,3,4		
Interiorization of two-levels of units	1,2,3,4	1,2,3,4		
Explicit reflection on the composite units	1,2,4	1,2,4		
Operate with composite units	2,3	2,3,4	3	
Disembedding	2,4	2,4		4
Drawing to demonstrate after the teacher's request	1,2,4	4		
Equipartitioning	3	3		
Drawing to check the answer		1,4		4
Coordinating two-levels of units in activity			1,2,4	2
Need drawings to solve the problem		2	1,2	2
Decaying composite units constructed during activity			2,4	2,4
Difficulty in keeping track of the multiple quantities	1	2	2,4	2,4
Iterative counting to construct composite unit			2	2,3
Adding/Subtracting different levels of units			2,4	2,4
No answer				1, 3

* 1: The Crate Problem; 2: The Chairs-in-Rows Problem; 3: The Bar Problem (including 5 tasks); 4: The Cupcake Problem

Although each two students shared indicators of the same level, they also demonstrated some intra-level differences such as in fluency in multiplicative operations and in the frequency of using drawings. Therefore, the levels of units coordination were identified as advanced MC2 (Roy), regular MC2 (Belle), regular MC1 (Sara), and early MC1 (Luke), respectively, from the one demonstrating the most sophisticated units coordination to the one with the lowest level of units coordination. The students' mental operations for each multiplicative concept (i.e., MC1 and MC2) with examples are presented in the following headings separately.

4.1.1.1 The MC2 students' mental operations

Roy and Belle demonstrated mostly the MC2 level indicators in their responses, with subtle differences when working on the problems in the Units Coordination Interview. For example, in the Cupcake Problem, they demonstrated similar performance by applying disembedding operation, assimilating two levels of units, and using composite units in further operations. First, they did the same calculations shown in Figure 4.1, and then Roy did the drawings when the teacher asked him to, and Belle, without the teacher's request, drew the hidden cupcakes in rows to further check her answer.

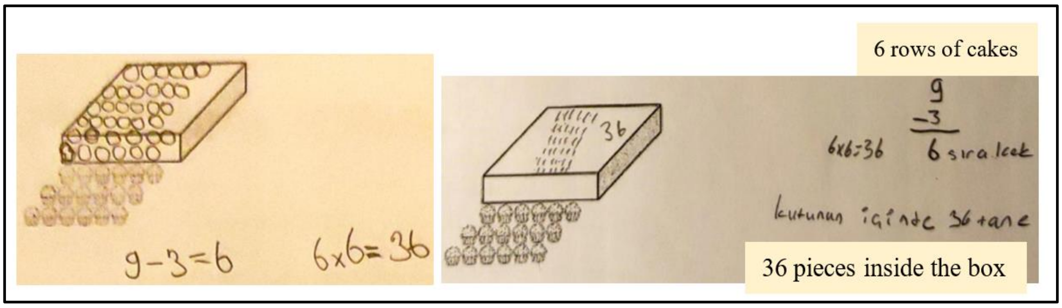


Figure 4.1. Roy's (left) and Belle's (right) solutions in the Cupcake Problem

They found the number of rows (second-level units) hidden in the box (i.e., disembedding the composite unit) and used this number to calculate the number of cupcakes (transition between units).

Another example that shows their common indicators and also the constraints in coordinating three levels of units is their performance in the Crate Problem:

> **The Crate Problem:** There are 6 chocolates in a package and 8 packages of chocolate in a box. A crate contains 5 boxes. How can you find how many chocolates are in a crate? Can you draw a picture to show how you find it? (Hackenberg & Lee, 2015).

Both students reached the final correct answer, which requires coordinating three-levels of units. They could coordinate three levels of units in activity and use two levels of units as given, which is an indicator of the MC2 level. However, both had difficulty initially in understanding the problem. For example, Belle needed to read the problem a few times, before saying: “I multiplied 6 and 8, and it is 48. Then I multiplied 48 and 5”. After she answered, she continued reading the problem and thinking over it. Through the teacher’s suggestion, she started to make drawings. In the drawing, she wrote 6 in each square, an indicator of the iteration of composite units in activity (see Figure 4.2). She said: “There are 40 packets and there are 6 in each so by multiplying 40 and 6, finding 240”.

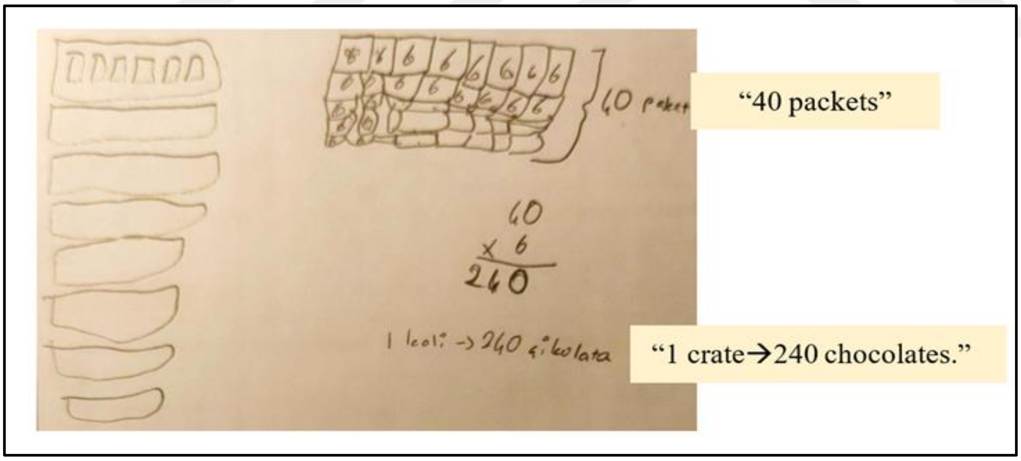


Figure 4.2. Belle’s solution in the Crate Problem

Similarly, Roy immediately multiplied 8 and 5 just after he read the problem. Then he changed his strategy in his second attempt through the teacher’s prompts. In the first attempt, he could not figure out the problem and could not coordinate three levels of units. In addition, he could not accurately state what 40 refers to. He indicated that there are 40 chocolates, instead of 40 packets of chocolates as follows:

> **Roy:** I multiply 8 and 5… one minute, (he is thinking and writing the multiplication). I multiply 8 and 5, it makes 40. Because there are 8 chocolates in one box, in one crate there are 40. I mean there are 40 chocolates in each crate.
>
> **Teacher:** Has it finished?
>
> **Roy:** Yes teacher (He still seems to be thinking)
>
> After the initial drawings, the teacher asked him to read the problem again, which made him change his solution by multiplying 8 by 6 and then 48 by 5 as follows:
>
> **Teacher:** Can you do drawings to demonstrate the situation?
>
> **Roy:** (He thought a little and drew) Now, there are 6 chocolates in a packet. It makes 48 chocolates in 8 packets. After that, 8 and … 5 are not multiplied teacher, we multiply 48 and 5, because there are 48 in one box (he is multiplying) 240
>
> **Teacher:** 240 what?
>
> **Roy:** Chocolates

Finding the answer 240 and his way of working on the solution demonstrated his coordinating three-levels of units in activity rather than being assimilated three levels of units, which indicates that he is not operating at the MC3 level.

An explicit difference between the mental operations of Roy and Belle emerged in the Chairs-in-Rows Problem:

> **The Chairs-in-Rows Problem:** There are 6 rows in a movie theater with 4 chairs in each row. 12 more chairs were brought to this hall. In the last case, how many rows can be made in total with 4 chairs in each row? In the last case, how many chairs are there in the hall? (Hackenberg & Lee, 2015)

Roy grasped the problem quickly and performed the operations by flexibly moving between the levels of units. He initially found the newly added number of rows by dividing 12 chairs (first level of units) into rows, each containing four chairs. Then he calculated the total number of rows by adding 3 new rows and 6 initial rows (operating with composite units). Lastly, he multiplied nine rows by four chairs (in each row) to find the total number of chairs. In contrast, Belle initially needed drawings to solve the problem (see Figure 4.3).

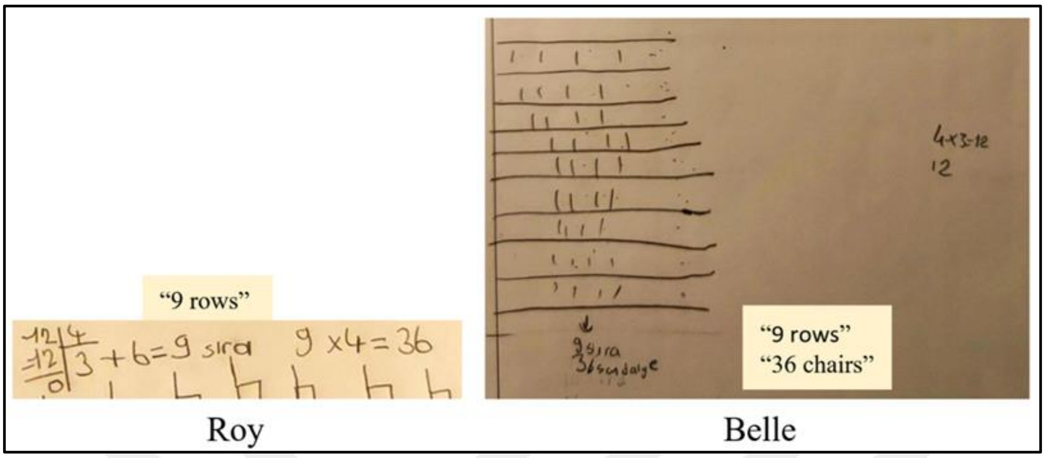


Figure 4.3. MC2 students' solutions in the Chairs-in-Rows Problem

Her drawings indicate that two levels of units are not given and are not explicit to her, which is an indicator of MC1. She made small errors and corrections in her drawings as follows:

> **Belle:** One minute…there is one more row… Because we were going to add three rows (she erased the last row). I have just realized it
>
> **Teacher:** How did you figure out that you need to add three rows quickly? Did you find it by drawings, or have you done mental calculations?
>
> **Belle:** Now, if there are 4 in each row, I found that we will put 3 rows in 12, so when we multiply 4 by 3, it becomes 12. (She did the multiplication that will answer 12 next to the drawing).
>
> **Teacher:** Okay, so you thought, what I multiply with 4 makes 12.
>
> **Belle:** Yes, by three. That's why we put 3 rows, 4 in each row as well makes 12, that's it. Now here are 9 rows.

As seen, after a while, she indicated that there would be 3 more rows with 12 chairs added. Then she counted the number of rows and wrote 36 chairs as the result. This mental operation shows that she operated with composite units with the help of the drawings.

In general, Roy's and Belle's performances, such as disembedding operations in the Cupcake Problem and coordinating three levels of units in activity in the Crate Problem put them at the MC2 level. Roy was fast and fluent in almost all problems, and Belle was slower in solving the problems and needed to make drawings to check the answer (e.g., Cupcake Problem) or do some parts of the operations (e.g., Chair-in-Rows Problem). Her reliance on drawings and activity in some instances is similar to an MC2 student who was identified by the raters as at "the lower end of other Stage 2 students" because she had more cognitive demands and relied on figural materials in some tasks (Norton et al., 2015, p. 57). Therefore, Belle was identified as being at the lower end of Roy at the MC2 level, and Roy was an advanced MC2 student who was in a different fluency in terms of mental operations with the different levels of units.

4.1.1.2 MC1 students' mental operations

Sara and Luke demonstrated the MC1 level indicators such as coordinating two-levels of units in activity (making drawings) and having difficulty keeping track of the constructed units (i.e., decaying composite units). They had similar difficulties and some differences in solving the problems.

In the Chairs-in-Rows and the Cupcake Problems, the students reached the correct solution in similar ways through drawings and the teacher-researcher's prompts. They also had similar challenges which are indicators of MC1. Both students made drawings to understand the problem or to correct the misunderstanding that appeared in their initial trials. They were confused about the units in their first attempts and were not able to follow the composite units they constructed in the activity.

For instance, in the Chair-in-Rows Problem, Sara said: "There were 12 rows and 4 chairs in each row," and she performed an addition of 12 and 4. Similarly, Luke did multiplication by saying: "There are 4 chairs in each row and I added 12 more chairs, so it is 48... I multiplied 12 by 4". This indicated that they were unable to accurately follow the level of units. In the second trial, Luke found the number of rows added (i.e., 3 rows) by quickly calculating from the number of chairs added (i.e., 12 is made up of three fours). However, subsequently, he added the values of 3 (the number of rows added) and 24 (the original number of chairs) and found 27. This showed that the composite units (3 rows) he was constructing in activity were decaying in his further operation. However, he needed to recall or construct the composite unit (3 rows) again to use it accurately in a new situation. This is a constraint that eTNS (early TNS or early MC1) students have as well (Ulrich & Wilkins, 2017).

Their final solutions and drawings are represented in Figure 4.4. They added three new rows of four chairs which they get from newly added 12 chairs. Finally, they relied on calculating the number of chairs by focusing on one level of units (either by counting by four or adding 24 chairs and 12 chairs). These demonstrated a coordination of two levels of units in activity (MC1).

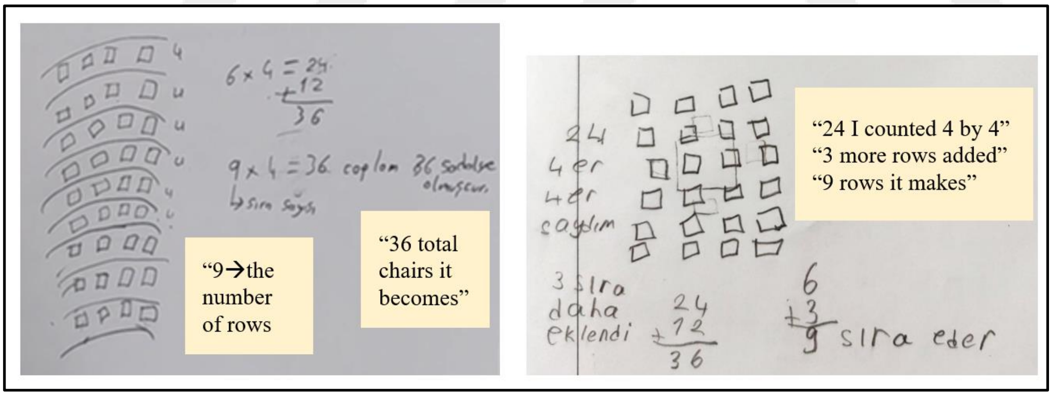


Figure 4.4. Sara's (left) and Luke's (right) solutions in the Chairs-in-Rows Problem

The differences between Luke and Sara's performances appeared in the Crate and the Bar Problems as well. Neither MC1 student could find the total number of chocolates in the Crate Problem. About the difference between them, Sara found the answer 40 by counting fives 8 times, and she named 40 chocolates rather than 40 packets of chocolates. On the other hand, Luke did not do any arithmetic operations or drawing. Sara at least demonstrated her multiplicative reasoning, whereas Luke displayed a lack of comprehension of the problem and a desire to move on.

In Bar Tasks 3 and 5, where the long bars were given unpartitioned (see Table 3.10), Sara incorrectly equipartitioned the long bar with respect to the given unit of the short bar, despite attempting to use her finger to ensure that each piece was equal. In contrast, Luke demonstrated the ability to perform equipartitioning correctly in Bar Task 3, where the length of the short bar is given, but not in Bar Task 5, where the length of the long bar is given (i.e., division is required). Additionally, in Bar Tasks 4 and 5, where inverse multiplicative reasoning is required, Luke employed a trial-and-error strategy by assigning random numbers for the length of the short bar. Conversely, Sara employed a division operation, taking into account the number of partitions involved in those tasks. In addition to these, Luke's struggle to understand the problems and lack of confidence even with the correct answers he found made us consider him at the lower end of Sara within the MC1 level.

4.1.2 Students' algebraic thinking

There were five problems in the Algebraic Thinking Interview before starting the teaching experiments (see Table 3.11). The Growing Caterpillar and the Bouncing Ball Problems were to examine the students' identification of the functional relationships between variables, in the form of $y = ax$, where it was presented in figures or tables, and their generalization and expression of these relationships, either verbally or in symbols. There was another problem addressing a functional relationship in the form of $y = ax + b$, the Penny Bank Problem. Only Belle, Sara, and Luke answered this problem because Roy could not see this problem in his pre-assessment. Roy was the first student to be interviewed, and this problem was not included in the initial form of the interview. In addition, it was not possible to do a follow-up interview with Roy about this issue at that time. The other two problems, The Caterpillar and Leaf and the Cord Problems were to assess the students' generalization and representation of the multiplicative relationships between known or unknown quantities using symbols.

Analysis of the students' responses in the Algebraic Thinking Interview revealed that MC2 students (Roy and Belle) demonstrated more advanced algebraic reasoning as compared to MC1 students (Sara and Luke). They also have a relational understanding of the equal sign, unlike MC1 students who demonstrated an operational view of the equal sign. Furthermore, students in the same group differed from each other in certain ways, as they did in units coordination. For example, Roy was more capable than Belle in terms of variable understanding and symbolic representation. A general overview of students' starting location in algebraic thinking and reasoning, including equal sign understanding and variable understanding, is shown in Figure 4.5.

	MC2		MC1	
	Roy	Belle	Sara	Luke
Equal Sign Understanding	Relational Understanding	Relational Understanding	Operational Understanding	Operational Understanding
Variable Understanding	"Letters as mathematical objects" *	"Letters as representing indeterminate quantities with fixed arbitrarily chosen values." *	"Letters as representing indeterminate quantities with fixed arbitrarily chosen values." *	"Letters as representing indeterminate quantities with fixed arbitrarily chosen values." *
Algebraic Thinking	Verbal generalization Symbolic representation in patterns Use of indeterminate quantities Standard algebraic thinking in patterns	Verbal generalization Contextual algebraic thinking	Recursive thinking Verbal generalization in patterns Factual algebraic thinking	Recursive thinking Factual Algebraic thinking

*Blanton et al (2017)

Figure 4.5. The analysis of students' initial performance in algebraic thinking

As summarized in Figure 4.5, only Roy achieved the use of symbolic representation, which is an indicator of standard algebraic thinking (Radford, 2014). However, he used symbolic expression only in problems including figural and tabular data as a pattern situation in a contextual situation (e.g., the Bouncing Ball and the Growing Caterpillar Problems). His verbal expression of the generalizations also included indeterminate quantities as an indicator of algebraic thinking as well. Similarly, Belle could generalize the relationships in pattern problems, but only verbally. She was unable to formulate the calculations by using indeterminate quantities in words or symbols in the Caterpillar and Leaf Problem, unlike Roy. Therefore, she demonstrated only contextual algebraic thinking in some instances where she could generalize the relationship between variables presented in a pattern situation.

MC1 students predominantly used recursive thinking. Therefore, they had difficulty finding the larger values in a pattern situation. This shows their factual algebraic thinking where dominant recursive thinking limits the students' generalizations. On the other hand, Sara showed some indicators of contextual algebraic thinking in certain instances, such as finding the larger steps in a pattern situation and using indeterminate quantities in her verbal expressions of the generalized relationship in the Bouncing Ball Problem. Therefore, she was explicitly a step ahead of Luke in terms of algebraic thinking and reasoning. The following two subsections explain the students' performances by giving specific examples.

4.1.2.1 The MC2 students' algebraic thinking

The analysis of the MC2 students' understanding of equal signs revealed that both students exhibited a relational view of equal signs, interpreting them as comparison symbols rather than operational ones. For instance, in grouping different kinds of symbols, including numbers, operational signs, and comparison signs, both students grouped the equal sign with the other comparison symbols (i.e., < , >). Additionally, both students correctly answered all of the missing value operations in their papers. Figure 4.6 illustrates the distinct symbols that were grouped by the students, along with a representative sample of the operations that they performed. The following paragraphs present the students’ performance in determining the relationships, expressing them verbally and symbolically, and overall understanding of variables.

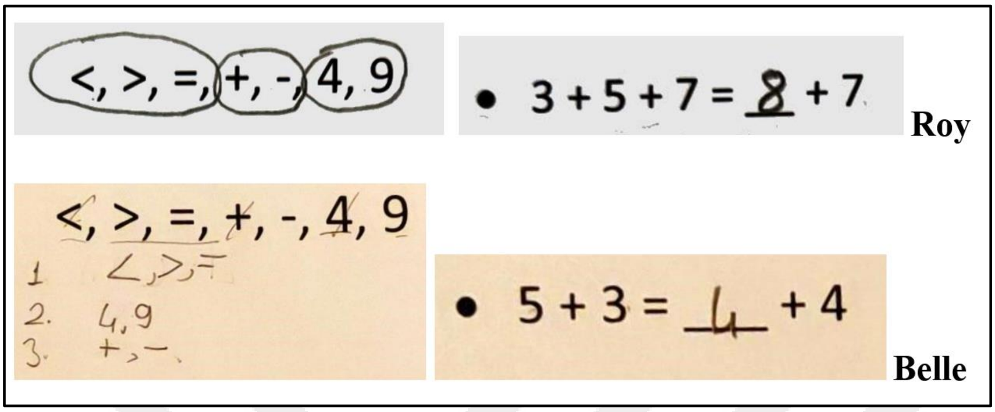


Figure 4.6. MC2 students’ understanding of equal signs

The MC2 students could verbally express the functional relationship between variables in the problems involving figural (e.g., the Growing Caterpillar) and tabular data (e.g., the Bouncing Ball). For instance, in the Bouncing Ball Problem, which requires understanding and expressing the covariational relationship between two variables (i.e., the drop height is twice of bouncing height), Roy indicated the relationship as “One is two multiples of the other” and Belle similarly indicated that “One is half of the other” by relying on the data given in the table. Similarly, in the Growing Caterpillar Problem, which presented figurative data such as each day the caterpillar grows two more body parts, the MC2 students verbally generated a formula. Roy's understanding of the variables and his verbal expression of the rule were more explicit than Belle's by saying: “We will multiply the number of days with 2”. Belle did not use an explicit expression such as “The length of the caterpillar is 2 times the number of days elapsed” as follows:

> **Belle:** (She is drawing until the fifth step using circles) …if this pattern continues, it will again grow by two. On the fourth day it will be 8, on the fifth day it will be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, so it will be 10.
>
> **Teacher:** Okay, then how can we calculate the size of the caterpillar for any given day?

Belle: Multiplying by 2... So, if it gets longer by two every day, we can find it by multiplying it mathematically

Roy explicitly used indeterminate quantities and used a correspondence approach in which one quantity is determined with respect to the other quantity (Confrey & Smith, 1994). Hence, his verbal generalization of the relationship was more sophisticated, compared to Belle.

In the symbolic representations of the relationships, Roy wrote accurate equations in both problems. In addition, he wrote equations in two forms by using both multiplication and division as inverse forms (see Figure 4.7).

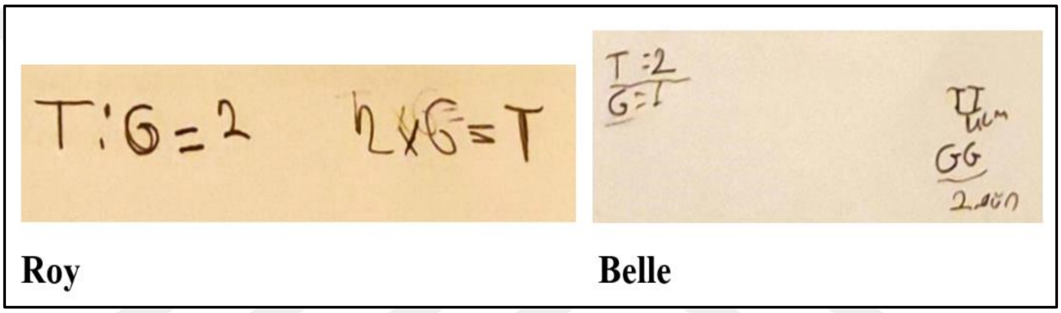


Figure 4.7. Roy’s and Belle’s symbolic representations in the Growing Caterpillar Problem

Roy’s symbolic representations in those problems demonstrated his structural way of thinking, indicating standard algebraic thinking (Radford, 2010). In contrast, Belle struggled to understand how to write a symbolic expression for the relationship between variables in both problems. For example, Belle wrote an incorrect equation in the Bouncing Ball Problem as $x = y$ and she could not write an explicit equation in the Growing Caterpillar Problem (see Figure 4.7).

Belle explained her symbolic expression for the relationship between the number of days (G) and the length of the caterpillar (T) as follows:

Belle: We can use two G's. G means 1, it represents the day. T ... for example if this is 1 G, this is 2 on the second day. On the second day, the size of the caterpillar... We called T as 2. The second day... The first day was 2, the second day will be 4, so we can put 2 T's. This is the size of a 4 cm caterpillar.

Belle could not also write a symbolic equation in the Penny Bank Problem involving a functional relationship in the form of $y = ax + b$ although she could formulate a rule for finding the amount of money saved for any given day elapsed. Although this problem was not included in Roy's interview, he answered a similar problem in the Variable Understanding Interview which requires them to formulate a rule using letters for calculating the length of a sapling where the initial length is L cm and it grows 2 cm each day as follows: "If the height of a sapling that grows 2 cm every day is L at the time of planting, what will be its height in 10 days?".

In the Sapling Problem, Roy accurately used indeterminate quantities and formulated the rule to calculate the length of the tenth day: "I multiply 10 and 2 and add L". However, Belle gave numerical values to L and she did not write accurate operations. As seen, Belle could not make sense of the fact that the letters were the symbols representing the varying numerical values. Her verbal expressions of the relationships in both problems and lacking an accurate symbolic representation showed her algebraic thinking at the contextual level (Radford, 2014). She was able to use the given data and take it to further steps through this generalization; however, she did not use indeterminate quantities to denote in an analytical way, which is one of the dimensions of algebraic thinking (Radford, 2014).

In another context involving a multiplicative relationship between two known quantities and requiring a generalization of that relationship to different values (i.e., the Caterpillar and Leaf Problem), Roy and Belle demonstrated different performances in terms of algebraic thinking. Roy used indeterminate quantities in his verbal generalization, indicating analytical thinking. The following discussion demonstrated his formulation of the general rule in this context:

> **Teacher:** Ali is keeping 2 caterpillars at home. Each day brings 6 leaves to feed these 2 caterpillars. What if there were 24 caterpillars?
>
> **Roy:** 24 caterpillars (He thought for a few seconds) …I could divide 24 by two, then multiply by 5. It is 60.
>
> …

Teacher: If I asked for more caterpillars, what would you do to solve it in a more general way? For example, if there were A caterpillars, how would you find the number of leaves for A caterpillars? Consider that A is any number. Could you find it when any number was given?

Roy: Any number… (he waited a few seconds). I could do this. I divide A by 2 and multiply by 5

As seen, he constructed a narrative formula by relying on the context, but he could not write the equation representing this formula. Therefore, he showed a transition between the levels, contextual and standard algebraic thinking due to his lack of symbolic representation.

Belle solved the problem the same way Roy did. However, unlike Roy, she did not express the relationship in a structural way. Her reasoning was more arithmetical. Although she swiftly calculated the number of leaves for any given number of caterpillars, she could not express this by operating with indeterminate quantities. When the teacher asked her to assign the letter "a" to the number of caterpillars, she gave numerical values to *a* by saying: "Can it be 18, because you said it can be any number". Her understanding of the variables as "fixed arbitrarily chosen numbers" (Blanton et al., 2017) prevented her from interpreting the general situation in this contextual problem.

Lastly, in the Cord Problem, the students were required to represent a multiplicative relationship in the 1:5 ratio between two unknown lengths of cords. Therefore, it was expected that they would write an equation such as $y = 5x$. Both students had difficulty in writing equations. Roy only wrote the ratio of 1/5 by indicating that 1 represents the length of one cord, and 5 represents the length of the longer cord without using symbols. He further gave numerical examples as convenient to this relationship such as 30 cm and 150 cm by indicating that this makes 1/5. Similarly, Belle gave a numerical value for the length of the shorter cord to find the length of the longer cord. Hence, she found a numerical value at the end as follows:

Teacher: Yes, if it is a, what will be Sinan's?... (Belle waited for a while) We called its length a, since we don't know, let it represent a number, a can be any number.

Belle: For example, let a be 5. Well, Sinan's cord is 5 times more than Zeynep's cord, so we multiply five by five. It makes 25.

This performance demonstrated her lack of understanding of variables too. This separates her from Roy’s way of thinking in which he could write a general ratio in this problem, and he indicated the computations in formulation verbally by using letters in the Caterpillar and Leaf Problem.

In summary, in expressing the general rule, Roy could generalize the functional relationship between two variables (e.g., Bouncing Ball and the Growing Caterpillar) and between two known quantities in multiplicative problems (Caterpillar and Leaf Problem) while Belle expressed the relationships verbally in only between two variables and she did not generate the rule in other problems (e.g., the Caterpillar and Leaf and the Cord Problems). In the case of the representation of the generalizations through symbols, which is another aspect of algebraic thinking and reasoning, the difference between the students was more explicit. Roy’s symbolic representation was limited to the functional relationship between two variables in which multiple numerical examples are given. He had difficulty in making sense of the relationship between two unknown quantities. On the other hand, Belle could not represent the relationships using symbols accurately in any given task. Her reliance on numerical examples to express the generic rules showed her factual algebraic thinking. In addition, she did not show any indication of an analytical or structural way of thinking. Belle apparently struggled to make sense of the variables and unknown quantities because she generally tended to give numerical values for the letters assigned to the variables.

