

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

**THE ASYMPTOTIC W_5 SYMMETRY IN THREE-DIMENSIONAL
CHERN-SIMONS SPIN-5 GRAVITATION THEORY AND BEYOND**

Ph.D. THESIS

Aytül FİLİZ

Department of Physics Engineering

Physics Engineering Programme

JULY 2024

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

**THE ASYMPTOTIC W_5 SYMMETRY IN THREE-DIMENSIONAL
CHERN-SIMONS SPIN-5 GRAVITATION THEORY AND BEYOND**

Ph.D. THESIS

**Aytül FİLİZ
(509122102)**

Department of Physics Engineering

Physics Engineering Programme

Thesis Advisor: Assoc. Prof. Hakkı Tunçay ÖZER

JULY 2024

İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

ÜÇ BOYUTLU CHERN-SIMONS SPİN-5 GRAVİTASYON KURAMINDA
ASİMPOTOTİK W_5 SİMETRİSİ VE ÖTESİ

DOKTORA TEZİ

Aytül FİLİZ
(509122102)

Fizik Mühendisliği Anabilim Dalı

Fizik Mühendisliği Programı

Tez Danışmanı: Doç. Dr. Hakkı Tunçay ÖZER

TEMMUZ 2024

Aytül FİLİZ, a Ph.D. student of ITU Graduate School with ID number 509122102, successfully defended the thesis entitled “THE ASYMPTOTIC W_5 SYMMETRY IN THREE-DIMENSIONAL CHERN-SIMONS SPIN-5 GRAVITATION THEORY AND BEYOND”, which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor : **Assoc. Prof. Hakkı Tunçay ÖZER**
Istanbul Technical University

Jury Members : **Prof. Tolga BİRKANDAN**
Istanbul Technical University

Prof. M. Hikmet YÜKSELİCİ
Yıldız Technical University

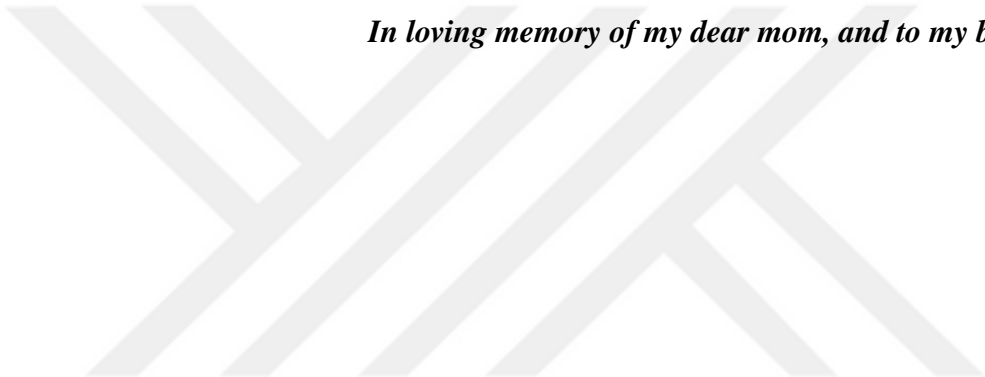
Prof. Neşe ÖZDEMİR
Istanbul Technical University

Assoc. Prof. Göksel DAYLAN ESMER
Istanbul University

Date of Submission : 13 June 2024

Date of Defense : 03 July 2024





In loving memory of my dear mom, and to my beloved dad,



FOREWORD

First and foremost, I would like to express my deepest gratitude to my thesis advisor Hakkı Tunçay ÖZER for providing me with invaluable guidance and endless support throughout my Ph.D. journey. I am grateful for his dedicated attitude, knowledge, kindness, respect, and patience. This thesis could not have been accomplished without his guidance and motivation. As a doctoral student, he has been the best advisor I could have asked for, and I would like to reiterate that I am truly grateful for his support in dealing with all the challenges I faced during this process. I hope our relationship continues forever, not only in the realm of physics but also in all aspects of life.

My thesis advisory committee has guided me in the best possible way throughout all these years. I owe many sincere thanks to Tolga BİRKANDAN and M. Hikmet YÜKSELİCİ for their precious contributions.

I would like to thank Istanbul Technical University Scientific Research Projects Department (ITU BAP-Project No:40199) for funding this work.

Special thanks to my dear friends Burçin DANACI, Gökhan TORUN, Nesrin ÇELEBİOĞLU and Sevcan TABANLI, who made me feel fortunate to have friends like them throughout this thesis study. Whenever I needed motivation and inspiration, especially when I felt disheartened, they always encouraged and motivated me. They made efforts to cheer me up during my toughest times. I miss our coffee breaks, and sweet conversations on the green grass dearly. I would like to thank all my colleagues and friends in the Department of Physics at Istanbul Technical University who contribute to creating a pleasant working atmosphere, including those whose names I cannot list, as well.

I would like to convey my sincere thanks to dear Göksel DAYLAN ESMER, who has supported me in every aspect since the day we met, always been by my side whenever I needed help, and even treated me like a mother after my mom's passing. I feel very fortunate to have her in my life.

I would also like to extend my deepest thanks to Şehsuvar ZEBİTAY, who has inspired countless students, including myself, to embark on a journey into physics and whom I look up to both academically and personally. Having the opportunity to know him is a tremendous privilege for me, and I always feel lucky to have been his student. His profound knowledge of physics and the confidence and courage he instills in his students are priceless.

Last but not least, I would like to wholeheartedly express my gratitude and dedicate this thesis to my family for their endless love and support. Without their unconditional love and unwavering belief in me, I wouldn't have reached this point. I always take pride in being their child. Undoubtedly, a child's first teachers are always the mother and father. On this occasion, I would like to once again thank my unique first teachers who instilled in me a curiosity for learning new things as a child and who have supported and stood by me throughout my academic journey, as well as in every aspect of my life. The person who most deserves to see the completion of this thesis is my dear mother,

who has dedicated years of effort to me. I wish she could have seen it finished. I hope she is watching me peacefully from somewhere. And lastly, I wish for my dear father, currently battling illness, to be with me for many years to come, witnessing countless beautiful moments. With the hope that the future brings happy and healthy days to all of us... I am grateful for your presence. Thank you all for filling my life with love, happiness, and endless beautiful memories.

June 2024

Aytül FİLİZ
(M.Sc. in Physics Engineering)



TABLE OF CONTENTS

	<u>Page</u>
FOREWORD.....	x
TABLE OF CONTENTS.....	xi
ABBREVIATIONS	xv
SYMBOLS.....	xvii
LIST OF TABLES	xix
SUMMARY	xxi
ÖZET	xxiii
1. INTRODUCTION.....	1
2. ASYMPTOTIC W_5 SYMMETRY OF THREE DIMENSIONAL CHERN-SIMONS HIGHER-SPIN AdS_3 GRAVITY.....	9
2.1 Gravity in Three Dimensions, a Review:.....	13
2.1.1 Connection to Chern - Simons theory	13
2.1.2 $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory	15
2.1.3 Adding a chemical potential to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory	17
2.1.4 Holonomies in $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory	18
2.2 Higher-spin Chern - Simons Theory	21
2.2.1 A calculation of $\mathfrak{sl}(5, \mathbb{R})$ commutators	21
2.2.2 Spin-5 AdS_3 solution and conformal Ward identities.....	22
2.2.3 Adding chemical potentials to $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern - Simons theory	24
2.2.4 Determining the $1/c$ -dependence of the structure constants.....	27
2.2.5 Holonomies in $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern - Simons theory	29
2.3 Summary and Conclusion	32
3. EXPLORING NEW BOUNDARY CONDITIONS FOR $N=(1,1)$ EXTENDED HIGHER-SPIN AdS_3 SUPERGRAVITY.....	35
3.1 Supergravity in Three Dimensions, a Review:.....	38
3.1.1 Connection to Chern - Simons theory	38
3.1.2 $\mathfrak{osp}(1 2) \oplus \mathfrak{osp}(1 2)$ Chern - Simons $\mathcal{N} = (1,1)$ supergravity for affine boundary	39
3.1.3 $\mathfrak{osp}(1 2) \oplus \mathfrak{osp}(1 2)$ Chern - Simons $\mathcal{N} = (1,1)$ supergravity for superconformal boundary	43
3.2 $\mathcal{N} = (1,1)$ $\mathfrak{osp}(3 2) \oplus \mathfrak{osp}(3 2)$ Higher - spin Chern - Simons Supergravity	44
3.2.1 For affine boundary	44
3.2.2 For superconformal boundary	51

3.3 Other Extended (Super)gravity Checks in the Most General Boundary Conditions	55
3.3.1 Avery - Poojary - Suryanarayana boundary conditions for $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)$	55
3.3.2 Avery - Poojary - Suryanarayana $\mathcal{N} = 1$ supergravity for $\mathfrak{osp}(1 2) \oplus \mathfrak{sl}(2)$	57
3.3.3 The on - shell action and the variational principle	59
3.3.4 $\mathfrak{osp}(1 2)$ - case :	60
3.3.5 $\mathfrak{osp}(3 2)$ - case :	60
3.4 Summary and Comments	60
4. $\mathcal{N}=(2,2)$ EXTENDED $\mathfrak{sl}(3 2)$ CHERN-SIMONS AdS_3 SUPERGRAVITY WITH NEW BOUNDARIES	63
4.1 Review of Chern - Simons Supergravity in Three Dimensions:.....	66
4.1.1 Connection to Chern - Simons theory	67
4.1.2 $\mathfrak{sl}(2 1) \oplus \mathfrak{sl}(2 1)$ Chern - Simons $\mathcal{N} = (2, 2)$ supergravity for affine boundary	68
4.1.3 $\mathfrak{sl}(2 1) \oplus \mathfrak{sl}(2 1)$ Chern - Simons $\mathcal{N} = (2, 2)$ supergravity for superconformal boundary	72
4.2 $\mathcal{N} = (2, 2)$ $\mathfrak{sl}(3 2) \oplus \mathfrak{sl}(3 2)$ Higher - spin Chern - Simons Supergravity	74
4.2.1 For affine boundary	74
4.2.2 For superconformal boundary	78
4.2.3 Remarks on charges and the nature of quantum gravity through holography	83
4.2.4 Remarks on the correspondence of higher-spin theories to the string theory and recent developments	85
4.3 Concluding Remarks.....	85
5. CONCLUSION	89
REFERENCES.....	93
APPENDICES	101
APPENDIX A : Notation and Convention	103
A.1 $\mathfrak{sl}(2, \mathbb{R})$ Generators	103
APPENDIX B : Spin-3 Case	105
B.1 $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ Higher-spin Chern-Simons Theory	105
B.1.1 $\mathfrak{sl}(3, \mathbb{R})$ generators as the principal embedding of $\mathfrak{sl}(2, \mathbb{R})$	105
B.1.2 Holonomies in $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ Chern - Simons theory	105
APPENDIX C : Spin-5 Extension	107
C.1 $\mathfrak{sl}(5, \mathbb{R})$ Generators as the Principal Embedding of $\mathfrak{sl}(2, \mathbb{R})$	107
C.2 Spin-5 Conformal Ward Identities	110
C.2.1 Solved parameters.....	110
C.2.2 Ward identities	114
C.3 Classical W_5 Symmetry OPE's from the Conformal Ward Identities	116
C.4 Chemical Potentials for $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern-Simons Theory	119
APPENDIX D : The $\mathfrak{sl}(3 2)$ Superconformal Case	125
D.1 The Temporal Evolution of the Twenty Four Independent Source Fields.....	125

D.2 The Gauge Transformations of the Twenty Four Independent Source Fields.....	127
D.3 The Gauge Transformations of the Twenty Four Chemical Potentials	129
D.4 The $\mathfrak{sl}(3 2)$ Lie Superalgebra Valued Gauge Parameter λ	132
CURRICULUM VITAE.....	135





ABBREVIATIONS

AdS	: Anti de-Sitter Space
ASA	: Asymptotic Symmetry Algebra
BTZ	: Bañados-Teitelboim-Zanelli
CFT	: Conformal Field Theory
CS	: Chern-Simons
OPE	: Operator Product Expansion
osp(N,R)	: Orthosymplectic Super Lie Algebra
SL(N,R)	: Special Linear Group
sl(N,R)	: Special Linear Algebra
SO(N,R)	: Special Orthogonal Group
so(N,R)	: Special Orthogonal Algebra
WZW	: Wess-Zumino-Witten



SYMBOLS

W	: W-algebra
$\mathfrak{hs}(\lambda)$	: λ valued higher spin algebra
c	: Central charge
s	: Spin
l	: AdS_3 radius
k	: Level of Chern-Simons theory / Chern-Simons level
G	: Gravitational constant
$\oplus$	: Direct sum of Lie algebras
S	: Action
e_μ^a	: Vielbein
ω_μ^a	: Spin connection
$\wedge$	: Wedge product
$[\cdot, \cdot]$	: Lie bracket
$\{\cdot, \cdot\}$	: Poisson bracket
F	: Field Strength
η_{ab}	: Invariant bilinear form
$g_{\mu\nu}$	: Metric tensor
ds^2	: Metric
$\mathcal{M}$	: Manifold
$\mathcal{O}(1)$	: Order of 1
λ	: Gauge parameter
δ_λ	: Infinitesimal gauge transformation
ε	: Gauge parameter
$\mathcal{Q}[\varepsilon]$	: Canonical boundary charge
μ, ν	: Chemical potentials
g	: Gauge function
$\mathcal{C}$	: Non-contractable cycle
$H_\phi(a), H_\tau(a)$	: Holonomies
w	: Holonomy matrix
Θ	: Holonomy invariants
β	: Inverse temperature
$S(\mathcal{L})$	: Entropy



LIST OF TABLES

	<u>Page</u>
Table 2.1 : Some important structure constants for classical and quantum asymptotic $\mathcal{W}_5$ symmetry algebra with $1/c$ correction.	28





THE ASYMPTOTIC W_5 SYMMETRY IN THREE-DIMENSIONAL CHERN-SIMONS SPIN-5 GRAVITATION THEORY AND BEYOND

SUMMARY

This thesis is devoted to contributing somewhat to the understanding of gravity, higher spin theories, and conformal field theories by exploring asymptotic symmetries in the light of *AdS/CFT* correspondence. The motivations are two-fold: firstly, understanding the theoretical description of higher spin theories is a highly interesting task, as they might have better quantum properties than Einstein's gravity (whose quantum mechanical behaviour is not understood). Hence, such theories are a promising generalization of gravity (besides supergravity and string theory). Secondly, how supersymmetry is incorporated into this bosonic theory.

The dissertation consists of six main parts:

In chapter 1, we give a brief introduction serving as an overview of the thesis, outlining the main objectives, scope, and significance of the research conducted.

In chapter 2, firstly, we briefly touch upon some technical details that form the basis of the theory addressed in the thesis. The first part of this chapter lays down the foundations for the mathematical framework in the subsequent sections. In this chapter, our primary goal is to thoroughly investigate some important aspects of three-dimensional anti-de Sitter (AdS_3) higher spin gravity with a negative cosmological constant within the framework of $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern-Simons gauge theory. We explicitly construct the asymptotic symmetry algebra as two copies of the classical W_5 algebra with central charge c . We derive an explicit solution that can be regarded as a spin-5 generalization of the BTZ solution, thanks to a method recently discovered. We show how the W_5 symmetry algebra and the higher spin Ward identities emerge from the bulk equations of motion coupled to spin-5 currents. Additionally, we indicate how to introduce the chemical potentials and holonomy conditions associated with these higher spin charges in AdS_3 higher spin gravity in a manner that preserves the asymptotic symmetry algebra. We give an explicit generalized entropy expression that is very important to establish a relationship among generalized spin-5 black hole thermodynamics, asymptotic symmetries and W_5 algebras. This higher spin asymptotic W_5 symmetry algebra that we calculated within the framework of the $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Lie algebra-valued Chern-Simons gauge theory with a negative cosmological constant contains a finite number of conformal primary spins $s : s = 2, 3, 4, 5$. Our results offer an important, concrete supporting example, providing valuable insights into the space-time geometries of three-dimensional anti-de Sitter (AdS_3) higher spin gravity, as formulated through the framework of $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ Chern-Simons gauge theories.

In Chapter 3, we carry out the supersymmetric extension of our work. We perform supersymmetric extension of our study including the supersymmetry transformation laws, and demonstrate how to recover super algebras from its asymptotic space-time symmetries. We present a candidate for $\mathcal{N} = (1, 1)$ extended higher spin AdS_3

supergravity with the most general boundary conditions. We eventually show that the asymptotic symmetry algebra consists of two copies of the $\mathfrak{osp}(3|2)_k$ affine algebra on the affine boundary and two copies of $\mathcal{SW}(\frac{3}{2}, 2)$ symmetry algebra on the superconformal boundary respectively in the presence of the most general boundary conditions. Furthermore, we impose some certain restrictions on gauge fields for the loosest set of boundary conditions and that leads us to the supersymmetric extension of the Brown-Henneaux boundary conditions. And moreover, we derive the chemical potentials associated with the source fields that manifest through the temporal components of the connection. We also check that the consistency of our construction in the presence of two different class of boundary conditions that appeared in the literature. We discuss conservation of the charges, and also show that the Chern - Simons action which is compatible with our boundary conditions leads to a finite, well - defined variational principle for these higher - spin fields.

In Chapter 4, we present the first example of $\mathcal{N} = (2, 2)$ formulation for the extended higher spin AdS_3 supergravity with the most general boundary conditions as an extension of the previous chapter. Using a method recently proposed, we construct a consistent class of the most general boundary conditions to extend it. An important consequence of our approach is that, for the loosest set of boundary conditions it ensures that their asymptotic symmetry algebras consist of two copies of the $\mathfrak{sl}(3|2)_k$. Moreover, we enjoin some certain restrictions on the gauge fields for the most general boundary conditions, leading to the supersymmetric extensions of the Brown and Henneaux boundary conditions. Based on these results, we finally find out that the asymptotic symmetry algebras are two copies of the super $\mathcal{W}_3$ algebra for $\mathcal{N} = (2, 2)$ extended higher spin supergravity theory in AdS_3 . We investigate whether or not that our theory falls under the same metric class. As discussed in the previous section, we examine the conservation of charges, and further illustrate that the Chern-Simons action, in accordance with our boundary conditions, results in a finite and well-defined variational principle for the higher spin fields. In this way, we establish an explicit holographic dictionary between Chern-Simons supergravity on AdS_3 and two-dimensional CFT_2 with superconformal symmetry. Therefore, it can be considered that our method provides a good laboratory for investigating the rich asymptotic structure of both extended gravity and extended supergravity.

The final chapter collects our results and provides guidance on interpreting the results of the dissertation in order to present the conclusions of our work effectively.

ÜÇ BOYUTLU CHERN-SIMONS SPİN-5 GRAVİTASYON KURAMINDA ASİMPOTOTİK w_5 SİMETRİSİ VE ÖTESİ

ÖZET

Genel Görelilik ve Kuantum Kuramı, evrenin işleyişini açıklayan iki temel kuramdır. Bunlardan biri, astronomik ölçekteki kütleçekimin baskın olduğu Merkür'ün perihelon problemi, gezegen yörüngeleri, galaksi çarpışmaları, genişleyen evrenin dinamiği ve kütleçekim dalgaları gibi konuları, diğeri ise atomik ölçekteki kütleçekim dışındaki tüm temel etkileşimleri standard model ile başarılı bir şekilde açıklar. Genel göreliliği kuantum mekaniği ilkeleri ile uzlaştırarak evreni tutarlı tek bir çerçevede açıklayabilen bir teorinin varlığı, modern fiziğin yanıtını aradığı en önemli problemlerden biridir. Bu amaçla gerçekleştirilmesi hedeflenen kuantum kütleçekim kuramı, karadeliklerin davranışı, evrenin kökeni gibi çok yüksek enerjili durumların ve uzayda çok küçük ölçeklerdeki diğer problemlerin aydınlatılmasına ışık tutacaktır. Sicim kuramı, süperkütleçekim kuramı ve yüksek spin kütleçekim kuramı, aranan bu kuram için en güçlü adaylardır.

Şüphesiz, modern kuramsal fiziğin son yıllardaki en büyük başarılarından bir tanesi de AdS/CFT dualitesinin keşfidir. Holografik ilkenin açık bir tezahürü olan bu meşhur dualitenin, genel görelilik, kuantum kütleçekimi, karadelikler, kuantum alan kuramı ve yüksek spin kuramlarının yanısıra, kuantum renk dinamiği, nükleer fizik, istatistiksel fizik ve yoğun madde fiziği gibi fiziğin farklı bir çok alanında önemli uygulamaları bulunmaktadır. Bu dualite esasen, d -boyutlu bir uzay-zamanda tanımlı kütleçekim kuramını, $(d - 1)$ -boyutlu bir ayar kuramıyla ilişkilendirir.

AdS_3/CFT_2 dualitesi ise, $2 + 1$ boyutlu, negatif kozmolojik sabitli, klasik kütleçekim kuramının iki boyutlu bir konformal alan kuramı (CFT_2) ile ilişkilendirildiği özel bir durumdur. Gravitasyon kuramlarının $2 + 1$ boyutta incelenmesi, karşılaşılan matematiksel zorlukların bertaraf edilebilmesi açısından çok daha elverişlidir. Çünkü $2 + 1$ boyutta, negatif kozmolojik sabitli, Einstein'ın klasik kütleçekim kuramı, matematiksel yapının önemli ölçüde sadeleştiği bir Chern - Simons ayar kuramı olarak yeniden formüle edilebilir (AdS_3 klasik kütleçekimi eylemi ve hareket denklemleri, uygun ayar grubu için Chern Simons ayar kuramına eşdeğerdir). Böylelikle, örneğin $3 + 1$ boyutlu kuramda hesaplanması zor olan kavramları $2 + 1$ boyutta farklı bir çerçevede hesaplamak ve $3 + 1$ boyuttaki benzer kavramların fiziği hakkında bilgi sahibi olmak mümkündür.

Genel görelilik kuramının, $2 + 1$ boyutta herhangi bir yerel yayılma serbestlik derecesine sahip olmayışı, kuramın geometrik yerine tamamen topolojik olarak ele alınmasına olanak tanır. $2 + 1$ boyutlu kuramda madde, uzay-zamanı sadece yerel olarak bükür ve gravitasyon alanın dinamik serbestlik derecesi sıfır olur. Bu durumda, gravitasyon dalga çözümleri, dolayısıyla graviton bulunmaz (Einstein-Hilbert eyleminin yeniden formülasyonu ile bu meselenin de aşıldığı iki temel yaklaşım söz konusudur) ve yalnızca global etkiler fiziksel olarak anlamlıdır. $2 + 1$ boyutta herhangi bir yerel

serbestlik derecesi bulunmamasına rağmen, kuramın dinamik içeriği sınır koşullarının varlığı sayesinde çok zengin ve ilgi çekicidir. Kuramın dinamik özellikleri, tamamen sınır koşulları tarafından yönetilir ve eylemin “iyi tanımlanmış bir varyasyon ilkesi”ni sağlaması için uygun sınır koşullarının seçimi oldukça önemlidir. Uygun sınır koşulları seçimiyle, sınırda yaşayan sonsuz sayıda serbestlik derecesi vardır. Bu koşullar, eylemin “iyi tanımlanmış bir varyasyon ilkesi”ne sahip olması için gereklidir, ancak seçimleri tek değildir. Buna göre sınırda kalan uygun ayar simetrisi, asimptotik simetrisi olarak adlandırılır.

AdS_3 'de Einstein'ın klasik kütleçekim kuramının asimptotik simetri cebri (bozonik) Virasoro cebri iki kopyası olduğu bilinmektedir. Buna göre, $sl(2;R) \oplus sl(2;R)$ ayar cebri üzerinde tanımlı bir Chern - Simons alan kuramı olan klasik Einstein kütleçekim kuramının diffeomorfizmaları ve Lorentz simetrisi, Chern - Simons kuramının ayar dönüşümleri olarak yeniden tanımlanır. Farklı bir ifadeyle, AdS_3 metriğinin asimptotik formunu koruyan ayar dönüşümlerinin üreticileri, dual konformal alan kuramında Virasoro cebri sağlar. Bu durum, hem AdS/CFT dualitesinin hem de holografik ilkenin açık bir gerçekleştirilmesidir. Bu nedenle, uygun sınır koşullarının seçimi ve bu seçim altında kuramın farklı açılardan asimptotik simetrisinin incelenmesi önemlidir.

Öte yandan, $2 + 1$ boyutlu kütleçekiminin önemli bir dayanak noktası da, BTZ karadelisi adı verilen $3 + 1$ boyutta Kerr karadelisine bir çok yönüyle benzer, karadelik çözümlerini içermesidir. BTZ karadelik çözümleri, karadeliklerin termodinamik özellikleri gibi kavramsal konuların test edilmesi için ideal araştırma alanlarıdır. BTZ karadelisi yerel olarak anti-de Sitter (AdS) uzayına izometriktir. Bu nedenle, AdS uzayının asimptotik simetrisinin araştırılması, karadelikler ve karadeliklerin termodinamik özellikleriyle ilgili (örneğin entropi gibi) anlaşılması zor kavramsal konular hakkında fikir sahibi olmak açısından da önem taşır.

Yüksek spin kütleçekim kuramları, kuantum kütleçekim kuramı ile ilgili beklentilerimizi karşılamayı hedefleyen, son yıllarda artan ilgiyle birlikte önem kazanan çalışma alanlarından biridir. Kuram, AdS/CFT dualitesi gibi holografi tabanlı tekniklerin araştırılması ve bu tekniklerin test edilmesi için ideal bir zemin sunar. İki büyük kütleçekim alanlarını içeren teori, klasik kütleçekimi için gravitonu bir spin-2 alanı şeklinde, temel durum olarak açıklar. Buradaki tutarlı ilişkilendirme, $sl(N;R)$ ayar cebri iki kopyası ile verilen bir Chern - Simons eylemi aracılığıyla sağlanır. Yüksek spin kuramlarının kütleli madde alanlarıyla tutarlı bir biçimde nasıl ilişkilendirildiği ise, Vasiliev tarafından sonsuz elemanlı bir yüksek spin alanları kümesi ile açıklığa kavuşturulmuştur. $2 + 1$ boyutlu Vasiliev kuramı, sonsuz boyutlu $hs(\lambda)$ ayar cebrine dayanan tek parametrelili bir kuram sınıfıdır. Bu tutarlı eşleştirme sonucunda, kuramın dinamik içeriğinin her iki çerçevede incelenmesi ve dual konformal alan kuramının asimptotik simetrisinin araştırılması oldukça ilgi çekici hale gelmiştir. Bu sayede, genişletme yapılarak daha yüksek spin alanları için ve daha yüksek boyutlar için fikir sahibi olmak mümkün olacaktır. Öte yandan, üç boyutlu $O(N)$ vektör modellerinin büyük N — limitini, AdS de yüksek spin kütleçekim kuramlarıyla ilişkilendiren dualite sayesinde de, AdS holografisi ve yüksek spin kütleçekim kuramı farklı bir bakış açısıyla araştırılmaya devam etmektedir.

Yüksek spin kütleçekim kuramının asimptotik simetrisi, Drinfeld - Sokolov Hamiltonyen indirgemesi ile elde edilir. Bu sayede saf kütleçekimi temsil eden $sl(2;R)$ durumunda Virasoro cebri veren asimptotik simetri cebirleri yüksek spin durumlarına genişletilebilir. Bu genişletme ile, yüksek spin durumları için, $sl(N;R)$ ayar cebri

dayanan dual konformal alan kuramının asimptotik simetrileri olarak klasik W_N -cebirleri karşımıza çıkar. Benzer şekilde, $hs(\lambda)$ ayar cebriyle tanımlı Chern - Simons kuramı için asimptotik simetriler dual konformal alan kuramında $W_\infty(\lambda)$ olarak bilinen özel bir W cebiridir.

Bu tezde üç boyutlu negatif kozmolojik sabitli AdS_3 yüksek spin kütleçekimi, spin-5 alanlarını da içerecek şekilde genişletilmiş ve asimptotik simetrileri ele alınmıştır. Öncelikle, $sl(5; \mathbb{R})$ ayar cebriyle dayanan ve dual konformal alan kuramının asimptotik simetrileri olan klasik W_5 -cebirleri açık bir biçimde kurulmuştur. Buradan $sl(5, \mathbb{R}) \oplus sl(5, \mathbb{R})$ Chern-Simons ayar kuramı çerçevesinde üç boyutlu negatif kozmolojik sabitli AdS_3 yüksek spin kütleçekiminin asimptotik simetri cebri, c merkezi yükü ile tanımlanan klasik W_5 simetri cebri için iki kopyası olarak hesaplanmıştır. BTZ karadelik çözümünün spin-5 genelleştirmesi olarak nitelendirilebilecek bir çözüm türetilmiştir. Spin-5 akılarıyla ilişkilendirilen bulk hareket denklemlerinden yüksek spin Ward özdeşlikleri hesaplanarak, asimptotik simetri cebri nasıl elde edildiği gösterilmiştir. Bu yüksek spin yükleriyle ilişkilendirilen ve asimptotik simetri cebri koruyan kimyasal potansiyeller hesaplanmış ve holonomi koşulları belirlenmiştir. Genelleştirilmiş spin-5 kara delik termodinamiği, asimptotik simetriler ve W_5 -cebirleri arasında bir ilişki kurmayı sağlayan genelleştirilmiş açık bir entropi ifadesi verilmiştir. Bu sayede, $sl(N, \mathbb{R}) \oplus sl(N, \mathbb{R})$ Chern-Simons teorileri aracılığıyla formüle edilen uzay-zaman geometrilerinin anlaşılmasına yönelik somut bir örnek sunulmuştur.

Kuramın asimptotik simetrilerini belirleyen sınır koşulları yıllar içerisinde bir çok kez yeniden düzenlenmiş ve farklı yeni sınır koşulları da önerilmiştir. Bu yeni sınır koşullarının özel hallerde, klasik AdS_3 kütle çekiminin asimptotik simetrisini Virasoro cebri için iki kopyası şeklinde veren Brown-Henneaux sınır koşullarına indirgenebildiği gösterilmiştir. O halde tüm bu bilgiler ışığında, kuramın dinamik içeriği olarak asimptotik sınır koşullarının seçimi, bu koşulların farklı açılardan değerlendirilmesi, örneğin düz bir uzayda incelenmesi, farklı yönlerinin ele alınması; örneğin kiral yüksek spin kütleçekimine uygulanması, hatta bu sınır koşullarının en genel şekilde ifade edilebilmesi, dahası tüm bu durumların süpersimetrik versiyonlarının elde edilip, bunların genişletilmesi ve sonuç olarak asimptotik simetrilerin tüm tönleriyle araştırılması, AdS/CFT dualitesi aracılığı ile her iki cephe açısından tutarlı bir eşleştirmenin yapılabilmesi bakımından son derece önemlidir.

Bu amaçla bu tezde, en genel sınır koşullarının varlığında kurama süpersimetri de dahil edilerek asimptotik simetriler ele alınmıştır. En genel sınır koşulları oluşturularak, $\mathcal{N} = (1, 1)$ genişletilmiş yüksek spin AdS_3 süperkütleçekim modeli inşa edilmiştir. Bu koşullar altında asimptotik simetri cebri için $osp(3|2)_k$ afin cebri için iki kopyasından oluştuğu gösterilmiştir. Ayrıca bu en genel sınır koşulları altında ayar alanlarına tutarlı bir takım kısıtlamalar getirilerek, Brown-Henneaux sınır koşullarının süpersimetrik genişletmesine ulaşılmıştır. Bunun yanı sıra literatürde yer alan iki farklı sınır koşulu sınıfı için modelin tutarlılığı kontrol edilmiştir. Yüklerin korunumu incelenmiş ve en genel sınır koşulları ile uyumlu Chern-Simons eyleminin, bu yüksek-spin alanları için sonlu, iyi tanımlı bir varyasyon ilkesini sağladığı gösterilmiştir. Son olarak, $\mathcal{N} = (1, 1)$ genişletilmiş yüksek spin süperkütleçekimi için asimptotik simetri cebri için $SW(\frac{3}{2}, 2)$ cebri için iki kopyasına indirgendiği gösterilmiştir.

Bu tez kapsamında, daha sonra süpersimetri miktarı artırılarak genişletilmiş yüksek spin AdS_3 süper kütleçekiminin $\mathcal{N} = (2, 2)$ formülasyonu sunulmuştur. AdS_3 'de $N = (2, 2)$ genişletilmiş yüksek spin süperkütleçekimi için, en genel sınır koşulları

elde edilmiştir. Ayrıca, kaynak alanların kurama nasıl dahil olduğu detaylı bir şekilde incelenip, holografik Ward özdeşlikleri hesaplanmıştır. Kuramın süpersimetri miktarının arttırılarak en genel koşullar altında incelenmesi bir takım teknik zorlukları beraberinde getirir de, asimptotik simetri cebriinin iki adet $sl(3|2)_k$ kopyasından oluştuğu gösterilmiştir. Yine en genel sınır koşulları altında ayar alanlarına bir takım sınırlamalar getirilerek, Brown-Henneaux sınır koşullarının süpersimetrik genişletmesine ulaşılmıştır. AdS_3 'de $\mathcal{N} = (2,2)$ genişletilmiş yüksek spin süper kütleçekimi kuramının asimptotik simetri cebriinin süper $\mathcal{W}_3$ cebriinin iki kopyasını içerdiği gösterilmiştir. Süperkonformal yapı belirginleştirilip, bulk akıları ile dönüşüm parametreleri arasında tutarlı bir ilişki kurulmuştur.

Bu sayede AdS_3/CFT_2 holografik dualitesi bağlamında, Chern - Simons ayar cebriyle tanımlı AdS_3 süperkütleçekimi ile sırasıyla $N = 1$ ve $N = 2$ süperkonformal simetrileri arasında (her biri için ayrı olarak) tutarlı, açık bir holografik sözlük oluşturulmuştur.



1. INTRODUCTION

General relativity and quantum field theory are two fundamental theories that explain the working of the universe. One of them, dominant in astronomical scales where gravity prevails, addresses topics such as the perihelion problem of Mercury, planetary orbits, galaxy collisions, dynamics of the expanding universe, and gravitational waves, while the other, at atomic scales, successfully describes all fundamental interactions except gravity through the standard model. Reconciling Einstein's classical theory of gravity with the principles of quantum mechanics and explaining the universe in a single framework is one of the greatest challenges faced by modern theoretical physicists. Despite current experimental technology remains far from detecting quantum gravitational effects, investigating a quantum theory of gravity is still an important theoretical task. On the other hand, albeit quantum gravity still maintains its mystery, significant experimental evidence such as black hole entropy, gravitational waves, the holographic principle, and the early universe, indicates its existence. A consistent theory of quantum gravity aimed at achieving this goal shed light on high-energy phenomena such as the behavior of black holes and the origin of the universe as well as on other problems at very small scales in space.

At the current stage, various competitive responses are being constructed to come through this challenge. In this context, string theory, supergravity theory, and higher spin gravity theories (HSGRAs) are the strongest candidates for this sought-after theory. While each of them is internally consistent, the resolution of the quantum gravity problem remains uncertain due to their open issues, and no definite answer has been reached so far. In this scenario, the holographic principle, one of the core concepts at the heart of quantum gravity, is a powerful tool for exploring ideas on this platform. Undoubtedly, one of the forefront breakthroughs in modern theoretical physics in the past few decades is the discovery of the celebrated AdS/CFT correspondence [1].

The AdS/CFT correspondence, particularly, is the first concrete realization of the holographic principle, has broad applications not only in general relativity, quantum

gravity, black holes, quantum field theory, and higher spin theories but also in various branches of physics such as quantum chromodynamics, nuclear physics, statistical physics, and condensed matter physics. Essentially, the AdS/CFT duality establishes a connection between gravitational theory defined in a d -dimensional asymptotically anti-de Sitter spacetime (AdS) and conformally invariant gauge theory formulated at the $(d - 1)$ -dimensional boundary of AdS.

To particularly consider this duality in $2 + 1$ dimensions offers substantial advantages in finding solutions to many problems that are difficult to solve in higher-dimensional counterparts. Because in $2 + 1$ spacetime dimensions, this correspondence provides a relatively simple platform for testing ideas and carrying out precise calculations in quantum gravitational models. The Hamiltonian analysis is more tractable in this case and the results obtained are valuable in terms of providing understanding into the disagreements in the literature for the four-dimensional case. Moreover, in a $2 + 1$ dimensional theory, matter curves spacetime only locally, and the dynamical degrees of freedom of the gravitational field vanish and thus no graviton. In other words, physically, this means that there are no local propagating degrees of freedom, which also implies the absence of gravitational waves in the theory. The fact that general relativity does not have any local propagating degrees of freedom in $2 + 1$ dimensions allows the theory to be considered purely topologically instead of geometrically. In this case, the theory can be reformulated as a Chern-Simons gauge theory where the mathematical structure significantly simplifies [2, 3]. The action and the equations of motion for AdS_3 are equivalent to a Chern-Simons theory under an appropriate gauge group. Accordingly, it is possible to overcome encountered conceptual difficulties that are challenging to calculate in a $2 + 1$ dimensional theory by employing a different framework in $2 + 1$ dimensions and gaining insights into the physics of similar concepts in $2 + 1$ dimensions. The absence of local degrees of freedom indicates that the theory relies solely on global effects. For instance, three-dimensional gravity with a negative cosmological constant admits an intriguing black hole solutions. These black holes resemble Anti-de Sitter spacetime locally, but globally, they exhibit the properties of a black hole; they possess an event horizon and are classically characterized by their mass and angular momentum. These global charges, which can be computed on the boundary of AdS, differ from those in the AdS vacuum.

Additionally, a significant aspect of the $2 + 1$ dimensional gravity is that it contains black hole solutions, which are similar in many ways to the $3 + 1$ dimensional Kerr black holes, called the BTZ black holes. BTZ solutions are ideal laboratories in this sense to examine difficult conceptual issues related to the thermodynamic properties of black holes such as entropy [4, 5].

Moreover, although there are no local degrees of freedom in $2 + 1$ dimensions, the dynamic content of the theory is rich and intriguing due to the existence of boundary conditions. The dynamic properties of the theory are completely controlled by the choice of boundary condition. The choice of appropriate boundary conditions are crucial for the action to yield a "well-defined variation principle. With the choice of suitable boundary conditions, there exist infinitely many degrees of freedom living on the boundary. These conditions are necessary for the action to have a "well-defined variation principle," but their choices are not unique. Accordingly, the appropriate gauge symmetries remaining at the boundary are referred to as asymptotic symmetries.

In addition, a notable feature of three-dimensional gravity is that the Chern-Simons theory induces a conformal field theory on a manifold with a boundary. Utilizing asymptotically AdS boundary conditions results in an asymptotic symmetry algebra relating different physical states with the same boundary conditions. As shown in the pioneering work of Brown and Henneaux, this asymptotic symmetry algebra comprises two copies of the Virasoro algebra with a classical central extension [6].

Based on the results of this study, the diffeomorphisms and Lorentz symmetries of classical gravity theory do match gauge transformations in the Chern-Simons theory defined on the $sl(2;R) \oplus sl(2;R)$ gauge algebra and in this case, the generators of gauge transformations preserving the asymptotic form of the AdS_3 metric yields the Virasoro algebra in the dual conformal field theory. This connection discovered by Brown and Henneaux can be considered as a precursor to AdS/CFT correspondence, and consequently, is to what is now known as holography. As a result of this study, the exploration of asymptotic symmetries of the theory from various perspectives under the choice of appropriate boundary conditions has become enticing. These boundary conditions have been extensively studied and modified over the years [7–17]. Asymptotic symmetries continue to be investigated for different new classes of boundary conditions beyond the Brown-Henneaux boundary conditions [18–20]. Moreover, the

asymptotic symmetries of the theory have been investigated for the most general set of boundary conditions, which can be reduced to the Brown and Henneaux boundary conditions in a special case and encompass other boundary conditions as well [21]. These boundary conditions have also been applied to flat space and chiral higher spin gravity [22, 23].

As previously emphasized, through the AdS_3/CFT_2 holographic duality, the dynamic properties of the theory are completely controlled by the choice of boundary conditions. As a result, the asymptotic symmetries arising from the appropriate choice of boundary conditions to have a well-defined variational principle are quite rich and interesting to study. In this sense, it is important to investigate the extended asymptotic symmetries of the theory, and also explore the asymptotic symmetries by incorporating supersymmetry under the most general supersymmetric boundary conditions, and whose dynamic content is completely shaped by the choice of boundary conditions, to ensure a consistent match in terms of both frames for the obtained results.

With the above considerations in mind, higher spin gravity is an area of research that has gained importance in recent years with increasing attention, aiming to meet our expectations regarding quantum gravity. The theory where holographic duality is realizable is ideal for understanding and testing holography-based techniques such as AdS/CFT . This includes extensions of gravity with at least one field having spin greater than two and takes the classical Gravity theory with spin-2 graviton as the ground state. The consistent matching here is achieved through a Chern-Simons action given by two copies of the $sl(N;R)$ gauge algebra. Also, the consistent association of higher spin theories with massive matter fields has been elucidated by Vasiliev through an infinite set of higher spin fields [24–31]. Also, it is possible to consistently truncate the Vasiliev theory to a finite N spin value in $2 + 1$ dimensions [32]. When it was shown that massless higher spin fields in $2 + 1$ dimensions consistently match with AdS_3 gravity, the investigation of the theory’s dynamic content in both frameworks and the exploration of the asymptotic symmetries of dual conformal field theory have become intriguing.

The $2 + 1$ dimensional Vasiliev theory is a one-parameter class of theories based on the infinite-dimensional $hs(\lambda)$ gauge algebra [33]. In this way, it is possible to develop understanding of higher spin fields and higher dimensions by expanding the framework.

On the other hand, the duality between the large N limit of three-dimensional $O(N)$ vector models and higher spin gravity theories in AdS , pioneered by Klebanov and Polyakov [34] and further developed by Sezgin and Sundell [35], continues to drive actively investigations into AdS holography and higher spin gravity theory from a different perspective.

For a certain N -limit, it was shown that the W_N minimal models are dual to Vasiliev type higher spin theory in AdS_3 [36]. Therefore, the boundary conformal field theory can be represented as a diagonal coset of WZW models. Considering all these developments from different perspectives, in the context of the AdS/CFT duality, there are still numerous questions to be answered for both sides of the coin.

The asymptotic symmetries of higher spin gravity theory are obtained through the Drinfeld-Sokolov Hamiltonian reduction [37, 38]. Therefore, asymptotic symmetry algebras that yield the Virasoro algebra for the case representing pure gravity, $sl(2; R)$, can be extended to higher spin cases. With this extension, classical W_N algebras emerge as the asymptotic symmetries of the dual conformal field theory based on the $sl(N; R)$ gauge algebra for higher spin cases [39–43]. Subsequently, these results have been applied to supergravity and higher spin gravity [44–47]. Additionally, the bosonic theories have been extended to their supersymmetric versions by cutting all odd spin generators and a copy of fermionic operators within $sl(N, N-1)$ [44, 45, 48–51] or $osp(N, N-1)$ [52] superalgebras via Chern-Simons theories. Similarly, for the $hs(\lambda)$ gauge algebra valued Chern-Simons theory, the asymptotic symmetries in the dual conformal field theory are a special W algebra known as $W_\infty(\lambda)$ [38, 53].

In the light of all the facts mentioned above, it is natural to extend AdS_3 gravity to include higher spin fields greater than two and explore its asymptotic symmetries. It is appropriate to state our purpose at this point. One of the main objectives of this thesis is to explicitly construct the W_5 symmetry algebra as asymptotic symmetries of the three-dimensional anti-de Sitter (AdS_3) higher spin gauge theory which is equivalent to the classical Einstein gravity. Thus, the theory, which considers classical Einstein gravity as its ground state, has been extended to encompass bosonic- spin-5 fields. Despite the technical difficulties of the calculations, this study is important in the literature for being the most comprehensive work that includes the largest number of bosonic spin fields within the theory. As detailed in Chapter 2, the classical W_5 algebras, which

are the asymptotic symmetries of the dual conformal field theory based on the $sl(5;R)$ gauge algebra, are established first. Then, in the $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern-Simons gauge theory framework, the asymptotic symmetry algebra of three-dimensional higher spin gravity with a negative cosmological constant is calculated as two copies of the classical W_5 symmetry algebra defined by the central charge c . Moreover, a solution has been derived which can be interpreted as a spin-5 generalization of the BTZ black hole solution. The emergence of W_5 symmetry and higher spin Ward identities from the bulk equations of motion coupled to spin-3, spin-4, and spin-5 currents has been demonstrated. This higher spin asymptotic W_5 symmetry algebra contains a finite number of conformal primary spins s : $s = 2, 3, 4, 5$. How to introduce chemical potentials and holonomy conditions associated with these higher spin charges in AdS_3 higher spin gravity in a manner that preserves the asymptotic symmetry algebra is also indicated. In addition, an explicit generalized entropy expression is presented, which is important for establishing a relationship among generalized spin-5 black hole thermodynamics, asymptotic symmetries, and W_5 algebras. These results furnish a significant, concrete promoting example to understand and interpret the space-time geometries in three-dimensional anti-de Sitter (AdS_3) higher spin gravity formulated via $sl(N, R) \oplus sl(N, R)$ Chern-Simons gauge theories.

More generally, it is also important to consider the most general boundary conditions for this theory, where dynamic properties are entirely shaped by the choice of boundary conditions, and examine the asymptotic symmetries under these conditions. With this motivation, in Chapter 3, asymptotic symmetries have been addressed by including supersymmetry in the presence of the most general boundary conditions. The $\mathcal{N} = (1, 1)$ extended higher spin AdS_3 supergravity model has been constructed by establishing the most general boundary conditions [54]. Consequently, it has been shown that the asymptotic symmetry algebra consists of two copies of the affine algebra $\mathfrak{osp}(3|2)_k$ under the most general boundary conditions. Besides, under these most general boundary conditions, a consistent set of constraints has been imposed on the gauge fields, leading to the supersymmetric extension of the Brown-Henneaux boundary conditions. The consistency of our approach has also been verified for two different classes of boundary conditions found in the literature. The conservation of charges has been investigated, demonstrating that the Chern-Simons action consistent with the

most general boundary conditions provides a finite, well-defined variational principle for these higher spin fields. As a final step for this chapter, it has been demonstrated that the asymptotic symmetry algebra for $\mathcal{N} = (1, 1)$ extended higher spin supergravity reduces to two copies of the $\mathcal{SW}(\frac{3}{2}, 2)$ algebra.

Within the scope of this thesis, in Chapter 4, the $\mathcal{N} = (2, 2)$ formulation of extended higher spin AdS_3 supergravity has been presented by subsequently increasing the amount of supersymmetry [55]. The most general boundary conditions have been obtained for $N = (2, 2)$ extended higher spin supergravity in AdS_3 . Additionally, a detailed examination of how source fields are incorporated into the theory has been conducted, and holographic Ward identities have been calculated. Although examining the theory with an increased amount of supersymmetry under the most general conditions poses some technical challenges, it has been shown that the asymptotic symmetry algebra consists of two copies of the $\mathfrak{sl}(3|2)_k$ algebra. Similarly, under the most general boundary conditions, a set of constraints has been imposed on the gauge fields, leading to the supersymmetric extension of the Brown-Henneaux boundary conditions. And as a result, it has been demonstrated that the asymptotic symmetry algebra of the $\mathcal{N} = (2, 2)$ extended higher spin supergravity theory in AdS_3 contains two copies of the super $\mathcal{W}_3$ algebra. The superconformal structure has been elucidated, and a consistent relationship between bulk currents and transformation parameters has been established. Thus, within the context of the AdS_3/CFT_2 holographic duality, a consistent and explicit holographic dictionary has been established separately between the $N = 1$ and $N = 2$ superconformal symmetries and three-dimensional supergravity defined by the Chern-Simons gauge algebra.

The various chapters of this thesis are based on the following publications:

Chapter 2, H.T. Özer and A. Filiz, On the explicit asymptotic $\mathcal{W}_5$ symmetry of 3D Chern-Simons higher spin AdS_3 gravity, J.Math.Phys. 59 (2018) 8, 083504 [43],

Chapter 3, H.T. Özer and A. Filiz, Exploring new boundary conditions for $N = (1, 1)$ extended higher spin AdS_3 supergravity, Eur.Phys.J.C 80 (2020) 11, 1072 [54],

Chapter 4, H.T. Özer and A. Filiz, $N = (2, 2)$ extended $\mathfrak{sl}(3|2)$ Chern-Simons AdS_3 supergravity with new boundaries, Eur. Phys. J. C 82, 472 (2022) [55].



2. ASYMPTOTIC W_5 SYMMETRY OF THREE DIMENSIONAL CHERN-SIMONS HIGHER-SPIN AdS_3 GRAVITY

Holographic dualities, which have found many remarkable applications that range from mathematics to modern theoretical physics during the last twenty years, claim an equivalence between gravitational theories defined in the bulk of some region of space-time and field theories on the boundary of that region. By now it is studied in many diverse subfields, and the literature on the subject has become enormous. The most famous realization of the holographic duality is the AdS/CFT correspondence that was first conjectured by Juan Maldacena [1] in 1997. To put it simply, the correspondence says that a theory of quantum gravity with a negative cosmological constant defined in AdS_5 is equivalent to a certain CFT_4 living on its boundary.

In the case of three dimensions, the space AdS_3 has the two-dimensional conformal group $SO(2,2)$ as isometry group. This group acts on the two-dimensional boundary of AdS_3 as the symmetry of the two-dimensional conformal field theory CFT_2 . It is well-known that CFT_2 has a fundamental role in modern physics. Therefore it has always been of interest to see to what degree a Virasoro algebra, which is its underlying symmetry algebra as the local extension of the global conformal algebra $SO(2,2)$, may be extended. In this context, $\mathcal{W}_N$ algebras are extensions of the Virasoro algebra by currents of higher spin [56–60]. These algebras are available for arbitrary values of the central charge c .

Another intriguing $\mathcal{W}$ algebra which is very closely linked to $\mathcal{W}_N$ algebra is $\mathcal{W}_\infty$ family. $\mathcal{W}_N$ algebra can then be commented in the context of a $\mathcal{W}_\infty$ algebra [33, 61–63]. Such an algebra has an infinite set of higher spin, $s \geq 1$ generators. It should be emphasized that the $\mathcal{W}_\infty$ has been studied for special values of parameters where the algebra linearizes. Besides, $\mathcal{W}_N$ algebras have been studied also for generic values of the parameters, see e.g. the review by Gaberdiel and Gopakumar [64]. The first marks of the connection between $\mathcal{W}_\infty$ and $\mathcal{W}_N$ is given in the pioneering work [63]. Based on the contemporary point of view, one can say that $\mathcal{W}_\infty$ is an algebra with two parameters

[53, 65, 66]. The connection between $\mathcal{W}_N$ algebras with one parameter, particularly central charge c , and $\mathcal{W}_\infty$ algebra family with two-parameters is analogous to the construction of the higher spin algebra $\mathfrak{hs}(\lambda)$ in Vasiliev theory [28] and its universal relation to $\mathfrak{sl}(N)$ algebras. In an exactly related context of the extended $\mathcal{W}$ algebras, one can say that the AdS_3 higher spin gravity [24, 25, 27] is an interesting extension of the pure Einstein gravity. Einstein gravity theory is a lot easier to handle in three dimensions than in higher dimensions, because it allows a reformulation in terms of a Chern-Simons gauge theory [2, 3]. It has also been shown by Brown and Henneaux in their seminal work [6] that the asymptotic symmetry of AdS_3 gravity is given by two copies of a classical Virasoro algebra. This result can be interpreted as a pioneer application of the AdS/CFT correspondence. Then application was generalized by Henneaux and Rey [38] and Campoleoni et al. [39, 40]. The authors provide a closed formula for the structure constants of all classical W_N algebras where the approach depends on obtaining the algebras in the Poisson bracket notation. Our equivalent approach is the Operator Product Expansion (OPE) notation]to higher spin extensions on AdS_3 .

Another route for the asymptotic symmetry algebra, instead of classical one, is the quantum $\mathcal{W}_N$ minimal models [67]. The quantum asymptotic symmetry algebra of the AdS_3 higher spin gravity can be seen in the context of the minimal model holographic dualities. In particular, the $\mathcal{W}_N$ minimal models, which describe a family of two-dimensional CFT_2 with finite values of parameter N and the central charge c are dual to higher spin theories in AdS_3 [36]. More importantly, the quantization problems are partially solved by Gaberdiel and Gopakumar in Ref. [66]. In this thesis, the quantum corrections to the classical $\mathcal{W}_\infty$ algebra have been determined by using the Jacobi identity as well as the representation theory of the $\mathcal{W}_N$ minimal models. It is very interesting to calculate related quantum corrections directly in the higher spin gravity theory.

The impressive progress in the classification of the asymptotic symmetry algebras in recent years indicates that it is necessary to describe the AdS_3 higher spin gravity theories for sufficiently large conformal spin values. Therefore, the higher spin extension of the AdS_3 higher spin gravity theories is still an open problem because it is not yet known at least whether to be consistent with both small and large central charges. The reason for this is fundamentally the appearance of the composite or nonlinear terms

in the related asymptotic symmetry algebras. We emphasize here that the appearance of these composite terms in the related asymptotic symmetry algebras have caused some serious problems from the proper quantum field theory point of view. That is, they mean that the semi-classical asymptotic symmetry algebra must be corrected in order to guarantee that the Jacobi identities hold. To this end, one straightforward way is to replace the usual semi - classical asymptotic symmetry algebra with a quantum version one [38]. Finally, a deformed version of usual semi-classical asymptotic symmetry algebra with new structure constants can be obtained. This will be the final form of the quantum asymptotic symmetry algebra of the AdS_3 higher spin gravity [56, 68–71].

The higher spin gravity theories have also come to play recently for their remarkable role in the context of AdS/CFT correspondence. In three dimensions, within the setting of AdS_3/CFT_2 duality, the difficulty of the higher spin gravity theories can be simplified as it is possible to truncate [39] from an infinite number of spin fields to a set of the finite spin fields in the $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ Chern-Simons theory. In this convention, the $\mathfrak{sl}(N, \mathbb{R})$ Lie algebra belongs to $SL(N, \mathbb{R})$ group. Specifically, in the $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern - Simons theory, the higher spin AdS_3 gravity has asymptotic classical $\mathcal{W}_5$ symmetry algebra, which is the focus of the thesis. We emphasize that the main idea of this type of works is that of Hamiltonian reduction through AdS - type boundary conditions, therefore $\mathcal{W}_N$ symmetry algebras should follow straightforwardly for sufficiently large values of N , in the $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ Chern - Simons theory. Therefore, this work is also concerned with two-dimensional CFT_2 that appears in the holographic duality with higher spin theory on three dimensional AdS_3 space. The CFT_2 can be formulated as asymptotic $\mathcal{W}_5$ symmetry algebra which is higher-spin generalisation of the Virasoro algebra describing conformal symmetry with higher spin charges.

Since the asymptotic conditions allow incorporating the chemical potentials conjugated to the higher spin charges, one has the chance to identify the thermodynamic properties of the higher spin gravity. A detailed recent discussion of higher spin black holes with chemical potentials are claimed and fulfilled in Ref. [72] and also proposed independently in the form of new black hole solutions with spin 3 fields by Gutperle and Kraus in the pioneering paper [73]. There and in following works involving the same authors the case of spin 3 black holes has been analysed in great detail, while generic

higher spin black holes have been discussed only marginally. This led to successive papers focussing on the spin 4 case (see e.g. Refs. [41, 74–78]), aiming at finding further support for the proposal of Ref. [73] in a wider context. Especially, one can note that the case of spin 4 black hole story has been generalized without complication to the $\mathfrak{sl}(4, \mathbb{R}) \oplus \mathfrak{sl}(4, \mathbb{R})$ case [41]. In the gauge theories chemical potentials are presented by adding the time-component of the gauge connection. Therefore, it is also possible to carry out these studies entirely by adding chemical potentials associated to higher spin charges. It is well known that the chemical potential associated to spin 2 charge in pure gravity defines the temperature of the black hole. The procedure of adding a chemical potential to the connection in the higher spin algebra $\mathfrak{sl}(3, \mathbb{R})$ gravity is discussed [72, 79] as an additional contributions to the thermal circles around the horizon of the black hole, and also showed that introducing of the chemical potential does not modify the asymptotic $\mathcal{W}_3$ symmetry algebra. Besides that, in the seminal work of Gutperle and Kraus [73] it was proposed that a higher spin black hole should be defined as a flat connection with a trivial holonomy along the thermal Euclidean cycle on the torus. The holonomy along the non-contractible spatial cycle on the other hand, was proposed to define different black hole solutions. We also emphasize here that in the literature on higher-spin black holes there are a few quantities that have been evaluated in the $\mathfrak{sl}(3, \mathbb{R})$ as well as in the $\mathfrak{sl}(4, \mathbb{R})$ case, but whose extension to the $\mathfrak{sl}(N, \mathbb{R})$ Chern-Simons theories has been only sketched.

Motivated by these developments in three-dimensional anti-de Sitter (AdS_3) higher spin gravity we construct a classical solution based on an $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Lie algebra valued Chern-Simons gauge theory with a negative cosmological constant. Using some technique recently found, we calculated explicitly a solution that can be interpreted as spin-5 generalization of Bañados–Teitelboim–Zanelli (BTZ) solution. That is, we have constructed a spin-5 extension of Einstein gravity in three dimensions.

This chapter has the following outline. In Sec. 2.1, we give a fundamental formulation of pure gravity within $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern-Simons theory in three-dimensions. Sec. 2.2 is particularly devoted to the case of spin-5 where we showed in this section the principal embedding of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{sl}(5, \mathbb{R})$, and also demonstrated how $\mathcal{W}_5$ symmetry and higher spin Ward identities arise from the bulk equations of motion coupled to spin s , ($s = 3, 4, 5$) currents. Finally, classical $\mathcal{W}_5$ symmetry

algebra as asymptotic spin-5 symmetry algebra, the chemical potentials appearing through the temporal components of the connection and holonomy conditions are obtained, and also the results are checked with the quantum $\mathcal{W}_5$ algebra. We conclude with a summary for our results and a few suggestions for future work. Appendix A.1, B.1 and C.1 collect our conventions for the $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{sl}(3, \mathbb{R})$ and $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra generators respectively. Appendix B.1.2 also contains holonomies for $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ Chern - Simons Theory. And conformal spin-5 Ward identities are introduced in Appendix C.2. The classical $\mathcal{W}_5$ symmetry algebra is presented in Appendix C.3. Finally, Appendix C.4 contains the chemical potentials for $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern - Simons Theory.

2.1 Gravity in Three Dimensions, a Review:

Three dimensions is a good candidate because of its topological nature, which is a consequence of the lack of degrees of freedom. Chern-Simons theory is a quantum theory in three dimensions that computes only topological invariants. It can be defined on any manifold, and the metric does not need to be specified as it is a topological theory. Thus the physical quantities do not depend on the local geometry. Chern-Simons gravity is also a gauge theory and used as an interesting playground for investigating the AdS_3/CFT_2 correspondence by Brown and Henneaux in their seminal work [6] in 1986. They showed that any quantum gravity theory with asymptotically AdS_3 boundary conditions in three-dimensions must be dual to CFT_2 at the meaning that asymptotic symmetries of AdS_3 are given by two duplicate copies of the Virasoro algebra and related the central charge c of the CFT_2 to the corresponding AdS_3 radius ℓ in the following way

$$c = \frac{3\ell}{2G}. \quad (2.1)$$

This Brown Henneaux – formula, as one entry in the AdS_3/CFT_2 dictionary, notes that the central charge on the left-hand side is defined in the CFT_2 , while the quantities at the right-hand side are defined in AdS_3 .

2.1.1 Connection to Chern - Simons theory

It is a striking fact that the vacuum Einstein AdS_3 gravity in three dimensions with a negative cosmological constant can be formulated as a Chern - Simons gauge theory, as

it was first proposed by Achucarro and Townsend in Ref. [2] and developed by Witten in Ref. [3]. One can start by defining 1-forms $(\mathcal{A}, \bar{\mathcal{A}})$ taking values in the gauge group's $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra, and the trace is taken over the algebra generators. The Chern-Simons action can be written in the form,

$$S = S_{CS}[\mathcal{A}] - S_{CS}[\bar{\mathcal{A}}] \quad (2.2)$$

where

$$S_{CS}[\mathcal{A}] = \frac{k}{4\pi} \int \text{tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right). \quad (2.3)$$

Here $k = \frac{\ell}{8G\text{tr}(\mathbf{L}_0\mathbf{L}_0)} = \frac{c}{12\text{tr}(\mathbf{L}_0\mathbf{L}_0)}$ is the level of the Chern-Simons theory depending on the AdS radius ℓ and the Newton's constant G with the related central charge c of the CFT_2 . Nevertheless, trace shows a metric on the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra. If $\mathbf{L}_a, (a = \pm 1, 0)$ are the generators of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra,

$$[\mathbf{L}_a, \mathbf{L}_b] = (a - b)\mathbf{L}_{a+b} \quad (2.4)$$

one can define the invariant bilinear form

$$\text{tr}(\mathbf{L}_a \mathbf{L}_b) = \frac{1}{2} \eta_{ab}. \quad (2.5)$$

For a negative cosmological constant $\mathcal{A}$ and $\bar{\mathcal{A}}$ are $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra valued one-form, and they depend on the vielbein e_μ^a and dual spin connection $\omega_\mu^a = e^{abc} \omega_{bc\mu}$ as follows

$$\mathcal{A} = \left(\omega_\mu^a + \frac{e_\mu^a}{\ell} \right) \mathbf{L}_a dx^\mu, \quad \bar{\mathcal{A}} = \left(\omega_\mu^a - \frac{e_\mu^a}{\ell} \right) \mathbf{L}_a dx^\mu. \quad (2.6)$$

The equations of motion for the Chern-Simons gauge theory give the flatness condition $F = \bar{F} = 0$ where

$$F = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0, \quad (2.7)$$

is the same as the Einstein's equation. $\mathcal{A}$ and $\bar{\mathcal{A}}$ are related to the metric $g_{\mu\nu}$ through the vielbein $e = \frac{\ell}{2}(\mathcal{A} - \bar{\mathcal{A}})$

$$g_{\mu\nu} = \frac{1}{2} \text{tr}(e_\mu e_\nu) \quad (2.8)$$

2.1.2 $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory

Spin 2 case is given by reviewing asymptotically AdS_3 boundary conditions for a $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory, and how to determine the asymptotic symmetry algebra using the methods described in Ref. [80]. By using the Fefferman - Graham expansion method, the most general solution of the Einstein's equation that is asymptotically AdS , is given by with a flat boundary metric [81]

$$ds^2 = l^2 \left\{ d\rho^2 - \frac{8\pi G}{l} \left(\mathcal{L}(dx^+)^2 + \tilde{\mathcal{L}}(dx^-)^2 \right) - \left(e^{2\rho} + \frac{64\pi^2 G^2}{l^2} \mathcal{L} \tilde{\mathcal{L}} e^{-2\rho} \right) dx^+ dx^- \right\} \quad (2.9)$$

where $(\rho, x^\pm \equiv \frac{t}{\ell} \pm \phi)$ shows the solid cylinder as the light - like coordinates and $\mathcal{L} \equiv \mathcal{L}(x^+), \tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}(x^-)$ are arbitrary functions of $x^\pm$. Therefore, one can write the light - like components of the gauge fields by using the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra generators

$$\mathcal{A} = b^{-1} a(x^+) b + b^{-1} db, \quad \bar{\mathcal{A}} = b \bar{a}(x^-) b^{-1} + b db^{-1}, \quad (2.10)$$

with $b = e^{\rho L_0}$, we obtain

$$a = \left(L_1 - \frac{2\pi}{k} \mathcal{L} L_{-1} \right) dx^+, \quad \bar{a} = - \left(L_{-1} - \frac{2\pi}{k} \tilde{\mathcal{L}} L_1 \right) dx^-. \quad (2.11)$$

A very important point in this description is that this theory has to be asymptotically AdS_3 , as required in Ref. [39] the boundary conditions have to be defined in a similar conditions

$$(\mathcal{A} - \mathcal{A}_{AdS_3}) \Big|_{\text{boundary}} = \mathcal{O}(1). \quad (2.12)$$

We will carry out such an analysis for a connection of the spin 2 charge, finding that the asymptotic symmetry can be identified with a classical $\mathcal{W}_2$ -algebra. After that, we will only work on the positive chiral components having x^+ , although the same can be done in the other one as well. If we expand $\lambda(x^+)$ in the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra, as we did also the connection, then

$$\lambda = \sum_{i=-1}^1 \epsilon_i L_i. \quad (2.13)$$

We are now interested in the transformation that preserve the structure of (4.9). Under an infinitesimal gauge transformation with gauge parameter λ, a which is equivalent to $\mathcal{A}$ transforms as the flatness condition:

$$\delta_\lambda a = d\lambda + [a, \lambda]. \quad (2.14)$$

Thus we have to impose that all terms proportional to L_0, L_1 vanish. These constraints can be solved to find ε_0 and ε_{-1} in terms of $\varepsilon_1 \equiv \varepsilon$ which is called the gauge parameter related to $\mathfrak{sl}(2, \mathbb{R})$ and superscripted primes denote ∂_{x^+} , one finds

$$\begin{aligned}\varepsilon_0 &= -\varepsilon', \\ \varepsilon_{-1} &= \frac{1}{2}\varepsilon'' - \frac{2\pi}{k}\varepsilon\mathcal{L}.\end{aligned}\tag{2.15}$$

and (2.13) is of the form:

$$\lambda(\varepsilon) = \varepsilon L_1 - \varepsilon' L_0 + \frac{1}{2}\left(\varepsilon'' - \frac{4\pi}{k}\varepsilon\mathcal{L}\right)L_{-1}.\tag{2.16}$$

Now, one can also determine how the function $\mathcal{L}$ transforms under this gauge transformation. This is given by

$$\mathcal{L} \rightarrow \mathcal{L} + \delta_\varepsilon \mathcal{L},\tag{2.17}$$

where

$$\delta_\varepsilon \mathcal{L} = 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon + \frac{k}{4\pi}\varepsilon'''.\tag{2.18}$$

As a final step, one now has to determine the canonical boundary charge $\mathcal{Q}[\varepsilon]$ that generates this transformation. Therefore, the corresponding variation of the boundary charge $\mathcal{Q}[\varepsilon]$ can be integrated which reads

$$\mathcal{Q}[\varepsilon] = \int dx^+ \varepsilon(x^+) \mathcal{L}(x^+).\tag{2.19}$$

This leads to Dirac bracket algebra by using $\delta_\varepsilon F = \{F, \mathcal{Q}[\varepsilon]\}$

$$\begin{aligned}\{\mathcal{L}(x^+), \mathcal{L}(y^+)\} &= 2\mathcal{L}(y^+)\delta'(x^+ - y^+) + \mathcal{L}'(y^+)\delta(x^+ - y^+) \\ &\quad + \frac{k}{4\pi}\delta'''(x^+ - y^+).\end{aligned}\tag{2.20}$$

One can also expand $\mathcal{L}(x^+)$ into Fourier modes $\mathcal{L}(x^+) = \sum_n L_n e^{-inx^+}$, and replacing $i\{\cdot, \cdot\} \rightarrow [\cdot, \cdot]$. A Virasoro algebra can then be defined as:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}.\tag{2.21}$$

We now consider the case that the conformal boundary is a complex cylinder, as in (2.9). We work in complex coordinates, $z(\bar{z}) \equiv \phi \pm i\frac{t}{\ell}$. Therefore, the Virasoro algebra in these space is equivalent to operator product algebra

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{\frac{c}{2}}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}}\tag{2.22}$$

where the central charge c is again (2.1) related to the level of the Chern-Simons theory as

$$c = 6k = \frac{3\ell}{2G}. \quad (2.23)$$

The same algebra is also realized as the asymptotic symmetry algebra of the anti-chiral connection $\bar{\mathcal{A}}$ having x^- in terms of the second copy of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra.

