PORTFOLIO RISK CALCULATION AND STOCHASTIC PORTFOLIO OPTIMIZATION BY A COPULA BASED APPROACH

by

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ABSTRACT

PORTFOLIO RISK CALCULATION AND STOCHASTIC PORTFOLIO OPTIMIZATION BY A COPULA BASED APPROACH

In this study we used copulas to calculate the risks of stock portfolios and developed a stochastic portfolio optimization model using copulas to find optimal portfolios. Copula is a multivariate distribution function supported in the unit hypercube. The main advantage of copula is that one can separate the marginals of a multivariate distribution from their dependence structure. Thus it is able to model the marginals separately and choose a copula to represent the dependence structure between them.

Since the portfolio return is a multivariate distribution of individual asset returns, the portfolio return distribution can be modeled by copulas. With this aim, we selected 15 stocks from New York Stock Exchange and constructed different portfolios. Then we modeled the distributions of individual stock returns and fitted a set of copulas to the joint return data. We found that Student-t and Generalized Hyperbolic distributions are very nice models for modeling individual asset returns. We also found that the t-copula is the best copula to represent the dependence structure between stock returns. Therefore we used this model to calculate the risks of portfolios and compared the results of this model with the results of the classical portfolio risk calculation methods.

After the risk calculation, we adopted the copula model to the classical Markowitz portfolio selection problem since the Markowitz optimal portfolio would no longer be optimal. Therefore we transformed the classical quadratic optimization problem into a stochastic optimization problem. We used Nelder-Mead simplex search algorithm to solve this problem and compared our findings with the solution of the classical model.

ÖZET

KOPULA TEMELLİ BİR YAKLAŞIM İLE PORTFÖY RİSK HESAPLAMASI VE RASTLANTISAL PORTFÖY ENİYİLEMESİ

Bu çalışmada hisse senedi portföylerinin risklerini hesaplamak için kopulaları kullandık ve en iyi portföyleri oluşturmak için kopulaları kullanarak rastlantısal bir portföy eniyileme modeli geliştirdik. Kopula birim hiperküpte destekli çok değişkenli bir dağılım işlevidir. Kopulanın esas üstünlüğü, isteyenin çok değişkenli bir dağılımın bileşenlerini bağımlılık yapılarından ayrıştırabilmesidir. Böylece bileşenlerin ayrı olarak modellenmesi ve bağımlılık yapısını temsil etmek için bir kopula seçilmesi mümkündür.

Portföy getirisi tek varlık getirilerinin çok değişkenli bir dağılımı olduğundan, portföy getiri dağılımı kopulalar ile modellenebilir. Bu amaçla, New York Hisse Senedi Borsası'ndan 15 adet hisse senedi seçtik ve çeşitli portföyler oluşturduk. Daha sonra tek hisse senetlerinin dağılımlarını modelledik ve birleşik getiri verisine bir kopulalar kümesini oturttuk. Student'in t ve Genelleştirilmiş Hiperbolik dağılımlarının tek varlık getirilerini modellemek için çok cazip modeller olduğunu gördük. Ayrıca t-kopulanın hisse senedi getirileri arasındaki bağımlılık yapısını temsil eden en iyi kopula olduğunu bulduk. Bu nedenle bu modeli portföylerin risklerini hesaplamak için kullandık ve bu modelin sonuçlarını klasik portföy risk hesaplama yöntemleriyle karşılaştırdık.

Risk hesaplamasından sonra, Markowitz en iyi portföyü artık en iyi olmayacağı için, kopula modelini klasik Markowitz portföy seçimi problemine benimsedik. Bu nedenle klasik karesel eniyileme problemini rastlantısal eniyileme problemine dönüştürdük. Bu problemi çözmek için Nelder-Mead simpleks arama algoritmasını kullandık ve bulgularımızı klasik modelin çözümüyle karşılatırdık.