Belle demonstrated an inability to interpret the letters as indeterminate quantities and varying unknowns by giving fixed values to the letters assigned to the variables. This showed that she had an understanding of “Letters representing variables with fixed but arbitrarily chosen values” (Blanton et al., 2017, p.194). In contrast, Roy’s operating with letters (i.e., indeterminate quantities) such as writing equations and their inverse forms demonstrated his higher level of understanding in the function context which is “Letters representing variables as mathematical objects” (p. 196).

4.1.2.2 The MC1 students’ algebraic thinking

The analysis of MC1 students’ understanding of equal signs revealed that both students held an operational view of equal signs, interpreting an equal sign only as an operational symbol. The grouping of symbols activity demonstrated that both students grouped the equal sign with the operational symbols (i.e., + and –). In addition, Luke considered that the result of an operation should come just after the equal sign in every missing value operation in his paper. Sara also performed the same way in some of the operations (see Figure 4.8).

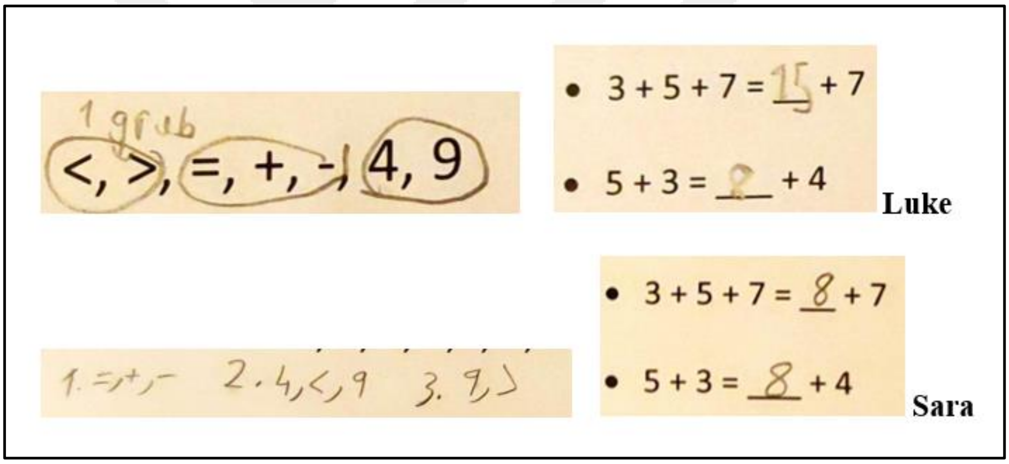


Figure 4.8. The MC1 students’ understanding of equal sign

The MC1 students relied more on recursive thinking about the relationship between two variables (e.g., the Growing Caterpillar Problem). In addition, Luke’s continual recursive thinking prevented him from generalizing the functional relationship given in the figures. In the problems involving tabular and figural data (e.g., the Bouncing Ball and the Growing Caterpillar Problems), they could express the relationship between two variables verbally by relying on the given data. For example, in the Bouncing Ball Problem, Sara said “It rises to half of the height from which it was left” while Luke stated, “This (the height of the ball’s rise) is half of this (the initial height)” by using the data in a table. Similarly, in the Growing Caterpillar Problem, Sara stated how she formulated the general rule for finding a value for any given day: “I could calculate it by skip counting because it increases by two; when you ask me the thing of what day, I could continue to count.” Similarly, Luke expressed the relationships in an arithmetic way: “It (the length of the caterpillar) increases by two”. In this problem, their recursive thinking became apparent, which is a characteristic of a factual algebraic thinker (Radford, 2014).

In generalizing the relationships between variables and formulating a rule using indeterminate quantities were not easy for the MC1 students, unlike the MC2 students. Sara could later explain that she could shortly do multiplication when asked for the hundredth day in the Growing Caterpillar Problem. On the other hand, Luke’s recursive thinking did not allow him to find the length of the caterpillar on the hundredth day although he kept counting by two for a while. Since he could not pursue counting recursively until the hundredth day, he asserted to divide 100 by two and said, “50, this is the result, I guess.” A parallel performance comparison appeared in the Penny Bank Problem, which included a functional relationship in the form of $y = ax + b$. Sara was able to determine the value of the saved money for any given day by formulating the rule, whereas Luke was unable to calculate the larger values because he relied solely on recursive operations, adding threes until he reached the requested value. Therefore, Luke stayed at the factual level regarding algebraic thinking while Sara demonstrated thinking at the contextual level.

In writing symbolic representations, both students wrote inaccurate equations by using T for the length of the caterpillar and G for the number of days – Sara: “$T + T = G$”, Luke: “$T = G$”. In the Bouncing Ball Problem, Sara wrote “$x = y$” whereas Luke did not even understand what writing an equation meant. Furthermore, neither student could represent the functional relationship between variables in symbols in the Penny Bank Problem ($y = ax + b$).

In the Caterpillar and Leaf Problem, both Sara and Luke had difficulty determining the relationship between known quantities (e.g., the number of leaves and the number of caterpillars) and finding the number of leaves for any given number of caterpillars. Hence, at this point, it was hard to expect them to express the situation algebraically. As conjectured, both MC1 students could not write the symbolic representation. Specifically, Sara could not figure out how to use the letter a that was assigned for the number of caterpillars: "I could say 36 for a". The difficulty in keeping track of multiple quantities in multiplicative relationships can be explained by their level of unit coordination, MC1. That is, if they could keep track of the units while making operations, we would expect them to express the relationship at least verbally. Although Luke did correct operations at first, the lack of awareness about what he was doing while counting by twos (finding the number of groups of two caterpillars) might have held him from going further and generating a rule.

Lastly, in the Cord Problem, neither student could write an equation representing the multiplicative relationship between two unknown quantities. When the teacher gave numerical values to the length of the shorter cord, Luke could accurately find the length of the longer cord. When the teacher named the length of the short cord by the letter a, he could not say $5a$ for the length of the longer cord, but he used another letter b to represent the length of the longer cord. Similarly, Sara indicated that she could not write an equation by using s and z to represent the cord lengths:

> **Teacher:** Well, let's consider Zeynep's cord length as a, could we express the length of Sinan's by using a?
>
> **Sara:** We cannot express it using a, it has to be another number, that is, a letter. Because this is shorter, and this is longer it is not equal. And each must have something else.

Her response indicated that she perceived the act of writing a relationship between two unknowns using mathematical equations to be a fundamentally different process such as writing them as if they were equal.

In summary, the MC1 students were inclined to recursive thinking in pattern situations. This created a constraint in some instances to formulate a general rule expressing the relationship between two variables. In addition, they struggled to understand what the letters that were assigned for the variables meant in the problems. Therefore, they could not operate with indeterminate quantities. Sara (regular MC1) partially demonstrated the contextual level of algebraic thinking because she could express indeterminate quantities in her generalization in the Bouncing Ball Problem and she could calculate the larger steps in the Growing Caterpillar Problem. On the other hand, Luke (early MC1) may be a factual algebraic thinker because he had difficulty recognizing the multiplicative relationship between quantities and did not generalize the functional relationships. Furthermore, their interpretations of letters in these problems, and inability to write equations using indeterminate quantities demonstrated their lack of understanding of letters in terms of variables. Sara interpreted letters to be given an arbitrary numerical value or to be the name of an object in many of the problems. Similarly, Luke tried to give random numerical values to the letters assigned for the variable quantities in the Variable Understanding Interview. Therefore, both students' views of letters were determined as "representing variables with fixed but arbitrarily chosen values" like Belle (Blanton et al., 2017).

4.2 Students' Progression Through Teaching Episodes

This section presents the students' progress in algebraic thinking throughout six teaching episodes. The analyses of the teaching episodes revealed findings into four main headings based on the objectives of the episodes. The first heading yielded from Episode 1 describes the students' learning progression in terms of comparing unknown quantities by using symbols. The second heading from Episode 2 outlines how the students represent the quantitative relationship, multiplicative or additive, between unknown quantities by writing equations. The third heading, comprising the objectives of Episodes 3 and 4, explains how the students identified and represented a functional relationship in the form of $y = ax$. Lastly, the fourth heading about the objectives of Episodes 5 and 6 describes how the students identified and represented a functional relationship in the form of $y = ax + b$. The findings for the teaching episodes are presented in the following sections in a structure that includes the intended processes of the teaching episodes, conjectures based on the literature, and outcomes of the teaching as briefly compared and contrasted with the literature and theory. This way of presentation will reveal testing of the conjectures specific to the teaching episode. The results also reveal revised conjectures for further implementation of the learning trajectory.

The flowchart below presents a sample organization of each main heading in this section by using the example of the first heading, a comparison of unknown quantities in Episode 1(see Figure 4.9). All four headings will continue with the same structure.

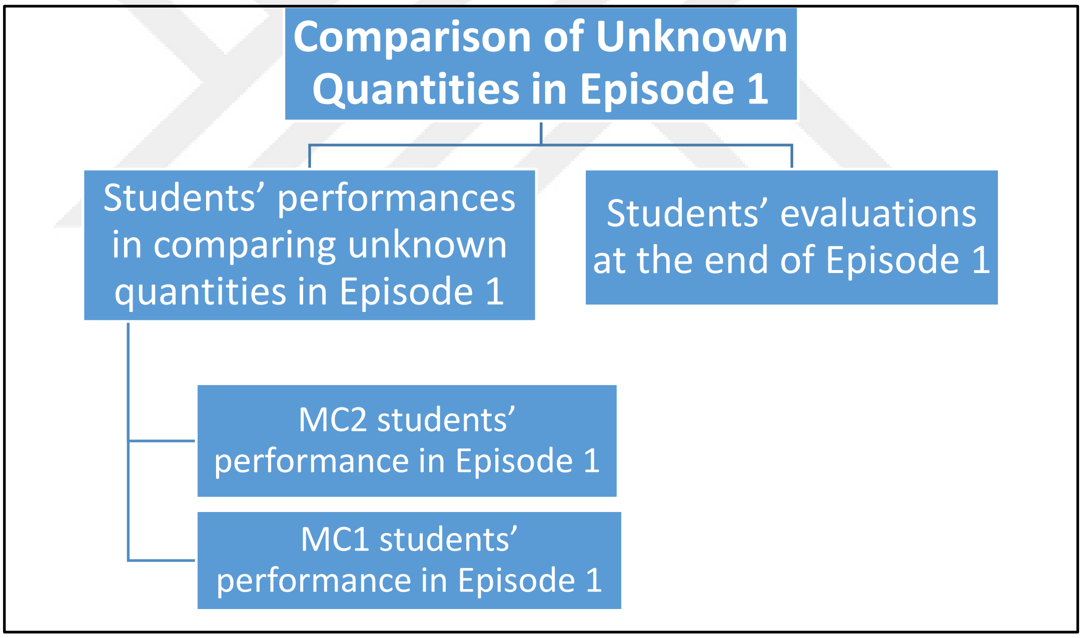


Figure 4.9 The flowchart of the presentation of findings for each heading

4.2.1 Comparison of unknown quantities in Episode 1

In the first part of the teaching experiment, the aim was to get students to encounter and compare unknown quantities and express them by using symbols. For this aim, in the first episode, the students worked on three tasks which asked them to compare unknown quantities that were represented in figures and express the comparison in different ways such as using verbal expressions, hypothetical values, and symbols.

In the first task of Episode 1, they began by comparing two pencils of different and unknown lengths. In the second task, they compared the other two pencils that were the same length. As an example, the first task with several discussion questions is shown in Figure 4.10.

TASK 1-1. Zarife found that the lengths of the two pencils were not equal. In what different ways can she describe the relationship between the lengths?

Scaffolding questions:

1. Guess the lengths of pencils? Are they the same length?
2. What else? Which values can we give for the lengths?
3. Can you fill in the table with your estimations?

Length of the Orange Pencil 		*Length of the Yellow Pencil* 

4. What is the unit of your values? (number, cm, kg…)
5. Compare your values in each situation. How can you represent this comparison by using mathematical symbols/comparison symbols?
6. For example, let's say the length of the longer pencil is *a cm* and the shorter one is *b cm*. *a* and *b* can take any value.
7. Now, can you represent the same relationship by using all these symbols? How can you represent it in another way? (a<b; b>a).

Figure 4.10 The first task in Episode 1: Comparing unknown quantities

The intended processes in the first two tasks were to let the students attain hypothetical values for the unknowns on a table, formulate a general case by interpreting the data on the table, and represent this comparison by using assigned letters and symbols. It is important to note that the students can prefer to express the comparison in the shortest way such as “the yellow (pencil) is greater/longer than the orange (pencil)” or “they are equal/same”. In those situations, the teacher emphasized the word “length” and prompted them to use it. In addition, she asked for the units such as “In what unit can it be?” to draw students’ attention to the meaning of units and quantities instead of figural objects. In this way, they focused on the units of the quantities, started using comparison symbols (i.e., > , <) and continued with using equal signs in the second task.

In the third task, the students worked on various comparison situations such as seesaw, and (un)balanced scales (see Figure 4.11).

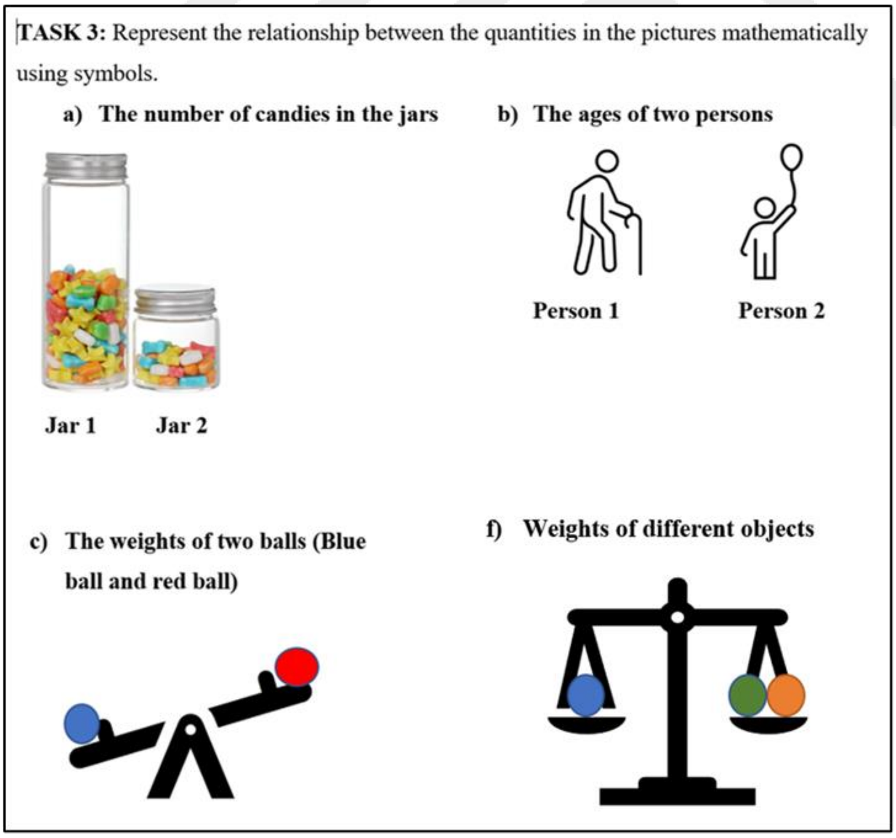


Figure 4.11 Sample questions in the third task of Episode 1

This task allowed them to practice the different contexts and situations in comparing quantities. In terms of the students' representation of the comparison between unknown quantities by using symbols in inequality or equality expressions, the conjecture was that all the students might express the comparison verbally or use numerical examples of the unknowns. Researchers indicated that students from grade 1 to grade 8 have some misinterpretations of the use of algebraic letters such as tending to give numerical values to the letters or interpret them as placeholders instead of interpreting them as generalized numbers (e.g., Ayala-Altamirano, 2022; Hackenberg & Lee, 2015; Küchemann, 1981; MacGregor & Stacey, 1997). Regarding the understanding of variables and unknown quantities, the teacher always asked what $x/a/c$ etc represents. She emphasized that the numbers they wrote on the tables were just their guesses.

In addition, as the MC1 and MC2 students participated in this study, it was hypothesized that students would continue to rely on numerical examples or interpret the letters as placeholders during this episode (Hackenberg & Lee, 2015). However, Roy's (advanced MC2) initial performance in the interviews assessing his algebraic thinking before the teaching experiments showed his use of letters in pattern situations to generalize the relationship as different than other students. Therefore, I expected to see some diversity in Roy's performance in this episode. In addition, I wrote symbolic expressions in different forms to develop equal sign understanding and to develop algebraic thinking in expressions of arithmetic operations. Figure 4.12 presents the task characteristics, intended processes, and certain conjectures.

EPISODE 1			
Descriptions of Tasks	**Intended Processes**	**Conjectures**	**Instructional Moves Aligning with the Conjectures**
Task 1: Comparison of two pencils in different lengths represented in a figure **Task 2:** Comparison of two pencils in the same lengths represented in a figure **Task 3:** Comparison of two or three unknown quantities represented in figures such as un/balanced scales, and jars of sugars	-Express comparison of unknown quantities verbally (e.g., it is longer/ heavier/ older than the other) -Attain hypothetical values for unknown quantities by using tables - Using letters and symbols to represent the comparison - Relational understanding of the equal sign	a) MC1 and MC2 students would compare the unknown quantities and express them verbally at the beginning of tasks. b) MC1 and MC2 students would attain values for each unknown instead of using literal symbols c) MC1 students would not understand how they represent the relationship using symbols. d) Roy or both MC2 students would use the assigned letters to represent the comparison with symbols towards the end of the episode. e) MC1 students would continue to assign values to unknown quantities instead of using symbols. f) MC1 and MC2 students would have difficulty in representing the comparison between three unknowns on un/balanced scales	**Conjectures b – c – e:** -Assign multiple values for each unknown on a table and discuss the generalized comparison -Discuss the comparison symbols in expressing the numerical situations in mathematical language (e.g., =, <, >) -Direct the student to use letters for unknowns by saying "let the length of yellow pencil '*a*' and the length of orange pencil '*b*'." **Conjecture f:** -Use a table to assign values to three unknowns on an un/balanced scale and discuss how to represent two unknown weights on one side in comparison to the other on the other side.

Figure 4.12 Description of Episode 1: Comparison of unknown quantities

4.2.1.1 Students' performances in comparing unknown quantities during Episode 1

The students showed mostly inter-level differences in using symbolic language to represent the comparison of unknown quantities besides that all had similar difficulties and accomplishments. Firstly, all students were able to verbally express a comparative relationship between two unknown quantities which can be visually observed in a figure (see Table 4.2). For example, they indicated that one length is longer than the other or the lengths of the pencils are equal/same.

Table 4.2 The Students' Performances during Episode 1

	MC2		**MC1**	
	Roy	**Belle**	**Sara**	**Luke**
Writing equation/inequality with more than two unknowns (using operations)	2	3	3	3
Writing equation/inequality with two unknowns (without operations)	2	2	2	2
Symmetric property in writing equation	2	3	3	3
Assigning letters for unknowns in the last task	3	1	3	3
Relational understanding of equal sign	1	1	1	-
Verbal expression of comparisons	1	1	1	1
Filling the table with appropriate values	1	1	2	2
Giving numerical values for unknowns	-	-	1	1
Writing equations by using object names or figures (Variables as objects) (e.g., yellow = green)	-	-	1	1

1: Without prompting; **2:** With probing questions, **3:** With leading questions

Note: The level of algebraic thinking increases with color darkness from the bottom to the top of the table

Besides, all students demonstrated difficulty in similar situations such as using comparison symbols with letters assigned to the unknown quantities and using operation signs to represent the quantitative relationships (e.g., $a + b = c$) as conjectured. Secondly, the students showed both inter-level and intra-level differences. MC2 students needed fewer prompts from the teacher as compared to MC1 students in general. Table 4.2 summarizes the findings about the students' performances in using symbolic language in comparison of unknown quantities by demonstrating both inter-level and intra-level differences. In this table, their performances were labeled based on whether they were prompted or given leading questions before answering the questions. The following sections present a detailed description of the findings for each group of students.

4.2.1.1.1 MC2 students' performance during Episode 1

MC2 students (Roy and Belle) demonstrated a higher capability to use letters/symbols representing unknown quantities through the end of the episode although they initially refrained from answering the questions. They were able to use the letters to represent unknown quantities in the last task of the episode (e.g., the number of candies in the jars, the age of people, and the weight of objects on a balance scale) on their first try without needing a prompt from the teacher. This is consistent with the conjectures for Roy (see Figure 4.12). However, Belle's progress was higher than previously conjectured (see Figure 4.12). She was close to Roy's performance.

In a scale model with three unknown quantities, representing this relationship was challenging for the MC2 students because it required using an operation with unknowns. This difficulty was a hypothesized outcome, but the prediction about their performance had not been specifically defined beforehand. Although they identified indeterminate quantities and flexibly expressed them in the previous tasks, they could not consider making operations with them. This might be more demanding for them as it requires analytical thinking (Radford, 2014). With the teacher's help which she gave hypothetical values for each unknown quantity in a table on the board, they could use addition operations in their symbolic representation (i.e., $t + b > g$) by generalizing the table of data on the board. Thus, through the numerical examples, they made a transition from determinate quantities to indeterminate ones. Through this task, they began to work on quantitative relationships in which an operation on two quantities determines the third (Thompson, 1990).

The intra-level differences among MC2 students emerged in comfort with answering the questions and expressing the relationship between unknown quantities instead of achieving lesson goals. Belle generally hesitated to answer questions. Roy was faster and more flexible in generating different expressions. Hence, Belle learned what to do after the teacher's prompts or Roy's answers during the discussion of the tasks. Then she could successfully adopt the procedure in the next tasks by flexibly applying what she learned/saw in the previous tasks. Further, Roy wrote a complete verbal expression in each task, in addition to the symbolic expressions, that showed the comparative relationship between the unknown quantities. Roy was also the only student who used the units of quantities next to the values or letters in the questions such as cm or kg. Just as their intra-level difference in unit coordination appeared through Roy's swiftness and Belle's hesitations, this Episode revealed their differences in the same way.

4.2.1.1.2 MC1 students' performance during Episode 1

The performance of the MC1 students (Sara and Luke) in Episode 1 highlighted several points. First, the MC1 students were inclined to give hypothetical values for the unknown quantities before expressing the relationship symbolically or verbally. Sara continued to write the comparative relationship by using symbols with hypothetical values (e.g., $25 > 10$ or $5 = 5$) even after the first two tasks were completed. This was the conjecture for all the students (see Figure 4.12) while the performance of the MC2 students in this episode was different. This demonstrates that the MC1 students were not ready to use indeterminate quantities as much as the MC2 students, which is an inevitable condition of algebraic thinking (Radford, 2010).

Second, Luke (early MC1) used comparison symbols (i.e., >, <) between the objects in figures or with the object names such as "*yellow = green*" instead of using assigned letters or values. This may demonstrate similarity to a variable understanding of assigned letters as labels or as representing objects instead of quantities, indicating a pre-variable understanding (Blanton et al., 2017). This was another conjecture that was actualized in this episode by only the MC1 students. Given that this performance continued in the third task, developing the variable understanding and letter use for unknowns may not be an easy process for the MC1 students.

Third, the MC1 students had difficulty in representing the quantitative relationship between three unknowns displayed on an unbalanced scale, like the MC2 students. Although they accurately interpreted this relationship, they could not use the operation signs with the letters assigned for the unknown weights. For instance, both MC1 students wrote "$t\,b\ >\ g$" rather than "$t\ +\ b\ >\ g$" in this task (see Figure 4.13).

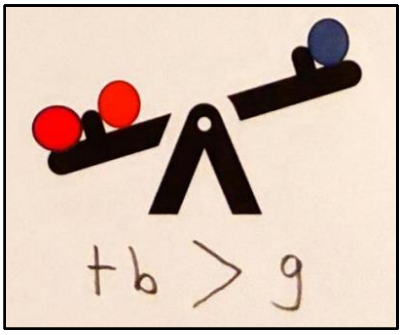


Figure 4.13. Luke's symbolic representation in an item of Task 3

Their representation demonstrates that they identified the relationship as combining both weights t and b is greater than the weight g. However, their inability to use an addition operation with indeterminate quantities demonstrated their lack of analytical thinking (Radford, 2010) and variable understanding (Ventura et al., 2021). These representations showed that the MC1 students made comparisons based on their observations and by labeling the objects rather than the quantities. After a discussion on the board and drawing table and the teacher's questions, they used operation signs with indeterminate quantities. Further, in these tasks including more than two unknown quantities on the scale figures, Sara attempted to give values for each unknown quantity (i.e., the weights of balls). She indicated that she guessed the values. With the help of the teacher’s prompts and questions, both MC1 students assigned letters for the unknown quantities, and they also used symmetry property to express the equalities in different forms (i.e., $a = b + c$; $b + c = a$).

There was not a clear distinction between the performances of the MC1 students in this episode. Their discussions complemented each other. For instance, one of them expressed her/his opinion then the other understood the point and developed it in further questions. Sara had a notable tendency to assign numerical values to unknown quantities compared to Luke. On the other hand, Luke stood out from Sara due to his ability to quickly apply what he learned during the episode. Although Luke initially struggled with using alphanumeric symbols in his representations, he quickly learned how to use them during classroom discussions and was able to put this knowledge into practice.

4.2.1.2 Students’ evaluations at the end of Episode 1

At the end of Episode 1, the students attended an end-of-lesson assessment evaluating their learning in the episode. Their understanding of equality and use of symbols for representing the comparison between unknowns were the focus of the assessment. The results of the final evaluation in Episode 1 showed that all the students learned to assign letters for unknown quantities and represent the relationships by using letters, operations, and comparison symbols.

In terms of equality understanding, MC2 students maintained their relational understanding of the equal sign. One of the MC1 students, Sara, demonstrated a relational understanding of the equal sign too although she incorrectly answered a similar question in the initial interview by focusing on the operational meaning of the equal sign. She seemed to have developed her understanding of equality by working on different equality and inequality situations during the episode. However, the other MC1 student, Luke, continued to have an operational view of the equal sign by writing an incorrect answer in a missing value equation (see Figure 4.14).

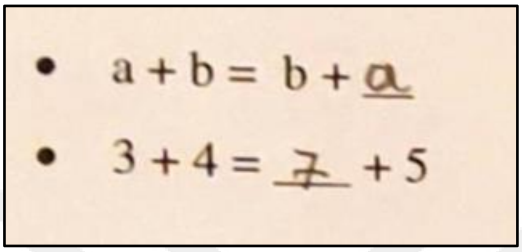
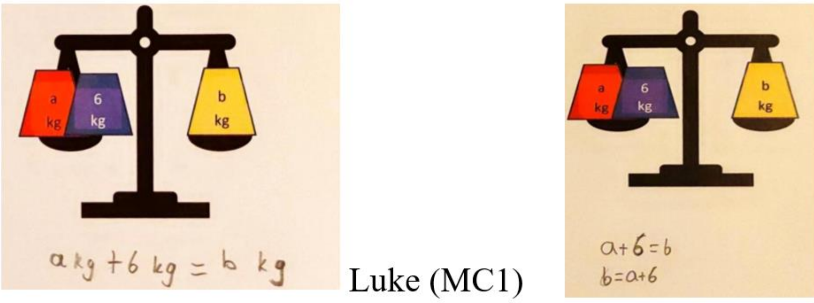


Figure 4.14. Luke's (early MC1) answer to question 1 in the quiz in Episode 1

His answer showed that he had taken the operational meaning of the equal sign, thinking that the answer comes right after an equal sign rather than thinking of its relational meaning (Knuth et al., 2006). This might be an important constraint in thinking structurally to represent the relationships in equations, as a dimension of algebraic thinking. Because of this, the teacher-researcher has decided to add a small task in the second episode including virtual simulations of balanced scales to enhance the understanding of equal signs.

In writing the symbolic representation for the relationship between quantities on the balance scale, the only difference between the two groups of students is that the MC2 students wrote the equations in two forms by using the symmetric property while the MC1 students did not write different forms although they could do it during the episode (see Figure 4.15).



Luke (MC1) Belle (MC2)

Figure 4.15. Written symbolic expressions by an MC2 and an MC1 student

Overall, key findings from Episode 1 in terms of the students' use of symbolic representations in comparing unknown quantities, and their understanding of equal signs are shown in Table 4.3.