2.1.3 Adding a chemical potential to $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory

The procedure of adding a chemical potential to the connection in pure $\mathfrak{sl}(2, \mathbb{R})$ gravity discussed in Ref. [72] is an additional contribution to the thermal circles around the horizon of the black hole. One can show that an addition of the chemical potential to the connection does not modify the asymptotic $\mathcal{W}_2$ symmetry algebra as in (2.18)-(2.23). To do this, one can propose the following boundary conditions for the connection $a(t, \phi)$ in the Chern - Simons formulation,

$$a(t, \phi) = a_\phi(t, \phi)d\phi + a_t(t, \phi)dt \quad (2.24)$$

where

$$a_\phi(t, \phi) = L_1 - \frac{2\pi}{k} \mathcal{L} L_{-1} \quad (2.25)$$

and

$$a_t(t, \phi) = \mu_2 L_1 + \sum_{i=-1}^0 v_2^{(i)} L_i. \quad (2.26)$$

Here μ_2 is in principle arbitrary functions of t and ϕ . We interpret the function μ_2 as a chemical potential. This means that we assume the chemical potential to be fixed at infinity, i.e. $\delta\mu_2 = 0$. The functions $v_2^{(0)}$ and $v_2^{(-1)}$ are fixed by the flatness condition (4.7). For fixed chemical potential μ_2 , the time evolution of canonical boundary charge $\mathcal{L}$ as well as $v_2^{(0)}$ and $v_2^{(-1)}$ can be written as

$$\begin{aligned} \partial_t \mathcal{L} &= 2\mathcal{L}\mu_2' + \mathcal{L}'\mu_2 + \frac{k}{4\pi}\mu_2''', \\ v_2^{(0)} &= -\mu_2', \\ v_2^{(-1)} &= \frac{1}{2}\mu_2'' - \frac{2\pi}{k}\mu_2\mathcal{L}, \end{aligned} \quad (2.27)$$

where prime denotes derivative with respect to ϕ , and the temporal connection a_t is of the form:

$$a_t = \mu_2 L_1 - \mu_2' L_0 + \frac{1}{2}\left(\mu_2'' - \frac{4\pi}{k}\mu_2\mathcal{L}\right)L_{-1}. \quad (2.28)$$

In many applications, however, the chemical potential μ_2 is constant, which can simplify most of the formulas, especially in AdS_3 higher spin gravity, considerably. Therefore, one can note that from now on one will assume that the chemical potential μ_2 as well as the canonical boundary charge $\mathcal{L}$ are constant. Under this assumption, the spatial connection a_ϕ and the temporal connection a_t simplify as

$$a_\phi = L_1 - \frac{2\pi}{k} \mathcal{L} L_{-1} \quad (2.29)$$

$$a_t = \mu_2 \left(L_1 - \frac{2\pi}{k} \mathcal{L} L_{-1} \right) \quad (2.30)$$

respectively. These manifestly solve the field equation (2.27), and describe black holes solutions carrying not only mass and angular momentum but also canonical charges of the pure gravity. As discussed in Ref. [72], and comparison with e.g. eq. (2.29) of Ref. [79] shows that the temporal component of the connection is linear in the chemical potential μ_2 . The same procedure can be followed to establish $\bar{a}$ with a constant chemical potential.

2.1.4 Holonomies in $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory

As we have seen, classical solutions of the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern - Simons theory are given by the flat connections, and thus locally may be written as pure gauges, i.e. $\mathcal{A} = g^{-1} dg$. Globally, however this is not true as the space-time may have some non-trivial topology, then the gauge function g is not single-valued. When the spacetime has a non-contractable cycle $\mathcal{C}$, the gauge function g attains a factor of holonomy $\mathcal{P}\exp(\oint_{\mathcal{C}} \mathcal{A}_\mu d\mu)$. This means that classical solutions in Chern-Simons theory are uniquely specified by the holonomies, which are gauge invariant quantities, around the cycles of the manifold which is a solid torus, up to an overall gauge transformation.

The solutions of the Chern Simons theory are, in Euclidean signature, specified by two cycles: The spatial cycle and the thermal cycle. The thermal cycle is contractable, that is, the holonomy around the thermal cycle is trivial, which is usually related to the inverse temperature of the solution. But the ϕ coordinate represents the non-contractable cycle. Thus, the flat connection a 's are uniquely specified by the non-trivial holonomy:

$$H_\phi(a) = b^{-1} e^w b, \quad 0 < \phi < 2\pi \quad (2.31)$$

where

$$\begin{aligned} w &= \oint_C a_\phi d\phi \\ &= 2\pi a_\phi \end{aligned} \quad (2.32)$$

is the holonomy matrix. For the Euclidean *BTZ* black hole, the ϕ -cycle is non-contractable, and will have a non-trivial holonomy matrix:

$$w = 2\pi a_\phi = \begin{pmatrix} 0 & -\frac{4\pi^2}{k}\mathcal{L} \\ -2\pi & 0 \end{pmatrix}. \quad (2.33)$$

One must emphasize here that what different values of $\mathcal{L}$ give rise to different solutions. Besides, in many cases it might be cumbersome or even impossible to evaluate the holonomy matrix exactly, as was done above. But often we do not need the full matrix and will be satisfied with knowledge of the eigenvalues of the exponentiated matrix e^w . The easiest way to solve the holonomy condition is by solving the characteristic polynomial. This means that one may express exponent matrix w , a general 2×2 matrix, as

$$w^2 = \Theta_0 + \Theta_1 w \quad (2.34)$$

where Θ 's are referred to as the *holonomy invariants*. To find this invariants one consider the general characteristic equation of exponent matrix w , explicitly

$$\begin{aligned} \lambda^2 &= 4\pi^2 Q_2 \\ &= -(\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2 \end{aligned} \quad (2.35)$$

where

$$\begin{aligned} Q_2 &= \frac{1}{2} \text{tr}[a_\phi^2] \\ &= \frac{2\pi}{k} \mathcal{L} \end{aligned} \quad (2.36)$$

is a second order *Casimir operator*-related parameter, and one can use the identities

$$\text{tr}[w] = \sum_i \lambda_i, \quad \text{det}[w] = \prod_i \lambda_i. \quad (2.37)$$

If λ_i is i^{th} eigenvalue of the exponentiated matrix w , and identify the *holonomy invariants* in (2.34) as

$$\begin{aligned} \Theta_0 &= -\text{det}[w] \\ &= -4\pi^2 Q_2, \\ \Theta_1 &= \text{tr}[w] \\ &= 0. \end{aligned} \quad (2.38)$$

Therefore, since $\mathcal{L}$ is non-negative, they are manifestly nontrivial. If a holonomy is trivial around a contractible cycle the exponentiated matrix e^w must have eigenvalues $\pm\pi i$, it follows that

$$\begin{aligned}\mathrm{tr}[w] &= 0, \\ \mathrm{tr}[w^2] + 2\pi^2 &= 0.\end{aligned}\tag{2.39}$$

Now when it comes to calculating the holonomy about contractible (thermal) cycle one may use the temporal component $a_t = \mu_2 a_\phi$ in (2.29), which fulfills the equation of motion $[a_t, a_\phi] = 0$ it is obvious that $a_t = f(a_\phi)$. Besides, it is important to recall that one usually works with a free periodicity on the temporal coordinate with Euclidean signature. This is usually done by first performing a Wick rotation ($t \rightarrow i\tau$) of the Lorentzian time coordinate t to Euclidean time τ which is then compactified. This changes the topology of the solid cylinder to that of a solid torus. One can introduce complex coordinates $(z, \bar{z})$ by analytically continuing the light-cone directions $x^\pm \rightarrow i\frac{\tau}{\ell} \pm \phi \equiv (z, -\bar{z})$. Then $z \simeq z + 2\pi\tau$ where τ is the modular parameter of the boundary torus, which is usually related to the inverse temperature β of the solution as $\tau = \frac{i\beta}{2\pi}$ and also compatible with chemical potential μ_2 in (2.30). Therefore, the trivial holonomy is

$$H_\tau(a) = b^{-1} e^w b, \tag{2.40}$$

where

$$\begin{aligned}w &= \oint_{\mathcal{C}} a_t dt \\ &= 2\pi\tau a_t\end{aligned}\tag{2.41}$$

is the holonomy matrix, and the eigenvalues of w will be $\pm\pi i$ as non-degenerate for the connection considered here. Hence, we can use (2.39), it follows that

$$\begin{aligned}\mathrm{tr}[w] &= 0, \\ \mathrm{tr}[w^2] + 2\pi^2 &= 0.\end{aligned}\tag{2.42}$$

Finally, while not $\mathbb{I}$, the exponentiated matrix $e^w = -\mathbb{I}$ is in the center of $\mathfrak{sl}(2, \mathbb{R})$, so it is a trivial holonomy through the periodicity of β . In order to solve the holonomy conditions, the entropy function can be written in the form

$$\mathcal{S}(\mathcal{L}) = 2\pi\sqrt{2\pi k\mathcal{L}} \tag{2.43}$$

such that

$$\begin{aligned}\tau &= i \frac{1}{4\pi^2} \frac{\partial \mathcal{S}}{\partial \mathcal{L}} \\ &= \frac{ik}{2} \frac{1}{\sqrt{2\pi k \mathcal{L}}} \quad .\end{aligned}\tag{2.44}$$

This solves the holonomy condition, $\text{tr}[w^2] + 2\pi^2 = 0$, with $\beta = \sqrt{\frac{\pi k}{2\mathcal{L}}}$.

2.2 Higher-spin Chern - Simons Theory

The Chern-Simons formalism in three dimensions lends itself to generalization to higher spin theories coupled to gravity by raising the gauge group to $\mathfrak{sl}(N, \mathbb{R})$. The $\mathfrak{sl}(N, \mathbb{R})$ Chern-Simons theory describes AdS_3 gravity theories coupled to a finite tower of integer spin- $s \leq N$. Therefore, an interesting extension of pure AdS gravity can be constructed by adding massless higher spin fields to the spectrum, which proves to be particularly easy in Chern - Simons formulation. The first nontrivial spin 3 extensions of higher spin theories coupled to the pure gravity can be seen in Refs. [38, 39], where a consistent set of asymptotic conditions for the theory is described. It can be shown that, instead of describing difficult nonlinear interactions of the higher spin fields, promoting the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ symmetry group of the Chern - Simons action to $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$, with $N > 2$, is sufficient to successfully describe a higher spin theory.

2.2.1 A calculation of $\mathfrak{sl}(5, \mathbb{R})$ commutators

The principal embedding of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra to $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra contains the spin 2 triplet $L_n, (n = \pm 1, 0)$, the spin 3 triplet $W_n^{(3)}, (n = \pm 2, \pm 1, 0)$, the spin 4 quartet $W_n^{(4)}, (n = \pm 3, \pm 2, \pm 1, 0)$ and the spin-5 quintet $W_n^{(5)}, (n = \pm 4, \pm 3, \pm 2, \pm 1, 0)$ can be represented in terms of the commutation relations for $L_n \equiv W_n^{(2)}$.

Let us now explain how $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra can be constructed. We proceed in two steps. First, we make the most general ansatz for the commutators between all the Lie algebra generators $W_n^{(s_i)}, W_m^{(s_j)}$ as follows

$$[L_n, W_m^{(s_i)}] = ((s_i - 1)n - m)W_{n+m}^{(s_i)}, (s_i = 2, 3, 4, 5) \tag{2.45}$$

$$[W_n^{(s_i)}, W_m^{(s_j)}] = \sum_t^{s_i+s_j} f_{s_i s_j}^t(n, m) W_{n+m}^{(t)}. \tag{2.46}$$

Our next goal is to find concrete expressions for the structure constants $f_{s_1 s_2}^{s_3}(n, m)$. It can be seen here that some structure constants does not survive because of the spin gradation conformity. This structure constants which are the polynomials in n and m indices are also standard and known in closed form and given as just the commutator structure of quasi primary field in a *CFT*, (see e.g. the book of Blumenhagen and Plauschinn [82] in Sec. 2.6.3) as $f_{s_i s_j}^{s_k}(n, m) \sim p_{ijk}(n, m)$, where

$$p_{ijk}(n, m) = \sum_{\substack{r, s \in \mathbb{Z}_0^+ \\ r+s = s_i + s_j - s_k - 1}} C_{r,s}^{ijk} \cdot \binom{-m + s_i - 1}{r} \cdot \binom{-n + s_j - 1}{s} \quad (2.47)$$

$$C_{r,s}^{ijk} = (-1)^r \frac{(2s_k - 1)!}{(s_i + s_j + s_k - 2)!} \prod_{t=0}^{s-1} (2s_i - 2 - r - t) \prod_{u=0}^{r-1} (2s_j - 2 - s - u). \quad (2.48)$$

Then, we used the associativity of the $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra with Jacobi identities under rescaling $f_{s_i s_j}^{s_k}(n, m) = \sigma_a p_{ijk}(n, m)$. Since this associativity condition is of central importance for this section, we will find 11 expressions for the structure constants $f_{s_1 s_2}^{s_3}(n, m)$ (Further details of our conventions can be found in Appendix C.1).

2.2.2 Spin-5 AdS_3 solution and conformal Ward identities

We now proceed to construct $\mathfrak{sl}(5, \mathbb{R})$ connections which describe spin-5 AdS_3 solution with non-zero higher spin charge. We consider the class of connection using the same gauge and notation as in Sec. 2.1.2, one can write down the spin-5 extension of (2.11) as Ref. [39]. In this part, we will now make a similar calculation for spin-5. We begin with the ansatz

$$a = \left(L_1 + \alpha \mathcal{L} L_{-1} + \beta \mathcal{W}_3 \tilde{W}_{-2}^{(3)} + \gamma \mathcal{W}_4 \tilde{W}_{-3}^{(4)} + \delta \mathcal{W}_5 \tilde{W}_{-4}^{(5)} \right) dx^+ \quad (2.49)$$

$$\bar{a} = - \left(L_{-1} + \alpha \tilde{\mathcal{L}} L_1 + \beta \tilde{\mathcal{W}}_3 \tilde{W}_2^{(3)} + \gamma \tilde{\mathcal{W}}_4 \tilde{W}_3^{(4)} + \delta \tilde{\mathcal{W}}_5 \tilde{W}_4^{(5)} \right) dx^- \quad (2.50)$$

where α, β, γ and δ are some scaling parameters to be determined later. After this, we analyze the holomorphic connection a . All resulting expressions involving the $\bar{A}$ sector can be obtained in the similar way. If we expand $\lambda(x^+)$ in the $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra, as we did also for the connection which is compatible with (2.49), then

$$\lambda = \sum_{i=-1}^1 \varepsilon_i L_i + \sum_{i=-2}^2 \chi_i \tilde{W}_i^{(3)} + \sum_{i=-3}^3 f_i \tilde{W}_i^{(4)} + \sum_{i=-4}^4 \eta_i \tilde{W}_i^{(5)}. \quad (2.51)$$

The flat connection condition (2.14) emerges the following constraints for 24 undetermined functions of $x^\pm$. There are 24 equations with 4 parameters $\varepsilon_1, \chi_2, f_3, \eta_4$

that can be chosen arbitrarily. For notational convenience, we write $\varepsilon_1 \equiv \varepsilon$, $\chi_2 \equiv \chi$, $f_3 \equiv f$, $\eta_4 \equiv \eta$, and show derivatives with respect to x^+ as primes for convenience, one finds

$$\varepsilon_0 = -\varepsilon' \quad (2.52)$$

$$\varepsilon_{-1} = \frac{69120\gamma f \sigma_1^2 \mathcal{W}_4}{49\sigma_2^2} + 24\beta \sigma_1 \chi \mathcal{W}_3 + \frac{3686400\delta \eta \sigma_1^3 \mathcal{W}_5}{343\sigma_2^2 \sigma_4^2} + \alpha \mathcal{L} \varepsilon + \frac{\varepsilon''}{2}, \quad (2.53)$$

and all other parameters are given in Appendix C.2.1. Then, one can now also determine how the functions $\mathcal{L}, \mathcal{W}_s$ ($s = 3, 4, 5$) transform under the gauge transformation in terms of $\varepsilon, \chi, f, \eta, \mathcal{L}, \mathcal{W}_s$ ($s = 3, 4, 5$) and their derivatives. The transformations of $\mathcal{L}, \mathcal{W}_s$ ($s = 3, 4, 5$) are given by

$$\mathcal{L} \rightarrow \mathcal{L} + \delta_\varepsilon \mathcal{L} + \delta_\chi \mathcal{L}, \quad (2.54)$$

$$\mathcal{W}_s \rightarrow \mathcal{W}_s + \delta_\varepsilon \mathcal{W}_s + \delta_\chi \mathcal{W}_s + \delta_f \mathcal{W}_s + \delta_\eta \mathcal{W}_s, \quad (s = 3, 4, 5) \quad (2.55)$$

where

$$\delta_\varepsilon \mathcal{L} = 2\mathcal{L}\varepsilon' + \mathcal{L}'\varepsilon + \frac{\varepsilon'''}{2\alpha} \quad (2.56)$$

here ε is the gauge parameter related to $\mathfrak{sl}(2, \mathbb{R})$ subgroup in $\mathfrak{sl}(5, \mathbb{R})$, which generate conformal transformations. Thus, we see that $\mathcal{L}$ can be identified with the CFT_2 stress tensor if we relate the Chern-Simons level and the central charge as $c = 6k$. Now, first observe that if we set from (2.18) and (2.23), we can write the first scaling parameter α as

$$\alpha = \frac{6}{c} \quad (2.57)$$

which is compatible with the central charge of the Virasoro algebra $c = 6k$. Finally we will set $\sigma_1 = -\frac{1}{30}$, $\sigma_2 = \frac{16}{\sqrt{105}}$, $\sigma_4 = \frac{\sqrt{15}}{7}$, and also $\beta = \frac{15}{c}$, $\gamma = \frac{28}{c}$ and $\delta = \frac{45}{c}$. In the light of these results, for the later calculations from now on we generalize the connection (2.49) for $\mathfrak{sl}(N, \mathbb{R})$ and $L_n \equiv W_n^{(2)}$ as

$$a = \left(W_1^{(2)} + \sum_{s=2}^N \frac{s(2s-1)}{c} \mathcal{W}_s W_{-s+1}^{(s)} \right) dx^+, \quad (2.58)$$

in order to match AdS_3 connection parameters to the classical $\mathcal{W}_5$ asymptotic symmetry algebra as in Appendix C.3. Then, the following variations

$$\delta_\varepsilon \mathcal{W}_s = s\mathcal{W}_s \varepsilon' + \varepsilon \mathcal{W}_s', \quad (s = 3, 4, 5) \quad (2.59)$$

define, that the each conformal field $\mathcal{W}_s$ has the conformal spin s , with $s = 3, 4, 5$. All variations which defines spin-5 conformal Ward identities are given in Appendix

C.2.2. As in the spin 2 case in (2.19), one can now determine the corresponding canonical boundary charges as

$$\begin{aligned} \mathcal{Q}[\varepsilon] = & \int dx^+ (\varepsilon(x^+) \mathcal{L}(x^+) + \chi(x^+) \mathcal{W}_3(x^+) + f(x^+) \mathcal{W}_4(x^+) \\ & + \eta(x^+) \mathcal{W}_5(x^+)). \end{aligned} \quad (2.60)$$

In fact, from these Ward identities, that is, from the variation of these fields one can derive their corresponding classical asymptotic $\mathcal{W}_5$ symmetry algebra as in Appendix C.3.

2.2.3 Adding chemical potentials to $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern - Simons theory

So far, we have not incorporated the chemical potentials in the $\mathfrak{sl}(5, \mathbb{R})$ Chern-Simons higher spin theory. We emphasize here that in the literature on higher-spin black holes there are a few quantities that have been evaluated in the $\mathfrak{sl}(N, \mathbb{R})$ case up to $N = 4$, but whose extension to the $\mathfrak{sl}(N, \mathbb{R})$ Chern-Simons theories, $N > 4$, has been only sketched. To do this, one must first concentrate on (2.51) and twenty solved free parameters in Appendix C.2.1 depending on ε, χ, f and η . Here $\mathcal{L}, \mathcal{W}_i$; ($i = 3, 4, 5$) and $\varepsilon, \chi, f, \eta$ stand for arbitrary functions of (t, ϕ) . The asymptotic behaviour of the spatial part of the connection is preserved under gauge transformations generated by λ in (2.51).

The procedure of adding a chemical potential to the connection in pure $\mathfrak{sl}(5, \mathbb{R})$ gravity discussed in Ref. [72] is an additional contribution to the thermal circles around the horizon of the black hole. One can show that an addition of the chemical potential to the connection does not modify the asymptotic $\mathcal{W}_5$ symmetry algebra as in Sec. 2.2.2. Notice that the chemical potentials μ_2, μ_3, μ_4 and μ_5 appear only in the components of the gauge fields along t , and they are assumed to be fixed at infinity, $\delta\mu_2 = \delta\mu_3 = \delta\mu_4 = \delta\mu_5 = 0$. As explained in Refs. [72] and [79], the chemical potentials are included in the temporal component of the gauge fields only, so that the asymptotic form of the the gauge fields with constant fields $\mathcal{L}, \mathcal{W}_i$; ($i = 3, 4, 5$) and chemical potentials μ_2, μ_3, μ_4 and μ_5 , provided that the spatial part of the connection is unchanged. To do this, one can propose the following boundary conditions for the connection $a(t, \phi)$ in the Chern - Simons formulation,

$$a(t, \phi) = a_\phi(t, \phi) d\phi + a_t(t, \phi) dt \quad (2.61)$$

where

$$\begin{aligned}
a_\phi(t, \phi) &= L_1 + \frac{6}{c} \mathcal{L} L_{-1} + \frac{15}{c} \mathcal{W}_3 W_{-2}^{(3)} + \frac{28}{c} \mathcal{W}_4 W_{-3}^{(4)} + \frac{45}{c} \mathcal{W}_5 W_{-4}^{(5)}, \\
a_t(t, \phi) &= \mu_2 L_1 + \mu_3 W_2^{(3)} + \mu_4 W_3^{(4)} + \mu_5 W_4^{(5)} + \sum_{i=-1}^0 v_2^{(i)} L_i \\
&\quad + \sum_{i=-2}^1 v_3^{(i)} W_i^{(3)} + \sum_{i=-3}^2 v_4^{(i)} W_i^{(4)} + \sum_{i=-4}^3 v_5^{(i)} W_i^{(5)} \\
&= a_t^{(\mu_2)} + a_t^{(\mu_3)} + a_t^{(\mu_4)} + a_t^{(\mu_5)}. \tag{2.62}
\end{aligned}$$

Here, μ_s 's are in principle arbitrary functions of t and ϕ . We can interpret these arbitrary functions as chemical potentials. This means that we assume those chemical potential to be fixed at infinity, i.e. $\delta\mu_2 = \delta\mu_3 = \delta\mu_4 = \delta\mu_5 = 0$. The remaining functions $v_s^{(i)}$'s are fixed by the flatness condition (4.7). For these fixed chemical potentials μ_s 's, the time evolution of canonical boundary charges $\mathcal{L}$ and $\mathcal{W}_s$'s as well as $v_s^{(i)}$'s is similar as in the procedure of the Sec. 2.1.3. The final form of the temporal connection a_t is given in Appendix C.4.

In many applications, however, the chemical potentials μ_s 's can be taken as a constant, which can simplify most of the formulas, especially in AdS_3 higher spin gravity, considerably. Therefore, one note that from now on one will assume that the chemical potential μ_s 's as well as the canonical boundary charge $\mathcal{L}$ and $\mathcal{W}_s$'s are constant. Under this assumption, the temporal connection a_t simplify as

$$\begin{aligned}
a_t^{(\mu_2)} &= \mu_2 \left(L_1 + \frac{6}{c} \mathcal{L} L_{-1} + \frac{15}{c} \mathcal{W}_3 W_{-2}^{(3)} + \frac{28}{c} \mathcal{W}_4 W_{-3}^{(4)} + \frac{45}{c} \mathcal{W}_5 W_{-4}^{(5)} \right) \\
a_t^{(\mu_3)} &= \mu_3 \left(W_2^{(3)} + \frac{12}{c} \mathcal{W}_3 L_{-1} + \left(\frac{36}{c^2} \mathcal{L}^2 + \frac{16}{c} \sqrt{\frac{15}{7}} \mathcal{W}_4 \right) W_{-2}^{(3)} \right. \\
&\quad + \left(\frac{96}{c^2} \sqrt{\frac{15}{7}} \mathcal{L} \mathcal{W}_3 + \frac{8\sqrt{15}}{c} \mathcal{W}_5 \right) W_{-3}^{(4)} + \left(\frac{1800}{7\sqrt{7}c^2} \mathcal{W}_3^2 \right. \\
&\quad + \left. \mathcal{W}_4 \frac{48\sqrt{15}}{c^2} \mathcal{L} \right) W_{-4}^{(5)} + \frac{8\sqrt{15}}{c} \mathcal{W}_4 W_{-2}^{(5)} + \frac{16}{c} \sqrt{\frac{15}{7}} \mathcal{W}_3 W_{-1}^{(4)} \\
&\quad \left. + \frac{12}{c} \mathcal{L} W_0^{(3)} \right)
\end{aligned}$$

$$\begin{aligned}
a_t^{(\mu_4)} &= \mu_4 \left(\mathcal{W}_3^{(4)} + \frac{18}{c} \mathcal{W}_4 \mathcal{L}_{-1} + \left(\frac{108}{c^2} \mathcal{L}^2 + \frac{2\sqrt{105}}{c} \mathcal{L}^2 \right) \mathcal{W}_{-1}^{(4)} \right. \\
&\quad + \left(\frac{540\sqrt{15}}{7c^2} \mathcal{L} \mathcal{W}_3 - \frac{30}{c} \sqrt{\frac{15}{7}} \mathcal{W}_5 \right) \mathcal{W}_{-2}^{(5)} + \left(\frac{576}{7c^2} \sqrt{\frac{15}{7}} \mathcal{L} \mathcal{W}_3 \right. \\
&\quad + \frac{45\sqrt{15}}{7c} \mathcal{W}_5 \left. \right) \mathcal{W}_{-2}^{(3)} + \left(\frac{216}{c^3} \mathcal{L}^3 + \frac{1500}{7c^2} \mathcal{W}_3^2 \right. \\
&\quad + \frac{44}{c^2} \sqrt{\frac{21}{5}} \mathcal{L} \mathcal{W}_4 \left. \right) \mathcal{W}_{-3}^{(4)} + \left(\frac{1620\sqrt{15}}{7c^3} \mathcal{L}^2 \mathcal{W}_3 + \frac{900}{\sqrt{7}c^2} \mathcal{W}_3 \mathcal{W}_4 \right. \\
&\quad - \frac{18\sqrt{105}}{c^2} \mathcal{L} \mathcal{W}_5 \left. \right) \mathcal{W}_{-4}^{(5)} + \frac{45\sqrt{15}}{7c} \mathcal{W}_3 \mathcal{W}_0^{(5)} + \frac{120}{7c} \sqrt{\frac{15}{7}} \mathcal{W}_3 \mathcal{W}_0^{(3)} \\
&\quad \left. + \frac{18}{c} \mathcal{L} \mathcal{W}_1^{(4)} \right) \\
a_t^{(\mu_5)} &= \mu_5 \left(\mathcal{W}_4^{(5)} + \frac{24}{c} \mathcal{W}_5 \mathcal{L}_{-1} + \left(\frac{216}{c^2} \mathcal{L}^2 - \frac{4\sqrt{105}}{c} \mathcal{W}_4 \right) \mathcal{W}_0^{(5)} \right. \\
&\quad + \left(\frac{1440}{7\sqrt{7}c^2} \mathcal{W}_3^2 + \frac{240\sqrt{15}}{7c^2} \mathcal{L} \mathcal{W}_4 \right) \mathcal{W}_{-2}^{(3)} + \left(\frac{576\sqrt{15}}{7c^2} \mathcal{L} \mathcal{W}_3 \right. \\
&\quad - \frac{4\sqrt{105}}{c} \mathcal{W}_5 \left. \right) \mathcal{W}_{-1}^{(4)} + \left(\frac{864}{c^3} \mathcal{L}^3 + \frac{2400}{7c^2} \mathcal{W}_3^2 - \frac{272}{c^2} \sqrt{\frac{15}{7}} \mathcal{L} \mathcal{W}_4 \right) \mathcal{W}_{-2}^{(5)} \\
&\quad + \left(\frac{1440\sqrt{15}}{7c^3} \mathcal{L}^2 \mathcal{W}_3 + \frac{112\sqrt{7}}{c^2} \mathcal{W}_3 \mathcal{W}_4 - \frac{104}{c^2} \sqrt{\frac{15}{7}} \mathcal{L} \mathcal{W}_5 \right) \mathcal{W}_{-3}^{(4)} \\
&\quad + \left(\frac{1296}{c^4} \mathcal{L}^4 - \frac{624}{c^3} \sqrt{\frac{15}{7}} \mathcal{L}^2 \mathcal{W}_4 + \frac{90000}{49c^3} \mathcal{L} \mathcal{W}_3^2 + \frac{420}{c^2} \mathcal{W}_4^2 \right. \\
&\quad - \frac{990}{\sqrt{7}c^2} \mathcal{W}_3 \mathcal{W}_5 \left. \right) \mathcal{W}_{-4}^{(5)} + \frac{8\sqrt{15}}{c} \mathcal{W}_3 \mathcal{W}_1^{(4)} + \frac{8\sqrt{15}}{c} \mathcal{W}_4 \mathcal{W}_0^{(3)} \\
&\quad \left. + \frac{24}{c} \mathcal{L} \mathcal{W}_2^{(5)} \right) \tag{2.63}
\end{aligned}$$

This manifestly solve the field equation (4.7), and describe black holes solutions carrying not only mass and angular momentum but also canonical charges of the gravity. As discussed in Ref. [72], and comparison with e.g. eqn. (2.29) of Ref. [79] shows that the temporal component of the connection is linear in the chemical potentials μ_s 's. The same procedure can be followed to establish $\bar{a}$ with a constant chemical potential. If one set $\mu_3 = \mu_4 = \mu_5 = 0$ and $\mathcal{W}_s = 0$; ($s = 3, 4, 5$), this connection reduces to the usual *BTZ* black hole. This solution is therefore interpreted as a generalization of the *BTZ* black hole to include nonzero spin s : ($s = 3, 4, 5$) charges and related chemical potentials μ_3 , μ_4 and μ_5 .

2.2.4 Determining the $1/c$ -dependence of the structure constants

So far, the asymptotic symmetry algebra we obtained is not the quantum algebra because of the appearance of the nonlinear terms in the $\mathcal{W}_5$ algebra, but rather one semi-classical algebra. We know that this fits with the fact that classical limit takes $c \rightarrow \infty$. For this, we are interested in the quantum mechanical version of the asymptotic symmetry algebra, hence we take into account normal ordering effects. We also emphasize here that we know that the normal ordered version of the algebra then becomes inconsistent for finite values of the central charge c and thus has to be modified. So we have to lift the semi-classical results to quantum results and finally end up with the quantum $\mathcal{W}_5$ asymptotic symmetry algebra. Finally, a deformed version of usual semi-classical asymptotic symmetry algebra with new arbitrary structure constants can be obtained. Refs. [70, 71]. This will be the final form of the quantum asymptotic symmetry algebra of the AdS_3 higher spin gravity.

In the shorthand notation, all important structure constants $\mathcal{C}_{s_1 s_2}^{s_3}$'s appearing in the classical asymptotic $\mathcal{W}_5$ symmetry algebra as in Appendix C.3 are given by schematically

$$\begin{aligned}
\mathcal{W}_3 * \mathcal{W}_3 &\sim \frac{c}{3} \mathbf{1} + \mathcal{C}_{33}^4 \mathcal{W}_4 \\
\mathcal{W}_3 * \mathcal{W}_4 &\sim \mathcal{C}_{34}^3 \mathcal{W}_3 + \mathcal{C}_{34}^5 \mathcal{W}_5 \\
\mathcal{W}_4 * \mathcal{W}_4 &\sim \frac{c}{4} \mathbf{1} + \mathcal{C}_{44}^4 \mathcal{W}_4 + \mathcal{C}_{44}^{[33]} [\mathcal{W}_3 \mathcal{W}_3] \\
\mathcal{W}_3 * \mathcal{W}_5 &\sim \mathcal{C}_{35}^4 \mathcal{W}_4 + \mathcal{C}_{35}^{[33]} [\mathcal{W}_3 \mathcal{W}_3] \\
\mathcal{W}_4 * \mathcal{W}_5 &\sim \mathcal{C}_{45}^3 \mathcal{W}_3 + \mathcal{C}_{45}^5 \mathcal{W}_5 + \mathcal{C}_{45}^{[34]} [\mathcal{W}_3 \mathcal{W}_4] + \mathcal{C}_{45}^{[34]'} \partial [\mathcal{W}_3 \mathcal{W}_4] \\
\mathcal{W}_5 * \mathcal{W}_5 &\sim \frac{c}{5} \mathbf{1} + \mathcal{C}_{55}^4 \mathcal{W}_4 + \mathcal{C}_{55}^{[33]} [\mathcal{W}_3 \mathcal{W}_3] + \mathcal{C}_{55}^{[35]} [\mathcal{W}_3 \mathcal{W}_5] \\
&\quad + \mathcal{C}_{55}^{[44]} [\mathcal{W}_4 \mathcal{W}_4] + \mathcal{C}_{55}^{[33]''} \partial^2 [\mathcal{W}_3 \mathcal{W}_3]
\end{aligned}$$

where we only write out the contributions of the Virasoro primaries and the composite primary fields to the singular part of OPEs. We must also emphasize here that the structure constants $\mathcal{C}_{s_1 s_2}^{s_3}$'s displayed in (57) are sufficient to obtain all the remaining one, especially given that in the semiclassical case all structure constants have been displayed explicitly. Nevertheless, some important structure constants for classical and

quantum asymptotic $\mathcal{W}_5$ symmetry algebra are given by

$$\mathcal{C}_{34}^3 = \frac{3}{4}\mathcal{C}_{33}^4, \quad \mathcal{C}_{35}^4 = \frac{4}{5}\mathcal{C}_{34}^5, \quad \mathcal{C}_{45}^3 = \frac{3}{5}\mathcal{C}_{34}^5 = \frac{3}{4}\mathcal{C}_{35}^4, \quad \mathcal{C}_{55}^4 = \frac{4}{5}\mathcal{C}_{45}^5 \quad (2.64)$$

Table 2.1 : Some important structure constants for classical and quantum asymptotic $\mathcal{W}_5$ symmetry algebra with $1/c$ correction.

classical algebra		quantum algebra
$\mathcal{C}_{33}^4$	$\frac{32}{\sqrt{105}}$	$\sqrt{\frac{1024(c+2)(c+23)}{3(5c+22)(7c+68)}}$
$\mathcal{C}_{34}^5$	$\frac{5\sqrt{15}}{7}$	$\sqrt{\frac{25(3c+116)(5c+22)}{(7c+68)(7c+114)}}$
$\mathcal{C}_{45}^5$	$\frac{11}{14} \frac{\sqrt{15}}{7}$	$\sqrt{\frac{75(11c^3+204c^2+9340c+70272)^2}{4(c+2)(c+23)(5c+22)(7c+68)(7c+114)^2}}$
$\mathcal{C}_{44}^{[33]}$	$\frac{45}{2c}$	$\frac{9(5c+22)}{2(c+2)(c+23)}$
$\mathcal{C}_{35}^{[33]}$	$\frac{24}{\sqrt{7}c}$	$\sqrt{\frac{432(2c-1)^2}{(c+2)(c+23)(3c+116)(7c+114)}}$
$\mathcal{C}_{45}^{[34]}$	$\frac{74}{\sqrt{7}c}$	$\sqrt{\frac{12(37c+334)^2}{(c+2)(c+23)(3c+116)(7c+114)}}$
$\mathcal{C}_{45}^{[34]}'$	$\frac{66}{\sqrt{7}c}$	$\sqrt{\frac{108(3c+116)}{(c+2)(c+23)(7c+114)}}$
$\mathcal{C}_{55}^{[33]}$	$\frac{181}{14c}$	$\frac{3(181c^3+14880c^2+248948c+1507824)}{2(c+2)(c+23)(3c+116)(7c+114)}$
$\mathcal{C}_{55}^{[44]}$	$\frac{448}{15c}$	$\frac{64(7c+114)}{(3c+116)(5c+22)}$

One can check as in the table 2.1 that if it can be taken into account the normal ordering effects in the quantum asymptotic $\mathcal{W}_5$ symmetry algebra, then the classical asymptotic $\mathcal{W}_5$ symmetry algebra can be obtained under $c \rightarrow \infty$ classical limit with $1/c$ correction as in the second column of the table 2.1, i.e $\mathcal{C}_{33}^4$

$$\begin{aligned} (\mathcal{C}_{33}^4)^2 &= \frac{1024(c+2)(c+23)}{3(5c+22)(7c+68)} \\ &= \frac{1024 + \frac{25600}{c} + \frac{47104}{c^2}}{105 + \frac{1482}{c} + \frac{4488}{c^2}} \end{aligned} \quad (2.65)$$

under $c \rightarrow \infty$ classical limit with $1/c$ correction

$$\mathcal{C}_{33}^4 = \frac{32}{\sqrt{105}} + \mathcal{O}\left(\frac{1}{c}\right), \quad (2.66)$$

which is compatible with table 2.1.

Finally, one can calculate the Operator Product Expansions of the quantum $\mathcal{W}_N$ asymptotic symmetry algebra by starting with the classical one of Chern-Simons theory

based on $\mathfrak{sl}(N, \mathbb{R})$ Lie algebra. The corrections for finite c in the nonlinear terms can also be determined recursively by solving some constraints for the Jacobi identities given in the Appendix C.3 and Ref. [70], especially for $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra.

2.2.5 Holonomies in $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern - Simons theory

The $\mathfrak{sl}(5, \mathbb{R})$ higher spin black hole solution will be characterised by independent global charges $\mathcal{W}_s$ and the parameters $\mu_s, (s = 2, 3, 4, 5)$ are the potentials conjugate to these charges. For $s = 2$, $\mathcal{W}_2$ is the stress tensor $\mathcal{L}$ in the boundary and μ_2 is the inverse temperature β . The charges are again recovered from the partition function

$$\mathcal{Z} = \text{tr} \left(e^{4\pi^2 i \sum_{s=2}^5 \mu_s \mathcal{W}_s} \right) \quad (2.67)$$

or taking the logarithmic of the partition function we find

$$\ln \mathcal{Z} = \mathcal{S} + 4\pi^2 i \sum_{s=2}^5 \mu_s \mathcal{W}_s \quad (2.68)$$

as

$$\mathcal{W}_s = -\frac{i}{4\pi^2} \frac{\partial \ln \mathcal{Z}}{\partial \mu_s}, \quad (2.69)$$

or equivalently

$$\mu_s = \frac{i}{4\pi^2} \frac{\partial \mathcal{S}}{\partial \mathcal{W}_s}, \quad (2.70)$$

A novel feature however, is that these expressions are seen to satisfy:

$$\frac{\partial \mathcal{W}_s}{\partial \mu_{s'}} = \frac{\partial \mathcal{W}_{s'}}{\partial \mu_s}, \quad s \neq s'. \quad (2.71)$$

This is called the *integrability conditions* which the black hole solution has to satisfy in order to associate to the black hole a partition function as in (B.10), and similar expressions for the barred sector. Another way to say this is that needs to be satisfied in order that (2.71) the thermodynamic quantities assigned to the black hole will obey the first law of thermodynamics.

As in Appendix B.1 for $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ case, let's recall the connection a_ϕ for $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern - Simons theory as:

$$a_\phi = \begin{pmatrix} 0 & \frac{Q_2}{10} & \frac{Q_3}{7\sqrt{6}} & \frac{Q_4}{24} - \frac{17Q_2^2}{1200} & \frac{Q_5}{24} - \frac{31Q_2Q_3}{840} \\ 2 & 0 & \frac{1}{10}\sqrt{\frac{3}{2}}Q_2 & \frac{Q_3}{14} & \frac{Q_4}{24} - \frac{17Q_2^2}{1200} \\ 0 & \sqrt{6} & 0 & \frac{1}{10}\sqrt{\frac{3}{2}}Q_2 & \frac{Q_3}{7\sqrt{6}} \\ 0 & 0 & \sqrt{6} & 0 & \frac{Q_2}{10} \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \quad (2.72)$$

depending on *Casimir operator*-related parameters $Q_s = \frac{1}{s} \text{tr}[a_\phi^s]$, ($s = 2, 3, 4, 5$) for $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra. The easiest way to solve the holonomy condition is by solving the characteristic polynomial. This means that one may express exponentiated matrix $w = 2\pi a_\phi$, a general 5×5 matrix as in (2.33)

$$w^5 = \Theta_0 + \Theta_1 w + \Theta_2 w^2 + \Theta_3 w^3 + \Theta_4 w^4. \quad (2.73)$$

To find the *holonomy invariants* Θ 's one consider the general characteristic equation of exponentiated matrix w , explicitly

$$\lambda^5 = 4\pi^2 \lambda^3 Q_2 + 8\pi^3 \lambda^2 Q_3 - 8\pi^4 \lambda Q_2^2 + 16\pi^4 \lambda Q_4 - 32\pi^5 Q_2 Q_3 + 32\pi^5 Q_5 \quad (2.74)$$

If λ_i is i^{th} eigenvalue of the exponentiated matrix w , and identify the *holonomy invariants* in (2.73) as

$$\begin{aligned} \Theta_0 &= \det[w] \\ &= 32\pi^5 (Q_5 - Q_2 Q_3) \\ &= \sum_{i < j < k < l < m} \lambda_i \lambda_j \lambda_k \lambda_l \lambda_m \end{aligned} \quad (2.75)$$

$$\begin{aligned} \Theta_1 &= \frac{1}{4} \text{tr}[w^4] - \frac{1}{8} \text{tr}[w^2]^2 \\ &= 8\pi^4 (2Q_4 - Q_2^2) \\ &= - \sum_{i < j < k < l} \lambda_i \lambda_j \lambda_k \lambda_l \end{aligned} \quad (2.76)$$

$$\begin{aligned} \Theta_2 &= -\frac{1}{3} \text{tr}[w^3] \\ &= 8\pi^3 Q_3 \\ &= \sum_{i < j < k} \lambda_i \lambda_j \lambda_k \end{aligned} \quad (2.77)$$

$$\begin{aligned} \Theta_3 &= -\frac{1}{2} \text{tr}[w^2] \\ &= 4\pi^2 Q_2 \\ &= - \sum_{i < j} \lambda_i \lambda_j \end{aligned} \quad (2.78)$$

$$\begin{aligned} \Theta_4 &= \text{tr}[w] \\ &= 0 \\ &= \sum_i \lambda_i \end{aligned} \quad (2.79)$$

If a holonomy is trivial around a contractible cycle the exponentiated matrix e^w must have eigenvalues $0, \pm 2\pi i, \pm 4\pi i$, it follows that

$$\begin{aligned}
\det[w] &= 0, \\
\text{tr}[w] &= 0, \\
\text{tr}[w^3] &= 0, \\
\text{tr}[w^5] &= 0, \\
\text{tr}[w^2] + 40\pi^2 &= 0, \\
\text{tr}[w^4] - 544\pi^4 &= 0.
\end{aligned} \tag{2.80}$$

Now when it comes to calculating the holonomy about contractible (thermal) cycle one may use the temporal component

$$a_t = \mu_2 a_\phi + \sum_{i=3}^5 \mu_i \left(a_\phi^{i-1} - \frac{1}{5} \text{tr}[a_\phi^{i-1}] \right) \tag{2.81}$$

which fullfills the equation of motion $[a_t, a_\phi] = 0$ it is obvious that $a_t = f(a_\phi)$. Therefore the trivial holonomy is

$$H_\tau(a) = b^{-1} e^w b, \tag{2.82}$$

where

$$w = \oint_C a_t dt = 2\pi\tau a_t \tag{2.83}$$

is the holonomy matrix with $\tau = \frac{i\beta}{2\pi}$.

One can also show that this holonomy matrix gives the same holonomy conditions (B.5) for the time circle with $w = 2\pi\tau a_t$ and similar expressions for the barred sector. Thus our final goal might be to solve the holonomy conditions to see whether this gives a consistent thermodynamics of the black hole with the phase structure. Therefore, the six equations of the holonomy conditions (B.5) can be explicitly written in terms of the eight parameters of the connection (2.69-2.70). To do this, the entropy function can be written in the form

$$\mathcal{S}(Q_2, Q_3, Q_4, Q_5) = 2\pi k \sqrt{Q_2} F(x, y, z) \tag{2.84}$$

with $F(0, 0, 0) = 1$, such that

$$\mu_s = i \frac{1}{2\pi k} \frac{\partial \mathcal{S}}{\partial Q_s}, \quad (s = 2, 3, 4, 5) \tag{2.85}$$

where we define three dimensionless parameters $x = \frac{Q_3}{Q_2^{3/2}}$, $y = \frac{Q_4}{Q_2^2}$ and $z = \frac{Q_5}{Q_2 Q_3}$. Then the simplest holonomy equation becomes

$$\begin{aligned}
1 = & F^2 - (3xF_x + 4yF_y + 2zF_z)^2 \\
& + \frac{8}{5}(5y-1)F_x^2 + \frac{4}{5}x(25z-6)F_xF_y \\
& + \frac{2}{5}(6x^2 + 30y-5)F_y^2 \\
& + \frac{2}{5}(46y-50z^2 + 82z-35)F_yF_z \\
& + \frac{4}{5x}(15x^2 + y(22-20z) + 4z-5)F_xF_z \\
& + \frac{2}{5x^2}(10(x^2+1)z + 4(5y-1)z^2 - 44yz + 4y(y+5) - 5)F_z^2. \quad (2.86)
\end{aligned}$$

But it might be cumbersome or even impossible to solve these equations for now, since they're non-linear partial differential equations. Therefore, we are thinking that this calculation can make up the content of one future article, and now we do not go into detail. Finally, in this context, one can check the entropy for the $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Lie algebra valued Chern-Simons gauge theory, which should be given in full generally by the total boundary term as

$$\begin{aligned}
S &= -2\pi i k \operatorname{tr}(a_t a_\phi) \\
&= -2\pi i k \sum_{n=2}^5 n \mu_n Q_n \quad (2.87)
\end{aligned}$$

and similar expressions for the barred sector.

2.3 Summary and Conclusion

In this chapter, we first reviewed a relation between AdS_3 and $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern-Simons theory. The Chern-Simons formulation of AdS_3 allows for a straightforward generalization to a higher spin theory. The higher spin gauge fields have no propagating degree of freedom, but we noted that there are a large class of interesting non-trivial solution. Specifically, AdS_3 in the presence of a tower of higher spin fields up to spin-5 is obtained by enlarging $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$. Finally, classical $\mathcal{W}_5$ symmetry algebra as spin-5 asymptotic symmetry algebra is obtained.

As in the Sec. 2.2, the asymptotic $\mathcal{W}_5$ symmetry algebra we obtained is not the final form of the related algebra of the AdS_3 spin-5 gravity since it is just effectively for

the Virasoro central charge c in the large values. On the other hand, we also indicated how to introduce chemical potentials and holonomy conditions associated with these higher spin charges in higher spin AdS_3 gravity in a manner that it preserves the asymptotic symmetry algebra [78]. However, from CFT_2 point of view, it is not clear how the introductions of the chemical potentials through the time component can be interpreted. One can proceed entirely in the same way for AdS_3 spin-5 gravity and thus determine the asymptotic symmetry algebra as quantum $\mathcal{W}_5$ algebra instead of rather than semiclassical one. Therefore, we can take into account normal ordering effects of the quantum $\mathcal{W}_5$ algebra as in Ref. [70]. So far we only discussed the classical $\mathcal{W}_5$ symmetry. One may wonder, whether the quantum case also admits a description in terms of an classical $\mathcal{W}_N$ symmetry, or a quantum version of it. These questions seem to fairly non-trivial, but it seems that the procedure does go through straightforwardly.



3. EXPLORING NEW BOUNDARY CONDITIONS FOR $N=(1,1)$ EXTENDED HIGHER-SPIN AdS_3 SUPERGRAVITY

The most common asymptotic symmetries of AdS_3 gravity with a negative cosmological constant in $3D$ are known as two copies of the Virasoro algebras and this has been written first by Brown and Henneaux in their seminal paper [6]. So this work is well known as both a pioneer of AdS_3/CFT_2 correspondence [1] and also a realization of the holographic principle [83]. One of the biggest breakthroughs in theoretical physics in the past few decades is undoubtedly the discovery of the AdS/CFT correspondence describing an equivalence between the Einstein gravity and a large N gauge field theory.

The pure Einstein gravity in this context is simply a Chern - Simons gauge theory, that is, it is rewritten as a gauge field theory, in such a way that the structure simplifies substantially. This recalls us that there are no local propagating degrees of freedom in the theory, and hence no graviton in three - dimensions. Therefore, this gauge theory is said purely topological and only global effects are of physical relevance. Finally, one must emphasize here that the dynamics of the theory is controlled entirely by the boundary conditions, because its dynamical content is far from insignificant due to the existence of boundary conditions. This fact was first discovered by Achucarro and Townsend [2], and subsequently developed by Witten [3]. The things found here is that the gravity action in three - dimensions and equations of motions are in the same class with a Chern - Simons theory for a suitable gauge group. Under a convenient choice of boundary conditions, there is an infinite number of degrees of freedom living on the boundary. These boundary conditions are required, but these conditions are not unique in the selections. The dynamical properties of the theory are also highly sensitive to the selection of these boundary conditions. Thus, the situations of asymptotic symmetries as the residual global gauge symmetries emerge.

In the case of AdS_3 gravity above with a negative cosmological constant in $3D$, the most famous of these boundary conditions is worked in the same paper [6]. These boundary conditions also contain BTZ black holes [4,5]. Besides, the Chern - Simons higher - spin

theories are purely bosonic theories [46, 47] as versions of Vasiliev higher - spin theories [28, 29] with higher - spin fields of integer spin, they are based on the $\mathfrak{sl}(N, \mathbb{R})$ algebras and higher - spin algebras $\mathfrak{hs}(\lambda)$ respectively. The Chern - Simons higher - spin theories can also be resulted with a realization of the classical W_N asymptotic symmetry algebras as in the related two - dimensional CFT's [38–43]. The validity of these results can be extended to supergravity theory [6, 44, 45], as well as a higher - spin theory [46, 47]. Beyond that a supersymmetric generalization of these bosonic theories can be achieved by considering Chern - Simons theories based on superalgebras such as $\mathfrak{sl}(N|N - 1)$, see e.g [44, 45, 48–51], or $\mathfrak{osp}(N|N - 1)$ [52] which can be obtained by truncating out all the odd spin generators and one copy of the fermionic operators in $\mathfrak{sl}(N|N - 1)$.

The critical role of boundary conditions in field theories and in particular in gravity theories has been well understood so far. In three dimensions, as previously mentioned, asymptotic AdS_3 boundary conditions by Brown and Henneaux [6] provided an important precursor to the AdS_3/CFT_2 . For a long time, these boundary conditions have been changed (see, eg [14, 15, 17–21]) and generalized (see, eg [7–13, 16]) many times. Our motivation is to construct a candidate solution for the most general $\mathcal{N} = (1, 1)$ extended higher - spin supergravity theory in AdS_3 . Our theory falls under the same metric class as the recently constructed most general AdS_3 boundary condition by Grumiller and Riegler [21]. They showed that all charges and chemical potentials that appear in the Chern-Simons formulation can also be seen in the metric formulation. It is also mentioned that the possibility to obtain asymptotic symmetries that are the affine version of the gauge algebras of the Chern-Simons theory is not a surprise given the relation between Chern-Simons theories and Wess-Zumino-Witten models [84]. If one examines the correspondence between 2+1 dimensional pure Chern-Simons gauge theories and 1+1 dimensional Wess-Zumino-Witten conformal field theories, it can be seen that this is the core of the relationship between these theories. This method recently has also been applied to flat-space [22] and chiral higher - spin gravity [23] which showed a new class of boundary conditions for higher - spin theories in AdS_3 . The simplest extension of the Grumiller and Riegler's analysis for the most general $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 2)$ extended higher spin supergravity theory in AdS_3 is the Valcárcel's paper [85] where the asymptotic symmetry algebra for the loosest set of boundary

conditions for (extended) supergravity has been obtained. This is an alternative to the non - chiral Drinfeld - Sokolov type boundary conditions. In particular, we first focus on the simplest example, $\mathcal{N} = (1, 1)$ Chern - Simons theory based on the superalgebra $\mathfrak{osp}(1|2)$. The related asymptotic symmetry algebra is two copies of the $\mathfrak{osp}(1|2)_k$ affine algebra. Then one can extend this study to the $\mathcal{N} = (1, 1)$ Chern - Simons theory based on the superalgebra $\mathfrak{osp}(3|2)$ and the symmetry algebras are given by two copies of the $\mathfrak{osp}(3|2)_k$ affine algebra. Furthermore, we also impose certain restrictions on the gauge fields on the most general boundary conditions and that leads to the supersymmetric extensions of the Brown - Henneaux boundary conditions. From these restrictions, we see that the asymptotic symmetry algebras reduce to two copies of the $\mathcal{SW}(\frac{3}{2}, 2)$ algebra for the most general $\mathcal{N} = (1, 1)$ extended higher - spin supergravity theory in AdS_3 . It will be also very interesting to show another class of boundary conditions that appeared in the literature (see, eg [15, 18–20]) for (super)gravity case, whose higher - spin generalization is less clear than the Grumiller - Riegler boundary conditions. Therefore, one can think that this method provides a good laboratory for investigating the rich asymptotic structure of extended supergravity.

The outline of this chapter is as follows. In the next section, we give briefly a fundamental formulation of $\mathcal{N} = (1, 1)$ supergravity in the perspective of $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ Chern - Simons gauge theory for both affine and superconformal boundaries respectively in three-dimensions. In section 3.2, we carry out our calculations to extend the theory to $\mathfrak{osp}(3|2) \oplus \mathfrak{osp}(3|2)$ higher - spin Chern - Simons supergravity in the presence of both affine and superconformal boundaries, in where we showed explicitly principal embedding of $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ and also demonstrated how asymptotic symmetry and higher - spin Ward identities arise from the bulk equations of motion coupled to spin s , ($s = \frac{3}{2}, 2, 2, \frac{5}{2}$) currents. This section is devoted to the case of classical two copies of the $\mathfrak{osp}(3|2)_k$ affine algebra on the affine boundary and $\mathcal{SW}(\frac{3}{2}, 2)$ symmetry algebra on the superconformal boundary as asymptotic symmetry algebras, and also the chemical potentials related to source fields appearing through the temporal components of the connection are obtained. In section 3.3, we present two most general boundary conditions for pure bosonic gravity and $\mathcal{N} = 1$ extended supergravity. That is, The Avery - Poojary - Suryanarayana gravity for $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)$ and $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)$ respectively with some further checks. On the other hand, it is

shown that the Chern - Simons action which compatible with our boundary conditions leads to a finite action and a well - defined variational principle for the higher - spin fields. The final section contains our summary and conclusion.

3.1 Supergravity in Three Dimensions, a Review:

In this section, we review the Chern - Simons formalism for higher - spin supergravity. In particular, we use this formalism to study supergravity in the $\mathfrak{osp}(1|2)$ superalgebra basis under the same metric class as the recently constructed most general AdS_3 boundary condition by Grumiller and Riegler [21].