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LIST OF SYMBOLS/ABBREVIATIONS

C	Copula function
$C_{\rho,n}$	CDF of n-dimensional normal copula
$c_{\rho,n}$	Density of n-dimensional normal copula
$C_{n,\rho,\upsilon}$	CDF of n-dimensional t-copula with υ degrees of freedom
$c_{n,\rho,\upsilon}$	Density of n-dimensional t-copula with υ degrees of freedom
$\hat{C}(u)$	Empirical copula
$\hat{c}(u)$	Empirical copula density
$\tilde{C}$	Subset of available copulas
$Cov(X,Y)$	Covariance of X and Y
∂	Partial derivative
e_i	Unit base vector
ES_α	ES at confidence level α
$f(w)$	Objective function of stochastic optimization for input vector w
$f_a(w)$	Function of approximate portfolio return
$f_e(w)$	Function of exact portfolio return
$\bar{f}$	Average of the objective values in the simplex
F	Multivariate distribution function
F_i	Distribution function of marginal i
F_X	CDF of random variable X
$F(w)$	Response value for input vector w
F_X^{-1}	Quantile function of random variable X
F_{best}	Response value of the best point in the simplex
F_{cont}	Response value of the contraction point in the simplex
F_{exp}	Response value of the expansion point in the simplex
F_{ref}	Response value of the reflection point in the simplex
F_{sworst}	Response value of the second worst point in the simplex
F_{worst}	Response value of the worst point in the simplex
G_X	Mapping function of X

K	Number of assets in the portfolio
L	Loss of the portfolio
$l(\theta)$	Log-likelihood function for copula (ML)
$l_i(\beta_i)$	Log-likelihood function for marginal i
$l(\alpha)$	Log-likelihood function for copula (IFM)
N_n	n-dimensional standard normal distribution
n_{days}	Time horizon for risk calculation
n_{inner}	Sample size for each simulation
n_{outer}	Number of replications of simulation
n_{par}	Number of parameters to estimate for marginal distributions
P_t	Value of portfolio at time t
R	Correlation matrix of asset returns
r	Expected return of portfolio
r_a^t	Arithmetic return of a portfolio from time zero to time t
r_f	Risk-free return
r_g^t	Geometric return of a portfolio from time zero to time t
r_i	Expected return of asset i
r_l^t	Logreturn of a portfolio from time zero to time t
r_m	Return of the market portfolio
S_f	Standard deviation of the objective values in the simplex
S_i	Sharpe ratio for asset i
T_v	t-distribution with v degrees of freedom
$T_{n,\rho,v}$	n-dimensional t-distribution with correlation matrix ρ and v degrees of freedom
$\bar{T}$	Survival function of the t-distribution
VaR_α	VaR at confidence level α
w	Weight vector of portfolio assets
w_i	Relative amount of asset i in the portfolio
W	ARCH or GARCH process
x_{cent}	Centroid of the simplex
x_{cont}	Contraction point in NMSS algorithm
x_{exp}	Expansion point in NMSS algorithm

x_{ref}	Reflection point in NMSS algorithm
x_{best}	Best point in the simplex
x_{sworst}	Second worst point in the simplex
x_{worst}	Worst point in the simplex
z_i	0-1 decision variable about asset i
$\nabla^2 f$	Hessian matrix of f
α	Confidence level for VaR and ES (Chapter 2), parameter vector of the copula for fitting (Chapter 3), reflection coefficient of NMSS (Chapter 5)
β	Regression coefficient (Chapter 3), parameter vector of marginal distributions (Chapter 4), contraction coefficient of NMSS (Chapter 5)
β_i	Parameter vector of marginal i for estimation (Chapter 3), systematic risk of asset i (Chapter 5)
β_0	Intercept for ARCH or GARCH process
β_{1i}	ARCH coefficient for $(t-i)^{th}$ return
β_{2i}	GARCH coefficient for $(t-i)^{th}$ return
γ	Risk measure (Chapter 1), expansion coefficient of NMSS (Chapter 5)
δ	Parameter vector for copula fitting with ML (Chapter3), shrinking factor of NMSS (Chapter 5)
δ_i	Maximum weight of asset i in the portfolio
ε_i	Minimum weight of asset i in the portfolio
$\varepsilon(w)$	Noise of the objective function $f(w)$
θ	The parameter of Archimedean copulas
λ_U	Upper tail dependence coefficient
λ_L	Lower tail dependence coefficient
λ_i	Penalty parameter for constraint i
μ	Mean vector of portfolio assets' returns
μ_i	Mean of the returns of asset i
μ_p	Mean of the portfolio returns