Table 4.3 Students' Challenges and Progress during and at the end of Episode 1

		MC2		MC1	
		Roy	**Belle**	**Sara**	**Luke**
Challenges	Using comparison symbols with hypothetical values			D	D
	Using comparison symbols with objects				D
	Operational view of equal sign			D	D, A
Progress-Advancement	A relational view of equal sign	D, A	D, A	A	
	Representing comparisons symbolically	D, A	D, A	A	A
	Using operation signs with the letters	A	A	A	A
	Writing the equations in two forms	A	A		

*D: During the Episode *A: At the end of the Episode

4.2.2 Representing an additive and multiplicative relationship between unknown quantities during Episode 2

The second step in the HLT was developing the students' expression of additive or multiplicative relationships between unknown quantities. Therefore, the intended process in Episode 2 was quantitative reasoning where students analyze the problem situation and determine the quantitative relationships, generalization, and symbolic representation of quantitative relationships. For this aim, the students worked on two tasks: one included a multiplicative relationship between two unknowns, and the other included an additive relationship between three unknowns (see Figure 4.16).

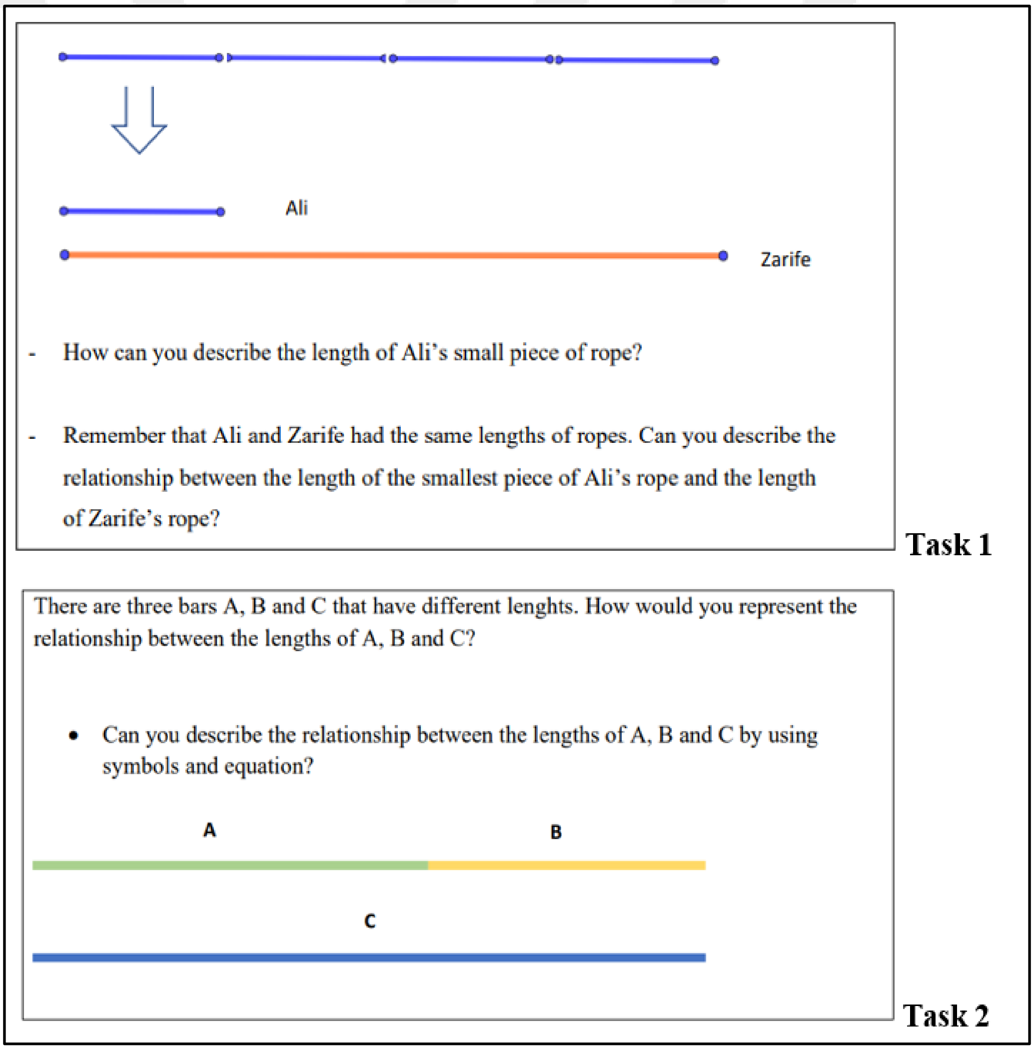


Figure 4.16 The tasks in Episode 2

First, the students worked on a task including a multiplicative relationship between two unknown lengths where one rope is four times the length of the other rope (see Figure 4.16). The second task involved an additive relationship between three unknown lengths where each one was represented by letters. Expressing the comparison of unknown quantities verbally (e.g., It is four times the other, it is one-fourth the other, the sum of A and B makes C) indicates the students' determination of quantitative relationships between two or more quantities as a pre-requisite step before constructing an algebraic equation.

Based on the literature and the learning outcomes of Episode 1, it was assumed that the MC2 students would use symbols to represent the relationships between unknown quantities with the help of prompts in both tasks. Although Hackenberg and Lee (2015) noted that MC2 students tend to use numerical examples, we did not make the same inference for the MC2 students in this study based on their performance in the previous episode. On the other hand, we hypothesized that MC1 students would require more guidance from the teacher to successfully transfer the relationship they have constructed arithmetically into an algebraic one using indeterminate quantities. We also conjectured that MC1 students would use numerical examples instead of letters as they are at a lower level than the MC2 students (e.g., Hackenberg & Lee, 2015). In addition, we hypothesized that MC1 students would perceive the multiplicative task as more challenging than the additive task since they were constructing their first multiplicative concept. The task descriptions, the intended processes, the conjectures, and instructional plans are shown in Figure 4.17.

As conjectured, the teacher used more scaffolding questions in the multiplicative task. In particular, the MC1 students required more guidance from the teacher in understanding the multiplicative relationship between the unknown quantities. Therefore, the teacher used scaffolding questions to move them to a higher step in the task and she also used leading questions giving numerical examples when the prompts were not enough for their understanding.

EPISODE 2			
Descriptions of Tasks	**Intended Processes**	**Conjectures**	**Instructional Moves Aligning with the Conjectures**
Task 1: Expressing the multiplicative relationship between two unknowns by using symbolic expressions **Task 2:** Expressing the additive relationship between three unknowns by using symbolic expressions	-Express comparison of unknown quantities verbally (e.g., It is four times the other, it is one-fourth the other, the sum of A and B makes C) - Attain hypothetical values for unknown quantities by using tables -Use letters to represent the unknowns -Use symbolic expressions to represent the relationships - Relational understanding of equal sign	a) MC1 and MC2 students would express the additive and multiplicative relationships verbally b) MC1 students use the letters to represent the lengths, but they would not write the equations c) MC1 students would assign values for the length of ropes such as 1 and 4 or 2 and 8. d) MC1 students would give numerical values to the unknowns and do operations, but they would not represent the additive and multiplicative relationship using symbolic expressions e) MC2 students would express the additive and multiplicative relationship verbally and symbolically by using letters, operations, and equality. f) MC2 students would write different algebraic expressions representing the same multiplicative relationship (e.g., 4 x a: it is the longest rope; 4 x r = longer rope; r + r + r + r = longer rope; 4 x r = s; s / 4 = r).	**Conjectures a – b – c – d:** -Ask them to use letters and describe the same thing by using symbols - Ask and discuss "Is there another way to represent the relationship (addition /division/ multiplication/subtraction)?" - Emphasize that we do not know the lengths. Ask and discuss the relationship between the assigned numbers. For example, ask: "Which operation can you do to find one?" **Conjecture e – f:** -Ask and discuss "How differently can you demonstrate the same relationship?"

Figure 4.17 Description of Episode 2

For example, when MC1 students' numerical examples were not congruent to the given multiplicative relationship, the teacher showed a pen to represent the longer rope and said to split it into 4 and put a small pen cap to represent the smaller length. She said, "This is 1 and this is four; this is 2 and this is 8; then if this is 3, what could this be?". Therefore, the teacher employed these orientations when the students were stuck during the task.

The teacher aimed to use tables to help students generalize based on different numerical values with a constant relationship. Even though the students used the correct symbolic representation on their first try (e.g., Task 2), the teachers still asked mediating questions, and used the table for generalization. Therefore, she used tables to help both their identification of the relationship and connect different representations for their further assistance.

In developing the students' algebraic thinking, the teacher asked them to use different representations of equations to express the same relationship. When they could not achieve this, she used leading questions. For example, in the multiplicative task, she asked about the length of the shorter rope after assigning a letter to the longer one and vice versa. In the additive task (i.e., $A + B = C$), she asked them to represent the lengths of B and A respectively aiming to use subtraction.

4.2.2.1 The students' performance during Episode 2

The students' performances on two tasks in this episode revealed distinct results because of the relationship involved in the problems as either an additive or a multiplicative one. Therefore, the observed differences between the students in terms of algebraic thinking are presented separately for each task. In perceiving the relationship between unknown quantities and representing them with symbols, all the students performed better in the second task, which covers an additive relationship, compared to the first task which involves a multiplicative one. They responded to the questions in representing the additive relationship mostly without needing prompts (see Table 4.4)

Table 4.4 The Students' Performance during Episode 2

		MC2		MC1	
		Roy	**Belle**	**Sara**	**Luke**
Multiplicative relationship between unknown quantities	Symbolic generalization (Standard algebraic thinking)	1	2	3	2
	Reversing the equation (Structural thinking)	1	-	-	-
	Verbal generalization (Contextual algebraic thinking)	1	2	-	-
	Identifying the relationship (Table)	1	2	3	3
	Inclination to use numeric examples	-	-	1	-
	Inclination to additive thinking	-	1	-	1
Additive relationship between unknown quantities	Symbolic generalization (Standard algebraic thinking)	1	1	1	1
	Reversing the equation (Structural thinking)	1	2	2	2
	Verbal generalization (Contextual algebraic thinking)	1	1	1	1
	Identifying the relationship (Table)	1	1	1	1

1: Without prompting; **2:** With probing questions, **3:** With leading questions

Note: The level of algebraic thinking increases with color darkness

The struggle to recognize the multiplicative relationship was evident in a series of processes starting from filling in the table with the assigned values for each unknown quantity to representing it by using symbols. All the students but Roy tended to use comparison symbols such as greater than and less than in expressing the relationship with symbols in their first attempt as they did in the first episode. Their first answer to express the relationship between two lengths of ropes was an expression involving the comparison symbols or terms such as longer and shorter. In addition to this commonality among the students, their performance in further sub-questions of the task such as filling the table and representing the relationship with symbols revealed differences in both inter-level and intra-level perspectives.

In addition to the general summary of findings in Table 4.4, these differences are presented in further sections, starting with MC2 students, in detail. Finally, the students' final evaluations in relation to the lesson objectives are presented.

4.2.2.1.1 MC2 students' performances during Episode 2

MC2 students required prompts to identify the multiplicative relationship between two lengths of ropes. They filled a table with appropriate values to generalize the relationship to indeterminate quantities, which was a conjectured outcome for only the MC1 students. Roy's performance, aside from his initial attempt in the first task, was more in line with the lesson conjectures for the MC2 students compared to Belle. Belle demonstrated a deviation from what was expected of her before the experiment.

In the given problem, the MC2 students initially expressed the relationship between the lengths of two ropes by stating that "The length of Zarife's rope is longer than the length of Ali's rope" and vice versa. However, when asked to fill the table according to the relationship in the problem, only Roy (advanced MC2) filled the table congruent to the multiplicative relationship with a ratio of 1: 4 between the quantities (see Figure 4.18, left). On the other hand, Belle (regular MC2) filled the table additively. In other words, she recorded her estimations for two quantities, such that the difference between them remains constant (see Figure 4.18, right). Belle stated that she gave such values by considering only one rope was longer than the other. After this, Roy explained his values by referring to the multiplicative relationship by saying: "I thought that the length of Ali's rope should be one-fourth of Zarife's rope." The tables and their explanations demonstrated that, in their first attempt, Belle did not recognize the multiplicative relationship between the unknown quantities while Roy expressed it using a ratio. The values assigned for the unknowns represented the difference in the mental operations of MC2 students. However, Belle could interpret the relationship correctly after hearing Roy's explanation of his table and the teacher's guiding questions.

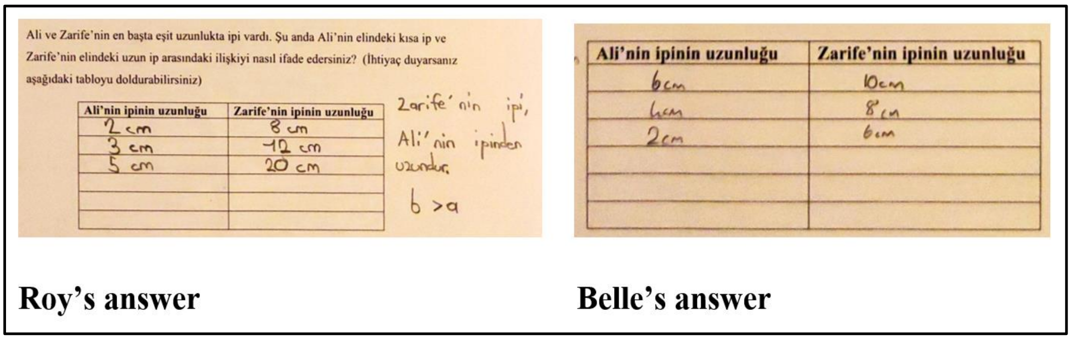


Figure 4.18. MC2 students' tables in Task 1 including multiplicative relationship

In representing the unknown quantities with symbols such as expressing one quantity in terms of the other by using the assigned letters, Roy successfully used the letters to represent the relationship in his first attempt (see Table 4.5). Belle could not write any symbolic expression initially. Although she understood that one unknown quantity should be multiplied by four and the other should be divided by four, she could not express it by using letters and operations. Hence, her verbal generalization of the multiplicative relationship did not advance to a symbolic level without the teacher's guidance. After the discussion of the first question requiring a symbolic representation, Belle understood how to use letters and operations together to express the recognized relationship between unknown quantities in further questions. In the last question, she could use symbolic language accurately to represent the multiplicative relationship between unknowns. Her final performance in representing the multiplicative relationship was consistent with the conjectures before the lesson. The teacher's guidance and discussion in the previous question helped her to achieve this as seen in Table 4.5. Compared to Belle, Roy used division to invert the equation he wrote using multiplication in the last question (e.g., $m = k \times 4$ and $m \div 4 = k$). His equivalent expressions of the relationship clearly show his algebraic thinking through the structures and relationships embedded in his operations and operations with indeterminate quantities (Kieran, 2022; Radford, 2014).

Table 4.5 MC2 Students' Symbolic Expressions for Task 1 in Episode 2

Questions	**Roy's written answers**	**Belle's written answers**
How can you represent the length of Ali's rope?	No answer	No answer
Let the length of Ali's rope be a, how could you express the length of Zarife's rope	$a \times 4 = b$ → the length of Zarife's rope	No answer
Let the length of Zarife's rope b, how could you express the length of Ali's rope	$b \div 4$	$b \div 4$ $4 \div b$ (deleted)
Let the length of Zarife's rope be m and the length of Ali's rope be k, how can you express the relationship between m and k?	$m = k \times 4$ $m \div 4 = k$	$k < m$ $m > k$ $k + k + k + k = m$ $m = k \times 4$

Belle represented the relationship by using two equivalent expressions too, but her equations included only multiplication or addition. This also showed her structural and analytical thinking like Roy. However, Roy's expressions are more sophisticated in terms of algebraic thinking by reversing the multiplicative relationship. Belle's equations represent her construction of the relationship between repeated addition and multiplication. In addition, she made a mistake in expressing the length of Ali's rope in the third question by confusing the quantities (see Table 4.5), which indicates a problem in recognizing the indeterminate quantities in the problem and structural thinking. In short, the diversity in the amount of guidance they needed from the teacher and in their equations represented the difference in their progression of algebraic thinking, indicating an intra-level difference.

In representing the additive relationship between three unknown lengths, the MC2 students correctly wrote equations by using symbols before filling the table as conjectured. However, their expressions of the equations involved some differences. Roy wrote both $a + b = c$ and $c - b = a$ which are the inverse forms of the same relationship as he did in the previous task. Therefore, Roy showed a solid understanding of the relationship. Belle wrote two equations as well, but her equations involved only the symmetric forms such as $A + B = C$ and $C = A + B$. Although the teacher asked her whether she could provide another equation to show the same relationship, she was unable to do so. This highlights Roy's sophisticated understanding of the relationships and operations with structures, as another evidence of intra-level difference.

In representing the difference between the lengths of C and A or the lengths of C and B (see Figure 4.16), Roy easily wrote the equations: $C - A = B$ and $C - B = A$. However, Belle had difficulty understanding the question and she could not answer without the teacher's help. After the teacher emphasized the term "difference" to assist her in remembering the subtraction operation, she could write an appropriate equation using symbols to show the difference between C and B.

Overall, Belle needed more prompts from the teacher to understand and represent the relationships as compared to Roy. She could represent the multiplicative relationship by using letters and equality with the help of the teacher's prompts and interpretations of Roy. On the other hand, Roy was quicker in understanding and expressing relationships. Additionally, his flexibility in writing equations in inverse forms displays him as a more analytical and structural thinker.

4.2.2.1.2 MC1 students' performance during Episode 2

In comparison to the MC2 students, the MC1 students, Sara and Luke, needed more leading questions to understand the multiplicative relationship between the unknown quantities. They had considerable difficulty in identifying the multiplicative relationship. They underperformed in this task regarding the conjectures, indicating that they could have been able to verbally express the multiplicative relationship and assign appropriate numerical values to the unknown quantities. Therefore, the discussion and the teacher's guidance were quite intensive in this part for their identification of the multiplicative relationship.

The MC1 students first indicated only the comparison between the long and short ropes by representing it through symbols such as $a < b$ like the MC2 students' initial answers. In the next step, in their tables, they did not assign numerical values as congruent to the given multiplicative relationship, which indicates their lack of recognition of the multiplicative relationship (see Figure 4.19).

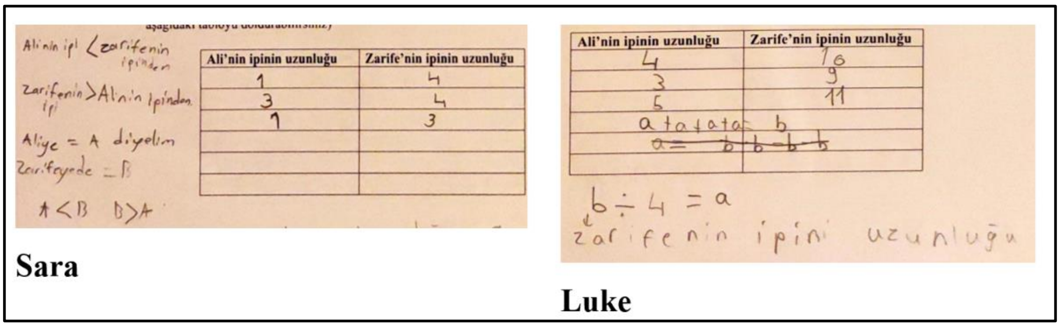


Figure 4.19. MC1 students' assigning of values to the lengths of ropes in Task 1

As seen in Figure 4.19 Luke filled in the table in such a way that the difference between two quantities is constant, which is six. He explained this constant difference: "I thought there was a difference of 6 because there is a big difference between two lengths". Luke here made an inappropriate additive comparison by guessing a difference between the lengths which is a less complex comparison in terms of units coordination compared to a multiplicative one (Ulrich, 2016b). On the other hand, Sara explained her values in the first row (i.e., 1 and 4) as follows: "The longer rope was divided by four". However, the other values on her table did not correspond to the same relationship. Instead, they appeared to be random numbers chosen simply for being longer or shorter. Therefore, her interpretation of the comparison corresponds to neither an additive nor multiplicative one.

The teacher helped MC1 students understand the multiplicative relationship between unknown quantities by showing them how to split the longer rope equally to generate the shorter rope, iterate the shorter rope to make the longer rope, and fill in a table with newly assigned numbers matching with the given multiplicative relationship (see Figure 4.20).

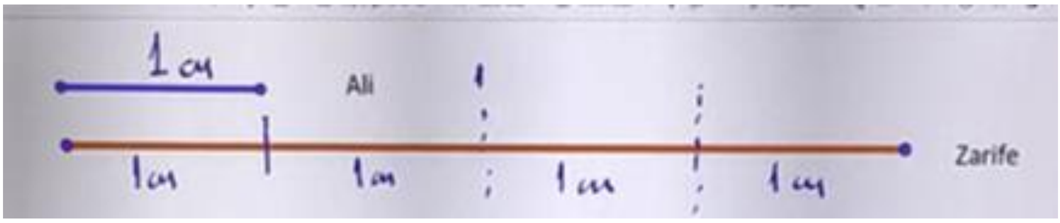


Figure 4.20. The teacher’s representation of Task 1 during the discussion

After splitting the longer rope and assigning values to the shorter rope, the teacher asked: “What length could Zarife’s rope (the longer one) be if Ali’s rope were 4 cm long?” The students could not answer the question, so she modified it: “If the longer rope was 4 cm long, what could be the length of the shorter rope?” Luke’s answer was 1 cm, and Sara agreed because the teacher had provided the same values earlier. Then the discussion continued:

> **Teacher:** If the longer rope was 8 cm long, what could be the possible length of the shorter rope?
>
> **Luke:** Could it be 2 cm?
>
> **Teacher:** Why?
>
> **Luke:** I thought when we multiply 4 (the first given value to the longer rope) and 2 it becomes 8 so Ali’s rope can be 2 cm long.
>
> **Teacher:** You think, if this is doubled (she shows the column of Zarife’s rope length), the other should be doubled too?
>
> **Luke:** Yes

Luke relied on the change in just one column (variable), showing basic recursive thinking. The teacher explained how the longer rope is divided into four equal parts, generating the shorter rope, and provided additional numeric examples by also requesting from the students. Luke began accurately determining values for the other variable multiplying by four, while Sara still concentrated on the change across the cells in a single column by thinking recursively for a little while longer. The teacher clarified the generation of the shorter rope from the longer one until Sara correctly listed the values in her table. Here, the teacher highlighted the relationship between two quantities that vary simultaneously, rather than focusing solely on the change in one variable. Filling the table with appropriate values did not help them much in expressing the multiplicative relationship by using indeterminate quantities because they relied on recursive thinking instead of identifying the relationship between the columns. However, covariational reasoning, which is an approach to functional thinking, was partially constituted by the help of the teacher.

After completing the table and discussing the relationship between two unknown quantities, the teacher attempted to ask questions by using indeterminate quantities such as: "What can we do to generate the length of Zarife's rope by using Ali's rope and what can we do to generate the length of Ali's rope by using Zarife's rope?". Initially, both MC1 students stated "Three more are required to obtain the longer rope" instead of stating multiplication by four as follows:

> **Luke:** We add three times the length of it (shorter rope) to get the longer one.
>
> **Sara:** We add 30 cm more to obtain the longer rope if the shorter rope, lets say to be 10 cm.

Their responses demonstrated a utilization of an additive comparison by focusing on the difference which is less demanding reasoning (Ulrich, 2016b). Researchers explained this constraint as resulting from a lack of splitting and disembedding operations which is a characteristic of MC1 students (Hackenberg, 2013; Steffe & Olive, 2010). Because they could not disembed the shorter bar from the longer one, they could not construct the multiplicative relationship between those unknown lengths. The teacher stressed the importance of starting from scratch, without assuming the presence of any rope in the beginning. After the discussion, Luke accurately explained how to perform operations using indeterminate quantities like: "We divide it by four" and "We multiply it by four".

In writing the symbolic representation of the multiplicative relationship, the MC1 students generated different expressions from the beginning to the end in which the variables were slightly integrated into the questions. For example, Luke wrote a sum of four identical values, $a + a + a + a = b$ after the discussion of the multiplicative relationship while Sara could not even write it using addition. She only wrote operations with numerical values such as $4 - 3 = 1$ on her first try. During the discussions, Sara first stated that $b = a \times 3$, demonstrating that she still thought about adding over the first rope to make a longer rope which was their constraint at the beginning of the lesson. However, she later corrected herself by writing $a \times 4$ to represent the length of the longer rope when the teacher asked additional questions (see Table 4.6).

Table 4.6 MC1 Students' Symbolic Representations of Multiplicative Relationships

Questions	**Sara's written answers**	**Luke's written answers**
How can you represent the short length of Ali's rope?	A = 1 (deleted) $5 < 10$ $A < Z$	$a = 1cm$
Let the length of Ali's rope a, how could you express the lengths of Zarife's rope	$a \times 4$	$a + a + a + a$
Let the length of Zarife's rope b, how could you express the lengths of Ali's rope	$b \div 4$ $b - 4$	$a = b - b - b - b$ $b \div 4$ $b - 4$
Let the length of Zarife's rope m and the length of Ali's rope be k, how can you express the relationship between m and k?	$k < m$ *(deleted)* $m > k$ *(deleted)* $k = m - 4$ $m = k \times 4$	$k \times 3 = m$ $k + k + k + k = m$

Although the MC1 students recognize and represent the multiplicative relationship with the leading questions of the teacher, they occasionally wrote incorrect equations for the relationships as seen in Table 4.6. For example, Luke attempted to write the relationship in a new form by incorrectly reversing the equation as $a = b - b - b - b$. This represents a lack of structural thinking in which he is not thoroughly aware of the relations between indeterminate quantities (Kieran, 2022). Because of this, he could not elaborate on the initial structure of representation. Both students also wrote $b - 4$ and $b \div 4$ together to represent the same relationship as different forms. Like using repeated addition, they might have attempted to the repeated subtraction to indicate the same relationship in the reverse form. This way of thinking may demonstrate their undeveloped multiplicative structures. It was a common point between the MC1 students that they could not write equivalent expressions to represent the multiplicative relationship. However, the MC2 students could accomplish this in their lessons.

The MC1 students could represent the additive relationship between three unknown quantities by using symbols and equations in their first attempts without completing the table (see Figure 4.21).

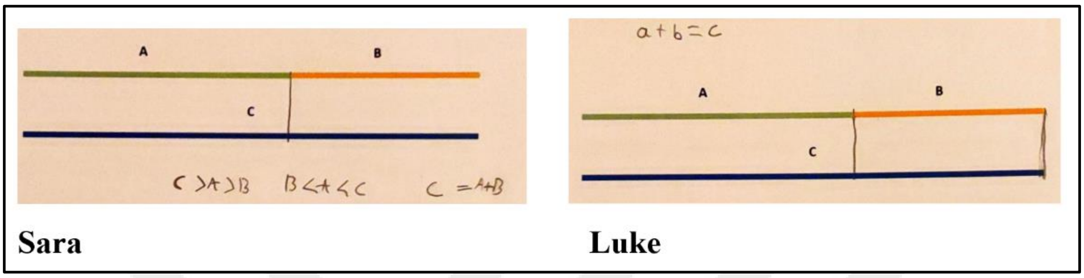


Figure 4.21. MC1 students' representation of an additive relationship

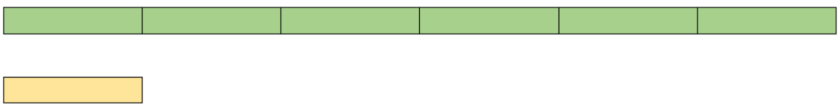
Therefore, they performed in the opposite direction to the assumptions. After they represented this relationship with symbols, they also filled in the table with values so that the sum of the values assigned for A and B was equal to the value assigned for C.

The MC1 students had difficulty understanding the question: "How can you express the difference between the lengths of A and C?" Sara wrote the equation $c - b = a$ while Luke wrote different expressions such as $A = 4$; $C = 8$; $A + A = C$ and $A < C$. The teacher showed how to write the opposite form of the same relationship to both students. In the next question about the difference between the lengths of B and C, Luke correctly wrote the equation $C - A = B$ using letters. However, Sara incorrectly wrote $A - C = B$ by confusing the lengths. The teacher intervened and helped her correct the equation.

Overall, the MC1 students had notable difficulty in understanding the multiplicative relationship between two unknown quantities compared to an additive relationship. They tended to think additively in problem situations and this lack of operating with multiplicative situations made it difficult for them to work with indeterminate quantities. Contrary to conjectures, the main trouble was about their understanding of the relationship within the problem situation instead of representing it using indeterminate quantities. The difference in their performance in additive and multiplicative tasks demonstrated that the main constraint in their algebraic performance resulted from the difficulty in identifying the relationships in the problem situations, not writing equations. Even so, writing equations demonstrated some other differences in their algebraic thinking as well. For example, the inability to reverse a multiplicative situation in an equation demonstrated their lack of structural thinking as a dimension of algebraic thinking (Kieran, 2022). The teacher needed to provide continuous support for them to accurately answer each question. Although they could reach the correct thinking in some instances, they mostly could not maintain this in further steps.

4.2.2.2 Students' evaluations at the end of Episode 2

In the end-of-lesson assessment of Episode 2, the given task asked the students to represent the multiplicative relationship between two lengths of sticks. The sticks were represented as a union of equal parts so that the students could see the multiplicative relationships between them (see Figure 4.22).