3.1.1 Connection to Chern - Simons theory

The three - dimensional Einstein - Hilbert action for $\mathcal{N} = (1, 1)$ supergravity, which is defined on any manifold $\mathcal{M}$, with a negative cosmological constant is classically equivalent to the Chern - Simons action, as it was first proposed by Achucarro and Townsend in [2] and developed by Witten in [3]. One can start by defining 1 - forms $(\Gamma, \bar{\Gamma})$ taking values in the gauge group's $\mathfrak{osp}(1|2)$ superalgebra, and also the supertrace str is taken over the superalgebra generators. The Chern - Simons action can be written in the form,

$$S = S_{\text{cs}}[\Gamma] - S_{\text{cs}}[\bar{\Gamma}] \quad (3.1)$$

where

$$S_{\text{cs}}[\Gamma] = \frac{k}{4\pi} \int_{\mathcal{M}} \langle \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \rangle. \quad (3.2)$$

Here $k = \frac{\ell}{4G\text{str}(\mathbf{L}_0\mathbf{L}_0)} = \frac{c}{6\text{str}(\mathbf{L}_0\mathbf{L}_0)}$ is the level of the Chern - Simons theory depending on the AdS radius ℓ and Newton's constant G with the related central charge c of the superconformal field theory. If $L_i, (i = \pm 1, 0)$ and $G_p, (p = \pm \frac{1}{2})$ are the generators of $\mathfrak{osp}(1|2)$ superalgebra. We have expressed $\mathfrak{osp}(1|2)$ superalgebra such that

$$[L_i, L_j] = (i - j)L_{i+j}, \quad [L_i, G_p] = \left(\frac{i}{2} - p\right)G_{i+p}, \quad \{G_p, G_q\} = -2L_{p+q}. \quad (3.3)$$

The equations of motion for the Chern - Simons gauge theory give the flatness condition $F = \bar{F} = 0$ where

$$F = d\Gamma + \Gamma \wedge \Gamma = 0 \quad (3.4)$$

is the same as Einstein's equation. Γ and $\bar{\Gamma}$ are related to the metric $g_{\mu\nu}$ through the veilbein $e = \frac{\ell}{2}(\Gamma - \bar{\Gamma})$

$$g_{\mu\nu} = \frac{1}{2} \text{str}(e_\mu e_\nu). \quad (3.5)$$

One can choose a radial gauge of the form

$$\Gamma = b^{-1}a(t, \phi)b + b^{-1}db, \quad \bar{\Gamma} = b\bar{a}(t, \phi)b^{-1} + bdb^{-1} \quad (3.6)$$

with state-independent group element as [21]

$$b(\rho) = e^{\text{L}_{-1}} e^{\rho \text{L}_0} \quad (3.7)$$

which manifests all the $\mathfrak{osp}(1|2)$ charges and chemical potentials and also the choice of b is irrelevant in the case of asymptotic symmetry, as long as $\delta b = 0$. Moreover, $a(t, \phi)$ and $\bar{a}(t, \phi)$ in the radial gauge are the $\mathfrak{osp}(1|2)$ superalgebra valued fields, which are independent from the radial coordinate, ρ as

$$a(t, \varphi) = a_t(t, \varphi)dt + a_\varphi(t, \varphi)d\varphi. \quad (3.8)$$

3.1.2 $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ Chern-Simons $\mathcal{N} = (1, 1)$ supergravity for affine boundary

The affine case is given by reviewing asymptotically AdS_3 boundary conditions for a $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ Chern-Simons theory, and how to determine the asymptotic symmetry algebra using the method described in [21]. Thus the most general solution of Einstein's equation that is asymptotically AdS_3 , as a generalization of Fefferman-Graham method is given by with a flat boundary metric

$$\begin{aligned} ds^2 = & d\rho^2 + 2 \left[e^\rho N_i^{(0)} + N_i^{(1)} + e^{-\rho} N_i^{(2)} + \mathcal{O}(e^{-2\rho}) \right] d\rho dx^i \\ & + \left[e^{2\rho} g_{ij}^{(0)} + e^\rho g_{ij}^{(1)} + g_{ij}^{(2)} + \mathcal{O}(e^{-\rho}) \right] dx^i dx^j. \end{aligned} \quad (3.9)$$

We need to choose the most general boundary conditions for $\mathcal{N} = (1, 1)$ supergravity such that they maintain this metric form. In the following, we only focus on the Γ -sector. Therefore, one can propose to write the components of the $\mathfrak{osp}(1|2)$ superalgebra valued connection in the form,

$$a_\varphi(t, \varphi) = \sum_{i=-1}^{+1} \alpha_i \mathcal{L}^i(t, \varphi) \text{L}_i + \sum_{p=-1/2}^{+1/2} \beta_p \mathcal{G}^p(t, \varphi) \text{G}_p \quad (3.10)$$

where (α_i, β_p) 's are some scaling parameters to be determined later and we have five functions: three bosonic $\mathcal{L}^i$ and two fermionic $\mathcal{G}^p$. They are usually called *charges* and also the time component of the connection $a(t, \varphi)$

$$a_t(t, \varphi) = \sum_{i=-1}^{+1} \mu^i(t, \varphi) L_i + \sum_{p=-1/2}^{+1/2} v^p(t, \varphi) G_p. \quad (3.11)$$

Here, the time component has in total of five independent functions (μ^i, v^p) . They are usually called *chemical potentials*. But, they are not allowed to vary

$$\delta a_\varphi = \sum_{i=-1}^{+1} \alpha_i \delta \mathcal{L}^i L_i + \sum_{p=-1/2}^{+1/2} \beta_p \delta \mathcal{G}^p G_p, \quad (3.12)$$

$$\delta a_t = 0. \quad (3.13)$$

The flat connection conditions (4.7) for fixed chemical potentials impose the following additional conditions as the temporal evolution of the five independent source fields $(\mathcal{L}^i, \mathcal{G}^p)$ as

$$\alpha_{\pm 1} \partial_t \mathcal{L}^{\pm 1} = \partial_\varphi \mu^{\pm 1} \mp \alpha_0 \mathcal{L}^0 \mu^{\pm 1} \pm \alpha_{\pm 1} \mathcal{L}^{\pm 1} \mu^0 - 2\beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} v^{\pm \frac{1}{2}}, \quad (3.14)$$

$$\begin{aligned} \frac{1}{2} \alpha_0 \partial_t \mathcal{L}^0 &= \frac{1}{2} \partial_\varphi \mu^0 + \alpha_{+1} \mathcal{L}^{+1} \mu^{-1} - \alpha_{-1} \mathcal{L}^{-1} \mu^{+1} - \beta_{+\frac{1}{2}} \mathcal{G}^{+\frac{1}{2}} v^{-\frac{1}{2}} \\ &\quad - \beta_{-\frac{1}{2}} \mathcal{G}^{-\frac{1}{2}} v^{+\frac{1}{2}}, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \beta_{\pm \frac{1}{2}} \partial_t \mathcal{G}^{\pm \frac{1}{2}} &= \partial_\varphi v^{\pm \frac{1}{2}} \pm \alpha_{\pm 1} \mathcal{L}^{\pm 1} v^{\mp \frac{1}{2}} \mp \frac{1}{2} \alpha_0 \mathcal{L}^0 v^{\pm \frac{1}{2}} \mp \beta_{\mp \frac{1}{2}} \mu^{\pm 1} \mathcal{G}^{\mp \frac{1}{2}} \\ &\quad \pm \frac{1}{2} \beta_{\pm \frac{1}{2}} \mu^0 \mathcal{G}^{\pm \frac{1}{2}}. \end{aligned} \quad (3.16)$$

After the temporal evolution of the source fields, one can start to compute the gauge transformations for asymptotic symmetry algebra by considering all transformations

$$\delta_\lambda \Gamma = d\lambda + [\Gamma, \lambda] \quad (3.17)$$

that preserve the boundary conditions with the gauge parameter in the $\mathfrak{osp}(1|2)$ superalgebra

$$\lambda = b^{-1} \left[\sum_{i=-1}^{+1} \varepsilon^i(t, \varphi) L_i + \sum_{p=-1/2}^{+1/2} \zeta^p(t, \varphi) G_p \right] b. \quad (3.18)$$

Here, the gauge parameter has in total of five arbitrary functions : bosonic ε^i and fermionic ζ^p on the boundary. The condition (4.19) impose that transformations on

the gauge are given by

$$\alpha_{\pm 1} \delta_\lambda \mathcal{L}^{\pm 1} = \partial_\varphi \varepsilon^{\pm 1} \mp \alpha_0 \mathcal{L}^0 \varepsilon^{\pm 1} \pm \alpha_{\pm 1} \mathcal{L}^{\pm 1} \varepsilon^0 - 2\beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \zeta^{\pm \frac{1}{2}}, \quad (3.19)$$

$$\begin{aligned} \frac{1}{2} \alpha_0 \delta_\lambda \mathcal{L}^0 &= \frac{1}{2} \partial_\varphi \varepsilon^0 + \alpha_{+1} \mathcal{L}^{+1} \varepsilon^{-1} - \alpha_{-1} \mathcal{L}^{-1} \varepsilon^{+1} - \beta_{+\frac{1}{2}} \mathcal{G}^{+\frac{1}{2}} \zeta^{-\frac{1}{2}} \\ &\quad - \beta_{-\frac{1}{2}} \mathcal{G}^{-\frac{1}{2}} \zeta^{+\frac{1}{2}}, \end{aligned} \quad (3.20)$$

$$\begin{aligned} \beta_{\pm \frac{1}{2}} \delta_\lambda \mathcal{G}^{\pm \frac{1}{2}} &= \partial_\varphi \zeta^{\pm \frac{1}{2}} \pm \alpha_{\pm 1} \mathcal{L}^{\pm 1} \zeta^{\mp \frac{1}{2}} \mp \frac{1}{2} \alpha_0 \mathcal{L}^0 \zeta^{\pm \frac{1}{2}} \mp \beta_{\mp \frac{1}{2}} \varepsilon^{\pm 1} \mathcal{G}^{\mp \frac{1}{2}} \\ &\quad \pm \frac{1}{2} \beta_{\pm \frac{1}{2}} \varepsilon^0 \mathcal{G}^{\pm \frac{1}{2}}. \end{aligned} \quad (3.21)$$

With analogical reasoning, they obey the following transformations

$$\alpha_{\pm 1} \delta_\lambda \mu^{\pm 1} = \partial_t \varepsilon^{\pm 1} \mp \alpha_0 \mu^0 \varepsilon^{\pm 1} \pm \alpha_{\pm 1} \mu^{\pm 1} \varepsilon^0 - 2\beta_{\pm \frac{1}{2}} \nu^{\pm \frac{1}{2}} \zeta^{\pm \frac{1}{2}}, \quad (3.22)$$

$$\begin{aligned} \frac{1}{2} \alpha_0 \delta_\lambda \mu^0 &= \frac{1}{2} \partial_t \varepsilon^0 + \alpha_{+1} \mu^{+1} \varepsilon^{-1} - \alpha_{-1} \mu^{-1} \varepsilon^{+1} - \beta_{+\frac{1}{2}} \nu^{+\frac{1}{2}} \zeta^{-\frac{1}{2}} \\ &\quad - \beta_{-\frac{1}{2}} \nu^{-\frac{1}{2}} \zeta^{+\frac{1}{2}}, \end{aligned} \quad (3.23)$$

$$\begin{aligned} \beta_{\pm \frac{1}{2}} \delta_\lambda \nu^{\pm \frac{1}{2}} &= \partial_t \zeta^{\pm \frac{1}{2}} \pm \alpha_{\pm 1} \mu^{\pm 1} \zeta^{\mp \frac{1}{2}} \mp \frac{1}{2} \alpha_0 \mu^0 \zeta^{\pm \frac{1}{2}} \mp \beta_{\mp \frac{1}{2}} \varepsilon^{\pm 1} \nu^{\mp \frac{1}{2}} \\ &\quad \pm \frac{1}{2} \beta_{\pm \frac{1}{2}} \varepsilon^0 \nu^{\pm \frac{1}{2}}. \end{aligned} \quad (3.24)$$

As a final step, one now has to determine the canonical boundary charge $\mathcal{Q}[\lambda]$ that generates the transformations (3.19)-(3.21). Therefore, the corresponding variation of the boundary charge $\mathcal{Q}[\lambda]$ [81, 86–88], to show the asymptotic symmetry algebra, is given by

$$\delta_\lambda \mathcal{Q} = \frac{k}{2\pi} \int d\varphi \, \text{str}(\lambda \delta \Gamma_\varphi). \quad (3.25)$$

The canonical boundary charge $\mathcal{Q}[\lambda]$ can be integrated which reads

$$\mathcal{Q}[\lambda] = \int d\varphi \left[\mathcal{L}^0 \varepsilon^0 + \mathcal{L}^{+1} \varepsilon^{-1} + \mathcal{L}^{-1} \varepsilon^{+1} + \mathcal{G}^{+\frac{1}{2}} \zeta^{-\frac{1}{2}} + \mathcal{G}^{-\frac{1}{2}} \zeta^{+\frac{1}{2}} \right]. \quad (3.26)$$

We now prefer to work in complex coordinates for the affine boundary, $z(\bar{z}) \equiv \varphi \pm i \frac{t}{\ell}$. After both the infinitesimal transformations and the canonical boundary charge have been determined, one can yield the Poisson bracket algebra by using the methods [89] with

$$\delta_\lambda F = \{F, \mathcal{Q}[\lambda]\} \quad (3.27)$$

for any phase space functional F :

$$\begin{aligned} \{\mathcal{L}^i(z_1), \mathcal{L}^j(z_2)\}_{PB} &= (i-j)\mathcal{L}^{i+j}(z_2)\delta(z_1-z_2) \\ &\quad - \frac{1}{\alpha_{+1}}\eta^{ij}\partial_\varphi\delta(z_1-z_2), \end{aligned} \quad (3.28)$$

$$\{\mathcal{L}^i(z_1), \mathcal{G}^p(z_2)\}_{PB} = \left(\frac{i}{2} - p\right)\mathcal{G}^{i+p}(z_2)\delta(z_1-z_2), \quad (3.29)$$

$$\begin{aligned} \{\mathcal{G}^p(z_1), \mathcal{G}^q(z_2)\}_{PB} &= -2\mathcal{L}^{p+q}(z_2)\delta(z_1-z_2) \\ &\quad - \frac{1}{\alpha_{+1}}\eta^{pq}\partial_\varphi\delta(z_1-z_2) \end{aligned} \quad (3.30)$$

where $\alpha_{+1} = -\frac{2\pi}{k}$ for the convention in the literature and η^{ij} is the bilinear form in the fundamental representation of $\mathfrak{osp}(1|2)$ superalgebra. One can also expand $\mathcal{L}(z)$ and $\mathcal{G}(z)$ charges into Fourier modes $\mathcal{L}^i(z) = \frac{1}{2\pi}\sum_n L_n^i z^{-n-1}$, and $\mathcal{G}^p(z) = \frac{1}{2\pi}\sum_r G_r^p z^{-r-\frac{1}{2}}$, and also replacing $i\{\cdot, \cdot\}_{PB} \rightarrow [\cdot, \cdot]$. A mode algebra can then be defined as:

$$[L_m^i, L_n^j] = (i-j)L_{m+n}^{i+j} + kn\eta^{ij}\delta_{m+n,0}, \quad (3.31)$$

$$[L_m^i, G_r^p] = \left(\frac{i}{2} - p\right)G_{m+r}^{i+p}, \quad (3.32)$$

$$\{G_r^p, G_s^q\} = -2L_{r+s}^{p+q} + ks\kappa^{pq}\delta_{r+s,0}. \quad (3.33)$$

Besides, this mode algebra in this space is equivalent to operator product algebra,

$$\mathcal{L}^i(z_1)\mathcal{L}^j(z_2) \sim \frac{\frac{k}{2}\eta^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}}\mathcal{L}^{i+j}(z_2), \quad (3.34)$$

$$\mathcal{L}^i(z_1)\mathcal{G}^p(z_2) \sim \frac{(\frac{i}{2} - p)}{z_{12}}\mathcal{G}^{i+p}(z_2), \quad (3.35)$$

$$\mathcal{G}^p(z_1)\mathcal{G}^q(z_2) \sim \frac{\frac{k}{2}\eta^{pq}}{z_{12}^2} + \frac{2}{z_{12}}\mathcal{L}^{p+q}(z_2) \quad (3.36)$$

where $z_{12} = z_1 - z_2$, or in the more compact form,

$$\mathfrak{J}^A(z_1)\mathfrak{J}^B(z_2) \sim \frac{\frac{k}{2}\eta^{AB}}{z_{12}^2} + \frac{\mathfrak{f}_{AB}^C\mathfrak{J}^C(z_2)}{z_{12}}. \quad (3.37)$$

Here, η^{AB} is the supertrace matrix and $\mathfrak{f}_C^{AB}$'s are the structure constants of the related algebra with $(A, B = 0, \pm 1, \pm \frac{1}{2})$, i.e., $\eta^{ip} = 0$ and $\mathfrak{f}_{i+j}^{ij} = (i-j)$. After repeating the same algebra for $\bar{\Gamma}$ -sector, one can say that the asymptotic symmetry algebra for the most general boundary conditions of $\mathcal{N} = (1, 1)$ supergravity is two copies of the affine $\mathfrak{osp}(1|2)_k$ algebra as in Ref. [85].

3.1.3 $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ Chern - Simons $\mathcal{N} = (1, 1)$ supergravity for superconformal boundary

Under the following restrictions as the Drinfeld - Sokolov highest weight gauge condition,

$$\mathcal{L}^0 = \mathcal{G}^{+\frac{1}{2}} = 0, \quad \mathcal{L}^{-1} = \mathcal{L}, \quad \mathcal{G}^{-\frac{1}{2}} = \mathcal{G}, \quad \alpha_{+1}\mathcal{L}^{+1} = 1 \quad (3.38)$$

on the boundary conditions with the $\mathfrak{osp}(1|2)$ superalgebra valued connection (4.13), one can get the superconformal boundary conditions as the supersymmetric extension of the Brown - Henneaux boundary conditions proposed in [6] for AdS_3 supergravity. Therefore we have the supersymmetric connection as,

$$a_\varphi = L_1 + \alpha_{-1}\mathcal{L}L_{-1} + \beta_{-\frac{1}{2}}\mathcal{G}G_{-\frac{1}{2}}, \quad (3.39)$$

$$a_t = \mu L_1 + \sum_{i=-1}^0 \mu^i L_i + \nu G_{\frac{1}{2}} + \nu^{-\frac{1}{2}} G_{-\frac{1}{2}} \quad (3.40)$$

where $\mu \equiv \mu^{+1}$ and $\nu \equiv \nu^{+\frac{1}{2}}$ can be interpreted as an independent *chemical potentials*. This means that we assume the chemical potential to be fixed at infinity, i.e. $\delta\mu = 0$. The functions μ^0, μ^{-1} and $\nu^{-\frac{1}{2}}$ are fixed by the flatness condition (4.7) as

$$\mu^0 = -\mu', \quad (3.41)$$

$$\mu^{-1} = \frac{1}{2}\mu'' + \alpha_{-1}\mathcal{L}\mu + \beta_{-\frac{1}{2}}\mathcal{G}\nu, \quad (3.42)$$

$$\nu^{-\frac{1}{2}} = -\nu' + \beta_{-\frac{1}{2}}\mathcal{G}\nu. \quad (3.43)$$

For the fixed chemical potentials μ and ν , the time evolution of canonical boundary charges $\mathcal{L}$ and $\mathcal{G}$ can be written as

$$\partial_t \mathcal{L} = -\frac{\mu'''}{2\alpha_{-1}} + 2\mathcal{L}\mu' + \mathcal{L}'\mu + \frac{\beta_{-\frac{1}{2}}}{\alpha_{-1}} \left(3\mathcal{G}\nu' + \mathcal{G}'\nu \right), \quad (3.44)$$

$$\partial_t \mathcal{G} = -\frac{\nu''}{\alpha_{-1}} - \frac{\alpha_{-1}}{\beta_{-\frac{1}{2}}}\mathcal{L}\nu + \frac{3}{2}\mu'\mathcal{G} + \mu\mathcal{G}' \quad (3.45)$$

where α_{-1} and $\beta_{-\frac{1}{2}}$ are some scaling parameters to be determined later. Now, we are in a position to work the superconformal asymptotic symmetry algebra under the Drinfeld - Sokolov reduction. This reduction implies that the only independent parameters $\varepsilon \equiv \varepsilon^{+1}$, and $\zeta \equiv \zeta^{+\frac{1}{2}}$. One can start to compute the gauge transformations for asymptotic symmetry algebra by considering all transformations (4.19) that preserve

the boundary conditions. with the gauge parameter in the $\mathfrak{osp}(1|2)$ superalgebra (4.20) as

$$\begin{aligned} \lambda = & b^{-1} \left[\varepsilon L_1 - \varepsilon' L_0 + \left(\frac{1}{2} \varepsilon'' + \alpha_{-1} \mathcal{L} \varepsilon + \beta_{-\frac{1}{2}} \mathcal{G} \zeta \right) L_{-1} + \zeta G_{+\frac{1}{2}} \right. \\ & \left. + \left(\beta_{-\frac{1}{2}} \mathcal{G} \varepsilon - \zeta' \right) G_{-\frac{1}{2}} \right] b. \end{aligned} \quad (3.46)$$

The flat connection condition (4.19) with $\alpha_{-1} = \frac{6}{c}$ and $\beta_{-\frac{1}{2}} = -\frac{3}{c}$ becomes:

$$\delta_\lambda \mathcal{L} = \frac{c}{12} \varepsilon''' + 2\mathcal{L} \varepsilon' + \mathcal{L}' \varepsilon - \frac{1}{2} \left(3\mathcal{G} \zeta' + \mathcal{G}' \zeta \right), \quad (3.47)$$

$$\delta_\lambda \mathcal{G} = \frac{c}{3} \zeta'' + 2\mathcal{L} \zeta + \frac{3}{2} \varepsilon' \mathcal{G} + \varepsilon \mathcal{G}'. \quad (3.48)$$

The variation of canonical boundary charge $\mathcal{Q}[\lambda]$ (4.29) can be integrated which reads

$$\mathcal{Q}[\lambda] = \int d\varphi [\mathcal{L} \varepsilon + \mathcal{G} \zeta]. \quad (3.49)$$

This leads to operator product expansions in the complex coordinates by using (4.31)

$$\mathcal{L}(z_1) \mathcal{L}(z_2) \sim \frac{\frac{c}{2}}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}}, \quad (3.50)$$

$$\mathcal{L}(z_1) \mathcal{G}(z_2) \sim \frac{\frac{3}{2}\mathcal{G}}{z_{12}^2} + \frac{\mathcal{G}'}{z_{12}}, \quad (3.51)$$

$$\mathcal{G}(z_1) \mathcal{G}(z_2) \sim \frac{\frac{2c}{3}}{z_{12}^3} + \frac{2\mathcal{L}}{z_{12}}. \quad (3.52)$$

After repeating the same algebra for $\bar{\Gamma}$ - sector, one can say that the asymptotic symmetry algebra for a set of boundary conditions of $\mathcal{N} = (1, 1)$ supergravity is two copies of the super - Virasoro algebra with central charge $c = 6k$.

3.2 $\mathcal{N} = (1, 1)$ $\mathfrak{osp}(3|2) \oplus \mathfrak{osp}(3|2)$ Higher - spin Chern - Simons Supergravity

3.2.1 For affine boundary

In this section, we will construct an extension of the $\mathcal{N} = (1, 1)$ higher - spin supergravity theory. To this end, $\mathfrak{osp}(3|2)$ algebra is defined by using an higher - spin extension of $\mathfrak{osp}(1|2)$ algebra which can be presented as a sub - superalgebra as the ordinary case of $\mathfrak{osp}(N|M)$ gauge algebra. If $L_i (i = \pm 1, 0)$, $G_p (p = \pm \frac{1}{2})$, $A_i (i = \pm 1, 0)$, and

$S_p (p = \pm \frac{1}{2}, \pm \frac{3}{2})$ are the generators of $\mathfrak{osp}(3|2)$ superalgebra, we have expressed $\mathfrak{osp}(3|2)$ superalgebra such that

$$[L_i, L_j] = (i - j)L_{i+j}, \quad (3.53)$$

$$[L_i, G_p] = \left(\frac{i}{2} - p\right)G_{i+p}, \quad (3.54)$$

$$\{G_p, G_q\} = \sigma_1 A_{p+q} + \sigma_2 L_{p+q}, \quad (3.55)$$

$$\{G_p, S_q\} = \left(\frac{3p}{2} - \frac{q}{2}\right) \left(\sigma_3 A_{p+q} + \sigma_4 L_{p+q}\right), \quad (3.56)$$

$$[L_i, A_j] = \left(\frac{i}{2} - j\right)A_{i+j}, \quad (3.57)$$

$$[L_i, S_p] = \left(\frac{3i}{2} - p\right)S_{i+p}, \quad (3.58)$$

$$[A_i, A_j] = (i - j)(\sigma_5 L_{i+j} + \sigma_6 A_{i+j}), \quad (3.59)$$

$$[A_i, G_p] = \sigma_7 S_{i+p} + \sigma_8 \left(\frac{i}{2} - p\right)G_{i+p}, \quad (3.60)$$

$$[A_i, S_p] = \sigma_9 \left(\frac{3i}{2} - p\right)S_{i+p} + \sigma_{10} \left(3i^2 - 2ip + p^2 - \frac{9}{4}\right)G_{i+p}, \quad (3.61)$$

$$\{S_p, S_q\} = \left(3p^2 - 4pq + 3q^2 - \frac{9}{2}\right) \left(\sigma_{11} A_{p+q} + \sigma_{12} L_{p+q}\right). \quad (3.62)$$

The super Jacobi identities give us the nontrivial relations for some constants σ'_i , ($i = 1, 2, \dots, 12$) appearing on the RHS of eq.(3.53) – (3.62) as,

$$\begin{aligned} \sigma_1 &= 0, \\ \sigma_2 &= -\sigma_3 \sigma_7, \\ \sigma_4 &= -\sigma_3 \sigma_8, \\ \sigma_5 &= \frac{1}{2}(\sigma_8 - 2\sigma_9)(\sigma_8 + \sigma_9), \\ \sigma_6 &= \frac{1}{2}(5\sigma_9 - \sigma_8), \\ \sigma_{10} &= -\frac{(\sigma_8 - \sigma_9)^2}{4\sigma_7}, \\ \sigma_{11} &= -\frac{\sigma_3(\sigma_8 - \sigma_9)}{4\sigma_7}, \\ \sigma_{12} &= -\frac{\sigma_3 \sigma_9(\sigma_9 - \sigma_8)}{4\sigma_7}. \end{aligned} \quad (3.63)$$

For the corresponding algebra, the resulting relations are as $\sigma_3 = 2$, $\sigma_7 = -1$, $\sigma_8 = 0$, $\sigma_9 = 1$. We are now ready to formulate most general boundary conditions for

asymptotically AdS_3 spacetimes:

$$a_\varphi = \alpha_i \mathcal{L}^i \mathbb{L}_i + \gamma_i \mathcal{A}^i \mathbb{A}_i + \beta_p \mathcal{G}^p \mathbb{G}_p + \tau_p \mathcal{S}^p \mathbb{S}_p, \quad (3.64)$$

$$a_t = \mu^i \mathbb{L}_i + \chi^i \mathbb{A}_i + f^p \mathbb{G}_p + v^p \mathbb{S}_p \quad (3.65)$$

where $(\alpha_i, \gamma_i, \beta_p, \tau_p)$'s are some scaling parameters to be determined later and we have twelve functions: six bosonic $(\mathcal{L}^i, \mathcal{A}^i)$ and six fermionic $(\mathcal{G}^p, \mathcal{S}^p)$ as the *charges* and also here, the time component has in total of twelve independent functions $(\mu^i, \chi^i, f^p, v^p)$ as the *chemical potentials*.

The flat connection conditions (4.7) for fixed chemical potentials impose the following additional conditions as the temporal evolution of the twelve independent source fields, $(\mathcal{L}^i, \mathcal{A}^i, \mathcal{G}^p, \mathcal{S}^p)$ as

$$\begin{aligned} \alpha_{\pm 1} \partial_t \mathcal{L}^{\pm 1} &= \partial_\varphi \mu^{\pm 1} \pm \alpha_{\pm 1} \mu^0 \mathcal{L}^{\pm 1} \mp \alpha_0 \mu^{\pm 1} \mathcal{L}^0 + 2\beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} v^{\pm \frac{1}{2}} \\ &\quad \mp \mathcal{A}^{\pm 1} \gamma_{\pm 1} \chi^0 \pm \mathcal{A}^0 \gamma_0 \chi^{\pm 1} + 3\tau_{\mp \frac{1}{2}} f^{\pm \frac{3}{2}} \mathcal{S}^{\mp \frac{1}{2}} \\ &\quad - 2\tau_{\pm \frac{1}{2}} f^{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} + 3\tau_{\pm \frac{3}{2}} f^{\mp \frac{1}{2}} \mathcal{S}^{\pm \frac{3}{2}}, \end{aligned} \quad (3.66)$$

$$\begin{aligned} \frac{1}{2} \alpha_0 \partial_t \mathcal{L}^0 &= \frac{1}{2} \partial_\varphi \mu^0 + \alpha_{+1} \mu^{-1} \mathcal{L}^1 - \alpha_{-1} \mu^{+1} \mathcal{L}^{-1} + \beta_{+\frac{1}{2}} \mathcal{G}^{+\frac{1}{2}} v^{-\frac{1}{2}} \\ &\quad + \beta_{-\frac{1}{2}} \mathcal{G}^{-\frac{1}{2}} v^{+\frac{1}{2}} - \mathcal{A}^{+1} \gamma_{+1} \chi^{-1} + \mathcal{A}^{-1} \gamma_{-1} \chi^{+1} \\ &\quad + \frac{9}{2} \tau_{-\frac{3}{2}} f^{+\frac{3}{2}} \mathcal{S}^{-\frac{3}{2}} - \frac{1}{2} \tau_{-\frac{1}{2}} f^{+\frac{1}{2}} \mathcal{S}^{-\frac{1}{2}} - \frac{1}{2} \tau_{+\frac{1}{2}} f^{-\frac{1}{2}} \mathcal{S}^{+\frac{1}{2}} \\ &\quad + \frac{9}{2} \tau_{+\frac{3}{2}} f^{-\frac{3}{2}} \mathcal{S}^{+\frac{3}{2}}, \end{aligned} \quad (3.67)$$

$$\begin{aligned} g_{\pm \frac{1}{2}} \partial_t \mathcal{G}^{\pm \frac{1}{2}} &= \partial_\varphi v^{\pm \frac{1}{2}} \mp \frac{1}{2} \alpha_0 v^{\pm \frac{1}{2}} \mathcal{L}^0 \pm \alpha_{\pm 1} v^{\mp \frac{1}{2}} \mathcal{L}^{\pm 1} \mp \beta_{\mp \frac{1}{2}} \mathcal{G}^{\mp \frac{1}{2}} \mu^{\pm 1} \\ &\quad \pm \frac{1}{2} \beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \mu^0 + \frac{3}{2} \mathcal{A}^{\mp 1} \gamma_{\mp 1} f^{\pm \frac{3}{2}} - \frac{1}{2} \mathcal{A}^0 \gamma_0 f^{\pm \frac{1}{2}} \\ &\quad + \frac{1}{2} \mathcal{A}^{\pm 1} \gamma_{\pm 1} f^{\mp \frac{1}{2}} - \frac{3}{2} \tau_{\pm \frac{3}{2}} \mathcal{S}^{\pm \frac{3}{2}} \chi^{\mp 1} + \frac{1}{2} \tau_{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \chi^0 \\ &\quad - \frac{1}{2} \tau_{\mp \frac{1}{2}} \mathcal{S}^{\mp \frac{1}{2}} \chi^{\pm 1}, \end{aligned} \quad (3.68)$$

$$\begin{aligned} \frac{1}{2} \gamma_0 \partial_t \mathcal{A}^0 &= \frac{1}{2} \partial_\varphi \chi^0 + \mathcal{A}^{+1} \gamma_{+1} \mu^{-1} - \mathcal{A}^{-1} \gamma_{-1} \mu^{+1} + \frac{5}{2} \mathcal{A}^{+1} \gamma_{+1} \chi^{-1} \\ &\quad - \frac{5}{2} \mathcal{A}^{-1} \gamma_{-1} \chi^{+1} - \beta_{-\frac{1}{2}} f^{+\frac{1}{2}} \mathcal{G}^{-\frac{1}{2}} + \beta_{+\frac{1}{2}} f^{-\frac{1}{2}} \mathcal{G}^{+\frac{1}{2}} \\ &\quad - \frac{9}{2} \tau_{-\frac{3}{2}} f^{+\frac{3}{2}} \mathcal{S}^{-\frac{3}{2}} + \frac{1}{2} \tau_{-\frac{1}{2}} f^{+\frac{1}{2}} \mathcal{S}^{-\frac{1}{2}} + \frac{1}{2} \tau_{+\frac{1}{2}} f^{-\frac{1}{2}} \mathcal{S}^{+\frac{1}{2}} \\ &\quad - \frac{9}{2} \tau_{+\frac{3}{2}} f^{-\frac{3}{2}} \mathcal{S}^{+\frac{3}{2}} - \tau_{+\frac{1}{2}} v^{-\frac{1}{2}} \mathcal{S}^{+\frac{1}{2}} + \tau_{-\frac{1}{2}} v^{+\frac{1}{2}} \mathcal{S}^{-\frac{1}{2}} \\ &\quad + \alpha_{+1} \chi^{-1} \mathcal{L}^{+1} - \alpha_{-1} \chi^{+1} \mathcal{L}^{-1}, \end{aligned} \quad (3.69)$$

$$\begin{aligned}
\gamma_{\pm 1} \partial_t \mathcal{A}^{\pm 1} &= \partial_\phi \chi^{\pm 1} \pm \mathcal{A}^{\pm 1} \gamma_{\pm 1} \mu^0 \mp \mathcal{A}^0 \gamma_0 \mu^{\pm 1} \pm \frac{5}{2} \mathcal{A}^{\pm 1} \gamma_{\pm 1} \chi^0 \\
&\mp \frac{5}{2} \mathcal{A}^0 \gamma_0 \chi^{\pm 1} \mp 3 \beta_{\mp \frac{1}{2}} f^{\pm \frac{3}{2}} \mathcal{G}^{-\frac{1}{2}} + \beta_{\pm \frac{1}{2}} f^{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \\
&- 3 \tau_{\mp \frac{1}{2}} f^{\pm \frac{3}{2}} \mathcal{S}^{\mp \frac{1}{2}} + 2 \tau_{\pm \frac{1}{2}} f^{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} - 3 \tau_{\pm \frac{3}{2}} f^{\mp \frac{1}{2}} \mathcal{S}^{\pm \frac{3}{2}} \\
&- 3 \tau_{\pm \frac{3}{2}} v^{\mp \frac{1}{2}} \mathcal{S}^{\pm \frac{3}{2}} + \tau_{\pm \frac{1}{2}} v^{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \pm \alpha_{\pm 1} \chi^0 \mathcal{L}^{\pm 1} \\
&\mp \alpha_0 \chi^{\pm 1} \mathcal{L}^0,
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
\tau_{\pm \frac{1}{2}} \partial_t \mathcal{S}^{\pm \frac{1}{2}} &= \partial_\phi f^{\pm \frac{1}{2}} - \mathcal{A}^{\pm 1} \gamma_{\pm 1} v^{\mp \frac{1}{2}} - \mathcal{A}^0 \gamma_0 v^{\pm \frac{1}{2}} \mp 3 \mathcal{A}^{\mp 1} \gamma_{\mp 1} f^{\pm \frac{3}{2}} \\
&\mp \frac{1}{2} \mathcal{A}^0 \gamma_0 f^{\pm \frac{1}{2}} \pm 2 \mathcal{A}^{\pm 1} \gamma_{\pm 1} f^{\mp \frac{1}{2}} \mp 3 \alpha_{\mp 1} f^{\pm \frac{3}{2}} \mathcal{L}^{\mp 1} \\
&- \frac{1}{2} \alpha_0 f^{\pm \frac{1}{2}} \mathcal{L}^0 \pm 2 \alpha_{\pm 1} f^{\mp \frac{1}{2}} \mathcal{L}^{\pm 1} + \beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \chi^0 \\
&+ \beta_{\mp \frac{1}{2}} \mathcal{G}^{\mp \frac{1}{2}} \chi^{\pm 1} \pm 3 \tau_{\pm \frac{3}{2}} \mu^{\mp 1} \mathcal{S}^{\pm \frac{3}{2}} \pm \frac{1}{2} \tau_{\pm \frac{1}{2}} \mu^0 \mathcal{S}^{\pm \frac{1}{2}} \\
&\mp 2 \tau_{\mp \frac{1}{2}} \mu^{\pm 1} \mathcal{S}^{\mp \frac{1}{2}} \pm \tau_{\pm \frac{3}{2}} \mathcal{S}^{\pm \frac{3}{2}} \chi^{\mp 1} \pm \frac{1}{2} \tau_{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \chi^0 \\
&\mp 2 \tau_{\mp \frac{1}{2}} \mathcal{S}^{\mp \frac{1}{2}} \chi^{\pm 1},
\end{aligned} \tag{3.71}$$

$$\begin{aligned}
\tau_{\pm \frac{3}{2}} \partial_t \mathcal{S}^{\pm \frac{3}{2}} &= \partial_\phi f^{\pm \frac{3}{2}} - \mathcal{A}^{\pm 1} \gamma_{\pm 1} v^{\pm \frac{1}{2}} \mp \frac{3}{2} \mathcal{A}^0 \gamma_0 f^{\pm \frac{3}{2}} \pm \mathcal{A}^{\pm 1} \gamma_{\pm 1} f^{\pm \frac{1}{2}} \\
&\mp \frac{3}{2} \alpha_0 f^{\pm \frac{3}{2}} \mathcal{L}^0 + \alpha_{\pm 1} f^{\pm \frac{1}{2}} \mathcal{L}^{\pm 1} + \beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \chi^{\pm 1} \\
&\pm \frac{3}{2} \tau_{\pm \frac{3}{2}} \mu^0 \mathcal{S}^{\pm \frac{3}{2}} - \tau_{\pm \frac{1}{2}} \mu^{\pm 1} \mathcal{S}^{\pm \frac{1}{2}} \pm \frac{3}{2} \tau_{\pm \frac{3}{2}} \mathcal{S}^{\pm \frac{3}{2}} \chi^0 \\
&\mp \tau_{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \chi^{\pm 1}.
\end{aligned} \tag{3.72}$$

After the temporal evolution of the source fields, one can start to compute the gauge transformations for asymptotic symmetry algebra by considering all transformations (4.19) that preserve the boundary conditions, with the gauge parameter λ in the $\mathfrak{osp}(3|2)$ superalgebra as,

$$\lambda = b^{-1} \left[\varepsilon^i L_i + \kappa^i A_i + \zeta^P G_P + \rho^P S_P \right] b. \tag{3.73}$$

Here, the gauge parameter has in total of twelve arbitrary functions: six bosonic $(\varepsilon^i, \kappa^i)$ and six fermionic (ζ^P, ρ^P) on the boundary. The condition (4.19) impose that

transformations on the gauge are given by

$$\begin{aligned}\alpha_{\pm 1}\delta_\lambda \mathcal{L}^{\pm 1} &= \partial_\varphi \varepsilon^{\pm 1} \pm \alpha_{\pm 1} \varepsilon^0 \mathcal{L}^{\pm 1} \mp \alpha_0 \varepsilon^{\pm 1} \mathcal{L}^0 + 2\beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \rho^{\pm \frac{1}{2}} \\ &\mp \mathcal{A}^{\pm 1} \gamma_{\pm 1} \kappa^0 \pm \mathcal{A}^0 \gamma_0 \kappa^{\pm 1} + 3\tau_{\mp \frac{1}{2}} \zeta^{\pm \frac{3}{2}} \mathcal{S}^{\mp \frac{1}{2}} - 2\tau_{\pm \frac{1}{2}} \zeta^{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \\ &+ 3\tau_{\pm \frac{3}{2}} \zeta^{\mp \frac{1}{2}} \mathcal{S}^{\pm \frac{3}{2}},\end{aligned}\quad (3.74)$$

$$\begin{aligned}\frac{1}{2}\alpha_0\delta_\lambda \mathcal{L}^0 &= \frac{1}{2}\partial_\varphi \varepsilon^0 + \alpha_{+1} \varepsilon^{-1} \mathcal{L}^1 - \alpha_{-1} \varepsilon^{+1} \mathcal{L}^{-1} + \beta_{+\frac{1}{2}} \mathcal{G}^{+\frac{1}{2}} \rho^{-\frac{1}{2}} \\ &+ \beta_{-\frac{1}{2}} \mathcal{G}^{-\frac{1}{2}} \rho^{+\frac{1}{2}} - \mathcal{A}^{+1} \gamma_{+1} \kappa^{-1} + \mathcal{A}^{-1} \gamma_{-1} \kappa^{+1} + \frac{9}{2} \tau_{-\frac{3}{2}} \zeta^{+\frac{3}{2}} \mathcal{S}^{-\frac{3}{2}} \\ &- \frac{1}{2} \tau_{-\frac{1}{2}} \zeta^{+\frac{1}{2}} \mathcal{S}^{-\frac{1}{2}} - \frac{1}{2} \tau_{+\frac{1}{2}} \zeta^{-\frac{1}{2}} \mathcal{S}^{+\frac{1}{2}} + \frac{9}{2} \tau_{+\frac{3}{2}} \zeta^{-\frac{3}{2}} \mathcal{S}^{+\frac{3}{2}},\end{aligned}\quad (3.75)$$

$$\begin{aligned}g_{\pm \frac{1}{2}}\delta_\lambda \mathcal{G}^{\pm \frac{1}{2}} &= \partial_\varphi \rho^{\pm \frac{1}{2}} \mp \frac{1}{2} \alpha_0 \rho^{\pm \frac{1}{2}} \mathcal{L}^0 \pm \alpha_{\pm 1} \rho^{\mp \frac{1}{2}} \mathcal{L}^{\pm 1} \mp \beta_{\mp \frac{1}{2}} \mathcal{G}^{\mp \frac{1}{2}} \varepsilon^{\pm 1} \\ &\pm \frac{1}{2} \beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \varepsilon^0 + \frac{3}{2} \mathcal{A}^{\mp 1} \gamma_{\mp 1} \zeta^{\pm \frac{3}{2}} - \frac{1}{2} \mathcal{A}^0 \gamma_0 \zeta^{\pm \frac{1}{2}} \\ &+ \frac{1}{2} \mathcal{A}^{\pm 1} \gamma_{\pm 1} \zeta^{\mp \frac{1}{2}} - \frac{3}{2} \tau_{\pm \frac{3}{2}} \mathcal{S}^{\pm \frac{3}{2}} \kappa^{\mp 1} + \frac{1}{2} \tau_{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \kappa^0 \\ &- \frac{1}{2} \tau_{\mp \frac{1}{2}} \mathcal{S}^{\mp \frac{1}{2}} \kappa^{\pm 1},\end{aligned}\quad (3.76)$$

$$\begin{aligned}\frac{1}{2}\gamma_0\delta_\lambda \mathcal{A}^0 &= \frac{1}{2}\partial_\varphi \kappa^0 + \mathcal{A}^{+1} \gamma_{+1} \varepsilon^{-1} - \mathcal{A}^{-1} \gamma_{-1} \varepsilon^{+1} + \frac{5}{2} \mathcal{A}^{+1} \gamma_{+1} \kappa^{-1} \\ &- \frac{5}{2} \mathcal{A}^{-1} \gamma_{-1} \kappa^{+1} - \beta_{-\frac{1}{2}} \zeta^{+\frac{1}{2}} \mathcal{G}^{-\frac{1}{2}} + \beta_{+\frac{1}{2}} \zeta^{-\frac{1}{2}} \mathcal{G}^{+\frac{1}{2}} \\ &- \frac{9}{2} \tau_{-\frac{3}{2}} \zeta^{+\frac{3}{2}} \mathcal{S}^{-\frac{3}{2}} + \frac{1}{2} \tau_{-\frac{1}{2}} \zeta^{+\frac{1}{2}} \mathcal{S}^{-\frac{1}{2}} + \frac{1}{2} \tau_{+\frac{1}{2}} \zeta^{-\frac{1}{2}} \mathcal{S}^{+\frac{1}{2}} \\ &- \frac{9}{2} \tau_{+\frac{3}{2}} \zeta^{-\frac{3}{2}} \mathcal{S}^{+\frac{3}{2}} - \tau_{+\frac{1}{2}} \rho^{-\frac{1}{2}} \mathcal{S}^{+\frac{1}{2}} + \tau_{-\frac{1}{2}} \rho^{+\frac{1}{2}} \mathcal{S}^{-\frac{1}{2}} + \alpha_{+1} \kappa^{-1} \mathcal{L}^{+1} \\ &- \alpha_{-1} \kappa^{+1} \mathcal{L}^{-1},\end{aligned}\quad (3.77)$$

$$\begin{aligned}\gamma_{\pm 1}\delta_\lambda \mathcal{A}^{\pm 1} &= \partial_\varphi \kappa^{\pm 1} \pm \mathcal{A}^{\pm 1} \gamma_{\pm 1} \varepsilon^0 \mp \mathcal{A}^0 \gamma_0 \varepsilon^{\pm 1} \pm \frac{5}{2} \mathcal{A}^{\pm 1} \gamma_{\pm 1} \kappa^0 \\ &\mp \frac{5}{2} \mathcal{A}^0 \gamma_0 \kappa^{\pm 1} \mp 3\beta_{\mp \frac{1}{2}} \zeta^{\pm \frac{3}{2}} \mathcal{G}^{-\frac{1}{2}} \pm \beta_{\pm \frac{1}{2}} \zeta^{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \\ &- 3\tau_{\mp \frac{1}{2}} \zeta^{\pm \frac{3}{2}} \mathcal{S}^{\mp \frac{1}{2}} + 2\tau_{\pm \frac{1}{2}} \zeta^{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} - 3\tau_{\pm \frac{3}{2}} \zeta^{\mp \frac{1}{2}} \mathcal{S}^{\pm \frac{3}{2}} \\ &- 3\tau_{\pm \frac{3}{2}} \rho^{\mp \frac{1}{2}} \mathcal{S}^{\pm \frac{3}{2}} + \tau_{\pm \frac{1}{2}} \rho^{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \pm \alpha_{\pm 1} \kappa^0 \mathcal{L}^{\pm 1} \\ &\mp \alpha_0 \kappa^{\pm 1} \mathcal{L}^0,\end{aligned}\quad (3.78)$$

$$\begin{aligned}\tau_{\pm \frac{1}{2}}\delta_\lambda \mathcal{S}^{\pm \frac{1}{2}} &= \partial_\varphi \zeta^{\pm \frac{1}{2}} - \mathcal{A}^{\pm 1} \gamma_{\pm 1} \rho^{\mp \frac{1}{2}} - \mathcal{A}^0 \gamma_0 \rho^{\pm \frac{1}{2}} \mp 3\mathcal{A}^{\mp 1} \gamma_{\mp 1} \zeta^{\pm \frac{3}{2}} \\ &\mp \frac{1}{2} \mathcal{A}^0 \gamma_0 \zeta^{\pm \frac{1}{2}} \pm 2\mathcal{A}^{\pm 1} \gamma_{\pm 1} \zeta^{\mp \frac{1}{2}} \mp 3\alpha_{\mp 1} \zeta^{\pm \frac{3}{2}} \mathcal{L}^{\mp 1} - \frac{1}{2} \alpha_0 \zeta^{\pm \frac{1}{2}} \mathcal{L}^0 \\ &\pm 2\alpha_{\pm 1} \zeta^{\mp \frac{1}{2}} \mathcal{L}^{\pm 1} + \beta_{\pm \frac{1}{2}} \mathcal{G}^{\pm \frac{1}{2}} \kappa^0 + \beta_{\mp \frac{1}{2}} \mathcal{G}^{\mp \frac{1}{2}} \kappa^{\pm 1} \pm 3\tau_{\pm \frac{3}{2}} \varepsilon^{\mp 1} \mathcal{S}^{\pm \frac{3}{2}} \\ &\pm \frac{1}{2} \tau_{\pm \frac{1}{2}} \varepsilon^0 \mathcal{S}^{\pm \frac{1}{2}} \mp 2\tau_{\mp \frac{1}{2}} \varepsilon^{\pm 1} \mathcal{S}^{\mp \frac{1}{2}} \pm \tau_{\pm \frac{3}{2}} \mathcal{S}^{\pm \frac{3}{2}} \kappa^{\mp 1} \\ &\pm \frac{1}{2} \tau_{\pm \frac{1}{2}} \mathcal{S}^{\pm \frac{1}{2}} \kappa^0 \mp 2\tau_{\mp \frac{1}{2}} \mathcal{S}^{\mp \frac{1}{2}} \kappa^{\pm 1},\end{aligned}\quad (3.79)$$

$$\begin{aligned}
\tau_{\pm\frac{3}{2}}\delta_\lambda\mathcal{S}^{\pm\frac{3}{2}} &= \partial_\phi\zeta^{\pm\frac{3}{2}} - \mathcal{A}^{\pm 1}\gamma_{\pm 1}\rho^{\pm\frac{1}{2}} \mp \frac{3}{2}\mathcal{A}^0\gamma_0\zeta^{\pm\frac{3}{2}} \pm \mathcal{A}^{\pm 1}\gamma_{\pm 1}\zeta^{\pm\frac{1}{2}} \\
&\mp \frac{3}{2}\alpha_0\zeta^{\pm\frac{3}{2}}\mathcal{L}^0 + \alpha_{\pm 1}\zeta^{\pm\frac{1}{2}}\mathcal{L}^{\pm 1} + \beta_{\pm\frac{1}{2}}\mathcal{G}^{\pm\frac{1}{2}}\kappa^{\pm 1} \pm \frac{3}{2}\tau_{\pm\frac{3}{2}}\varepsilon^0\mathcal{S}^{\pm\frac{3}{2}} \\
&- \tau_{\pm\frac{1}{2}}\varepsilon^{\pm 1}\mathcal{S}^{\pm\frac{1}{2}} \pm \frac{3}{2}\tau_{\pm\frac{3}{2}}\mathcal{S}^{\pm\frac{3}{2}}\kappa^0 \mp \tau_{\pm\frac{1}{2}}\mathcal{S}^{\pm\frac{1}{2}}\kappa^{\pm 1}.
\end{aligned} \tag{3.80}$$

With analogical reasoning, they obey the following transformations

$$\begin{aligned}
\alpha_{\pm 1}\delta_\lambda\mu^{\pm 1} &= \partial_t\varepsilon^{\pm 1} \pm \alpha_{\pm 1}\varepsilon^0\mu^{\pm 1} \mp \alpha_0\varepsilon^{\pm 1}\mu^0 + 2\beta_{\pm\frac{1}{2}}f^{\pm\frac{1}{2}}\rho^{\pm\frac{1}{2}} \mp \chi^{\pm 1}\gamma_{\pm 1}\kappa^0 \\
&\mp \chi^{\pm 1}\gamma_{\pm 1}\kappa^0 \pm \chi^0\gamma_0\kappa^{\pm 1} + 3\tau_{\mp\frac{1}{2}}\zeta^{\pm\frac{3}{2}}v^{\mp\frac{1}{2}} - 2\tau_{\pm\frac{1}{2}}\zeta^{\pm\frac{1}{2}}v^{\pm\frac{1}{2}} \\
&+ 3\tau_{\pm\frac{3}{2}}\zeta^{\mp\frac{1}{2}}v^{\pm\frac{3}{2}},
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
\frac{1}{2}\alpha_0\delta_\lambda\mu^0 &= \frac{1}{2}\partial_t\varepsilon^0 + \alpha_{+1}\varepsilon^{-1}\mu^1 - \alpha_{-1}\varepsilon^{+1}\mu^{-1} + \beta_{+\frac{1}{2}}f^{+\frac{1}{2}}\rho^{-\frac{1}{2}} \\
&+ \beta_{-\frac{1}{2}}f^{-\frac{1}{2}}\rho^{+\frac{1}{2}} - \chi^{+1}\gamma_{+1}\kappa^{-1} + \chi^{-1}\gamma_{-1}\kappa^{+1} + \frac{9}{2}\tau_{-\frac{3}{2}}\zeta^{+\frac{3}{2}}v^{-\frac{3}{2}} \\
&- \frac{1}{2}\tau_{-\frac{1}{2}}\zeta^{+\frac{1}{2}}v^{-\frac{1}{2}} - \frac{1}{2}\tau_{+\frac{1}{2}}\zeta^{-\frac{1}{2}}v^{+\frac{1}{2}} + \frac{9}{2}\tau_{+\frac{3}{2}}\zeta^{-\frac{3}{2}}v^{+\frac{3}{2}},
\end{aligned} \tag{3.82}$$

$$\begin{aligned}
g_{\pm\frac{1}{2}}\delta_\lambda f^{\pm\frac{1}{2}} &= \partial_t\rho^{\pm\frac{1}{2}} \mp \frac{1}{2}\alpha_0\rho^{\pm\frac{1}{2}}\mu^0 \pm \alpha_{\pm 1}\rho^{\mp\frac{1}{2}}\mu^{\pm 1} \mp \beta_{\mp\frac{1}{2}}f^{\mp\frac{1}{2}}\varepsilon^{\pm 1} \\
&\pm \frac{1}{2}\beta_{\pm\frac{1}{2}}f^{\pm\frac{1}{2}}\varepsilon^0 + \frac{3}{2}\chi^{\mp 1}\gamma_{\mp 1}\zeta^{\pm\frac{3}{2}} - \frac{1}{2}\chi^0\gamma_0\zeta^{\pm\frac{1}{2}} + \frac{1}{2}\chi^{\pm 1}\gamma_{\pm 1}\zeta^{\mp\frac{1}{2}} \\
&- \frac{3}{2}\tau_{\pm\frac{3}{2}}v^{\pm\frac{3}{2}}\kappa^{\mp 1} + \frac{1}{2}\tau_{\pm\frac{1}{2}}v^{\pm\frac{1}{2}}\kappa^0 - \frac{1}{2}\tau_{\mp\frac{1}{2}}v^{\mp\frac{1}{2}}\kappa^{\pm 1},
\end{aligned} \tag{3.83}$$

$$\begin{aligned}
\frac{1}{2}\gamma_0\delta_\lambda\chi^0 &= \frac{1}{2}\partial_t\kappa^0 + \chi^{+1}\gamma_{+1}\varepsilon^{-1} - \chi^{-1}\gamma_{-1}\varepsilon^{+1} + \frac{5}{2}\chi^{+1}\gamma_{+1}\kappa^{-1} \\
&- \frac{5}{2}\chi^{-1}\gamma_{-1}\kappa^{+1} - \beta_{-\frac{1}{2}}\zeta^{+\frac{1}{2}}f^{-\frac{1}{2}} + \beta_{+\frac{1}{2}}\zeta^{-\frac{1}{2}}f^{+\frac{1}{2}} \\
&- \frac{9}{2}\tau_{-\frac{3}{2}}\zeta^{+\frac{3}{2}}v^{-\frac{3}{2}} + \frac{1}{2}\tau_{-\frac{1}{2}}\zeta^{+\frac{1}{2}}v^{-\frac{1}{2}} + \frac{1}{2}\tau_{+\frac{1}{2}}\zeta^{-\frac{1}{2}}v^{+\frac{1}{2}} \\
&- \frac{9}{2}\tau_{+\frac{3}{2}}\zeta^{-\frac{3}{2}}v^{+\frac{3}{2}} - \tau_{+\frac{1}{2}}\rho^{-\frac{1}{2}}v^{+\frac{1}{2}} + \tau_{-\frac{1}{2}}\rho^{+\frac{1}{2}}v^{-\frac{1}{2}} \\
&+ \alpha_{+1}\kappa^{-1}\mu^{+1} - \alpha_{-1}\kappa^{+1}\mu^{-1},
\end{aligned} \tag{3.84}$$

$$\begin{aligned}
\gamma_{\pm 1}\delta_\lambda\chi^{\pm 1} &= \partial_t\kappa^{\pm 1} \pm \chi^{\pm 1}\gamma_{\pm 1}\varepsilon^0 \mp \chi^0\gamma_0\varepsilon^{\pm 1} \pm \frac{5}{2}\chi^{\pm 1}\gamma_{\pm 1}\kappa^0 \\
&\mp \frac{5}{2}\chi^0\gamma_0\kappa^{\pm 1} \mp 3\beta_{\mp\frac{1}{2}}\zeta^{\pm\frac{3}{2}}f^{-\frac{1}{2}} \pm \beta_{\pm\frac{1}{2}}\zeta^{\pm\frac{1}{2}}f^{\pm\frac{1}{2}} \\
&- 3\tau_{\mp\frac{1}{2}}\zeta^{\pm\frac{3}{2}}v^{\mp\frac{1}{2}} + 2\tau_{\pm\frac{1}{2}}\zeta^{\pm\frac{1}{2}}v^{\pm\frac{1}{2}} - 3\tau_{\pm\frac{3}{2}}\zeta^{\mp\frac{1}{2}}v^{\pm\frac{3}{2}} \\
&- 3\tau_{\pm\frac{3}{2}}\rho^{\mp\frac{1}{2}}v^{\pm\frac{3}{2}} + \tau_{\pm\frac{1}{2}}\rho^{\pm\frac{1}{2}}v^{\pm\frac{1}{2}} \pm \alpha_{\pm 1}\kappa^0\mu^{\pm 1} \mp \alpha_0\kappa^{\pm 1}\mu^0,
\end{aligned} \tag{3.85}$$

$$\begin{aligned}
\tau_{\pm\frac{1}{2}}\delta_\lambda v^{\pm\frac{1}{2}} &= \partial_t \zeta^{\pm\frac{1}{2}} - \chi^{\pm 1} \gamma_{\pm 1} \rho^{\mp\frac{1}{2}} - \chi^0 \gamma_0 \rho^{\pm\frac{1}{2}} \mp 3\chi^{\mp 1} \gamma_{\mp 1} \zeta^{\pm\frac{3}{2}} \mp \frac{1}{2} \chi^0 \gamma_0 \zeta^{\pm\frac{1}{2}} \\
&\quad \pm 2\chi^{\pm 1} \gamma_{\pm 1} \zeta^{\mp\frac{1}{2}} \mp 3\alpha_{\mp 1} \zeta^{\pm\frac{3}{2}} \mu^{\mp 1} - \frac{1}{2} \alpha_0 \zeta^{\pm\frac{1}{2}} \mu^0 \pm 2\alpha_{\pm 1} \zeta^{\mp\frac{1}{2}} \mu^{\pm 1} \\
&\quad + \beta_{\pm\frac{1}{2}} f^{\pm\frac{1}{2}} \kappa^0 + \beta_{\mp\frac{1}{2}} f^{\mp\frac{1}{2}} \kappa^{\pm 1} \pm 3\tau_{\pm\frac{3}{2}} \varepsilon^{\mp 1} v^{\pm\frac{3}{2}} \pm \frac{1}{2} \tau_{\pm\frac{1}{2}} \varepsilon^0 v^{\pm\frac{1}{2}} \\
&\quad \mp 2\tau_{\mp\frac{1}{2}} \varepsilon^{\pm 1} v^{\mp\frac{1}{2}} \pm \tau_{\pm\frac{3}{2}} v^{\pm\frac{3}{2}} \kappa^{\mp 1} \pm \frac{1}{2} \tau_{\pm\frac{1}{2}} v^{\pm\frac{1}{2}} \kappa^0 \\
&\quad \mp 2\tau_{\mp\frac{1}{2}} v^{\mp\frac{1}{2}} \kappa^{\pm 1}, \tag{3.86}
\end{aligned}$$

$$\begin{aligned}
\tau_{\pm\frac{3}{2}}\delta_\lambda v^{\pm\frac{3}{2}} &= \partial_t \zeta^{\pm\frac{3}{2}} - \chi^{\pm 1} \gamma_{\pm 1} \rho^{\pm\frac{1}{2}} \mp \frac{3}{2} \chi^0 \gamma_0 \zeta^{\pm\frac{3}{2}} \pm \chi^{\pm 1} \gamma_{\pm 1} \zeta^{\pm\frac{1}{2}} \mp \frac{3}{2} \alpha_0 \zeta^{\pm\frac{3}{2}} \mu^0 \\
&\quad + \alpha_{\pm 1} \zeta^{\pm\frac{1}{2}} \mu^{\pm 1} + \beta_{\pm\frac{1}{2}} f^{\pm\frac{1}{2}} \kappa^{\pm 1} \pm \frac{3}{2} \tau_{\pm\frac{3}{2}} \varepsilon^0 v^{\pm\frac{3}{2}} - \tau_{\pm\frac{1}{2}} \varepsilon^{\pm 1} v^{\pm\frac{1}{2}} \\
&\quad \pm \frac{3}{2} \tau_{\pm\frac{3}{2}} v^{\pm\frac{3}{2}} \kappa^0 \mp \tau_{\pm\frac{1}{2}} v^{\pm\frac{1}{2}} \kappa^{\pm 1}. \tag{3.87}
\end{aligned}$$

As in the $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ case one can now determine the canonical boundary charges $\mathcal{Q}[\lambda]$ that generates the transformations (3.74)-(3.80). Therefore, the corresponding variation of the boundary charge $\mathcal{Q}[\lambda]$ [81, 86–88], to show the asymptotic symmetry algebra, is given by (4.29). The canonical boundary charge $\mathcal{Q}[\lambda]$ can be integrated which reads

$$\mathcal{Q}[\lambda] = \int d\varphi \left[\mathcal{L}^i \varepsilon^{-i} + \mathcal{A}^i \kappa^{-i} + \mathcal{G}^p \zeta^{-p} + \mathcal{S}^p \rho^{-p} \right]. \tag{3.88}$$

Having determined both the infinitesimal transformations and the canonical boundary charges as the generators of the asymptotic symmetry algebra, one can also get the Poisson bracket algebra by using the methods [89] with (4.31) again for any phase space functional F , the operator product algebra can then be defined as,

$$\mathcal{L}^i(z_1) \mathcal{L}^j(z_2) \sim \frac{\frac{k}{2} \eta^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}} \mathcal{L}^{i+j}, \tag{3.89}$$

$$\mathcal{L}^i(z_1) \mathcal{G}^p(z_2) \sim \frac{(\frac{i}{2} - p)}{z_{12}} \mathcal{G}^{i+p}, \tag{3.90}$$

$$\mathcal{G}^p(z_1) \mathcal{G}^q(z_2) \sim \frac{\frac{k}{2} \eta^{pq}}{z_{12}^2} + \frac{1}{z_{12}} (\sigma_1 \mathcal{A}^{p+q} + \sigma_2 \mathcal{L}^{p+q}), \tag{3.91}$$

$$\mathcal{G}^p(z_1) \mathcal{S}^q(z_2) \sim \frac{(\frac{3p}{2} - \frac{q}{2})}{z_{12}} (\sigma_3 \mathcal{A}^{p+q} + \sigma_4 \mathcal{L}^{p+q}), \tag{3.92}$$

$$\mathcal{L}^i(z_1) \mathcal{A}^j(z_2) \sim \frac{(\frac{i}{2} - j)}{z_{12}} \mathcal{A}^{i+j}, \tag{3.93}$$

$$\mathcal{L}^i(z_1) \mathcal{S}^j(z_2) \sim \frac{(\frac{3i}{2} - j)}{z_{12}} \mathcal{S}^{i+j}, \tag{3.94}$$

$$\mathcal{A}^i(z_1) \mathcal{A}^j(z_2) \sim \frac{\frac{k}{2} \eta^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}} (\sigma_5 \mathcal{L}^{i+j} + \sigma_6 \mathcal{A}^{i+j}), \tag{3.95}$$

$$\mathcal{A}^i(z_1)\mathcal{G}^p(z_2) \sim \frac{1}{z_{12}} \left(\sigma_7 \mathcal{S}_{i+p} + \sigma_8 \left(\frac{i}{2} - p \right) \mathcal{G}_{i+p} \right), \quad (3.96)$$

$$\begin{aligned} \mathcal{A}^i(z_1)\mathcal{S}^p(z_2) &\sim \frac{1}{z_{12}} \left(\sigma_9 \left(\frac{3i}{2} - p \right) \mathcal{S}_{i+p} \right. \\ &\quad \left. + \sigma_{10} \left(3i^2 - 2ip + p^2 - \frac{9}{4} \right) \mathcal{G}_{i+p} \right), \end{aligned} \quad (3.97)$$

$$\begin{aligned} \mathcal{S}^p(z_1)\mathcal{S}^q(z_2) &\sim \frac{\frac{k}{2}\eta^{pq}}{z_{12}^2} + \frac{1}{z_{12}} \left(\left(3p^2 - 4pq + 3q^2 - \frac{9}{2} \right) (\sigma_{11}\mathcal{A}_{p+q} \right. \\ &\quad \left. + \sigma_{12}\mathcal{L}_{p+q}) \right) \end{aligned} \quad (3.98)$$

where $z_{12} = z_1 - z_2$, or in the more compact form,

$$\mathfrak{J}^A(z_1)\mathfrak{J}^B(z_2) \sim \frac{\frac{k}{2}\eta^{AB}}{z_{12}^2} + \frac{\mathfrak{f}_{\mathcal{C}}^{AB}\mathfrak{J}^C(z_2)}{z_{12}}. \quad (3.99)$$

Here, η^{AB} is the supertrace matrix and $\mathfrak{f}_{\mathcal{C}}^{AB}$'s are the structure constants of the related algebra with $(A, B = 0, \pm 1, \pm \frac{1}{2}, 0, \pm 1, \pm \frac{1}{2}, \pm \frac{3}{2})$, i.e, $\eta^{ip} = 0$ and $\mathfrak{f}_{i+j}^{ij} = (i - j)$. After repeating the same algebra for $\bar{\Gamma}$ -sector, one can say that the asymptotic symmetry algebra for the most general boundary conditions of $\mathcal{N} = (1, 1)$ supergravity is two copies of the affine $\mathfrak{osp}(3|2)_k$ algebra.