ρ	Correlation matrix of the normal copula and the t-copula
ρ_{ij}	Correlation between asset i and asset j
ρ_S	Spearman's rho rank correlation coefficient
$\hat{\rho_S}$	Sample estimator of ρ_S
$\rho(X,Y)$	Correlation between X and Y
σ_i	Standard deviation of the returns of asset i
σ_{ij}	Covariance between asset i and asset j
σ_p	Standard deviation of the portfolio returns
${}^{t\|t-1}\sigma^2$	Conditional variance
Σ	Covariance of portfolio assets' returns
${}^{t\|t-1}\Sigma$	Conditional covariance matrix
τ	Kendal's tau rank correlation coefficient
$\hat{\tau}$	Sample estimator of τ
υ	Degrees of freedom of the t-distribution, t-copula and χ^2 distribution
Φ	CDF of the standard normal distribution
$\Phi_{\rho,n}$	CDF of n-dimensional standard normal distribution
φ	The generator of Arhimedean copulas
χ^2_υ	Chi-square distribution with υ degrees of freedom
ω	Vector of copula parameters to estimate with minimization of L^2 distance

APT	Arbitrage Pricing Theory
ARCH	Autoregressive Conditional Heteroskedasticity
CAPM	Capital Asset Pricing Model
CCC-GARCH	Constant Conditional Correlation GARCH
CML	Canonical Maximum Likelihood (Chapter 3), Capital Market Line (Chapter 5)
df	Degrees of freedom
ES	Expected Shortfall
EVT	Extreme Value Theory
GARCH	Generalized Autoregressive Conditional Heteroskedasticity

GHD	Generalized Hyperbolic Distribution
IFM	Inference Functions for Margins
ISE	İstanbul Stock Exchange
MIQP	Mixed Integer Quadratic Programming
ML	Maximum Likelihood
MRP	Minimum risk portfolio
MVD	Multivariate Distribution
NMSS	Nelder-Mead Simplex Search
pdf	Probability density function
QP	Quadratic Programming
SD	Standard Deviation
SE	Standard Error
SCL	Security Characteristic Line
SML	Security Market Line
TP	Tangency Portfolio
VaR	Value-at-Risk

1. INTRODUCTION

Today the borders of the financial markets expanded so that people in different parts of the world can invest into the markets of other countries. This makes the financial markets become more dependent to each other and the system more complex. For example a shock in one of the financial markets also affects the other countries' markets. In this complex system, investors are concerned with the risk of their investments and they want to construct portfolios to take less risk and get more profits. Therefore people have developed different models to calculate the risk of financial assets and use it in portfolio selection.

Markowitz [1] became a pioneer for the portfolio risk calculation and portfolio optimization with his seminal paper "Portfolio Selection" in 1952. He introduced the mean-variance rule by which one can diversify to minimize the risk and maximize the expected return of a portfolio assuming that the portfolio logreturns follow a multinormal distribution. Many researchers were inspired by this approach and they developed several models based on his classical model. Sharpe [2] developed Capital Asset Pricing Model (CAPM) and Ross [3] developed Arbitrage Pricing Theory (APT). However the classical model of Markowitz measures the risk of a portfolio by the "variance of expected returns" and this risk measure became inadequate to describe the market risk.