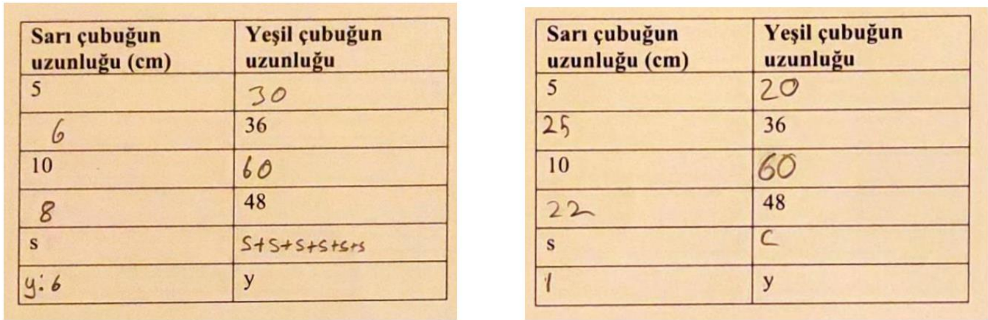
Question 1: Based on the relationship in the picture above, fill in the table.

Length of yellow	Length of green bar
5	
	36
10	
	48
s	
	y

*What is the relationship between the length of the yellow bar and the lengths of the green bar?

Figure 4.22. The questions of the post-assessment in Episode 2

Like previous tasks in the episode, the students had to complete a table with various values and use assigned letters for the lengths to write an equation. The students displayed varying levels of performance when filling out the table in the first question. Figure 4.23 represents two examples of them.



Sarı çubuğun uzunluğu (cm)	Yeşil çubuğun uzunluğu
5	30
6	36
10	60
8	48
s	s+s+s+s+s+s
y:6	y

Belle

Sarı çubuğun uzunluğu (cm)	Yeşil çubuğun uzunluğu
5	20
25	36
10	60
22	48
s	c
1	y

Sara

Figure 4.23. Belle's (left) and Sara's (right) tables in the end-of-lesson assessment

Specifically, Roy (advanced MC2), Belle (regular MC2), and Luke (early MC1) accurately recorded the numerical values in the table by taking into account the multiplicative relationship of 1/6. However, Luke left a cell empty in the table where he was asked to write the length of the yellow stick when the length of the green stick was 48. This required him to perform a division operation. In addition, Sara, another MC1 student, provided random estimates for the stick values in her table (see Figure 4.23), similar to what she did during the lesson. Sara's values showed that she could not identify a multiplicative relationship between the lengths initially. On the other hand, two MC2 students and one MC1 student (Luke) could identify the multiplicative relationship accurately in filling the tables.

In using the letters in the last two rows of the table, Belle was the only one who used the assigned letter to further operate on to express the other unknown. As seen in her table (see Figure 4.23), she wrote $s + s + s + s + s + s$ to express the length of the green stick when the letter s represents the length of the yellow stick. On the other hand, the other students only used the letters s and y as seen in Sara's table without representing the length of one stick by using the assigned letter for the length of the other stick. Therefore, Belle expressed the relationship between two unknowns by using letters starting from this step.

In writing equations to represent this multiplicative relationship, the more sophisticated responses came from the MC2 students. They wrote different accurate symbolic expressions to represent the multiplicative relationship between two unknown quantities. Both students could write various forms of equations such as inversing the equation. Belle also wrote the equation by using addition (see Table 4.7). On the other hand, MC1 students performed differently. Sara's expressions demonstrated that she could understand the multiplicative relationship accurately unlike her performance in completing the table. However, she was confused about the correct letters representing the length of different sticks in writing the equation in the last question (see Table 4.7).

Table 4.7 The Students' Expressions of the Relationship in the End-of-Lesson Assessment

Students	Writing the relationship between the length of yellow and green sticks? (Question 1)	Writing the relationship between a and b using equations? (Question 2)
Roy	$s \times 6 = y$ $y \div 6 = s$	$a \times 6 = b$ $b = a \times 6$ $b \div 6 = a$
Belle	The length of the yellow bar is one-sixth of the length of the green bar.	$a + a + a + a + a + a = b$ $a \times 6 = b$ $b \div 6 = a$
Sara	Green = A Yellow = b $A = b \times 6$	$b = a \div 6$
Luke	$y \times 5 = s$	$a \times 5 = b$ $b \div 5 = a$

Luke wrote an equation involving an incorrect multiplicative relationship. He wrote the equation $y \div 5 = s$ rather than $y \div 6 = s$. He skipped counting the small road in interpreting the multiplicative relationship as he did during the episode. Those errors showed their trouble in keeping track of the following operations after they constructed each quantity or variable in problems (Ulrich, 2016b).

Furthermore, MC1 students' final performances demonstrated that Sara's progression in algebraic thinking is more evident than Luke's. Although both students attempted to the same mistake by confusing the ratio between the unknown quantities during the episode, Sara abandoned this in the final assessment by interpreting the multiplicative relationship accurately, disregarding the confusion with literal symbols.

4.2.3 Representing functional relationships between variables in the form of $y = ax$ during Episodes 3 and 4

In the third and fourth episodes, the aim was to introduce the tasks for developing the students' functional thinking. These tasks involved identifying the variables in contextual problems, identifying the relationship between variables and representing the relationship using symbols. In these episodes, the problems included the functional relationship in the form of $y = ax$ where there is one independent variable, one dependent variable and a constant rate of change.

In Episode 3, there were two main tasks including discrete variables such as the relationship between the number of legs and the number of chairs or the relationship between the number of ears and the number of people, which were introduced for generalizing the functional relationships in elementary years by Blanton et al. (2011). The first task and several questions related to it are presented in Figure 4.24. In Episode 4, the same form of functional relationship was introduced in different contexts such as the relationship between the number of months and the amount of saved money or the number of tables and the number of people seated around the tables. The main part of each problem is presented in Figure 4.25.

TASK1 (Chair and legs): Suppose that you have some chairs, and each chair has four legs. How would you describe the relationship between the number of chairs and the corresponding number of chair legs?

Additional Questions:

1. How many legs would there be if there were 1/2/3/5/100 chairs?
2. Can you fill in the table by considering the relationship between the number of chairs and the corresponding number of chair legs?

Number of chairs	1	2	3	4	5	100
Number of legs						

3. Can you write a formula/rule representing this relationship?

Figure 4.24 A sample task in Episode 3

The Saving Money Problem: Suppose that you save 5 Lira each month. How would you describe the relationship between the number of months and the total amount of money you saved?

Can you interpret the new relationship between time (month) and the total amount of money saved represented in the table below? Please fill in the blank spaces in the table.

Time (month)	1	2	3	4	5		100
The amount of money saved (TL)	12	24	36	48		84	

The Pool Problem: 2 tons of water flows into an empty pool in 1 hour from a tap that flows at a constant speed. How would you describe the relationship between the amount of water in this pool and the elapsed time (hours)?

The Birthday Party 1 Problem: Suppose that at a birthday party, two people can be seated at a square table. If the desks are joined end to end, no one sits on the ends, one person can sit on each of two sides of a desk, how would you describe the relationship between the number of tables and the total number of people who have been seated?

Figure 4.25 Main problems in Episode 4

The intended processes in these episodes were identifying problem variables, using function tables, identifying and generalizing the functional relationship in the table of data (covariational or correspondence thinking), and using equations and letters to represent the functional relationship (see Figure 4.26). Specifically, it is aimed to support the students' recognition of the variables in contextual problems, understanding how two variables vary depending on each other, and representing this relationship between two variables using symbols.

Based on the literature and the learning outcomes of Episodes 1 and 2, it was hypothesized that the MC2 students would identify the relationships between two variables after drawing the tables and would represent them using symbols and equations. Zwanch (2022a) observed that ENS students (MC2) could successfully and quickly use symbolic representations for generalizations in all tasks. In addition, Hackenberg and Lee (2015) found that some MC2 students could write equations using whole-number coefficients to represent the multiplicative relationship between two unknown quantities without inversing the equation. Based on the MC2 students' written equations in Episode 2, including inverse forms (i.e., both multiplication and division), it was inferred that they would also inverse the equations in representing the functional relationships between variables.

On the other hand, it was conjectured that the MC1 students' tendency towards recursive thinking (Zwanch, 2022a) may be a constraint for them to identify and express the functional relationship indicating the covariation between two variables. Therefore, they would require additional scaffolding questions to develop functional thinking by interpreting the function tables and contextual situations. Hackenberg (2013) reported that MC1 students had difficulty in generalizations and writing equations representing the multiplicative relationship because of a lack of disembedding operation. She observed that only a few MC1 students were able to write equations with the interviewer's support. Hence, it was assumed that MC1 students would require more teacher support in the lesson, including generating and interpreting tables together and connecting variable quantities on the tables. Furthermore, incorporating drawings to help in problem comprehension would assist students in visualizing the relationship and lower the task's complexity.

EPISODES 3-4			
Descriptions of Tasks	**Intended Processes**	**Conjectures**	**Instructional Moves Aligning with the Conjectures**
3-1) Chair and Legs: The relationship between the number of chairs and the number of legs **3-2) The number and the number:** dogs-legs, people-ears, people-noses **4-1) The Saving Money Problem:** The relationship between time and the total amount of money saved **4-2) The Pool Problem:** The relationship between the amount of water in a pool and the elapsed time **4-3) The Birthday Party 1 Problem:** The relationship between the number of tables and the number of people who are seated.	-Identify the variables - Use function tables to represent the data by assigning values for variables -Identify and generalize the functional relationship in tables of data (covariational or correspondence thinking) - Use equations and letters to represent the functional relationship.	a) MC2 students would calculate any corresponding value in function tables. b) MC1 students would not calculate the larger values in function tables because of recursive thinking. c) MC2 students would indicate the functional relationship verbally by using indeterminate quantities and write equations by using symbols. d) MC1 students would not indicate the functional relationship by using indeterminate quantities and letters. e) MC1 students would have difficulty understanding the problem about the relationship between the number of tables and the number of people seated around the tables	**Conjecture a – c:** -Ask them to use different strategies and explain the relationship by using different expressions and equations (Developing structural thinking) **Conjecture b – d:** -Fill the table together on the board and ask about the relationship between two variables. Indicate the names of each variable in discussing each case. Let the students interpret the change in both variables simultaneously. **Conjecture e:** Ask them to draw models to represent each situation. Show one table, two tables, and three tables on the board respectively, and ask them to interpret the situation.

Figure 4.26 Description of Episodes 3 and 4: Functional thinking ($y = ax$)

During the actual learning process, the teacher guided the learning process based on conjectures and literature. In representing the functional relationship, the teacher paid more attention to the students' forms of thinking (e.g., recursive and covariational) and their generalization processes. Therefore, filling the tables and interpreting the data within them had the most labor-intensive processes in each task. In the case of recursive thinking, she forced them to calculate the bigger steps in the table. In addition, she drew their attention to how the two variables covary by emphasizing the changes in both columns/rows simultaneously. For example, she indicated how one variable changes as the other variable increases by any amount.

In using symbolic representations and writing equations, the teacher always referred to the process where the students calculated the corresponding values for each variable in tables when they had difficulty in writing equations. She also asked the students or demonstrated by herself the equivalent expressions such as $d = 3 \times c$, $3 \times c = d$, and $d = c \times 3$. In this way, she aimed to develop their understanding of equivalence and properties of operations by using algebraic reasoning. Moreover, she did not rush the students to use inverse forms of equations when she felt that they were not ready yet. In these situations, she also expected the students to learn from each other by listening to more developed thinking processes and operations. Students' performance during Episodes 3 and 4

Talking about the general findings, the students' performances on two episodes revealed distinct patterns for each group of students (see Table 4.8). The MC2 students demonstrated functional thinking by completing all steps of tasks successfully with no or few probing questions On the other hand, MC1 students had difficulty in completing each step of the tasks by needing probing or leading questions. In addition, they also relied on recursive (Sara and Luke) and additive thinking (Luke) as different than the MC2 students. In the following sections, MC2 and MC1 students' progression in generalizing and representing the functional relationship is presented respectively.

Table 4.8 The Students' Performance in Episodes 3 And 4 (y = ax)

	MC2		MC1	
	Roy	**Belle**	**Sara**	**Luke**
Symbolic representation of functional relationships	1	1	2	2
Reversing the equation	1	2	2	2
Correspondence approach	1	1	2	2
Covariational approach	1	-	3	3
Verbal generalization	1	2	3	2
Generalization after filling the table	-	1	2	2
Recursive thinking	-	-	1	1
Additive thinking	-	-	-	1

1: Without prompting; **2:** With probing questions, **3:** With leading questions

Note: The level of algebraic thinking increases with color darkness from the bottom to the top of the table.

4.2.3.1.1 MC2 students' performances during Episodes 3 and 4

Roy started to express the functional relationship by using indeterminate quantities from the beginning of Episode 3. For example, in the Chairs and Legs Problem, he wrote: "The number of chairs × the chair leg = The total number of legs". This expression displays his analytical thinking although he did not write the variables in an appropriate way such as the number of legs on a chair. From his equation including verbal statements of variables, it was clear that he formulated the functional relationship as follows: When multiplying the number of chairs by the number of legs on each chair, the total number of legs is obtained.

Similarly, he could express the relationship verbally in a flexible way in further tasks by mostly using an operational formula. For example, in the Pool Problem (Episode 4, Task 2), they were given another table of data in which they needed to identify the relationship between the elapsed time (in hours) and the total amount of water collected in a pool. After he filled in the table with appropriate values according to the relationships between variables (see Figure 4.27) he explained what he recognized about the relationship between variables as follows:

> **Roy:** I divided 5 by 1; I divided 10 by 2 and it is 5; I divided 15 by 3, it is 5. I recognized it is always five. To find the total amount of water in the pool, the time should be multiplied by 5. (It is how he wrote in Figure 4.27).

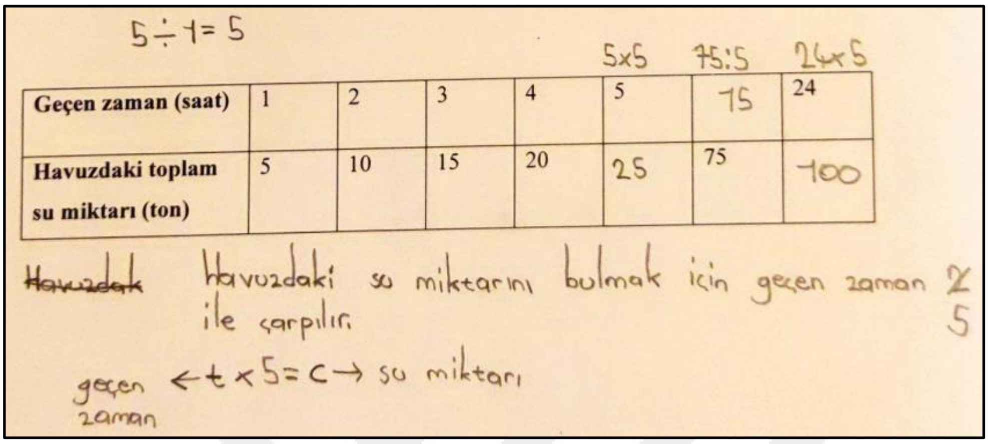


Figure 4.27. Roy’s operations in the Pool Problem

His recognition of the functional relationship generally represented a correspondence approach in which his formula defined one variable in terms of the other variable (Smith, 2008) such as “if I multiply the number of hours by five, I find the amount of water”. Roy’s corresponding thinking in identifying the relationship between two variables allowed him to write the symbolic equations quickly before writing a verbal generalization during Episode 4. Further, he sometimes stated the relationship in the form of covariation such as “As the number of tables increases by one, the number of people increases by two” (the Birthday Party 1 Problem in Episode 4). Therefore, his algebraic performance regarding the generalization of the functional relationship was as conjectured before the teaching experiments.

Unlike Roy, Belle had difficulty in expressing the relationship between two variables verbally or symbolically at the beginning of the tasks although she could identify the relationship by writing the appropriate values in function tables. She needed to fill the tables to understand the relationship between the variables in these Episodes. For example, in the Chairs and Legs Problem (Episode 3), she explained the rule of filling the table by *multiplying by four*, indicating that she could mentally identify the relationship. It resembles how she checked her responses through drawings in the unit coordination tasks although she constructed the correct reasoning without relying on drawings as an MC2 student. In all cases, interactions with Belle revealed that she required supporting materials, such as drawings, numerical examples, and tables to confirm the relationships she had formulated in her mind. She asked more questions to the teacher such as “the number of legs mean for just one chair?”. After the teacher's discussion with her, she could express the relationships such as “the number of legs is always four times the number of chairs”, indicating a correspondence approach of functional relationship. She showed her algebraic thinking after reflecting upon the contextual situation and taking scaffolding from the teacher while Roy did not require extra work on the problem or probing questions to identify the relationship (see Table 4.8).

MC2 students’ performance in using symbolic representations showed some differences from each other like in their identification of the relationship. After the first task in Episode 3, both students quickly began writing equations to represent the functional relationships. Although Roy used indeterminate quantities in the first task (the Chairs and Legs Problem) and filled the table correctly, he confused the names of variables in writing the equation. He considered that the number of legs means the number of legs on one chair, and he wrote: “c × d = the number of legs” (c: the number of chairs and d: the number of legs). He then changed his equation and wrote: “d ÷ c = the number of legs in the chair.” Considering his understanding of variable names, he accurately formulated the relationships in symbols. After a small perturbation in writing a symbolic representation, he reconstructed his understanding in the abstract level by relating different representations of the relationship (von Glasersfeld, 1995). In the following tasks, he wrote all the functional relationships in symbols and equations correctly by using assimilated structures he just reflected upon. Additionally, he wrote the inverse forms of the equations after the teacher asked him to write the relationship in a different way such as writing both $a \times 2 = c$ and $c \div 2 = a$ for the relationship between the number of ears and the number of people in the People and Ears Problem.

Belle spent more time understanding the problems and the relationships in each task. After discussing the problems, interpreting the table of values, and reflecting upon her rule in writing each value on the tables, she could more flexibly move to writing equations. Like in the previous episodes, she learned much from Roy's responses and she flexibly applied what she learned in the following tasks. For example, when Roy expressed the relationship between the amount of water and the time in the Pool Problem, she could write an equation representing this relationship as $h \div t = 5$. She could write the symbolic expressions and equations in all the tasks more easily than verbalizing the relationship. Her performance in Episode 4 demonstrated that she is very good at using symbolic language after she identified or verbalized the relationship between the variables. She learned to write the inverse forms of equations in Episode 3 after the teacher's request and Roy's responses. In Episode 4, she wrote inverse forms of her equations before being prompted by the teacher.

All those performances on tasks involving functional relationships demonstrated that Roy could transition to standard algebraic thinking (Radford, 2010) while Belle required assistance to reach that level from the contextual level. Additionally, Belle invested more time in comprehending the relationships and generating the formulas. This pattern is consistent between the two students since the beginning of Episode 1

4.2.3.1.2 MC1 students' performance during Episodes 3 and 4

The MC1 students demonstrated a partial progression in algebraic thinking in terms of using function tables, generalizing the relationship, and using symbolic representation along Episodes 3 and 4. Their thinking was often supported by the teacher's scaffolding and leading questions in most of the tasks so that they could move to the other steps of the tasks. Their performance on the first task in Episode 3 (i.e., the Chairs and Legs Problem) was the least developed one compared to the other tasks in Episodes 3 and 4. Therefore, it is better to present their performance and constraints in this task first to display their progress along these two episodes.

In expressing the relationship between the variables in the Chairs and Legs Problem (Episode 3), the MC1 students encountered failures and some difficulties compared to the MC2 students as conjectured. Neither of them could identify the functional relationship even after filling a table together. Sara initially attempted to use comparison symbols such as $a > b$. Luke misinterpreted the multiplicative relationship between two variables by considering the difference between them would be constant by saying: "If there is one chair it has four legs, if there are two chairs there are five legs. There is always a particular difference". Focusing on an additive comparison is a less complex operation than a multiplicative comparison in terms of units coordination (Ulrich, 2015b). Therefore, Luke's failure in interpretation of the relationship might have resulted from his operations at the MC1 level which allows him to make an additive comparison but not for multiplicative one.

Because they failed in the identification of the relationship, the teacher filled the table together with them by discussing the relationship between the variables. It helped Luke to calculate a bigger step in this pattern situation (e.g., the total number of legs of a hundred chairs). Luke could indicate the relationship by formulating the rule by saying: "We always multiply the number of chairs by four" and Sara listened to him. A remarkable characteristic of Sara in formulating the rule and generalizing the relationship was her reliance on recursive thinking. She expressed the relationship by saying "We find it by counting by fours" or "It increases four by four". This is another indicator of their operations relying on additive thinking like Luke's additive comparison. This signified their algebraic thinking at the factual level in their first attempt to identify a functional relationship which they could not make generalizations between covarying quantities (Radford, 2010). Hence, Sara needed further support from the teacher to understand the functional relationship through covarying quantities.

Writing equations to represent this functional relationship was another struggle for the MC1 students. Although the teacher discussed the relationship by constructing a function table and they were able to identify the function rule as they found the other steps of the table, Sara wrote $c = d$ as an inappropriate equation to represent this relationship, using assigned letters for each variable. On the other hand, Luke wrote two equations, $c \times d = 4$ and $c \times d = 12$, by taking d as a constant value, 4, representing the number of legs in one chair and taking c as the number of chairs. Therefore, he got different values for the product of $c \times d$. At this point, the teacher reminded them that they had multiplied the number of chairs by four to find the total number of legs previously in filling the table. She led them to think operationally with assigned letters for each variable by connecting their operations with new equations. After the first task in Episode 3, they flexibly applied what they learned beforehand and wrote correct equations in the other tasks which are very similar to the first task (see Figure 4.28).

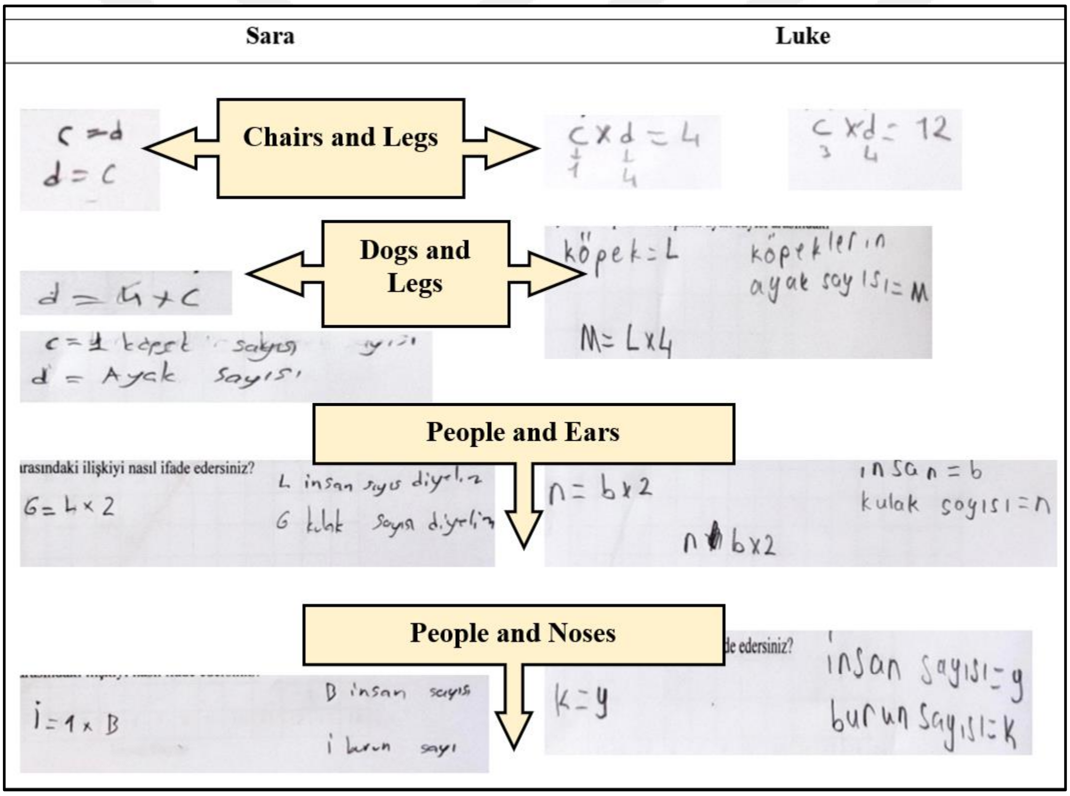


Figure 4.28. MC1 students' symbolic representations for the tasks in Episode 3

The tasks in Episode 4 included different contextual situations and continuous variables such as the amount of time, the amount of money, and the amount of water in a pool. The students were required to fill the tables initially and then express the relationship. Both MC1 students were able to fill the tables accurately and calculate the bigger steps by generating an arithmetic formula. Luke's written expressions were explicating an arithmetic rule coming out from the function table while Sara started to indicate the change in each variable as related to each other as shown in Table 4.9. Luke's written expression represents that he used indeterminate quantities (analytical thinking) and adapted functional thinking with a correspondence approach by coming up with a formula. On the other hand, Sara indicated a covariational approach in her written expression of the functional rule (see Table 4.9) and a correspondence approach in her verbal explanation.

Table 4.9 MC1 Students' Responses in the Pool Problem in Episode 4

	Written Answers	**Explanations** **(How did you find it?)**
Sara	"As the hour goes by one, the water in the pool increases by five tons" $d \times 5 = k$ $k \div 5 = d$ **d:** the elapsed time **k:** The amount of water in the pool	"I always looked at the bottom (second row/variable of the table), but I also looked at the top to see if it was going the same way. I just multiplied it by five."
Luke	"If we multiply the time by five, we find the amount of water in the pool" $b \times 5 = g$ $g \div 5 = b$ $g = 5 \times b$ $b = 5 \div g$ **b:** the elapsed time **g:** The amount of water in the pool	"I figured it out by looking at the top and bottom (the first row/variable and second row of the table)." **Teacher:** How did you get 15? "First, I went five by five and got 15. Then I multiplied it to make sure it was right."

Considering that Sara focused on the recursive pattern in her first attempts in these episodes, her expression in this task can be an indicator of Sara's progression in functional thinking by including both covariation and correspondence approaches.

During Episode 4, MC1 students' writing equations also demonstrated significant progress. After they identified the functional relationships, they wrote symbolic representations more flexibly. Luke figured out the equations after he generalized the arithmetic rule operating between two variables. Sara similarly applied what she learned in the previous lessons and swiftly wrote equations according to what she did in filling the table of values. Moreover, Luke could inverse the equation that he wrote using multiplication after the first task in Episode 4. In further tasks, both students could write the inverse forms of the equations (e.g., $b \times 5 = g$ and $g \div 5 = b$) by including some errors (see Figure 4.29).

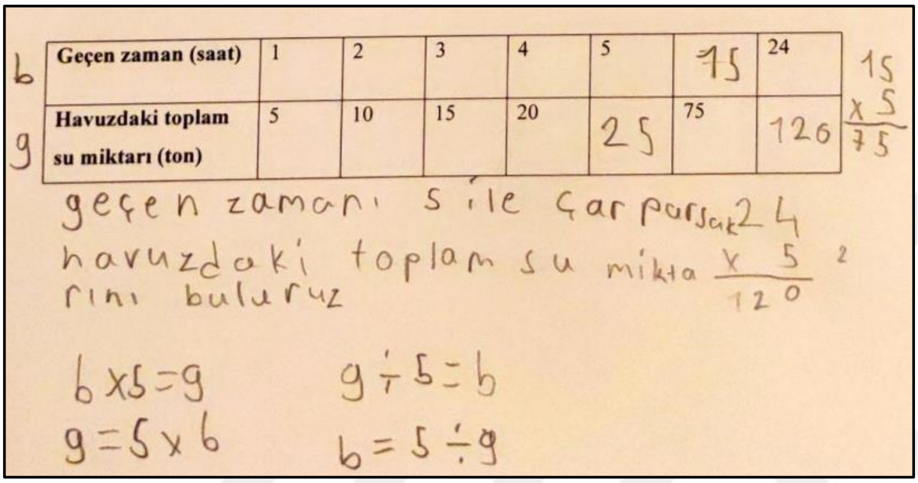


Figure 4.29. Luke's expressions in the Pool Problem in Episode 4

For example, Luke wrote four equations, as shown in Figure 4.29, to represent the functional relationship between the time and the amount of water in the pool. Although three of the equations structurally represent the same relationship, one of them, $b = 5 \div g$, contains an error in terms of this specific relationship. This shows that Luke still needs support or intervention in writing equations because he may not be able to keep track of the relationship between variables, as in the coordination of composite units.

The findings from Episodes 3 and 4 primarily demonstrate the students' progress in functional thinking, as previously anticipated before the teaching experiments. As conjectured, they needed leading and scaffolding from the teacher more than the MC2 students. Due to their progress in algebraic thinking with the teacher's support, students may make mistakes, such as the one seen in Luke's equations in Figure 4.29 at the end of Episode 4. This indicates a need for further reflection on operations and outputs to facilitate the assimilation of generalizations and the creation of equations using appropriate variables. Furthermore, Sara demonstrated bigger progress in Episode 4 while she had fallen behind Luke in Episode 3 in terms of generalizations and expression of the relationships. Therefore, in Episode 4 the gap between Luke and Sara was almost closed with small differences in the way of functional thinking.