3.2.2 For superconformal boundary

Under the following restrictions as the Drinfeld-Sokolov highest weight gauge condition,

$$\begin{aligned} \mathcal{L}^0 = \mathcal{A}^0 = \mathcal{A}^{+1} = \mathcal{G}^{+\frac{1}{2}} = \mathcal{S}^{+\frac{1}{2}} = \mathcal{S}^{+\frac{3}{2}} = 0, \\ \mathcal{L}^{-1} = \mathcal{L}, \quad \mathcal{A}^{-1} = \mathcal{A}, \quad \mathcal{G}^{-\frac{1}{2}} = \mathcal{G}, \quad \mathcal{S}^{-\frac{3}{2}} = \mathcal{S}, \quad \alpha_{+1}\mathcal{L}^{+1} = 1 \end{aligned} \quad (3.100)$$

on the boundary conditions with the $\mathfrak{osp}(3|2)$ superalgebra valued connection (4.13), one can get the superconformal boundary conditions as the supersymmetric extension of the Brown - Henneaux boundary conditions proposed in [6, 45] for AdS_3 supergravity. Therefore we have the supersymmetric connection as,

$$a_\varphi = \mathcal{L}_1 + \alpha_{-1}\mathcal{L}\mathcal{L}_{-1} + \gamma_{-1}\mathcal{A}\mathcal{L}_{-1} + \beta_{-\frac{1}{2}}\mathcal{G}\mathcal{G}_{-\frac{1}{2}} + \tau_{-\frac{3}{2}}\mathcal{S}\mathcal{S}_{-\frac{3}{2}}, \quad (3.101)$$

$$\begin{aligned} a_t = & \mu\mathcal{L}_1 + \chi\mathcal{A}_1 + f\mathcal{G}_{+\frac{1}{2}} + \nu\mathcal{S}_{+\frac{3}{2}} + \sum_{i=-1}^0 \mu^i\mathcal{L}_i + \sum_{i=-1}^0 \chi^i\mathcal{A}_i + f^{-\frac{1}{2}}\mathcal{G}_{-\frac{1}{2}} \\ & + \sum_{p=-\frac{3}{2}}^{\frac{1}{2}} \nu^p\mathcal{S}_p \end{aligned} \quad (3.102)$$

where $\mu \equiv \mu^{+1}$, $\chi \equiv \chi^{+1}$, $v \equiv v^{+\frac{1}{2}}$, and $f \equiv f^{+\frac{3}{2}}$ can be interpreted as the independent *chemical potentials*. One can choose for the corresponding asymptotic symmetry algebra, the resulting relations as $\sigma_3 = \frac{2i}{\sqrt{5}}$, $\sigma_7 = i\sqrt{5}$, $\sigma_8 = 0$, $\sigma_9 = 1$, appearing on the RHS of eq.(3.53) – (3.62) in the super Jacobi identities of the $\mathfrak{osp}(3|2)$ superalgebra. The functions, except the *chemical potentials* are fixed by the flatness condition (4.7). For the fixed chemical potentials, the time evolution of canonical boundary charges can be written as,

$$\begin{aligned} \partial_t \mathcal{L} = & \frac{\mu'''}{2\alpha_{-1}} + \mu \mathcal{L}' + 2\mathcal{L}\mu' + \frac{\gamma_{-1}\chi \mathcal{A}'}{\alpha_{-1}} + \frac{2\mathcal{A}\gamma_{-1}\chi'}{\alpha_{-1}} - \frac{3\beta_{-\frac{1}{2}}\mathcal{G}f'}{\alpha_{-1}} \\ & - \frac{\beta_{-\frac{1}{2}}f\mathcal{G}'}{\alpha_{-1}} - \frac{9\delta_{-\frac{3}{2}}v\mathcal{S}'}{10\alpha_{-1}} - \frac{3\delta_{-\frac{3}{2}}\mathcal{S}v'}{2\alpha_{-1}}, \end{aligned} \quad (3.103)$$

$$\begin{aligned} \partial_t \mathcal{G} = & \frac{3i\gamma_{-1}v\mathcal{A}'}{2\sqrt{5}\beta_{-\frac{1}{2}}} + \frac{2i\mathcal{A}\gamma_{-1}v'}{\sqrt{5}\beta_{-\frac{1}{2}}} - \frac{f''}{\beta_{-\frac{1}{2}}} - \frac{\alpha_{-1}f\mathcal{L}}{\beta_{-\frac{1}{2}}} + \mu\mathcal{G}' + \frac{3\mathcal{G}\mu'}{2} \\ & + \frac{3i\delta_{-\frac{3}{2}}\mathcal{S}\chi}{2\sqrt{5}\beta_{-\frac{1}{2}}}, \end{aligned} \quad (3.104)$$

$$\begin{aligned} \partial_t \mathcal{A} = & \mu\mathcal{A}' - \frac{5}{2}i\chi\mathcal{A}' - \frac{9\mathcal{A}\beta_{-\frac{1}{2}}\mathcal{G}v}{\sqrt{5}} + 2\mathcal{A}\mu' - 5i\mathcal{A}\chi' + \frac{\chi'''}{2\gamma_{-1}} \\ & + \frac{3i\delta_{-\frac{3}{2}}f\mathcal{S}}{\sqrt{5}\gamma_{-1}} - \frac{3i\beta_{-\frac{1}{2}}v\mathcal{G}''}{2\sqrt{5}\gamma_{-1}} - \frac{4i\beta_{-\frac{1}{2}}\mathcal{G}'v'}{\sqrt{5}\gamma_{-1}} - \frac{3i\beta_{-\frac{1}{2}}\mathcal{G}v''}{\sqrt{5}\gamma_{-1}} \\ & - \frac{9i\alpha_{-1}\beta_{-\frac{1}{2}}\mathcal{G}v\mathcal{L}}{2\sqrt{5}\gamma_{-1}} + \frac{9i\delta_{-\frac{3}{2}}v\mathcal{S}'}{10\gamma_{-1}} + \frac{3i\delta_{-\frac{3}{2}}\mathcal{S}v'}{2\gamma_{-1}} + \frac{\alpha_{-1}\chi\mathcal{L}'}{\gamma_{-1}} \\ & + \frac{2\alpha_{-1}\mathcal{L}\chi'}{\gamma_{-1}}, \end{aligned} \quad (3.105)$$

$$\begin{aligned} \partial_t \mathcal{S} = & \frac{i\gamma_{-1}v\mathcal{A}''}{2\delta_{-\frac{3}{2}}} + \frac{5i\gamma_{-1}\mathcal{A}'v'}{3\delta_{-\frac{3}{2}}} - \frac{i\sqrt{5}\gamma_{-1}f\mathcal{A}'}{3\delta_{-\frac{3}{2}}} + \frac{5i\mathcal{A}\gamma_{-1}v''}{3\delta_{-\frac{3}{2}}} \\ & - \frac{4i\sqrt{5}\mathcal{A}\gamma_{-1}f'}{3\delta_{-\frac{3}{2}}} - \frac{3\sqrt{5}\mathcal{A}\beta_{-\frac{1}{2}}\gamma_{-1}\mathcal{G}\chi}{\delta_{-\frac{3}{2}}} + \frac{3i\alpha_{-1}\mathcal{A}\gamma_{-1}v\mathcal{L}}{\delta_{-\frac{3}{2}}} - \frac{v^{(4)}}{6\delta_{-\frac{3}{2}}} \\ & - \frac{i\sqrt{5}\beta_{-\frac{1}{2}}\chi\mathcal{G}''}{6\delta_{-\frac{3}{2}}} - \frac{7\beta_{-\frac{1}{2}}^2\mathcal{G}v\mathcal{G}'}{2\delta_{-\frac{3}{2}}} - \frac{2i\sqrt{5}\beta_{-\frac{1}{2}}\mathcal{G}'\chi'}{3\delta_{-\frac{3}{2}}} - \frac{i\sqrt{5}\beta_{-\frac{1}{2}}\mathcal{G}\chi''}{\delta_{-\frac{3}{2}}} \\ & - \frac{3i\sqrt{5}\alpha_{-1}\beta_{-\frac{1}{2}}\mathcal{G}\chi\mathcal{L}}{2\delta_{-\frac{3}{2}}} + \mu\mathcal{S}' - i\chi\mathcal{S}' + \frac{5\mathcal{S}\mu'}{2} - \frac{5}{2}i\mathcal{S}\chi' - \frac{\alpha_{-1}v\mathcal{L}''}{2\delta_{-\frac{3}{2}}} \\ & - \frac{3\alpha_{-1}^2v\mathcal{L}^2}{2\delta_{-\frac{3}{2}}} - \frac{5\alpha_{-1}v'\mathcal{L}'}{3\delta_{-\frac{3}{2}}} - \frac{5\alpha_{-1}\mathcal{L}v''}{3\delta_{-\frac{3}{2}}} \end{aligned} \quad (3.106)$$

where $\alpha_{-1}, \gamma_{-1}, \beta_{-\frac{1}{2}}$ and $\tau_{-\frac{3}{2}}$ are some scaling parameters to be determined later. Now, we are in a position to work the superconformal asymptotic symmetry

algebra under the Drinfeld - Sokolov reduction. This reduction implies that the only independent parameters $\varepsilon \equiv \varepsilon^{+1}$, $\kappa \equiv \kappa^{+1}$, $\zeta \equiv \zeta^{+\frac{1}{2}}$ and $\rho \equiv \rho^{+\frac{3}{2}}$. One can start to compute the gauge transformations for asymptotic symmetry algebra by considering all transformations (4.19) that preserve the boundary conditions with the gauge parameter λ in the $\mathfrak{osp}(3|2)$ superalgebra (3.53) – (3.62) as

$$\begin{aligned} \lambda = & b^{-1} \left[\varepsilon L_{+1} - \varepsilon' L_0 + \left(\mathcal{A} \gamma_{-1} \kappa - \beta_{-\frac{1}{2}} \zeta \mathcal{G} - \frac{9}{10} \delta_{-\frac{3}{2}} \rho \mathcal{S} + \alpha_{-1} \mathcal{L} \varepsilon \right. \right. \\ & + \frac{\varepsilon''}{2} \Big) L_{-1} + \zeta \mathcal{G}_{+\frac{1}{2}} + \left(\frac{3i\gamma_{-1} \rho \mathcal{A}}{2\sqrt{5}} - \zeta' + \beta_{-\frac{1}{2}} \mathcal{G} \varepsilon \right) \mathcal{G}_{-\frac{1}{2}} \\ & + \kappa \mathcal{A}_{+1} + \left(-\kappa' + \frac{3i\beta_{-\frac{1}{2}} \rho \mathcal{G}}{\sqrt{5}} \right) \mathcal{A}_0 + \left(-\frac{5}{2} i \mathcal{A} \gamma_{-1} \kappa + \mathcal{A} \gamma_{-1} \varepsilon - \frac{3i\beta_{-\frac{1}{2}} \rho \mathcal{G}'}{2\sqrt{5}} \right. \\ & - \frac{1}{2} i \sqrt{5} \beta_{-\frac{1}{2}} \mathcal{G} \rho' + \frac{\kappa''}{2} + \frac{9}{10} i \delta_{-\frac{3}{2}} \rho \mathcal{S} + \alpha_{-1} \kappa \mathcal{L} \Big) \mathcal{A}_{-1} + \rho \mathcal{S}_{\frac{3}{2}} - \rho' \mathcal{S}_{\frac{1}{2}} \\ & + \left(-\frac{3}{2} i \mathcal{A} \gamma_{-1} \rho + \frac{1}{2} i \sqrt{5} \beta_{-\frac{1}{2}} \mathcal{G} \kappa + \frac{\rho''}{2} + \frac{3}{2} \alpha_{-1} \rho \mathcal{L} \right) \mathcal{S}_{-\frac{1}{2}} + \left(\frac{1}{2} i \gamma_{-1} \rho \mathcal{A}' \right. \\ & - \frac{1}{3} i \sqrt{5} \mathcal{A} \gamma_{-1} \zeta + \frac{7}{6} i \mathcal{A} \gamma_{-1} \rho' + \beta_{-\frac{1}{2}}^2 - \mathcal{G}^2 \rho - \frac{1}{6} i \sqrt{5} \beta_{-\frac{1}{2}} \kappa \mathcal{G}' \\ & - \frac{1}{2} i \sqrt{5} \beta_{-\frac{1}{2}} \mathcal{G} \kappa' - \frac{\rho^{(3)}}{6} - i \delta_{-\frac{3}{2}} \kappa \mathcal{S} + \delta_{-\frac{3}{2}} \mathcal{S} \varepsilon - \frac{1}{2} \alpha_{-1} \rho \mathcal{L}' \\ & \left. \left. - \frac{7}{6} \alpha_{-1} \mathcal{L} \rho' \right) \mathcal{S}_{-\frac{3}{2}} \right] b. \end{aligned} \quad (3.107)$$

The condition (4.19) impose that transformations on the gauge with $\alpha_{-1} = \gamma_{-1} = \frac{6}{c}$, $\beta_{-\frac{1}{2}} = -\frac{3}{c}$ and $\tau_{-\frac{3}{2}} = -\frac{10}{c}$ are given by

$$\delta_\lambda \mathcal{L} = \frac{c\varepsilon'''}{12} + \varepsilon \mathcal{L}' + 2\mathcal{L} \varepsilon' + \kappa \mathcal{A}' + 2\mathcal{A} \kappa' + \frac{\zeta \mathcal{G}'}{2} + \frac{3\mathcal{G} \zeta'}{2} + \frac{3\rho \mathcal{S}'}{2} + \frac{5\mathcal{S} \rho'}{2}, \quad (3.108)$$

$$\delta_\lambda \mathcal{G} = -\frac{3i\rho \mathcal{A}'}{\sqrt{5}} - \frac{4i\mathcal{A} \rho'}{\sqrt{5}} + \frac{c\zeta''}{3} + \varepsilon \mathcal{G}' + \frac{3\mathcal{G} \varepsilon'}{2} + i\sqrt{5} \kappa \mathcal{S} + 2\zeta \mathcal{L}, \quad (3.109)$$

$$\begin{aligned} \delta_\lambda \mathcal{A} = & -\frac{5}{2} i \kappa \mathcal{A}' + \varepsilon \mathcal{A}' - 5i \mathcal{A} \kappa' + 2\mathcal{A} \varepsilon' + \frac{27\mathcal{A} \mathcal{G} \rho}{\sqrt{5}c} + \frac{27i\mathcal{G} \rho \mathcal{L}}{2\sqrt{5}c} \\ & + \frac{c\kappa'''}{12} + \frac{3i\rho \mathcal{G}''}{4\sqrt{5}} + \frac{2i\mathcal{G}' \rho'}{\sqrt{5}} + \frac{3i\mathcal{G} \rho''}{2\sqrt{5}} - \frac{3}{2} i \rho \mathcal{S}' - i\sqrt{5} \zeta \mathcal{S} \\ & - \frac{5}{2} i \mathcal{S} \rho' + \kappa \mathcal{L}' + 2\mathcal{L} \kappa', \end{aligned} \quad (3.110)$$

$$\begin{aligned}
\delta_\lambda \mathcal{S} = & -\frac{3}{10}i\rho\mathcal{A}'' + \frac{i\zeta\mathcal{A}'}{\sqrt{5}} - i\mathcal{A}'\rho' + \frac{4i\mathcal{A}\zeta'}{\sqrt{5}} - i\mathcal{A}\rho'' - \frac{27\mathcal{A}\mathcal{G}\kappa}{\sqrt{5}c} \\
& - \frac{54i\mathcal{A}\rho\mathcal{L}}{5c} + \frac{63\mathcal{G}\rho\mathcal{G}'}{20c} - \frac{27i\mathcal{G}\kappa\mathcal{L}}{2\sqrt{5}c} + \frac{c\rho^{(4)}}{60} + \frac{27\rho\mathcal{L}^2}{5c} - \frac{i\kappa\mathcal{G}''}{4\sqrt{5}} \\
& - \frac{i\mathcal{G}'\kappa'}{\sqrt{5}} - \frac{3i\mathcal{G}\kappa''}{2\sqrt{5}} - i\kappa\mathcal{S}' + \varepsilon\mathcal{S}' \\
& - \frac{5}{2}i\mathcal{S}\kappa' + \frac{5\mathcal{S}\varepsilon'}{2} + \frac{3\rho\mathcal{L}''}{10} + \rho'\mathcal{L}' + \mathcal{L}\rho''.
\end{aligned} \tag{3.111}$$

The variation of canonical boundary charge $\mathcal{Q}[\lambda]$ (4.29) can be integrated which reads

$$\mathcal{Q}[\lambda] = \int d\varphi [\mathcal{L}\varepsilon + \mathcal{A}\kappa + \mathcal{G}\zeta + \mathcal{S}\rho]. \tag{3.112}$$

This leads to operator product expansions in the complex coordinates by using (4.31)

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{\frac{c}{2}}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}}, \tag{3.113}$$

$$\mathcal{L}(z_1)\mathcal{G}(z_2) \sim \frac{\frac{3}{2}\mathcal{G}}{z_{12}^2} + \frac{\mathcal{G}'}{z_{12}}, \tag{3.114}$$

$$\mathcal{G}(z_1)\mathcal{G}(z_2) \sim \frac{\frac{2c}{3}}{z_{12}^3} + \frac{2\mathcal{L}}{z_{12}}, \tag{3.115}$$

$$\mathcal{L}(z_1)\mathcal{A}(z_2) \sim \frac{2\mathcal{A}}{z_{12}^2} + \frac{\mathcal{A}'}{z_{12}}, \tag{3.116}$$

$$\mathcal{L}(z_1)\mathcal{S}(z_2) \sim \frac{\frac{5}{2}\mathcal{S}}{z_{12}^2} + \frac{\mathcal{S}'}{z_{12}}, \tag{3.117}$$

$$\mathcal{G}(z_1)\mathcal{A}(z_2) \sim -\frac{i\sqrt{5}\mathcal{A}}{z_{12}}, \tag{3.118}$$

$$\mathcal{G}(z_1)\mathcal{S}(z_2) \sim \frac{\frac{4i}{\sqrt{5}}\mathcal{A}}{z_{12}^2} - \frac{\frac{i}{\sqrt{5}}\mathcal{A}'}{z_{12}}, \tag{3.119}$$

$$\mathcal{A}(z_1)\mathcal{A}(z_2) \sim \frac{\frac{c}{2}}{z_{12}^4} + \frac{2\mathcal{L} - 5i\mathcal{A}}{z_{12}^2} + \frac{\mathcal{L} - \frac{5i}{2}\mathcal{A}}{z_{12}}, \tag{3.120}$$

$$\begin{aligned}
\mathcal{A}(z_1)\mathcal{S}(z_2) \sim & -\frac{\frac{3i}{\sqrt{5}}\mathcal{G}}{z_{12}^3} - \frac{\frac{i\mathcal{G}'}{\sqrt{5}} + \frac{5i\mathcal{S}}{2}}{z_{12}^2} \\
& - \frac{1}{z_{12}} \left(\frac{27\mathcal{A}\mathcal{G}}{\sqrt{5}c} + \frac{27i\mathcal{G}\mathcal{L}}{2\sqrt{5}c} + i\mathcal{S}' + \frac{i\mathcal{G}''}{4\sqrt{5}} \right),
\end{aligned} \tag{3.121}$$

$$\begin{aligned}
\mathcal{S}(z_1)\mathcal{S}(z_2) \sim & \frac{\frac{2c}{5}}{z_{12}^5} + \frac{2(\mathcal{L} - i\mathcal{A})}{z_{12}^3} + \frac{\mathcal{L}' - i\mathcal{A}'}{z_{12}^2} \\
& + \frac{1}{z_{12}} \left(\frac{27\mathcal{L}^2}{5c} - \frac{54i\mathcal{A}\mathcal{L}}{5c} + \frac{63\mathcal{G}\mathcal{G}'}{20c} - \frac{3i\mathcal{A}''}{10} + \frac{3\mathcal{L}''}{10} \right).
\end{aligned} \tag{3.122}$$

After repeating the same algebra for $\bar{\Gamma}$ - sector, one can say that the asymptotic symmetry algebra for a set of boundary conditions of $\mathcal{N} = (1, 1)$ supergravity is two copies of the

$\mathcal{SW}(\frac{3}{2}, 2)$ algebra with central charge $c = 6k$. Finally, one can also take into account normal ordering (quantum) effects of this algebra as in Ref's. [90, 91].

3.3 Other Extended (Super)gravity Checks in the Most General Boundary Conditions

In this section, it will be very interesting to show another class of boundary conditions that appeared in the literature (see, eg [15, 18–20]) for gravity case, whose higher - spin generalization is less clear than the Grumiller - Riegler boundary conditions. Because of the possibility to obtain asymptotic symmetries that are the affine version of the gauge algebras of the Chern-Simons theory is not a surprise given the relation between Chern-Simons theories and Wess-Zumino-Witten models [84]. Therefore, we present two most general boundary conditions for pure bosonic gravity and $\mathcal{N} = 1$ extended supergravity. That is, The Avery - Poojary - Suryanarayana gravity for $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)$ and $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)$ respectively. In order to check, we compare the most general boundary conditions used so far with the Avery - Poojary - Suryanarayana boundary conditions for $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)$ pure gravity in section 3.3.1 and $\mathcal{N} = 1$ supergravity for $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)$ in section 3.3.2 respectively after we have worked with the Brown - Henneaux boundary conditions for $\mathcal{N} = 1$ supergravity for $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)$ in section 3.2.2. We also mention the checks that our boundary conditions passed and discuss conservation of the charges and consistency of the variational principle $\mathcal{N} = (1, 1)$ supergravity for $\mathfrak{osp}(1|2)$ in section 3.3.4 and $\mathfrak{osp}(3|2)$ in section 3.3.5 respectively.

3.3.1 Avery - Poojary - Suryanarayana boundary conditions for

$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)$

The Avery - Poojary - Suryanarayana boundary conditions proposed in [20] are a generalization of the Brown - Henneaux boundary conditions and that can be obtained by choosing for Γ - sector the Brown - Henneaux boundary conditions while leaving all charges to vary for the $\bar{\Gamma}$ - sector. Consequently, these lead to an interesting set of asymptotic symmetries in the form of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)_k$. To this end, we recapitulate Avery - Poojary - Suryanarayana gravity for $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)$ from the most general Grumiller - Riegler gravity for $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ as in [21]. Therefore, the most general

connections for Grumiller - Riegler boundary conditions is given by

$$\Gamma - \text{sector} : \quad a_\varphi = \alpha_i \mathcal{L}^i L_i, \quad a_t = \mu^i L_i, \quad (3.123)$$

$$\bar{\Gamma} - \text{sector} : \quad \bar{a}_\varphi = -\alpha_i \bar{\mathcal{L}}^i L_i, \quad \bar{a}_t = \bar{\mu}^i L_i. \quad (3.124)$$

Besides, in the Avery - Poojary - Suryanarayana gravity the boundary conditions is given by

$$\Gamma - \text{sector} : \quad a_\varphi = L_1 - \kappa L_{-1}, \quad a_t = L_1 - \kappa L_{-1}, \quad (3.125)$$

$$\bar{\Gamma} - \text{sector} : \quad \bar{a}_\varphi = L_{-1} - \bar{\kappa} L_1 + f^i L_i, \quad \bar{a}_t = -L_{-1} + \bar{\kappa} L_1 + f^i L_i. \quad (3.126)$$

In this gravity language, this amounts to the following restrictions on the charge and the chemical potential:

$\Gamma - \text{sector} :$

$$\begin{aligned} \mathcal{L}^0 &= 0, & \mathcal{L}^{-1} &= \mathcal{L} = -\frac{\kappa}{\alpha_{-1}}, & \mathcal{L}^1 &= \frac{1}{\alpha_1}, \\ \mu^0 &= 0, & \mu^1 &= \mu = 1, & \mu^{-1} &= \kappa \end{aligned} \quad (3.127)$$

$\bar{\Gamma} - \text{sector} :$

$$\begin{aligned} \bar{\mathcal{L}}^0 &= -\frac{f^0}{\alpha_0}, & \bar{\mathcal{L}}^{-1} &= -\frac{1}{\alpha_{-1}}(1 + f^{-1}), & \bar{\mathcal{L}}^1 &= \frac{1}{\alpha_1}(\bar{\kappa} - f^1), \\ \bar{\mu}^0 &= f^0, & \bar{\mu}^1 &= \mu = \bar{\kappa} + f^1, & \bar{\mu}^{-1} &= f^{-1} - 1, \\ \bar{\mathcal{L}}^a &= \mathcal{T}^a \end{aligned} \quad (3.128)$$

Next we want to find two sets of gauge transformations that preserve the boundary conditions. First, we define the gauge parameter λ for Γ - sector as:

$$a_\varphi = L_1 - \alpha_{-1} \mathcal{L} L_{-1}, \quad \lambda = \varepsilon^i L_i, \quad (3.129)$$

where $\alpha_{-1} = \frac{6}{c}$. If $L_i, (i = \pm 1, 0)$ are the generators of $\mathfrak{sl}(2, \mathbb{R})$ algebra such that $[L_i, L_j] = (i - j)L_{i+j}$ that is, this is the Brown - Henneaux boundary condition that produce the following asymptotic symmetry algebra :

$$\mathcal{L}(z_1) \mathcal{L}(z_2) \sim \frac{c}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\partial \mathcal{L}}{z_{12}} \quad (3.130)$$

with central charge $c = 6k$. Second, we define the gauge parameter $\bar{\lambda}$ for $\bar{\Gamma}$ - sector as:

$$\bar{a}_\varphi = L_{-1} - \alpha_{-1} \mathcal{L} L_1 + \beta_i \mathcal{T}^i T_0^i, \quad \bar{\lambda} = \eta^i L_i + \sigma^i T_0^i. \quad (3.131)$$

where $\alpha_{-1} = \frac{6}{c}$, $\beta_{-1} = \beta_1 = -\frac{2}{k}$, and $\beta_0 = \frac{4}{k}$. If T_n^a , ($a = \pm 1, 0$) are the generators of $\mathfrak{sl}(2)$ current algebra. We have expressed $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)$ algebra such that

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m}, \\ [L_n, T_m^a] &= -mT_{n+m}^a, \\ [T_n^a, T_m^b] &= (a-b)T_{n+m}^{a+b} \end{aligned} \quad (3.132)$$

that is, this is an affine boundary condition that produce the following $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2)_k$ asymptotic symmetry algebra :

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{c}{2z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}}, \quad (3.133)$$

$$\mathcal{L}(z_1)\mathcal{T}^a(z_2) \sim \frac{\mathcal{T}^a}{z_{12}^2} + \frac{\mathcal{T}^{a'}}{z_{12}}, \quad (3.134)$$

$$\mathcal{T}^a(z_1)\mathcal{T}^b(z_2) \sim \frac{\frac{k}{2}\eta^{ab}}{z_{12}^2} + \frac{(a-b)}{z_{12}}\mathcal{T}^{a+b}(z_2) \quad (3.135)$$

with central charge $c = 6k$ and performing a Sugawara shift [21],

$$\mathcal{L} \rightarrow \mathcal{L}^{-1} + \frac{1}{k}(\mathcal{T}^0\mathcal{T}^0 - \mathcal{T}^+\mathcal{T}^-). \quad (3.136)$$

3.3.2 Avery - Poojary - Suryanarayana $\mathcal{N} = 1$ supergravity for

$\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)$

Inspired by previous calculations, in this section, we calculate Avery - Poojary - Suryanarayana $\mathcal{N} = 1$ gravity for $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)$ from the most general Grumiller - Riegler $\mathcal{N} = (1, 1)$ supergravity for $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$. Therefore, the most general connections for Grumiller - Riegler boundary conditions is given by

$$\Gamma - \text{sector} : \quad a_\varphi = \alpha_i \mathcal{L}^i L_i + \beta_p \mathcal{G}^p G_p, \quad (3.137)$$

$$a_t = \mu^i L_i + v^p G_p, \quad (3.138)$$

$$\bar{\Gamma} - \text{sector} : \quad \bar{a}_\varphi = -(\alpha_i \bar{\mathcal{L}}^i L_i + \beta_p \bar{\mathcal{G}}^p G_p), \quad (3.139)$$

$$\bar{a}_t = \bar{\mu}^i L_i + \bar{v}^p G_p. \quad (3.140)$$

Besides, in the Avery - Poojary - Suryanarayana gravity the boundary conditions is given by

$$\Gamma - \text{sector} : a_\varphi = L_1 - \kappa L_{-1} - \omega G_{-\frac{1}{2}}, \quad (3.141)$$

$$a_t = L_1 - \kappa L_{-1} - \omega G_{-\frac{1}{2}}, \quad (3.142)$$

$$\bar{\Gamma} - \text{sector} : \bar{a}_\varphi = L_{-1} - \bar{\kappa} L_1 - \bar{\omega} G_{\frac{1}{2}} + f^i L_i + g^p G_p, \quad (3.143)$$

$$\bar{a}_t = -L_{-1} + \bar{\kappa} L_1 + \bar{\omega} G_{\frac{1}{2}} + f^i L_i + g^p G_p. \quad (3.144)$$

In this gravity language, this amounts to the following restrictions on the charges and the chemical potentials:

$\Gamma - \text{sector} :$

$$\begin{aligned} \mathcal{L}^0 &= 0, & \mathcal{L}^{-1} &= \mathcal{L} = -\frac{\kappa}{\alpha_{-1}}, & \mathcal{L}^1 &= \frac{1}{\alpha_1}, \\ \mathcal{G}^{-\frac{1}{2}} &= -\frac{\omega}{\beta_{-\frac{1}{2}}}, & \mathcal{G}^{\frac{1}{2}} &= 0, & & \\ \mu^0 &= 0, & \mu^1 &= \mu = 1, & \mu^{-1} &= \kappa, \\ \nu^{-\frac{1}{2}} &= -\omega, & \nu^{\frac{1}{2}} &= 0. & & \end{aligned} \quad (3.145)$$

$\bar{\Gamma} - \text{sector} :$

$$\begin{aligned} \bar{\mathcal{L}}^0 &= -\frac{f^0}{\alpha_0}, & \bar{\mathcal{L}}^{-1} &= -\frac{1}{\alpha_{-1}}(1 + f^{-1}), & \bar{\mathcal{L}}^1 &= \frac{1}{\alpha_1}(\bar{\kappa} - f^1), \\ \bar{\mathcal{G}}^{-\frac{1}{2}} &= \frac{g^{-\frac{1}{2}}}{\beta_{-\frac{1}{2}}}, & \bar{\mathcal{G}}^{\frac{1}{2}} &= \frac{\bar{\omega} - g^{\frac{1}{2}}}{\beta_{\frac{1}{2}}}, & & \\ \bar{\mu}^0 &= f^0, & \bar{\mu}^1 &= \mu = \bar{\kappa} + f^1, & \bar{\mu}^{-1} &= f^{-1} - 1, \\ \bar{\nu}^{-\frac{1}{2}} &= g^{-\frac{1}{2}}, & \bar{\nu}^{\frac{1}{2}} &= \bar{\omega} + g^{\frac{1}{2}}. & & \end{aligned} \quad (3.146)$$

Next, we want to find two sets of gauge transformations that preserve the boundary conditions. First, we define the gauge parameter λ for Γ - sector as:

$$a_\varphi = L_1 - \alpha_{-1} \mathcal{L} L_{-1} - \beta_{-\frac{1}{2}} \mathcal{G} G_{-\frac{1}{2}}, \quad \lambda = \varepsilon^i L_i + \zeta^p G_p. \quad (3.147)$$

This is the Brown - Henneaux boundary condition for the $\Gamma - \text{sector}$ that produces one copy of the asymptotic symmetry algebra as in section 4.1.3. Second, we define the gauge parameter $\bar{\lambda}$ for $\bar{\Gamma} - \text{sector}$ as:

$$\bar{a}_\varphi = L_{-1} - \alpha_{-1} \mathcal{L} L_1 - \beta_{-\frac{1}{2}} \mathcal{G} G_{\frac{1}{2}} + \gamma_i \mathcal{T}^i T_0^i, \quad \bar{\lambda} = \varepsilon^i L_i + \zeta^p G_p + \sigma^i T_0^i. \quad (3.148)$$

where $\alpha_{-1} = \frac{6}{c}$, $\beta_{-\frac{1}{2}} = -\frac{3}{c}$, $\gamma_{-1} = \gamma_1 = -\frac{2}{k}$, and $\gamma_0 = \frac{4}{k}$. If $\mathcal{T}_n^a$, ($a = \pm 1, 0$) are the generators of $\mathfrak{sl}(2)$ current algebra. We have expressed $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)$ superalgebra

such that

$$[L_n, L_m] = (n - m)L_{n+m}, \quad (3.149)$$

$$[L_n, G_p] = \left(\frac{n}{2} - p\right)G_{n+p}, \quad (3.150)$$

$$\{G_p, G_q\} = -2L_{p+q}, \quad (3.151)$$

$$[L_n, T_m^a] = -mT_{n+m}^a, \quad (3.152)$$

$$[T_n^a, T_m^b] = (a - b)T_{n+m}^{a+b}. \quad (3.153)$$

that is, this is an affine boundary condition that produces the following $\mathfrak{osp}(1|2) \oplus \mathfrak{sl}(2)_k$ asymptotic symmetry algebra :

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{c}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}}, \quad (3.154)$$

$$\mathcal{L}(z_1)\mathcal{G}(z_2) \sim \frac{\frac{3}{2}\mathcal{G}}{z_{12}^2} + \frac{\mathcal{G}'}{z_{12}}, \quad \mathcal{G}(z_1)\mathcal{G}(z_2) \sim \frac{\frac{2c}{3}}{z_{12}^3} + \frac{2\mathcal{L}}{z_{12}}, \quad (3.155)$$

$$\mathcal{L}(z_1)\mathcal{T}^a(z_2) \sim \frac{T^a}{z_{12}^2} + \frac{T^{a'}}{z_{12}}, \quad \mathcal{T}^a(z_1)\mathcal{T}^b(z_2) \sim \frac{\frac{k}{2}\eta^{ab}}{z_{12}^2} + \frac{(a-b)}{z_{12}}\mathcal{T}^{a+b}. \quad (3.156)$$

with central charge $c = 6k$, and the same Sugawara shift (3.136).

3.3.3 The on - shell action and the variational principle

It is well known that on manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ the Chern - Simons action (3.2) in general is neither differentiable nor gauge invariant. Therefore, it is important to show that the boundary conditions admit a well-defined variational principle. To see this, take a closer look at the variation of the Chern - Simons action,

$$\delta S_{\text{cs}}[\Gamma] = \frac{k}{2\pi} \int_{\mathcal{M}} \langle \delta\Gamma \wedge F \rangle + \frac{k}{4\pi} \int_{\partial\mathcal{M}} \langle \delta\Gamma \wedge \Gamma \rangle, \quad (3.157)$$

where, due that on - shell $F = 0$, the on - shell bulk part is zero and we are left with only a boundary term :

$$\delta S_{\text{cs}}[\Gamma] = \frac{k}{4\pi} \int_{\partial\mathcal{M}} dt d\varphi \langle \Gamma_\varphi \wedge \delta\Gamma_t - \Gamma_t \wedge \delta\Gamma_\varphi \rangle. \quad (3.158)$$

Therefore, it would be good to have an appropriate way of extending the possible consistent boundary conditions. A consistent way is to add a boundary term $S_{\text{b}}[\Gamma]$ to the Chern - Simons action. So, the total action will then be of the form :

$$S_{\text{tot}} = S_{\text{cs}} + \frac{k}{4\pi} \int_{\partial\mathcal{M}} dt d\varphi \langle \Gamma_t \wedge \Gamma_\varphi \rangle. \quad (3.159)$$

3.3.4 $\mathfrak{osp}(1|2)$ - case :

Then, considering a solution of the form (3.137), for our boundary conditions is very easy to see that the radial dependent group element plays no role in this analysis, in fact plugging (3.137) into (3.159) we have :

$$\begin{aligned}\delta S_{\text{tot}}[\Gamma] &= -\frac{k}{2\pi} \int_{\partial\mathcal{M}} dt d\varphi \langle a_\varphi \delta a_t \rangle \\ &= \frac{k}{2\pi} \int_{\partial\mathcal{M}} dt d\varphi \langle \mathcal{L}^i \delta \mu^{-i} + \mathcal{G}^p \delta v^{-p} \rangle \\ &= 0\end{aligned}\tag{3.160}$$

because of $\delta a_t = 0$. A similar calculation should be made for the barred sector. This solution ensures for that $\alpha_1 = -\frac{2\pi}{k}$, $\alpha_0 = \frac{4\pi}{k}$, $\alpha_{-1} = -\frac{2\pi}{k}$, $\beta_{\frac{1}{2}} = \frac{\pi}{k}$ and $\beta_{-\frac{1}{2}} = -\frac{\pi}{k}$ in (4.13).

3.3.5 $\mathfrak{osp}(3|2)$ - case :

After the calculation of $\mathfrak{osp}(1|2)$ - case, it could be an interesting straightforward exercise to establish a well-defined variational principle also for $\mathcal{N} = (1, 1)$ extended higher - spin supergravity with two copies of the $\mathfrak{osp}(3|2)_k$ affine algebra as follows. To this end, considering a solution of the form (3.64), plugging these into (3.159) we have :

$$\begin{aligned}\delta S_{\text{tot}}[\Gamma] &= \frac{k}{2\pi} \int_{\partial\mathcal{M}} dt d\varphi \langle \tilde{\mathcal{L}}^i \delta \mu^{-i} + \tilde{\mathcal{A}}^i \delta \chi^{-i} + \mathcal{G}^p \delta f^{-p} + \mathcal{S}^p \delta v^{-p} \rangle \\ &= 0\end{aligned}\tag{3.161}$$

because of $\delta a_t = 0$, under the re-definitions of $\tilde{\mathcal{L}}^i = 3\mathcal{L}^i + 5\mathcal{A}^i$ and $\tilde{\mathcal{A}}^i = 5\mathcal{L}^i + 3\mathcal{A}^i$, A similar calculation should be made for the barred sector. This solution finally ensures for that $\alpha_1 = -\frac{2\pi}{k}$, $\alpha_0 = \frac{4\pi}{k}$, $\alpha_{-1} = -\frac{2\pi}{k}$, $\beta_{\frac{1}{2}} = \frac{\pi}{3k}$, $\beta_{-\frac{1}{2}} = -\frac{\pi}{3k}$, $\gamma_1 = -\frac{2\gamma}{k}$, $\alpha_0 = \frac{4\pi}{k}$, $\gamma_{-1} = -\frac{2\pi}{k}$, $\tau_{\frac{3}{2}} = \frac{\pi}{9k}$, $\tau_{\frac{1}{2}} = \frac{\pi}{3k}$, $\tau_{-\frac{1}{2}} = -\frac{\pi}{3k}$ and $\tau_{-\frac{3}{2}} = \frac{\pi}{9k}$ in (3.64).

3.4 Summary and Comments

In this chapter, a relation between AdS_3 and $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ Chern - Simons theory was first reviewed. The Chern - Simons formulation of AdS_3 allows for a straightforward generalization to a higher - spin theory as in the bosonic cases. The higher - spin gauge fields have no propagating degrees of freedom, but we noted that there can be a large class of interesting non-trivial solutions. Specifically, AdS_3 in the presence of a tower of higher - spin fields up to spin $\frac{5}{2}$ is obtained by enlarging $\mathfrak{osp}(1|2) \oplus \mathfrak{osp}(1|2)$ to

$\mathfrak{osp}(3|2) \oplus \mathfrak{osp}(3|2)$ for the most general $\mathcal{N} = (1, 1)$ extended higher - spin supergravity theory in AdS_3 . Finally, classical two copies of the $\mathfrak{osp}(3|2)_k$ affine algebra on the affine boundary and two copies of $\mathcal{SW}(\frac{3}{2}, 2)$ symmetry algebra on the superconformal boundary as asymptotic symmetry algebras, and also the chemical potentials related to source fields appearing through the temporal components of the connection are obtained. Finally, higher - spin generalization of other class of boundary conditions, in particular for the Avery - Poojary - Suryanarayana gravity, that appeared in the literature for gravity case is also shown and checked. On the other hand, it is shown that the Chern - Simons action which compatible with our boundary conditions leads to a finite action and a well - defined variational principle for the higher - spin fields. Therefore, one can think that this method provides a good laboratory for investigating the rich asymptotic structure of extended supergravity.

It could be rewarding to cast our results in the metric formulation, since this may help in lifting our boundary conditions to higher dimensions where Chern - Simons formulation exists. Such generalizations are interesting in itself and have the potential for novel applications in holography. Finally, in our discussion of the most general boundary conditions of (super)gravity, we have barely scratched the boundary. Therefore, there could be numerous open questions that call for further investigations. For instance, which other boundary conditions can be obtained from a similar starting point? Or how to explain the puzzling result that the related geometries appear to have an entropy?

In conclusion, we consider it gratifying that the asymptotically AdS story of the most general boundary conditions initiated recently by Grumiller and Riegler to inspire new and surprising developments even in the simple cases of three - dimensional gravity.

Our results presented in this chapter can be extended in various ways. One possible extension is by enlarging $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ to $\mathfrak{sl}(3|2) \oplus \mathfrak{sl}(3|2)$ supergravity for the most general $\mathcal{N} = (2, 2)$ extended higher - spin supergravity theory in AdS_3 . In this context, one can finally emphasise here that the rather comprehensive analysis of the $\mathfrak{sl}(3|2) \oplus \mathfrak{sl}(3|2)$ theory for the Brown - Henneaux boundary conditions has been already given in Ref. [44]. The details of this possible extension will be examined in next chapter.



4. $N=(2,2)$ EXTENDED $sl(3|2)$ CHERN-SIMONS AdS_3 SUPERGRAVITY WITH NEW BOUNDARIES

In modern theoretical physics, unquestionably, one of the forefront achievements in the past few decades is the discovery of the AdS/CFT correspondence, which was first presented concretely by Maldacena [1] in late 1997. This remarkable duality has profound implications ranging from a better understanding of many aspects of theoretical (and even experimental) physics, especially general relativity, quantum gravity, quantum field theory, higher spin theory, and black holes. Moreover, the AdS/CFT correspondence is an important manifestation of the holographic principle that posits a relation between a certain classical gravitational theory and a lower-dimensional non-gravitational one. The AdS_3/CFT_2 correspondence which is also a useful testing arena in this respect, implies an equivalence between pure Einstein AdS_3 gravity with a negative cosmological constant in 3D and a 2-dimensional conformal field theory. As a matter of fact, the main advantage of this eminent correspondence in three dimensions is to allow Einstein's gravity can be reformulated as a Chern - Simons gauge theory in such a way that all the structure is considerably simplified [2, 3]. What they discovered in their pioneer work is that, in three dimensions, the action and equations of motions are equivalent to a Chern - Simons theory for an appropriate gauge group.

Despite the simplicity owing to its topological nature, besides being a very rich and spectacular theory, it is well known that three dimensional gravity has outstanding holographic properties. In this context, the striking feature of 3 - dimensional Einstein's Gravity is the absence of any local, propagating degrees of freedom, which means that any negatively curved Einstein space is locally AdS_3 . There are only global degrees of freedom and hence no graviton in three dimensions. Notwithstanding that there are no local propagating degrees of freedom in the theory, its dynamic content is far from being insignificant due to the existence of boundary conditions. In other words, this means that the theory is wholly determined by global effects, since general relativity turns into a topological field theory, whose dynamics can be portrayed holographically by a 2 - dimensional conformal field theory at the boundary. That is a Chern - Simons theory

in an equivalent formulation. At this point, it would be fair to say that the dynamics of the theory is totally presided over by boundary conditions. Furthermore, one should call attention here that there is an infinite number of degrees of freedom living on the boundary under an appropriate choice of boundary conditions. These boundary conditions are requisite in order to provide that the action has a well-defined variational principle. Nevertheless, their choice is not unique. Essentially, the dynamic features of the theory take shape according to the choice of these boundary conditions. So the residual gauge symmetry on the boundary within this framework emerges as global symmetry (asymptotic symmetry).

One of the most crucial results this story aforementioned tells us is that the asymptotic boundary conditions play a vital role in AdS_3 gravity. In their seminal paper [6], Brown and Henneaux proposed that under a convenient choice of boundary conditions, asymptotic symmetry algebras of AdS_3 gravity yields two copies of the Virasoro algebras with a classical central extension. The reason why this significant result is pointed out as a pioneer work is the fact that it is actually the first realization of AdS/CFT correspondence, and also an important realization of holographic duality. Incidentally, another notable strength of $2+1$ dimensional gravity is that it also contains black hole solutions such as the famous BTZ black hole in which Einstein equations admit in the presence of the negative cosmological constant [4,5]. Parenthetically, since the BTZ geometry is algebraically simple, it has provided quite a useful playground for studying features of black holes, which has a tremendous importance in exploring both classical and quantum gravitational physics.

It is worth mentioning that the Chern - Simons formulation of higher spin gravity in three dimensions has attracted more attention by the discovery of the Chern - Simons theories based on gauge algebras such as $\mathfrak{sl}(N|\mathbb{R})$ and $hs(\lambda)$ are versions of Vasiliev higher spin theories [28,29] and also these are purely bosonic theories [46,47] with higher spin fields of integer spin. Moreover, the Chern - Simons higher spin theories could pan out with a realization of the classical $\mathcal{W}_N$ asymptotic symmetry algebras as in the related two-dimensional CFT 's [38–43]. The promising results obtained from this perspective are adapted to extend the theory to the supergravity [44,45] as well as higher spin theory [46,47]. Additionally, a supersymmetric generalization of these bosonic theories can be accomplished by keeping in view Chern - Simons theories based

on superalgebras such as $\mathfrak{sl}(N|N-1)$, see, e.g. [44, 45, 48–51], or $\mathfrak{osp}(N|N-1)$ [52] which can be obtained by truncating out all the odd spin generators and one copy of the fermionic operators in $\mathfrak{sl}(N|N-1)$.

As already noted, one can casually state that three dimensional gravity is all about the choice of boundary conditions. More precisely, the specification of boundary conditions is pivotal in comprehending how a theory that (locally) admits only a single solution. This analysis was first carried out by Brown and Henneaux [6] in their famous paper. Their study has also been encouraging to propose new sets of boundary conditions by sparking a vigorous research area which has gained in breadth over the years modifying [14, 15, 17–21, 92] and generalizing [7–13, 16] these bc's. In [21], Grumiller and Riegler have considered the most general AdS_3 boundary conditions, as a consequence, they have derived the asymptotic symmetry algebra consists of two $\mathfrak{sl}(2)_k$ current algebras. Furthermore, they have recovered all other previously found boundary conditions, imposing some certain restrictions to their most general boundary conditions. It is pertinent to address that there are several papers recently inspired by them, i.e., flat space [22] and chiral higher spin gravity [23], which is shown a new class of boundary conditions for higher spin theories in AdS_3 . The simplest extension of Grumiller and Riegler's procedure for the most general $\mathcal{N} = (1, 1)$, and $\mathcal{N} = (2, 2)$ extended higher spin supergravity is introduced by Valcarcel [85] where the asymptotic symmetry algebra for the loosest set of boundary conditions for (extended) supergravity obtained. The most general $\mathcal{N} = (1, 1)$ extended AdS_3 higher spin supergravity theory has been similarly presented, including further in [54].

This chapter is concerned with the previously unresolved phenomenon; we construct a candidate solution for the most general $\mathcal{N} = (2, 2)$ extended higher spin supergravity theory in AdS_3 . We show that our theory falls under the same metric class as [22], in which it is seen that the metric formulation could include even both charge and chemical potentials which are present in the Chern - Simons formalism. This can be considered as an alternative solution to the non - chiral Drinfeld - Sokolov type boundary conditions. Firstly, we focus on the simplest case $\mathcal{N} = (2, 2)$ Chern - Simons theory based on the $\mathfrak{sl}(2|1)_k$ superalgebra. The related asymptotic symmetry algebra is two copies of the $\mathfrak{sl}(2|1)_k$ affine algebra. Then, we are dealing with the extended $\mathcal{N} = (2, 2)$ Chern - Simons theory based on $\mathfrak{sl}(3|2)$ superalgebra, as a result, we obtain asymptotic

symmetry algebra consists of two copies of the $\mathfrak{sl}(3|2)_k$ affine algebra. Additionally, we impose some certain restrictions to the gauge fields on the most general boundary conditions, which leading us to the supersymmetric extensions of the Brown - Henneaux boundary conditions. We also show that the asymptotic symmetry algebras are reduced to two copies of the super $\mathcal{W}_3$ algebra for the most general $\mathcal{N} = (2,2)$ extended higher spin AdS_3 supergravity theory. It is useful to indicate that it would be an interesting problem in its own right perform a different class of boundary conditions for (super)gravity that emerges in the literature (see, e.g. [15, 18–20]), since their higher spin generalization is not as clear as Grumiller and Riegler’s boundary conditions. In light of all these results, it is inevitable to say that this method yields an excellent laboratory to investigate the rich asymptotic structure of extended higher spin supergravity.

This chapter is organized as follows. We first introduce a fundamental formulation of $\mathcal{N} = (2,2)$ supergravity as $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern - Simons gauge theory for both affine and superconformal boundaries, respectively. Then, in section 4.2, we maintain our calculations to extend the theory $\mathfrak{sl}(3|2) \oplus \mathfrak{sl}(3|2)$ higher - spin Chern - Simons supergravity in the case of both affine and superconformal boundaries, in where we reveal explicitly principal embedding of $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ and also we come up with how asymptotic symmetry and higher spin Ward identities arise from these bulk equations of motion coupled to spin s , ($s = 1, \frac{3}{2}, \frac{3}{2}, 2, 2, \frac{5}{2}, \frac{5}{2}, 3$) currents. We dedicate this section to perform the asymptotic symmetry algebras as classical two copies of the $\mathfrak{sl}(3|2)_k$ affine algebra on the affine boundary and the super $\mathcal{W}_3$ symmetry algebra on the superconformal boundary, respectively. Besides, we describe the chemical potentials related to source fields appearing through the temporal components of the connection. In the final section, we conclude with a discussion, open issues and future research directions.

4.1 Review of Chern - Simons Supergravity in Three Dimensions:

In this section, we give a brief discussion for AdS_3 higher spin supergravity based on Chern - Simons formalism. We especially employ this formulation to analyze AdS_3 supergravity in the presence of $\mathfrak{sl}(2|1)$ superalgebra basis, belonging to the same metric

class as Grumiller and Riegler's recently proposed, the most general AdS_3 boundary conditions [21].

4.1.1 Connection to Chern - Simons theory

In three dimensions, Einstein - Hilbert action for $\mathcal{N} = (2, 2)$ supergravity with a negative cosmological constant, can be defined in an equivalent Chern - Simons formulation over a spacetime manifold $\mathcal{M}$ as

$$S = S_{CS}[\Gamma] - S_{CS}[\bar{\Gamma}] \quad (4.1)$$

where

$$S_{CS}[\Gamma] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathfrak{str} \left(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right) \quad (4.2)$$

which was first noted by Achucarro and Townsend [2] and further developed by Witten [3].

The Chern-Simons level k will eventually be related to ratio of AdS_3 radius ℓ and Newton constant G , and also the related central charge c of the superconformal field theory as $k = \frac{\ell}{8G\mathfrak{str}(\mathbb{L}_0\mathbb{L}_0)} = \frac{c}{12\mathfrak{str}(\mathbb{L}_0\mathbb{L}_0)}$. Notice that while the 1-forms $(\Gamma, \bar{\Gamma})$ connections are defined as to take values in the gauge group of $\mathfrak{sl}(2|1)$ superalgebra, the supertrace $\mathfrak{str}$ which shows a metric on the $\mathfrak{sl}(2|1)$ Lie superalgebra, is taken over the superalgebra generators.

It is convenient to get started with standard basis for $\mathfrak{sl}(2|1)$ Lie superalgebra. We denote the bosonic generators by $\mathbb{L}_i$ ($i = \pm 1, 0$), J and the fermionic ones by G_r^M ($r = \pm \frac{1}{2}, M = \pm$), whose commutations relations read

$$[\mathbb{L}_i, \mathbb{L}_j] = (i - j)\mathbb{L}_{i+j}, \quad (4.3)$$

$$[\mathbb{L}_i, G_r^\pm] = \left(\frac{i}{2} - r \right) G_{i+r}^\pm, \quad (4.4)$$

$$[J, G_r^\pm] = \pm G_r^\pm, \quad (4.5)$$

$$\{G_r^\pm, G_s^\mp\} = 2\mathbb{L}_{r+s} \pm (r - s)J \quad (4.6)$$

except for zero commutators.

The Chern - Simons equations of motions, also known as the flatness conditions correspond to vanishing field strengths; $F = \bar{F} = 0$ where

$$F = d\Gamma + \Gamma \wedge \Gamma = 0, \quad \bar{F} = d\bar{\Gamma} + \bar{\Gamma} \wedge \bar{\Gamma} = 0 \quad (4.7)$$

which is equivalent to Einstein's equation. The relation to the Einstein's equation is made by expressing Lie algebra valued generalizations of the vielbein and spin connection in terms of the gauge connections. Then, one can obtain the metric $g_{\mu\nu}$ from the vielbein $e = \frac{\ell}{2}(\Gamma - \bar{\Gamma})$ in the usual fashion

$$g_{\mu\nu} = \frac{1}{2} \text{str}(e_\mu e_\nu). \quad (4.8)$$

By the choice of the radial gauge, asymptotically AdS_3 connections can be taken to have the form

$$\Gamma = b^{-1}a(t, \phi)b + b^{-1}db, \quad \bar{\Gamma} = b\bar{a}(t, \phi)b^{-1} + bdb^{-1} \quad (4.9)$$

with state - independent group element (called Grumiller-Riegler gauge);

$$b = e^{L_{-1}} e^{\rho L_0} \quad (4.10)$$

which yields a more general metric and means that it includes all $\mathfrak{sl}(2|1)$ charges and chemical potentials can be chosen accordingly. At this point, it is important to note that as long as $\delta b = 0$, the choice of b is irrelevant for asymptotic symmetries. Unlike the standard choice of b , this freedom enables a more general metric. Therefore, it is crucial to choose the most general boundary conditions preserving this most general metric form for supergravity.

Further, in the radial gauge $a(t, \phi)$ and $\bar{a}(t, \phi)$ connections are the $\mathfrak{sl}(2|1)$ Lie superalgebra valued fields which are independent of a radial coordinate as

$$\begin{aligned} a(t, \phi) &= a_t(t, \phi)dt + a_\phi(t, \phi)d\phi \\ \bar{a}(t, \phi) &= \bar{a}_t(t, \phi)dt + \bar{a}_\phi(t, \phi)d\phi \end{aligned} \quad (4.11)$$

Hereafter only focused on the unbarred sector, since the analysis of the barred sector works in complete analogy yielding the same outcomes with the barred sector and it can be figured out by the same algorithm thanks to the procedure used.

4.1.2 $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern - Simons $\mathcal{N} = (2, 2)$ supergravity for affine boundary

We begin by reviewing asymptotically AdS_3 boundary conditions for $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern - Simons theory in the affine case. We present how the procedure mentioned in [21] can be used to evaluate the asymptotic symmetry algebra. According to the

results obtained, the most general solution of Einstein's equation that is asymptotically AdS_3 is defined by the following general metric form:

$$ds^2 = d\rho^2 + 2 \left[e^\rho N_i^{(0)} + N_i^{(1)} + e^{-\rho} N_i^{(2)} + \mathcal{O}(e^{-2\rho}) \right] d\rho dx^i + \left[e^{2\rho} g_{ij}^{(0)} + e^\rho g_{ij}^{(1)} + g_{ij}^{(2)} + \mathcal{O}(e^{-\rho}) \right] dx^i dx^j. \quad (4.12)$$

So, it is important to define the most general $\mathcal{N} = (2, 2)$ supergravity boundary conditions which preserve this form of the metric.

We start by proposing $\mathfrak{sl}(2|1)$ Lie superalgebra valued a_φ component of the gauge connection in the form:

$$a_\varphi = \rho \mathcal{J} + \gamma_i \mathcal{L}^i L_i + \sigma_M^p \mathcal{G}_M^p \mathcal{G}_p^M \quad (4.13)$$

where $\rho = \frac{1}{k}$, $\frac{\gamma_0}{2} = -\gamma_{\pm 1} = \frac{2}{k}$, $\sigma_\pm^{-\frac{1}{2}} = -\sigma_\pm^{\frac{1}{2}} = \frac{1}{k}$ are some scaling parameters to be identified later. We have eight state-dependent functions consisting of four bosonic $(\mathcal{J}, \mathcal{L}^i)$ and four fermionic $\mathcal{G}_M^p$, usually called as *charges*. The time component a_t of the connection $a(t, \varphi)$ can be given as

$$a_t = \eta \mathcal{J} + \mu^i L_i + v_M^p \mathcal{G}_p^M. \quad (4.14)$$

In this case, we have eight independent functions (η, μ^i, v_M^p) , as chemical potentials which are not allowed to vary, $\delta a_t = 0$.

Using the flatness conditions (4.7), the equations of motions for fixed chemical potentials impose the following additional conditions on the charges $(\mathcal{J}, \mathcal{L}^i, \mathcal{G}_M^p)$:

$$2\partial_t \mathcal{L}^0 = \frac{k}{2} \partial_\varphi \mu^0 + 2\mathcal{L}^{+1} \mu^{-1} + 2\mathcal{L}^{-1} \mu^{+1} - \mathcal{G}_-^{+\frac{1}{2}} v_+^{-\frac{1}{2}} + \mathcal{G}_-^{-\frac{1}{2}} v_+^{+\frac{1}{2}} - \mathcal{G}_+^{+\frac{1}{2}} v_-^{-\frac{1}{2}} + \mathcal{G}_+^{-\frac{1}{2}} v_-^{+\frac{1}{2}}, \quad (4.15)$$

$$\partial_t \mathcal{L}^{\pm 1} = -\frac{k}{2} \partial_\varphi \mu^{\pm 1} \pm \mathcal{L}^0 \mu^{\pm 1} \pm \mathcal{L}^{\pm 1} \mu^0 + \mathcal{G}_+^{\pm \frac{1}{2}} v_-^{\pm \frac{1}{2}} + \mathcal{G}_-^{\pm \frac{1}{2}} v_+^{\pm \frac{1}{2}}, \quad (4.16)$$

$$\partial_t \mathcal{G}_\pm^{\pm \frac{1}{2}} = \pm k \partial_\varphi v_\pm^{\pm \frac{1}{2}} \pm 2\mathcal{L}^\pm v_\pm^{\mp \frac{1}{2}} + 2\mathcal{L}^0 v_\pm^{\pm \frac{1}{2}} \pm \mathcal{G}_\pm^{\mp \frac{1}{2}} \mu^\pm \pm \frac{1}{2} \mu^0 \mathcal{G}_\pm^{\pm \frac{1}{2}} \mp \mathcal{J} v_\pm^{\pm \frac{1}{2}} - \eta \mathcal{G}_\pm^{\pm \frac{1}{2}}, \quad (4.17)$$

$$\partial_t \mathcal{J} = k \partial_\varphi \eta + \mathcal{G}_-^{+\frac{1}{2}} v_+^{-\frac{1}{2}} + \mathcal{G}_-^{-\frac{1}{2}} v_+^{+\frac{1}{2}} - \mathcal{G}_+^{+\frac{1}{2}} v_-^{-\frac{1}{2}} - \mathcal{G}_+^{-\frac{1}{2}} v_-^{+\frac{1}{2}}, \quad (4.18)$$

that represents the temporal evolution of the eight state-dependent source fields.

We want to derive asymptotic symmetry algebra for the most general boundary conditions through a canonical analysis. That's why we embark on by considering all

gauge transformations:

$$\delta_\lambda \Gamma = d\lambda + [\Gamma, \lambda] \quad (4.19)$$

which preserve the most general boundary conditions. At this point, it would be appropriate to single out the gauge parameter in terms of the $\mathfrak{sl}(2|1)$ Lie superalgebra basis

$$\lambda = b^{-1} \left[\rho J + \varepsilon^i L_i + \zeta_M^P G_P^M \right] b. \quad (4.20)$$

Note that the gauge parameter includes four bosonic ρ, ε^i and four fermionic ζ_M^P , arbitrary functions of boundary coordinates. And also, we are concerned with the gauge parameters that satisfy (4.19). One can now determine the boundary preserving gauge transformations. Accordingly, the infinitesimal gauge transformations are given by;

$$\begin{aligned} 2\partial_t \mathcal{L}^0 &= \frac{k}{2} \partial_\varphi \varepsilon^0 + 2\mathcal{L}^{+1} \varepsilon^{-1} + 2\mathcal{L}^{-1} \varepsilon^{+1} - \mathcal{G}_-^{+\frac{1}{2}} \zeta_+^{-\frac{1}{2}} + \mathcal{G}_-^{-\frac{1}{2}} \zeta_+^{+\frac{1}{2}} \\ &\quad - \mathcal{G}_+^{+\frac{1}{2}} \zeta_-^{-\frac{1}{2}} + \mathcal{G}_+^{-\frac{1}{2}} \zeta_-^{+\frac{1}{2}}, \end{aligned} \quad (4.21)$$

$$\partial_t \mathcal{L}^{\pm 1} = -\frac{k}{2} \partial_\varphi \varepsilon^{\pm 1} \pm \mathcal{L}^0 \varepsilon^{\pm 1} \pm \mathcal{L}^{\pm 1} \varepsilon^0 + \mathcal{G}_+^{\pm \frac{1}{2}} \zeta_{\mp}^{\pm \frac{1}{2}} + \mathcal{G}_-^{\pm \frac{1}{2}} \zeta_{\pm}^{\pm \frac{1}{2}}, \quad (4.22)$$

$$\begin{aligned} \partial_t \mathcal{G}_\pm^{\pm \frac{1}{2}} &= \pm k \partial_\varphi \zeta_\pm^{\pm \frac{1}{2}} \pm 2\mathcal{L}^\pm \zeta_\pm^{\mp \frac{1}{2}} + 2\mathcal{L}^0 \zeta_\pm^{\pm \frac{1}{2}} \pm \mathcal{G}_\pm^{\mp \frac{1}{2}} \varepsilon^\pm \pm \frac{1}{2} \varepsilon^0 \mathcal{G}_\pm^{\pm \frac{1}{2}} \\ &\quad \mp \mathcal{J} \zeta_\pm^{\pm \frac{1}{2}} - \rho \mathcal{G}_\pm^{\pm \frac{1}{2}}, \end{aligned} \quad (4.23)$$

$$\partial_t \mathcal{J} = k \partial_\varphi \rho + \mathcal{G}_-^{+\frac{1}{2}} \zeta_+^{-\frac{1}{2}} + \mathcal{G}_-^{-\frac{1}{2}} \zeta_+^{+\frac{1}{2}} - \mathcal{G}_+^{+\frac{1}{2}} \zeta_-^{-\frac{1}{2}} - \mathcal{G}_+^{-\frac{1}{2}} \zeta_-^{+\frac{1}{2}}. \quad (4.24)$$

One can also derive the following constraints for the chemical potentials analogously.

$$\begin{aligned} 2\partial_t \mu^0 &= \frac{k}{2} \partial_\varphi \varepsilon^0 + 2\mu^{+1} \varepsilon^{-1} + 2\mu^{-1} \varepsilon^{+1} - \mathbf{v}_-^{+\frac{1}{2}} \zeta_+^{-\frac{1}{2}} + \mathbf{v}_-^{-\frac{1}{2}} \zeta_+^{+\frac{1}{2}} \\ &\quad - \mathbf{v}_+^{+\frac{1}{2}} \zeta_-^{-\frac{1}{2}} + \mathbf{v}_+^{-\frac{1}{2}} \zeta_-^{+\frac{1}{2}}, \end{aligned} \quad (4.25)$$

$$\partial_t \mu^{\pm 1} = -\frac{k}{2} \partial_\varphi \varepsilon^{\pm 1} \pm \mu^0 \varepsilon^{\pm 1} \pm \mu^{\pm 1} \varepsilon^0 + \mathbf{v}_+^{\pm \frac{1}{2}} \zeta_{\mp}^{\pm \frac{1}{2}} + \mathbf{v}_-^{\pm \frac{1}{2}} \zeta_{\pm}^{\pm \frac{1}{2}}, \quad (4.26)$$

$$\begin{aligned} \partial_t \mathbf{v}_\pm^{\pm \frac{1}{2}} &= \pm k \partial_\varphi \zeta_\pm^{\pm \frac{1}{2}} \pm 2\mu^\pm \zeta_\pm^{\mp \frac{1}{2}} + 2\mu^0 \zeta_\pm^{\pm \frac{1}{2}} \pm \mathbf{v}_\pm^{\mp \frac{1}{2}} \varepsilon^\pm \pm \frac{1}{2} \varepsilon^0 \mathbf{v}_\pm^{\pm \frac{1}{2}} \\ &\quad \mp \eta \zeta_\pm^{\pm \frac{1}{2}} - \rho \mathbf{v}_\pm^{\pm \frac{1}{2}}, \end{aligned} \quad (4.27)$$

$$\partial_t \eta = k \partial_\varphi \rho + \mathbf{v}_-^{+\frac{1}{2}} \zeta_+^{-\frac{1}{2}} + \mathbf{v}_-^{-\frac{1}{2}} \zeta_+^{+\frac{1}{2}} - \mathbf{v}_+^{+\frac{1}{2}} \zeta_-^{-\frac{1}{2}} - \mathbf{v}_+^{-\frac{1}{2}} \zeta_-^{+\frac{1}{2}}. \quad (4.28)$$

As a final step, the canonical boundary charge $\mathcal{Q}[\lambda]$ that generates the transformations (4.21)-(4.24) can be defined. For this purpose, the variation of the canonical boundary charge $\mathcal{Q}[\lambda]$ [81, 86–88] leading the asymptotic symmetry algebra is given by

$$\delta_\lambda \mathcal{Q} = \frac{k}{2\pi} \int d\varphi \, \text{str}(\lambda \delta \Gamma_\varphi). \quad (4.29)$$

Hence, the variation of the canonical boundary charge $\mathcal{Q}[\lambda]$ can be functionally integrated to yield

$$\mathcal{Q}[\lambda] = \int d\phi \left[\mathcal{J}\rho + \mathcal{L}^i \varepsilon^{-i} + \mathcal{G}_M^p \zeta_M^{-p} \right]. \quad (4.30)$$

After both having determined the infinitesimal transformations and the canonical boundary charge, now we are in a position to derive the asymptotic symmetry algebra using the standart method [89], which can be obtained through the following relation

$$\delta_\lambda F = \{F, \mathcal{Q}[\lambda]\} \quad (4.31)$$

for any phase space functional F . The Poisson brackets of all fields can be calculated as

$$\{\mathcal{L}^i(z_1), \mathcal{L}^j(z_2)\}_{PB} = (i-j)\mathcal{L}^{i+j}(z_2)\delta(z_1-z_2) - k\eta_2^{ij}\partial_\phi\delta(z_1-z_2), \quad (4.32)$$

$$\{\mathcal{L}^i(z_1), \mathcal{G}_\pm^p(z_2)\}_{PB} = \left(\frac{i}{2} - p\right)\mathcal{G}_\pm^{i+p}(z_2)\delta(z_1-z_2), \quad (4.33)$$

$$\{\mathcal{J}(z_1), \mathcal{G}_\pm^p(z_2)\}_{PB} = \pm\mathcal{G}_\pm^p(z_2)\delta(z_1-z_2), \quad (4.34)$$

$$\{\mathcal{J}(z_1), \mathcal{J}(z_2)\}_{PB} = -k\eta\partial_\phi\delta(z_1-z_2), \quad (4.35)$$

$$\begin{aligned} \{\mathcal{G}_\pm^p(z_1), \mathcal{G}_\pm^q(z_2)\}_{PB} &= \left(2\mathcal{L}^{p+q}(z_2) \pm (p-q)\mathcal{J}\right)\delta(z_1-z_2) \\ &\quad + k\eta_{\frac{3}{2}}^{pq}\partial_\phi\delta(z_1-z_2). \end{aligned} \quad (4.36)$$

where $\eta = \text{str}(\mathcal{J}\mathcal{J})$, $\eta_2^{ij} = \text{str}(\mathcal{L}_i\mathcal{L}_j)$ and $\eta_{\frac{3}{2}}^{pq} = \text{str}(\mathcal{G}_p^M\mathcal{G}_q^M)$ are the bilinear forms in the fundamental representation of $\mathfrak{sl}(2|1)$ Lie superalgebra. $\mathcal{J}(z)$ and $\mathcal{G}_M^p(z)$ charges are also identified as the generators of the asymptotic symmetry algebra. Finally, the operator product algebra can be written as

$$\mathcal{L}^i(z_1)\mathcal{L}^j(z_2) \sim \frac{\frac{k}{2}\eta_2^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}}\mathcal{L}^{i+j}(z_2), \quad (4.37)$$

$$\mathcal{L}^i(z_1)\mathcal{G}_\pm^p(z_2) \sim \frac{(\frac{i}{2} - p)}{z_{12}}\mathcal{G}_\pm^{i+p}(z_2), \quad (4.38)$$

$$\mathcal{J}(z_1)\mathcal{G}_\pm^p(z_2) \sim \pm\frac{\mathcal{G}_\pm^p(z_2)}{z_{12}}, \quad (4.39)$$

$$\mathcal{J}(z_1)\mathcal{J}(z_2) \sim \frac{\frac{k}{2}\eta}{z_{12}^2}, \quad (4.40)$$

$$\mathcal{G}_\pm^p(z_1)\mathcal{G}_\pm^q(z_2) \sim \frac{\frac{k}{2}\eta_{\frac{3}{2}}^{pq}}{z_{12}^2} + \frac{2}{z_{12}}\left(\mathcal{L}^{p+q}(z_2) \pm \frac{(p-q)}{2}\mathcal{J}(z_2)\right). \quad (4.41)$$

where $z_{12} = z_1 - z_2$, or in the more compact form,

$$\mathfrak{J}^A(z_1)\mathfrak{J}^B(z_2) \sim \frac{\frac{k}{2}\eta^{AB}}{z_{12}^2} + \frac{\mathfrak{f}_{\mathcal{C}}^{AB}\mathfrak{J}^{\mathcal{C}}(z_2)}{z_{12}}. \quad (4.42)$$

Note that η^{AB} is the supertrace matrix and f_C^{AB} 's are the structure constants of the related algebra with $(A, B = 0, \pm 1, \pm \frac{1}{2})$, i.e, $\eta^{ip} = 0$ and $f_{i+j}^{ij} = (i - j)$. Lastly, by repeating the same analysis for the barred sector also, the asymptotic symmetry algebra of $\mathcal{N} = (2, 2)$ supergravity for the loosest set of boundary conditions is given by two copies of the affine $\mathfrak{sl}(2|1)_k$ algebra.