By the 1980's, financial institutions started to search more sophisticated risk measures since the markets were becoming more volatile and the market risk was increasing. Some institutions used Value-at-Risk (VaR) as a new risk measure and implemented sophisticated VaR metrics during the 1980's, but they were only known to professionals within those institutions. During the early 1990's, concerns about the increasing number of financial derivatives and publicized losses spurred the field of financial risk management. JP Morgan popularized VaR to professionals by publishing its *RiskMetrics Technical Document* in 1994 with the methods to measure the portfolio risk and it was recognized by the Basle Committee, which authorized its use by banks for performing regulatory capital calculations [4]. Although Artzner [5] showed that VaR is not a coherent risk measure but the Expected Shortfall (ES) is, VaR is more popular in financial applications.

In RiskMetrics, the market risk of a portfolio can be calculated in three ways; covariance approach, historical simulation and Monte Carlo simulation. The covariance approach is an analytical method and it is very similar to the classical Markowitz model. In historical simulation, the future returns of financial assets are obtained by sampling from their past returns, and applying them to the current price level of assets to obtain different price scenarios. Monte Carlo method generates random market scenarios assuming that the risk factors follow a multivariate normal distribution. Then for each scenario, the profits and the losses of the portfolio are computed and the corresponding VaR (or any other risk measure) is calculated.

The classical approaches above have several deficiencies to estimate the portfolio risk. The Markowitz model is easy to use for portfolio risk calculation because of the nice properties of multinormal distribution. However especially in the last decades the asset returns have shown that they are far from the normal distribution, i.e. they have fat tails and high kurtosis. Also the dependence between the asset returns is assumed to be linear in the multinormal model and it does not take into account the dependencies such as rank correlation and tail dependence. However the extreme co-movements in the stock markets indicate that the dependence between the asset returns are not linear and there exist tail dependencies. Although historical simulation is able to capture the extreme returns better than the multinormal model, the empirical distributions lack of data especially in the tails. Therefore these classical models estimate inaccurate risks. People looked for other models to overcome these problems and they showed that the problems with the classical models can be effectively solved by the copula method.

The idea of copulas was introduced by Sklar [6] in 1959 but it has been used only since the last decade in finance. A copula is a multivariate distribution function defined on the n-dimendional unit hypercube $[0, 1]^n$ with uniformly distributed marginals. Thus the individual marginals of a multivariate distribution can be separated from their dependence structure, and the dependence structure which is represented by a copula. The advantage of the copula is that one can construct a multivariate distribution without concerning the marginals. In terms of finance, the copula method can be applied to model the return distribution of a portfolio of financial assets since the portfolio returns follow a multivariate distribution of individual asset returns. Thus individual asset return distributions can be modeled separately and the dependence between them can be represented by a suitable copula to form the portfolio return distribution.

The aim of this study is two fold; first we are interested in modeling the multivariate return distributions of stock portfolios with the copula method to calculate accurate portfolio risks. For this aim we fitted different copulas to joint return data of arbitrarily chosen stock portfolios and determined suitable copulas for each of them. Then with Monte Carlo simulations we produced several scenarios and calculated the VaR and ES for those portfolios and compared the results with the results of the classical methods.

The classical Markowitz portfolio selection is based on the assumption of multinormally distributed returns. Since the multinormal model is inadequate for risk estimation, the Markowitz optimal portfolio might not be optimal any more. Thus the second aim of the study is to construct optimal portfolios using this alternative risk calculation model.

For fitting models and calculating the risks of different methods, we used the price data of 15 stocks traded in New York Stock Exchange (NYSE) obtained from http://finance.yahoo.com. The statistical software R [7] is used for implementing the methods and analyzing the results. We also benefited from its useful add-on packages for some fitting procedures [8, 9, 10, 11]. The codes are ran on a notebook with 2 GHz Pentium Core 2 Duo processor having 2 GB RAM under Windows Vista.