4.2.3.2 Students' evaluations at the end of Episodes 3 and 4

In the final assessments of Episodes 3 and 4, both groups of students demonstrated that they achieved the general aims of the lessons such as filling the table according to the given relationship, making generalizations, and writing equations. However, several differences were observed between and within the groups such as writing inverse forms of equations and representing a variable symbolically in terms of the other variable.

One difference between the students' progress in algebraic thinking was their verbal expressions of generalized relationships. In the final assessment of Episode 4, only Luke expressed the relationship by writing the operational rule between the variables as a correspondence approach. In the problem, they were asked to write an equation to represent the relationship between the distance traveled by the cyclist and the time elapsed if she rides 4 km in 1 hour at a constant speed. Luke wrote: "If we multiply the time by four, we calculate the distance traveled". Other students, Roy (advanced MC2), Belle (regular MC2), and Sara (regular MC1), wrote the relationship covariationally such as "For every hour that passes, the distance traveled increases by 4 km" (Roy). Here, expressing the relationship in a narrative form that differs from how the rule was generated to calculate the dependent variable in filling the table may demonstrate a more sophisticated understanding of the functional relationship. These students used different ways to express the same relationship, indicating the extension of connections.

In addition, writing equations to represent the functional relationships demonstrated variations across groups. For example, in the final assessment of Episode 3, the students were required to fill a function table according to a given relationship between the number of cows and the amount of food they consumed daily and write symbolic expressions for the variables and the relationship (see Figure 4.30).

Farm Problem: On a farm, each cow consumes 3 kg of food daily.

a) Can you fill the function table below?

The number of cows	1	2		7			a
The amount of water			60		120	180	

b) Assuming there are x cows on the farm, what is the total amount of water they consume?

c) Write an equation that represents the relationship between x and y, where y is the daily amount of food consumed by all cows on the farm and x is the number of cows.

Figure 4.30. The problem in the end-of-lesson assessment of Episode 3

In this problem, the MC2 students wrote different forms of equations such as $d \div 3 = c$ and $c \times 3 = d$ while the MC1 students only wrote the multiplication form. The MC1 students were able to write inverse forms of equations first in Episode 4 while the MC2 students wrote the inverse forms starting from Episode 3. Moreover, in the same assessment paper, only Roy wrote $a \times 3$ to express the amount of food given to the stock animals while the number of cows was represented by a. The other three students assigned a different letter to the dependent variable although they could write equations when two different letters were assigned to two variables. This demonstrated Roy's structural thinking, like in writing the inverse forms of the equations.

In the Cyclist Problem at the end of Episode 4, all students wrote equations in inverse forms, including multiplication and division. However, the use of division for this relationship differed between the MC2 and MC1 students. (see Table 4.10).

Table 4.10 Students' Written Equations in the Final Assessment of Episode 4

Roy	**Belle**	**Sara**	**Luke**
s: Time	a: Time	s: Time	ç: Time
y: Distance traveled.	b: Distance traveled.	k: Distance traveled.	k: Distance traveled.
$s \times 4 = y$	$a \times 4 = b$	$s \times 4 = k$	$ç \times 4 = k$
$y \div s = 4$	$4 \times a = b$	$k \div 4 = s$	$k \div 4 = ç$
	$b = 4 \times a$		
	$b = a \times 4$		
	$b \div 4 = a$		
	$b \div a = 4$		

The MC2 students wrote an extra equation showing that dividing the distance traveled by the time was 4 (e.g., $y \div s = 4$). On the other hand, the MC1 students wrote only the equation showing that dividing the distance traveled by 4 was the time elapsed (e.g., $k \div 4 = s$) by using division. The former equation demonstrates structural and covariational thinking more explicitly because it represents the covariation between variables through doing operations with variable quantities and finding the rate of change. In the latter, the students do an operation with only one variable quantity to find the other, which may be more visible when reading the table of values without using the structures in operations. Even so, the MC1 students demonstrated an improvement over previous episodes by writing equations in inverse forms.

4.2.4 Representing functional relationships between variables in the form of y = ax + b in Episodes 5 and 6

In the fifth and sixth episodes, the goal was to present various contextual problems involving functional relationships in the form of $y = ax + b$. The tasks in these episodes required the students to identify the problem variables, construct a function table, identify and generalize the functional relationship in the table of data, and express the functional relationship between two variables in different ways, like in Episodes 3 and 4. Differently, in this step of the HLT, the problems included, in addition to dependent and independent variables, a constant value that represents the initial value of the dependent variable.

In Episode 5, there were two problems with the same contexts as the problems in the previous episode to ensure a gradual transition between the forms of functional relationships (see Figure 4.31).

The Birthday Problem: At a birthday party, square tables were arranged end to end to form a long row with only one person seated on each side. How can you express the relationship between the number of tables and the total number of people sitting at these tables? (After the tables are joined, one person sits at each end.)

The Bank Account Problem: Ali has 30 liras in his bank account. Ali decides to deposit 15 liras into his account every month.

a) Fill in the table below according to the information above.

Elapsed Time (the number of months)	1	2	3	4	5	30
The amount of money saved in the account (Liras)						

b) Let's call the number of months (elapsed time) *a*. How would you express the total amount of money saved in the account?

c) Can you write an equation that shows the relationship between the total amount of money saved in the account and the elapsed time (number of months)?

Figure 4.31. The main parts of the tasks in Episode 5

Episode 5 began by revisiting the Birthday Party Problem, previously discussed in Episode 4. However, this time the seating arrangement was different, with two people sitting at the ends of tables. The students were required again to show the relationship between the number of people seated and the number of tables put end to end. The Bank Account Problem reappears, similar to the previous episode as well. However, this time, the problem involved an initial sum of money that increases at a constant rate each month.

In Episode 6, students continued to work with different contextual problems including functional relationships between variables in the form of $y = x \pm a$ (see Figure 4.32).

The People and Hats Problem: Think about a hat with a height of 20 cm. How would you describe the relationship between a person's height without a hat and with a hat? (Carraher et al., 2006)

Customer height (cm)	160	161	162	163	164	180
The customer's height with a hat (cm)						

The Credit Card Problem: Ali is getting a new credit card. In the bank's promotion, the bank pays for the first purchase of 20 liras.
Fill in the table below according to the information above.

The initial amount spent	50	60	70	80	250	500
Total debt amount						

-Write an equation that shows the relationship between the initial expenditure amount and the total debt amount.

The Sapling Problem: The height of a tree sapling grows 2 cm every day from the day it is planted. Since the tree sapling was 35 cm when it was first planted, draw a table expressing the relationship between the elapsed time (days) and the height of the sapling.
-Express the relationship between the elapsed time (days) and the height of the sapling.

Figure 4.32 The main tasks in Episode 6

Additionally, this episode involved the Sapling Problem with a functional relationship in the form of $y = ax + b$ due to the students' difficulties with similar problems in the previous episode. The first problem involved the relationship between a person's height without a hat and with a hat with 20 cm height, which was taken from the study of Carraher et al. (2006). The second problem (the Credit Card) was about a credit card promotion of a constant amount given for any initial expenditure. Finally, the Sapling Problem, which was similar to the Bank Account Problem, involved a relationship between the height of a tree sapling, whose initial height is 35 cm, and which grows 2 cm each day, and the elapsed time.

The intended processes in Episodes 5 and 6 were quite the same as those in Episodes 3 and 4 (see Figure 4.33). Episodes 3 and 4 have successfully equipped MC2 students with the ability to identify relationships between two variables and accurately fill function tables. However, it was hypothesized that identifying the constant value in problems involving functional relationships in the form of $y = ax + b$ may pose a challenge to students, after they have worked on the problem in the other form in previous episodes ($y = ax$). To address this perturbation, probing questions such as "Can you check your formula for any given value in the table?" were ready to use in these episodes. On the other hand, it was conjectured that the MC1 students would not easily handle this situation. Hence, there were additional scaffolding questions to ask them which were about understanding the problem, filling the table and writing the equations. The teacher was also alerted to the MC1 students' inclination towards recursive thinking (Zwanch, 2022a) when identifying functional relationships.

As conjectured, the teacher took a more active role during the episodes with the MC1 students, guiding them to understand the problem and identify the relationships during the learning process. Her questions effectively challenged the erroneous reasoning and responses given by the MC1 students. She needed to make the students falsify their reasoning by recalling the variables, questioning, and checking the answers numerically. The teacher generally falsified the erroneous generalizations of the students in the first part of each task. She was more dominant in recognizing the relationship between variables because the students struggled to understand how to calculate the value of one variable for the larger value of another variable.

EPISODES 5-6			
Descriptions of Tasks	**Intended Processes**	**Conjectures**	**Instructional Moves Aligning with the Conjectures**
5-1) The Birthday Party 2 Problem: The relationship between the number of tables and the number of people who are seated. **5-2) The Bank Account Problem:** The relationship between time and the total amount of money saved in the account **6-1) The People and Hats Problem:** The relationship between a person's height without a hat and with a hat **6-2) The Credit Card Problem:** The relationship between the initial amount expenditure and total debt amount **6-3) The Sapling Problem:** The relationship between the elapsed time (days) and the height of the sapling.	-Identify the variables and constant values in functional relationships - Use function tables to represent the data by assigning values for variables -Identify and generalize the functional relationship in the table of data (covariational or correspondence thinking) - Use symbolic expressions to represent the functional relationships	a) MC2 students would calculate any corresponding value in the function tables and indicate the functional relationship verbally by using indeterminate quantities b) MC1 students would not calculate the larger values in the function tables because of ignoring the constant value and they would not indicate the functional relationship by using indeterminate quantities and letters. c) MC1 and MC2 students would have difficulty in writing the equations representing the functional relationship such as ignoring the constant value d) MC1 students would have difficulty in writing the equations representing the functional relationship in the form of $y = x \pm a$ which is different than the previous problems	**Conjectures a** -Ask them to explain the relationship by using different expressions and equations (Developing structural thinking) **Conjectures b** - Fill in the table together on the board and ask the relationship between two variables. Let the students interpret the change in both variables simultaneously. **Conjectures c** -Ask them to pay attention to how they fill in the table and what operation they did in calculating one variable by using the value of another variable. **Conjectures d** -Discuss the meaning of problem variables, pay attention to the table of values, and highlight the covariation between the variables.

Figure 4.33 Description of Episodes 5 and 6: Functional thinking ($y = ax + b$)

Subsequently, the teacher summarized their answers and discussed them on the board without any interruption. On the contrary, she used fewer probing questions with MC2 students. She used generally small questions to trigger MC2 students to think about their erroneous answers. This approach helped them in a short amount of time. She emphasized what changes and what is constant in each question for both groups of students. The descriptions of tasks, intended processes, conjectures and instructional moves are summarized in Figure 4.33.

4.2.4.1 Students' Performance during Episodes 5 and 6

The conjectures for Episodes 5 and 6 indicated that both groups of students would encounter difficulties in different parts of the problems that involve functional relationships in the form of $y = ax + b$. This difficulty is due to the constant value which is additively involved in the relationship, and which the students are not familiar with. As conjectured, this novel situation in the contextual problems created constraints for the students in determining the functional relationship and filling the tables in some problems. This was evident in their erroneous values written in the tables or their overthinking to find the larger values for variables.

In parallel to the conjectures for the MC2 students, they had difficulty in this type of functional relationship, but not in writing the equation, it starts with their identification of the relationship in the Bank Account Problem. In contrast to the conjectures, they had this difficulty only in the Bank Account Problem. They exhibited minor difficulties in the identification of the functional relationships between variables in the Bank Account Problem. Roy was usually faster than Belle at identifying relationships and writing equations, like in the previous episodes. He needed less help from the teacher and could correct his own mistakes. When they misunderstood the problem or could not determine the larger value in the function table, the teacher's prompts or the students' own attempts swiftly resolved the issue. Roy represented the most sophisticated performance amongst all students also during Episodes 5 and 6 with almost no help from the teacher (see Table 4.11).

Table 4.11 The Students' Performance in Representing Functional Relationships in Episodes 5 and 6

	MC2		MC1	
	Roy	**Belle**	**Sara**	**Luke**
Symbolic representation of functional rel.	1	1	3	3
Reversing the equation	1	1	-	2
Correspondence approach	1	1	3	3
Covariational approach	1	-	3	3
Verbal generalization	1	2	3	3
Generalization after filling the table	1	1	3	3
Recursive thinking	-	-	1	1

1: Without prompting; **2:** With probing questions, **3:** With leading questions

Note: The level of algebraic thinking increases with color darkness from the bottom to the top of the table.

The MC1 students struggled to determine the relationship between variables in both problems in Episode 5. They had major difficulties in the identification of the functional relationships between variables in the Bank Account Problem. Luke, who attended Episode 6 alone, had a similar difficulty only with the Sapling Problem in Episode 6, which is similar to the Bank Account Problem. In these problems, the teacher directed students in key steps, such as determining the relationships between two variables by generating a rule. Therefore, the teacher provided more than scaffolding and prompting in these instances. Furthermore, their reliance on recursive thinking continued in these episodes.

4.2.4.1.1 MC2 students' performance during Episodes 5 and 6

Roy could identify the functional relationships between the variables in each problem by making generalizations to larger values for the independent variables. He needed to think a little more about the Birthday Party 2 and Bank Account Problems in Episode 5 while he quickly identified and expressed the relationships in other problems. In the Birthday Party 2 Problem, he had difficulty understanding the table arrangement. He understood the relationship after creating visual representations of the table arrangement with the teacher's suggestion. In the Bank Account Problem, he calculated the consecutive cells by adding the rate of change each month, recursively. He recognized the initial amount of money by himself after thinking for a while and he could calculate each cell by formulating the rule. He explained his reasoning by saying "There is 30 liras in the beginning." (see Table 4.12)

Table 4.12 MC2 Students' Identification of Relationships in Episodes 5 And 6

Problems	**Roy**	**Belle**
The Birthday Party 2 (Episode 5) $y = 2x + 2$	After drawing visuals he formulated a rule "I multiplied 100 by 2 and then added 2 people. I also added 2 because 2 people were sitting at the ends."	After drawing visuals she formulated a rule "If we multiply the number of tables by 2 and add the people on the edge, I mean 2 people, we get the total number of people sitting."
The Bank Account (Episode 5) $y = 30 + 15x$	"If we multiply the elapsed time by 15 and add 30, we can find the amount of money saved."	The teacher showed the calculations
People and Hats (Episode 6) $y = x + 20$	"The customer's height increases by 20 cm when wearing the hat."	She wrote the symbolic representation before filling the table ($a + 20 = b$)
Credit Card (Episode 6) $y = x - 20$	"Since the bank pays 20 liras, we subtract 20 liras. The higher the amount of spending, the higher the amount of debt."	"Total debt is 20 liras less than the initial expenditure"
Sapling (Episode 6) $y = 2x + 35$	"150 times 2 in parentheses and then plus 35."	"First I multiplied 150 by 2, I got 300, then I added it to 35, I got 335."

Roy was able to identify and express the functional relationships between the problem variables using a variety of verbal expressions and justifications. In the

Birthday Party 2 Problem, he explained why he added two after the multiplication by emphasizing the additional two people at the ends of the table arrangement. Similarly, in the Sapling Problem, he indicated the parentheses in his verbal expression, which shows the logic in his generalization and formulation of the rule using structures in the problems (see Table 4.12).

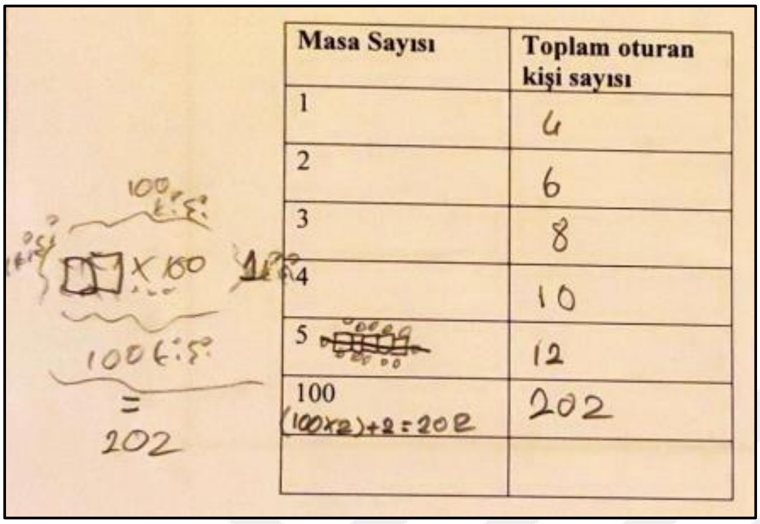
Roy's verbal expressions of the relationship were more varied than Belle's. For example, he indicated a covariational relationship between the height of a customer and the height of a customer with a hat (Episode 6) as follows:

> **Teacher:** Did you always add 20? (Asking about the function table in the People and Hats Problem)
>
> **Roy:** yes teacher, there is a 20 cm difference between them.
>
> **Teacher:** How else can we say? What does it mean that there is a 20 difference between them?
>
> **Roy:** So it's 20 cm more.
>
> **Teacher:** Which one? The height of the customer with the hat is 20 cm
>
> **Belle:** Big
>
> **Teacher:** … bigger than the height of the normal customer. It always adds 20, so can we say that it is always 20 more. (Yes) What else can we say?
>
> **Roy:** Teacher, for example, as the height of the customer increases by one, the height of the customer with the hat also increases by one.

As seen, Roy used various verbal expressions indicating the same relationships like writing equivalent expressions. His last expression represents his covariational thinking as well such as "as one increases by one, the other increases by one too." His effort for generating multiple expressions was appreciated.

Belle had considerable difficulty in identifying and generalizing the relationship in the Bank Account Problem. In the other problems, she could identify the relationships by herself without any help. As shown in Table 4.12, she could express the generalized relationships using narrated formulas (e.g., the Sapling Problem), write equations immediately (the People and Hats Problem), or provide the rationale behind her formulated rules (the Birthday Party 2 Problem).

Belle had a good start in Episode 5 with the Birthday Party 2 Problem. She immediately started drawing tables and could identify the relationship more quickly than Roy (see Figure 4.34).



Masa Sayısı	Toplam oturan kişi sayısı
1	4
2	6
3	8
4	10
5	12
100 (100x2)+2=202	202

Figure 4.34. Belle’s drawings in the Birthday Party 2 Problem

Drawings probably helped her to figure out the problem because she could fill the table with correct values on her first try, unlike the other three students. When the teacher asked her to explain how she recognized this relationship and found the number of people for a hundred tables, she indicated that drawings helped them figure out the relationship as follows:

> **Belle:** When the tables are brought end to end, 2 people sit at each table, but one person sits at each end. So if we multiply the number of tables by 2 and add the people on the edge, I mean 2 people, we get the total number of people sitting.

She both used indeterminate quantities and explained the reasoning behind her calculations as the indicators of algebraic thinking (Kieran, 2004). Although the constant value in the functional relationship (i.e., 2 people on the edges) created some constraints, it was seen that drawing sample situations, such as 1, 2, and 3 tables, helped the students make generalizations. This shows that this problem was a good starter for this kind of relationship.

Like Roy, Belle encountered trouble with the Bank Account Problem in Episode 5. She calculated the smaller values by recursively adding the constant rate of change. However, she required assistance and discussion with the teacher to calculate the larger value in the function table. The teacher demonstrated all the operations for each column, and this discussion lasted almost ten minutes with minimal contribution from Belle. Consequently, this calculation was teacher-directed.

In expressing the generalizations using equations, the MC2 students showed a higher level of performance than had been conjectured. After identifying the relationships by filling the function tables and expressing them verbally, they quickly transformed the formulations of the rules into equations using letters (see Table 4.13).

Table 4.13 MC2 Students' Written Symbolic Representations

Problems	**Roy**	**Belle**
The Birthday Party 2 (Episode 5) $y = 2x + 2$	$(m \times 2) + 2 = k$	$(m \times 2) + 2 = k$ $k = (m \times 2) + 2$
The Bank Account (Episode 5) $y = 30 + 15x$	(a × 15) + 30 = b b: the amount of money a: the number of months	$30 + b \times 5 = a$ a: the amount of money b: the number of months
People and Hats (Episode 6) $y = x + 20$	$b - 20 = a$ $b - a = 20$	$a + 20 = b$; $20 + b = a$ $b = a + 20$; $b = 20 + a$ $b - 20 = a$; $a = b - 20$
Credit Card (Episode 6) $y = x - 20$	$b - 20 = c$ $c + 20 = b$ $b - c = 20$	$a - 20 = b$; $b + 20 = a$ $20 + b = a$; $a = 20 + b$ $a = b + 20$
Sapling (Episode 6) $y = 2x + 35$	$(d \times 2) + 35 = e$	$(a \times 2) + 35 = b$

The MC2 students could represent the symbolic representations in different forms by using inverse operations (e.g., addition and subtraction) or using commutative property (i.e., $b = 20 + a$ and $b = a + 20$) or symmetry property of addition (i.e., $b = 20 + a$ and $a + 20 = b$). Writing equivalent expressions using the same relationships between variables demonstrated their algebraic thinking emphasizing their structural and analytical thinking.

As seen in Table 4.13, both students used parentheses in the relationships including more than one operation. However, Belle did not use parentheses in the Bank Account Problem in which she had difficulty understanding the relationship. The parentheses can show a higher level of abstraction by objectifying each output of an operation. In the Birthday Party 2 Problem, Belle wrote the number of people sitting at the sides of the tables as the first expression ($m \times 2$) in parentheses (see Table 4.13), showing that it is the first output that depends on the number of tables before adding two more people at the ends. This represents an invented relationship through structural thinking and objectification of each process of operations. This is also evident in Belle's drawings in the identification of this relationship (see Figure 4.34). In her drawings to find the number of people for one hundred tables, she first showed her understanding of the formula on the drawings by writing "×100" near the table figures and representing "plus ones" at the ends, indicating multiplication of the number of tables by 100 and adding two people at the ends. This showed her generalization process and algebraic thinking.

In addition to using parentheses, the MC2 students also showed their structural thinking as an important indicator of algebraic thinking by rewriting the equations in different forms such as reversing the operations or using the properties of operations. As seen in Table 4.13, Belle demonstrated a tendency in most of the problems by trying to write as many equations as possible. However, Roy's equations demonstrated more sophistication in his thinking, particularly evident in the Credit Card and the People and Hats Problems. Belle wrote each equation by operating with an indeterminate and a determinate quantity on one side which shows the reasoning

"One variable is 20 less than the other and the other variable 20 more than the first one". On the other hand, Roy also formulated an equation that included an operation with two indeterminate quantities on one side (i.e., $b - a = 20$), which represented that he considered the difference between two indeterminate quantities. He showed this transformation in both problems (see Table 4.13) while Belle did not consider this form, even though she wrote more equations.

4.2.4.1.2 MC1 students' performances in Episodes 5 and 6

In this part, Sara (regular MC1) participated only in Episode 5. Therefore, Sara's performance was analyzed through two tasks while Luke's performance was analyzed through five tasks. The comparison of the progress of MC1 students could only be done through two tasks in Episode 5.

As conjectured, both MC1 students had difficulty understanding the relationships in the problems in Episode 5. Therefore, the initial parts of each task in Episode 5 took more time, including writing the following corresponding values in a function table, finding a larger value in the function table, and formulating a rule for finding the value of the dependent variable that corresponds to any value of the independent variable. It was seen that they were still inclined to think recursively in interpreting the function tables. Hence, they needed to discuss with the teacher to calculate the number of people seated on one hundred tables (the Birthday Party 2 Problem) or to find the amount of money saved after 30 months (the Bank Account Problem).

In the Birthday Party 2 Problem, the MC1 students could fill the table correctly on their second attempt until the value of five (for the number of tables), after the teacher explained the table arrangement (see Figure 4.35). They explained how they found the values on the table recursively, they indicated "it increases two by two". Therefore, this thinking way created a constraint for finding the number of people for one hundred tables and formulating the rule, as factual algebraic thinkers have (Radford, 2010).

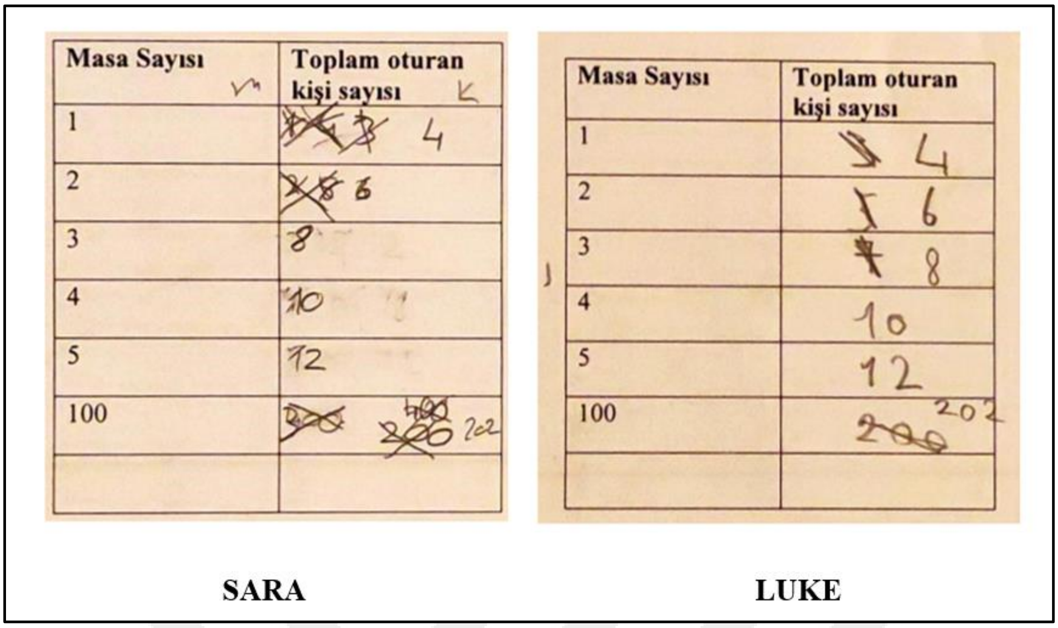


Figure 4.35. MC1 students' function table in the Birthday Party 2 Problem

Both students thought that the rule was multiplying by two. However, the teacher interfered to show that the rule did not work in the given situations as follows:

> **Teacher:** Think about the relationship between the number of tables and the people seated. What happens to the number of tables and what happens to the number of people? How we can find when it says one hundred tables?
>
> **Sara:** I multiplied by two and found like this…For what number of tables… (she thinks). As Luke said, the numbers go as 2-4-6-…ahh sorry, they go as 4-6-8-10-12, it increases two by two. I multiplied 100 by two and it is 200.
>
> **Teacher:** You said, it is found when we multiply by 2. What do you think Luke?
>
> **Luke:** Teacher, I also did 200 at first. Then I thought it was wrong because at first, it increased by 4, then it increased by two by two.
>
> **Teacher:** That is, when there is zero tables, zero people... it does not start directly from 2 people. How else do we know that your first situation is wrong? Come and draw the table arrangement on the board.

After they drew the table arrangement for several situations and the teacher explained the case through the drawings by relating it to the previous problem in Episode 4, Sara answered 202 by formulating the rule correctly. Luke also explained how she formulated the rule after her answer. However, in this process they drew the table arrangement twice with the teacher and the teacher explained how the people sit around the tables in each situation to help them to understand the relationship.

Similarly, in the Bank Account Problem, they filled the table by adding 15 and stated the relationship by thinking recursively in the same way:

> **Luke:** We add 15 as each month passes
>
> **Sara:** As the month passes, because the amount of money increases by 15 liras, we find the relationship by adding 15 liras over 30 liras.

Consequently, they formulated the rule incorrectly, as evidenced by the multiplication of 15 and 30 to determine the amount of money saved after 30 months. This approach ignored the constant value of money at the beginning. The students indicated that the amount of money could be found by multiplying by 15. Their focus was on the recursive increase in the amount of money, rather than on the role of the constant initial amount of 30 liras in generalizing the relationship. This showed their incorrect generalization of the relationship. In this section, the teacher stepped in and illustrated that their understanding was flawed by repeatedly adding 15 until the 30th month. The students initially presumed that they had made a calculation error (Sara) or had multiplied by an incorrect number (Luke). Eventually, the teacher demonstrated the operations in each column, emphasizing the initial amount of money as a constant. This indicates that they were unable to identify the functional relationship in the form of $y = ax + b$ with any degree of support.

In writing equations, Luke's performance was better than Sara's and better than his performance in identifying relationships. In the Birthday Party 1 Problem, he could easily write a symbolic expression for the relationship between the number of tables and the number of people without any correction or prompt from the teacher (i.e., $m \times 2 + 2 = k$). He also explained his expression: "I multiplied m and 2 and then added 2," indicating his operations with indeterminate quantities. However, in the Bank Account problem, Luke's first answer was $a \times 15$ as he had generalized the relationship earlier in the problem, which is consistent with the conjectures. The teacher reminded him that when they multiplied the number of months by 15, the output value was erroneous. He quickly corrected his expression by adding 30 at the end of the expression (i.e., $c \times 15 + 30 = b$) and said: “We multiply 30 and 15 and then add 30 because in the beginning there is an amount of money of 30 liras.”