4.1.3 $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern - Simons $\mathcal{N} = (2, 2)$ supergravity for superconformal boundary

In this section, our aim is to look into the asymptotic symmetry algebra for the supersymmetric extension of the Brown-Henneaux boundary conditions. We start on by imposing the Drinfeld-Sokolov highest weight gauge condition on the $\mathfrak{sl}(2|1)$ Lie superalgebra valued connection (4.13), in order to further restrict the coefficients. So, the Drinfeld-Sokolov reduction sets the fields such that

$$\mathcal{L}^0 = \mathcal{G}_{\pm}^{+\frac{1}{2}} = 0, \quad \mathcal{L}^{-1} = \mathcal{L}, \quad \mathcal{G}_{\pm}^{-\frac{1}{2}} = \mathcal{G}_{\pm}, \quad \gamma_{+1} \mathcal{L}^{+1} = 1. \quad (4.43)$$

Meanwhile, it is worth noting that the super-conformal boundary conditions are the supersymmetric extension of the well-known Brown-Henneaux boundary conditions proposed in [6] for AdS_3 supergravity. As a result, the supersymmetric gauge connection takes the form

$$a_{\varphi} = L_{+1} + \gamma_{-1} \mathcal{L} L_{-1} + \sigma_{\pm}^{-\frac{1}{2}} \mathcal{G}_{\pm} G_{-\frac{1}{2}}^{\pm} + \rho \mathcal{J} J, \quad (4.44)$$

where $\gamma_{-1} = -\frac{1}{k}$, $\rho = \frac{1}{2k}$, $\sigma_{+}^{-\frac{1}{2}} = \frac{1}{2k}$, and $\sigma_{-}^{-\frac{1}{2}} = -\frac{1}{2k}$'s are some scaling parameters and we have four functions: two bosonic $(\mathcal{J}, \mathcal{L})$ and two fermionic $\mathcal{G}_{\pm}$ as *charges*. After performing these steps, we are now close to acquire the superconformal asymptotic symmetry algebra. Following the results implied by the Drinfeld-Sokolov reduction, the gauge parameter λ has only four independent functions $(\rho, \varepsilon \equiv \varepsilon^{+1}, \zeta_{\pm} \equiv \zeta_{\pm}^{+\frac{1}{2}})$ and given as

$$\begin{aligned} \lambda = & b^{-1} \left[\varepsilon L_1 - \varepsilon' L_0 + \left(\frac{1}{2} \varepsilon'' + \frac{1}{k} \mathcal{L} \varepsilon + \frac{1}{2k} \mathcal{G}_{+} \zeta_{+} + \frac{1}{2k} \mathcal{G}_{-} \zeta_{-} \right) L_{-1} \right. \\ & + \zeta_{+} G_{\frac{1}{2}}^{+} + \zeta_{-} G_{\frac{1}{2}}^{-} - \left(\frac{1}{2k} \mathcal{G}_{-} \varepsilon + \frac{1}{2k} \mathcal{J} \zeta_{+} + \zeta_{+}' \right) G_{-\frac{1}{2}}^{-} \\ & \left. - \left(\frac{1}{2k} \mathcal{G}_{+} \varepsilon + \frac{1}{2k} \mathcal{J} \zeta_{-} + \zeta_{-}' \right) G_{-\frac{1}{2}}^{+} + \rho J \right] b. \end{aligned} \quad (4.45)$$

Substituting this gauge parameter in the transformation of the fields expression (4.19), we obtain the infinitesimal gauge transformations:

$$\delta_\lambda \mathcal{J} = \frac{1}{2k} \rho' - \mathcal{G}_+ \zeta_+ + \mathcal{G}_- \zeta_-, \quad (4.46)$$

$$\begin{aligned} \delta_\lambda \mathcal{L} = & \frac{k}{2} \varepsilon''' + 2\mathcal{L} \varepsilon' + \mathcal{L}' \varepsilon + \frac{3}{2} \mathcal{G}_+ \zeta'_+ + \frac{3}{2} \mathcal{G}_- \zeta'_- + \frac{1}{2} \mathcal{G}'_+ \zeta_+ + \frac{1}{2} \mathcal{G}'_- \zeta_- \\ & + \frac{1}{2k} (\mathcal{J} \mathcal{G}_+ \zeta_+ + \mathcal{J} \mathcal{G}_- \zeta_-), \end{aligned} \quad (4.47)$$

$$\begin{aligned} \delta_\lambda \mathcal{G}_\pm = & \frac{1}{2k} \zeta''_\mp + 2 \left(\mathcal{L} + \frac{1}{4k} (\mathcal{J} \mathcal{J}) \right) \zeta_\pm \pm \left(\rho - \frac{1}{2k} \mathcal{J} \varepsilon \right) \mathcal{G}_\pm \mp \zeta_\pm \mathcal{J}' + \frac{3}{2} \mathcal{G}_\pm \varepsilon' \\ & + \varepsilon \mathcal{G}'_\pm \mp \mathcal{J} \zeta'_\mp. \end{aligned} \quad (4.48)$$

In fact, these gauge transformations give a cue for the asymptotic symmetry algebra [45]. By taking forward to see the asymptotic symmetries, one can integrate the variation of the canonical boundary charges, i.e., $\delta_\lambda \mathcal{Q}$ expression (4.29) such that

$$\mathcal{Q}[\lambda] = \int d\varphi [\mathcal{L} \varepsilon + \mathcal{G}_M \zeta_M + \mathcal{J} \rho]. \quad (4.49)$$

But, these canonical boundary charges do not give a convenient asymptotic operator product algebra for $\mathcal{N} = (2, 2)$ superconformal boundary,

$$\mathcal{J}(z_1) \mathcal{J}(z_2) \sim \frac{2k}{z_{12}^2}, \quad (4.50)$$

$$\mathcal{J}(z_1) \mathcal{G}_\pm(z_2) \sim \frac{\mp \mathcal{G}_\pm}{z_{12}^2}, \quad (4.51)$$

$$\mathcal{L}(z_1) \mathcal{L}(z_2) \sim \frac{3k}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}' - \frac{\mathcal{G}_+ \mathcal{G}_-}{k}}{z_{12}}, \quad (4.52)$$

$$\mathcal{L}(z_1) \mathcal{J}(z_2) \sim 0 \quad (4.53)$$

$$\mathcal{L}(z_1) \mathcal{G}_\pm(z_2) \sim \frac{\frac{3}{2} \mathcal{G}_\pm}{z_{12}^2} + \frac{\mathcal{G}'_\pm \pm \frac{\mathcal{J} \mathcal{G}_\pm}{2k}}{z_{12}}, \quad (4.54)$$

$$\mathcal{G}_\pm(z_1) \mathcal{G}_\pm(z_2) \sim \frac{\mp 4k}{z_{12}^3} - \frac{2\mathcal{J}}{z_{12}^2} + \frac{\mp 2}{z_{12}} \left(\mathcal{L} + \frac{\mathcal{J} \mathcal{J}}{4k} \pm \frac{\mathcal{J}'}{2} \right). \quad (4.55)$$

because $\mathcal{L}(z)$ and $\mathcal{J}(z)$ do not transform like a primary conformal field, besides there exist some nonlinear terms such as $(\mathcal{J} \mathcal{J})(z)$, $(\mathcal{G}_+ \mathcal{G}_-)(z)$ and also $(\mathcal{J} \mathcal{G}_\pm)(z)$. Therefore, it is necessary to perform a shift on the boundary charge $\mathcal{L}$ and also make a redefinition on the gauge parameter ρ as follows:

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{3}{2c} (\mathcal{J} \mathcal{J}), \quad \rho \rightarrow \rho + \frac{3}{c} \mathcal{J} \varepsilon. \quad (4.56)$$

Before closing this section, one should also emphasize that these new variables do not affect the boundary charges. Thus, this leads to operator product expansions of

the convenient asymptotic symmetry algebra for $\mathcal{N} = (2, 2)$ superconformal boundary with a set of conformal generators $\mathcal{G}_\pm \rightarrow \mathcal{G}^+ \pm \mathcal{G}^-$ in the complex coordinates by using (4.31)

$$\mathcal{J}(z_1)\mathcal{J}(z_2) \sim \frac{\frac{c}{3}}{z_{12}^2}, \quad (4.57)$$

$$\mathcal{J}(z_1)\mathcal{G}^\pm(z_2) \sim \frac{\pm\mathcal{G}^\pm}{z_{12}^2}, \quad (4.58)$$

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{\frac{c}{2}}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}}, \quad (4.59)$$

$$\mathcal{L}(z_1)\mathcal{J}(z_2) \sim \frac{\mathcal{J}}{z_{12}^2} + \frac{\mathcal{J}'}{z_{12}}, \quad (4.60)$$

$$\mathcal{L}(z_1)\mathcal{G}^\pm(z_2) \sim \frac{\frac{3}{2}\mathcal{G}^\pm}{z_{12}^2} + \frac{\mathcal{G}^{\pm'}}{z_{12}}, \quad (4.61)$$

$$\mathcal{G}^\pm(z_1)\mathcal{G}^\mp(z_2) \sim \frac{\frac{2c}{3}}{z_{12}^3} + \frac{2\mathcal{J}}{z_{12}^2} + \frac{1}{z_{12}}(2\mathcal{L} \pm \mathcal{J}'). \quad (4.62)$$

When the same analysis repeated for the barred sector, it is seen that the asymptotic symmetry algebra for the loosest set of boundary conditions of $\mathcal{N} = (2, 2)$ supergravity, consists of two copies of the super-Virasoro algebra with central charge $c = 6k$.

4.2 $\mathcal{N} = (2, 2)$ $\mathfrak{sl}(3|2) \oplus \mathfrak{sl}(3|2)$ Higher - spin Chern - Simons Supergravity

Finally, after having laid the groundwork by performing the canonical analysis of the simplest case, now we are in a position to present the extended $\mathcal{N} = (2, 2)$ higher - spin Chern - Simons supergravity theory based on $\mathfrak{sl}(3|2)_k$ superalgebra.

4.2.1 For affine boundary

The objective of this section is to construct $\mathcal{N} = (2, 2)$ extended higher-spin AdS_3 supergravity as $\mathfrak{sl}(3|2) \oplus \mathfrak{sl}(3|2)$ Chern - Simons gauge theory on the affine boundary. We proceed our calculations to elucidate the asymptotic symmetry algebra for the loosest set of boundary conditions.

As already stated in the previous section, we consider the principal embedding of $\mathfrak{sl}(2|1)$ into $\mathfrak{sl}(3|2)$ as a sub-algebra, giving rise to the asymptotic symmetry, where the even - graded sector of the superalgebra decomposes into spin-2, the $\mathfrak{sl}(2)$ generators $L_i, (i = \pm 1, 0)$, a spin-1 element J , a spin-2 multiplet $A_i, (i = \pm 1, 0)$ and a spin-3 multiplet $W_i, (i = \pm 2, \pm 1, 0)$. All these generators together span the bosonic sub-algebra

$\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{u}(1)$. Furthermore, the odd - graded elements decompose in two spin $\frac{3}{2}$ multiplets $G_r^M, (r = \pm\frac{1}{2}), (M = \pm)$ and two spin $\frac{5}{2}$ multiplets $S_r^M, (r = \pm\frac{3}{2}, \pm\frac{1}{2}), (M = \pm)$. Then, the bosonic sector of this algebra is given as follows:

$$\begin{aligned}
[L_i, L_j] &= (i-j)L_{i+j}, \\
[L_i, A_j] &= (i-j)A_{i+j}, \\
[L_i, W_j] &= (2i-j)W_{i+j}, \\
[A_i, A_j] &= (i-j)L_{i+j}, \\
[A_i, W_j] &= (2i-j)W_{i+j}, \\
[W_i, W_j] &= -\frac{1}{6}(i-j)(2i^2 + 2j^2 - ij - 8)(L_{i+j} + A_{i+j}).
\end{aligned} \tag{4.63}$$

Additionally, the explicit commutation relations between the bosonic and fermionic sectors are given by

$$[L_i, G_p^\pm] = \left(\frac{i}{2} - p\right) G_{i+p}^\pm, \tag{4.64}$$

$$[L_i, S_p^\pm] = \left(\frac{3i}{2} - p\right) S_{i+p}^\pm, \tag{4.65}$$

$$[J, G_r^\pm] = \pm G_r^\pm, \tag{4.66}$$

$$[J, S_r^\pm] = \pm S_r^\pm, \tag{4.67}$$

$$[A_i, G_r^\pm] = \frac{5}{3}\left(\frac{i}{2} - r\right) G_{i+r}^\pm \pm \frac{4}{3} S_{i+r}^\pm, \tag{4.68}$$

$$[A_i, S_r^\pm] = \frac{1}{3}\left(\frac{3i}{2} - r\right) S_{i+r}^\pm \mp \frac{1}{3}\left(3i^2 - 2ir + r^2 - \frac{9}{4}\right) G_{i+r}^\pm, \tag{4.69}$$

$$[W_i, G_r^\pm] = -\frac{4}{3}\left(\frac{i}{2} - 2r\right) S_{i+r}^\pm, \tag{4.70}$$

$$\begin{aligned}
[W_i, S_r^\pm] &= \mp \frac{1}{3}\left(2r^2 - 2ri + i^2 - \frac{5}{2}\right) S_{i+r}^\pm \\
&\quad - \frac{1}{6}\left(4r^3 - 3r^2i + 2ri^2 - i^3 - 9r + \frac{19}{4}i\right) G_{i+r}^\pm.
\end{aligned} \tag{4.71}$$

And lastly, the fermionic sector satisfy the following anti - commutation relations

$$\{G_r^\pm, G_s^\mp\} = 2L_{r+s} \pm (r-s)J \tag{4.72}$$

$$\{G_r^\pm, S_s^\pm\} = -\frac{3}{2}W_{r+s}^\mp + \frac{3}{4}(3r-s)A_{r+s} - \frac{5}{4}(3r-s)L_{r+s}, \tag{4.73}$$

$$\begin{aligned}
\{S_r^\pm, S_s^\mp\} &= -\frac{3}{4}(r-s)W_{r+s} + \frac{1}{8}\left(3s^2 - 4rs + 3r^2 - \frac{9}{2}\right)(L_{r+s} - 3A_{r+s}) \\
&\quad - \frac{1}{4}(r-s)\left(r^2 + s^2 - \frac{5}{2}\right)J
\end{aligned} \tag{4.74}$$

except for zero commutators.

Having discussed the principle embedding of $\mathfrak{sl}(2|1)$ into $\mathfrak{sl}(3|2)$, we are now able to formulate the most general boundary conditions for asymptotically AdS_3 spacetimes. In accordance with this purpose it is useful to define the gauge connection as

$$a_\varphi = \rho \mathcal{J} + \gamma_i \mathcal{L}^i + \vartheta_i \mathcal{A}^i + \omega_i \mathcal{W}^i + \sigma_M^p \mathcal{G}_M^p + \tau_M^p \mathcal{S}_M^p \quad (4.75)$$

$$a_t = \eta \mathcal{J} + \mu^i \mathcal{L}_i + \xi^i \mathcal{A}_i + f^i \mathcal{W}_i + \nu_M^p \mathcal{G}_M^p + \psi_M^p \mathcal{S}_M^p \quad (4.76)$$

where

$$\begin{aligned} \rho &= \sigma_M^{\frac{1}{2}} = -\sigma_M^{\frac{1}{2}} = \frac{1}{k} \\ 2\gamma_1 &= 2\gamma_{-1} - 2\gamma_0 = 2\vartheta_1 = 2\vartheta_{-1} = -\vartheta_0 = \frac{4}{k} \\ 3\tau_M^{-\frac{3}{2}} &= -3\sigma_M^{\frac{3}{2}} = \tau_M^{\frac{1}{2}} = -\sigma_M^{\frac{1}{2}} = \frac{8}{3k} \\ 6\omega_{-2} = 6\omega_2 &= -\frac{3}{2}\omega_{-1} = -\frac{3}{2}\omega_1 = \omega_0 = \frac{3}{k} \end{aligned}$$

are some scaling parameters. As a result, we have twenty four functions; twelve bosonic $(\mathcal{J}, \mathcal{L}^i, \mathcal{A}^i, \mathcal{W}^i)$ and twelve fermionic $(\mathcal{G}_M^p, \mathcal{S}_M^p)$ as *charges*. Also, we have in total twenty four independent functions $(\eta, \mu^i, \xi^i, f^i, \nu_M^p, \psi_M^p)$ too, as *chemical potentials* for the time component.

In the presence of the loosest set of boundary conditions, thanks to the flatness conditions (4.7), the equations of motion for fixed chemical potentials impose the additional conditions as the temporal evolution of the twenty four independent source fields $(\mathcal{J}, \mathcal{L}^i, \mathcal{A}^i, \mathcal{W}^i, \mathcal{G}_M^p, \mathcal{S}_M^p)$ as calculated in Appendix D.1. Thus, the consequences of our calculations to derive the relevant superalgebra for the loosest set of boundary conditions can now be evaluated through a canonical analysis. We now consider the boundary preserving gauge transformations (encompassing all) (4.19) generated by the $\mathfrak{sl}(3|2)$ Lie superalgebra-valued gauge parameter λ , which we choose as

$$\lambda = b^{-1} \left[\rho \mathcal{J} + \varepsilon^i \mathcal{L}_i + \phi^i \mathcal{A}_i + v^i \mathcal{W}_i + \varsigma_M^p \mathcal{G}_M^p + \omega_M^p \mathcal{S}_M^p \right] b. \quad (4.77)$$

Note that there are in total twenty four arbitrary functions on the boundary, consist of twelve bosonic $(\rho, \varepsilon^i, \phi^i, v^i)$ and twelve fermionic $(\varsigma_M^p, \omega_M^p)$. Inserting this expression into (4.19) imposes the gauge transformations. For further details, see Appendix D.2. Analogously, the gauge transformations for the chemical potentials are calculated whose details can again be found in Appendix D.3. Following a similar approach as previous section, we act on to make out the canonical boundary charges $\mathcal{Q}[\lambda]$ that

generates the transformations (D.12)-(D.22). As is well known, it is convenient to express the variation of the canonical boundary charge $\delta_\lambda \mathcal{Q}$ (4.29), to reach out the asymptotic symmetry algebra [81, 86–88]. Hence, the canonical boundary charge $\mathcal{Q}[\lambda]$ can be obtained which reads

$$\mathcal{Q}[\lambda] = \int d\varphi [\mathcal{J}\rho + \mathcal{L}^i \varepsilon^{-i} + \mathcal{A}^i \phi^{-i} + \mathcal{W}^i v^{-i} + \mathcal{G}_M^p \varsigma^{-p} + \mathcal{S}_M^p \omega^{-p}]. \quad (4.78)$$

The next step to derive the asymptotic symmetry algebra is to calculate Poisson bracket algebra using the standard method [89], which is acquired by the relation (4.31) given for any phase space functional F .

In light of the above statement, the operator product algebra for the bosonic sector is then achieved as

$$\mathcal{L}^i(z_1) \mathcal{L}^j(z_2) \sim \frac{\frac{k}{2} \eta_2^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}} \mathcal{L}^{i+j}, \quad (4.79)$$

$$\mathcal{J}(z_1) \mathcal{J}(z_2) \sim \frac{\frac{k}{2} \eta}{z_{12}^2}, \quad (4.80)$$

$$\mathcal{L}^i(z_1) \mathcal{A}^j(z_2) \sim \frac{(i-j)}{z_{12}} \mathcal{A}^{i+j}, \quad (4.81)$$

$$\mathcal{L}^i(z_1) \mathcal{W}^j(z_2) \sim \frac{(2i-j)}{z_{12}} \mathcal{W}^{i+j}, \quad (4.82)$$

$$\mathcal{A}^i(z_1) \mathcal{A}^j(z_2) \sim \frac{\frac{k}{2} \eta_2^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}} \mathcal{A}^{i+j}, \quad (4.83)$$

$$\mathcal{A}^i(z_1) \mathcal{W}^j(z_2) \sim \frac{(2i-j)}{z_{12}} \mathcal{A}^{i+j}, \quad (4.84)$$

$$\begin{aligned} \mathcal{W}^i(z_1) \mathcal{W}^j(z_2) &\sim \frac{\frac{k}{2} \eta_3^{ij}}{z_{12}^2} \\ &+ \frac{1}{z_{12}} \left(\frac{1}{3} (i-j) (2i^2 - ij + 2j^2 - 8) (\mathcal{A}^{i+j} + \mathcal{L}^{i+j}) \right). \end{aligned} \quad (4.85)$$

Furthermore, the explicit operator product algebra between the bosonic and fermionic sectors is given by

$$\mathcal{J}(z_1) \mathcal{G}_\pm^p(z_2) \sim \pm \frac{\mathcal{G}_\pm^p}{z_{12}}, \quad (4.86)$$

$$\mathcal{L}^i(z_1) \mathcal{G}_\pm^p(z_2) \sim \frac{(\frac{i}{2} - p)}{z_{12}} \mathcal{G}_\pm^{i+p}, \quad (4.87)$$

$$\mathcal{L}^i(z_1) \mathcal{S}_\pm^p(z_2) \sim \frac{(\frac{3i}{2} - p)}{z_{12}} \mathcal{S}_\pm^{i+p}, \quad (4.88)$$

$$\mathcal{J}(z_1) \mathcal{S}_\pm^p(z_2) \sim \pm \frac{\frac{k}{2}}{z_{12}} \mathcal{S}_\pm^{p+i}, \quad (4.89)$$

$$\mathcal{G}_{\pm}^p(z_1)\mathcal{A}^i(z_2), \sim \mp \frac{\mathcal{S}_{\pm}^p}{z_{12}}, \quad (4.90)$$

$$\mathcal{G}_{\pm}^p(z_1)\mathcal{W}^i(z_2) \sim -\frac{\frac{4}{3}(2i-\frac{p}{2})}{z_{12}}\mathcal{S}_{\mp}^{p+i}, \quad (4.91)$$

$$\begin{aligned} \mathcal{A}(z_1)\mathcal{S}_{\pm}^p(z_2) &\sim \frac{1}{z_{12}}\left(\frac{1}{3}\left(\frac{3i}{2}-p\right)\mathcal{S}_{\pm}^{i+p}\right. \\ &\quad \left. \mp \frac{1}{4}\left(3i^2-2ip+p^2-\frac{9}{4}\right)\mathcal{G}_{\pm}^{i+p}\right), \end{aligned} \quad (4.92)$$

$$\begin{aligned} \mathcal{S}_{\pm}^p(z_1)\mathcal{W}^i(z_2) &\sim \mp \frac{1}{z_{12}}\left(\frac{1}{3}\left(i^2-2ip+2p^2-\frac{5}{2}\right)\mathcal{S}_{\mp}^{p+i}\right. \\ &\quad \left. + \frac{1}{8}\left(4p^3-i^3+2i^2p-3ip^2-9p+\frac{19}{4}i\right)\mathcal{G}_{\mp}^{p+i}\right). \end{aligned} \quad (4.93)$$

Finally, the explicit operator product algebra for the fermionic sector yields

$$\mathcal{G}_{\pm}^p(z_1)\mathcal{G}_{\pm}^q(z_2) \sim \frac{\frac{k}{2}\eta_{\frac{3}{2}}^{pq}}{z_{12}^2} + \frac{2}{z_{12}}\left(\mathcal{L}^{p+q} + \frac{5}{3}\mathcal{A}^{p+q} \pm \frac{(p-q)}{2}\mathcal{J}\right), \quad (4.94)$$

$$\mathcal{G}_{\pm}^p(z_1)\mathcal{S}_{\pm}^q(z_2) \sim \frac{2}{z_{12}}\left(\frac{3}{4}\mathcal{W}^{p+q} - \left(\frac{3p}{2} - \frac{q}{2}\right)\mathcal{A}^{p+q}\right), \quad (4.95)$$

$$\begin{aligned} \mathcal{S}_{\pm}^p(z_1)\mathcal{S}_{\pm}^q(z_2) &\sim \frac{\frac{k}{2}\eta_{\frac{5}{2}}^{pq}}{z_{12}^2} \pm \frac{1}{z_{12}}\left(\frac{1}{8}\left(3p^2-4pq+3q^2-\frac{9}{2}\right)(\mathcal{A}^{p+q}+3\mathcal{L}^{p+q})\right. \\ &\quad \left. \mp \frac{3}{4}(p-q)\mathcal{W}^{p+q} \pm \frac{3}{16}(p-q)\left(p^2+q^2-\frac{5}{2}\right)\mathcal{J}\right), \end{aligned} \quad (4.96)$$

where $z_{12} = z_1 - z_2$, or in the more compact form,

$$\mathfrak{J}^A(z_1)\mathfrak{J}^B(z_2) \sim \frac{\frac{k}{2}\eta^{AB}}{z_{12}^2} + \frac{\mathfrak{f}_{\mathcal{C}}^{AB}\mathfrak{J}^{\mathcal{C}}(z_2)}{z_{12}}. \quad (4.97)$$

Note that η^{AB} is the supertrace matrix and $\mathfrak{f}_{\mathcal{C}}^{AB}$'s are the structure constants of the related algebra with $(A, B = 0, \pm 1, \pm \frac{1}{2}, 0, \pm 1, \pm \frac{1}{2}, \pm \frac{3}{2})$, i.e. $\eta^{ip} = 0$ and $\mathfrak{f}_{i+j}^{ij} = (i-j)$.

Since the barred sector is completely analogous, the same results are obtained. Eventually, it follows that the asymptotic symmetry algebra for the loosest set of boundary conditions of $\mathcal{N} = (2, 2)$ supergravity is two copies of the affine $\mathfrak{sl}(3|2)_k$ algebra.

4.2.2 For superconformal boundary

As already discussed, it is convenient to point out that the super-conformal boundary conditions are the supersymmetric extension of the well-known Brown-Henneaux boundary conditions, presented in [6] for AdS_3 supergravity. In this section, our main goal is to construct the asymptotic symmetry algebra for the most general

boundary conditions as the supersymmetric extension of the Brown-Henneaux boundary conditions. In accordance for this purpose, we launch into our section by imposing the Drinfeld-Sokolov highest weight gauge condition on the $\mathfrak{sl}(3|2)$ Lie superalgebra valued connection (4.75), setting the fields as

$$\begin{aligned}\mathcal{L}^0 = \mathcal{A}^0 = \mathcal{A}^{+1} = \mathcal{G}_M^{+\frac{1}{2}} = \mathcal{S}_M^{+\frac{1}{2}} = \mathcal{S}_M^{+\frac{3}{2}} = 0, \\ \mathcal{L}^{-1} = \mathcal{L}, \quad \mathcal{A}^{-1} = \mathcal{A}, \quad \mathcal{G}_M^{-\frac{1}{2}} = \mathcal{G}_M, \quad \mathcal{S}_M^{-\frac{3}{2}} = \mathcal{S}_M, \quad \gamma_{+1}\mathcal{L}^{+1} = 1.\end{aligned}\quad (4.98)$$

Correspondingly, the supersymmetric gauge connection is taken to be

$$\begin{aligned}a_\varphi = & \mathcal{L}_1 + \gamma_{-1}\mathcal{L}\mathcal{L}_{-1} + \vartheta_{-1}\mathcal{A}\mathcal{A}_{-1} + \omega_{-2}\mathcal{W}\mathcal{W}_{-2} + \rho\mathcal{J}\mathcal{J} \\ & + \sigma_M^{-\frac{1}{2}}\mathcal{G}_M\mathcal{G}_{-\frac{1}{2}}^M + \tau_M^{-\frac{3}{2}}\mathcal{S}_M\mathcal{S}_{-\frac{3}{2}}^M,\end{aligned}\quad (4.99)$$

$$\begin{aligned}a_t = & \eta\mathcal{J} + \mu\mathcal{L}_1 + \xi\mathcal{A}_1 + f\mathcal{W}_2 + v_M\mathcal{G}_{+\frac{1}{2}}^M + \psi_M\mathcal{S}_{+\frac{3}{2}}^M \\ & + \sum_{i=-1}^0 \mu^i\mathcal{L}_i + \sum_{i=-1}^0 \xi^i\mathcal{A}_i + v_M^{-\frac{1}{2}}\mathcal{G}_{-\frac{1}{2}}^M + \sum_{p=-\frac{3}{2}}^{\frac{1}{2}} \psi_M^p\mathcal{S}_p^M,\end{aligned}\quad (4.100)$$

where η , $\mu \equiv \mu^{+1}$, $\xi \equiv \xi^{+1}$, $f \equiv f^{+2}$, $v_M \equiv v_M^{+\frac{1}{2}}$, and $\psi_M \equiv \psi_M^{+\frac{3}{2}}$ can be interpreted as the independent *chemical potentials*.

When we request to gather up our next steps yielding the asymptotic symmetry algebra in a one paragraph, it is an appropriate option to give the following brief summary. The all functions except the *chemical potentials* can be fixed by the flatness conditions (4.7) in the usual manner. The equations of motion for the fixed chemical potentials can also be obtained conventionally as the time evolution of the canonical boundary charges. But unfortunately, moving from the $\mathfrak{sl}(2|1)$ -case to $\mathfrak{sl}(3|2)$ -extension brings along the technical cumbersome although we have overcome. Our preference is to ignore presenting these calculations here, because they take up too much space. We make choice to spare space for the calculations of the gauge parameter λ and further.

In line with all these results to be obtained, we are now able to derive the superconformal asymptotic symmetry algebra. Using the Drinfeld-Sokolov reduction we have only six independent parameters as ρ , $\varepsilon \equiv \varepsilon^{+1}$, $\phi \equiv \phi^{+1}$, $v \equiv v^{+2}$, $\varsigma_M \equiv \varsigma_M^{+\frac{1}{2}}$ and $\omega_M \equiv \omega_M^{+\frac{3}{2}}$. It is now possible to compute the gauge transformations by considering all transformations (4.19) that preserve the boundary conditions with the $\mathfrak{sl}(3|2)$ Lie superalgebra valued gauge parameter λ . See Appendix D.4, for more details.

Since we are dealing with the asymptotic symmetries, it is natural to demand obtaining the canonical boundary charges $\mathcal{Q}[\lambda]$. So, the variation of the canonical boundary charge, i.e., $\delta_\lambda \mathcal{Q}$ (4.29) can be integrated to yield

$$\mathcal{Q}[\lambda] = \int d\phi [\mathcal{J}\rho + \mathcal{L}\varepsilon + \mathcal{A}\phi + \mathcal{W}v + \mathcal{G}_M \zeta_M + \mathcal{S}_M \omega_M]. \quad (4.101)$$

But, these canonical boundary charges do not give a convenient asymptotic operator product algebra in the complex coordinates by using (4.31) for $\mathcal{N} = (2, 2)$ superconformal boundary,

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{3k}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}' - \frac{\mathcal{G}_+\mathcal{G}_-}{k}}{z_{12}} \quad (4.102)$$

$$\mathcal{L}(z_1)\mathcal{J}(z_2) \sim 0 \quad (4.103)$$

$$\mathcal{L}(z_1)\mathcal{G}_\pm(z_2) \sim \frac{3\mathcal{G}_\pm}{2z_{12}^2} + \frac{\mathcal{G}'_\pm \pm \frac{\mathcal{J}\mathcal{G}_\pm}{2k}}{z_{12}} \quad (4.104)$$

$$\mathcal{L}(z_1)\mathcal{A}(z_2) \sim \frac{\mathcal{A}'}{z_{12}} + \frac{2\mathcal{A}}{z_{12}^2} \quad (4.105)$$

$$\mathcal{L}(z_1)\mathcal{S}_\pm(z_2) \sim \frac{5\mathcal{S}_\pm}{2z_{12}^2} + \frac{\mathcal{S}'_\pm \pm \frac{\mathcal{J}\mathcal{S}_\pm}{2k}}{z_{12}} \quad (4.106)$$

$$\mathcal{L}(z_1)\mathcal{W}(z_2) \sim +\frac{3\mathcal{W}}{z_{12}^2} + \frac{1}{z_{12}} \left(\mathcal{W}' + \frac{\mathcal{G}_+\mathcal{S}_-}{k} - \frac{\mathcal{S}_+\mathcal{G}_-}{k} \right) \quad (4.107)$$

$$\mathcal{J}(z_1)\mathcal{J}(z_2) \sim \frac{2k}{z_{12}^2} \quad (4.108)$$

$$\mathcal{J}(z_1)\mathcal{G}_\pm(z_2) \sim \mp \frac{\mathcal{G}_\pm}{z_{12}} \quad (4.109)$$

$$\mathcal{J}(z_1)\mathcal{A}(z_2) \sim 0 \quad (4.110)$$

$$\mathcal{J}(z_1)\mathcal{S}_\pm(z_2) \sim \mp \frac{\mathcal{S}_\pm}{z_{12}} \quad (4.111)$$

$$\mathcal{J}(z_1)\mathcal{W}(z_2) \sim 0 \quad (4.112)$$

$$\mathcal{G}_\pm(z_1)\mathcal{G}_\pm(z_2) \sim \frac{\mp 4k}{z_{12}^3} - \frac{2\mathcal{J}}{z_{12}^2} + \frac{\mp 2}{z_{12}} \left(\mathcal{L} + \frac{\mathcal{J}\mathcal{J}}{4k} \pm \frac{\mathcal{J}'}{2} \mp \frac{10\mathcal{A}}{3} \right) \quad (4.113)$$

$$\mathcal{G}_\pm(z_1)\mathcal{G}_\mp(z_2) \sim 0 \quad (4.114)$$

$$\mathcal{G}_\pm(z_1)\mathcal{A}(z_2) \sim \mp \frac{15\mathcal{S}_\pm}{4z_{12}} \quad (4.115)$$

$$\mathcal{G}_\pm(z_1)\mathcal{S}_\pm(z_2) \sim \pm \frac{16\mathcal{A}}{15z_{12}^2} \pm \frac{1}{z_{12}} \left(\frac{4\mathcal{A}'}{15} \pm \frac{8\mathcal{A}\mathcal{J}}{15k} \mp 4\mathcal{W} \right) \quad (4.116)$$

$$\mathcal{G}_\pm(z_1)\mathcal{S}_\mp(z_2) \sim 0 \quad (4.117)$$

$$\mathcal{G}_\pm(z_1)\mathcal{W}(z_2) \sim -\frac{5\mathcal{S}_\mp}{4z_{12}^2} - \frac{1}{z_{12}} \left(\frac{\mathcal{S}'_\mp}{4} - \frac{2\mathcal{A}\mathcal{G}_\mp}{15k} \mp \frac{\mathcal{J}\mathcal{S}_\mp}{2k} \right) \quad (4.118)$$

$$\mathcal{A}(z_1)\mathcal{A}(z_2) \sim \frac{3k}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}' - \frac{\mathcal{G}_+\mathcal{G}_-}{4k}}{z_{12}} \quad (4.119)$$

$$\mathcal{S}_\pm(z_1)\mathcal{S}_\mp(z_2) \sim -\frac{2\mathcal{G}_\mp\mathcal{G}'_\mp}{5kz_{12}} \quad (4.120)$$

$$\begin{aligned} \mathcal{A}(z_1)\mathcal{S}_\pm(z_2) &\sim \frac{4\mathcal{G}_+}{5z_{12}^3} + \frac{1}{z_{12}^2} \left(\frac{5\mathcal{S}_\pm}{6} + \frac{4\mathcal{G}'_\pm}{15} \pm \frac{2\mathcal{G}_\pm\mathcal{J}}{15k} \right) \\ &+ \frac{1}{z_{12}} \left(\frac{\mathcal{G}_\pm\mathcal{J}^2}{60k^2} - \frac{3\mathcal{A}\mathcal{G}_\pm}{5k} \pm \frac{\mathcal{G}_\pm\mathcal{J}'}{30k} \pm \frac{\mathcal{J}\mathcal{G}'_\pm}{15k} \right. \\ &\left. + \frac{3\mathcal{G}_\pm\mathcal{L}}{5k} \pm \frac{\mathcal{J}\mathcal{S}_\pm}{6k} + \frac{\mathcal{S}'_\pm}{3} + \frac{\mathcal{G}''_\pm}{15} \right) \end{aligned} \quad (4.121)$$

$$\begin{aligned} \mathcal{W}(z_1)\mathcal{W}(z_2) &\sim \frac{2k}{z_{12}^6} + \frac{2(\mathcal{A} + \mathcal{L})}{z_{12}^4} - \frac{1}{z_{12}^3} \left(\mathcal{A}' + \frac{13\mathcal{G}_-\mathcal{G}_+}{12k} + \mathcal{L}' \right) \\ &+ \frac{1}{z_{12}^2} \left(\frac{3\mathcal{A}''}{10} - \frac{13\mathcal{G}_+\mathcal{G}'_-}{24k} - \frac{13\mathcal{G}_-\mathcal{G}'_+}{24k} + \frac{16\mathcal{A}^2}{15kz_{12}^2} \right. \\ &+ \frac{32\mathcal{A}\mathcal{L}}{15kz_{12}^2} + \frac{16\mathcal{L}^2}{15k} + \frac{3\mathcal{L}''}{10} \left. \right) - \frac{1}{z_{12}} \left(\frac{\mathcal{G}_-\mathcal{G}_+\mathcal{J}^2}{30k^3} \right. \\ &+ \frac{32\mathcal{A}\mathcal{G}_-\mathcal{G}_+}{45k^2} - \frac{\mathcal{A}^{(3)}}{15} - \frac{\mathcal{G}_+\mathcal{J}\mathcal{S}_-}{3k^2} + \frac{\mathcal{G}_-\mathcal{J}\mathcal{S}_+}{3k^2} \\ &- \frac{\mathcal{G}_+\mathcal{J}\mathcal{G}'_-}{15k^2} + \frac{\mathcal{G}_-\mathcal{J}\mathcal{G}'_+}{15k^2} + \frac{16\mathcal{G}_-\mathcal{G}_+\mathcal{L}}{15k^2} - \frac{16\mathcal{L}\mathcal{A}'}{15k} \\ &- \frac{16\mathcal{A}\mathcal{A}'}{15k} - \frac{16\mathcal{A}\mathcal{L}'}{15k} - \frac{5\mathcal{S}_+\mathcal{G}'_-}{12k} + \frac{\mathcal{G}_+\mathcal{S}'_-}{4k} - \frac{5\mathcal{S}_-\mathcal{G}'_+}{12k} \\ &+ \frac{\mathcal{G}_-\mathcal{S}'_+}{4k} + \frac{7\mathcal{G}'_-\mathcal{G}'_+}{30k} + \frac{11\mathcal{G}_+\mathcal{G}''_-}{60k} + \frac{11\mathcal{G}_-\mathcal{G}''_+}{60k} + \frac{10\mathcal{S}_-\mathcal{S}_+}{k} \\ &\left. - \frac{16\mathcal{L}\mathcal{L}'}{15k} - \frac{\mathcal{L}^{(3)}}{15} \right) \end{aligned} \quad (4.122)$$

$$\begin{aligned} \mathcal{S}_\pm(z_1)\mathcal{W}(z_2) &\sim \mp \frac{3\mathcal{G}_\mp}{4z_{12}^4} \pm \frac{1}{z_{12}^3} \left(\frac{5\mathcal{S}_\mp}{4} \mp \frac{\mathcal{G}_\mp\mathcal{J}}{4k} - \frac{\mathcal{G}'_\mp}{4} \right) \mp \frac{1}{z_{12}^2} \left(\frac{\mathcal{G}''_\mp}{16} \right. \\ &+ \frac{3\mathcal{G}_\mp\mathcal{J}^2}{64k^2} + \frac{91\mathcal{A}\mathcal{G}_\mp}{240k} \pm \frac{5\mathcal{G}_\mp\mathcal{J}'}{32k} \pm \frac{\mathcal{J}\mathcal{G}'_\mp}{16k} + \frac{11\mathcal{G}_\mp\mathcal{L}}{16k} \\ &\mp \frac{3\mathcal{J}\mathcal{S}_\mp}{8k} - \frac{\mathcal{S}'_\mp}{2} \left. \right) - \frac{1}{z_{12}} \left(\frac{\mathcal{G}_\mp\mathcal{J}^3}{160k^3} \mp \frac{\mathcal{G}_\mp^{(3)}}{80} + \frac{13\mathcal{A}\mathcal{G}_\mp\mathcal{J}}{120k^2} \right. \\ &\pm \frac{3\mathcal{J}^2\mathcal{G}'_-}{320k^2} \pm \frac{9\mathcal{G}_\mp\mathcal{J}\mathcal{J}'}{160k^2} \pm \frac{9\mathcal{G}_\mp\mathcal{J}\mathcal{L}}{40k^2} \mp \frac{\mathcal{J}^2\mathcal{S}_\mp}{16k^2} \pm \\ &\frac{13\mathcal{G}_\mp\mathcal{A}'}{80k} \pm \frac{13\mathcal{A}\mathcal{G}'_\mp}{80k} \mp \frac{11\mathcal{A}\mathcal{S}_\mp}{4k} + \frac{9\mathcal{G}_\mp\mathcal{J}''}{160k} + \frac{7\mathcal{G}'_\mp\mathcal{J}'}{160k} \\ &+ \frac{\mathcal{J}\mathcal{G}''_\mp}{80k} - \frac{5\mathcal{G}_\mp\mathcal{W}}{2k} \pm \frac{27\mathcal{G}_\mp\mathcal{L}'}{80k} \pm \frac{19\mathcal{L}\mathcal{G}'_\mp}{80k} - \frac{\mathcal{S}_\mp\mathcal{J}'}{4k} \\ &\left. - \frac{\mathcal{J}\mathcal{S}'_\mp}{8k} \mp \frac{5\mathcal{S}_\mp\mathcal{L}}{4k} \mp \frac{\mathcal{S}''_\mp}{8} \right) \end{aligned} \quad (4.123)$$

$$\begin{aligned}
\mathcal{S}_\pm(z_1)\mathcal{S}_\pm(z_2) \sim & \frac{12k}{5z_{12}^5} \pm \frac{6\mathcal{J}}{5z_{12}^4} + \frac{1}{z_{12}^3} \left(\frac{2\mathcal{A}}{3} + \frac{3\mathcal{J}^2}{10k} + 2\mathcal{L} \right) \\
& + \frac{1}{z_{12}^2} \left(\frac{\mathcal{A}'}{3} \pm \frac{\mathcal{J}''}{5} \pm \frac{3\mathcal{J}'}{5} \pm \frac{\mathcal{J}^3}{20k^2} \pm \frac{\mathcal{A}\mathcal{J}}{3k} \right. \\
& - \frac{13\mathcal{G}_-\mathcal{G}_+}{20k} + \frac{3\mathcal{J}\mathcal{J}'}{10k} \pm \frac{\mathcal{J}\mathcal{L}}{k} \mp 4\mathcal{W} + \mathcal{L}' \Big) \\
& + \frac{1}{z_{12}} \left(\frac{\mathcal{A}''}{10} \pm \frac{\mathcal{J}^{(3)}}{20} + \frac{\mathcal{J}^4}{160k^3} + \frac{\mathcal{A}\mathcal{J}^2}{12k^2} \right. \\
& \mp \frac{\mathcal{G}_-\mathcal{G}_+\mathcal{J}}{5k^2} + \frac{\mathcal{J}^2\mathcal{L}}{4k^2} \pm \frac{3\mathcal{J}^2\mathcal{J}'}{40k^2} - \frac{3\mathcal{A}^2}{2k} \pm \frac{\mathcal{J}\mathcal{A}'}{6k} \\
& \pm \frac{\mathcal{A}\mathcal{J}'}{6k} + \frac{3\mathcal{A}\mathcal{L}}{5k} \mp \frac{2\mathcal{G}_+\mathcal{S}_-}{k} \pm \frac{2\mathcal{G}_-\mathcal{S}_+}{k} - \frac{9\mathcal{G}_\pm\mathcal{G}'_\pm}{20k} \\
& - \frac{\mathcal{G}_-\mathcal{G}'_+}{5k} + \frac{\mathcal{J}\mathcal{J}''}{10k} \pm \frac{\mathcal{L}\mathcal{J}'}{2k} + \frac{3\mathcal{J}'^2}{40k} - \frac{2\mathcal{J}\mathcal{W}}{k} \\
& \left. \pm \frac{\mathcal{J}\mathcal{L}'}{2k} + \frac{9\mathcal{L}^2}{10k} \mp 2\mathcal{W}' + \frac{3\mathcal{L}''}{10} \right) \quad (4.124)
\end{aligned}$$

because some boundary charges do not transform like a primary conformal field, and also there exist some nonlinear terms such as $(\mathcal{J}\mathcal{J})(z)$, $(\mathcal{G}_+\mathcal{G}_-)(z)$, and $(\mathcal{J}\mathcal{G}_\pm)(z)$, as already discussed in the previous section (4.1.3). Therefore, it is required to consider some redefinitions on the boundary charges and gauge parameters as

$$\gamma_{-1}\mathcal{L} \rightarrow \frac{6}{c} \left(\mathcal{L} - \frac{3}{2c} (\mathcal{J}\mathcal{J}) + \frac{\kappa}{2} \mathcal{A} \right), \quad \varepsilon \rightarrow \varepsilon + \frac{\kappa}{2} \left(\phi + \frac{6}{c} v \mathcal{J} \right) \quad (4.125)$$

$$\vartheta_{-1}\mathcal{A} \rightarrow -\frac{9\kappa}{5c} \mathcal{A}, \quad \phi \rightarrow \phi - \frac{3\kappa}{10} \left(\phi + \frac{6}{c} v \mathcal{J} \right) \quad (4.126)$$

$$\omega_{-2}\mathcal{W} \rightarrow \frac{3\kappa}{5c} \left(\mathcal{W} - \frac{6}{c} \mathcal{J}\mathcal{A} \right), \quad v \rightarrow \frac{3\kappa}{10} v \quad (4.127)$$

$$\rho \mathcal{J} \rightarrow \frac{3}{c} \mathcal{J}, \quad \rho \rightarrow \rho + \frac{3}{c} \left(\varepsilon \mathcal{J} + 2v \mathcal{A} \right) \quad (4.128)$$

$$\sigma_\pm^{-\frac{1}{2}} \mathcal{G}_\pm \rightarrow \mp \frac{3}{c} \mathcal{G}_\pm, \quad \varsigma_\pm \rightarrow \pm \varsigma_\pm \quad (4.129)$$

$$\tau_\pm^{-\frac{3}{2}} \mathcal{S}_\pm \rightarrow \pm \frac{4\kappa}{5c} \mathcal{S}_\pm, \quad \omega_\pm \rightarrow \mp \frac{2\kappa}{5} \omega_\pm \quad (4.130)$$

where $\kappa = \pm \frac{5i}{2}$, which is defined to make a relation with the notation in [93] at the classical level.

It is important to emphasize that these new variables do not affect the boundary charges. Finally, this leads to operator product expansions of convenient asymptotic symmetry algebra for $\mathcal{N} = (2, 2)$ superconformal boundary with a set of conformal generators $\mathcal{G}_\pm \rightarrow \mathcal{G}^+ \pm \mathcal{G}^-$ and $\mathcal{S}_\pm \rightarrow \mathcal{S}^+ \pm \mathcal{S}^-$ in the complex coordinates by using (4.31). After repeating the same procedure for the barred-sector, one can say that the asymptotic

symmetry algebra for the loosest set of boundary conditions of $\mathcal{N} = (2, 2)$ supergravity is two copies of the super $\mathcal{W}_3$ algebra with central charge $c = 6k$. In this chapter, we do not explicitly carry out the whole computation to obtain the classical $\mathcal{N} = (2, 2)$ super $\mathcal{W}_3$ algebra (see [93] for the entire quantum $\mathcal{N} = 2$ super $\mathcal{W}_3$ algebra and [94] for the classical case). Recently, a detailed derivation of the asymptotic symmetry algebra is also given in [95] for the $\mathfrak{sl}(3|2)$ case.

4.2.3 Remarks on charges and the nature of quantum gravity through holography

Before we move on to the last section, we make some significant remarks on the physical interpretation for the charges of the extended algebras. As previously mentioned, when one analyzes a theory with a boundary, it is essential to consider boundary conditions for the dynamical variables. In the covariant approach, the boundary conditions imposed on conformal gravity such that it entails to having a well - defined variational principle give the boundary conditions preserving transformations, and subsequently the asymptotic symmetry algebra. These conditions can be physically motivated, or determined to make into the theory self - consistent. As is well - known, if the original action is not differentiable in the presence of the boundary conditions in question, a boundary term is needed to be added it. In this context, the canonical boundary charges are defined as the boundary terms added to the smeared generators of the gauge transformations to make them differentiable. In order to be endowed with the canonical boundary charges related to the mass and angular momentum of the BTZ black hole for pure gravity case, one would in principle work with the metric formulation or use the Chern - Simons description through the proposed holonomies which contain the gauge - invariant information [43, 79]. It is also worth noting within this framework that the charges for extended algebras give rise to the symmetries of the dual two - dimensional CFT and so by evaluating the charges and their corresponding algebras, one can determine what chiral algebra the dual CFT has. In other words, the charges of these extended algebras have a physical meaning as conserved charges of the putative dual CFTs, and as such, one can define generalized partition functions with chemical potentials that couple to these conserved charges. Moreover, these could be corresponding to the black hole solutions that carry higher spin charges in AdS_3 . Chiral and anti - chiral functions appearing in the canonical boundary charges characterize the physical state space in the black hole solutions. These

are the vacuum expectation values of the corresponding operators in the holographic interpretation.

In addition, it would be appropriate to point out that these higher spin theories have been studied quite a bit over the last few years, and it is by now pretty clear which higher spin theory yields which two - dimensional chiral algebra. In brief: if the higher spin theory is described by a Chern - Simons theory based on Lie algebra g , then the dual CFT will be the Hamiltonian reduction of g . Which Hamiltonian reduction appears depends on the boundary conditions one imposes on the Chern - Simons theory, but also this has been studied in various examples by now.

Having studied the asymptotic symmetry algebra realization of the classical extended $\mathcal{N} = (2,2)$ superalgebra in terms of higher - spin fields on AdS_3 with the most general boundary conditions, the corresponding dual field theory being two classical copies of the $\mathfrak{sl}(3|2)_k$ is highly suggestive because the physical Hilbert space, belongs to representations of two copies of the related superalgebra, as in the bosonic case. Therefore, it is of interest to investigate the holographic implications of our new encouraging boundary conditions, and also, this super extension will be novel in that it follows the boundary conditions discussed in [21].

It is also convenient to note that the choice of various boundary conditions enriches the holographic interpretation. Most prevalent works in the literature based on AdS_3/CFT_2 duality are those belonging to the class of Brown - Henneaux boundary conditions, obtained via Drinfeld - Sokolov reduction in the highest-weight gauge of the Chern-Simons theory. The holographic interpretation of each boundary condition is idiosyncratic for both Brown - Henneaux boundary conditions [6] and its (extended) supersymmetric extensions [5], and also other boundary conditions that have been proposed by altering and generalizing the Brown - Henneaux boundary conditions in the last years [18–20]. The most general boundary conditions and their supersymmetric extensions make it possible to look at the holographic picture from a much broader perspective in this sense. From this point of view, it is considerable to impose the most general boundary conditions, explore the asymptotic structure for (extended) supergravity and obtain new algebras. It is also important to check out whether these most general boundary conditions and their (extended) supersymmetric extensions

encompassing or not the Brown - Henneaux boundary conditions in terms of consistency as a first test, or even other boundary conditions in the literature.

Furthermore, the analysis we performed is essentially classical. However, it is also possible to examine the corresponding dual conformal field theory at the quantum level. The usual way in this case is to compute Poisson brackets and then quantize them canonically by replacing with the commutators. An additional point worth noting here is that to make a relation with the holographic description, one needs to extract the semiclassical limit of the operator product expansion relations. This procedure involves taking a large - c and large - current limit, a procedure that is more subtle than a naive expansion in $1/c$.

4.2.4 Remarks on the correspondence of higher-spin theories to the string theory and recent developments

It is pertinent to discuss how a supergravity theory constructed in this manner can be physically interesting. Infact, it is not exactly known how these higher spin theories (and their dual *CFTs*) sit inside string theory (and their dual *CFTs*) in general. Fortunately, there is one example in the literature that is understood well: this is the $\mathcal{N} = 4$ higher spin theory in the limit $\lambda \rightarrow 0$ which sits inside superstring theory on $AdS_3 \times S^3 \times T^4$ at the tensionless point, the dual *CFT* is then exactly the symmetric orbifold of T^4 . This was originally explained in [96]; more recently, the relevant string theory has been managed to construct in detail, see in particular [97, 98] and [99].

It would also be intriguing to see what the analogue of [96] should be for other theories, particularly with less supersymmetry. As a result, there can be made some suggestions in this direction for the questions such as: “Can one identify the higher spin theory corresponding to the theory in [100]?”, or, alternatively, “Can one see what string theory could correspond to higher spin theories?”. So, it could correspond to a limit of our higher spin theory at this very point.

4.3 Concluding Remarks

In this chapter, it is clearly put forward that the Chern - Simons formulation of AdS_3 (super)gravity also allows a more convenient generalization of higher spin theories for fermionic states as well as for bosonic states. Furthermore, we also confirm explicitly

that although the higher spin fields do not propagate any degrees of freedom, there exists a large class of intriguing nontrivial solutions. Specifically, we build up a candidate solution for $\mathcal{N} = (2, 2)$ extended higher spin AdS_3 supergravity and scrutinize its asymptotic symmetries.

To summarize, we give a brief discussion for AdS_3 higher spin supergravity based on Chern - Simons formulation. We first work out $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern - Simons $\mathcal{N} = (2, 2)$ supergravity theory in detail. Then, we construct AdS_3 higher spin supergravity enlarging $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ to $\mathfrak{sl}(3|3) \oplus \mathfrak{sl}(3|2)$ in the presence of a tower of higher-spin fields up to spin-3. Thereafter we obtain two classical copies of the $\mathfrak{sl}(3|2)_k$ affine algebra on the affine boundary and two copies of super $\mathcal{W}_3$ symmetry algebra on the superconformal boundary as asymptotic symmetry algebras. We also go through the chemical potentials related to source fields appearing through the temporal components of the connection. On the other hand, we see that Chern - Simons action is compatible with our boundary conditions, resulting in a finite effect for higher spin fields and a well-defined variational principle. Consequently, this method can be considered as a good laboratory for researching the fertile asymptotic structure of extended supergravity. It also might be worthwhile to translate our outcomes into the metric formulation language because it will lift our boundary to higher dimensions, where is a Chern - Simons theory.

The results in this chapter leave some further investigations which we put in order a few here:

It is a natural question to ask if another class of boundary conditions appearing in literature (see, e.g. [15, 18–20]), whose higher-spin generalization is not as clear as the Grumiller and Riegler 's, are consistent with these most general ones, and it would be interesting to examine this. Besides, it has arousing curiosity to get two copies of $\mathcal{N} = (2, 2)$ warped superconformal algebras for the supersymmetric boundary conditions of [19] and also to check the supersymmetric extension of the Avery -Poojary -Suryanarayana boundary conditions [20, 54]. In this point, it is also worth noting that the boundary conditions are more restrictive than the pure bosonic case [21]. While the first motivation is to extend them, it is worth mentioning that these generalizations about boundaries would also have a good potential for novel holographic applications.

Finally, we close scratching the limits in our debate of the most general boundary conditions of supergravity. Many open questions still exist for further investigations, e.g., what other boundary conditions from a similar starting point can be attained? Or how can be explained the puzzling result that the related geometries have an entropy? Overall, we think it is satisfying to see that even in specific instances of three dimensional gravity, the asymptotically AdS tale of the most general boundaries recently set forth by Grumiller and Riegler inspires new and unexpected innovations.

Last but not least, according to [21] and [54] there is $\mathcal{N} = (2, 2)$ extended supergravity with new boundaries. However, it is an open problem to decide whether we end up to its enough higher order $\mathcal{N}$ extension by taking the Grumiller-Riegler method as we perform in this chapter. Therefore, our results presented in this chapter can be extended in various ways. One possible extension is $\mathcal{N} = 3$ supergravity theory in AdS_3 . In this context, the details of this possible extension will be examined in our forthcoming works.



5. CONCLUSION

In the scope of this thesis, based on the relationship between AdS_3 and the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern-Simons theory, we have obtained an explicit asymptotically Anti-de Sitter spin-5 solution as asymptotic symmetries of higher spin gauge theories coupled to three-dimensional gravity with a negative cosmological constant. In three dimensions, despite the absence of any propagating degrees of freedom, we have demonstrated the existence of an intriguing nontrivial solution. The derivation of higher spin Ward identities, as well as the asymptotic W_5 symmetry algebra, can be elucidated as evidence for our results as an extension of the AdS_3 gravity to spin-5 higher spin gravity theory. By virtue of the $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern-Simons gauge theory, we showed that the asymptotic symmetry algebra comprises two instances of the classical W_5 symmetry algebra, defined by the central charge c . We aspire to gain a more profound conceptual understanding of the holonomy prescription and chemical potentials. To this end, we have found the chemical potentials appearing through the temporal components of the connection and explicitly calculated the holonomy conditions associated with these higher spin charges that preserve the asymptotic symmetry algebra. We gave an explicit generalized entropy expression that is very important to establish a relationship among generalized spin-5 black hole thermodynamics, asymptotic symmetries and W_5 algebras.

However, it remains unclear from the perspective of CFT_2 how the introduction of chemical potentials through the time component can be interpreted. It is possible to proceed in a similar manner for AdS_3 spin-5 gravity and thus determine the asymptotic symmetry algebra as the quantum W_5 algebra, rather than the semiclassical one. Consequently, we can consider the normal ordering effects of the quantum W_5 algebra. Up to this point, we have only addressed the classical W_5 symmetry. One may question whether the quantum case also allows for a description in terms of a classical W_N symmetry or a quantum version thereof. While these questions appear to be quite challenging, it seems that the procedure can be straightforwardly implemented. Moreover, acquiring the solution for the entropy expression we articulated

for spin-5 would certainly be contributory for interpreting higher-spin AdS_3 gravity with Chern-Simons formulation.

In order to go beyond the bosonic gauge fields, we have obtained that within the inclusion of fermionic fields to the theory, the Chern-Simons formulation allows for a direct generalization of higher spin theories. We performed supersymmetric generalization of our work including the supersymmetry transformation laws, and demonstrate how to recover super algebras from its asymptotic spacetime symmetries. We presented a candidate for $\mathcal{N} = (1, 1)$ extended higher spin AdS_3 supergravity with the most general boundary conditions as a generalization of the Grumiller and Riegler boundary conditions.

To summarize, we showed that asymptotic symmetry algebras for the most general set of boundary conditions involve two copies of the $\mathfrak{osp}(3|2)_k$ affine algebra on the affine boundary and two copies of $\mathcal{SW}(\frac{3}{2}, 2)$ symmetry algebra on the superconformal boundary respectively. Afterwards, when considering the most general boundary conditions, we reached the supersymmetric extensions of the Brown-Henneaux boundary conditions by imposing some certain restrictions on the gauge fields. And also, we obtained the chemical potentials related to source fields appearing through the temporal components of the connection. We checked that the consistency of our construction in the presence of two different class of boundary conditions that appeared in the literature. We discussed conservation of the charges, and also showed that the Chern - Simons action which is compatible with our boundary conditions leads to a finite, well - defined variational principle for the higher - spin fields. In this regard, this method can serve as a tool for conducting in-depth analysis of extended supergravity.

And as the final main section of our thesis, we advanced our previous work one step further. We develop a candidate solution for $\mathcal{N} = (2, 2)$ extended higher spin AdS_3 supergravity. This time, we have acquired two classical copies of the $\mathfrak{sl}(3|2)_k$ affine algebra on the affine boundary and two copies of the super $\mathcal{W}_3$ symmetry algebra on the superconformal boundary, thereby constituting the asymptotic symmetry algebras. Furthermore, once again, we derive the chemical potentials associated with source fields that emerge through the temporal components of the connection.

It is a natural question to ask whether this study is compatible with different classes of boundary conditions found in the literature. At this point, it's important to note that the boundary conditions are more restrictive than the pure bosonic case.

As another future direction, it could be interesting to express our results in the metric formulation language. Besides, such generalizations are inherently intriguing and hold promise for new applications in holography. In our discussion of the most general boundary conditions in (super)gravity, there are numerous unanswered questions warranting further investigation. For instance, what other boundary conditions could be derived from a similar starting point? Or, how can we explain the perplexing observation that the associated geometries seem to exhibit entropy? While the goal may seem to be expansion, it's worth mentioning that these expansions about boundaries also hold good potential for novel holographic applications. In general, it's rewarding to witness how the asymptotically AdS scenario presented by Grumiller and Riegler for the most comprehensive boundaries stimulates fresh and unexpected developments, even within specific cases of three-dimensional gravity.

In other words: we aimed to deepen our understanding of higher spin (super)gravity by following six interconnected key themes that form the backbone of our study. In this way, we have tried to answer the following questions:

- i. How can the classical theory of gravity, which can be reformulated as a gauge theory through the Chern-Simons formalism, be used as an intriguing platform to investigate the AdS/CFT holographic duality?
- ii. How can one generalize the Chern-Simons formalism to the higher spin theories by extending the set of appropriate gauge groups?
- iii. Do the boundary conditions lead to a well-defined variational principle?
- iv. What is the explicit relation between higher spin algebras and the symmetries of gravitational theory?
- v. How achieve the AdS_3 Chern-Simons formalism to allow a direct generalization to the higher spin (super)gravity theories as in the bosonic case?
- vi. In this case, what are the asymptotic symmetries of the theory?