The thesis is organized as follows: the formal definitions of VaR and ES are given with the basic definitions of financial risk in Chapter 2. We also explain the classical portfolio risk calculation methods within this chapter and prove that the portfolio risk of the classical Markowitz model is an upper bound for the exact portfolio risk which one can calculate only by simulation. The copula method is introduced in Chapter 3 by giving the essentials, i.e. copula families, parameter estimation of copulas and simulation from copulas. We present the results of copula fitting to NYSE data. We calculate the risks of different portfolios by the copula method and compare the results with the results of the classical methods in Chapter 4. The classical Markowitz portfolio optimization problem and CAPM are explained in Chapter 5. We redefine the portfolio optimization problem as a copula based stochastic optimization problem. Using heuristic optimization we solve it and compare the results with the results of the classical approach. Finally in Chapter 6, we draw the conclusions and discuss possible future work.

2. FINANCIAL RISK AND CLASSICAL RISK CALCULATION METHODS

2.1. The Concept of Financial Risk

Risk is a concept that denotes a potential negative impact to an asset or some characteristic of value that may arise from some present process or future event. From the economic point of view, risk is any event or action that may adversely affect an organization's ability to achieve its objectives and execute its strategies [12]. In finance, financial risk is essentially any risk associated with any form of financing.

Risk has two components: *uncertainty* and *exposure*. Uncertainty is referred to the probability of facing the risk. Exposure is the amount of the potential loss if the risk has been faced. For a portfolio of financial assets, the risk comprises of the *systematic risk*, also known as *undiversifiable risk*, and the *unsystematic risk* which is also known as *idiosyncratic risk*, *specific risk* or *diversifiable risk*[13].

Systematic risk is the *market risk* or the risk that cannot be diversified away. It refers to the movements of the whole economy. Even if we have a perfectly diversified portfolio, there is some risk that we cannot avoid. However the systematic risk is not the same for all securities or portfolios. Different companies respond differently to a recession or a booming economy. For example both the automotive industry and the food industry will be affected in a case of recession. But food industry possibly will not be affected as much as automotive industry [14].

Unsystematic risk is the risk associated with individual assets and it differs from asset to asset. Unlike the systematic risk, unsystematic risk can be diversified away by including a number of assets in the portfolio [14].

2.2. Risk Measures

A risk measure is defined as a mapping from a set of random variables to the real numbers. In finance, it is used to determine the amount of cash that is required to make the risk acceptable to the regulator. A risk metric is an interpretation of such a measure. Risk metrics take one of the three forms [13]:

- Metrics that quantify the exposure,
- Metrics that quantify uncertainty,
- Metrics that quantify exposure and uncertainty in some combined manner.

2.2.1. Coherent Risk Measures

According to Artzner et al. [5], a risk measure γ is coherent if it satisfies the properties of monotonicity, sub-additivity, homogeneity, and translational invariance.

- **Monotonicity**: If $X \geq Y$ then $\gamma(X) \leq \gamma(Y)$
- **Sub-additivity**: $\gamma(X+Y) \leq \gamma(X) + \gamma(Y)$
- **Positive Homogeneity**: For all $\lambda \geq 0$, $\gamma(\lambda X) = \lambda\gamma(X)$
- **Translational Invariance**: For all $a \in \mathbb{R}$, $\gamma(X+a) = \gamma(X) - a$

If we explain these four properties in words, monotonicity implies that if the returns of portfolio X are higher than the returns of portfolio Y for all possible risk factor return scenarios, then the risk of portfolio X is less than the risk of portfolio Y. Sub-additivity implies that the risk of the portfolio that includes X and Y is never greater than the risk of X plus the risk of Y. In other words, the risk of the sum of individual assets is smaller than or equal to the sum of their individual risks. Homogeneity implies that if the amount of every position in a portfolio is increased by a certain rate, the risk of the portfolio will increase with the same rate, i.e. twice as large if the positions are doubled. What translational invariance implies is that adding cash into a portfolio decreases its risk by the same amount.