On the other hand, Sara wrote several erroneous expressions before the teacher helped her to correct her expressions in both problems. For example, in the Birthday Party 1 Problem, she tried the expressions $m = k$, mk, $m = \times 2$, $m = \times 2 = + 2 = k$ (m: the number of tables and k: the number of people). Similarly, in the Bank Account Problem, she tried different expressions such as $a = 30 + 15$, $a = 30 + 15$, $a \times 30$ at the beginning. In each attempt, the teacher falsified her reasoning by giving an example from the function table, and then she could write the correct expression at the end. Her erroneous expressions demonstrated that she could not operate with indeterminate quantities when there was more than one operation between the problem quantities.

Luke was the only MC1 student to attend Episode 6 and to continue with the remaining three problems. He demonstrated better performance in the People and Hats and the Credit Card Problems, which included only one operation in the function rule. In these problems, he was able to identify the relationship and express it verbally and symbolically, although he did encounter a few minor challenges, such as misunderstanding the problem (the Credit Card Problem), and writing an incorrect equation before filling the table (the People and Hats Problem). In these problems, the act of filling in the function table proved to be an effective method of rapidly identifying the relationships. For instance, in the Credit Card Problem, Luke was able to express the relationship as follows: "We find the total amount of debt if we subtract 20 from α," where α was assigned the value of the amount of expense. His verbal expression included a statement indicating an operation that he did while constructing a table of values.

In the Sapling problem, Luke could not calculate the length of the sapling in the 150th day after it was planted, indicating that he could not generalize the relationship. Furthermore, he demonstrated some confusion regarding the quantities such as multiplying 2 (the amount of the growth in cm each day) and 35 (the length of the sapling at the beginning) and adding 150 (the number of days passed) and 35. His answers given during the discussion on the calculation of sapling height on the 150th day after planting are listed as follows:

1) "Can it be like multiplying 150 and 2? It is 300."

2) "What if we add 35 to the amount of elapsed days (150)? Then we add 2 to 185, it is 187."

3) "I found the first three days by increasing two by two."

4) "What if we multiply 35 by the elapsed time and add 2?"

5) "Is it okey to multiply 35 and 2?"

6) "Could it be 160. I found until the 10th day by adding two. At last, I found 55. 55 and 55 makes 110. It makes 165, one more 55 it has."

7) "(Instead of adding two recursively) I can multiply 150 and 35 or multiply 35 and 2."

Because the problem included both an addition to the initial amount and calculating the amount of growth in any given day it required more than one operation to calculate the dependent variable. Hence, Luke had difficulty in keeping track of the multiple units such as the length at the beginning, the amount of growth in a day, and the amount of growth in given days and he could not go further.

Luke could represent the functional relationships in the Birthday Party 2 and Credit Card Problems using equations, as he was able to identify the relationships. One noteworthy observation is that he had difficulty in writing the equation in the reversed form (i.e., $g = a + 20$ and $g - 20 = a$). The teacher prompted him to engage in reversible thinking and asked him whether he could utilize subtraction in formulating the equation in the People and Hats Problem while he wrote symmetric forms of equations (i.e., $\alpha + 20 = g$ and $g = \alpha + 20$). He incorrectly wrote $20 - g = a$. Thereafter, the teacher let him try his new equation in the function table to check whether it worked or not and she provided numerical examples. After this, Luke realized his mistake and revised the equation correctly.

Conclusively, the MC1 students performed better in the previous episodes than Episodes 5 and 6, which included the functional relationships in the form of $y = ax + b$. Therefore, there was no notable progress in their algebraic thinking in these episodes. In this type of functional relationship (i.e., $y = ax + b$), they demonstrated difficulty in identifying the relationship by integrating the initial value (constant value). They made many mistakes until they reached the correct expressions of the relationships with the teacher's help in most of the tasks. Especially in the Bank Account ($y = 30 + 15x$) and the Sapling Problems ($y = 35 + 2x$), the teacher directed the MC1 students more than scaffoldings and prompts. The initial amount presented a challenge, as it required integrating different quantities into the problem such as the amount of growth in each day, and the number of days passed in the Sapling problem. Furthermore, their reliance on recursive thinking persisted in these episodes.

4.2.4.2 Students' evaluations at the end of Episodes 5 and 6

At the end of Episode 5, the students worked on the Pool Problem that involved the functional relationship in the form of $y = ax + b$ like the Bank Account and the Sapling Problems. At the end of Episode 6, they worked on the Bouncing Ball 2 Problem that involved the functional relationship in the form of $y = x \pm b$ like the People and Hats and the Credit Card Problems (see Figure 4.36). Both problems required the students to fill the given tables, generalize the relationship between variables, and write a symbolic representation in further steps of the problems.

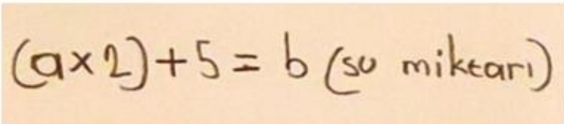
The Pool Problem (Episode 5): There are 5 tons of water in a swimming pool. A faucet running at a constant speed adds 2 tons of water into the pool in 1 hour Write an equation that shows the relationship between the amount of water in the pool and the elapsed time (hours).

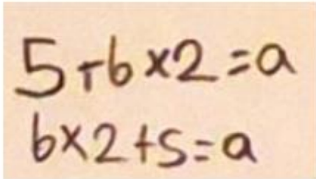
The Bouncing Ball 2 (Episode 6): Elif drops a ball from different heights and it bounces on the ground. She prepared a table in which she recorded the height from which the ball was dropped and how much the ball rose after the bounce.

The height the ball was dropped (cm)	40	50	60	70	300
The rise after the bounce (cm)	20	30	40	50	

Figure 4.36 The main parts of the post-assessment problems in Episodes 5 and 6

In the end-of-lesson assessments of Episodes 5 and 6, the gap between the students' performances in identifying and generalizing the functional relationships in the form of $y = ax + b$ and writing equations was almost closed with several differences. The MC2 students could fill in the function tables accurately and find the larger value by making the correct operations, which shows their accurate generalizations in both assessments. In parallel, they could also represent functional relationships through symbolic expressions. They demonstrated quite the same performance in their assessments. One difference between them was their equations in the Pool Problem. Like in Episode 5, Roy used parentheses in writing the equation in the Pool Problem. In contrast, Belle did not use parentheses in her equation. (Figure 4.37). However, it does not demonstrate in general that Belle did not achieve this level of abstraction because she used parentheses in the Birthday Party 2 Problem in Episode 5 (i.e., $(m \times 2) + 2 = k$). This may be because she did not focus on it or forgot it in the final assessment. In parallel, both students wrote all the equation forms for the additive relationship between two variables in the Bouncing Ball 2 Problem by inversing the initial equations. This also shows their abstraction and structural thinking.



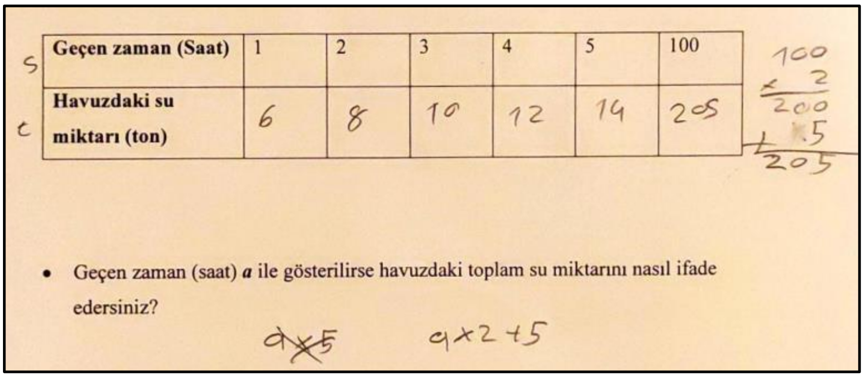


Roy	**Belle**
a: Elapsed time	***a:*** The amount of water
b: The amount of water	***b:*** Elapsed time

Figure 4.37. MC2 students' equations in the Pool Problem

Another difference between the MC2 students was their verbal expressions of the relationship in the Bouncing Ball 2 Problem. Roy indicated the relationship as: "A difference of 20 exists between the height at which the ball was left from and the amount of rise of the ball after the hit on the ground" ($d - c = 20$) while Belle indicated the relationship as: "The height at which the ball was dropped is 20 cm greater than the amount of rise of the ball after its hit on the ground." ($a = b + 20$). In Roy's expression, which indicates how he initially determined the relationship, there is a direct operation between two problem variables. In contrast, Belle initially determined how to find the value of one variable in terms of the other variable, including a direct operation between one variable and one known quantity. Considering this, Roy's generalization encompasses analytical thinking to a greater extent than Belle's, even though both students were able to correctly write all the forms of equations in the further step.

Although the MC1 students had difficulty in understanding and generalizing the functional relationships in each task during the episodes, end-of-lesson assessments showed that they could generalize the functional relationship both operationally and symbolically. In the Pool Problem, both students could calculate the larger value in the function table, which indicates their generalization of a functional relationship in the form of $y = ax + b$. However, Sara made some initial mistakes such as writing erroneous values in the function table and confusing the operation in writing symbolic representation (see Figure 4.38).



Geçen zaman (Saat)	1	2	3	4	5	100
Havuzdaki su miktarı (ton)	6	8	10	12	14	205

- Geçen zaman (saat) ***a*** ile gösterilirse havuzdaki toplam su miktarını nasıl ifade edersiniz?

Figure 4.38. Sara’s answers in the Pool Problem

Given that she was able to identify the larger value in the table, the initial erroneous values were regarded as computational errors. Furthermore, she was able to revise his symbolic expression after a small confusion.

In Episode 6, Luke demonstrated success in tasks involving the functional relationship in the form of $y = x \pm a$, as evidenced by his performance on the People and Hats and Credit Card Problems. The end-of-lesson assessment included a problem of a similar nature. Consequently, he successfully completed each step in the Bouncing Ball 2 Problem. He once again wrote the equations in the symmetric forms (i.e., $b - 20 = g$ and $g = b - 20$). As he did in the previous tasks in the episode, he did not reverse the operation in the end-of-lesson assessment.

In general, the nuances between the group of students’ performances in algebraic thinking (i.e., MC1 and MC2) appeared in their written expressions of the relationships and in the forms of the equations they wrote. While all students could verbalize the functional relationship, there were variations in the written expressions. For instance, Luke (early MC1) consistently employed a language that emphasized the operation he used to determine one variable in terms of the other such as “We find the amount of rise of the ball if we subtract 20 from the height the ball was dropped.” On the other hand, the expressions of Belle and Roy focused on the explicit relationship between variables, as exemplified by Roy's statement: "The height at which the ball was dropped is 20 cm greater than the amount of rise of the ball after its hit on the ground." Their approach to verbal generalizations of the relationships represents a novel construction of the relationship expressed in a manner distinct from how they do calculations in the table. In addition, the variation and comparison between the variables were more visible as compared to Luke's. This was also evident in their symbolic representations. While the MC2 students could write the equations in the reverse form, the MC1 students were only able to write symmetric forms of equations, which does not require the construction of a new expression or the structure of the relationship.

4.3 General Progress of Students in Algebraic Thinking

This section presents the general progress of each student, interrelating it with their unit coordination levels and with their performances before the teaching episodes. As Sara was unable to attend the final episode and evaluation, her performance was evaluated based on her performances before the teaching experiments and the last episode she attended.

In the post-assessment when the teaching episodes ended, students answered six questions. Four were from the Algebraic Thinking Interview: the Caterpillar and Leaf, the Growing Caterpillar, the Penny Bank, and the Cord problems. One question was from the Units Coordination Interview: the Crate problem. Lastly, the students also calculated two missing value operations to be assessed for the equal sign understanding and defined the concept of variable. As distinct understandings and thinking processes were addressed by particular problems, this section presents the students' final evaluations in different headings, including equal sign understanding, variable understanding, algebraic thinking, and units coordination.

4.3.1 Equal sign understanding

Students' responses to the missing value operations in the final assessment showed that their understanding of the equal sign was the same as in the pre-assessment. Roy and Belle demonstrated again a relational understanding of the equal sign while Luke showed an operational understanding. Belle used a compensation strategy to find the missing value in one of the operations, as different than her response in the pre-assessment. This confirms that she showed a more advanced understanding of equal signs. In contrast, Luke thought that the result of the operation should come just after the equal sign, ignoring the quantities on the right side (see Figure 4.39).

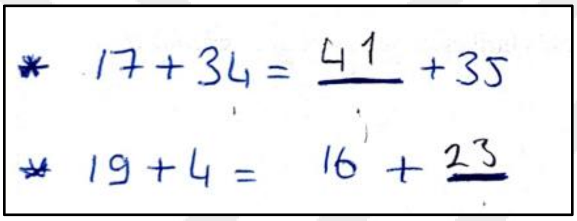


Figure 4.39. Luke’s operational view of equal sign in the missing value operations

His response showed that his operational view of the equal sign persisted even after they had been presented with a minor equality activity represented in balance scales through virtual manipulatives in Episode 2.

Consequently, these six teaching episodes, which were designed to facilitate algebraic thinking rather than directly addressing the understanding of the equal sign, did not facilitate Luke’s understanding of equality and the equal sign in numerical operations.

4.3.2 Variable understanding

During the final assessment, the students were asked to define the concept of variable. Their definitions commonly involved the expression “continuously changing things” as seen in Table 4.14.

Table 4.14 Students' Descriptions of Variables

Roy	Indeterminate, continuously changing. For example, it can be expressed in symbols.
Belle	Constantly changing things (Can you give an example?) For example, Ali puts 3 TL in his penny bank every day… they are variables, they change every day. (What changes every day?) He puts 3 liras in his penny bank every day. For example, on the first day, he puts 3 TL, on the second day 6 TL, on the third day 9 TL. They change in this way. The number of days also changes there.
Luke	Always taking different values. Constantly changing values.

In the pre-assessment, they could not give accurate definitions because they had heard the variable concept for the first time. Their definitions in the post-assessment demonstrated that they emphasized the meaning of variables as constantly changing things, as in the problems involving functional relationships. In addition, Roy also indicated the indeterminacy in his definition. In this way, his understanding seemed more comprehensive than that of his peers.

Other than answering this definition question, the students could identify the problem variables correctly in each problem while assigning the letters for the variables in the final assessment. In the pre-assessment results, they had missed some of them while indicating the problem variables, for example in the following problem.

> A tree sapling grows taller by a certain amount every day from the day it is planted. What do we need to know to calculate the height of a tree sapling on a given day? Can you indicate the problem variables?

In response to this question, Roy identified "how much it grows each day" and "the number of days elapsed" as the things we need to know. Belle indicated that we need to know how much it grows each day and the first length of the sapling. On the other hand, Luke only identified "how much it grows each day" as the problem's important element. Comparing their answers in the pre-assessment, they demonstrated achievement in interpreting the problem variables and understanding them in the function context.

Furthermore, all students could write the inverse form of the relationships in the Cord Problem in the post-assessment by operating with the letters assigned for the variables in different ways. This represents an understanding of variables as mathematical objects, referring to Level 6 according to Blanton et al. (2017). Roy demonstrated the same performance in the Growing Caterpillar problem in the pre-assessment which was in a function context. Therefore, his progress may not be very significant compared to other students, Belle and Luke, who demonstrated an understanding of variables as quantities with fixed arbitrarily chosen values, which refers to Level 4 (Blanton et al., 2017) in the pre-assessment findings. Considering the evidence of operating with letters differently for the same relationship, they demonstrated the same level of understanding of variables in the post-assessment. However, Roy's use of inverse operations in each problem may show his understanding as more advanced than others because he could flexibly think of them without prompting.

4.3.3 Algebraic thinking

In the final assessment, three problems addressed the students' generalizations of functional relationships between variables and their symbolic representations (the Growing Caterpillar, the Caterpillar and Leaf, and the Penny Bank Problems). Each problem had special features that either helped students understand the problem or challenged them to complete the steps. In the Growing Caterpillar Problem, students saw figural data showing that the body length of a caterpillar increased by two parts each day. In the Caterpillar and Leaf problem, there were no figures or tables. The given information was that "two caterpillars eat 6 leaves." Finding the amount for one caterpillar would help the students write an equation representing the relationship between the number of caterpillars and the number of leaves. Lastly, the Penny Bank Problem involved a saving money context where Ali puts each day three TL into his penny bank, with an initial amount of 10 TL. Therefore, students needed to interpret the constant initial value in addition to the problem variables. The general performances of students in each problem are represented in Table 4.15.

Table 4.15 Pre and Post-Assessment Comparisons in Functional Relationships

Growing Caterpillar						
	Roy		Belle		Luke	
	Pre	Post	Pre	Post	Pre	Post
Inverse form of equation	1	1	0	0	0	1
Symbolic representation	1	1	0	1	0	1
Verbal generalization	1	1	1	1	1	1
Finding a larger/any value	1	1*	1	1	0	1
*Roy was confused at first then the teacher prompted her he said 202						
Caterpillar and Leaf						
	Roy		Belle		Luke	
	Pre	Post	Pre	Post	Pre	Post
Inverse form of equation	0	1	0	0	0	0
Symbolic representation	0	1	0	1	0	1
Verbal generalization	1	-	0	1	0	1
Finding a larger/any value	1	1	1	1	0	1*
*Luke's calculations were corrected through teacher prompts						
Penny Bank						
	Roy		Belle		Luke	
	Pre*	Post	Pre	Post	Pre	Post
Symbolic representation	-	1	0	1	0	0
Verbal generalization	-	1	1	1	0	1
Finding a larger/any value	-	1	1	1	0	0
*This problem was not included in Roy's pre-assessment.						

In determining a functional relationship and representing it through symbols, the MC2 students demonstrated the standard algebraic thinking level by generalizing the relationship and writing the function rule through an equation in all the problems. Roy wrote the inverse forms of all the equations that he could write in the form of

$y = ax$, while Belle did not include the inverse forms of the equations in the problems including functional relationships. Considering the pre-assessment outcomes, Roy demonstrated progress in writing symbolic representations in the problems except the Growing Caterpillar Problem in which he had already written equations in the pre-assessment. On the other hand, Belle demonstrated a significant improvement by writing equations in any problem type, which she had been unable to do in the pre-assessment (see Table 4.15).

As an MC1 student, Luke demonstrated a distinct performance in the post-assessment, although he exhibited notable progress. The post-assessment performance indicated that he could generalize functional relationships and represent these relationships through equations in the form of $y = ax$, as different than the pre-assessment performance. However, in formulating the relationship, he sometimes exhibited confusion. For instance, in the Caterpillar and Leaf Problem, he initially determined that the number of leaves required to feed 12 caterpillars was 72. Then the teacher helped him to correct his understanding, as follows:

> **Luke:** If we multiply the number of caterpillars by 6, we find how many leaves they eat.
>
> …
>
> **Teacher:** You mean, one caterpillar eats 6 leaves
>
> **Luke:** No, three leaves
>
> **Teacher:** You understood that one caterpillar eats three leaves, then you say 12 caterpillars eat 12 times 6, 72 leaves.
>
> **Luke:** No, (he corrected his writings) 12 times three, 36 leaves.
>
> **Teacher:** You can change it in symbol form as well.

As seen, the teacher's intervention was limited to a prompting question, which emphasized the units. This allowed Luke to resolve his confusion regarding units. After he could simplify the units by noting the number of leaves for a single caterpillar, the verbal generalization and symbolic representations were accurately conveyed as he correctly identified the variables and assigned the letters for each (see Figure 4.40).

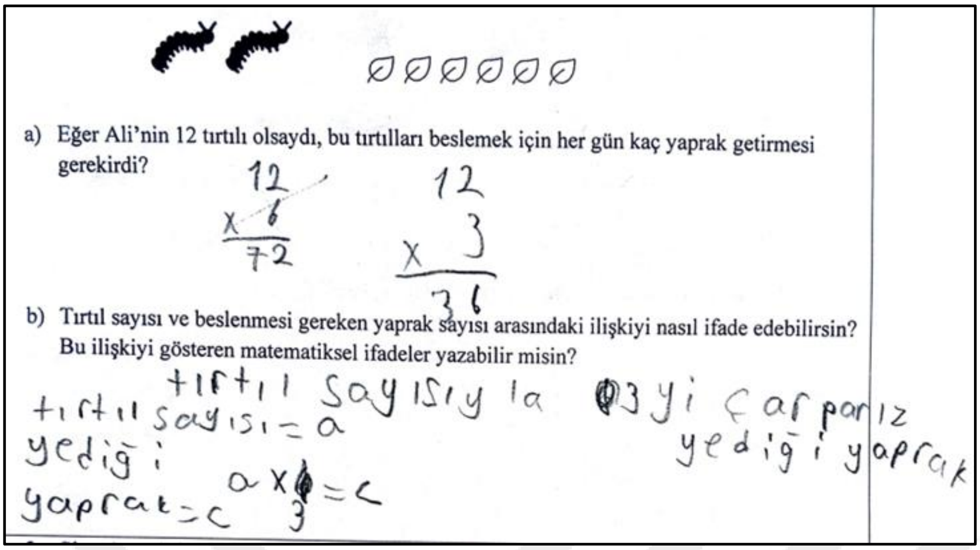


Figure 4.40. Luke's responses in the Caterpillar and Leaf Problem

Students demonstrated progress in representing a multiplicative relationship between two unknown quantities in symbols (the Cord Problem). While none of them were able to write an equation representing the relationship between two cord lengths where one of them is five times the other in pre-assessment interviews, all of them (except Sara) could write correct equations (see Table 4.16).

Table 4.16 Pre and Post-Assessment Comparisons in the Cord Problem

Cord Problem						
	Roy		Belle		Luke	
	Pre	Post	Pre	Post	Pre	Post
The inverse form of the equation	0	1	0	1	0	1
Symbolic representation	0	1	0	1	0	1
Accurate drawing	1	1	0	0	0	0

Furthermore, these students demonstrated their ability to write the inverse forms of equations by employing the division operation in the second one (see Figure 4.41).

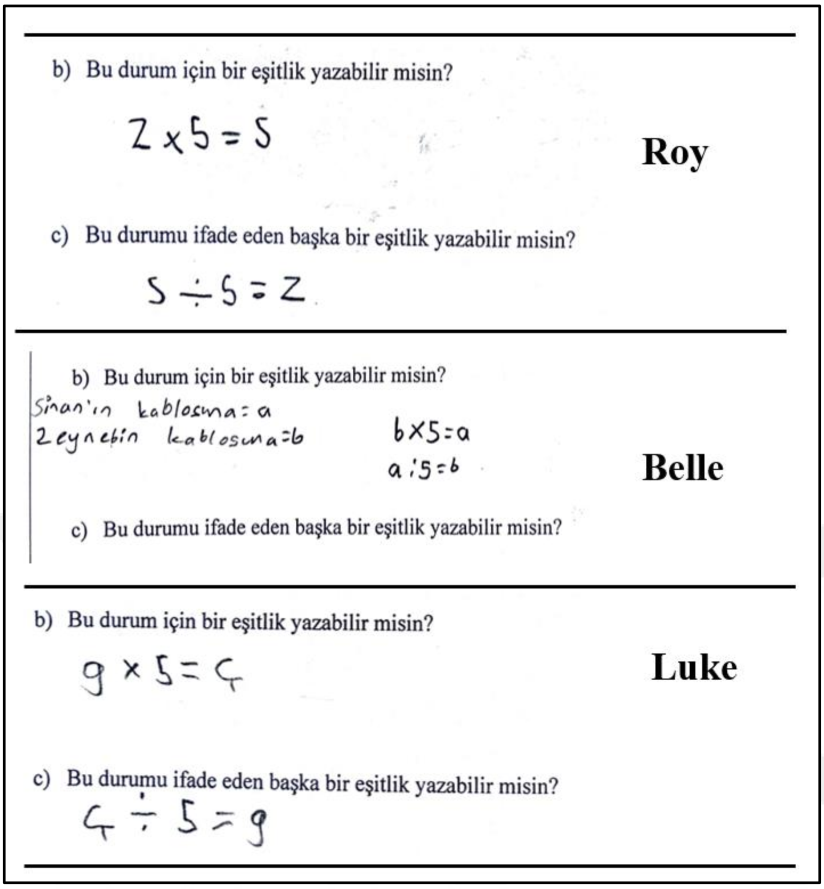


Figure 4.41. Written equations in the Cord Problem

Belle was able to write this inverse form without being asked to do so. Although Belle and Luke did not write the inverse forms of the equations in other problems, they showed their ability to think structurally in the context of similar relationships in this problem (i.e., $y = ax$). It was observed that the form of relationship, $y = ax$, between either variables or unknown quantities, can be interpreted and represented by the students in company with structural thinking after the teaching episodes.

As seen in general, notable differences between the MC2 and MC1 students appeared in the functional relationships in the form of $y = ax + b$ that was represented in the Penny Bank Problem. While Luke could not generalize and represent the functional relationship between the amount of money and the number of days elapsed, both MC2 students could generalize the relationship by calculating the larger values and writing the equations representing this relationship. It is noteworthy that the MC2 students (Roy and Belle) employed parentheses to separate different quantities in the functional relationship. Both Roy and Belle wrote the multiplication of the number of days elapsed and three (the rate of change) in parentheses and, in this way, separated the constant value (the initial amount of money) from this multiplication such as $(x.3) + 10 = y$. This indicates that they were aware of the meanings of different quantities, whether known or unknown, in the formulation. In addition, the quantity in the parentheses shows another constructed quantity with which they could perform further operations. This performance demonstrates a parallelism between their mental operations in terms of units coordination and writing equations.

4.3.4 Units Coordination

The post-assessment included two problems to assess the students' units coordination as well, the Crate and Cord problems. The Cord Problem included the examination of both writing equations to represent the multiplicative relationship between two unknowns and representing the unknown quantities in figures as convenient to the multiplicative relationship.

In the pre-assessment of units coordination, both MC2 students could find the correct answer of the Crate Problem by having some difficulty in keeping track of the quantities. In the final assessment, they could find the correct answer again. However, at this time, both students could accurately indicate each quantity and units more flexibly. Luke, as an MC1 student, could not find the correct answer in both assessments.

In the Cord Problem, Roy could draw the figures of cords by considering the multiplicative relationship between each other. He began by drawing a longer cord length, then partitioned the line into five equal parts. This resulted in a remainder part at the end of the longer line he previously drew, which he then erased. This process allowed him to construct a small unit for the length of the shorter cord. On the other hand, Belle and Luke did not draw the cords in a way that suited the given multiplicative relationship. They drew the cords in random lengths as one is a little bit longer than the other as they did in the pre-assessment. These findings indicate that there has been no observable improvement in their units coordination in this process.

CHAPTER 5

DISCUSSION AND CONCLUSION

The goal of this study was to investigate fifth-grade students' progress in algebraic thinking with the potential and plausible interaction with their units coordination. To accomplish this goal, I designed an HLT, with the support of several mathematics educators, targeting the students' generalizations and symbolic representations (Kaput, 2008). The HLT also targeted functional thinking that provides an abundant context for generalization (Blanton & Kaput, 2011). Hence, the HLT united the generalization, a core aspect of algebraic reasoning (Kaput, 2008), and functional thinking which is a dimension of algebraic thinking (Kieran, 2022) and a strand of algebraic reasoning (Kaput et al., 2008). The analysis of teaching episodes and the comparison of pre- and post-assessments of students in terms of their progress in algebraic thinking revealed findings worthy of further discussion from both algebraic thinking and units coordination perspectives in the context of mathematics education.

In light of the findings, this chapter first discusses the main constraints the students encountered and the main differences between the students in their progress in algebraic thinking during the teaching episodes. It includes the students' performances in generalizing and symbolizing the relationship between unknown quantities or between variables, key thinking processes supporting the students' algebraic reasoning in different tasks, and instructional decisions that would influence further teaching processes. Then the conclusion section presents the overall argument about the findings and mentions the theoretical and practical implications, the limitations of the study, and the suggestions for further research.

5.1 The Discussion of the Progress of Students in Algebraic Thinking

From the beginning of the teaching episodes, all students demonstrated a degree of progress in algebraic thinking and reasoning by taking distinct paths. In Episode 1, the students learned to compare unknown quantities through hypothetical values, use letters for unknown quantities, and use comparison symbols, including equal signs, to represent the relationship between unknown quantities. In Episode 2, they continued to write equations by using literal symbols they assigned to the unknowns. In addition, Episode 2 presented both additive and multiplicative relationships between unknown quantities. After getting used to working with letters, tables, and equations to represent the relationship between unknown quantities, since Episode 3, they began to interpret the functional relationship between variables through contextual problems. They worked on many contextual problems involving either the $y = ax$ or $y = ax + b$ forms of linear relationship.