And it should also be noted that there are still many open questions to be answered for both sides of the coin in the context of AdS_3 higher spin gravity within the AdS/CFT duality framework.



REFERENCES

- [1] **Maldacena, J.M.** (1999). The Large N limit of superconformal field theories and supergravity, *Int. J. Theor. Phys.*, *38*, 1113.
- [2] **Achucarro, A. and Townsend, P.K.** (1986). A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories, *Phys. Lett. B*, *180*, 89.
- [3] **Witten, E.** (1988). (2+1)-Dimensional Gravity as an Exactly Soluble System, *Nucl. Phys. B*, *311*, 46.
- [4] **Banados, M., Teitelboim, C. and Zanelli, J.** (1992). The Black hole in three-dimensional space-time, *Phys. Rev. Lett.*, *69*, 1849–1851, hep-th/9204099.
- [5] **Banados, M., Henneaux, M., Teitelboim, C. and Zanelli, J.** (1993). Geometry of the (2+1) black hole, *Phys. Rev. D*, *48*, 1506–1525, [Erratum: Phys.Rev.D *88*, 069902 (2013)], gr-qc/9302012.
- [6] **Brown, J.D. and Henneaux, M.** (1986). Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity, *Commun. Math. Phys.*, *104*, 207–226.
- [7] **Henneaux, M., Martinez, C., Troncoso, R. and Zanelli, J.** (2002). Black holes and asymptotics of 2+1 gravity coupled to a scalar field, *Phys. Rev. D*, *65*, 104007, hep-th/0201170.
- [8] **Henneaux, M., Martinez, C., Troncoso, R. and Zanelli, J.** (2004). Asymptotically anti-de Sitter spacetimes and scalar fields with a logarithmic branch, *Phys. Rev. D*, *70*, 044034, hep-th/0404236.
- [9] **Grumiller, D. and Johansson, N.** (2009). Consistent boundary conditions for cosmological topologically massive gravity at the chiral point, *Int. J. Mod. Phys. D*, *17*, 2367–2372, 0808.2575.
- [10] **Henneaux, M., Martinez, C. and Troncoso, R.** (2009). Asymptotically anti-de Sitter spacetimes in topologically massive gravity, *Phys. Rev. D*, *79*, 081502, 0901.2874.
- [11] **Oliva, J., Tempo, D. and Troncoso, R.** (2009). Three-dimensional black holes, gravitational solitons, kinks and wormholes for BHT massive gravity, *JHEP*, *07*, 011, 0905.1545.
- [12] **Barnich, G. and Lambert, P.H.** (2013). Einstein-Yang-Mills theory: Asymptotic symmetries, *Phys. Rev. D*, *88*, 103006, 1310.2698.

- [13] **Bunster, C. and Pérez, A.** (2015). Superselection rule for the cosmological constant in three-dimensional spacetime, *Phys. Rev. D*, *91*(2), 024029, 1412.1492.
- [14] **Afshar, H., Detournay, S., Grumiller, D. and Oblak, B.** (2016). Near-Horizon Geometry and Warped Conformal Symmetry, *JHEP*, *03*, 187, 1512.08233.
- [15] **Afshar, H., Detournay, S., Grumiller, D., Merbis, W., Perez, A., Tempo, D. and Troncoso, R.** (2016). Soft Heisenberg hair on black holes in three dimensions, *Phys. Rev. D*, *93*(10), 101503, 1603.04824.
- [16] **Perez, A., Riquelme, M., Tempo, D. and Troncoso, R.** (2016). Asymptotic structure of the Einstein-Maxwell theory on AdS_3 , *JHEP*, *02*, 015, 1512.01576.
- [17] **Pérez, A., Tempo, D. and Troncoso, R.** (2016). Boundary conditions for General Relativity on AdS_3 and the KdV hierarchy, *JHEP*, *06*, 103, 1605.04490.
- [18] **Compère, G., Song, W. and Strominger, A.** (2013). New Boundary Conditions for AdS_3 , *JHEP*, *05*, 152, 1303.2662.
- [19] **Troessaert, C.** (2013). Enhanced asymptotic symmetry algebra of AdS_3 , *JHEP*, *08*, 044, 1303.3296.
- [20] **Avery, S.G., Poojary, R.R. and Suryanarayana, N.V.** (2014). An $sl(2, \mathbb{R})$ current algebra from AdS_3 gravity, *JHEP*, *01*, 144, 1304.4252.
- [21] **Grumiller, D. and Riegler, M.** (2016). Most general AdS_3 boundary conditions, *JHEP*, *10*, 023, 1608.01308.
- [22] **Grumiller, D., Merbis, W. and Riegler, M.** (2017). Most general flat space boundary conditions in three-dimensional Einstein gravity, *Class. Quant. Grav.*, *34*(18), 184001, 1704.07419.
- [23] **Krishnan, C. and Raju, A.** (2017). Chiral Higher Spin Gravity, *Phys. Rev. D*, *95*(12), 126004, 1703.01769.
- [24] **Fradkin, E.S. and Vasiliev, M.A.** (1987). Cubic Interaction in Extended Theories of Massless Higher Spin Fields, *Nucl. Phys. B*, *291*, 141–171.
- [25] **Fradkin, E.S. and Vasiliev, M.A.** (1987). On the Gravitational Interaction of Massless Higher Spin Fields, *Phys. Lett. B*, *189*, 89–95.
- [26] **Vasiliev, M.A.** (1990). Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions, *Phys. Lett. B*, *243*, 378–382.
- [27] **Vasiliev, M.A.** (1992). More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions, *Phys. Lett. B*, *285*, 225–234.
- [28] **Vasiliev, M.A.** (1999). Higher spin gauge theories: Star product and AdS space, 533–610, hep-th/9910096.
- [29] **Vasiliev, M.A.** (2000). Higher spin symmetries, star product and relativistic equations in AdS space, hep-th/0002183.

- [30] **Vasiliev, M.A.** (2001). Progress in higher spin gauge theories, *International Conference on Quantization, Gauge Theory, and Strings: Conference Dedicated to the Memory of Professor Efim Fradkin*, pp.452–471, hep-th/0104246.
- [31] **Bekaert, X., Cnockaert, S., Iazeolla, C. and Vasiliev, M.A.** (2004). Nonlinear higher spin theories in various dimensions, *1st Solvay Workshop on Higher Spin Gauge Theories*, pp.132–197, hep-th/0503128.
- [32] **Aragone, C. and Deser, S.** (1980). Higher Spin Vierbein Gauge Fermions and Hypergravities, *Nucl. Phys. B*, 170, 329–352.
- [33] **Pope, C.N., Romans, L.J. and Shen, X.** (1990). $W(\infty)$ and the Racah-wigner Algebra, *Nucl. Phys. B*, 339, 191–221.
- [34] **Klebanov, I.R. and Polyakov, A.M.** (2002). AdS dual of the critical $O(N)$ vector model, *Phys. Lett. B*, 550, 213–219, hep-th/0210114.
- [35] **Sezgin, E. and Sundell, P.** (2005). Holography in 4D (super) higher spin theories and a test via cubic scalar couplings, *JHEP*, 07, 044, hep-th/0305040.
- [36] **Gaberdiel, M.R. and Gopakumar, R.** (2011). An AdS_3 Dual for Minimal Model CFTs, *Phys. Rev. D*, 83, 066007, 1011.2986.
- [37] **Bershadsky, M.** (1991). Conformal field theories via Hamiltonian reduction, *Commun. Math. Phys.*, 139, 71–82.
- [38] **Henneaux, M. and Rey, S.J.** (2010). Nonlinear W_{∞} as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity, *JHEP*, 12, 007, 1008.4579.
- [39] **Campoleoni, A., Fredenhagen, S., Pfenninger, S. and Theisen, S.** (2010). Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields, *JHEP*, 11, 007, 1008.4744.
- [40] **Campoleoni, A., Fredenhagen, S. and Pfenninger, S.** (2011). Asymptotic W-symmetries in three-dimensional higher-spin gauge theories, *JHEP*, 09, 113, 1107.0290.
- [41] **Tan, H.S.** (2012). Aspects of Three-dimensional Spin-4 Gravity, *JHEP*, 02, 035, 1111.2834.
- [42] **Campoleoni, A., Francia, D. and Heissenberg, C.** (2018). Asymptotic symmetries and charges at null infinity: from low to high spins, *EPJ Web Conf.*, 191, 06011, 1808.01542.
- [43] **Özer, H.T. and Filiz, A.** (2018). On the explicit asymptotic W_5 symmetry of 3D Chern-Simons higher spin AdS_3 gravity, *J. Math. Phys.*, 59(8), 083504, 1707.09514.
- [44] **Tan, H.S.** (2012). Exploring Three-dimensional Higher-Spin Supergravity based on $sl(N \mid N - 1)$ Chern-Simons theories, *JHEP*, 11, 063, 1208.2277.

- [45] **Banados, M., Bautier, K., Coussaert, O., Henneaux, M. and Ortiz, M.** (1998). Anti-de Sitter / CFT correspondence in three-dimensional supergravity, *Phys. Rev. D*, 58, 085020, hep-th/9805165.
- [46] **Bergshoeff, E., Blencowe, M.P. and Stelle, K.S.** (1990). Area Preserving Diffeomorphisms and Higher Spin Algebra, *Commun. Math. Phys.*, 128, 213.
- [47] **Blencowe, M.P.** (1989). A Consistent Interacting Massless Higher Spin Field Theory in $D = (2+1)$, *Class. Quant. Grav.*, 6, 443.
- [48] **Candu, C. and Gaberdiel, M.R.** (2013). Supersymmetric holography on AdS_3 , *JHEP*, 09, 071, 1203.1939.
- [49] **Henneaux, M., Lucena Gómez, G., Park, J. and Rey, S.J.** (2012). Super-W(infinity) Asymptotic Symmetry of Higher-Spin AdS_3 Supergravity, *JHEP*, 06, 037, 1203.5152.
- [50] **Peng, C.** (2013). Dualities from higher-spin supergravity, *JHEP*, 03, 054, 1211.6748.
- [51] **Hanaki, K. and Peng, C.** (2013). Symmetries of Holographic Super-Minimal Models, *JHEP*, 08, 030, 1203.5768.
- [52] **Chen, B., Long, J. and Wang, Y.N.** (2013). Conical Defects, Black Holes and Higher Spin (Super-)Symmetry, *JHEP*, 06, 025, 1303.0109.
- [53] **Gaberdiel, M.R. and Hartman, T.** (2011). Symmetries of Holographic Minimal Models, *JHEP*, 05, 031, 1101.2910.
- [54] **Özer, H.T. and Filiz, A.** (2020). Exploring new boundary conditions for $\mathcal{N} = (1,1)$ extended higher-spin AdS_3 supergravity, *Eur. Phys. J. C*, 80(11), 1072, 1907.06104.
- [55] **Özer, H.T. and Filiz, A.** (2022). $\mathcal{N} = (2,2)$ extended $\mathfrak{sl}(3|2)$ Chern–Simons AdS_3 supergravity with new boundaries, *Eur. Phys. J. C*, 82(5), 472, 2107.11069.
- [56] **Zamolodchikov, A.B.** (1985). Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory, *Theor. Math. Phys.*, 65, 1205–1213.
- [57] **Fateev, V.A. and Lukyanov, S.L.** (1988). The Models of Two-Dimensional Conformal Quantum Field Theory with $Z(n)$ Symmetry, *Int. J. Mod. Phys. A*, 3, 507.
- [58] **Lukyanov, S.L. and Fateev, V.A.** (1988). Conformally Invariant Models of Two-dimensional QFT With $Z(N)$ Symmetry, *Sov. Phys. JETP*, 67, 447.
- [59] **Lukyanov, S.L. and Fateev, V.A.** (1990). Additional symmetries and exactly soluble models in two-dimensional conformal field theory, *Sov.Sci.Rev.,Rev. A Phys.*, 15, 1.
- [60] **Fateev, V.A. and Lukyanov, S.L.** (1990). *Sov.Sci.Rev.*, 15, 1.

- [61] **Pope, C., Romans, L. and Shen, X.** The Complete Structure of $\mathcal{W}_\infty$.
- [62] **Pope, C.N., Romans, L.J. and Shen, X.** (1990). A New Higher Spin Algebra and the Lone Star Product, *Phys. Lett. B*, 242, 401–406.
- [63] **Lu, H., Pope, C.N., Shen, X. and Wang, X.J.** (1991). The Complete structure of $W(N)$ from $W(\text{infinity})$ at $c = -2$, *Phys. Lett. B*, 267, 356–361.
- [64] **Gaberdiel, M.R. and Gopakumar, R.** (2013). Minimal Model Holography, *J. Phys. A*, 46, 214002, 1207.6697.
- [65] **Gaberdiel, M.R., Gopakumar, R., Hartman, T. and Raju, S.** (2011). Partition Functions of Holographic Minimal Models, *JHEP*, 08, 077, 1106.1897.
- [66] **Gaberdiel, M.R. and Gopakumar, R.** (2012). Triality in Minimal Model Holography, *JHEP*, 07, 127, 1205.2472.
- [67] **Belavin, A.A., Polyakov, A.M. and Zamolodchikov, A.B.** (1984). Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory, *Nucl. Phys. B*, 241, 333–380.
- [68] **Mathieu, P.** (1988). Extended Classical Conformal Algebras and the Second Hamiltonian Structure of Lax Equations, *Phys. Lett. B*, 208, 101–106.
- [69] **Bakas, I.** (1989). Hamiltonian Reduction and Conformal Symmetries in Two-dimensions, *Phys. Lett. B*, 219, 283–290.
- [70] **Ozer, H.T.** (2017). On the Casimir $\mathcal{WA}_N$ algebras as the truncated $\mathcal{W}_\infty$ algebra, *Int. J. Geom. Meth. Mod. Phys.*, 14, 1750121, 1503.04541.
- [71] **Hornfeck, K.** (1993). W algebras with set of primary fields of dimensions (3, 4, 5) and (3, 4, 5, 6), *Nucl. Phys. B*, 407, 237–246, hep-th/9212104.
- [72] **Henneaux, M., Perez, A., Tempo, D. and Troncoso, R.** (2013). Chemical potentials in three-dimensional higher spin anti-de Sitter gravity, *JHEP*, 12, 048, 1309.4362.
- [73] **Gutperle, M. and Kraus, P.** (2011). Higher Spin Black Holes, *JHEP*, 05, 022, 1103.4304.
- [74] **Chen, B., Long, J. and Wang, Y.n.** (2012). Black holes in Truncated Higher Spin AdS_3 Gravity, *JHEP*, 12, 052, 1209.6185.
- [75] **Chen, B., Long, J. and Wang, Y.N.** (2013). Phase Structure of Higher Spin Black Hole, *JHEP*, 03, 017, 1212.6593.
- [76] **Ferlino, M., Hollowood, T. and Kumar, S.P.** (2013). Asymptotic symmetries and thermodynamics of higher spin black holes in AdS_3 , *Phys. Rev. D*, 88, 066010, 1305.2011.
- [77] **Beccaria, M. and Macorini, G.** (2014). Analysis of higher spin black holes with spin-4 chemical potential, *JHEP*, 07, 047, 1312.5599.
- [78] **de Boer, J. and Jottar, J.I.** (2016). Boundary conditions and partition functions in higher spin AdS_3/CFT_2 , *JHEP*, 04, 107, 1407.3844.

- [79] **Bunster, C., Henneaux, M., Perez, A., Tempo, D. and Troncoso, R.** (2014). Generalized Black Holes in Three-dimensional Spacetime, *JHEP*, 05, 031, 1404.3305.
- [80] **Coussaert, O., Henneaux, M. and van Driel, P.** (1995). The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant, *Class. Quant. Grav.*, 12, 2961–2966, gr-qc/9506019.
- [81] **Banados, M.** (1999). Three-dimensional quantum geometry and black holes, *AIP Conf. Proc.*, 484(1), 147–169, hep-th/9901148.
- [82] **Blumenhagen, R. and Plauschinn, E.** (2009). *Introduction to conformal field theory: with applications to String theory*, volume 779.
- [83] **Susskind, L.** (1995). The World as a hologram, *J. Math. Phys.*, 36, 6377–6396, hep-th/9409089.
- [84] **Elitzur, S., Moore, G.W., Schwimmer, A. and Seiberg, N.** (1989). Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory, *Nucl. Phys. B*, 326, 108–134.
- [85] **Valcárcel, C.E.** (2019). New boundary conditions for (extended) AdS₃ supergravity, *Class. Quant. Grav.*, 36, 065002, 1812.02799.
- [86] **Carlip, S.** (2005). Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole, *Class. Quant. Grav.*, 22, R85–R124, gr-qc/0503022.
- [87] **Banados, M.** (1996). Global charges in Chern-Simons field theory and the (2+1) black hole, *Phys. Rev. D*, 52, 5816–5825, hep-th/9405171.
- [88] **Banados, M., Brotz, T. and Ortiz, M.E.** (1999). Boundary dynamics and the statistical mechanics of the (2+1)-dimensional black hole, *Nucl. Phys. B*, 545, 340–370, hep-th/9802076.
- [89] **Blagojevic, Milutin, S.**, (2002). *Gravitation and gauge symmetries*, IOP publishing, Bristol.
- [90] **Figueroa-O’Farrill, J.M.** (1997). A Note on the extended superconformal algebras associated with manifolds of exceptional holonomy, *Phys. Lett. B*, 392, 77–84, hep-th/9609113.
- [91] **Figueroa-O’Farrill, J.M. and Schrans, S.** (1991). Extended superconformal algebras, *Phys. Lett. B*, 257, 69–73.
- [92] **Donnay, L., Giribet, G., Gonzalez, H.A. and Pino, M.** (2016). Supertranslations and Superrotations at the Black Hole Horizon, *Phys. Rev. Lett.*, 116(9), 091101, 1511.08687.
- [93] **Romans, L.J.** (1992). The N=2 superW(3) algebra, *Nucl. Phys. B*, 369, 403–432.
- [94] **Lu, H., Pope, C.N., Romans, L.J., Shen, X. and Wang, X.J.** (1991). Polyakov construction of the N=2 superW(3) algebra, *Phys. Lett. B*, 264, 91–100.

- [95] **Castro, A., Faraggi, A. and Osorio, I.** (2020). A note on the $\mathcal{N} = 2$ super- $\mathcal{W}_3$ holographic dictionary, *JHEP*, *12*, 177, 2008.05541.
- [96] **Gaberdiel, M.R. and Gopakumar, R.** (2014). Higher Spins & Strings, *JHEP*, *11*, 044, 1406.6103.
- [97] **Gaberdiel, M.R. and Gopakumar, R.** (2018). Tensionless string spectra on AdS_3 , *JHEP*, *05*, 085, 1803.04423.
- [98] **Eberhardt, L., Gaberdiel, M.R. and Gopakumar, R.** (2019). The Worldsheet Dual of the Symmetric Product CFT, *JHEP*, *04*, 103, 1812.01007.
- [99] **Eberhardt, L., Gaberdiel, M.R. and Gopakumar, R.** (2020). Deriving the $\text{AdS}_3/\text{CFT}_2$ correspondence, *JHEP*, *02*, 136, 1911.00378.
- [100] **Datta, S., Eberhardt, L. and Gaberdiel, M.R.** (2018). Stringy $\mathcal{N} = (2,2)$ holography for AdS_3 , *JHEP*, *01*, 146, 1709.06393.





APPENDICES

APPENDIX A : Notation and Convention

APPENDIX B : Spin-3 Case

APPENDIX C : Spin-5 Extension

APPENDIX D : The $\mathfrak{sl}(3|2)$ Superconformal Case





APPENDIX A : Notation and Convention

A.1 $\mathfrak{sl}(2, \mathbb{R})$ Generators

Below, we denote the three generators of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra, which will be assumed to be described by a set of matrices L_a with $a = \pm 1, 0$ given by

$$L_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (\text{A.1})$$

which admit an invariant bilinear form

$$\eta_{ab} = \left(\begin{array}{c|ccc} & L_{-1} & L_0 & L_1 \\ \hline L_{-1} & 0 & 0 & -1 \\ L_0 & 0 & \frac{1}{2} & 0 \\ L_1 & -1 & 0 & 0 \end{array} \right). \quad (\text{A.2})$$



APPENDIX B : Spin-3 Case

B.1 $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ Higher-spin Chern-Simons Theory

The spin-3 case is given by promoting $\mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{sl}(3, \mathbb{R})$.

B.1.1 $\mathfrak{sl}(3, \mathbb{R})$ generators as the principal embedding of $\mathfrak{sl}(2, \mathbb{R})$

To construct this principal embedding, we denote the eight generators of $\mathfrak{sl}(3, \mathbb{R})$ Lie algebra, which will be assumed to be described by a set of matrices. The principal embedding of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra to $\mathfrak{sl}(3, \mathbb{R})$ Lie algebra contains the spin-2 triplet L_n with $n = \pm 1, 0$, and the spin-3 triplet $W_n^{(3)}$ and can be represented in terms of the commutation relations for $L_n \equiv W_n^{(2)}$. Below, we collect the eight $\mathfrak{sl}(3, \mathbb{R})$ generators:

$$\begin{aligned} L_0 &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & W_0^{(3)} &= 2\sqrt{-\frac{\sigma}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ L_1 &= \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & W_1^{(3)} &= \sqrt{-6\sigma} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\ L_{-1} &= \sqrt{2} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, & W_2^{(3)} &= 4\sqrt{-3\sigma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (\text{B.1})$$

and

$$W_{-m}^{(s)} = (-1)^m W_m^{(s)\dagger}$$

B.1.2 Holonomies in $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ Chern - Simons theory

Let's recall the connection a_ϕ for $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ Chern - Simons Theory as:

$$a_\phi = \begin{pmatrix} 0 & \frac{Q_2}{2\sqrt{2}} & \frac{Q_3}{2} \\ \sqrt{2} & 0 & \frac{Q_2}{2\sqrt{2}} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad (\text{B.2})$$

depending on *Casimir operator*-related parameters $Q_s = \frac{1}{s} \text{tr}[a_\phi^s]$, ($s = 2, 3$) for $\mathfrak{sl}(3, \mathbb{R})$ Lie algebra. Now when it comes to calculating the holonomy about contractible (thermal) cycle one may be used the temporal component as:

$$a_t = \mu_2 a_\phi + \mu_3 \left(a_\phi^2 - \frac{1}{3} \text{tr}[a_\phi^2] \right) \quad (\text{B.3})$$

which fulfills the equation of motion $[a_t, a_\phi] = 0$ it is obvious that $a_t = f(a_\phi)$. Therefore the trivial holonomy is

$$H_\tau(a) = b^{-1} e^w b, \quad (\text{B.4})$$

where $w = \oint_C a_t dt = 2\pi\tau a_t$ is the holonomy matrix with $\tau = \frac{i\beta}{2\pi}$. If a holonomy is trivial around a contractible cycle the exponentiated matrix e^w must have eigenvalues $0, \pm 2\pi i$, it follows that

$$\text{det}[w] = 0, \quad \text{tr}[w^2] + 8\pi^2 = 0. \quad (\text{B.5})$$

These conditions are written as:

$$0 = 27\mu_2^3 Q_3 + 18\mu_3\mu_2^2 Q_2^2 + 27\mu_3^2\mu_2 Q_2 Q_3 - 2\mu_3^3 Q_2^3 + 27\mu_3^3 Q_3^2 \quad (\text{B.6})$$

$$0 = 3 + 3\mu_2^2 Q_2 \tau^2 + \mu_3^2 Q_2^2 \tau^2 + 9\mu_2\mu_3 Q_3 \tau^2. \quad (\text{B.7})$$

In order to solve the holonomy conditions, the entropy function can be written in the form

$$\mathcal{S}(Q_2, Q_3) = 2\pi k \sqrt{Q_2} F(z) \quad (\text{B.8})$$

with $F(0) = 1$, we define a dimensionless parameter $z = \frac{27}{2} \frac{Q_3}{Q_2^{3/2}}$ such that

$$\mu_s = i \frac{1}{2\pi k} \frac{\partial \mathcal{S}}{\partial Q_s}, \quad (s = 2, 3). \quad (\text{B.9})$$

Then the holonomy equations become

$$\begin{aligned} 1 &= F^2 - 36z(z-2)(F')^2, \\ 0 &= F^3 - 18(z-2)F'(6zF'(2(z-2)F' - F) + F^2) \end{aligned} \quad (\text{B.10})$$

and similar expressions for the barred sector. This is compatible with the seminal work of Gutperle and Kraus. [73]

APPENDIX C : Spin-5 Extension

C.1 $\mathfrak{sl}(5, \mathbb{R})$ Generators as the Principal Embedding of $\mathfrak{sl}(2, \mathbb{R})$

Throughout this work we used the following matrix representations of $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra. Below, we denote the twenty-four generators of $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra, which will be assumed to be described by a set of matrices in Sec. 2.2.1. The principal embedding of $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra to $\mathfrak{sl}(5, \mathbb{R})$ Lie algebra contains the spin-2 triplet L_n with $n = \pm 1, 0$, the spin-3 triplet $W_n^{(3)}$ with $n = \pm 2, \pm 1, 0$, the spin-4 quartet $W_n^{(4)}$ with $n = \pm 3, \pm 2, \pm 1, 0$ and the spin-5 quintet $W_n^{(5)}$ with $n = \pm 4, \pm 3, \pm 2, \pm 1, 0$ and can be represented in terms of the commutation relations for $L_n \equiv W_n^{(2)}$ as follows

$$[L_n, W_m^{(s)}] = ((s-1)n - m)W_{n+m}^{(s)}, \quad (s = 2, 3, 4, 5) \quad (C.1)$$

$$[W_n^{(3)}, W_m^{(3)}] = f_{33}^2(n, m)L_{n+m} + f_{33}^4(n, m)W_{n+m}^{(4)} \quad (C.2)$$

$$[W_n^{(3)}, W_m^{(4)}] = f_{34}^3(n, m)W_{n+m}^{(3)} + f_{34}^5(n, m)W_{n+m}^{(5)} \quad (C.3)$$

$$[W_n^{(4)}, W_m^{(4)}] = f_{44}^2(n, m)L_{n+m} + f_{44}^4(n, m)W_{n+m}^{(4)} \quad (C.4)$$

$$[W_n^{(3)}, W_m^{(5)}] = f_{35}^4(n, m)W_{n+m}^{(4)} \quad (C.5)$$

$$[W_n^{(4)}, W_m^{(5)}] = f_{45}^3(n, m)W_{n+m}^{(3)} + f_{45}^5(n, m)W_{n+m}^{(5)} \quad (C.6)$$

$$[W_n^{(5)}, W_m^{(5)}] = f_{55}^2(n, m)L_{n+m} + f_{55}^4(n, m)W_{n+m}^{(4)}. \quad (C.7)$$

where

$$f_{33}^2(n, m) = \sigma_1(n - m)(2n^2 - nm + 2m^2 - 8) \quad (C.8)$$

$$f_{33}^4(n, m) = \sigma_2(n - m) \quad (C.9)$$

$$f_{34}^3(n, m) = \sigma_3(m^3 - 3m^2n + 5mn^2 - 9m - 5n^3 + 17n) \quad (C.10)$$

$$f_{34}^5(n, m) = \sigma_4(3n - 2m) \quad (C.11)$$

$$f_{44}^2(n, m) = \sigma_5(n - m)(3m^4 - 2m^3n + 4m^2n^2 - 39m^2 - 2mn^3 + 20mn + 3n^4 - 39n^2 + 108) \quad (C.12)$$

$$f_{44}^4(n, m) = \sigma_6(n - m)(m^2 + n^2 - mn - 7) \quad (C.13)$$

$$f_{35}^4(n, m) = \sigma_7(2m^3 - 9m^2n + 21mn^2 - 32 - 28n^3m + 88n) \quad (C.14)$$

$$f_{45}^3(n, m) = \sigma_8(3m^5 - 10m^4n + 20m^3n^2 - 75m^3 - 30m^2n^3 + 220m^2n + 35mn^4 - 355mn^2 + 432m - 28n^5 + 340n^3 - 792n) \quad (C.15)$$

$$f_{45}^5(n, m) = \sigma_9(5m^3 - 15m^2n + 21mn^2 - 59m - 14n^3 + 86n) \quad (C.16)$$

$$\begin{aligned}
f_{55}^2(n, m) = & \sigma_{10}(n-m)(4m^6 - 3m^5n + 6m^4n^2 - 116m^4 \\
& - 4m^3n^3 + 79m^3n + 6m^2n^4 - 156m^2n^2 \\
& + 976m^2 - 3mn^5 + 79mn^3 \\
& - 508mn + 4n^6 - 116n^4 + 976n^2 - 2304) \quad (C.17)
\end{aligned}$$

$$\begin{aligned}
f_{55}^4(n, m) = & \sigma_{11}(n-m)(14m^4 - 21m^3n + 29m^2n^2 \\
& - 310m^2 - 21mn^3 + 315mn + 14n^4 \\
& - 310n^2 + 1376) \quad (C.18)
\end{aligned}$$

with

$$\begin{aligned}
\sigma_3 &= -\frac{64\sigma_1}{49\sigma_2}, & \sigma_5 &= \frac{64\sigma_1^2}{49\sigma_2^2}, & \sigma_6 &= \frac{8\sigma_1}{7\sigma_2}, \\
\sigma_7 &= -\frac{5\sigma_1}{49\sigma_4}, & \sigma_8 &= -\frac{320\sigma_1^2}{2401\sigma_2^2\sigma_4}, & \sigma_9 &= \frac{8\sigma_1}{49\sigma_2}, \\
\sigma_{10} &= \frac{320\sigma_1^3}{2401\sigma_2^2\sigma_4^2}, & \sigma_{11} &= -\frac{40\sigma_1^2}{2401\sigma_2\sigma_4^2},
\end{aligned} \quad (C.19)$$

where σ_i 's are the 11 arbitrary parameters, depending on three arbitrary parameters, mainly σ_1, σ_2 and σ_4 which can be changed by rescaling $W_n^{(s)}$. Below, we collect the twenty-four $\mathfrak{sl}(5, \mathbb{R})$ generators:

$$L_0 = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix},$$

$$L_{-1} = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_0^{(3)} = \gamma_1 \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix},$$

$$W_1^{(3)} = \frac{\gamma_1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \end{pmatrix}, \quad W_2^{(3)} = 2\gamma_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
W_0^{(4)} &= \gamma_2 \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & W_1^{(4)} &= \frac{2\gamma_2}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{pmatrix}, \\
W_2^{(4)} &= \frac{5\gamma_2}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}, & W_3^{(4)} &= 10\gamma_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \\
W_0^{(5)} &= \gamma_3 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & W_1^{(5)} &= \frac{5\gamma_3}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \\
W_2^{(5)} &= \frac{5\gamma_3}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}, & W_3^{(5)} &= \frac{35\gamma_3}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \\
W_4^{(5)} &= 70\gamma_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.20}
\end{aligned}$$

where $\gamma_1 = \sqrt{-\frac{20}{7}\sigma_1}$, $\gamma_2 = \frac{480\sigma_1}{7\sigma_2}$, $\gamma_3 = \frac{192}{49\sigma_2\sigma_4}\sqrt{-\frac{5}{7}\sigma_1^3}$ and $W_{-m}^{(s)} = (-1)^m W_m^{(s)\dagger}$, which admit an invariant bilinear form η_{ab} corresponding to the values of σ_i 's specified at Sec. 2.2.2.2:

$$\eta_{ab} = \left(\begin{array}{c|c|c|c} \Delta_{3 \times 3} & 0 & 0 & 0 \\ \hline 0 & \Delta_{5 \times 5} & 0 & 0 \\ \hline 0 & 0 & \Delta_{7 \times 7} & 0 \\ \hline 0 & 0 & 0 & \Delta_{9 \times 9} \end{array} \right) \tag{C.21}$$

where

$$\Delta_{3 \times 3} = \begin{pmatrix} 0 & 0 & -20 \\ 0 & 10 & 0 \\ -20 & 0 & 0 \end{pmatrix}, \quad \Delta_{5 \times 5} = \begin{pmatrix} 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.22})$$

$$\Delta_{7 \times 7} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{30}{7} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{7} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{14} & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{7} & 0 & 0 & 0 & 0 & 0 \\ -\frac{30}{7} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.23})$$

and

$$\Delta_{9 \times 9} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{4}{105} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C.24})$$

C.2 Spin-5 Conformal Ward Identities

C.2.1 Solved parameters

In this appendix, we give a collection of eighteen solved remaining parameters as depending on ε, χ, f and η parameters which were discussed in Sec. 2.2.2 of the main text. One finds

$$\chi_1 = -\chi' \quad (\text{C.25})$$

$$\chi_0 = \frac{3840\beta f \sigma_1 \mathcal{W}_3}{49\sigma_2} + \frac{\chi''}{2} + \frac{460800\gamma \eta \sigma_1^2 \mathcal{W}_4}{343\sigma_2^2 \sigma_4} + 2\alpha \chi \mathcal{L}0 \quad (\text{C.26})$$

$$\begin{aligned} \chi_{-1} = & -\frac{2560\beta \sigma_1 \mathcal{W}_3 f'}{49\sigma_2} - \frac{1280\beta f \sigma_1 \mathcal{W}_3'}{49\sigma_2} - \frac{\chi^{(3)}}{6} - \frac{38400\gamma \sigma_1^2 \mathcal{W}_4 \eta'}{49\sigma_2^2 \sigma_4} \\ & - \frac{153600\gamma \eta \sigma_1^2 \mathcal{W}_4'}{343\sigma_2^2 \sigma_4} - \frac{2}{3}\alpha \chi \mathcal{L}' - \frac{5}{3}\alpha \mathcal{L} \chi' \end{aligned} \quad (\text{C.27})$$

$$\begin{aligned}
\chi_{-2} = & \frac{832\beta\sigma_1\mathcal{W}_3f''}{49\sigma_2} + \frac{960\beta\sigma_1f'\mathcal{W}_3'}{49\sigma_2} + \frac{3072\alpha\beta f\sigma_1\mathcal{W}_3\mathcal{W}_2}{49\sigma_2} \\
& + \frac{320\beta f\sigma_1\mathcal{W}_3''}{49\sigma_2} + \frac{230400\delta f\sigma_1^2\mathcal{W}_5}{343\sigma_2^2\sigma_4} + \frac{\chi^{(4)}}{24} + \alpha^2\chi\mathcal{W}_2^2 \\
& + \frac{2304000\alpha\gamma\eta\sigma_1^2\mathcal{W}_4\mathcal{W}_2}{2401\sigma_2^2\sigma_4} + \frac{2}{3}\alpha\mathcal{W}_2\chi'' + \frac{7}{12}\alpha\chi'\mathcal{W}_2' \\
& + \frac{1}{6}\alpha\chi\mathcal{W}_2'' + \frac{92160\beta^2\eta\sigma_1^2\mathcal{W}_3^2}{343\sigma_2\sigma_4} + \beta\mathcal{W}_3\varepsilon + \frac{556800\gamma\sigma_1^2\mathcal{W}_4\eta''}{2401\sigma_2^2\sigma_4} \\
& + \frac{105600\gamma\sigma_1^2\eta'\mathcal{W}_4'}{343\sigma_2^2\sigma_4} + \frac{38400\gamma\eta\sigma_1^2\mathcal{W}_4''}{343\sigma_2^2\sigma_4} + \frac{1920\gamma\sigma_1\chi\mathcal{W}_4}{49\sigma_2} \quad (C.28)
\end{aligned}$$

$$f_2 = -f' \quad (C.29)$$

$$f_1 = \frac{f''}{2} + 3\alpha f\mathcal{L} + \frac{240\beta\eta\sigma_1\mathcal{W}_3}{7\sigma_4} \quad (C.30)$$

$$f_0 = -\frac{f^{(3)}}{6} - \frac{8}{3}\alpha\mathcal{L}f' - \alpha f\mathcal{L}' - \frac{180\beta\sigma_1\mathcal{W}_3\eta'}{7\sigma_4} - \frac{80\beta\eta\sigma_1\mathcal{W}_3'}{7\sigma_4} \quad (C.31)$$

$$\begin{aligned}
f_{-1} = & \frac{f^{(4)}}{24} + \frac{7}{6}\alpha\mathcal{L}f'' + \frac{11}{12}\alpha f'\mathcal{L}' + \frac{240\gamma f\sigma_1\mathcal{W}_4}{7\sigma_2} + 3\alpha^2 f\mathcal{L}^2 \\
& + \frac{1}{4}\alpha f\mathcal{L}'' + \frac{465\beta\sigma_1\mathcal{W}_3\eta''}{49\sigma_4} + \frac{65\beta\sigma_1\eta'\mathcal{W}_3'}{7\sigma_4} + \frac{20\beta\eta\sigma_1\mathcal{W}_3''}{7\sigma_4} \\
& + \beta\sigma_2\chi\mathcal{W}_3 - \frac{19200\delta\eta\sigma_1^2\mathcal{W}_5}{49\sigma_2\sigma_4^2} + \frac{2880\alpha\beta\eta\sigma_1\mathcal{W}_3\mathcal{L}}{49\sigma_4} \quad (C.32)
\end{aligned}$$

$$\begin{aligned}
f_{-2} = & -\frac{f^{(5)}}{120} - \frac{1}{3}\alpha f^{(3)}\mathcal{L} - \frac{5}{12}\alpha f''\mathcal{L}' - \frac{144\gamma\sigma_1\mathcal{W}_4f'}{7\sigma_2} - \frac{11}{5}\alpha^2\mathcal{L}^2f' \\
& - \frac{7}{30}\alpha f'\mathcal{L}'' - \frac{48\gamma f\sigma_1\mathcal{W}_4'}{7\sigma_2} - \frac{1}{20}\alpha f\mathcal{L}^{(3)} - \frac{9}{5}\alpha^2 f\mathcal{L}\mathcal{L}' \\
& - \frac{113\beta\eta^{(3)}\sigma_1\mathcal{W}_3}{49\sigma_4} - \frac{4}{5}\beta\sigma_2\mathcal{W}_3\chi' - \frac{1}{5}\beta\sigma_2\chi\mathcal{W}_3' + \frac{9600\delta\sigma_1^2\mathcal{W}_5\eta'}{49\sigma_2\sigma_4^2} \\
& + \frac{3840\delta\eta\sigma_1^2\mathcal{W}_5'}{49\sigma_2\sigma_4^2} - \frac{736\alpha\beta\eta\sigma_1\mathcal{W}_3\mathcal{L}'}{49\sigma_4} - \frac{1772\alpha\beta\sigma_1\mathcal{W}_3\mathcal{L}\eta'}{49\sigma_4} \\
& - \frac{912\alpha\beta\eta\sigma_1\mathcal{L}\mathcal{W}_3'}{49\sigma_4} \quad (C.33)
\end{aligned}$$

$$\begin{aligned}
f_{-3} = & \frac{f^{(6)}}{720} + \frac{5}{72}\alpha f^{(4)}\mathcal{L} + \frac{1}{8}\alpha f^{(3)}\mathcal{L}' + \frac{40\gamma\sigma_1\mathcal{W}_4f''}{7\sigma_2} + \frac{34}{45}\alpha^2\mathcal{L}^2f'' \\
& + \frac{13}{120}\alpha f''\mathcal{L}'' + \frac{32\gamma\sigma_1f'\mathcal{W}_4'}{7\sigma_2} + \frac{17}{360}\alpha\mathcal{L}^{(3)}f' + \frac{241}{180}\alpha^2\mathcal{L}f'\mathcal{L}' \\
& + \frac{200}{7}\beta^2 f\sigma_1\mathcal{W}_3^2 + \frac{8\gamma f\sigma_1\mathcal{W}_4''}{7\sigma_2} + \frac{1}{120}\alpha f\mathcal{L}^{(4)} + \alpha^3 f\mathcal{L}^3 \\
& + \frac{23}{60}\alpha^2 f\mathcal{L}\mathcal{L}'' + \frac{3}{10}\alpha^2 f(\mathcal{L}')^2 + \frac{176\alpha\gamma f\sigma_1\mathcal{W}_4\mathcal{L}}{7\sigma_2} \\
& + \frac{3840\beta\gamma\eta\sigma_1^2\mathcal{W}_3\mathcal{W}_4}{7\sigma_2\sigma_4} + \frac{41\beta\eta^{(4)}\sigma_1\mathcal{W}_3}{98\sigma_4} + \frac{99\beta\eta^{(3)}\sigma_1\mathcal{W}_3'}{98\sigma_4} \\
& + \frac{101\beta\sigma_1\eta''\mathcal{W}_3''}{98\sigma_4} + \frac{\beta\sigma_1\mathcal{W}_3^{(3)}\eta'}{2\sigma_4} + \frac{2\beta\eta\sigma_1\mathcal{W}_3^{(4)}}{21\sigma_4}
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{10}\beta\sigma_2\mathcal{W}_3\chi'' + \frac{1}{6}\beta\sigma_2\chi'\mathcal{W}_3' + \frac{1}{30}\beta\sigma_2\chi\mathcal{W}_3'' + \gamma\mathcal{W}_4\epsilon \\
& - \frac{16000\delta\sigma_1^2\mathcal{W}_5\eta''}{343\sigma_2\sigma_4^2} - \frac{320\delta\sigma_1^2\eta'\mathcal{W}_5'}{7\sigma_2\sigma_4^2} - \frac{640\delta\eta\sigma_1^2\mathcal{W}_5''}{49\sigma_2\sigma_4^2} \\
& + \frac{80\delta\sigma_1\chi\mathcal{W}_5}{7\sigma_4} + \frac{1200\alpha^2\beta\eta\sigma_1\mathcal{W}_3\mathcal{L}^2}{49\sigma_4} + \frac{136\alpha\beta\eta\sigma_1\mathcal{W}_3\mathcal{L}''}{49\sigma_4} \\
& + \frac{468\alpha\beta\sigma_1\mathcal{W}_3\eta'\mathcal{L}'}{49\sigma_4} + \frac{824\alpha\beta\eta\sigma_1\mathcal{W}_3'\mathcal{L}'}{147\sigma_4} + \frac{517\alpha\beta\sigma_1\mathcal{W}_3\mathcal{L}\eta''}{49\sigma_4} \\
& + \frac{599\alpha\beta\sigma_1\mathcal{L}\eta'\mathcal{W}_3'}{49\sigma_4} + \frac{596\alpha\beta\eta\sigma_1\mathcal{L}\mathcal{W}_3''}{147\sigma_4} + \alpha\beta\sigma_2\chi\mathcal{W}_3\mathcal{L} \\
& - \frac{83200\alpha\delta\eta\sigma_1^2\mathcal{W}_5\mathcal{L}}{343\sigma_2\sigma_4^2}
\end{aligned} \tag{C.34}$$

$$\eta_3 = -\eta' \tag{C.35}$$

$$\eta_2 = \frac{\eta''}{2} + 4\alpha\eta\mathcal{L} \tag{C.36}$$

$$\eta_1 = -\frac{\eta^{(3)}}{6} - \frac{4}{3}\alpha\eta\mathcal{L}' - \frac{11}{3}\alpha\mathcal{L}\eta' \tag{C.37}$$

$$\begin{aligned}
\eta_0 = & 3\beta f\sigma_4\mathcal{W}_3 + \frac{\eta^{(4)}}{24} - \frac{480\gamma\eta\sigma_1\mathcal{W}_4}{7\sigma_2} + 6\alpha^2\eta\mathcal{L}^2 \\
& + \frac{1}{3}\alpha\eta\mathcal{L}'' + \frac{5}{4}\alpha\eta'\mathcal{L}' + \frac{5}{3}\alpha\mathcal{L}\eta''
\end{aligned} \tag{C.38}$$

$$\begin{aligned}
\eta_{-1} = & -\frac{13}{5}\beta\sigma_4\mathcal{W}_3f' - \frac{3}{5}\beta f\sigma_4\mathcal{W}_3' - \frac{\eta^{(5)}}{120} + \frac{48\gamma\sigma_1\mathcal{W}_4\eta'}{\sigma_2} \\
& + \frac{96\gamma\eta\sigma_1\mathcal{W}_4'}{7\sigma_2} - \frac{1}{15}\alpha\eta\mathcal{L}^{(3)} - \frac{73}{15}\alpha^2\mathcal{L}^2\eta' - \frac{19}{60}\alpha\eta'\mathcal{L}'' \\
& - \frac{56}{15}\alpha^2\eta\mathcal{L}\mathcal{L}' - \frac{7}{12}\alpha\eta''\mathcal{L}' - \frac{1}{2}\alpha\eta^{(3)}\mathcal{L}
\end{aligned} \tag{C.39}$$

$$\begin{aligned}
\eta_{-2} = & \frac{11}{10}\beta\sigma_4\mathcal{W}_3f'' + \frac{8}{15}\beta\sigma_4f'\mathcal{W}_3' + \frac{1}{10}\beta f\sigma_4\mathcal{W}_3'' - \frac{320\delta f\sigma_1\mathcal{W}_5}{7\sigma_2} \\
& + 6\alpha\beta f\sigma_4\mathcal{W}_3\mathcal{L} + \frac{\eta^{(6)}}{720} + \frac{320}{7}\beta^2\eta\sigma_1\mathcal{W}_3^2 - \frac{792\gamma\sigma_1\mathcal{W}_4\eta''}{49\sigma_2} \\
& - \frac{72\gamma\sigma_1\eta'\mathcal{W}_4'}{7\sigma_2} - \frac{16\gamma\eta\sigma_1\mathcal{W}_4''}{7\sigma_2} + 2\gamma\sigma_4\chi\mathcal{W}_4 + \frac{1}{90}\alpha\eta\mathcal{L}^{(4)} \\
& + 4\alpha^3\eta\mathcal{L}^3 + \frac{23}{360}\alpha\mathcal{L}^{(3)}\eta' + \frac{173}{90}\alpha^2\mathcal{L}^2\eta'' + \frac{38}{45}\alpha^2\eta\mathcal{L}\mathcal{L}'' \\
& + \frac{3}{20}\alpha\eta''\mathcal{L}'' + \frac{277}{90}\alpha^2\mathcal{L}\eta'\mathcal{L}' + \frac{28}{45}\alpha^2\eta(\mathcal{L}')^2 + \frac{13}{72}\alpha\eta^{(3)}\mathcal{L}' \\
& + \frac{1}{9}\alpha\eta^{(4)}\mathcal{L} - \frac{5440\alpha\gamma\eta\sigma_1\mathcal{W}_4\mathcal{L}}{49\sigma_2}
\end{aligned} \tag{C.40}$$

$$\begin{aligned}
\eta_{-3} = & -\frac{3}{10}\beta f^{(3)}\sigma_4\mathcal{W}_3 - \frac{7}{30}\beta\sigma_4f''\mathcal{W}_3' - \frac{19}{210}\beta\sigma_4f'\mathcal{W}_3'' + \frac{1280\delta\sigma_1\mathcal{W}_5f'}{49\sigma_2} \\
& - \frac{149}{35}\alpha\beta\sigma_4\mathcal{W}_3\mathcal{L}f' - \frac{1}{70}\beta f\sigma_4\mathcal{W}_3^{(3)} + \frac{320\delta f\sigma_1\mathcal{W}_5'}{49\sigma_2} - \frac{12}{7}\alpha\beta f\sigma_4\mathcal{W}_3\mathcal{L}' \\
& - \frac{39}{35}\alpha\beta f\sigma_4\mathcal{L}\mathcal{W}_3' - \frac{\eta^{(7)}}{5040} - \frac{200}{7}\beta^2\sigma_1\mathcal{W}_3^2\eta' - \frac{160}{7}\beta^2\eta\sigma_1\mathcal{W}_3\mathcal{W}_3'
\end{aligned}$$

$$\begin{aligned}
& + \frac{1192\gamma\eta^{(3)}\sigma_1\mathcal{W}_4}{343\sigma_2} + \frac{1296\gamma\sigma_1\eta''\mathcal{W}_4'}{343\sigma_2} + \frac{88\gamma\sigma_1\eta'\mathcal{W}_4''}{49\sigma_2} \\
& + \frac{16\gamma\eta\sigma_1\mathcal{W}_4^{(3)}}{49\sigma_2} - \frac{11}{7}\gamma\sigma_4\mathcal{W}_4\chi' - \frac{2}{7}\gamma\sigma_4\chi\mathcal{W}_4' - \frac{1}{630}\alpha\eta\mathcal{L}^{(5)} \\
& - \frac{3}{280}\alpha\mathcal{L}^{(4)}\eta' - \frac{93}{35}\alpha^3\mathcal{L}^3\eta' - \frac{47}{315}\alpha^2\eta\mathcal{L}\mathcal{L}^{(3)} - \frac{11}{360}\alpha\mathcal{L}^{(3)}\eta'' \\
& - \frac{22}{45}\alpha^2\eta^{(3)}\mathcal{L}^2 - \frac{877\alpha^2\mathcal{L}\eta'\mathcal{L}''}{1260} - \frac{17}{360}\alpha\eta^{(3)}\mathcal{L}'' - \frac{223}{180}\alpha^2\mathcal{L}\eta''\mathcal{L}' \\
& - \frac{37}{70}\alpha^2\eta'(\mathcal{L}')^2 - \frac{1}{24}\alpha\eta^{(4)}\mathcal{L}' + \frac{8640\alpha\gamma\eta\sigma_1\mathcal{W}_4\mathcal{L}'}{343\sigma_2} - \frac{116}{35}\alpha^3\eta\mathcal{L}^2\mathcal{L}' \\
& - \frac{94}{315}\alpha^2\eta\mathcal{L}'\mathcal{L}'' - \frac{7}{360}\alpha\eta^{(5)}\mathcal{L} + \frac{21296\alpha\gamma\sigma_1\mathcal{W}_4\mathcal{L}\eta'}{343\sigma_2} \\
& + \frac{7456\alpha\gamma\eta\sigma_1\mathcal{L}\mathcal{W}_4'}{343\sigma_2}
\end{aligned} \tag{C.41}$$

$$\begin{aligned}
\eta_{-4} = & \mathcal{L}^4\eta\alpha^4 + \frac{62}{63}\mathcal{L}\eta(\mathcal{L}')^2\alpha^3 + \frac{2747\mathcal{L}^2\mathcal{L}'\eta'\alpha^3}{1260} \\
& + \frac{197}{315}\mathcal{L}^2\eta\mathcal{L}''\alpha^3 + \frac{256}{315}\mathcal{L}^3\eta''\alpha^3 + \frac{47\eta(\mathcal{L}'')^2\alpha^2}{1260} \\
& + 3f\mathcal{L}^2\beta\mathcal{W}_3\sigma_4\alpha^2 + \frac{517\mathcal{L}'\eta'\mathcal{L}''\alpha^2}{2016} + \frac{2227(\mathcal{L}')^2\eta''\alpha^2}{10080} \\
& + \frac{88}{315}\mathcal{L}\mathcal{L}''\eta''\alpha^2 + \frac{47}{840}\eta\mathcal{L}'\mathcal{L}^{(3)}\alpha^2 + \frac{613\mathcal{L}\eta'\mathcal{L}^{(3)}\alpha^2}{5040} \\
& + \frac{29}{90}\mathcal{L}\mathcal{L}'\eta^{(3)}\alpha^2 + \frac{3}{140}\mathcal{L}\eta\mathcal{L}^{(4)}\alpha^2 + \frac{4}{45}\mathcal{L}^2\eta^{(4)}\alpha^2 \\
& - \frac{2080\mathcal{L}^2\gamma\eta\mathcal{W}_4\sigma_1\alpha^2}{49\sigma_2} + \frac{2000}{49}\mathcal{L}\beta^2\eta\mathcal{W}_3^2\sigma_1\alpha + 2\mathcal{L}\gamma\chi\mathcal{W}_4\sigma_4\alpha \\
& + \frac{253}{210}\beta\mathcal{W}_3\sigma_4f'\mathcal{L}'\alpha - \frac{4792\gamma\mathcal{W}_4\sigma_1\mathcal{L}'\eta'\alpha}{343\sigma_2} + \frac{169}{210}\mathcal{L}\beta\sigma_4f'\mathcal{W}_3'\alpha \\
& + \frac{99}{280}f\beta\sigma_4\mathcal{L}'\mathcal{W}_3'\alpha - \frac{2012\gamma\eta\sigma_1\mathcal{L}'\mathcal{W}_4'\alpha}{343\sigma_2} - \frac{4476\mathcal{L}\gamma\sigma_1\eta'\mathcal{W}_4'\alpha}{343\sigma_2} \\
& + \frac{146}{105}\mathcal{L}\beta\mathcal{W}_3\sigma_4f''\alpha + \frac{19}{56}f\beta\mathcal{W}_3\sigma_4\mathcal{L}''\alpha - \frac{1360\gamma\eta\mathcal{W}_4\sigma_1\mathcal{L}''\alpha}{343\sigma_2} \\
& - \frac{5448\mathcal{L}\gamma\mathcal{W}_4\sigma_1\eta''\alpha}{343\sigma_2} + \frac{23}{140}f\mathcal{L}\beta\sigma_4\mathcal{W}_3''\alpha - \frac{1128\mathcal{L}\gamma\eta\sigma_1\mathcal{W}_4''\alpha}{343\sigma_2} \\
& + \frac{7}{720}\mathcal{L}^{(3)}\eta^{(3)}\alpha + \frac{13\eta''\mathcal{L}^{(4)}\alpha}{2520} + \frac{1}{90}\mathcal{L}''\eta^{(4)}\alpha + \frac{31\eta'\mathcal{L}^{(5)}\alpha}{20160} \\
& + \frac{11\mathcal{L}'\eta^{(5)}\alpha}{1440} + \frac{\eta\mathcal{L}^{(6)}\alpha}{5040} + \frac{1}{360}\mathcal{L}\eta^{(6)}\alpha - \frac{32f\mathcal{L}\delta\mathcal{W}_5\sigma_1\alpha}{\sigma_2} \\
& + \frac{20}{7}\beta^2\eta\sigma_1(\mathcal{W}_3')^2 + \delta\epsilon\mathcal{W}_5 + \frac{1}{2}\beta^2\chi\mathcal{W}_3^2\sigma_2\sigma_4 \\
& + \frac{480f\beta\gamma\mathcal{W}_3\mathcal{W}_4\sigma_1\sigma_4}{7\sigma_2} + \frac{205}{14}\beta^2\mathcal{W}_3\sigma_1\eta'\mathcal{W}_3' + \frac{13}{56}\gamma\sigma_4\chi'\mathcal{W}_4' \\
& - \frac{200\delta\sigma_1f'\mathcal{W}_5'}{49\sigma_2} - \frac{328\delta\mathcal{W}_5\sigma_1f''}{49\sigma_2} + \frac{815}{98}\beta^2\mathcal{W}_3^2\sigma_1\eta'' \\
& + \frac{4}{7}\gamma\mathcal{W}_4\sigma_4\chi'' + \frac{30}{7}\beta^2\eta\mathcal{W}_3\sigma_1\mathcal{W}_3'' + \frac{17}{420}\beta\sigma_4f''\mathcal{W}_3'
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{28} \gamma \chi \sigma_4 \mathcal{W}_4'' - \frac{239 \gamma \sigma_1 \eta'' \mathcal{W}_4''}{343 \sigma_2} - \frac{40 f \delta \sigma_1 \mathcal{W}_5''}{49 \sigma_2} \\
& + \frac{1}{15} \beta \sigma_4 \mathcal{W}_3' f^{(3)} - \frac{311 \gamma \sigma_1 \mathcal{W}_4' \eta^{(3)}}{343 \sigma_2} + \frac{11}{840} \beta \sigma_4 f' \mathcal{W}_3^{(3)} \\
& - \frac{13 \gamma \sigma_1 \eta' \mathcal{W}_4^{(3)}}{49 \sigma_2} + \frac{7}{120} \beta \mathcal{W}_3 \sigma_4 f^{(4)} - \frac{184 \gamma \mathcal{W}_4 \sigma_1 \eta^{(4)}}{343 \sigma_2} \\
& + \frac{1}{560} f \beta \sigma_4 \mathcal{W}_3^{(4)} - \frac{2 \gamma \eta \sigma_1 \mathcal{W}_4^{(4)}}{49 \sigma_2} + \frac{\eta^{(8)}}{40320} \\
& - \frac{21120 \beta \delta \eta \mathcal{W}_3 \mathcal{W}_5 \sigma_1^2}{49 \sigma_2 \sigma_4}
\end{aligned} \tag{C.42}$$

which we will refer to as the twenty auxiliary equations.

C.2.2 Ward identities

In this appendix, we provide the detailed derivation of the spin-5 conformal Ward identities in AdS_3 , discussed in Sec. 2.2.2, expressed as:

$$\delta_\varepsilon \mathcal{W}_s = s \mathcal{W}_s \varepsilon' + \varepsilon \mathcal{W}_s' \quad \text{where,} \quad (s = 3, 4, 5) \tag{C.43}$$

This defines that each conformal field $\mathcal{W}_s$ carries conformal spin- s , with $s = 3, 4, 5$.

$$\begin{aligned}
\delta_\chi \mathcal{W}_3 = & \frac{c}{360} \chi^{(5)} + \frac{\chi'' \mathcal{L}'}{2} + \frac{1}{3} \chi^{(3)} \mathcal{L} + \chi' \left(\frac{32 \mathcal{L}^2}{5c} + \frac{32 \mathcal{W}_4}{\sqrt{105}} + \frac{3 \mathcal{L}''}{10} \right) \\
& + \chi \left(\frac{32 \mathcal{L} \mathcal{L}'}{5c} + \frac{16 \mathcal{W}_4'}{\sqrt{105}} + \frac{\mathcal{L}^{(3)}}{15} \right)
\end{aligned} \tag{C.44}$$

where χ is the gauge parameter related to the spin-3 charge in $\mathfrak{sl}(5, \mathbb{R})$, which defines spin-3 conformal Ward identity.

$$\begin{aligned}
\delta_\chi \mathcal{W}_4 = & + \frac{4 \chi^{(3)} \mathcal{W}_3}{\sqrt{105}} + \frac{4 \chi'' \mathcal{W}_3'}{\sqrt{105}} + \chi' \left(\frac{416}{7c} \sqrt{\frac{3}{35}} \mathcal{L} \mathcal{W}_3 + \frac{4}{7} \sqrt{\frac{3}{35}} \mathcal{W}_3'' \right. \\
& + \left. \frac{5 \sqrt{15} \mathcal{W}_5}{7} \right) + \chi \left(\frac{40 \sqrt{\frac{15}{7}} \mathcal{W}_3 \mathcal{L}'}{7c} + \frac{144 \sqrt{\frac{3}{35}} \mathcal{L} \mathcal{W}_3'}{7c} + \frac{2}{7} \sqrt{15} \mathcal{W}_5' \right. \\
& + \left. \frac{2 \mathcal{W}_3^{(3)}}{7 \sqrt{105}} \right)
\end{aligned} \tag{C.45}$$

$$\begin{aligned}
\delta_f \mathcal{W}_4 = & \frac{c f^{(7)}}{20160} + \frac{1}{60} f^{(5)} \mathcal{L} + \frac{1}{24} f^{(4)} \mathcal{L}' + f^{(3)} \left(\frac{7 \mathcal{L}^2}{5c} + \frac{1}{2} \sqrt{\frac{3}{35}} \mathcal{W}_4 + \frac{\mathcal{L}''}{20} \right) \\
& + f'' \left(\frac{21 \mathcal{L} \mathcal{L}'}{5c} + \frac{3}{4} \sqrt{\frac{3}{35}} \mathcal{W}_4' + \frac{\mathcal{L}^{(3)}}{30} \right) + f' \left(\frac{864 \mathcal{L}^3}{35c^2} + \frac{45 \mathcal{W}_3^2}{2c} + \frac{88 \mathcal{L} \mathcal{L}''}{35c} \right. \\
& + \left. \frac{59 (\mathcal{L}')^2}{28c} + \frac{4 \mathcal{W}_4 \sqrt{\frac{21}{5}} \mathcal{L}}{c} + \frac{1}{4} \sqrt{\frac{5}{21}} \mathcal{W}_4'' + \frac{\mathcal{L}^{(4)}}{84} \right) + f \left(\frac{1296 \mathcal{L}^2 \mathcal{L}'}{35c^2} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{45\mathcal{W}_3\mathcal{W}_3'}{2c} + \frac{39\mathcal{L}^{(3)}\mathcal{L}}{70c} + \frac{2\sqrt{\frac{21}{5}}\mathcal{W}_4\mathcal{L}'}{c} + \frac{177\mathcal{L}'\mathcal{L}''}{140c} + \frac{2\sqrt{\frac{21}{5}}\mathcal{L}\mathcal{W}_4'}{c} \\
& + \frac{\mathcal{W}_4^{(3)}}{4\sqrt{105}} + \frac{\mathcal{L}^{(5)}}{560} \Big) \tag{C.46}
\end{aligned}$$

where f is the gauge parameter related to the spin-4 charge in $\mathfrak{sl}(5, \mathbb{R})$, which defines spin-4 conformal Ward identity.

$$\begin{aligned}
\delta_\chi \mathcal{W}_5 = & \frac{2}{7}\sqrt{\frac{5}{3}}\chi^{(3)}\mathcal{W}_4 + \frac{1}{14}\sqrt{15}\chi''\mathcal{W}_4' + \chi' \left(\frac{24\mathcal{W}_3^2}{\sqrt{7}c} + \frac{32\sqrt{15}\mathcal{W}_4\mathcal{L}}{7c} \right. \\
& + \frac{1}{14}\sqrt{\frac{5}{3}}\mathcal{W}_4'' \Big) + \chi \left(\frac{16\mathcal{W}_3\mathcal{W}_3'}{\sqrt{7}c} + \frac{32\mathcal{W}_4\mathcal{L}'}{\sqrt{15}c} + \frac{128\mathcal{L}\mathcal{W}_4'}{7\sqrt{15}c} \right. \\
& + \frac{\mathcal{W}_4^{(3)}}{21\sqrt{15}} \Big) \tag{C.47}
\end{aligned}$$

$$\begin{aligned}
\delta_f \mathcal{W}_5 = & \frac{1}{56}\sqrt{\frac{3}{5}}f^{(5)}\mathcal{W}_3 + \frac{1}{56}\sqrt{\frac{5}{3}}f^{(4)}\mathcal{W}_3' + f^{(3)} \left(\frac{33\sqrt{15}\mathcal{W}_3\mathcal{L}}{49c} \right. \\
& + \frac{1}{196}\sqrt{15}\mathcal{W}_3'' - \frac{11}{28}\sqrt{\frac{5}{21}}\mathcal{W}_5 \Big) + f'' \left(\frac{48\sqrt{15}\mathcal{W}_3\mathcal{L}'}{49c} \right. \\
& + \frac{34\sqrt{15}\mathcal{L}\mathcal{W}_3'}{49c} - \frac{11}{14}\sqrt{\frac{3}{35}}\mathcal{W}_5' + \frac{1}{392}\sqrt{15}\mathcal{W}_3^{(3)} \Big) \\
& + f' \left(\frac{4896}{49c^2}\sqrt{\frac{3}{5}}\mathcal{W}_3\mathcal{L}^2 + \frac{74\mathcal{W}_3\mathcal{W}_4}{\sqrt{7}c} + \frac{563}{196c}\sqrt{\frac{3}{5}}\mathcal{W}_3\mathcal{L}'' \right. \\
& + \frac{397}{196c}\sqrt{\frac{5}{3}}\mathcal{W}_3'\mathcal{L}' + \frac{89}{98c}\sqrt{\frac{5}{3}}\mathcal{L}\mathcal{W}_3'' - \frac{58}{7c}\sqrt{\frac{15}{7}}\mathcal{W}_5\mathcal{L} \\
& - \frac{1}{14}\sqrt{\frac{15}{7}}\mathcal{W}_5'' + \frac{5\sqrt{\frac{5}{3}}\mathcal{W}_3^{(4)}}{2352} \Big) + f \left(\frac{1728}{49c^2}\sqrt{\frac{3}{5}}\mathcal{L}^2\mathcal{W}_3' \right. \\
& + \mathcal{W}_3\frac{4752}{49c^2}\sqrt{\frac{3}{5}}\mathcal{L}\mathcal{L}' + \frac{6\sqrt{7}\mathcal{W}_4\mathcal{W}_3'}{c} + \frac{24\mathcal{W}_3\mathcal{W}_4'}{\sqrt{7}c} \\
& + \frac{123}{196c}\sqrt{\frac{3}{5}}\mathcal{W}_3\mathcal{L}^{(3)} + \frac{97\sqrt{\frac{3}{5}}\mathcal{W}_3'\mathcal{L}''}{98c} + \frac{29\sqrt{15}\mathcal{W}_3''\mathcal{L}'}{196c} \\
& - \frac{4}{c}\sqrt{\frac{15}{7}}\mathcal{W}_5\mathcal{L}' - \frac{118}{7c}\sqrt{\frac{3}{35}}\mathcal{L}\mathcal{W}_5' + \frac{5\sqrt{15}\mathcal{W}_3^{(3)}\mathcal{L}}{98c} \\
& - \frac{1}{28}\sqrt{\frac{5}{21}}\mathcal{W}_5^{(3)} + \frac{\mathcal{W}_3^{(5)}}{784\sqrt{15}} \Big) \tag{C.48}
\end{aligned}$$

$$\begin{aligned}
\delta_\eta \mathcal{W}_5 = & \frac{c\eta^{(9)}}{1814400} + \frac{\eta^{(7)}\mathcal{L}}{2520} + \frac{1}{720}\eta^{(6)}\mathcal{L}' + \eta^{(5)}\left(\frac{13\mathcal{L}^2}{150c}\right. \\
& - \frac{11\mathcal{W}_4}{140\sqrt{105}} + \frac{\mathcal{L}''}{400}\Big) + \eta^{(4)}\left(\frac{13\mathcal{L}\mathcal{L}'}{30c} - \frac{11\mathcal{W}_4'}{56\sqrt{105}}\right. \\
& + \frac{\mathcal{L}^{(3)}}{360}\Big) + \eta^{(3)}\left(\frac{656\mathcal{L}^3}{105c^2} + \frac{181\mathcal{W}_3^2}{84c} + \frac{109\mathcal{L}\mathcal{L}''}{210c}\right. \\
& + \frac{73(\mathcal{L}')^2}{168c} - \frac{374\mathcal{W}_4\mathcal{L}}{21\sqrt{105}c} - \frac{11}{252}\sqrt{\frac{5}{21}}\mathcal{W}_4'' + \frac{\mathcal{L}^{(4)}}{504}\Big) \\
& + \eta''\left(\frac{984\mathcal{L}^2\mathcal{L}'}{35c^2} + \frac{181\mathcal{W}_3\mathcal{W}_3'}{28c} + \frac{29\mathcal{L}^{(3)}\mathcal{L}}{84c} - \frac{187\mathcal{W}_4\mathcal{L}'}{7\sqrt{105}c}\right. \\
& + \frac{219\mathcal{L}'\mathcal{L}''}{280c} - \frac{187\mathcal{L}\mathcal{W}_4'}{7\sqrt{105}c} - \frac{11\mathcal{W}_4^{(3)}}{84\sqrt{105}} + \frac{\mathcal{L}^{(5)}}{1120}\Big) \\
& + \eta\left(\frac{36864\mathcal{L}^3\mathcal{L}'}{175c^3} - \frac{11008\mathcal{L}^2\mathcal{W}_4'}{35\sqrt{105}c^2} - \frac{22016\mathcal{W}_4\mathcal{L}\mathcal{L}'}{35\sqrt{105}c^2}\right. \\
& + \frac{496\mathcal{W}_3^2\mathcal{L}'}{7c^2} + \frac{496(\mathcal{L}')^3}{105c^2} + \frac{1952\mathcal{L}^{(3)}\mathcal{L}^2}{525c^2} \\
& + \frac{8864\mathcal{L}\mathcal{L}'\mathcal{L}''}{525c^2} + \frac{992\mathcal{W}_3\mathcal{L}\mathcal{W}_3'}{7c^2} + \frac{5\mathcal{W}_3'\mathcal{W}_3''}{3c} - \frac{30\mathcal{W}_5\mathcal{W}_3'}{\sqrt{7}c} \\
& + \frac{448\mathcal{W}_4\mathcal{W}_4'}{15c} - \frac{30\mathcal{W}_3\mathcal{W}_5'}{\sqrt{7}c} + \frac{19\mathcal{W}_3\mathcal{W}_3^{(3)}}{21c} + \frac{29\mathcal{L}^{(5)}\mathcal{L}}{1575c} \\
& - \frac{368\mathcal{W}_4\mathcal{L}^{(3)}}{105\sqrt{105}c} - \frac{281\mathcal{W}_4'\mathcal{L}''}{35\sqrt{105}c} - \frac{157\mathcal{W}_4''\mathcal{L}'}{21\sqrt{105}c} + \frac{13\mathcal{L}^{(4)}\mathcal{L}'}{210c} \\
& + \frac{47\mathcal{L}^{(3)}\mathcal{L}''}{450c} - \frac{62\mathcal{W}_4^{(3)}\mathcal{L}}{21\sqrt{105}c} - \frac{\mathcal{W}_4^{(5)}}{180\sqrt{105}} + \frac{\mathcal{L}^{(7)}}{37800}\Big) \tag{C.49}
\end{aligned}$$

where η is the gauge parameter related to the spin-5 charge in $\mathfrak{sl}(5, \mathbb{R})$, which defines spin-5 conformal Ward identities.