The risks of financial assets can be calculated with different measures. One of the measures is the standard deviation. This is the easiest risk measure to use for the risks of individual assets or portfolios. Despite its simplicity, it is not an ideal risk measure since it penalizes the profits as much as the losses. More reliable risk measures are *Value-at-Risk* and *Expected Shortfall*.

2.2.2. Value-at-Risk

VaR is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over some investment horizon. VaR answers the question: "How much can I lose with α per cent probability over a given time horizon?" [15]. If we denote the losses as positive, the mathematical definition of VaR is:

"Given some confidence level $\alpha \in (0,1)$ the VaR of the portfolio at confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is not larger than $1-\alpha$" [12].

$$VaR_\alpha = \inf\{l \in \mathbb{R}, P(L > l) \leq 1-\alpha\} = \inf\{l \in \mathbb{R}, F_L(l) \geq \alpha\} \tag{2.1}$$

As it can be seen from the above definition, VaR_α is the α quantile of the loss distribution. It is also possible to say that VaR_α is the loss described by the $1-\alpha$ quantile of the return distribution.

It is important for the risk measurement theory that Artzner et al. [5] showed that VaR does not always satisfy the sub-additivity property, thus it is not a coherent risk measure. This means that it is possible to construct two portfolios X and Y such that:

$$VaR(X+Y) \geq VaR(X) + VaR(Y) \tag{2.2}$$

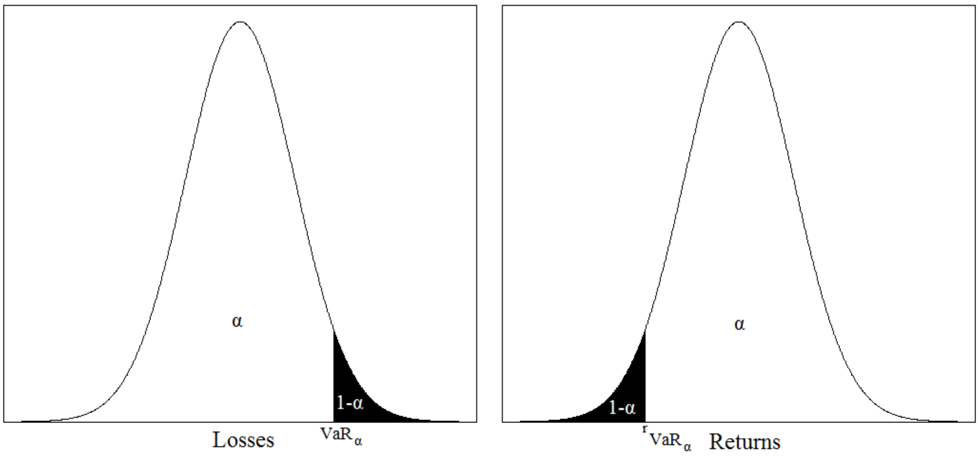

Figure 2.1. VaR for loss (left) and return (right) distributions

This is unexpected because it is hoped that portfolio diversification would reduce the risk. They also showed that ES satisfies all the properties of coherent risk measures. However, VaR is still a popular risk measure and widely used in practical applications.

2.2.3. Expected Shortfall

ES, or alternatively Conditional Value at Risk (CVaR), is the expected amount of loss of a position or a portfolio given that it has exceeded the VaR in some investment horizon under a given confidence level. It is defined mathematically as:

$$ES_\alpha = E(L|L > VaR_\alpha) \tag{2.3}$$

Some properties of ES are:

- Opposite to VaR, ES is a coherent risk measure,
- ES_α increases as α increases,
- ES_0 equals the expected value of the portfolio,
- For a given portfolio, $ES_\alpha \geq VaR_\alpha$.