During the course of the teaching episodes, the MC2 and the MC1 students demonstrated varying levels of proficiency in algebraic thinking, as evidenced by their differing approaches to algebraic reasoning. Therefore, we observed both inter-level and intra-level differences in addition to inter-level and intra-level commonalities in their performances. In the beginning, the students started with a notable disparity in their performance levels in specific processes, including functional thinking, working with indeterminate quantities (algebraic thinking), identifying functional relationships in the form of $y = ax + b$, and the way of writing symbolic representations. However, the gap between MC2 students (i.e., an intra-level disparity) was almost eliminated by the end of the study (see Figure 5.1). Intra-level differences between MC2 students had projected into their swiftness and comfort with algebraic tasks, comparing unknowns, expressing them in different ways, and writing symbolic representations. On the other hand, the inter-level differences between MC2 and MC1 students maintained in specific areas such as generalizing the functional relationships in the form of $y = ax + b$ and the way of writing equations.

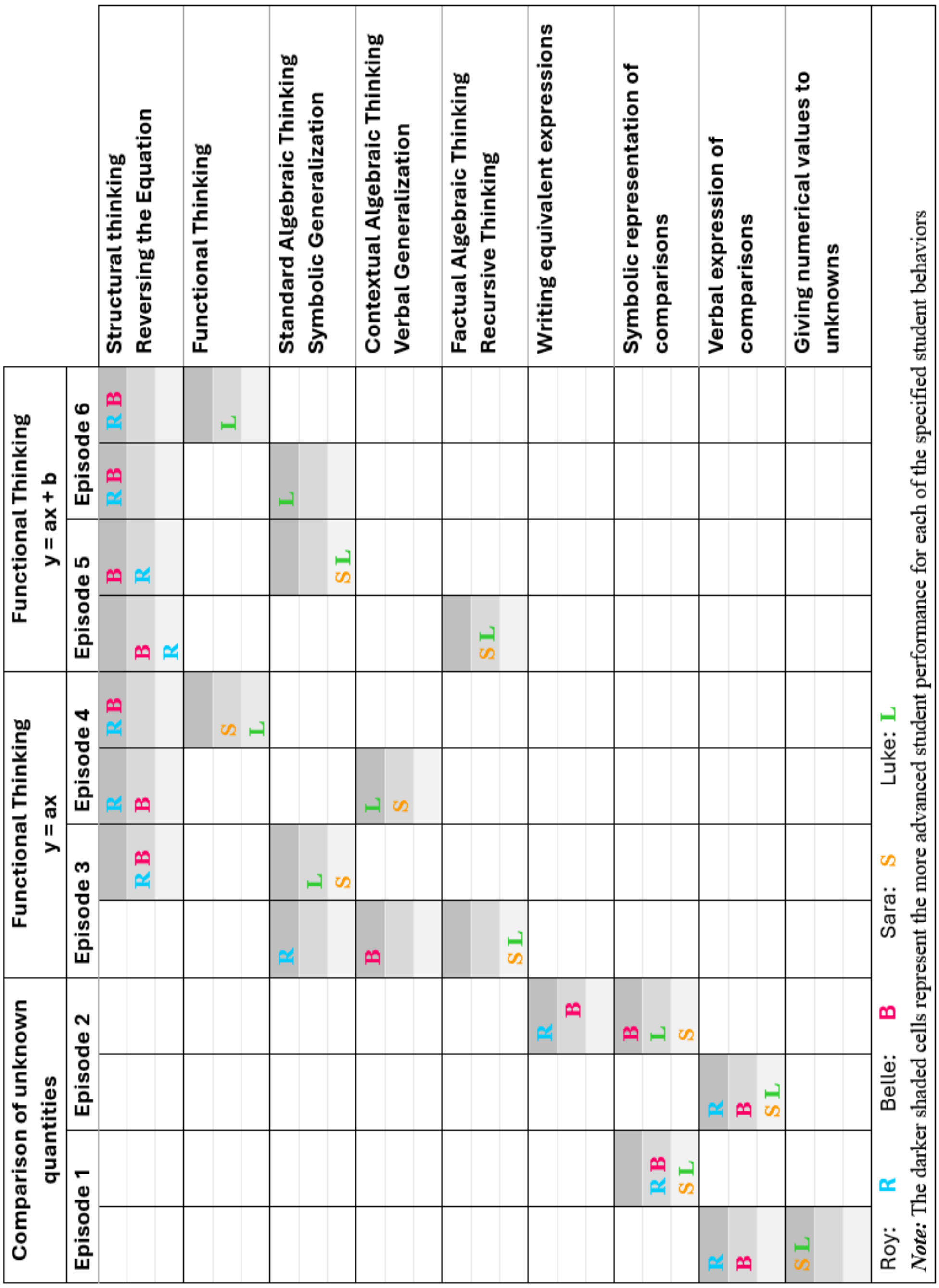


Figure 5.1 Students' progress from Episode 1 to Episode 6

Figure 5.1 demonstrates the path of progress of each student in different contexts such as the comparison of unknown quantities which includes representing additive and multiplicative relationships and generalizing functional relationships in different forms. The colored letters R, B, S, and L represent each student's performance points in the episodes, and each episode was divided into two parts, representing the first half and the second half of the episodes. Therefore, the progress of each student from the first half to the second half in each episode can be observed through these colored letters in Figure 5.1. In addition, the vertical difference between the points in the same cell in Figure 5.1, shaded with different levels of darkness, represents the differences in the students' performances. For example, a student who answered questions without help or with less need for teacher prompting and made fewer mistakes in reaching the correct answer had a higher position in the same cell than the other student who reached the same level of algebraic thinking. As shown in Figure 5.1, MC2 students demonstrated the indicators of algebraic thinking at times by writing equivalent expressions, taking fewer prompts, and using the indeterminate quantities more explicitly at the end of Episode 2, which is a shorter period. In addition, the MC1 students attended recursive thinking multiple times between Episodes 3 and 6 while the MC2 students started at least from the contextual level of algebraic thinking by formulating the function rule verbally or symbolically. The MC2 students demonstrated their structural thinking in these episodes more clearly by justifying their equations or reversing the equations. The progress points of students in Figure 5.1 demonstrated that MC2 students' progress in algebraic thinking is more stable and faster across the contexts and different tasks while the MC1 students' progress fluctuates between different contexts.

In the progress of the students in representing the functional relationships, there is a fluctuating point in Episode 5 for each student. This demonstrated that the functional relationship in the form of $y = ax + b$ (Episode 5) was a more challenging context for each student than the types $y = ax$ (Episodes 3 and 4) and $y = x \pm b$ (Episode 6). This was the context in Episode 5 where both MC1 students could not achieve the generalization by themselves. They again tried to find the values through a recursive approach. The difficulty that the MC2 students had in Episode 5 did not last much. They could figure out the relationship with little prompts while the MC1 students received more guidance from the teacher.

The following sections explain how the students progressed during the study and in what aspects their progress demonstrated an interaction between their algebraic thinking and units coordination. Therefore, four main headings emerged, indicating specific patterns and landmarks that the students demonstrated as they progressed in their algebraic thinking. These headings are interpreting the indeterminate quantities, interpreting the multiplicative relationships between unknowns, functional thinking, and the way of writing equations.

5.1.1 The differences in interpreting indeterminate quantities

At the beginning of the study, during Episodes 1 and 2, the students were confronted for the first time with the challenge of interpreting the relationships between indeterminate quantities. During this process, the students exhibited distinct patterns of cognitive processes in interpreting indeterminate quantities and performing operations with them. Furthermore, the intra-level differences were more apparent during these episodes, particularly for the MC2 students, which declined towards the end of the study. On the other hand, the discrepancy between the MC2 and MC1 students, in terms of inter-level differences, persisted to a greater extent in some aspects.

From the beginning of the study, including the pre-assessment results, Roy (advanced MC2) showed the most promising performance. In Episode 1, he was the only student to think about writing the units for each quantity, whereas Belle (regular MC2) did not pay attention to this as the MC1 students did. For example, when comparing two pencils of unknown length, Roy asked whether he should write them in cm. This represents his recognition of a quantity by assigning an appropriate unit (Thompson, 1990). This could be related to his higher level of mental operations with units as we had observed in his Units Coordination Interview. Consideration of the units of unknown lengths may be an indicator of emerging quantification, and awareness of the nature of the quantity, which are important processes for quantitative reasoning and units coordination (Olive & Çağlayan, 2008; Thompson, 1990).

Giving hypothetical values to unknowns was another distinguishing characteristic between students in terms of algebraic thinking and working with indeterminate quantities. A notable difference between the MC1 and MC2 students was the MC1 students' tendency to assign numerical values to unknown quantities to use with comparison symbols in Episode 1. This demonstrated that MC1 students were not ready to use indeterminate quantities as much as MC2 students in Episode 1, which is an inevitable condition of algebraic thinking (Radford, 2010). From the perspective of variable understanding (Blanton et al., 2017; Küchnemann, 1981), assigning numerical values to letters or unknowns is regarded as the lowest level in variable understanding. Furthermore, Hackenberg and Lee (2015) observed that relying on specific cases or numbers when representing a relationship was a behavior of MC2 students. However, in this study, only MC1 students used specific numbers to interpret the relationships between unknown quantities during the teaching episodes in the context of a comparison of unknown quantities and writing them in symbols. Although Belle (regular MC2) showed a tendency to assign numerical values to unknowns in the pre-assessment interviews, she did not do so during the teaching episodes. Therefore, it is possible that this is an impermanent tendency in MC2 students, as seen in Hackenberg and Lee's (2015) study, which could disappear with appropriate instructional decisions.

A further significant challenge encountered by all students in Episode 1 was the performance of operations with unknowns. At this stage, the construction of tables of values and the interpretation of the relationships between numerical values resolved the situation. This way of thinking simplified the process by comparing or performing operations with given hypothetical numerical values rather than unknown quantities. This reduced the level of analyticity and algebraic thinking (Radford, 2010; Smith & Thompson, 2008). Smith and Thompson (2008) posited that the comparison of numerical values or the identification of relationships between specific numbers could only provide additional information to the given generality, which includes a "little sense" (p.111) in terms of algebraic thinking. The recommended approach in algebra is to identify relationships between quantities where we do not know their "specific measure" (Smith & Thompson, 2008, p. 111). The refraining from assigning numerical values to indeterminate quantities and the practice of comparing unknown quantities by leaving them indeterminate offers an opportunity for the development of further symbolic representations (Carraher et al., 2008). The interpretation of the multiple scenarios through the table of numerical values allowed the students to generalize what operations they needed to do with given hypothetical values to show the relationship or comparison between the unknowns. This provided an emergent understanding of variables and functions as well.

5.1.2 The differences in interpreting the multiplicative and additive relationships between unknowns

Students demonstrated a significant difficulty in representing the multiplicative relationship between two unknown quantities (i.e., $m = 4k$) more than the additive relationship between three unknowns (i.e., $A + B = C$). The challenge in representing the multiplicative relationship between two unknowns was evident in the processes of constructing a table of values not involving a ratio of 1: 4 and erroneous symbolic equations. During these processes, the intra-level differences were still apparent. There were notable student performances for discussing in terms of both algebraic thinking and units coordination.

The students encountered representing a multiplicative relationship for the first time in Episode 2 where one of two equal-length ropes was divided into four equal parts and the relationship between the shorter and the longer ropes was investigated. In representing the multiplicative relationship in the tables of values, only Roy (advanced MC2) could construct a table in a manner consistent with the correct multiplicative relationship, such that one length is one-fourth of the other length. Consistently, he could also write a correct equation in two forms (i.e., $m = k \times 4$; $m \div 4 = k$). In their tabular representations, Belle (regular MC2) and Luke (regular MC1) wrote the hypothetical values for the lengths of each rope in such a way that the difference between the rope lengths is four, incorrectly indicating additive reasoning. Sara (MC1) assigned arbitrary hypothetical values to the lengths of the ropes to demonstrate that one was longer than the other. Her performance did not demonstrate the ability to perform either an additive or a multiplicative comparison. Giving numerical examples to unknowns is a common behavior among MC2 students in other studies (e.g., Hackenberg & Lee, 2015; Hackenberg et al., 2017). Here, both MC2 (Belle) and MC1 students gave numerical values to unknowns at the beginning of Episode 2.

Ulrich (2016a) remarked that multiplicative comparison would not be swift until the MC2 level and would not be assimilatory until the MC3 although it is attainable starting from the MC1 level. Roy's quick interpretation of the relationship and his ability to inverse the relationships in writing equations seemed supported by the characteristics of his multiplicative concepts. Researchers observed that MC2 students rarely wrote the inverse forms of the equations and could not use reciprocal reasoning (e.g., Hackenberg & Lee, 2015; Hackenberg & Sevinç, 2022) while MC3 students could inverse the written equations representing the multiplicative relationships. Therefore, Roy, as an advanced MC2, demonstrated a higher performance than the students in the upper grades in the other studies (e.g., Hackenberg & Lee, 2015; Hackenberg & Sevinç, 2022). In addition, performing an additive comparison is a less complex operation than performing a multiplicative comparison in terms of units coordination (Ulrich, 2016a). Considering Belle's slightly lower performance in the Units Coordination Interview compared to Roy, this performance in Episode 2 may also reflect the intra-level difference in their mental operations.

As one of the unknown quantities was constructed by equally partitioning the other unknown quantity in the problem, it required disembedding and splitting operations (Hackenberg, 2010; Steffe & Olive, 2010) to generate one unknown quantity by using the other. These operations allow the students to handle more complex tasks (Steffe & Olive, 2010). Roy’s identification of the multiplicative relationship might be evidence of his splitting operation which is related to multiplication, division, and ratio (Confrey & Smith, 1995). On the other hand, a reliance on additive thinking and focusing on a hypothetical difference between the unknowns, instead of a ratio between them, might represent Belle’s and Luke’s dominant operation of counting.

Belle (regular MC2) could interpret the relationship correctly after hearing Roy’s (advanced MC2) explanation of his table and the teacher’s guiding questions. Nevertheless, her equations continue to demonstrate the disparity between her cognitive processes and those of Roy. She wrote an equation to represent the relationship both additively and multiplicatively (e.g., $k + k + k + k = m$ and $m = k \times 4$). The equation involving addition may demonstrate how she gradually constructed the multiplicative relationship. On the other hand, Roy’s equations, the inverse form of the equation (i.e., using both multiplication and division), displayed his flexibility in identifying the multiplicative relationship between two unknowns.

The MC1 students, Luke and Sara, faced significant challenges in maintaining their progress. The intra-level differences between MC1 students in interpreting multiplicative relationships were apparently more deterministic in this process. Despite the teacher's provision of prompts and guidance, the students continued to interpret the relationship in an additive manner, as evidenced by Luke's response: "We add three times the length of the shorter rope to get the longer one." Following the provision of robust guidance and directions, Luke (regular MC1) could write an accurate equation while Sara (early MC1) experienced difficulty in formulating the equation. On the other hand, in Episode 3, Luke continued to fill the table additively in the first task by interpreting the relationship between the number of chairs and the number of legs as follows: “If there is one chair it has four legs, if there are two chairs there are five legs. There is always a particular difference." He thought that one increase in the number of chairs corresponds to one increase in the number of legs. Conversely and interestingly, Sara was able to identify the correct multiplicative relationship between the variables since Episode 3.

Zwanch and Wilkins (2021) observed that more than half of the sixth and seventh graders demonstrated MC1-level units coordination and additive reasoning rather than multiplicative reasoning. This indicates that a focus on the difference between two quantities is more common among middle school students as I observed in my students at the beginning of Episode 2. Ulrich (2016a) indicated that additive or multiplicative comparison tasks require the students to conceive the numbers, to be compared, as distinct quantities, namely composite units. This is initially possible at the TNS (i.e., MC1) and upper stages. Accordingly, the comparison tasks, which included both additive and multiplicative unknowns in the teaching episodes, were considered appropriate for the study participants, who had reached the lowest level of multiplicative concepts (i.e., MC1). In subsequent episodes, the students demonstrated notable advancement in their ability to identify and represent multiplicative relationships. This also demonstrated the efficacy of the teaching episodes.

5.1.3 The differences in functional thinking

Students in elementary grades can demonstrate recursive, covariational, or correspondence approaches in interpreting the functional relationships between covarying quantities (Blanton, Brizuela, et al., 2015; Blanton, Stephens, et al., 2015). In the recursive pattern approach, the students can identify the change/variation in only one quantity while in covariational thinking, they can describe the functional relationship between two covarying quantities verbally (e.g., each chair makes four more legs) (Blanton, Stephens, et al., 2017). In the correspondence approach, one quantity is determined with respect to the other quantity (Confrey & Smith, 1994) such as "multiplying the number of chairs by four gives the number of legs.

Recursive thinking in functional relationships is seen as an inevitable step before developing other functional thinking approaches, that is, covariation, and correspondence (Blanton, Brizuela, et al., 2015).

Attending the recursive approach inhibits students from seeing the general in a problem or pattern situation (Orton & Orton, 1999; Zazkis & Liljedahl, 2002). MC1 students, Sara and Luke, demonstrated a clear example of this process in different tasks such as in determining the multiplicative relationship between two unknowns (Episode 2) and in determining the functional relationships between variables (Episodes 3-6). Because they relied on recursive thinking in interpreting the table of values, they could not generalize the relationship between two unknown quantities or two variables, especially at the beginning of teaching episodes. Many researchers have observed the tendency of elementary and middle school students to engage in recursive thinking (e.g., Blanton, Brizuela, et al., 2015; Lannin, 2005; Orton & Orton, 1999; Zazkis & Liljedahl, 2002), as we also observed in the MC1 students in this study. This created a constraint for them to find a larger value of a dependent variable in the problems and generalize these relationships as conjectured. However, the teacher's guidance and prompts helped them determine the functional relationships between two variables in the form of $y = ax$ in further problems towards the end of teaching episodes.

A notable difference between MC2 and MC1 students during the teaching episodes was MC1 students' reliance on recursive thinking. Similarly, Zwanch (2022a) observed MC1 students' reliance on recursive thinking in generalization tasks. This may reflect the interaction between units coordination and algebraic thinking. Focusing on the difference in only one variable in a pattern situation demonstrates a lack of multiplicative thinking between two variables and a priority on recursive reasoning (Orton & Orton, 1999; Zazkis & Liljedahl, 2002). Although all the students in this study had difficulty in describing the relationship in tables and symbolic expressions in the first multiplicative task in the teaching episodes (Episode 2), the MC2 students began to think in a way of covariational or correspondence approach in the further tasks including a functional relationship. The

MC1 students continued recursive thinking and got additional support from the teacher. Consequently, the ability to maintain recursive thinking at the outset of each problem and to seek assistance from the teacher proved to be distinguishing factors in the performance of MC1 students as compared to that of MC2 students in this study. MC2 students were able to quickly adapt to the covariational and correspondence thinking in different problems.

Relying on recursive thinking may demonstrate an inclination toward additive reasoning more than multiplicative one. This is emphasized as an inclination of students who are at a level lower than MC2 (Ulrich, 2016a; Zwanch & Wilkins, 2021). However, MC1 students (Luke and Sara) could also find the larger values in pattern situations through teacher prompts and guidance. Especially, the contextual problems including functional relationships between variables provided them to formulate a general rule as a correspondence approach in and after Episode 3 where they worked on the problems. This shows a similarity with the findings of Pinto and Canadas (2021) who observed that most of the fifth graders demonstrated a correspondence approach in finding the generality in problems while only a few of them represented covariational thinking. They indicated that expressing the covariational relationship is a more sophisticated and less common way than the correspondence approach in which the fifth graders generalize the rule through the operations they did on the numerical examples.

In short, the findings indicate that there is an interaction between units coordination and students' interpretation of multiplicative relationships between unknowns or variables. This was evidenced by the MC1 students' predominant use of recursive thinking in the majority of the problems, in comparison to the MC2 students. Moreover, the MC2 students demonstrated a capacity for functional thinking that allowed them to adapt to different situations and contexts. In contrast, the MC1 students required more guidance from their teachers to adjust to novel situations and contexts, as the importance of teachers' prompts is emphasized in these cases (Hackenberg & Sevinc, 2024). Nevertheless, the capacity of the MC1 students to generalize the multiplicative relationships between variables ($y = ax$) towards the end of the episodes and in the final evaluation demonstrated the success of the teaching episodes in developing algebraic thinking. The interaction of units coordination determined the path of learning of each student and created a disparity in their performances during the episodes. However, this gap in algebraic thinking narrowed towards the end of the study.

5.1.4 The differences in writing equations

The teaching episodes in this study revealed that the students' level of units coordination determined how much guidance they needed from the teacher during the identification of the relationships between variables or unknown quantities rather than during the writing of symbolic representations. For example, the MC1 students exerted the majority of their effort in identifying the general rule in various problems. In contrast, the MC2 students could determine the relationship between variables more easily, with minimal prompting, or even without prompting. After determining a specific relationship, both groups of students could easily transform this narrative formula into an equation. For instance, in the fifth episode, Luke (early MC1) demonstrated an understanding of the general rule by finding a larger value in the Bank Account Problem through the teacher's prompting and the use of the table of values. Subsequently, he was able to rapidly construct the symbolic equation by assigning letters to the quantities involved in his operations in the table. Therefore, the writing equation process was faster and more flexible for both groups of students than generating the rule in words or arithmetic operations in the table.

As mentioned above, when the students were unable to determine a larger value in a function table, indicating a lack of ability to generalize the relationship, they also demonstrated difficulty in writing an equation. Therefore, the MC1 students had more difficulty in writing equations than the MC2 students because they had more difficulty in determining the relationships between the problem variables. On the other hand, Zwanch (2022a) explained the interaction between students' algebraic thinking and their units coordination (i.e., number sequences) through the ability to write algebraic equations rather than the method that the students used in generalizations. She observed the diverse methods of generalization employed by students from different units coordination levels in response to a problem requiring the identification of the number of squares on the border of a large square. In other words, the method of generalization (i.e., how they find the number of squares around a larger square) did not differ by students' level of units coordination. On the other hand, she observed a reduced ability to write symbolic representations as the students' level of units coordination decreased. For example, none of the MC1 students (TNS) and only a few of the advanced MC1 (aTNS) students could write a symbolic representation. This conflicting case between this study and Zwanch's (2022a) study might be due to the different contexts addressed in the studies such as problem types and the grade level of students.

In short, in this study, the students who had lower levels of mental operations with units and composite units, such as not being able to keep track of the newly constructed units and operate with them in further steps, similarly demonstrated limitations in understanding the problem quantities or variables in generalizing the relationship narratively or arithmetically. Eventually, this affected their performance primarily in the identification and generalization of relationships, before writing an equation representing these relationships. From the aforementioned points, it can be posited that the interaction between the students' multiplicative concepts and their algebraic thinking may be limited to the process of identifying relationships between problem variables or quantities, rather than directly to their symbolic representation of the relationships.

How the students write their equations may be further evidence to support the claims made in this study. For example, the contextual problems including the functional relationships in the form of $y = ax + b$ were the most challenging ones for the students in the identification of the relationship between variables (e.g., the Birthday Party 2 Problem, the Bank Account Problem, and the Sapling Problem). This was evident in their overthinking in finding the larger values in function tables and the amount of guidance requested from the teacher. The MC2 students' equations in these problems included parentheses as different than the MC1 students (as seen Tin able 4.16). Using parenthesis may demonstrate newly constructed composite units in students' formulations (Olive & Çağlayan, 2008) and the steps in students' mmindswhile they are formulating a rule (Radford, 2010). This might also represent the students' assimilation of each operation with problem quantities. For example, the MC2 students quantified the amount of money increased after "a" month (i.e., $(a \times 15) + 30$ in the Bank Account Problem) or the number of people seated on the sides of "m" tables (i.e., $(m \times 2) + 2$ in the Birthday Party 2 Problem) as newly constructed units. Olive and Çağlayan (2008) observed that some eighth graders could appropriately write equations to represent a word problem including multiple unknowns in which they used parenthesis and products to show a new "composed quantity" (p. 11), a monetary value, formed by the production of two different quantities such as the number of dimes and the value of one dime (e.g., $0.1(n + 1)$). This use of parenthesis seems to be similar to Roy's (advanced MC2) and Belle's (regular MC2) way of writing equations in the problems including the form of $y = ax + b$. Hence this might represent the support of MC2 students' construction and coordination of composite units in their structural and analytical thinking during the generalizations of the relationships.

Using parentheses in their equations might be also the students' justification which is a significant element in generalizations (e.g., Ellis, 2007; Lannin, 2005). Using parenthesis justifies how they constructed these relationships. For example, Belle's drawings in the Birthday Party 2 Problem when formulating how many people are seated around any number of tables demonstrated within which steps she formulated the rule, $(m \times 2) + 2$. Therefore, her using parenthesis in the symbolic expression might be an explanation for supporting her symbolic generalization and the order of her operations. Radford (2010) interprets the use of parentheses (or brackets) in equations as a reflection of the story of the students' thinking during the generation of the formula by indicating:

> "…the formula is not an abstract symbolic calculating artefact but rather a story that narrates, in a highly condensed manner, the students' mathematical experience." (p.10)

Regarding this, the MC2 students' mathematical experience with the function contexts involving multiple operations with variables seemed more sophisticated than the MC1 students. Their written formulas including parentheses supported this. Although this form of function rules (i.e., $y = ax + b$) presented a challenge for all the students in this study in terms of generalizing relationships, the MC1 students had more difficulty by not being able to generalize these relationships by themselves. Therefore, this distinction between the MC1 and MC2 students might be related to their operations with units and quantities, as a mathematical experience during the generalization of the functional relationships and representing the generalizations through symbols. Therefore, this mathematical experience in generating a formula Radford's (2010) may rely on students' understanding of quantities and the relationships of quantities, which foster students' understanding of functional relationships (Ellis, 2011).

Given the above discussion, it is obvious that the way of writing a symbolic expression and providing some justification for the generalizations is very important besides just writing a symbolic representation. Researchers indicated that using algebraic notation by itself does not mean thinking algebraically, and also the absence of algebraic notation does not mean the lack of algebraic thinking (e.g., Radford, 2010; Zazkis & Liljedahl, 2002). Zazkis and Liljedahl (2002) observed that there is an inconsistency between the students' verbal expression of generality and expressing it in symbolic notations. Forming an algebraic formula may be achieved by some other means rather than employing generality. For example, students can reach a formula through trial-and-error or repeated-guess while changing some of the terms in the formula until getting the correct result. However, this generation of a formula does not refer to "an analytical way of thinking about indeterminate quantities" which is "the chief characteristic of algebraic thinking" (Radford, 2010, p.9).

5.2 Conclusion

The findings and the discussion of findings provided several insights about learning and teaching algebraic thinking. In different steps of this study, from pre-assessment to the last teaching episode, specific thinking patterns of students and some task characteristics, on the one hand, created constraints for going further in algebraic thinking, and on the other hand, supported the improvement of students' algebraic thinking. Therefore, one of the concluding remarks is about the interaction between fifth-grade students' progress in algebraic thinking and their units coordination levels (i.e., multiplicative concepts). The second conclusion pertains to the characteristics of the HLT and the actual learning process that facilitated the students' algebraic thinking by highlighting their units coordination such as the types and sequence of learning tasks, the manner of teacher's intervention and the interactions among students.

The first conclusion was that there might be an indirect interaction between the students' units coordination levels and their progress in algebraic thinking regarding symbolic representations of the relationships. The most direct relationship was between the students' levels of units coordination and their generalization of relationships where they found the larger values in a pattern situation or verbalized the general rule by using indeterminate quantities. As they could identify the problem quantities and operate with these quantities, where they construct new composite units in multiplicative situations, their generalizations of the relationships became more possible during teaching episodes. Otherwise, they needed the teacher's prompting, additional visual representations, or more numerical examples. The need for these types of support was evident in numerous instances for the MC1 students and in the novel problem situations (e.g., The Bank Account Problem 2) for the MC2 students. When the students could not recognize the problem quantities, or find the larger values in a pattern situation, they were not able to create a symbolic expression. Therefore, their units coordination was in an interaction more with their ability to generalize the relationships than with their ability to write a symbolic representation in this learning process. As Radford (2010) asserted, an algebraic formula embedded a narrative story of generalizations, as a product of algebraic thinking. Therefore, only the generalizations that have a story in mind can be transformed into an algebraic formula.

Zwanch (2022a) indicated an inconsistency between her observation of aTNS (advanced MC1) students' performance in generalizations and the students' performances in an early algebra intervention study conducted by Blanton et al. (2019). Zwanch (2022a) observed that aTNS students could achieve oral generalization while they did not demonstrate sufficient symbolic representations. On the other hand, Blanton et al. (2019) observed that both control and experimental group students were more successful in representing the functional relationship by using variable notation as compared to using a verbal description. This made them reconsider the question of whether the students are ready for variable/symbolic notations in primary years. This is assumed to be a consequence of instructional interventions, which are also considered to be an important factor that can affect students' cognitive and algebraic performances (Blanton et al., 2019; Zwanch, 2022a). The findings of the current study are more consistent with the findings of Blanton et al. (2019) in terms of the difficulty in students' verbal generalizations. The majority of the effort of the students appeared in their generalization process rather than symbolic representations. This would again be explained by the influence of units coordination on generalizing the relationships in a narrative form rather than on the ability to write symbolic representations.