C.3 Classical $\mathcal{W}_5$ Symmetry OPE's from the Conformal Ward Identities

In this part we will give in detail obtained the classical $\mathcal{W}_5$ symmetry OPEs from the conformal Ward identities

$$\mathcal{L}(z_1)\mathcal{L}(z_2) \sim \frac{\frac{c}{2}}{z_{12}^4} + \frac{2\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}} \tag{C.50}$$

$$\mathcal{L}(z_1)\mathcal{W}_s(z_2) \sim \frac{s\mathcal{W}_s}{z_{12}^2} + \frac{\mathcal{W}_s'}{z_{12}}, \quad (s = 3, 4, 5) \tag{C.51}$$

$$\begin{aligned}\mathcal{W}_3(z_1)\mathcal{W}_3(z_2) &\sim \frac{\frac{c}{3}}{z_{12}^6} + \frac{2\mathcal{L}}{z_{12}^4} + \frac{\mathcal{L}'}{z_{12}^3} + \frac{1}{z_{12}^2} \left(\frac{32\mathcal{L}^2}{5c} + \frac{32\mathcal{W}_4}{\sqrt{105}} + \frac{3\mathcal{L}''}{10} \right) \\ &+ \frac{1}{z_{12}} \left(\frac{32\mathcal{L}\mathcal{L}'}{5c} + \frac{16\mathcal{W}_4'}{\sqrt{105}} + \frac{\mathcal{L}^{(3)}}{15} \right)\end{aligned}\quad (\text{C.52})$$

$$\begin{aligned}\mathcal{W}_3(z_1)\mathcal{W}_4(z_2) &\sim \frac{8}{\sqrt{105}} \left(\frac{3\mathcal{W}_3}{z_{12}^4} + \frac{\mathcal{W}_3'}{z_{12}^3} \right) + \frac{1}{z_{12}^2} \left(\frac{416}{7c} \sqrt{\frac{3}{35}} \mathcal{L}\mathcal{W}_3 \right. \\ &+ \frac{4}{7} \sqrt{\frac{3}{35}} \mathcal{W}_3'' + \frac{5}{7} \sqrt{15} \mathcal{W}_5 \left. \right) + \frac{1}{z_{12}} \left(\frac{40}{7c} \sqrt{\frac{15}{7}} \mathcal{W}_3 \mathcal{L}' \right. \\ &+ \frac{144}{7c} \sqrt{\frac{3}{35}} \mathcal{L}\mathcal{W}_3' + \frac{2}{7} \sqrt{15} \mathcal{W}_5' + \frac{2\mathcal{W}_3^{(3)}}{7\sqrt{105}} \left. \right)\end{aligned}\quad (\text{C.53})$$

$$\begin{aligned}\mathcal{W}_4(z_1)\mathcal{W}_4(z_2) &\sim \frac{\frac{c}{4}}{z_{12}^8} + \frac{2\mathcal{L}}{z_{12}^6} + \frac{\mathcal{L}'}{z_{12}^5} + \frac{1}{z_{12}^4} \left(3\sqrt{\frac{3}{35}} \mathcal{W}_4 + \frac{42}{5c} \mathcal{L}^2 \right. \\ &+ \frac{3}{10} \mathcal{L}'' \left. \right) + \frac{1}{z_{12}^3} \left(\frac{3}{2} \sqrt{\frac{3}{35}} \mathcal{W}_4' + \frac{42}{5c} \mathcal{L}\mathcal{L}' + \frac{\mathcal{L}^{(3)}}{15} \right) \\ &+ \frac{1}{z_{12}^2} \left(\frac{864}{35c^2} \mathcal{L}^3 + \frac{59}{28c} (\mathcal{L}')^2 + \frac{88}{35c} \mathcal{L}\mathcal{L}'' + \frac{45}{2c} \mathcal{W}_3^2 \right. \\ &+ \frac{4}{c} \sqrt{\frac{21}{5}} \mathcal{L}\mathcal{W}_4 + \frac{1}{4} \sqrt{\frac{5}{21}} \mathcal{W}_4'' + \frac{\mathcal{L}^{(4)}}{84} \left. \right) + \frac{1}{z_{12}} \left(\frac{1296}{35c^2} \mathcal{L}^3 \mathcal{L}' \right. \\ &+ \frac{177}{140c} \mathcal{L}' \mathcal{L}'' + \frac{45}{2c} \mathcal{W}_3 \mathcal{W}_3' + \frac{2}{c} \sqrt{\frac{21}{5}} \mathcal{L}\mathcal{W}_4' + \frac{2}{c} \sqrt{\frac{21}{5}} \mathcal{W}_4 \mathcal{L}' \\ &+ \frac{39}{70c} \mathcal{L}\mathcal{L}^{(3)} + \frac{\mathcal{W}_4^{(3)}}{4\sqrt{105}} + \frac{\mathcal{L}^{(5)}}{560} \left. \right)\end{aligned}\quad (\text{C.54})$$

$$\begin{aligned}\mathcal{W}_3(z_1)\mathcal{W}_5(z_2) &\sim \frac{\sqrt{15}}{7} \left(\frac{4\mathcal{W}_4}{z_{12}^4} + \frac{\mathcal{W}_4'}{z_{12}^3} \right) + \frac{1}{z_{12}^2} \left(\frac{24\mathcal{W}_3^2}{\sqrt{7}c} + \frac{32\sqrt{15}\mathcal{L}\mathcal{W}_4}{7c} \right. \\ &+ \frac{1}{14} \sqrt{\frac{5}{3}} \mathcal{W}_4'' \left. \right) + \frac{1}{z_{12}} \left(\frac{32\mathcal{W}_4 \mathcal{L}'}{\sqrt{15}c} + \frac{16\mathcal{W}_3 \mathcal{W}_3'}{\sqrt{7}c} + \frac{128\mathcal{L}\mathcal{W}_4'}{7\sqrt{15}c} \right. \\ &+ \frac{\mathcal{W}_4^{(3)}}{21\sqrt{15}} \left. \right)\end{aligned}\quad (\text{C.55})$$

$$\begin{aligned}\mathcal{W}_4(z_1)\mathcal{W}_5(z_2) &\sim \frac{\sqrt{15}}{7} \left(+ \frac{3\mathcal{W}_3}{z_{12}^6} + \frac{\mathcal{W}_3'}{z_{12}^5} \right) + \frac{1}{z_{12}^4} \left(\frac{198\sqrt{15}\mathcal{L}\mathcal{W}_3}{49c} \right. \\ &+ \frac{3}{98} \sqrt{15} \mathcal{W}_3'' - \frac{11}{14} \sqrt{\frac{15}{7}} \mathcal{W}_5 \left. \right) + \frac{1}{z_{12}^3} \left(\frac{96\sqrt{15}\mathcal{W}_3 \mathcal{L}'}{49c} \right. \\ &+ \frac{68\sqrt{15}\mathcal{L}\mathcal{W}_3'}{49c} - \frac{11}{7} \sqrt{\frac{3}{35}} \mathcal{W}_5' + \frac{1}{196} \sqrt{15} \mathcal{W}_3^{(3)} \left. \right) \\ &+ \frac{1}{z_{12}^2} \left(\frac{4896\sqrt{\frac{3}{5}} \mathcal{W}_3 \mathcal{L}^2}{49c^2} + \frac{397\sqrt{\frac{5}{3}} \mathcal{L}' \mathcal{W}_3'}{196c} + \frac{89\sqrt{\frac{5}{3}} \mathcal{L}\mathcal{W}_3''}{98c} \right.\end{aligned}$$

$$\begin{aligned}
& + \frac{563\sqrt{\frac{3}{5}}\mathcal{W}_3\mathcal{L}''}{196c} - \frac{58\sqrt{\frac{15}{7}}\mathcal{W}_5\mathcal{L}}{7c} + \frac{74\mathcal{W}_3\mathcal{W}_4}{\sqrt{7}c} - \frac{1}{14}\sqrt{\frac{15}{7}}\mathcal{W}_5'' \\
& + \frac{5\sqrt{\frac{5}{3}}\mathcal{W}_3^{(4)}}{2352} \Big) + \frac{1}{z_{12}} \Big(\frac{1728\sqrt{\frac{3}{5}}\mathcal{L}^2\mathcal{W}_3'}{49c^2} + \frac{4752\sqrt{\frac{3}{5}}\mathcal{W}_3\mathcal{L}\mathcal{L}'}{49c^2} \\
& + \frac{97\sqrt{\frac{3}{5}}\mathcal{W}_3'\mathcal{L}''}{98c} + \frac{29\sqrt{15}\mathcal{L}'\mathcal{W}_3''}{196c} - \frac{118\sqrt{\frac{3}{35}}\mathcal{L}\mathcal{W}_5'}{7c} - \frac{4\sqrt{\frac{15}{7}}\mathcal{W}_5\mathcal{L}'}{c} \\
& + \frac{6\sqrt{7}\mathcal{W}_4\mathcal{W}_3'}{c} + \frac{24\mathcal{W}_3\mathcal{W}_4'}{\sqrt{7}c} + \frac{5\sqrt{15}\mathcal{W}_3^{(3)}\mathcal{L}}{98c} + \frac{123\sqrt{\frac{3}{5}}\mathcal{W}_3\mathcal{L}^{(3)}}{196c} \\
& - \frac{1}{28}\sqrt{\frac{5}{21}}\mathcal{W}_5^{(3)} + \frac{\mathcal{W}_3^{(5)}}{784\sqrt{15}} \Big) \tag{C.56}
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_5(z_1)\mathcal{W}_5(z_2) \sim & \frac{\frac{c}{5}}{z_{12}^{10}} + \frac{2\mathcal{L}}{z_{12}^8} + \frac{\mathcal{L}'}{z_{12}^7} + \frac{1}{z_{12}^6} \Big(\frac{52\mathcal{L}^2}{5c} - \frac{22}{7}\sqrt{\frac{3}{35}}\mathcal{W}_4 + \frac{3\mathcal{L}''}{10} \Big) \\
& + \frac{1}{z_{12}^5} \Big(\frac{52\mathcal{L}\mathcal{L}'}{5c} - \frac{11}{7}\sqrt{\frac{3}{35}}\mathcal{W}_4' + \frac{\mathcal{L}^{(3)}}{15} \Big) \\
& + \frac{1}{z_{12}^4} \Big(\frac{1312\mathcal{L}^3}{35c^2} + \frac{73(\mathcal{L}')^2}{28c} + \frac{109\mathcal{L}\mathcal{L}''}{35c} - \frac{748\mathcal{W}_4\mathcal{L}}{7\sqrt{105}c} \\
& + \frac{181\mathcal{W}_3^2}{14c} - \frac{11}{42}\sqrt{\frac{5}{21}}\mathcal{W}_4'' + \frac{\mathcal{L}^{(4)}}{84} \Big) + \frac{1}{z_{12}^3} \Big(\frac{1968\mathcal{L}^2\mathcal{L}'}{35c^2} \\
& + \frac{219\mathcal{L}'\mathcal{L}''}{140c} - \frac{374\mathcal{L}\mathcal{W}_4'}{7\sqrt{105}c} - \frac{374\mathcal{W}_4\mathcal{L}'}{7\sqrt{105}c} + \frac{181\mathcal{W}_3\mathcal{W}_3'}{14c} \\
& + \frac{29\mathcal{L}^{(3)}\mathcal{L}}{42c} - \frac{11\mathcal{W}_4^{(3)}}{42\sqrt{105}} + \frac{\mathcal{L}^{(5)}}{560} \Big) + \frac{1}{z_{12}^2} \Big(\frac{18432\mathcal{L}^4}{175c^3} \\
& + \frac{592\mathcal{L}(\mathcal{L}')^2}{21c^2} + \frac{8824\mathcal{L}^2\mathcal{L}''}{525c^2} - \frac{22016\mathcal{W}_4\mathcal{L}^2}{35\sqrt{105}c^2} + \frac{992\mathcal{W}_3^2\mathcal{L}}{7c^2} \\
& + \frac{35(\mathcal{W}_3')^2}{12c} - \frac{188\mathcal{L}'\mathcal{W}_4'}{7\sqrt{105}c} + \frac{25\mathcal{L}^{(3)}\mathcal{L}'}{72c} - \frac{311\mathcal{L}\mathcal{W}_4''}{21\sqrt{105}c} \\
& + \frac{47(\mathcal{L}'')^2}{200c} - \frac{557\mathcal{W}_4\mathcal{L}''}{35\sqrt{105}c} + \frac{111\mathcal{W}_3\mathcal{W}_3''}{28c} + \frac{31\mathcal{L}^{(4)}\mathcal{L}}{252c} \\
& + \frac{448\mathcal{W}_4^2}{15c} - \frac{60\mathcal{W}_3\mathcal{W}_5}{\sqrt{7}c} - \frac{\mathcal{W}_4^{(4)}}{24\sqrt{105}} + \frac{\mathcal{L}^{(6)}}{4320} \Big) \\
& + \frac{1}{z_{12}} \Big(\frac{36864\mathcal{L}^3\mathcal{L}'}{175c^3} - \frac{11008\mathcal{L}^2\mathcal{W}_4'}{35\sqrt{105}c^2} - \frac{22016\mathcal{W}_4\mathcal{L}\mathcal{L}'}{35\sqrt{105}c^2} \\
& + \frac{496\mathcal{W}_3^2\mathcal{L}'}{7c^2} + \frac{496(\mathcal{L}')^3}{105c^2} + \frac{1952\mathcal{L}^{(3)}\mathcal{L}^2}{525c^2} + \frac{8864\mathcal{L}\mathcal{L}'\mathcal{L}''}{525c^2} \\
& + \frac{992\mathcal{W}_3\mathcal{L}\mathcal{W}_3'}{7c^2} + \frac{5\mathcal{W}_3'\mathcal{W}_3''}{3c} - \frac{30\mathcal{W}_5\mathcal{W}_3'}{\sqrt{7}c} + \frac{448\mathcal{W}_4\mathcal{W}_4'}{15c}
\end{aligned}$$

$$\begin{aligned}
& -\frac{30\mathcal{W}_3\mathcal{W}'_5}{\sqrt{7}c} + \frac{19\mathcal{W}_3\mathcal{W}_3^{(3)}}{21c} + \frac{29\mathcal{L}^{(5)}\mathcal{L}}{1575c} - \frac{368\mathcal{W}_4\mathcal{L}^{(3)}}{105\sqrt{105}c} \\
& -\frac{281\mathcal{W}'_4\mathcal{L}''}{35\sqrt{105}c} - \frac{157\mathcal{W}_4''\mathcal{L}'}{21\sqrt{105}c} + \frac{13\mathcal{L}^{(4)}\mathcal{L}'}{210c} + \frac{47\mathcal{L}^{(3)}\mathcal{L}''}{450c} \\
& -\frac{62\mathcal{W}_4^{(3)}\mathcal{L}}{21\sqrt{105}c} - \frac{\mathcal{W}_4^{(5)}}{180\sqrt{105}} + \frac{\mathcal{L}^{(7)}}{37800}
\end{aligned} \tag{C.57}$$

C.4 Chemical Potentials for $\mathfrak{sl}(5, \mathbb{R}) \oplus \mathfrak{sl}(5, \mathbb{R})$ Chern-Simons Theory

The final form of the spatial and temporal connection with the spin-5 chemical potential in Sec. 2.2.3 is given by

$$\begin{aligned}
a_\phi(t, \phi) &= L_1 + \frac{6}{c}\mathcal{L}L_{-1} + \frac{15}{c}\mathcal{W}_3W_{-2}^{(3)} + \frac{28}{c}\mathcal{W}_4W_{-3}^{(4)} + \frac{45}{c}\mathcal{W}_5W_{-4}^{(5)}, \\
a_t(t, \phi) &= \mu_2 L_1 + \mu_3 W_2^{(3)} + \mu_4 W_3^{(4)} + \mu_5 W_4^{(5)} \\
&+ \sum_{i=-1}^0 v_2^{(i)} L_i + \sum_{i=-2}^1 v_3^{(i)} W_i^{(3)} + \sum_{i=-3}^2 v_4^{(i)} W_i^{(4)} + \sum_{i=-4}^3 v_5^{(i)} W_i^{(5)} \\
&= a_t^{(\mu_2)} + a_t^{(\mu_3)} + a_t^{(\mu_4)} + a_t^{(\mu_5)}
\end{aligned} \tag{C.58}$$

where

$$\begin{aligned}
a_t^{(\mu_2)} &= \mu_2 \left(L_1 + \frac{6}{c}\mathcal{L}L_{-1} + \frac{15}{c}\mathcal{W}_3W_{-2}^{(3)} + \frac{28}{c}\mathcal{W}_4W_{-3}^{(4)} + \frac{45}{c}\mathcal{W}_5W_{-4}^{(5)} \right) \\
&- \mu_2' L_0 + \frac{1}{2}\mu_2'' L_{-1}
\end{aligned} \tag{C.59}$$

$$\begin{aligned}
a_t^{(\mu_3)} &= \mu_3 \left(W_2^{(3)} + \frac{12}{c}\mathcal{W}_3L_{-1} + \left(\frac{36}{c^2}\mathcal{L}^2 + \frac{16}{c}\sqrt{\frac{15}{7}}\mathcal{W}_4 + \frac{1}{c}\mathcal{L}'' \right) W_{-2}^{(3)} \right. \\
&+ \left(\frac{1800}{7\sqrt{7}c^2}\mathcal{W}_3^2 + \frac{48\sqrt{15}}{c^2}\mathcal{L}\mathcal{W}_4 + \frac{\sqrt{15}}{7c}\mathcal{W}_4'' \right) W_{-4}^{(5)} + \left(\frac{96}{c^2}\sqrt{\frac{15}{7}}\mathcal{L}\mathcal{W}_3 \right. \\
&+ \left. \frac{8}{\sqrt{105}c}\mathcal{W}_3'' + \frac{8\sqrt{15}}{c}\mathcal{W}_5 \right) W_{-3}^{(4)} - \frac{8\sqrt{15}}{7c}\mathcal{W}_4'W_{-3}^{(5)} - \frac{16}{c}\sqrt{\frac{3}{35}}\mathcal{W}_3'W_{-2}^{(4)} \\
&+ \frac{8\sqrt{15}}{c}\mathcal{W}_4W_{-2}^{(5)} + \frac{16}{c}\sqrt{\frac{15}{7}}\mathcal{W}_3W_{-1}^{(4)} - \frac{4}{c}\mathcal{L}'W_{-1}^{(3)} + \frac{12}{c}\mathcal{L}W_0^{(3)} \Big) \\
&+ \mu_3' \left(\frac{13\sqrt{15}}{14c}\mathcal{W}_4'W_{-4}^{(5)} + \frac{8}{c}\sqrt{\frac{5}{21}}\mathcal{W}_3'W_{-3}^{(4)} - \frac{44\sqrt{15}}{7c}\mathcal{W}_4W_{-3}^{(5)} \right. \\
&- \frac{64}{c}\sqrt{\frac{3}{35}}\mathcal{W}_3W_{-2}^{(4)} + \frac{7}{2c}\mathcal{L}'W_{-2}^{(3)} - \frac{10}{c}\mathcal{L}W_{-1}^{(3)} - W_1^{(3)} \Big) \\
&+ \mu_3'' \left(\frac{16\sqrt{15}}{7c}\mathcal{W}_4W_{-4}^{(5)} + \frac{24}{c}\sqrt{\frac{3}{35}}\mathcal{W}_3W_{-3}^{(4)} + \frac{4}{c}\mathcal{L}W_{-2}^{(3)} + \frac{1}{2}W_0^{(3)} \right) \\
&- \frac{\mu_3^{(3)}}{6}W_{-1}^{(3)} + \frac{\mu_3^{(4)}}{24}W_{-2}^{(3)}
\end{aligned} \tag{C.60}$$

$$\begin{aligned}
a_t^{(\mu_4)} = & \mu_4 \left(w_3^{(4)} + \frac{18}{c} w_4 L_{-1} + \left(\frac{108}{c^2} \mathcal{L}^2 + \frac{3}{2c} \mathcal{L}'' \right. \right. \\
& + \left. \frac{2\sqrt{105}}{c} w_4 \right) w_{-1}^{(4)} + \left(\frac{3\sqrt{15}}{784c} w_3^{(4)} + \frac{1620\sqrt{15}}{7c^3} \mathcal{L}^2 w_3 \right. \\
& + \frac{900}{\sqrt{7}c^2} w_3 w_4 + \frac{207\sqrt{15}}{98c^2} \mathcal{L} w_3'' + \frac{855\sqrt{15}}{196c^2} w_3 \mathcal{L}'' \\
& + \frac{891\sqrt{15}}{196c^2} \mathcal{L}' w_3' - \frac{18\sqrt{105}}{c^2} \mathcal{L} w_5 - \frac{15}{28c} \sqrt{\frac{15}{7}} w_5'' \Big) w_{-4}^{(5)} \\
& - \frac{15}{28c} \sqrt{\frac{15}{7}} w_5'' \Big) w_{-4}^{(5)} - \left(\frac{3}{10c} \mathcal{L}^{(3)} + \frac{324}{5c^2} \mathcal{L} \mathcal{L}' \right. \\
& + \frac{2}{c} \sqrt{\frac{21}{5}} w_4' \Big) w_{-2}^{(4)} + \left(-\frac{3\sqrt{15}}{98c} w_3^{(3)} - \frac{702\sqrt{15}}{49c^2} \mathcal{L} w_3' \right. \\
& - \frac{1080\sqrt{15}}{49c^2} w_3 \mathcal{L}' + \frac{30}{7c} \sqrt{\frac{15}{7}} w_5' \Big) w_{-3}^{(5)} + \left(\frac{3\sqrt{15}}{14c} w_3'' \right. \\
& + \frac{540\sqrt{15}}{7c^2} \mathcal{L} w_3 - \frac{30}{c} \sqrt{\frac{15}{7}} w_5 \Big) w_{-2}^{(5)} + \left(\frac{45\sqrt{15}}{7c} w_5 \right. \\
& + \frac{10}{7c} \sqrt{\frac{15}{7}} w_3'' + \frac{576}{7c^2} \sqrt{\frac{15}{7}} \mathcal{L} w_3 \Big) w_{-2}^{(3)} + \frac{45}{7c} \sqrt{15} w_3 w_0^{(5)} \\
& + \left(\frac{216}{c^3} \mathcal{L}^3 + \frac{54}{5c^2} \mathcal{L}'^2 + \frac{1500}{7c^2} w_3^2 + \frac{69}{5c^2} \mathcal{L} \mathcal{L}'' + \frac{1}{c} \sqrt{\frac{7}{15}} w_4'' \right. \\
& + \frac{44}{c^2} \sqrt{\frac{21}{5}} \mathcal{L} w_4 + \frac{1}{20c} \mathcal{L}^{(4)} \Big) w_{-3}^{(4)} - \frac{9}{7c} \sqrt{15} w_3' w_{-1}^{(5)} \\
& - \frac{6}{c} \mathcal{L}' w_0^{(4)} + \frac{18}{c} \mathcal{L} w_1^{(4)} - \frac{40}{7c} \sqrt{\frac{15}{7}} w_3' w_{-1}^{(3)} + \frac{120}{7c} \sqrt{\frac{15}{7}} w_3 w_0^{(3)} \Big) \\
& - \mu_4' \left(w_2^{(4)} + \frac{16}{c} \mathcal{L} w_0^{(4)} - \left(\frac{11\sqrt{15}}{392c} w_3^{(3)} + \frac{507\sqrt{15}}{49c^2} \mathcal{L} w_3' \right. \right. \\
& + \frac{759\sqrt{15}}{49c^2} w_3 \mathcal{L}' - \frac{75}{28c} \sqrt{\frac{15}{7}} w_5' \Big) w_{-4}^{(5)} + \left(\frac{17}{60c} \mathcal{L}^{(3)} \right. \\
& + \frac{241}{5c^2} \mathcal{L} \mathcal{L}' + \frac{4}{c} \sqrt{\frac{7}{15}} w_4' \Big) w_{-3}^{(4)} + \left(-\frac{19\sqrt{15}}{98c} w_3'' \right. \\
& - \frac{2682\sqrt{15}}{49c^2} \mathcal{L} w_3 + \frac{120}{7c} \sqrt{\frac{15}{7}} w_5 \Big) w_{-3}^{(5)} - \frac{80}{7c} \sqrt{\frac{15}{7}} w_3 w_{-1}^{(3)} \\
& - \frac{39\sqrt{15} w_3 w_{-1}^{(5)}}{7c} + \frac{11 \mathcal{L}' w_{-1}^{(4)}}{2c} - \left(\frac{396}{5c^2} \mathcal{L}^2 + \frac{6}{c} \sqrt{\frac{21}{5}} w_4 \right. \\
& + \frac{7}{5c} \mathcal{L}'' \Big) w_{-2}^{(4)} + \frac{8\sqrt{15}}{7c} w_3' w_{-2}^{(5)} + \frac{30}{7c} \sqrt{\frac{15}{7}} w_3' w_{-2}^{(3)} \Big) \\
& + \mu_4'' \left(\frac{1}{2} w_1^{(4)} + \frac{7}{5c} \mathcal{L}'' \right) w_{-2}^{(4)} + \left(\frac{136}{5c^2} \mathcal{L}^2 + \frac{13}{20c} \mathcal{L}'' \right. \\
& + \frac{1}{c} \sqrt{\frac{35}{3}} w_4 \Big) w_{-3}^{(4)} + \left(\frac{17\sqrt{15}}{196c} w_3'' + \frac{7}{5c} \mathcal{L}'' \right) w_{-2}^{(4)}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{17\sqrt{15}}{196c} \mathcal{W}_3'' + \frac{7}{5c} \mathcal{L}'' \right) \mathcal{W}_{-2}^{(4)} + \frac{876\sqrt{15}}{49c^2} \mathcal{L} \mathcal{W}_3 \\
& - \frac{123}{28c} \sqrt{\frac{15}{7}} \mathcal{W}_5 \mathcal{W}_{-4}^{(5)} - \frac{5}{2c} \mathcal{L}' \mathcal{W}_{-2}^{(4)} + \frac{7}{c} \mathcal{L} \mathcal{W}_{-1}^{(4)} \\
& - \frac{\sqrt{15}}{2c} \mathcal{W}_3' \mathcal{W}_{-3}^{(5)} + \frac{33}{14c} \sqrt{15} \mathcal{W}_3 \mathcal{W}_{-2}^{(5)} + \frac{26}{7c} \sqrt{\frac{15}{7}} \mathcal{W}_3 \mathcal{W}_{-2}^{(3)} \\
& + \mu_4^{(3)} \left(-\frac{1}{6} \mathcal{W}_0^{(4)} + \frac{3}{4c} \mathcal{L}' \mathcal{W}_{-3}^{(4)} - \frac{2}{c} \mathcal{L} \mathcal{W}_{-2}^{(4)} + \frac{\sqrt{15}}{7c} \mathcal{W}_3' \mathcal{W}_{-4}^{(5)} \right. \\
& \left. - \frac{9\sqrt{15}}{14c} \mathcal{W}_3 \mathcal{W}_{-3}^{(5)} \right) + \mu_4^{(4)} \left(\frac{1}{24} \mathcal{W}_{-1}^{(4)} + \frac{5}{12c} \mathcal{L} \mathcal{W}_{-3}^{(4)} \right. \\
& \left. + \frac{\sqrt{15}}{8c} \mathcal{W}_3 \mathcal{W}_{-4}^{(5)} \right) - \frac{\mu_4^{(5)}}{120} \mathcal{W}_{-2}^{(4)} + \frac{\mu_4^{(6)}}{720} \mathcal{W}_{-3}^{(4)} \tag{C.61}
\end{aligned}$$

$$\begin{aligned}
a_i^{(\mu_5)} = & \mu_5 \left(\mathcal{W}_4^{(5)} + \frac{24}{c} \mathcal{W}_5 \mathcal{L}_{-1} + \left(\frac{216}{c^2} \mathcal{L}^2 + \frac{2}{c} \mathcal{L}'' - \frac{4\sqrt{105}}{c} \mathcal{W}_4 \right) \mathcal{W}_0^{(5)} \right. \\
& + \left(\frac{1440}{7\sqrt{7}c^2} \mathcal{W}_3^2 + \frac{240\sqrt{15}}{7c^2} \mathcal{L} \mathcal{W}_4 + \frac{2}{c} \sqrt{\frac{5}{3}} \mathcal{W}_4'' \right) \mathcal{W}_{-2}^{(3)} \\
& + \left(-\frac{2}{5} \mathcal{L}^{(3)} - \frac{672}{5c^2} \mathcal{L} \mathcal{L}' + \frac{4}{c} \sqrt{\frac{21}{5}} \mathcal{W}_4' \right) \mathcal{W}_{-1}^{(5)} + \left(-\frac{2}{\sqrt{15}c} \mathcal{W}_3^{(3)} \right. \\
& + \frac{4}{c} \sqrt{\frac{21}{5}} \mathcal{W}_5' - \frac{736}{7c^2} \sqrt{\frac{3}{5}} \mathcal{W}_3 \mathcal{L}' - \frac{912}{7c^2} \sqrt{\frac{3}{5}} \mathcal{L} \mathcal{W}_3' \left. \right) \mathcal{W}_{-2}^{(4)} \\
& + \left(-\frac{1}{105c} \mathcal{L}^{(5)} + \frac{2}{\sqrt{105}c} \mathcal{W}_4^{(3)} - \frac{25056}{35c^3} \mathcal{L}^2 \mathcal{L}' - \frac{188}{35c^2} \mathcal{L} \mathcal{L}^{(3)} \right. \\
& - \frac{376}{35c^2} \mathcal{L}' \mathcal{L}'' - \frac{1200}{7c^2} \mathcal{W}_3 \mathcal{W}_3' + \frac{432}{7c^2} \sqrt{\frac{15}{7}} \mathcal{W}_4 \mathcal{L}' \\
& + \frac{1864}{7c^2} \sqrt{\frac{3}{35}} \mathcal{L} \mathcal{W}_4' \left. \right) \mathcal{W}_{-3}^{(5)} + \left(\frac{90000}{49c^3} \mathcal{L} \mathcal{W}_3^2 + \frac{1296}{c^4} \mathcal{L}^4 \right. \\
& + \frac{47}{35c^2} \mathcal{L}''^2 + \frac{150}{7c^2} \mathcal{W}_3'^2 + \frac{420}{c^2} \mathcal{W}_4^2 + \frac{1}{840c} \mathcal{L}^{(6)} - \frac{1}{4\sqrt{105}c} \mathcal{W}_4^{(4)} \\
& + \frac{4728}{35c^3} \mathcal{L}^2 \mathcal{L}'' + \frac{1488}{7c^3} \mathcal{L} \mathcal{L}'^2 + \frac{27}{35c^2} \mathcal{L} \mathcal{L}^{(4)} + \frac{141}{70c^2} \mathcal{L}' \mathcal{L}^{(3)} \\
& + \frac{225}{7c^2} \mathcal{W}_3 \mathcal{W}_3'' - \frac{990}{\sqrt{7}c^2} \mathcal{W}_3 \mathcal{W}_5 - \frac{624}{c^3} \sqrt{\frac{15}{7}} \mathcal{L}^2 \mathcal{W}_4 - \frac{68}{7c^2} \sqrt{\frac{15}{7}} \mathcal{W}_4 \mathcal{L}'' \\
& - \frac{282}{7c^2} \sqrt{\frac{3}{35}} \mathcal{L} \mathcal{W}_4'' - \frac{503}{7c^2} \sqrt{\frac{3}{35}} \mathcal{L}' \mathcal{W}_4' \left. \right) \mathcal{W}_{-4}^{(5)} + \left(\frac{1}{3\sqrt{15}c} \mathcal{W}_3^{(4)} \right. \\
& + \frac{1440\sqrt{15}}{7c^3} \mathcal{L}^2 \mathcal{W}_3 + \frac{596}{7\sqrt{15}c^2} \mathcal{L} \mathcal{W}_3'' + \frac{824}{7\sqrt{15}c^2} \mathcal{L}' \mathcal{W}_3' \\
& - \frac{2}{c} \sqrt{\frac{7}{15}} \mathcal{W}_5'' + \frac{136}{7c^2} \sqrt{\frac{3}{5}} \mathcal{W}_3 \mathcal{L}'' - \frac{104}{c^2} \sqrt{\frac{15}{7}} \mathcal{L} \mathcal{W}_5 \\
& \left. + \frac{112\sqrt{7}}{c^2} \mathcal{W}_3 \mathcal{W}_4 \right) \mathcal{W}_{-3}^{(4)} + \left(\frac{864}{c^3} \mathcal{L}^3 + \frac{112}{5c^2} \mathcal{L}'^2 + \frac{2400}{7c^2} \mathcal{W}_3^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{15c} \mathcal{L}^{(4)} - \frac{2}{c} \sqrt{\frac{7}{15}} \mathcal{W}_4'' - \frac{272}{c^2} \sqrt{\frac{15}{7}} \mathcal{L} \mathcal{W}_4 \\
& + \frac{152}{5c^2} \mathcal{L} \mathcal{L}'' \Big) \mathcal{W}_{-2}^{(5)} + \left(-\frac{4\sqrt{105}}{c} \mathcal{W}_5 + \frac{576\sqrt{15}}{7c^2} \mathcal{L} \mathcal{W}_3 \right. \\
& + \frac{2}{c} \sqrt{\frac{5}{3}} \mathcal{W}_3'' \Big) \mathcal{W}_{-1}^{(4)} - \frac{8}{c} \mathcal{L}' \mathcal{W}_1^{(5)} + \frac{24}{c} \mathcal{L} \mathcal{W}_2^{(5)} + \frac{8\sqrt{15}}{c} \mathcal{W}_3 \mathcal{W}_1^{(4)} \\
& + \frac{8\sqrt{15}}{c} \mathcal{W}_4 \mathcal{W}_0^{(3)} - \frac{8}{c} \sqrt{\frac{5}{3}} \mathcal{W}_3 \mathcal{W}_0^{(4)} - \frac{8}{c} \sqrt{\frac{5}{3}} \mathcal{W}_4 \mathcal{W}_{-1}^{(3)} \Big) \\
& + \mu_5' \left(-\mathcal{W}_3^{(5)} + \left(-\frac{876}{5c^2} \mathcal{L}^2 - \frac{19}{10c} \mathcal{L}'' + \frac{14}{c} \sqrt{\frac{21}{5}} \mathcal{W}_4 \right) \mathcal{W}_{-1}^{(5)} \right. \\
& + \left(\frac{7}{4\sqrt{15}c} \mathcal{W}_3^{(3)} - \frac{7}{c} \sqrt{\frac{7}{15}} \mathcal{W}_5' + \frac{468}{7c^2} \sqrt{\frac{3}{5}} \mathcal{W}_3 \mathcal{L}' \right. \\
& + \frac{599}{7c^2} \sqrt{\frac{3}{5}} \mathcal{L} \mathcal{W}_3' \Big) \mathcal{W}_{-3}^{(4)} + \left(\frac{31}{3360c} \mathcal{L}^{(5)} - \frac{13}{8\sqrt{105}c} \mathcal{W}_4^{(3)} \right. \\
& + \frac{16482}{35c^3} \mathcal{L}^2 \mathcal{L}' + \frac{613}{140c^2} \mathcal{L} \mathcal{L}^{(3)} + \frac{517}{56c^2} \mathcal{L}' \mathcal{L}'' \\
& + \frac{3075}{28c^2} \mathcal{W}_3 \mathcal{W}_3' - \frac{1119}{7c^2} \sqrt{\frac{3}{35}} \mathcal{L} \mathcal{W}_4' - \frac{1198}{7c^2} \sqrt{\frac{3}{35}} \mathcal{W}_4 \mathcal{L}' \Big) \mathcal{W}_{-4}^{(5)} \\
& + \left(-\frac{20088}{35c^3} \mathcal{L}^3 - \frac{666}{35c^2} \mathcal{L}^2 - \frac{1500}{7c^2} \mathcal{W}_3^2 - \frac{9}{140c} \mathcal{L}^{(4)} \right. \\
& + \frac{11}{\sqrt{105}c} \mathcal{W}_4'' - \frac{877}{35c^2} \mathcal{L} \mathcal{L}'' + \frac{5324}{7c^2} \sqrt{\frac{3}{35}} \mathcal{L} \mathcal{W}_4 \Big) \mathcal{W}_{-3}^{(5)} \\
& + \left(-\frac{17}{2\sqrt{15}c} \mathcal{W}_3'' + \frac{2\sqrt{105}}{c} \mathcal{W}_5 - \frac{1772}{7c^2} \sqrt{\frac{3}{5}} \mathcal{L} \mathcal{W}_3 \right) \mathcal{W}_{-2}^{(4)} \\
& - \frac{14}{c} \sqrt{\frac{5}{3}} \mathcal{W}_4 \mathcal{W}_{-1}^{(3)} + \frac{15\mathcal{L}'}{2c} \mathcal{W}_0^{(5)} + \left(\frac{23}{60c} \mathcal{L}^{(3)} \right. \\
& - \frac{3}{c} \sqrt{\frac{21}{5}} \mathcal{W}_4' + \frac{554}{5c^2} \mathcal{L} \mathcal{L}' \Big) \mathcal{W}_{-2}^{(5)} - \frac{22}{c} \mathcal{L} \mathcal{W}_1^{(5)} - \frac{6\sqrt{15}}{c} \mathcal{W}_3 \mathcal{W}_0^{(4)} \\
& + \frac{11}{2c} \sqrt{\frac{5}{3}} \mathcal{W}_4' \mathcal{W}_{-2}^{(3)} + \frac{13}{2c} \sqrt{\frac{5}{3}} \mathcal{W}_3' \mathcal{W}_{-1}^{(4)} \Big) + \mu_5'' \left(\frac{1}{2} \mathcal{W}_2^{(5)} + \left(-\frac{11}{60c} \mathcal{L}^{(3)} \right. \right. \\
& - \frac{223}{5c^2} \mathcal{L} \mathcal{L}' + \frac{54}{7c} \sqrt{\frac{3}{35}} \mathcal{W}_4' \Big) \mathcal{W}_{-3}^{(5)} + \left(\frac{346}{5c^2} \mathcal{L}^2 + \frac{9}{10c} \mathcal{L}'' \right. \\
& - \frac{33}{c} \sqrt{\frac{3}{35}} \mathcal{W}_4 \Big) \mathcal{W}_{-2}^{(5)} + \left(\frac{101}{28\sqrt{15}c} \mathcal{W}_3'' - \frac{10}{c} \sqrt{\frac{5}{21}} \mathcal{W}_5 \right. \\
& + \frac{517}{7c^2} \sqrt{\frac{3}{5}} \mathcal{L} \mathcal{W}_3 \Big) \mathcal{W}_{-3}^{(4)} + \left(\frac{6144}{35c^3} \mathcal{L}^3 + \frac{2227}{280c^2} \mathcal{L}^2 + \frac{12225}{196c^2} \mathcal{W}_3^2 \right. \\
& + \frac{13}{420c} \mathcal{L}^{(4)} - \frac{239}{56\sqrt{105}c} \mathcal{W}_4'' + \frac{352}{35c^2} \mathcal{L} \mathcal{L}'' \\
& - \frac{1362}{7c^2} \sqrt{\frac{3}{35}} \mathcal{L} \mathcal{W}_4 \Big) \mathcal{W}_{-4}^{(5)} - \frac{7\mathcal{L}'}{2c} \mathcal{W}_{-1}^{(5)} - \frac{92\mathcal{W}_3'}{7\sqrt{15}c} \mathcal{W}_{-2}^{(4)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{10}{c} \mathcal{L} W_0^{(5)} + \frac{31\sqrt{15}}{14c} \mathcal{W}_3 W_{-1}^{(4)} + \frac{29}{7c} \sqrt{\frac{5}{3}} \mathcal{W}_4 W_{-2}^{(3)} \\
& + \mu_5^{(3)} \left(-\frac{1}{6} W_1^{(5)} - \frac{3\mathcal{L}}{c} W_{-1}^{(5)} + \left(-\frac{88}{5c^2} \mathcal{L}^2 \right. \right. \\
& \left. \left. - \frac{17}{60c} \mathcal{L}'' + \frac{149}{7\sqrt{105}c} \mathcal{W}_4 \right) W_{-3}^{(5)} + \left(\frac{7}{120c} \mathcal{L}^{(3)} \right. \right. \\
& \left. \left. - \frac{311}{56\sqrt{105}c} \mathcal{W}_4' + \frac{58}{5c^2} \mathcal{L} \mathcal{L}' \right) W_{-4}^{(5)} - \frac{113\mathcal{W}_3}{14\sqrt{15}c} W_{-2}^{(4)} \right. \\
& \left. + \frac{13\mathcal{L}'}{12c} W_{-2}^{(5)} + \frac{33}{28c} \sqrt{\frac{3}{5}} \mathcal{W}_3' W_{-3}^{(4)} \right) + \mu_5^{(4)} \left(\frac{1}{24} W_0^{(5)} \right. \\
& \left. + \frac{2\mathcal{L}}{3c} W_{-2}^{(5)} + \left(\frac{16}{5c^2} \mathcal{L}^2 + \frac{1}{15c} \mathcal{L}'' - \frac{23}{7\sqrt{105}c} \mathcal{W}_4 \right) W_{-4}^{(5)} \right. \\
& \left. - \frac{1}{4c} \mathcal{L}' W_{-3}^{(5)} + \frac{41}{28\sqrt{15}c} \mathcal{W}_3 W_{-3}^{(4)} \right) + \mu_5^{(5)} \left(-\frac{1}{120} W_{-1}^{(5)} \right. \\
& \left. - \frac{7\mathcal{L}}{60c} W_{-3}^{(5)} + \frac{11\mathcal{L}'}{240c} W_{-4}^{(5)} \right) + \mu_5^{(6)} \left(\frac{1}{720} W_{-2}^{(5)} + \frac{\mathcal{L}}{60c} W_{-4}^{(5)} \right) \\
& - \frac{\mu_5^{(7)}}{5040} W_{-3}^{(5)} + \frac{\mu_5^{(8)}}{40320} W_{-4}^{(5)}
\end{aligned} \tag{C.62}$$



APPENDIX D : The $\mathfrak{sl}(3|2)$ Superconformal Case

D.1 The Temporal Evolution of the Twenty Four Independent Source Fields

In this appendix, we collect explicit calculations of additional conditions that the equations of motions impose for fixed chemical potentials as the temporal evolution of the twenty four source fields.

$$\begin{aligned}\partial_t \mathcal{J} = & k\partial_\phi \eta + \mathcal{G}_2^{\frac{1}{2}} v_1^{-\frac{1}{2}} + \mathcal{G}_2^{-\frac{1}{2}} v_1^{\frac{1}{2}} - \mathcal{G}_1^{\frac{1}{2}} v_2^{-\frac{1}{2}} - \mathcal{G}_1^{-\frac{1}{2}} v_2^{\frac{1}{2}} + \frac{4}{3} \mathcal{S}_2^{\frac{3}{2}} \psi_1^{-\frac{3}{2}} \\ & + \frac{4}{3} \mathcal{S}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} + \frac{4}{3} \mathcal{S}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} + \frac{4}{3} \mathcal{S}_2^{-\frac{3}{2}} \psi_1^{\frac{3}{2}} - \frac{4}{3} \mathcal{S}_1^{\frac{3}{2}} \psi_2^{-\frac{3}{2}} - \frac{4}{3} \mathcal{S}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} \\ & - \frac{4}{3} \mathcal{S}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} - \frac{4}{3} \mathcal{S}_1^{-\frac{3}{2}} \psi_2^{\frac{3}{2}}\end{aligned}\quad (\text{D.1})$$

$$\begin{aligned}\partial_t \mathcal{L}^0 = & \frac{k}{4} \partial_\phi \mu^0 - \frac{1}{2} \mathcal{G}_2^{\frac{1}{2}} v_1^{-\frac{1}{2}} + \frac{1}{2} \mathcal{G}_2^{-\frac{1}{2}} v_1^{\frac{1}{2}} - \frac{1}{2} \mathcal{G}_1^{\frac{1}{2}} v_2^{-\frac{1}{2}} + \frac{1}{2} \mathcal{G}_1^{-\frac{1}{2}} v_2^{\frac{1}{2}} - \frac{5}{8} \mathcal{G}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} \\ & - \frac{5}{8} \mathcal{G}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} + \frac{5}{8} \mathcal{G}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} + \frac{5}{8} \mathcal{G}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} - \frac{5}{3} v_1^{-\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}} - \frac{5}{3} v_1^{\frac{1}{2}} \mathcal{S}_2^{-\frac{1}{2}} + \frac{5}{3} v_2^{-\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}} \\ & + \frac{5}{3} v_2^{\frac{1}{2}} \mathcal{S}_1^{-\frac{1}{2}} + \frac{1}{2} \mathcal{S}_2^{\frac{3}{2}} \psi_1^{-\frac{3}{2}} - \frac{1}{6} \mathcal{S}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} - \frac{1}{2} \mathcal{S}_2^{-\frac{3}{2}} \psi_1^{\frac{3}{2}} + \frac{1}{2} \mathcal{S}_1^{\frac{3}{2}} \psi_2^{-\frac{3}{2}} + \frac{1}{6} \mathcal{S}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} \\ & - \frac{1}{6} \mathcal{S}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} - \frac{1}{2} \mathcal{S}_1^{-\frac{3}{2}} \psi_2^{\frac{3}{2}} - \mathcal{A}^1 \xi^{-1} + f^2 \mathcal{W}^{-2} + \frac{1}{2} f^1 \mathcal{W}^{-1} - \frac{1}{2} f^{-1} \mathcal{W}^1 \\ & - f^{-2} \mathcal{W}^2 - \mu^{-1} \mathcal{L}^1 + \mu^1 \mathcal{L}^{-1} + \frac{1}{6} \mathcal{S}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} + \mathcal{A}^{-1} \xi^1\end{aligned}\quad (\text{D.2})$$

$$\begin{aligned}\partial_t \mathcal{L}^{\pm 1} = & -\frac{k}{2} \partial_\phi \mu^{\pm 1} \pm \mathcal{G}_2^{\pm \frac{1}{2}} v_1^{\pm \frac{1}{2}} \pm \mathcal{G}_1^{\pm \frac{1}{2}} v_2^{\pm \frac{1}{2}} + \frac{15}{8} \mathcal{G}_2^{\mp \frac{1}{2}} \psi_1^{\pm \frac{3}{2}} + \frac{5}{8} \mathcal{G}_2^{\pm \frac{1}{2}} \psi_1^{\pm \frac{1}{2}} \\ & - \frac{15}{8} \mathcal{G}_2^{\frac{1}{2}} \psi_2^{\pm \frac{3}{2}} - \frac{5}{3} v_1^{\pm \frac{1}{2}} \mathcal{S}_2^{\pm \frac{1}{2}} - \frac{5}{3} v_1^{\mp \frac{1}{2}} \mathcal{S}_2^{\pm \frac{3}{2}} + \frac{5}{3} v_2^{\pm \frac{1}{2}} \mathcal{S}_1^{\pm \frac{1}{2}} + \frac{5}{3} v_2^{\mp \frac{1}{2}} \mathcal{S}_1^{\pm \frac{3}{2}} \\ & \mp \mathcal{S}_2^{\mp \frac{1}{2}} \psi_1^{\pm \frac{3}{2}} \mp \frac{2}{3} \mathcal{S}_2^{\pm \frac{1}{2}} \psi_1^{\pm \frac{1}{2}} \mp \mathcal{S}_1^{\mp \frac{1}{2}} \psi_2^{\pm \frac{3}{2}} + \frac{2}{3} \mathcal{S}_1^{\pm \frac{1}{2}} \psi_2^{\pm \frac{1}{2}} \mp \frac{1}{3} \mathcal{S}_1^{\pm \frac{3}{2}} \psi_2^{\mp \frac{1}{2}} \\ & \pm 2 \mathcal{A}^0 \xi^{\pm 1} \pm \mathcal{A}^{\pm 1} \xi^0 - \frac{1}{2} f^{\mp 1} \mathcal{W}^{\pm 2} \pm f^0 \mathcal{W}^{\pm 1} \pm 2 f^{\pm 2} \mathcal{W}^{\mp 1} \\ & \pm 2 \mu^{\pm 1} \mathcal{L}^0 \pm \mu^0 \mathcal{L}^{\pm 1} - \frac{5}{8} \mathcal{G}_1^{\pm \frac{1}{2}} \psi_2^{\pm \frac{1}{2}} \mp \frac{1}{3} \mathcal{S}_2^{\pm \frac{3}{2}} \psi_1^{\mp \frac{1}{2}} \pm \frac{3}{2} f^{\pm 1} \mathcal{W}^0\end{aligned}\quad (\text{D.3})$$

$$\begin{aligned}\partial_t \mathcal{A}^0 = & \frac{k}{4} \partial_\phi \xi_0 + \frac{3}{8} \mathcal{G}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} + \frac{3}{8} \mathcal{G}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} - \frac{3}{8} \mathcal{G}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} - \frac{3}{8} \mathcal{G}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} + v_1^{-\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}} \\ & + v_1^{\frac{1}{2}} \mathcal{S}_2^{-\frac{1}{2}} - v_2^{\frac{1}{2}} \mathcal{S}_1^{-\frac{1}{2}} - \frac{3}{2} \mathcal{S}_2^{\frac{3}{2}} \psi_1^{-\frac{3}{2}} - \frac{1}{2} \mathcal{S}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} + \frac{1}{2} \mathcal{S}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} + \frac{3}{2} \mathcal{S}_2^{-\frac{3}{2}} \psi_1^{\frac{3}{2}} \\ & - \frac{3}{2} \mathcal{S}_1^{\frac{3}{2}} \psi_2^{-\frac{3}{2}} - \frac{1}{2} \mathcal{S}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} + \frac{1}{2} \mathcal{S}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} + \frac{3}{2} \mathcal{S}_1^{-\frac{3}{2}} \psi_2^{\frac{3}{2}} - \mathcal{A}^1 \mu^{-1} + \mathcal{A}^{-1} \mu^1\end{aligned}$$

$$+f^2\mathcal{W}^{-2}+\frac{1}{2}f^1\mathcal{W}^{-1}-\frac{1}{2}f^{-1}\mathcal{W}^1-\xi^{-1}\mathcal{L}^1+\xi^1\mathcal{L}^{-1}-v_2^{-\frac{1}{2}}\mathcal{S}_1^{\frac{1}{2}}-f^{-2}\mathcal{W}^2 \quad (\text{D.4})$$

$$\begin{aligned} \partial_t \mathcal{A}^{\pm 1} = & -\frac{k}{2}\partial_\phi \xi^{\pm 1} - \frac{9}{8}\mathcal{G}_2^{\mp \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} \pm \frac{3}{8}\mathcal{G}_2^{\pm \frac{1}{2}}\psi_1^{\pm \frac{1}{2}} + \frac{9}{8}\mathcal{G}_1^{\mp \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} \mp \frac{3}{8}\mathcal{G}_1^{\pm \frac{1}{2}}\psi_2^{\pm \frac{1}{2}} \\ & + v_1^{\pm \frac{1}{2}}\mathcal{S}_2^{\pm \frac{1}{2}} - v_2^{\pm \frac{1}{2}}\mathcal{S}_1^{\pm \frac{1}{2}} - v_2^{\mp \frac{1}{2}}\mathcal{S}_1^{\pm \frac{3}{2}} \pm 3\mathcal{S}_2^{\mp \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} \pm 2\mathcal{S}_2^{\pm \frac{1}{2}}\psi_1^{\pm \frac{1}{2}} \pm \mathcal{S}_2^{\pm \frac{3}{2}}\psi_1^{\mp \frac{1}{2}} \\ & \pm 3\mathcal{S}_1^{\mp \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} \pm \mathcal{S}_1^{\pm \frac{3}{2}}\psi_2^{\mp \frac{1}{2}} \pm 2\mathcal{A}^0\mu^{\pm 1} \pm \mathcal{A}^{\pm 1}\mu^0 \pm \frac{1}{2}f^{\mp 1}\mathcal{W}^{\pm 2} \pm f^0\mathcal{W}^{\pm 1} \\ & \pm \frac{3}{2}f^{\pm 1}\mathcal{W}^0 \pm 2f^{\pm 2}\mathcal{W}^{\mp 1} \pm \xi^0\mathcal{L}^{\pm 1} + v_1^{\mp \frac{1}{2}}\mathcal{S}_2^{\pm \frac{3}{2}} \pm 2\mathcal{S}_1^{\pm \frac{1}{2}}\psi_2^{\pm \frac{1}{2}} \\ & \pm 2\xi^{\pm 1}\mathcal{L}^0 \end{aligned} \quad (\text{D.5})$$

$$\begin{aligned} \partial_t \mathcal{W}^0 = & \frac{k}{3}\partial_\phi f^0 + \frac{1}{2}\mathcal{G}_2^{\frac{1}{2}}\psi_1^{-\frac{1}{2}} - \frac{1}{2}\mathcal{G}_2^{-\frac{1}{2}}\psi_1^{\frac{1}{2}} + \frac{1}{2}\mathcal{G}_1^{\frac{1}{2}}\psi_2^{-\frac{1}{2}} - \frac{1}{2}\mathcal{G}_1^{-\frac{1}{2}}\psi_2^{\frac{1}{2}} + \frac{4}{3}v_1^{-\frac{1}{2}}\mathcal{S}_2^{\frac{1}{2}} \\ & - \frac{4}{3}v_1^{\frac{1}{2}}\mathcal{S}_2^{-\frac{1}{2}} - \frac{4}{3}v_2^{\frac{1}{2}}\mathcal{S}_1^{-\frac{1}{2}} + \frac{2}{3}\mathcal{S}_2^{\frac{3}{2}}\psi_1^{-\frac{3}{2}} - \frac{2}{3}\mathcal{S}_2^{\frac{1}{2}}\psi_1^{-\frac{1}{2}} - \frac{2}{3}\mathcal{S}_2^{-\frac{1}{2}}\psi_1^{\frac{1}{2}} + \frac{2}{3}\mathcal{S}_2^{-\frac{3}{2}}\psi_1^{\frac{3}{2}} \\ & - \frac{2}{3}\mathcal{S}_1^{\frac{3}{2}}\psi_2^{-\frac{3}{2}} - \frac{2}{3}\mathcal{S}_1^{-\frac{3}{2}}\psi_2^{\frac{3}{2}} - 2\mathcal{A}^1f^{-1} + 2\mathcal{A}^{-1}f^1 - 2f^{-1}\mathcal{L}^1 + 2f^1\mathcal{L}^{-1} \\ & - 2\mu^{-1}\mathcal{W}^1 + 2\mu^1\mathcal{W}^{-1} + 2\xi^1\mathcal{W}^{-1} + \frac{4}{3}v_2^{-\frac{1}{2}}\mathcal{S}_1^{\frac{1}{2}} + \frac{2}{3}\mathcal{S}_1^{\frac{1}{2}}\psi_2^{-\frac{1}{2}} + \frac{2}{3}\mathcal{S}_1^{-\frac{1}{2}}\psi_2^{\frac{1}{2}} \\ & - 2\xi^{-1}\mathcal{W}^1 \end{aligned} \quad (\text{D.6})$$

$$\begin{aligned} \partial_t \mathcal{W}^{\pm 1} = & -\frac{k}{2}\partial_\phi f^{\pm 1} \pm \frac{3}{4}\mathcal{G}_2^{\mp \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} \mp \frac{3}{4}\mathcal{G}_2^{\pm \frac{1}{2}}\psi_1^{\pm \frac{1}{2}} \pm \frac{3}{4}\mathcal{G}_1^{\mp \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} \mp \frac{3}{4}\mathcal{G}_1^{\pm \frac{1}{2}}\psi_2^{\pm \frac{1}{2}} \\ & \mp 2v_1^{\pm \frac{1}{2}}\mathcal{S}_2^{\pm \frac{1}{2}} \mp 2v_2^{\pm \frac{1}{2}}\mathcal{S}_1^{\pm \frac{1}{2}} \pm \frac{2}{3}v_2^{\mp \frac{1}{2}}\mathcal{S}_1^{\pm \frac{3}{2}} \mp 2\mathcal{S}_2^{\mp \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} - \frac{2}{3}\mathcal{S}_2^{\pm \frac{3}{2}}\psi_1^{\mp \frac{1}{2}} \\ & \pm 2\mathcal{S}_1^{\mp \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} + \frac{2}{3}\mathcal{S}_1^{\pm \frac{3}{2}}\psi_2^{\mp \frac{1}{2}} \pm 2\mathcal{A}^0f^{\pm 1} \pm 2\mathcal{A}^{\pm 1}f^0 \mp 4f^{\pm 2}\mathcal{L}^{\mp 1} \pm 2f^0\mathcal{L}^{\pm 1} \\ & \pm 2f^{\pm 1}\mathcal{L}^0 \pm 3\mu^{\pm 1}\mathcal{W}^0 \pm \mu^0\mathcal{W}^{\pm 1} \pm 3\xi^{\pm 1}\mathcal{W}^0 \pm \xi^0\mathcal{W}^{\pm 1} \mp \xi^{\mp 1}\mathcal{W}^{\pm 2} \\ & \pm \frac{2}{3}v_1^{\mp \frac{1}{2}}\mathcal{S}_2^{\pm \frac{3}{2}} \mp 4\mathcal{A}^1f^{\pm 2} + \mu^{\mp 1}\mathcal{W}^{\pm 2} \end{aligned} \quad (\text{D.7})$$

$$\begin{aligned} \partial_t \mathcal{W}^{\pm 2} = & 2k\partial_\phi f^{\pm 2} \pm 3\mathcal{G}_2^{\pm \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} \pm 3\mathcal{G}_1^{\pm \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} \mp \frac{8}{3}v_1^{\pm \frac{1}{2}}\mathcal{S}_2^{\pm \frac{3}{2}} \mp \frac{8}{3}v_2^{\pm \frac{1}{2}}\mathcal{S}_1^{\pm \frac{3}{2}} \\ & + 4\mathcal{S}_2^{\pm \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} - 4\mathcal{S}_1^{\pm \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} - \frac{4}{3}\mathcal{S}_1^{\pm \frac{3}{2}}\psi_2^{\pm \frac{1}{2}} \mp 16\mathcal{A}^0f^{\pm 2} \mp 4\mathcal{A}^{\pm 1}f^{\pm 1} \\ & + 16f^{\pm 2}\mathcal{L}^0 \mp 4f^{\pm 1}\mathcal{L}^{\pm 1} \pm 2\mu^0\mathcal{W}^{\pm 2} \pm 4\xi^{\pm 1}\mathcal{W}^{\pm 1} \pm 2\xi^0\mathcal{W}^{\pm 2} \\ & \mp \frac{4}{3}\mathcal{S}_2^{\pm \frac{3}{2}}\psi_1^{\pm \frac{1}{2}} \pm 4\mu^{\pm 1}\mathcal{W}^{\pm 1} \end{aligned} \quad (\text{D.8})$$

$$\begin{aligned} \partial_t \mathcal{G}_M^{\pm \frac{1}{2}} = & \mp k\partial_\phi v_M^{\pm \frac{1}{2}} + \frac{10}{3}\mathcal{A}^0v_M^{\mp \frac{1}{2}} + \frac{10}{3}\mathcal{A}^{\pm 1}v_M^{\mp \frac{1}{2}} \mp 4\mathcal{A}^{\mp 1}\psi_M^{\pm \frac{3}{2}} \mp \frac{8}{3}\mathcal{A}^0\psi_M^{\pm \frac{1}{2}} \\ & \mp \frac{4}{3}\mathcal{A}^{\pm 1}\psi_M^{\mp \frac{1}{2}} \mp \frac{16}{9}f^0\mathcal{S}_M^{\pm \frac{1}{2}} \mp \frac{8}{3}f^{\pm 1}\mathcal{S}_M^{\mp \frac{1}{2}} \mp \frac{32}{9}f^{\pm 2}\mathcal{S}_M^{\mp \frac{3}{2}} - \eta\mathcal{G}_M^{\pm \frac{1}{2}} \\ & \pm \frac{1}{2}\mu^0\mathcal{G}_M^{\pm \frac{1}{2}} \pm \mu^{\pm 1}\mathcal{G}_M^{\mp \frac{1}{2}} \pm \frac{5}{6}\xi^0\mathcal{G}_M^{\pm \frac{1}{2}} \mp \mathcal{J}v_M^{\pm \frac{1}{2}} \mp \frac{16}{9}\xi^{\mp 1}\mathcal{S}_M^{\pm \frac{3}{2}} - \frac{16}{9}\xi^0\mathcal{S}_M^{\pm \frac{1}{2}} \\ & - \frac{16}{9}\xi^{\pm 1}\mathcal{S}_M^{\mp \frac{1}{2}} - 2\mathcal{W}^{\mp 1}\psi_M^{\pm \frac{3}{2}} - 2\mathcal{W}^0\psi_M^{\pm \frac{1}{2}} - 2\mathcal{W}^{\pm 2}\psi_M^{\mp \frac{3}{2}} + 2\mathcal{L}^0v_M^{\pm \frac{1}{2}} \\ & + 2\mathcal{L}^{\pm 1}v_M^{\mp \frac{1}{2}} \mp \frac{8}{9}f^{\mp 1}\mathcal{S}_M^{\pm \frac{3}{2}} \pm \frac{5}{3}\xi^{\pm 1}\mathcal{G}_M^{\mp \frac{1}{2}} - 2\mathcal{W}^{\pm 1}\psi_M^{\mp \frac{1}{2}} \end{aligned} \quad (\text{D.9})$$

$$\begin{aligned}
\partial_t S_M^{\pm\frac{1}{2}} = & \mp \frac{3}{8} k \psi_M^{\pm\frac{1}{2}} \mp 2 \mathcal{A}^0 v_M^{\pm\frac{1}{2}} \pm \mathcal{A}^{\pm 1} v_M^{\mp\frac{1}{2}} - \frac{3}{4} \mathcal{A}^{\mp 1} \psi_M^{\pm\frac{3}{2}} + \frac{1}{4} \mathcal{A}^0 \psi_M^{\pm\frac{1}{2}} \\
& + \frac{1}{2} \mathcal{A}^{\pm 1} \psi_M^{\mp\frac{1}{2}} \mp \frac{3}{4} f^{\pm 1} \mathcal{G}_M^{\mp\frac{1}{2}} - \frac{2}{3} f^{\mp 1} S_M^{\pm\frac{3}{2}} - \frac{2}{3} f^0 S_M^{\pm\frac{1}{2}} + \frac{4}{3} f^{\pm 2} S_M^{\mp\frac{3}{2}} \\
& - \frac{1}{2} \xi^0 \mathcal{G}_M^{\pm\frac{1}{2}} + \frac{1}{2} \xi^{\pm 1} \mathcal{G}_M^{\mp\frac{1}{2}} \pm \mu^{\mp 1} S_M^{\pm\frac{3}{2}} \pm \frac{1}{2} \mu^0 S_M^{\pm\frac{1}{2}} - 2 \mu^{\pm 1} S_M^{\mp\frac{1}{2}} \mp \frac{1}{3} \xi^{\mp 1} S_M^{\pm\frac{3}{2}} \\
& \pm \frac{1}{6} \xi^0 S_M^{\pm\frac{1}{2}} \pm \frac{2}{3} \xi^{\pm 1} S_M^{\mp\frac{1}{2}} + \frac{3}{2} \mathcal{W}^{\mp 1} \psi_M^{\pm\frac{3}{2}} \mp \frac{3}{4} \mathcal{W}^0 \psi_M^{\pm\frac{1}{2}} \pm \frac{3}{4} \mathcal{W}^{\pm 2} \psi_M^{\mp\frac{3}{2}} \\
& - \frac{9}{4} \mathcal{L}^{\mp 1} \psi_M^{\pm\frac{3}{2}} + \frac{3}{4} \mathcal{L}^0 \psi_M^{\pm\frac{1}{2}} - \eta S_M^{\pm\frac{1}{2}} \mp \frac{1}{2} f^0 \mathcal{G}_M^{\pm\frac{1}{2}} \mp \frac{3}{8} \mathcal{J} \psi_M^{\pm\frac{1}{2}} \pm \frac{3}{2} \mathcal{W}^0 v_M^{\pm\frac{1}{2}} \\
& - \frac{3}{2} \mathcal{W}^{\pm 1} v_M^{\mp\frac{1}{2}} + \frac{3}{2} \mathcal{L}^{\pm 1} \psi_M^{\mp\frac{1}{2}} \tag{D.10}
\end{aligned}$$

$$\begin{aligned}
\partial_t S_M^{\pm\frac{3}{2}} = & \pm \frac{9}{8} k \psi_M^{\pm\frac{3}{2}} \mp 3 \mathcal{A}^{-1} v_M^{\pm\frac{1}{2}} - \frac{9}{4} \mathcal{A}^0 \psi_M^{\pm\frac{3}{2}} - \frac{3}{4} \mathcal{A}^{\pm 1} \psi_M^{\pm\frac{1}{2}} \pm \frac{3}{4} f^{\pm 1} \mathcal{G}_M^{\pm\frac{1}{2}} \\
& \pm 3 f^{\pm 2} \mathcal{G}_M^{\mp\frac{1}{2}} + 2 f^{\pm 1} S_M^{\pm\frac{1}{2}} + 4 f^{\pm 2} S_M^{\mp\frac{1}{2}} + \frac{3}{2} \xi^{\pm 1} \mathcal{G}_M^{\pm\frac{1}{2}} \pm \frac{9}{8} \mathcal{J} \psi_M^{\pm\frac{3}{2}} - \eta S_M^{\pm\frac{3}{2}} \\
& \pm \frac{3}{2} \mu^0 S_M^{\pm\frac{3}{2}} \pm 3 \mu^{\pm 1} S_M^{\pm\frac{1}{2}} \pm \xi^{\pm 1} S_M^{\pm\frac{1}{2}} - \frac{3}{2} \mathcal{W}^{\pm 1} v_M^{\pm\frac{1}{2}} - \frac{3}{2} \mathcal{W}^{\pm 2} v_M^{\mp\frac{1}{2}} \\
& + \frac{9}{4} \mathcal{W}^0 \psi_M^{\pm\frac{3}{2}} \mp \frac{3}{2} \mathcal{W}^{\pm 1} \psi_M^{\pm\frac{1}{2}} \mp \frac{3}{4} \mathcal{W}^{\pm 2} \psi_M^{\mp\frac{1}{2}} - \frac{27}{4} \mathcal{L}^0 \psi_M^{\pm\frac{3}{2}} - \frac{9}{4} \mathcal{L}^{\pm 1} \psi_M^{\pm\frac{1}{2}} \\
& + \frac{2}{3} f^0 S_M^{\pm\frac{3}{2}} \pm \frac{1}{2} \xi^0 S_M^{\pm\frac{3}{2}} \tag{D.11}
\end{aligned}$$

D.2 The Gauge Transformations of the Twenty Four Independent Source Fields

In this appendix, we present in more detail the gauge transformations of the twenty four independent source fields as:

$$\begin{aligned}
\partial_\lambda \mathcal{J} = & k \partial_\varphi \rho + \mathcal{G}_2^{\frac{1}{2}} v_1^{-\frac{1}{2}} + \mathcal{G}_2^{-\frac{1}{2}} v_1^{\frac{1}{2}} - \mathcal{G}_1^{\frac{1}{2}} v_2^{-\frac{1}{2}} - \mathcal{G}_1^{-\frac{1}{2}} v_2^{\frac{1}{2}} + \frac{4}{3} S_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} \\
& + \frac{4}{3} S_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} + \frac{4}{3} S_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} + \frac{4}{3} S_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} - \frac{4}{3} S_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} - \frac{4}{3} S_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} \\
& - \frac{4}{3} S_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{4}{3} S_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}}, \tag{D.12}
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda \mathcal{L}^0 = & \frac{k}{4} \partial_\varphi \varepsilon^0 - \frac{1}{2} \mathcal{G}_2^{\frac{1}{2}} v_1^{-\frac{1}{2}} + \frac{1}{2} \mathcal{G}_2^{-\frac{1}{2}} v_1^{\frac{1}{2}} - \frac{1}{2} \mathcal{G}_1^{\frac{1}{2}} v_2^{-\frac{1}{2}} + \frac{1}{2} \mathcal{G}_1^{-\frac{1}{2}} v_2^{\frac{1}{2}} - \frac{5}{8} \mathcal{G}_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} \\
& - \frac{5}{8} \mathcal{G}_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} + \frac{5}{8} \mathcal{G}_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} + \frac{5}{8} \mathcal{G}_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{5}{3} v_1^{-\frac{1}{2}} S_2^{\frac{1}{2}} - \frac{5}{3} v_1^{\frac{1}{2}} S_2^{-\frac{1}{2}} + \frac{5}{3} v_2^{-\frac{1}{2}} S_1^{\frac{1}{2}} \\
& + \frac{5}{3} v_2^{\frac{1}{2}} S_1^{-\frac{1}{2}} + \frac{1}{2} S_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} + \frac{1}{6} S_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} - \frac{1}{6} S_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} - \frac{1}{2} S_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} + \frac{1}{2} S_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} \\
& + \frac{1}{6} S_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} - \frac{1}{6} S_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{1}{2} S_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}} - \mathcal{A}^1 \phi^{-1} + \mathcal{A}^{-1} \phi^1 + v^2 \mathcal{W}^{-2} \\
& + \frac{1}{2} v^1 \mathcal{W}^{-1} - \frac{1}{2} v^{-1} \mathcal{W}^1 - v^{-2} \mathcal{W}^2 - \varepsilon^{-1} \mathcal{L}^1 + \varepsilon^1 \mathcal{L}^{-1} \tag{D.13}
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda \mathcal{L}^{\pm 1} = & -\frac{k}{2} \partial_\phi \varepsilon^{\pm 1} \pm \mathcal{G}_2^{\pm \frac{1}{2}} v_1^{\pm \frac{1}{2}} \pm \mathcal{G}_1^{\pm \frac{1}{2}} v_2^{\pm \frac{1}{2}} + \frac{15}{8} \mathcal{G}_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} + \frac{5}{8} \mathcal{G}_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} \\
& - \frac{15}{8} \mathcal{G}_2^{\frac{1}{2}} \omega_2^{\pm \frac{3}{2}} - \frac{5}{8} \mathcal{G}_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} - \frac{5}{3} v_1^{\pm \frac{1}{2}} \mathcal{S}_2^{\pm \frac{1}{2}} - \frac{5}{3} v_1^{\mp \frac{1}{2}} \mathcal{S}_2^{\pm \frac{3}{2}} + \frac{5}{3} v_2^{\pm \frac{1}{2}} \mathcal{S}_1^{\pm \frac{1}{2}} \\
& + \frac{5}{3} v_2^{\mp \frac{1}{2}} \mathcal{S}_1^{\pm \frac{3}{2}} \mp \mathcal{S}_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \mp \frac{2}{3} \mathcal{S}_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} \mp \frac{1}{3} \mathcal{S}_2^{\pm \frac{3}{2}} \omega_1^{\mp \frac{1}{2}} \mp \mathcal{S}_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} + \frac{2}{3} \mathcal{S}_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} \\
& \mp \frac{1}{3} \mathcal{S}_1^{\pm \frac{3}{2}} \omega_2^{\mp \frac{1}{2}} \pm 2\mathcal{A}^0 \phi^{\pm 1} \pm \mathcal{A}^{\pm 1} \phi^0 - \frac{1}{2} v^{\mp 1} \mathcal{W}^{\pm 2} \pm v^0 \mathcal{W}^{\pm 1} \pm \frac{3}{2} v^{\pm 1} \mathcal{W}^0 \\
& \pm 2v^{\pm 2} \mathcal{W}^{\mp 1} \pm 2\varepsilon^{\pm 1} \mathcal{L}^0 \pm \varepsilon^0 \mathcal{L}^{\pm 1}
\end{aligned} \tag{D.14}$$

$$\begin{aligned}
\partial_\lambda \mathcal{A}^0 = & \frac{k}{4} \partial_\phi \phi_0 + \frac{3}{8} \mathcal{G}_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} + \frac{3}{8} \mathcal{G}_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} - \frac{3}{8} \mathcal{G}_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} - \frac{3}{8} \mathcal{G}_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} + v_1^{-\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}} \\
& + v_1^{\frac{1}{2}} \mathcal{S}_2^{-\frac{1}{2}} - v_2^{-\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}} - v_2^{\frac{1}{2}} \mathcal{S}_1^{-\frac{1}{2}} - \frac{3}{2} \mathcal{S}_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} - \frac{1}{2} \mathcal{S}_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} + \frac{1}{2} \mathcal{S}_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} \\
& + \frac{3}{2} \mathcal{S}_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} - \frac{3}{2} \mathcal{S}_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} - \frac{1}{2} \mathcal{S}_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} + \frac{1}{2} \mathcal{S}_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} + \frac{3}{2} \mathcal{S}_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}} - \mathcal{A}^1 \varepsilon^{-1} \\
& + \mathcal{A}^{-1} \varepsilon^1 + v^2 \mathcal{W}^{-2} + \frac{1}{2} v^1 \mathcal{W}^{-1} - \frac{1}{2} v^{-1} \mathcal{W}^1 - v^{-2} \mathcal{W}^2 \\
& - \phi^{-1} \mathcal{L}^1 + \phi^1 \mathcal{L}^{-1}
\end{aligned} \tag{D.15}$$

$$\begin{aligned}
\partial_\lambda \mathcal{A}^{\pm 1} = & -\frac{k}{2} \partial_\phi \phi^{\pm 1} - \frac{9}{8} \mathcal{G}_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \pm \frac{3}{8} \mathcal{G}_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} + \frac{9}{8} \mathcal{G}_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} \mp \frac{3}{8} \mathcal{G}_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} \\
& + v_1^{\pm \frac{1}{2}} \mathcal{S}_2^{\pm \frac{1}{2}} + v_1^{\mp \frac{1}{2}} \mathcal{S}_2^{\pm \frac{3}{2}} - v_2^{\pm \frac{1}{2}} \mathcal{S}_1^{\pm \frac{1}{2}} - v_2^{\mp \frac{1}{2}} \mathcal{S}_1^{\pm \frac{3}{2}} \pm 3\mathcal{S}_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \pm 2\mathcal{S}_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} \\
& \pm \mathcal{S}_2^{\pm \frac{3}{2}} \omega_1^{\mp \frac{1}{2}} \pm 3\mathcal{S}_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} \pm 2\mathcal{S}_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} \pm \mathcal{S}_1^{\pm \frac{3}{2}} \omega_2^{\mp \frac{1}{2}} \pm 2\mathcal{A}^0 \varepsilon^{\pm 1} \pm \mathcal{A}^{\pm 1} \varepsilon^0 \\
& \pm \frac{1}{2} v^{\mp 1} \mathcal{W}^{\pm 2} \pm v^0 \mathcal{W}^{\pm 1} \pm \frac{3}{2} v^{\pm 1} \mathcal{W}^0 \pm 2v^{\pm 2} \mathcal{W}^{\mp 1} \\
& \pm 2\phi^{\pm 1} \mathcal{L}^0 \pm \phi^0 \mathcal{L}^{\pm 1}
\end{aligned} \tag{D.16}$$

$$\begin{aligned}
\partial_\lambda \mathcal{W}^0 = & \frac{k}{3} \partial_\phi v^0 + \frac{1}{2} \mathcal{G}_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} - \frac{1}{2} \mathcal{G}_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} + \frac{1}{2} \mathcal{G}_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} - \frac{1}{2} \mathcal{G}_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} + \frac{4}{3} v_1^{-\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}} \\
& - \frac{4}{3} v_1^{\frac{1}{2}} \mathcal{S}_2^{-\frac{1}{2}} + \frac{4}{3} v_2^{-\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}} - \frac{4}{3} v_2^{\frac{1}{2}} \mathcal{S}_1^{-\frac{1}{2}} + \frac{2}{3} \mathcal{S}_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} - \frac{2}{3} \mathcal{S}_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} - \frac{2}{3} \mathcal{S}_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} \\
& + \frac{2}{3} \mathcal{S}_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} - \frac{2}{3} \mathcal{S}_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} + \frac{2}{3} \mathcal{S}_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} + \frac{2}{3} \mathcal{S}_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{2}{3} \mathcal{S}_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}} - 2\mathcal{A}^1 v^{-1} \\
& + 2\mathcal{A}^{-1} v^1 - 2v^{-1} \mathcal{L}^1 + 2v^1 \mathcal{L}^{-1} - 2\varepsilon^{-1} \mathcal{W}^1 + 2\varepsilon^1 \mathcal{W}^{-1} - 2\phi^{-1} \mathcal{W}^1 \\
& + 2\phi^1 \mathcal{W}^{-1}
\end{aligned} \tag{D.17}$$

$$\begin{aligned}
\partial_\lambda \mathcal{W}^{\pm 1} = & -\frac{k}{2} \partial_\phi v^{\pm 1} \pm \frac{3}{4} \mathcal{G}_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \mp \frac{3}{4} \mathcal{G}_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} \pm \frac{3}{4} \mathcal{G}_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} \mp \frac{3}{4} \mathcal{G}_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} \\
& \mp 2v_1^{\pm \frac{1}{2}} \mathcal{S}_2^{\pm \frac{1}{2}} \pm \frac{2}{3} v_1^{\mp \frac{1}{2}} \mathcal{S}_2^{\pm \frac{3}{2}} \mp 2v_2^{\pm \frac{1}{2}} \mathcal{S}_1^{\pm \frac{1}{2}} \pm \frac{2}{3} v_2^{\mp \frac{1}{2}} \mathcal{S}_1^{\pm \frac{3}{2}} \mp 2\mathcal{S}_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \\
& - \frac{2}{3} \mathcal{S}_2^{\pm \frac{3}{2}} \omega_1^{\mp \frac{1}{2}} \pm 2\mathcal{S}_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} + \frac{2}{3} \mathcal{S}_1^{\pm \frac{3}{2}} \omega_2^{\mp \frac{1}{2}} \mp 4\mathcal{A}^1 v^{\pm 2} \pm 2\mathcal{A}^0 v^{\pm 1} \\
& \pm 2\mathcal{A}^{\pm 1} v^0 \mp 4v^{\pm 2} \mathcal{L}^{\mp 1} \pm 2v^0 \mathcal{L}^{\pm 1} \pm 2v^{\pm 1} \mathcal{L}^0 \pm 3\varepsilon^{\pm 1} \mathcal{W}^0 \pm \varepsilon^0 \mathcal{W}^{\pm 1} \\
& + \varepsilon^{\mp 1} \mathcal{W}^{\pm 2} \pm 3\phi^{\pm 1} \mathcal{W}^0 \pm \phi^0 \mathcal{W}^{\pm 1} \mp \phi^{\mp 1} \mathcal{W}^{\pm 2}
\end{aligned} \tag{D.18}$$

$$\begin{aligned}
\partial_\lambda \mathcal{W}^{\pm 2} = & 2k\partial_\phi v^{\pm 2} \pm 3\mathcal{G}_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \pm 3\mathcal{G}_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} \mp \frac{8}{3} v_1^{\pm \frac{1}{2}} \mathcal{S}_2^{\pm \frac{3}{2}} \mp \frac{8}{3} v_2^{\pm \frac{1}{2}} \mathcal{S}_1^{\pm \frac{3}{2}} \\
& + 4\mathcal{S}_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \mp \frac{4}{3} \mathcal{S}_2^{\pm \frac{3}{2}} \omega_1^{\pm \frac{1}{2}} - 4\mathcal{S}_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} - \frac{4}{3} \mathcal{S}_1^{\pm \frac{3}{2}} \omega_2^{\pm \frac{1}{2}} \mp 16\mathcal{A}^0 v^{\pm 2} \\
& \mp 4\mathcal{A}^{\pm 1} v^{\pm 1} + 16v^{\pm 2} \mathcal{L}^0 \mp 4v^{\pm 1} \mathcal{L}^{\pm 1} \pm 4\varepsilon^{\pm 1} \mathcal{W}^{\pm 1} \pm 2\varepsilon^0 \mathcal{W}^{\pm 2} \\
& \pm 4\phi^{\pm 1} \mathcal{W}^{\pm 1} \pm 2\phi^0 \mathcal{W}^{\pm 2}
\end{aligned} \tag{D.19}$$

$$\begin{aligned}
\partial_\lambda \mathcal{G}_M^{\pm \frac{1}{2}} = & \mp k\partial_\phi v_M^{\pm \frac{1}{2}} + \frac{10}{3} \mathcal{A}^0 v_M^{\mp \frac{1}{2}} + \frac{10}{3} \mathcal{A}^{\pm 1} v_M^{\mp \frac{1}{2}} \mp 4\mathcal{A}^{\mp 1} \omega_M^{\pm \frac{3}{2}} \mp \frac{8}{3} \mathcal{A}^0 \omega_M^{\pm \frac{1}{2}} \\
& \mp \frac{4}{3} \mathcal{A}^{\pm 1} \omega_M^{\mp \frac{1}{2}} \mp \frac{8}{9} v^{\mp 1} \mathcal{S}_M^{\pm \frac{3}{2}} \mp \frac{16}{9} v^0 \mathcal{S}_M^{\pm \frac{1}{2}} \mp \frac{8}{3} v^{\pm 1} \mathcal{S}_M^{\mp \frac{1}{2}} \mp \frac{32}{9} v^{\pm 2} \mathcal{S}_M^{\mp \frac{3}{2}} \\
& - \rho \mathcal{G}_M^{\pm \frac{1}{2}} \pm \frac{1}{2} \varepsilon^0 \mathcal{G}_M^{\pm \frac{1}{2}} \pm \varepsilon^{\pm 1} \mathcal{G}_M^{\mp \frac{1}{2}} \pm \frac{5}{6} \phi^0 \mathcal{G}_M^{\pm \frac{1}{2}} \pm \frac{5}{3} \phi^{\pm 1} \mathcal{G}_M^{\mp \frac{1}{2}} \mp \mathcal{J} v_M^{\pm \frac{1}{2}} \\
& \mp \frac{16}{9} \phi^{\mp 1} \mathcal{S}_M^{\pm \frac{3}{2}} - \frac{16}{9} \phi^0 \mathcal{S}_M^{\pm \frac{1}{2}} - \frac{16}{9} \phi^{\pm 1} \mathcal{S}_M^{\mp \frac{1}{2}} - 2\mathcal{W}^{\mp 1} \omega_M^{\pm \frac{3}{2}} - 2\mathcal{W}^0 \omega_M^{\pm \frac{1}{2}} \\
& - 2\mathcal{W}^{\pm 1} \omega_M^{\mp \frac{1}{2}} - 2\mathcal{W}^{\pm 2} \omega_M^{\mp \frac{3}{2}} + 2\mathcal{L}^0 v_M^{\pm \frac{1}{2}} + 2\mathcal{L}^{\pm 1} v_M^{\mp \frac{1}{2}}
\end{aligned} \tag{D.20}$$

$$\begin{aligned}
\partial_\lambda \mathcal{S}_M^{\pm \frac{1}{2}} = & \mp \frac{3}{8} k \omega_M^{\pm \frac{1}{2}} \mp 2\mathcal{A}^0 v_M^{\pm \frac{1}{2}} \pm \mathcal{A}^{\pm 1} v_M^{\mp \frac{1}{2}} - \frac{3}{4} \mathcal{A}^{\mp 1} \omega_M^{\pm \frac{3}{2}} + \frac{1}{4} \mathcal{A}^0 \omega_M^{\pm \frac{1}{2}} \\
& + \frac{1}{2} \mathcal{A}^{\pm 1} \omega_M^{\mp \frac{1}{2}} \mp \frac{1}{2} v^0 \mathcal{G}_M^{\pm \frac{1}{2}} \mp \frac{3}{4} v^{\pm 1} \mathcal{G}_M^{\mp \frac{1}{2}} - \frac{2}{3} v^{\mp 1} \mathcal{S}_M^{\pm \frac{3}{2}} - \frac{2}{3} v^0 \mathcal{S}_M^{\pm \frac{1}{2}} + \frac{4}{3} v^{\pm 2} \mathcal{S}_M^{\mp \frac{3}{2}} \\
& - \frac{1}{2} \phi^0 \mathcal{G}_M^{\pm \frac{1}{2}} + \frac{1}{2} \phi^{\pm 1} \mathcal{G}_M^{\mp \frac{1}{2}} \mp \frac{3}{8} \mathcal{J} \omega_M^{\pm \frac{1}{2}} - \rho \mathcal{S}_M^{\pm \frac{1}{2}} \pm \varepsilon^{\mp 1} \mathcal{S}_M^{\pm \frac{3}{2}} \pm \frac{1}{2} \varepsilon^0 \mathcal{S}_M^{\pm \frac{1}{2}} - 2\varepsilon^{\pm 1} \mathcal{S}_M^{\mp \frac{1}{2}} \\
& \mp \frac{1}{3} \phi^{\mp 1} \mathcal{S}_M^{\pm \frac{3}{2}} \pm \frac{1}{6} \phi^0 \mathcal{S}_M^{\pm \frac{1}{2}} \pm \frac{2}{3} \phi^{\pm 1} \mathcal{S}_M^{\mp \frac{1}{2}} \pm \frac{3}{2} \mathcal{W}^0 v_M^{\pm \frac{1}{2}} - \frac{3}{2} \mathcal{W}^{\pm 1} v_M^{\mp \frac{1}{2}} + \frac{3}{2} \mathcal{W}^{\mp 1} \omega_M^{\pm \frac{3}{2}} \\
& \mp \frac{3}{4} \mathcal{W}^0 \omega_M^{\pm \frac{1}{2}} \pm \frac{3}{4} \mathcal{W}^{\pm 2} \omega_M^{\mp \frac{3}{2}} - \frac{9}{4} \mathcal{L}^{\mp 1} \omega_M^{\pm \frac{3}{2}} + \frac{3}{4} \mathcal{L}^0 \omega_M^{\pm \frac{1}{2}} + \frac{3}{2} \mathcal{L}^{\pm 1} \omega_M^{\mp \frac{1}{2}}
\end{aligned} \tag{D.21}$$

$$\begin{aligned}
\partial_\lambda \mathcal{S}_M^{\pm \frac{3}{2}} = & \pm \frac{9}{8} k \omega_M^{\pm \frac{3}{2}} \mp 3\mathcal{A}^{-1} v_M^{\pm \frac{1}{2}} - \frac{9}{4} \mathcal{A}^0 \omega_M^{\pm \frac{3}{2}} - \frac{3}{4} \mathcal{A}^{\pm 1} \omega_M^{\pm \frac{1}{2}} \pm \frac{3}{4} v^{\pm 1} \mathcal{G}_M^{\pm \frac{1}{2}} \\
& \pm 3v^{\pm 2} \mathcal{G}_M^{\mp \frac{1}{2}} + \frac{2}{3} v^0 \mathcal{S}_M^{\pm \frac{3}{2}} + 2v^{\pm 1} \mathcal{S}_M^{\pm \frac{1}{2}} + 4v^{\pm 2} \mathcal{S}_M^{\mp \frac{1}{2}} + \frac{3}{2} \phi^{\pm 1} \mathcal{G}_M^{\pm \frac{1}{2}} \pm \frac{9}{8} \mathcal{J} \omega_M^{\pm \frac{3}{2}} \\
& - \rho \mathcal{S}_M^{\pm \frac{3}{2}} \pm \frac{3}{2} \varepsilon^0 \mathcal{S}_M^{\pm \frac{3}{2}} \pm 3\varepsilon^{\pm 1} \mathcal{S}_M^{\pm \frac{1}{2}} \pm \frac{1}{2} \phi^0 \mathcal{S}_M^{\pm \frac{3}{2}} \pm \phi^{\pm 1} \mathcal{S}_M^{\pm \frac{1}{2}} - \frac{3}{2} \mathcal{W}^{\pm 1} v_M^{\pm \frac{1}{2}} \\
& - \frac{3}{2} \mathcal{W}^{\pm 2} v_M^{\mp \frac{1}{2}} + \frac{9}{4} \mathcal{W}^0 \omega_M^{\pm \frac{3}{2}} \mp \frac{3}{2} \mathcal{W}^{\pm 1} \omega_M^{\pm \frac{1}{2}} \mp \frac{3}{4} \mathcal{W}^{\pm 2} \omega_M^{\mp \frac{1}{2}} - \frac{27}{4} \mathcal{L}^0 \omega_M^{\pm \frac{3}{2}} \\
& - \frac{9}{4} \mathcal{L}^{\pm 1} \omega_M^{\pm \frac{1}{2}}.
\end{aligned} \tag{D.22}$$

D.3 The Gauge Transformations of the Twenty Four Chemical Potentials

This appendix consists of the gauge transformations for the twenty four chemical potentials.

$$\begin{aligned}
\partial_\lambda \eta = & k\partial_\phi \rho + v_2^{\frac{1}{2}} v_1^{-\frac{1}{2}} + v_2^{-\frac{1}{2}} v_1^{\frac{1}{2}} - v_1^{\frac{1}{2}} v_2^{-\frac{1}{2}} - v_1^{-\frac{1}{2}} v_2^{\frac{1}{2}} + \frac{4}{3} \psi_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} \\
& + \frac{4}{3} \psi_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} + \frac{4}{3} \psi_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} + \frac{4}{3} \psi_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} - \frac{4}{3} \psi_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} - \frac{4}{3} \psi_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} \\
& - \frac{4}{3} \psi_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{4}{3} \psi_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}}
\end{aligned} \tag{D.23}$$

$$\begin{aligned}
\partial_\lambda \mu^0 = & \frac{k}{4} \partial_\phi \varepsilon^0 - \frac{1}{2} v_2^{\frac{1}{2}} v_1^{-\frac{1}{2}} + \frac{1}{2} v_2^{-\frac{1}{2}} v_1^{\frac{1}{2}} - \frac{1}{2} v_1^{\frac{1}{2}} v_2^{-\frac{1}{2}} + \frac{1}{2} v_1^{-\frac{1}{2}} v_2^{\frac{1}{2}} \\
& - \frac{5}{8} v_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} - \frac{5}{8} v_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} + \frac{5}{8} v_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} + \frac{5}{8} v_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{5}{3} v_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} - \frac{5}{3} v_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} \\
& + \frac{5}{3} v_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} + \frac{5}{3} v_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} + \frac{1}{2} \psi_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} + \frac{1}{6} \psi_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} - \frac{1}{6} \psi_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} - \frac{1}{2} \psi_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} \\
& + \frac{1}{2} \psi_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} + \frac{1}{6} \psi_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} - \frac{1}{6} \psi_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{1}{2} \psi_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}} - \xi^1 \phi^{-1} + \xi^{-1} \phi^1 \\
& + v^2 f^{-2} + \frac{1}{2} v^1 f^{-1} - \frac{1}{2} v^{-1} f^1 - v^{-2} f^2 - \varepsilon^{-1} \mu^1 + \varepsilon^1 \mu^{-1} \quad (D.24)
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda \mu^{\pm 1} = & -\frac{k}{2} \partial_\phi \varepsilon^{\pm 1} \pm v_2^{\pm \frac{1}{2}} v_1^{\pm \frac{1}{2}} \pm v_1^{\pm \frac{1}{2}} v_2^{\pm \frac{1}{2}} + \frac{15}{8} v_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} + \frac{5}{8} v_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} \\
& - \frac{15}{8} v_2^{\frac{1}{2}} \omega_2^{\pm \frac{3}{2}} - \frac{5}{8} v_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} - \frac{5}{3} v_1^{\pm \frac{1}{2}} \psi_2^{\pm \frac{1}{2}} - \frac{5}{3} v_1^{\mp \frac{1}{2}} \psi_2^{\pm \frac{3}{2}} + \frac{5}{3} v_2^{\pm \frac{1}{2}} \psi_1^{\pm \frac{1}{2}} \\
& + \frac{5}{3} v_2^{\mp \frac{1}{2}} \psi_1^{\pm \frac{3}{2}} \mp \psi_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \mp \frac{2}{3} \psi_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} \mp \frac{1}{3} \psi_2^{\pm \frac{3}{2}} \omega_1^{\mp \frac{1}{2}} \mp \psi_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} \\
& + \frac{2}{3} \psi_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} \mp \frac{1}{3} \psi_1^{\pm \frac{3}{2}} \omega_2^{\mp \frac{1}{2}} \pm 2 \xi^0 \phi^{\pm 1} \pm \xi^{\pm 1} \phi^0 - \frac{1}{2} v^{\mp 1} f^{\pm 2} \pm v^0 f^{\pm 1} \\
& \pm \frac{3}{2} v^{\pm 1} f^0 \pm 2 v^{\pm 2} f^{\mp 1} \pm 2 \varepsilon^{\pm 1} \mu^0 \pm \varepsilon^0 \mu^{\pm 1} \quad (D.25)
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda \xi^0 = & \frac{k}{4} \partial_\phi \phi_0 + \frac{3}{8} v_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} + \frac{3}{8} v_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} - \frac{3}{8} v_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} - \frac{3}{8} v_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} + v_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} \\
& + v_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} - v_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} - v_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} - \frac{3}{2} \psi_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} - \frac{1}{2} \psi_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} + \frac{1}{2} \psi_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} \\
& + \frac{3}{2} \psi_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} - \frac{3}{2} \psi_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} - \frac{1}{2} \psi_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} + \frac{1}{2} \psi_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} + \frac{3}{2} \psi_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}} \\
& - \xi^1 \varepsilon^{-1} + \xi^{-1} \varepsilon^1 + v^2 f^{-2} + \frac{1}{2} v^1 f^{-1} - \frac{1}{2} v^{-1} f^1 - v^{-2} f^2 - \phi^{-1} \mu^1 \\
& + \phi^1 \mu^{-1} \quad (D.26)
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda \xi^{\pm 1} = & -\frac{k}{2} \partial_\phi \phi^{\pm 1} - \frac{9}{8} v_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \pm \frac{3}{8} v_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} + \frac{9}{8} v_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} \mp \frac{3}{8} v_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} \\
& + v_1^{\pm \frac{1}{2}} \psi_2^{\pm \frac{1}{2}} + v_1^{\mp \frac{1}{2}} \psi_2^{\pm \frac{3}{2}} - v_2^{\pm \frac{1}{2}} \psi_1^{\pm \frac{1}{2}} - v_2^{\mp \frac{1}{2}} \psi_1^{\pm \frac{3}{2}} \pm 3 \psi_2^{\mp \frac{1}{2}} \omega_1^{\pm \frac{3}{2}} \pm 2 \psi_2^{\pm \frac{1}{2}} \omega_1^{\pm \frac{1}{2}} \\
& \pm \psi_2^{\pm \frac{3}{2}} \omega_1^{\mp \frac{1}{2}} \pm 3 \psi_1^{\mp \frac{1}{2}} \omega_2^{\pm \frac{3}{2}} \pm 2 \psi_1^{\pm \frac{1}{2}} \omega_2^{\pm \frac{1}{2}} \pm \psi_1^{\pm \frac{3}{2}} \omega_2^{\mp \frac{1}{2}} \pm 2 \xi^0 \varepsilon^{\pm 1} \pm \xi^{\pm 1} \varepsilon^0 \\
& \pm \frac{1}{2} v^{\mp 1} f^{\pm 2} \pm v^0 f^{\pm 1} \pm \frac{3}{2} v^{\pm 1} f^0 \pm 2 v^{\pm 2} f^{\mp 1} \pm 2 \phi^{\pm 1} \mu^0 \pm \phi^0 \mu^{\pm 1} \quad (D.27)
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda f^0 = & \frac{k}{3} \partial_\phi v^0 + \frac{1}{2} v_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} - \frac{1}{2} v_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} + \frac{1}{2} v_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} - \frac{1}{2} v_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} \\
& + \frac{4}{3} v_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} - \frac{4}{3} v_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} + \frac{4}{3} v_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} - \frac{4}{3} v_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} + \frac{2}{3} \psi_2^{\frac{3}{2}} \omega_1^{-\frac{3}{2}} - \frac{2}{3} \psi_2^{\frac{1}{2}} \omega_1^{-\frac{1}{2}} \\
& - \frac{2}{3} \psi_2^{-\frac{1}{2}} \omega_1^{\frac{1}{2}} + \frac{2}{3} \psi_2^{-\frac{3}{2}} \omega_1^{\frac{3}{2}} - \frac{2}{3} \psi_1^{\frac{3}{2}} \omega_2^{-\frac{3}{2}} + \frac{2}{3} \psi_1^{\frac{1}{2}} \omega_2^{-\frac{1}{2}} + \frac{2}{3} \psi_1^{-\frac{1}{2}} \omega_2^{\frac{1}{2}} - \frac{2}{3} \psi_1^{-\frac{3}{2}} \omega_2^{\frac{3}{2}} \\
& - 2 \xi^1 v^{-1} + 2 \xi^{-1} v^1 - 2 v^{-1} \mu^1 + 2 v^1 \mu^{-1} - 2 \varepsilon^{-1} f^1 + 2 \varepsilon^1 f^{-1} - 2 \phi^{-1} f^1 \\
& + 2 \phi^1 f^{-1} \quad (D.28)
\end{aligned}$$

$$\begin{aligned}
\partial_\lambda f^{\pm 1} = & -\frac{k}{2}\partial_\phi v^{\pm 1} \pm \frac{3}{4}v_2^{\mp \frac{1}{2}}\omega_1^{\pm \frac{3}{2}} \mp \frac{3}{4}v_2^{\pm \frac{1}{2}}\omega_1^{\pm \frac{1}{2}} \pm \frac{3}{4}v_1^{\mp \frac{1}{2}}\omega_2^{\pm \frac{3}{2}} \mp \frac{3}{4}v_1^{\pm \frac{1}{2}}\omega_2^{\pm \frac{1}{2}} \\
& \mp 2v_1^{\pm \frac{1}{2}}\psi_2^{\pm \frac{1}{2}} \pm \frac{2}{3}v_1^{\mp \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} \mp 2v_2^{\pm \frac{1}{2}}\psi_1^{\pm \frac{1}{2}} \pm \frac{2}{3}v_2^{\mp \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} \mp 2\psi_2^{\mp \frac{1}{2}}\omega_1^{\pm \frac{3}{2}} \\
& -\frac{2}{3}\psi_2^{\pm \frac{3}{2}}\omega_1^{\mp \frac{1}{2}} \pm 2\psi_1^{\mp \frac{1}{2}}\omega_2^{\pm \frac{3}{2}} + \frac{2}{3}\psi_1^{\pm \frac{3}{2}}\omega_2^{\mp \frac{1}{2}} \mp 4\xi^1 v^{\pm 2} \pm 2\xi^0 v^{\pm 1} \pm 2\xi^{\pm 1} v^0 \\
& \mp 4v^{\pm 2}\mu^{\mp 1} \pm 2v^0\mu^{\pm 1} \pm 2v^{\pm 1}\mu^0 \pm 3\varepsilon^{\pm 1}f^0 \pm \varepsilon^0 f^{\pm 1} + \varepsilon^{\mp 1}f^{\pm 2} \pm 3\phi^{\pm 1}f^0 \\
& \pm \phi^0 f^{\pm 1} \mp \phi^{\mp 1}f^{\pm 2}
\end{aligned} \tag{D.29}$$

$$\begin{aligned}
\partial_\lambda f^{\pm 2} = & 2k\partial_\phi v^{\pm 2} \pm 3v_2^{\pm \frac{1}{2}}\omega_1^{\pm \frac{3}{2}} \pm 3v_1^{\pm \frac{1}{2}}\omega_2^{\pm \frac{3}{2}} \mp \frac{8}{3}v_1^{\pm \frac{1}{2}}\psi_2^{\pm \frac{3}{2}} \mp \frac{8}{3}v_2^{\pm \frac{1}{2}}\psi_1^{\pm \frac{3}{2}} \\
& + 4\psi_2^{\pm \frac{1}{2}}\omega_1^{\pm \frac{3}{2}} \mp \frac{4}{3}\psi_2^{\pm \frac{3}{2}}\omega_1^{\pm \frac{1}{2}} - 4\psi_1^{\pm \frac{1}{2}}\omega_2^{\pm \frac{3}{2}} - \frac{4}{3}\psi_1^{\pm \frac{3}{2}}\omega_2^{\pm \frac{1}{2}} \mp 16\xi^0 v^{\pm 2} \\
& \mp 4\xi^{\pm 1} v^{\pm 1} + 16v^{\pm 2}\mu^0 \mp 4v^{\pm 1}\mu^{\pm 1} \pm 4\varepsilon^{\pm 1}f^{\pm 1} \pm 2\varepsilon^0 f^{\pm 2} \pm 4\phi^{\pm 1}f^{\pm 1} \\
& \pm 2\phi^0 f^{\pm 2}
\end{aligned} \tag{D.30}$$

$$\begin{aligned}
\partial_\lambda v_M^{\pm \frac{1}{2}} = & \mp k\partial_\phi v_M^{\pm \frac{1}{2}} + \frac{10}{3}\xi^0 v_M^{\mp \frac{1}{2}} + \frac{10}{3}\xi^{\pm 1} v_M^{\mp \frac{1}{2}} \mp 4\xi^{\mp 1}\omega_M^{\pm \frac{3}{2}} \mp \frac{8}{3}\xi^0 \omega_M^{\pm \frac{1}{2}} \\
& \mp \frac{4}{3}\xi^{\pm 1}\omega_M^{\mp \frac{1}{2}} \mp \frac{8}{9}v^{\mp 1}\psi_M^{\pm \frac{3}{2}} \mp \frac{16}{9}v^0\psi_M^{\pm \frac{1}{2}} \mp \frac{8}{3}v^{\pm 1}\psi_M^{\mp \frac{1}{2}} \mp \frac{32}{9}v^{\pm 2}\psi_M^{\mp \frac{3}{2}} \\
& -\rho v_M^{\pm \frac{1}{2}} \pm \frac{1}{2}\varepsilon^0 v_M^{\pm \frac{1}{2}} \pm \varepsilon^{\pm 1} v_M^{\mp \frac{1}{2}} \pm \frac{5}{6}\phi^0 v_M^{\pm \frac{1}{2}} \pm \frac{5}{3}\phi^{\pm 1} v_M^{\mp \frac{1}{2}} \mp \eta v_M^{\pm \frac{1}{2}} \\
& \mp \frac{16}{9}\phi^{\mp 1}\psi_M^{\pm \frac{3}{2}} - \frac{16}{9}\phi^0\psi_M^{\pm \frac{1}{2}} - \frac{16}{9}\phi^{\pm 1}\psi_M^{\mp \frac{1}{2}} - 2f^{\mp 1}\omega_M^{\pm \frac{3}{2}} - 2f^0\omega_M^{\pm \frac{1}{2}} \\
& - 2f^{\pm 1}\omega_M^{\mp \frac{1}{2}} - 2f^{\pm 2}\omega_M^{\mp \frac{3}{2}} + 2\mu^0 v_M^{\pm \frac{1}{2}} + 2\mu^{\pm 1} v_M^{\mp \frac{1}{2}}
\end{aligned} \tag{D.31}$$

$$\begin{aligned}
\partial_\lambda \psi_M^{\pm \frac{1}{2}} = & \mp \frac{3}{8}k\omega_M^{\pm \frac{1}{2}} \mp 2\xi^0 v_M^{\pm \frac{1}{2}} \pm \xi^{\pm 1} v_M^{\mp \frac{1}{2}} - \frac{3}{4}\xi^{\mp 1}\omega_M^{\pm \frac{3}{2}} + \frac{1}{4}\xi^0 \omega_M^{\pm \frac{1}{2}} + \frac{1}{2}\xi^{\pm 1}\omega_M^{\mp \frac{1}{2}} \\
& \mp \frac{1}{2}v^0 v_M^{\pm \frac{1}{2}} \mp \frac{3}{4}v^{\pm 1} v_M^{\mp \frac{1}{2}} - \frac{2}{3}v^{\mp 1}\psi_M^{\pm \frac{3}{2}} - \frac{2}{3}v^0\psi_M^{\pm \frac{1}{2}} + \frac{4}{3}v^{\pm 2}\psi_M^{\mp \frac{3}{2}} \\
& - \frac{1}{2}\phi^0 v_M^{\pm \frac{1}{2}} + \frac{1}{2}\phi^{\pm 1} v_M^{\mp \frac{1}{2}} \mp \frac{3}{8}\eta \omega_M^{\pm \frac{1}{2}} - \rho \psi_M^{\pm \frac{1}{2}} \pm \varepsilon^{\mp 1}\psi_M^{\pm \frac{3}{2}} \pm \frac{1}{2}\varepsilon^0 \psi_M^{\pm \frac{1}{2}} - 2\varepsilon^{\pm 1}\psi_M^{\mp \frac{1}{2}} \\
& \mp \frac{1}{3}\phi^{\mp 1}\psi_M^{\pm \frac{3}{2}} \pm \frac{1}{6}\phi^0\psi_M^{\pm \frac{1}{2}} \pm \frac{2}{3}\phi^{\pm 1}\psi_M^{\mp \frac{1}{2}} \pm \frac{3}{2}f^0 v_M^{\pm \frac{1}{2}} - \frac{3}{2}f^{\pm 1} v_M^{\mp \frac{1}{2}} + \frac{3}{2}f^{\mp 1}\omega_M^{\pm \frac{3}{2}} \\
& \mp \frac{3}{4}f^0 \omega_M^{\pm \frac{1}{2}} \pm \frac{3}{4}f^{\pm 2}\omega_M^{\mp \frac{3}{2}} - \frac{9}{4}\mu^{\mp 1}\omega_M^{\pm \frac{3}{2}} + \frac{3}{4}\mu^0 \omega_M^{\pm \frac{1}{2}} + \frac{3}{2}\mu^{\pm 1}\omega_M^{\mp \frac{1}{2}}
\end{aligned} \tag{D.32}$$

$$\begin{aligned}
\partial_\lambda \psi_M^{\pm \frac{3}{2}} = & \pm \frac{9}{8}k\omega_M^{\pm \frac{3}{2}} \mp 3\xi^{-1} v_M^{\pm \frac{1}{2}} - \frac{9}{4}\xi^0 \omega_M^{\pm \frac{3}{2}} - \frac{3}{4}\xi^{\pm 1}\omega_M^{\pm \frac{1}{2}} \pm \frac{3}{4}v^{\pm 1} v_M^{\pm \frac{1}{2}} \\
& \pm 3v^{\pm 2} v_M^{\mp \frac{1}{2}} + \frac{2}{3}v^0 \psi_M^{\pm \frac{3}{2}} + 2v^{\pm 1}\psi_M^{\pm \frac{1}{2}} + 4v^{\pm 2}\psi_M^{\mp \frac{1}{2}} + \frac{3}{2}\phi^{\pm 1} v_M^{\pm \frac{1}{2}} \pm \frac{9}{8}\eta \omega_M^{\pm \frac{3}{2}} \\
& - \rho \psi_M^{\pm \frac{3}{2}} \pm \frac{3}{2}\varepsilon^0 \psi_M^{\pm \frac{3}{2}} \pm 3\varepsilon^{\pm 1}\psi_M^{\pm \frac{1}{2}} \pm \frac{1}{2}\phi^0 \psi_M^{\pm \frac{3}{2}} \pm \phi^{\pm 1}\psi_M^{\pm \frac{1}{2}} - \frac{3}{2}f^{\pm 1} v_M^{\pm \frac{1}{2}} \\
& - \frac{3}{2}f^{\pm 2} v_M^{\mp \frac{1}{2}} + \frac{9}{4}f^0 \omega_M^{\pm \frac{3}{2}} \mp \frac{3}{2}f^{\pm 1}\omega_M^{\pm \frac{1}{2}} \mp \frac{3}{4}f^{\pm 2}\omega_M^{\mp \frac{1}{2}} - \frac{27}{4}\mu^0 \omega_M^{\pm \frac{3}{2}} \\
& - \frac{9}{4}\mu^{\pm 1}\omega_M^{\pm \frac{1}{2}}.
\end{aligned} \tag{D.33}$$

D.4 The $\mathfrak{sl}(3|2)$ Lie Superalgebra Valued Gauge Parameter λ

This appendix contains an explicit calculated form of the $\mathfrak{sl}(3|2)$ Lie superalgebra valued gauge parameter λ as:

$$\begin{aligned}
\lambda = & b^{-1} \left[\rho J + \varepsilon L_1 + \phi A_1 + v W_2 + \zeta_- G_{\frac{1}{2}}^- + \zeta_+ G_{\frac{1}{2}}^+ + \omega_- S_{\frac{3}{2}}^- + \omega_+ S_{\frac{3}{2}}^+ \right. \\
& + \left(-\frac{\varepsilon \mathcal{G}_-}{2k} - \frac{5\phi \mathcal{G}_-}{6k} - \frac{20v \mathcal{S}_-}{3k} - \frac{2\mathcal{A}\omega_-}{k} + \frac{\mathcal{J}\zeta_-}{2k} - \zeta'_- \right) G_{-\frac{1}{2}}^- \\
& + \left(\frac{\varepsilon \mathcal{G}_+}{2k} + \frac{5\phi \mathcal{G}_+}{6k} - \frac{20v \mathcal{S}_+}{3k} + \frac{2\mathcal{A}\omega_+}{k} - \frac{\mathcal{J}\zeta_+}{2k} - \zeta'_+ \right) G_{-\frac{1}{2}}^+ \\
& + \frac{1}{4} L_0 \left(-\frac{15\omega_- \mathcal{G}_+}{2k} - \frac{15\mathcal{G}_- \omega_+}{2k} - 4\varepsilon' \right) - W_1 v' + \frac{1}{4} A_0 \left(\frac{9\omega_- \mathcal{G}_+}{2k} \right. \\
& + \left. \frac{9\mathcal{G}_- \omega_+}{2k} - 4\phi' \right) + S_{\frac{1}{2}}^- \left(\frac{4v \mathcal{G}_-}{3k} + \frac{\mathcal{J}\omega_-}{2k} - \omega'_- \right) + S_{\frac{1}{2}}^+ \left(-\frac{4v \mathcal{G}_+}{3k} \right. \\
& - \left. \frac{\mathcal{J}\omega_+}{2k} - \omega'_+ \right) + \frac{1}{8} L_{-1} \left(\frac{8\mathcal{L}\varepsilon}{k} - \frac{80Wv}{k} + \frac{8\mathcal{A}\phi}{k} - \frac{5\mathcal{J}\omega_- \mathcal{G}_+}{2k^2} \right. \\
& - \frac{4\zeta_- \mathcal{G}_+}{k} + \frac{15\omega_- \mathcal{S}_+}{k} + \frac{5\mathcal{J}\mathcal{G}_- \omega_+}{2k^2} + \frac{15\mathcal{S}_- \omega_+}{k} + \frac{4\mathcal{G}_- \zeta_+}{k} + \frac{15\omega_+ \mathcal{G}'_-}{2k} \\
& + \left. \frac{25\mathcal{G}_+ \omega'_-}{2k} + \frac{15\omega_- \mathcal{G}'_+}{2k} + \frac{25\mathcal{G}_- \omega'_+}{2k} + 4\varepsilon'' \right) + \frac{1}{4} W_0 \left(\frac{8\mathcal{A}v}{k} + \frac{8\mathcal{L}v}{k} \right. \\
& + \left. \frac{3\omega_- \mathcal{G}_+}{2k} - \frac{3\mathcal{G}_- \omega_+}{2k} + 2v'' \right) + \frac{1}{8} A_{-1} \left(\frac{8\mathcal{A}\varepsilon}{k} - \frac{80Wv}{k} + \frac{8\mathcal{L}\phi}{k} \right. \\
& + \frac{3\mathcal{J}\omega_- \mathcal{G}_+}{2k^2} - \frac{45\omega_- \mathcal{S}_+}{k} - \frac{3\mathcal{J}\mathcal{G}_- \omega_+}{2k^2} - \frac{45\mathcal{S}_- \omega_+}{k} - \frac{9\omega_+ \mathcal{G}'_-}{2k} \\
& - \left. \frac{15\mathcal{G}_+ \omega'_-}{2k} - \frac{9\omega_- \mathcal{G}'_+}{2k} - \frac{15\mathcal{G}_- \omega'_+}{2k} + 4\phi'' \right) + \frac{1}{6} S_{-\frac{1}{2}}^- \left(\frac{3\omega_- \mathcal{J}^2}{4k^2} \right. \\
& + \frac{2v \mathcal{G}_- \mathcal{J}}{k^2} - \frac{3\omega'_- \mathcal{J}}{k} + \frac{2\phi \mathcal{G}_-}{k} - \frac{20v \mathcal{S}_-}{k} + \frac{3\mathcal{A}\omega_-}{k} \\
& + \left. \frac{9\mathcal{L}\omega_-}{k} - \frac{3\omega_- \mathcal{J}'}{2k} - \frac{7\mathcal{G}_- v'}{k} - \frac{4v \mathcal{G}'_-}{k} + 3\omega''_- \right) + \frac{1}{6} S_{-\frac{1}{2}}^+ \left(\frac{3\omega_+ \mathcal{J}^2}{4k^2} \right. \\
& + \frac{2v \mathcal{G}_+ \mathcal{J}}{k^2} + \frac{3\omega'_+ \mathcal{J}}{k} + \frac{2\phi \mathcal{G}_+}{k} + \frac{20v \mathcal{S}_+}{k} + \frac{3\mathcal{A}\omega_+}{k} \\
& + \left. \frac{9\mathcal{L}\omega_+}{k} + \frac{3\omega_+ \mathcal{J}'}{2k} + \frac{7\mathcal{G}_+ v'}{k} + \frac{4v \mathcal{G}'_+}{k} + 3\omega''_+ \right) + \frac{1}{12} W_{-1} \left(\frac{8v \mathcal{G}_- \mathcal{G}_+}{k^2} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{3\mathcal{J}\omega_-\mathcal{G}_+}{2k^2} - \frac{9\omega'_-\mathcal{G}_+}{2k} + \frac{15\omega_-\mathcal{S}_+}{k} + \frac{3\mathcal{J}\mathcal{G}_-\omega_+}{2k^2} - \frac{15\mathcal{S}_-\omega_+}{k} \\
& - \frac{8v\mathcal{A}'}{k} - \frac{8v\mathcal{L}'}{k} - \frac{20\mathcal{A}v'}{k} - \frac{20\mathcal{L}v'}{k} + \frac{3\omega_+\mathcal{G}'_-}{2k} - \frac{3\omega_-\mathcal{G}'_+}{2k} \\
& + \frac{9\mathcal{G}_-\omega'_+}{2k} - 2v^{(3)} \Big) + \frac{1}{54}S_{-\frac{3}{2}}^- \left(\frac{9\omega_-\mathcal{J}^3}{8k^3} + \frac{3v\mathcal{G}_-\mathcal{J}^2}{k^3} - \frac{27\omega'_-\mathcal{J}^2}{4k^2} \right. \\
& + \frac{3\phi\mathcal{G}_-\mathcal{J}}{k^2} - \frac{30v\mathcal{S}_-\mathcal{J}}{k^2} + \frac{21\mathcal{A}\omega_-\mathcal{J}}{2k^2} + \frac{63\mathcal{L}\omega_-\mathcal{J}}{2k^2} - \frac{27\omega_-\mathcal{J}'\mathcal{J}}{4k^2} \\
& - \frac{33\mathcal{G}_-v'\mathcal{J}}{2k^2} - \frac{12v\mathcal{G}'_-\mathcal{J}}{k^2} + \frac{27\omega''_-\mathcal{J}}{2k} + \frac{40\mathcal{A}v\mathcal{G}_-}{k^2} + \frac{72\mathcal{L}v\mathcal{G}_-}{k^2} \\
& - \frac{90\varepsilon\mathcal{S}_-}{k} - \frac{30\phi\mathcal{S}_-}{k} - \frac{180\mathcal{W}\omega_-}{k} + \frac{24\mathcal{A}\zeta_-}{k} + \frac{18\mathcal{G}_-\omega_-\mathcal{G}_+}{k^2} \\
& + \frac{9\mathcal{G}_-^2\omega_+}{k^2} - \frac{9\omega_-\mathcal{A}'}{k} - \frac{6v\mathcal{G}_-\mathcal{J}'}{k^2} - \frac{27\omega_-\mathcal{L}'}{k} + \frac{120\mathcal{S}_-v'}{k} \\
& - \frac{18\mathcal{G}_-\phi'}{k} - \frac{6\phi\mathcal{G}'_-}{k} + \frac{33v'\mathcal{G}'_-}{k} + \frac{60v\mathcal{S}'_-}{k} - \frac{21\mathcal{A}\omega'_-}{k} - \frac{63\mathcal{L}\omega'_-}{k} \\
& + \frac{27\mathcal{J}'\omega'_-}{2k} + \frac{9\omega_-\mathcal{J}''}{2k} + \frac{27\mathcal{G}_-v''}{k} + \frac{12v\mathcal{G}''_-}{k} - 9\omega_-(^{(3)}) \Big) \\
& + \frac{1}{54}S_{-\frac{3}{2}}^+ \left(-\frac{9\omega_+\mathcal{J}^3}{8k^3} - \frac{3v\mathcal{G}_+\mathcal{J}^2}{k^3} - \frac{27\omega'_+\mathcal{J}^2}{4k^2} - \frac{3\phi\mathcal{G}_+\mathcal{J}}{k^2} \right. \\
& - \frac{30v\mathcal{S}_+\mathcal{J}}{k^2} - \frac{21\mathcal{A}\omega_+\mathcal{J}}{2k^2} - \frac{63\mathcal{L}\omega_+\mathcal{J}}{2k^2} - \frac{27\omega_+\mathcal{J}'\mathcal{J}}{4k^2} - \frac{33\mathcal{G}_+v'\mathcal{J}}{2k^2} \\
& - \frac{12v\mathcal{G}'_+\mathcal{J}}{k^2} - \frac{27\omega''_+\mathcal{J}}{2k} + \frac{9\omega_-\mathcal{G}_+^2}{k^2} - \frac{40\mathcal{A}v\mathcal{G}_+}{k^2} - \frac{72\mathcal{L}v\mathcal{G}_+}{k^2} \\
& - \frac{90\varepsilon\mathcal{S}_+}{k} - \frac{30\phi\mathcal{S}_+}{k} + \frac{180\mathcal{W}\omega_+}{k} + \frac{18\mathcal{G}_-\mathcal{G}_+\omega_+}{k^2} - \frac{24\mathcal{A}\zeta_+}{k} \\
& - \frac{9\omega_+\mathcal{A}'}{k} - \frac{6v\mathcal{G}_+\mathcal{J}'}{k^2} - \frac{27\omega_+\mathcal{L}'}{k} - \frac{120\mathcal{S}_+v'}{k} - \frac{18\mathcal{G}_+\phi'}{k} - \frac{6\phi\mathcal{G}'_+}{k} \\
& - \frac{33v'\mathcal{G}'_+}{k} - \frac{60v\mathcal{S}'_+}{k} - \frac{21\mathcal{A}\omega'_+}{k} - \frac{63\mathcal{L}\omega'_+}{k} - \frac{27\mathcal{J}'\omega'_+}{2k} - \frac{9\omega_+\mathcal{J}''}{2k} \\
& - \frac{27\mathcal{G}_+v''}{k} - \frac{12v\mathcal{G}''_+}{k} - 9\omega_+^{(3)} \Big) + \frac{1}{48}W_{-2} \left(\frac{48v\mathcal{A}^2}{k^2} + \frac{96\mathcal{L}v\mathcal{A}}{k^2} \right. \\
& + \frac{27\omega_-\mathcal{G}_+\mathcal{A}}{2k^2} - \frac{27\mathcal{G}_-\omega_+\mathcal{A}}{2k^2} + \frac{32v''\mathcal{A}}{k} + \frac{120\mathcal{W}\varepsilon}{k} + \frac{48\mathcal{L}^2v}{k^2} \\
& + \frac{120\mathcal{W}\phi}{k} + \frac{10v\mathcal{S}_-\mathcal{G}_+}{k^2} + \frac{9\mathcal{J}^2\omega_-\mathcal{G}_+}{8k^3} + \frac{45\mathcal{L}\omega_-\mathcal{G}_+}{2k^2} + \frac{10v\mathcal{G}_-\mathcal{S}_+}{k^2} \\
& + \frac{15\mathcal{J}\omega_-\mathcal{S}_+}{k^2} - \frac{30\zeta_-\mathcal{S}_+}{k} - \frac{9\mathcal{J}^2\mathcal{G}_-\omega_+}{8k^3} - \frac{45\mathcal{L}\mathcal{G}_-\omega_+}{2k^2} + \frac{15\mathcal{J}\mathcal{S}_-\omega_+}{k^2} \\
& - \frac{30\mathcal{S}_-\zeta_+}{k} - \frac{15\omega_-\mathcal{G}_+\mathcal{J}'}{4k^2} - \frac{15\mathcal{G}_-\omega_+\mathcal{J}'}{4k^2} - \frac{29\mathcal{G}_-v'\mathcal{J}}{k^2} + \frac{28\mathcal{A}'v'}{k} \\
& + \frac{28\mathcal{L}'v'}{k} - \frac{14v\mathcal{G}_+\mathcal{G}'_-}{k^2} - \frac{3\mathcal{J}\omega_+\mathcal{G}'_-}{2k^2} + \frac{15\omega_+\mathcal{S}'_-}{k} - \frac{6\mathcal{J}\mathcal{G}_+\omega'_-}{k^2} \\
& - \frac{45\mathcal{S}_+\omega'_-}{k} - \frac{14v\mathcal{G}_-\mathcal{G}'_+}{k^2} - \frac{3\mathcal{J}\omega_-\mathcal{G}'_+}{2k^2} + \frac{6\omega'_-\mathcal{G}'_+}{k} - \frac{15\omega_-\mathcal{S}'_+}{k}
\end{aligned}$$

$$\begin{aligned}
& -\frac{6\mathcal{J}\mathcal{G}_-\omega'_+}{k^2} + \frac{45\mathcal{S}_-\omega'_+}{k} - \frac{6\mathcal{G}'_-\omega'_+}{k} + \frac{8v\mathcal{A}''}{k} + \frac{8v\mathcal{L}''}{k} + \frac{32\mathcal{L}v''}{k} \\
& -\frac{3\omega_+\mathcal{G}''_-}{2k} + \frac{9\mathcal{G}_+\omega''_-}{k} + \frac{3\omega_-\mathcal{G}''_+}{2k} - \frac{9\mathcal{G}_-\omega''_+}{k} + 2v^{(4)} \bigg) \Big] b. \tag{D.34}
\end{aligned}$$



CURRICULUM VITAE

Name Surname: Aytül FİLİZ

EDUCATION:

- **B.Sc.:** Istanbul University, Faculty of Science, Department of Physics
- **B.Sc.:** Istanbul University, Faculty of Science, Department of Mathematics (Double Major Degree)
- **M.Sc.:** Istanbul Technical University, Faculty of Science and Letters, Department of Physics Engineering

PROFESSIONAL EXPERIENCE:

- 2010-2022, Research and Teaching Assistant at Istanbul Technical University, Physics Engineering Department

PUBLICATIONS AND PRESENTATIONS ON THE THESIS:

- H.T. Özer and **A. Filiz**, On the explicit asymptotic $\mathcal{W}_5$ symmetry of 3D Chern-Simons higher spin AdS_3 gravity, J.Math.Phys. 59 (2018) 8, 083504,
- H.T. Özer and **A. Filiz**, Exploring new boundary conditions for $N = (1, 1)$ extended higher-spin AdS_3 supergravity, Eur.Phys.J.C 80 (2020) 11, 1072,
- H.T. Özer and **A. Filiz**, $N = (2, 2)$ extended $\text{sl}(3|2)$ Chern–Simons AdS_3 supergravity with new boundaries, Eur. Phys. J. C 82, 472 (2022).
- **A. Filiz**, Asymptotic $\mathcal{W}_N$ Symmetry of AdS_3 Higher Spin Gravity, 16th Workshop on Quantization, Dualities and Integrable Systems (2018, QDIS16)