Furthermore, the MC2 students demonstrated key dimensions of algebraic thinking in their different forms of written symbolic expressions (e.g., structural thinking) and in their verbal generalizations which explicitly included the indeterminate quantities and different forms of functional thinking. Therefore, their development in algebraic thinking was more salient through their performance in generalizations and symbolic representations than the MC1 students' development. Therefore, another concluding remark for the interaction between the students' progress in algebraic thinking and their units coordination is structural thinking and the sophisticated expressions of the generalizations of the MC2 students.

The second main conclusion is about the characteristics of the HLT that aimed to improve the students' algebraic thinking. It was seen that the contextual problems providing a pattern situation helped the students' understanding of problem variables and the relationships. As suggested by early algebra researchers (e.g., Blanton et al., 2011; Blanton, Stephens, et al., 2015; Carraher et al., 2008), using contextual problems by incorporating different representations such as tables and visuals could provide an effective learning environment for algebraic reasoning. For example, the students in the current study, even in the pre-assessment interview, could identify the relationships when there is a table of values or figural patterns such as in the Growing Caterpillar problem. Therefore, a learning process starting with pattern situations and incorporating the different representations such as figures and tables appeared to be an effective aspect of the HLT for developing students' algebraic thinking. Therefore, the problems involving multiplicative relationships between unknown quantities and disembedding operation, such as the first task in Episode 2, could be moved to further episodes after the problems involving functional relationships in the form of y =ax, given the difficulty the MC1 students had. When the students with lower levels of unit coordination began the learning process with problems involving composite units and fewer operations, they could have the scaffolding they needed.

Furthermore, the students' interaction with the tasks, the teacher, and their peers enhanced their learning as remarked by other researchers (e.g., Cobb et al., 1993; Hackenberg & Sevinç, 2024; Steffe & Olive, 2010; von Glasersfeld, 2001). For example, the teacher's request to invert the equations (i.e., $y = 4x$ and $x = y \div 4$) enhanced the students' structural thinking, especially the MC2 students, and allowed them to focus on the relationships between variables. This concluded that when the students were able to create equations representing the specific relationships, asking them to think about the different forms of the same relationships or to create equivalent expressions supported their analytical and structural thinking, which are the dimensions of algebraic thinking. This, in turn, contributed to the students' algebraic reasoning in different problems and contexts. In addition, the interaction between the students facilitated their learning during the episodes. Verbalizing their thinking and sharing their answers to questions in words helped their peers to learn and adapt to a similar way of thinking in further tasks.

Finally, we filled in the empty parts of the claim (i.e., parentheses) indicated on page 64 to provide a new perspective. In summary, this study concluded that a learning process including [contextual problems starting with pattern situations, including different representations, teacher prompting, and student interaction] develops students' algebraic thinking. When students [operate with the MC2], they can [flexibly develop their algebraic thinking in different contexts such as generalizing the multiplicative relationship between unknown quantities and representing functional relationships in the form of $y = ax + b$]. Students who [operate with the MC1] would progress in algebraic thinking when the learning activities start with [pattern situations and include functional relationships in the form of $y = ax$].

5.2.1 Implications

The investigation in this study resulted in both theoretical and practical implications. The theoretical implication of this study provides a new perspective on the interaction between the units coordination and algebraic thinking. The practical implication of this study highlights new insights into the instructional decisions about teaching algebra.

Regarding the first implication, this study offers a new way of looking at how units coordination and algebraic reasoning interact, as a theoretical implication. It showed that the fifth-grade students who demonstrated the MC1 and MC2 level of units coordination could progress in algebraic thinking to the end of writing equations representing the relationships between variables or between unknown quantities.

They could accomplish many tasks by experiencing significant processes in algebraic reasoning, such as identifying the problem variables, constructing tables of values, generalizing the relationships between variables, and writing symbolic expressions to represent these relationships. This could happen through some differences in their actual learning trajectories. The characteristics of students' mental operations in terms of units coordination explained the way they generalized, the amount and the extent of the support they received from visual materials and teacher prompts, and the types of tasks they could complete in these processes.

Furthermore, the analysis of students' mental operations provided new insights into the view of the students' categorization in terms of units coordination. Although the differences between fifth-grade students' mental operations were primarily based on their multiplicative concepts, MC1 and MC2, there were also notable intra-level differences between the students' mental operations within the same level (as seen in section 4.1.1). This difference was similarly observed in students' performance in interpreting the relationships between variables or unknown quantities. In other words, the discrepancy between MC2 students' mental operations observed in the pre-assessment interviews was also observed in their progress in algebraic thinking, especially in the first tasks of each teaching episode. Consequently, the unit coordination levels in terms of multiplicative concepts were not viewed as comprising a set of distinct and rigid categories. Rather, they are considered as levels (e.g., MC1, MC2, and MC3) within which different mental operations are still involved at each level, and these operations may be in continual flux for each student. Therefore, this within-level variation might have the potential to illustrate the nuances of algebraic thinking and reasoning. This illustrates how units coordination facilitates the processes and structures inherent to algebraic thinking.

This analysis also revealed practical implications for mathematics educators in research and instruction incorporating units coordination and algebraic thinking. First, the substantial progress of the MC1 and MC2 students from the pre-assessment to post-assessment suggests that early algebra interventions based on functional thinking in the form of $y = ax$, which involves only one operation with problem quantities, would be appropriate for incorporating the processes of algebraic reasoning. Zwanch (2022a) remarked on the inconsistent findings about advanced MC1 students’ (aTNS, the terminology in her research) generalizations in her study and elementary year students’ advancement in algebraic reasoning in the study of Blanton et al. (2019). She suggested more research on how specific instructions could affect the MC1 students’ performance on symbolic representations of generalizations to clarify this inconsistency. Considering this, the current study provides meaningful evidence and explanation by emphasizing that a learning process starting with pattern situations and incorporating different representations such as figures and tables would be a good start for developing the students' algebraic reasoning. Similarly, Kieran (2022) suggested implementing further research to investigate the inconsistencies between verbal and symbolic generalizations. The discussion of the patterns in students’ performance of generalizing the relationships and transforming this relationship into a symbolic expression in the current study provides a novel interpretation by incorporating the students’ abilities to operate with units.

Furthermore, the findings of this study suggest that problems including patterns and supporting materials for understanding the problem such as function tables, and figures for growing patterns would allow students, even at the early levels of units coordination, to gradually integrate algebraic reasoning, make sense of variables with given data, and move to the use of symbolic representations, as observed in other early algebra studies (e.g., Blanton et al., 2019; Carraher et al., 2006). The use of supporting materials also highlights the importance of multiple representations in algebraic reasoning (Brizuela & Ernest, 2008). Making transitions between different representations (or notations) within a mathematical thinking system such as tables, verbal expressions, and symbolic notations allows students to create meaningful communication and construct relationships (Brenner et al., 1997; Brizuela & Ernest, 2008; Kaput, 1991).

5.2.2 Limitations and recommendations for further research

This study involved an implementation of the HLT through six teaching episodes over three weeks. The duration of each teaching episode ranged from 60 to 75 minutes. Teaching episodes were carried out in two different groups with two students in each group. Students in each group had similar mental operations (e.g., advanced MC2 and regular MC2). In this way, there were two groups of students attending teaching episodes. Regarding this implementation of the data collection process, this study included some limitations and corresponding recommendations for future research.

Although the main focus is to investigate the interaction between the students' units coordination levels and their progress in algebraic thinking through the intervention of the HLT rather than investigate the intervention itself, the duration of teaching episodes can still be a limitation. Observing an improvement in students' algebraic thinking and reasoning would be more trustworthy in a longer period of intervention. For example, practicing different contexts by using a greater number of problems and providing more extensive discussions for each learning goal would help students make more progress in algebraic thinking, especially for the MC1 students. MC1 students could attend additional episodes to improve their generalization of functional relationships in the form of $y = ax + b$ because they had difficulty in this context. In addition, it would extend the duration of the intervention. However, the time frame in this study for conducting the HLT corresponds to two weeks of mathematics lessons in a real classroom environment according to the Turkish mathematics curriculum. Therefore, a three-week-long intervention in the current HLT may be a logical and appropriate amount of time to spend in real practice. Nevertheless, implementing longer-term teaching experiments, evaluating their findings, and comparing them with the current ones may also be a suggestion for future research.

The second limitation is the number of students who participated in this study. In order to conduct a more in-depth analysis and to better observe each student's thought processes, it could be more appropriate to work with fewer students. On the other hand, being able to observe students with more diverse mental operations in terms of units coordination might provide more evidence about the interaction between units coordination and progress in algebraic thinking. In addition, observing the same situation in a classroom setting with more students and also from diverse backgrounds might be another suggestion for further studies. The current study involved a relatively homogeneous group. Therefore, providing a heterogeneous group involving students with different levels of units coordination such as MC1, MC2, and MC3 could provide substantial information about the interaction between units coordination and the development of algebraic thinking. All these conditions would ensure the practicality of the HLT and the findings. Implementing and testing the HLT with appropriate revisions, such as moving the multiplicative task in Episode 2 after the tasks involving functional relationships, in a real classroom setting could be the next step in this study.

In addition to the previous recommendations for future research regarding the implementation of the HLT with different students, a final recommendation is related to the consideration of the intra-level differences between students at the same level of units coordination. At the beginning of the study, I didn't consider the intra-level differences for selecting the students, and I did not conjecture that these intra-level differences could be effective in the progress of algebraic thinking. However, the findings demonstrated that the intra-level differences could play a role in the development of students' algebraic thinking. Therefore, new research should consider the within-level differences in multiplicative concepts to discuss the potential interaction between students' units coordination and any addressed mathematical performance. In this way, future learning sequences could be developed by considering intra-level differences in students' units coordination.

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APPENDICES

A. Metu Human Subjects Ethics Committee Approval

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
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Sayı: 28620816 /

14 NİSAN 2022

Konu : Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi : İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Dr.Öğretim Üyesi Şerife SEVİNÇ

Danışmanlığını yürüttüğünüz Fatma ACAR'ın "Fonksiyonel İlişkileri Odak Alan Cebir Öğretiminin Beşinci Sınıf Öğrencilerine Uygulanması" başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **204-ODTUİAEK-2022** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

[redacted]

Prof.Dr. Mine MISIRLISOY
İAEK Başkan

B. Official Permissions from The Ministry of National Education

T.C.
İSTANBUL VALİLİĞİ
İl Milli Eğitim Müdürlüğü

Sayı : E-59090411-20-49911414 18/05/2022
Konu : Anket ve Araştırma İzni (Fatma ACAR)

VALİLİK MAKAMINA

İlgi : a) Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 21.01.2020 tarihli ve 2020/2 sayılı genelgesi.
b) Orta Doğu Teknik Üniversitesinin bilinmeyen tarihli ve 300 sayılı yazısı.
c) Müdürlüğümüz Araştırma ve Anket Komisyonunun 17.05.2022 tarihli tutanağı.

Araştırma Konusu	: Fonksiyonel İlişkileri Odak Alan Cebir Öğretiminin Beşinci Sınıf Öğrencilerine Uygulanması
Araştırma Türü	: Anket
Araştırma Yeri	[redacted]
Araştırma Yapılacak Kişiler	: Ortaokul Öğrencileri
Araştırmanın Süresi	: 2021 - 2022 Eğitim ve Öğretim Yılı

Yukarıda bilgileri verilen araştırmanın; 6698 sayılı Kişisel Verilerin Korunması Kanununa aykırı olarak kişisel veri istenmemesi, öğrenci velilerinden açık rıza onayı alınması, yüz yüze eğitime geçmiş olan kurumlarımızda, Covid-19 tedbirlerinin araştırmacı ve ilgili kurum idarelerince alınması, bilimsel amaç dışında kullanılmaması, bir örneği Müdürlüğümüzde muhafaza edilen mühürlü ve imzalı veri toplama araçlarının kurumlarımıza araştırmacı tarafından ulaştırılarak uygulanması, katılımcıların gönüllülük esasına göre seçilmesi, araştırma sonuç raporunun kamuoyuyla paylaşılmaması ve araştırma bittikten sonra 2 (iki) hafta içerisinde Müdürlüğümüze gönderilmesi, okul idarelerinin denetim, gözetim ve sorumluluğunda, eğitim ve öğretimi aksatmayacak şekilde, ilgi (a) genelge esasları dâhilinde uygulanması kaydıyla Müdürlüğümüzce uygun görülmektedir.

Makamınızca da uygun görüldüğü takdirde olurlarınıza arz ederim.

Levent YAZICI
İl Milli Eğitim Müdürü

OLUR
18/05/2022
Niyazi ERTEN
Vali a.
Vali Yardımcısı

Ek:
1- İlgi (b) Yazı ve Ekleri (4 Sayfa)
2- İlgi (c) Tutanak (1 Sayfa)

Bu belge güvenli elektronik imza ile imzalanmıştır.

Adres : Binbirdirek Mah. İmran Öktem Cad. No: 1 Sultanahmet Fatih İstanbul
Telefon : 0212 384 36 30
E-posta : stratejigelistirme34@meb.gov.tr
Kep Adresi : meb@hs01.kep.tr

Belge Doğrulama : https://www.turkiye.gov.tr/meb-ebys
Bilgi İçin : Aykut ÇELİK
Unvanı : Büro Hizmetleri
İnternet Adresi : http://istanbul.meb.gov. tr/

Bu evrak güvenli elektronik imza ile imzalanmıştır. https://evraksorgu.meb.gov.tr adresinden 8e23-e9f7-3882-9967-feef kodu ile teyit edilebilir.

C. Parent Permission Form

Sayın Veli;

Çocuğunuzun katılacağı bu çalışma, "Fonksiyonel İlişkileri Odak Alan Cebir Öğretiminin Beşinci Sınıf Öğrencilerine Uygulanması." adıyla, 6-17 Haziran, 2022 tarihleri arasında okul saatleri dışında yapılacak bir araştırma uygulamasıdır.

Araştırmanın Hedefi: beşinci sınıf öğrencilerinin cebirsel akıl yürütmelerini fonksiyonel ilişkileri yorumlama, değişken kavramını anlama ve sembollerle ifade etme kapsamında geliştirmeye yönelik bir öğrenme rotası tasarlamaktır. Bu kapsamda sizinle bazı değerlendirmeler ve öğretim uygulaması amacıyla çeşitli görüşmeler yapmayı amaçlıyoruz Çalışmanın çıktılarının, ileriki yıllarda matematik eğitimi alanında ve özellikle erken dönem cebir öğretimi konusunda yapılacak araştırmalar için yol gösterici olacağı beklenmektedir.

Araştırma Uygulaması: Görüşme ve öğretim uygulama şeklindedir.

Araştırma T.C. Millî Eğitim Bakanlığı'nın ve okul yönetiminin de izni ile gerçekleşmektedir. Araştırma uygulamasına katılım tamamıyla gönüllülük esasına dayalı olmaktadır. Çocuğunuz çalışmaya katılıp katılmamakta özgürdür. Araştırma çocuğunuz için herhangi bir istenmeyen etki ya da risk taşımamaktadır. Çocuğunuzun katılımı **tamamen sizin isteğinize bağlıdır**, reddedebilir ya da herhangi bir aşamasında ayrılabilirsiniz. Araştırmaya katılmamama veya araştırmadan ayrılma durumunda öğrencilerin akademik başarıları, okul ve öğretmenleriyle olan ilişkileri etkilemeyecektir.

Çalışmada öğrencilerden kimlik belirleyici hiçbir bilgi istenmemektedir. Cevaplar tamamıyla gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir.

Uygulamalar, genel olarak kişisel rahatsızlık verecek sorular ve durumlar içermemektedir. Ancak, katılım sırasında sorulardan ya da herhangi başka bir nedenden çocuğunuz kendisini rahatsız hissederse cevaplama işini yarıda bırakıp çıkmakta özgürdür. Bu durumda rahatsızlığın giderilmesi için gereken yardım sağlanacaktır. Çocuğunuz çalışmaya katıldıktan sonra istediği an vazgeçebilir. Böyle bir durumda veri toplama aracını uygulayan kişiye, çalışmayı tamamlamayacağını söylemesi yeterli olacaktır. Görüşmelere katılmamak ya da katıldıktan sonra vazgeçmek çocuğunuza hiçbir sorumluluk getirmeyecektir.

Onay vermeden önce sormak istediğiniz herhangi bir konu varsa sormaktan çekinmeyiniz. Çalışma bittikten sonra bizlere telefon veya e-posta ile ulaşarak soru sorabilir, sonuçlar hakkında bilgi isteyebilirsiniz. Saygılarımızla,

Araştırmacı : Fatma Acar

İletişim bilgileri : 0541[redacted] / [redacted]

Velisi bulunduğum .... sınıfı .... numaralı öğrencisi ....'in yukarıda açıklanan araştırmaya katılmasına izin veriyorum. (Lütfen formu imzaladıktan sonra çocuğunuzla okula geri gönderiniz*).

..../...../...........

İmza: İsim-Soyisim

Veli Adı-Soyadı :

Telefon Numarası:

D. The HLT

EPISODE 1: Comparison of unknown quantities using equality and inequality
Learning Outcomes
• Express the comparison of unknown quantities verbally (e.g., it is longer/heavier/older than the other) • Attain hypothetical values for unknown quantities by using tables. • Assing letters/symbols to represent an unknown quantity. • Use letters/symbols to represent the comparison between unknown quantities using equality and inequality. • Understand the relational meaning of the equal sign.
Tasks and Their Structures
Task 1: Expressing the multiplicative relationship between two unknowns by using symbolic expressions **Task 2:** Expressing the additive relationship between three unknowns by using symbolic expressions
Conjectures
a) MC1 and MC2 students would compare the unknown quantities and express them verbally at the beginning of tasks. b) MC1 and MC2 students would attain values for each unknown instead of using literal symbols c) MC1 students would not understand how they represent the relationship using symbols. d) Roy or both MC2 students would use the assigned letters to represent the comparison with symbols towards the end of the episode. e) MC1 students would continue to assign values to unknown quantities instead of using symbols. f) MC1 and MC2 students would have difficulty in representing the comparison between three unknowns on un/balanced scales
Instructional Moves Aligning with the Conjectures
• **Conjectures b – c – e** -Assign multiple values for each unknown on a table and discuss the generalized comparison -Discuss the comparison symbols in expressing the numerical situations in mathematical language (e.g., =, <, >)

The HLT Table (Continued)

-Direct the student to use letters for unknowns by saying “let the length of yellow pencil ‘*a’ a*nd the length of orange pencil ‘*b*’.”

- **Conjecture f**

-Use a table to assign values to three unknowns on an un/balanced scale and discuss how to represent two unknown weights on one side in comparison to the other on the other side.

EPISODE 2: Comparison of unknown quantities with additive and multiplicative relationships using equality

Learning Outcomes

- Recognize multiplicative and additive relationships between unknown quantities through models.
- Create different scenarios by attaining hypothetical values to unknown quantities having multiplicative or additive relationships by using tables.
- Assing letters/symbols to represent an unknown quantity.
- Express the multiplicative and additive relationship between unknown quantities verbally.
- Express the multiplicative and additive relationship between unknown quantities using symbols.
- Show the relational meaning of the equal sign.

Tasks and Their Structures

Task 1: Expressing the multiplicative relationship between two unknowns by using symbolic expressions.

Task 2: Expressing the additive relationship between three unknowns by using symbolic expressions

- Including contextual models and scenarios (Reality and level principle)
- Including a comparison of two or more unknown quantities
- Allowing using tables of hypothetical values for the unknowns
- Including quantitative reasoning through the multiplicative and additive relationships (Intertwinement principle)
- Generalizing from hypothetical values to symbols (Level principle)

Conjectures

a) MC1 and MC2 students would express the additive and multiplicative relationships verbally

b) MC1 students use the letters to represent the lengths, but they would not write the equations

c) MC1 students would assign values for the length of ropes like 1 and 4 or 2 and 8.

The HLT Table (Continued)
d) MC1 students would give numerical values to the unknowns and do operations, but they would not represent the additive and multiplicative relationship using symbolic expressions e) MC2 students would express the additive and multiplicative relationship verbally and symbolically by using letters, operations, and equality. f) MC2 students would write different algebraic expressions representing the same multiplicative relationship (e.g., 4 x a: it is the longest rope; 4 x r = longer rope; r + r + r + r = longer rope; 4 x r = s; s / 4 = r).
Instructional Moves Aligning with the Conjectures
• **Conjectures a – b – c – d:** -Ask them to use letters and describe the same thing by using symbols - Ask and discuss "Is there another way to represent the relationship (addition /division/ multiplication/subtraction)?" - Emphasize that we do not know the lengths. Ask and discuss the relationship between the assigned numbers. For example, ask: "Which operation can you do to find one?" • **Conjecture e – f:** -Ask and discuss "How differently can you demonstrate the same relationship?"
EPISODES 3-4: Representing functional relationships between variables in the form of $y = ax$
Learning Outcomes
• Identify the problem variables. • Construct a function table. • Identify and generalize the functional relationship in the table of data. • Understand and express the functional relationship between two variables through recursive, covariational, and correspondence approach • Represent the functional relationship using equation and variable notation. • Connection between multiple representation of functional relationship (verbal, table and symbolic)
Tasks and Their Structures
3-1) The Chair and Legs Problem: The relationship between the number of chairs and the number of legs ($y = 4x$) **3-2) The contexts like the first task:** The number of dogs / people/ and the number of legs/ ears /noses (e.g., $y = 2x$ and $y = x$) **4-1) The Saving Money Problem**: The relationship between time and the total amount of money ($y = 5x$)

The HLT Table (Continued)
4-2) The Pool Problem: The relationship between the amount of water in a pool and the elapsed time ($y = 2x$) **4-3) The Birthday Party 1 Problem:** The relationship between the number of tables and the number of people who are seated ($y = 2x$) • Including contextual problems (Reality principle) • Relationship between discrete quantities (Episode 3) • Relationship between continuous quantities (Episode 4) • Using function tables to represent the data before generalization (Level principle)
Conjectures
a) MC2 students would calculate any corresponding value in function tables. b) MC1 students would not calculate the larger values in function tables because of recursive thinking. c) MC2 students would indicate the functional relationship verbally by using indeterminate quantities and write equations by using symbols. d) MC1 students would not indicate the functional relationship by using indeterminate quantities and letters. e) MC1 students would have difficulty understanding the problem about the relationship between the number of tables and the number of people seated around the tables
Instructional Moves Aligning with the Conjectures
• Conjecture a – c: -Ask them to use different strategies and explain the relationship by using different expressions and equations (Developing structural thinking) • Conjecture b – d: -Fill the table together on the board and ask about the relationship between two variables. Indicate the names of each variable in discussing each case. Let the students interpret the change in both variables simultaneously. • Conjecture e: -Ask them to draw models to represent each situation. Show one table, two tables, and three tables on the board respectively, and ask them to interpret the situation.

EPISODES 5-6: Representing functional relationships between variables in the form of $y = ax + b$
Learning Outcomes
• Identify the variables and the constant term in the problem • Construct a function table.

The HLT Table (Continued)

- Identify and generalize the functional relationship in the table of data.
- Understand and express the functional relationship between two variables through recursive, covariational, and correspondence approach
- Represent the functional relationship using equation and variable notation.
- Connection between multiple representation of functional relationship (verbal, table and symbolic)

Tasks and Their Structures

5-1) The Birthday Party 2 Problem: The relationship between the number of tables and the number of people who are seated ($y = 2x + 2$)

5-2) The Bank Account Problem: The relationship between time and the total amount of money saved in the account ($y = 15x + 30$)

6-1) The People and Hats Problem: The relationship between a person's height without a hat and with a hat ($y = x + 20$)

6-2) The Credit Card Problem: The relationship between the initial amount expenditure and total debt amount ($y = x - 20$)

6-3) The Sapling Problem: The relationship between the elapsed time (days) and the height of the sapling. ($y = 2x + 35$)

- Including contextual problems (Reality principle)
- Relationship between discrete quantities (Episode 5)
- Relationship between continuous quantities (Episode 5-6)
- Using function tables to represent the data before generalization (Level principle)

Conjectures

a) MC2 students would calculate any corresponding value in the function tables and indicate the functional relationship verbally by using indeterminate quantities

b) MC1 students would not calculate the larger values in the function tables because of ignoring the constant value and they would not indicate the functional relationship by using indeterminate quantities and letters.

c) MC1 and MC2 students would have difficulty in writing the equations representing the functional relationship such as ignoring the constant value

d) MC1 students would have difficulty in writing the equations representing the functional relationship in the form of $y = x \pm a$ which is different than the previous problems

The HLT Table (Continued)
Instructional Moves Aligning with the Conjectures
• Conjectures a -Ask them to explain the relationship by using different expressions and equations (Developing structural thinking) • Conjectures b - Fill in the table together on the board and ask the relationship between two variables. Let the students interpret the change in both variables simultaneously. • Conjectures c -Ask them to pay attention to how they fill in the table and what operation they did in calculating one variable by using the value of another variable. • Conjectures d -Discuss the meaning of problem variables, pay attention to the table of values, and highlight the covariation

E. Post-Assessment Questions Including Sample Data of a Student

GENEL DEĞERLENDİRME

1. Ali evde 2 tırtıl beslemektedir. Bu 2 tırtılın beslenmesi için her gün 6 yaprak getirmektedir.

a) Eğer Ali'nin 12 tırtılı olsaydı, bu tırtılları beslemek için her gün kaç yaprak getirmesi gerekirdi?

||||||||||||| 3 3 3 3 3 3   36   3x12=36 hergün getirmesi gereken yaprak sayısı

b) Tırtıl sayısı ve beslenmesi gereken yaprak sayısı arasındaki ilişkiyi nasıl ifade edebilirsin? Bu ilişkiyi gösteren matematiksel ifadeler yazabilir misin?

Tırtıl sayısı=a

a×3

2. Sinan'ın telefonunun şarj kablosu belli bir uzunluktadır. Sinan'ın şarj kablosu Zeynep'in şarj kablosunun 5 katı uzunluğundadır.

a) Bu durumu çizerek gösterebilir misin?

b) Bu durum için bir eşitlik yazabilir misin?

Sinan'ın kablosuna: a
Zeynebin kablosuna=b

b×5=a
a:5=b

c) Bu durumu ifade eden başka bir eşitlik yazabilir misin?

3. Aşağıdaki resimlerde bir tırtılın her gün vücut büyüklüğündeki değişim görülmektedir. Buna göre soruları cevaplayınız.

a) Herhangi bir gün için tırtılın boyunu nasıl hesaplayabiliriz?

Tırtılın boyu = t

Geçen gün = g

g × 2 = t

b) 100. gün tırtılın boyu hesaplandığında başı hariç kaç parçası vardır?

100 × 2 = 200

b) Tırtılın boyu ile boyunun ölçüldüğü gün arasında nasıl bir ilişki vardır? Eşitlikle gösterebilir misiniz?

4. Ali'nin kumbarasında 10 lirası vardır. Ali her gün kumbarasına 3 lira koyarak para biriktirmeye karar veriyor. Aşağıdaki soruları bu bilgiye göre cevaplayınız.

a) Ali'nin 8 gün sonra kumbarasında kaç lirası olur? Nasıl bulduğunuzu gösteriniz.

8×3=24+10=34

(8×3)24+10=34

b) Geçen gün sayısı ile kumbarada biriken para miktarı arasındaki ilişkiyi nasıl ifade edersiniz?

Geçen gün=a

(a×3)+10.

c) Geçen gün sayısı ve kumbarada biriken para miktarını gösteren bir tablo çizer misiniz?

Gün Sayısı	1	2	3	4	10	X
Biriken Para (TL)	13	16	19	22	40	(X.3)+10

d) Gün sayısını x ile gösterdiğimizde, kumbarada biriken parayı nasıl ifade edebiliriz?

(X.3)+10

e) Kumbarada biriken parayı y ile, geçen gün sayısını x ile gösterdiğimizde bu ikisi arasındaki ilişkiyi eşitlik yazarak ifade edebilir misiniz?

(X.3)+10=y

5. Bir pakette 6 adet çikolata vardır. 8 paket çikolata ile bir kutu oluşmaktadır. 5 kutu ise bir koli yapmaktadır. Bir kolide kaç tane çikolata olduğunu nasıl bulursunuz? Nasıl bulacağınızı anlatan bir resim/model çizebilir misiniz?

- Çözümünüzü kağıt üzerinde gösterip açıklar mısınız?

□□□□□□□ 6×8=48 bir kutudaki çikolata sayısı

×8

×5 48 × 5 = 240 bir kolideki çikolata sayısı

6) * 17+34= 51/16 +35 34 +17 = 51

* 19+4= 16+7 (23)

CURRICULUM VITAE

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EDUCATION

Degree	Institution	Year of Graduation
MS	Boğaziçi University Primary Education	2018
BS	Boğaziçi University Elementary Mathematics Education	2013
High School	Prof. Dr. Necati Erşen AÖL, Sivas	2007

FOREIGN LANGUAGES

Advanced English

PUBLICATIONS

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