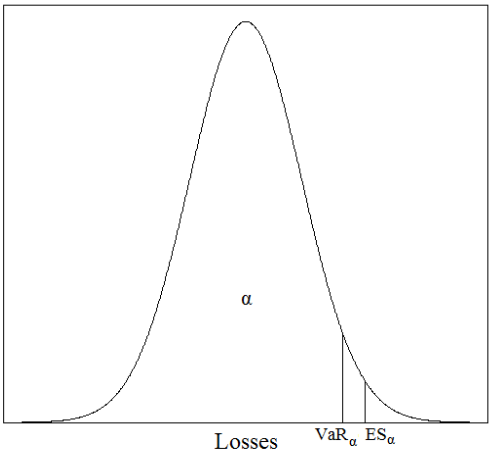

Figure 2.2. ES for loss distribution

2.3. Classical Methods for Portfolio Risk Calculation

Classical methods for portfolio risk calculation can be generalized as parametric and non-parametric methods. For calculating the risk of a portfolio with a parametric method, the multivariate return distribution of portfolio assets must be modeled. In non-parametric methods, the return distributions are not modeled and historical returns are used to calculate the risk.

The returns of financial assets can be calculated in three ways; arithmetic returns, geometric returns, logreturns. If we denote P_1 as the value of a portfolio at time one, then the three returns can be defined as:

$$r_a^1 = \frac{P_1 - P_0}{P_0} = \frac{P_1}{P_0} - 1 \tag{2.4}$$

$$r_g^1 = \frac{P_1}{P_0} \tag{2.5}$$

$$r_l^1 = \log(r_g^1) = \log\left(\frac{P_1}{P_0}\right) = \log(P_1) - \log(P_0) \tag{2.6}$$

where P_0 is the value of the portfolio at time zero. From time zero to time t, the arithmetic return can be written as:

$$r_a^t = \frac{P_t}{P_0} - 1 = \frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{t-2}}\frac{P_{t-2}}{P_{t-3}} \cdots \frac{P_1}{P_0} - 1 = \prod_{i=1}^{t} \frac{P_i}{P_{i-1}} - 1 \tag{2.7}$$

which shows that arithmetic returns cannot be written as the sum of the past but it can be written in terms of the product of the past geometric returns. The logreturn for the same time interval is:

$$r_l^t = \log\left(\frac{P_t}{P_0}\right) = \log\left(\frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{t-2}} \cdots \frac{P_1}{P_0}\right) = \sum_{i=1}^{t} \log\left(\frac{P_i}{P_{i-1}}\right) = \sum_{i=1}^{t} r_l^i \tag{2.8}$$

which shows in this case that logreturns can be summed over a time horizon. This makes the calculation easy if we deal with the logreturns. However when the returns are very close to zero, the difference between the arithmetic returns and the logreturns is small.

In the following sections the classical methods for portfolio risk calculation will be explained. “Return” refers to the term “logreturn”.

2.3.1. Approximate Multinormal Model

This method is also called “variance-covariance” or “mean-variance” approach. It was introduced by Markowitz [1] and was popularized by J.P Morgan in the early 1990’s when they published the *RiskMetrics Technical Document* [15]. This method is the basic parametric approach for portfolio risk calculation. In this method, individual asset returns are assumed to be i.i.d. normal random variables and the portfolio return is assumed to be the weighted sum of the individual asset returns. Thus the portfolio returns follow a multinormal distribution with a mean vector and a covariance matrix. The reason why this method is called “Approximate Multinormal Model” is that the portfolio logreturn is calculated by summing the weighted logreturns of the individual assets although this summing can be done only for the arithmetic returns. Thus the logreturns are approximated as the arithmetic returns since they are very close to each other around zero.

The model is given by Equations (2.9) to (2.12). X_i denotes the random variable for the return distribution of asset i with parameters μ_i and σ_i. Its relative amount (weight) is w_i. X_p is the random variable for the return distribution of the portfolio and it is the weighted sum of the asset returns. Thus X_p follows a multinormal distribution with mean vector μ^T and covariance matrix Σ. The diagonals of Σ are the variances of the marginal distributions. Non-diagonal elements are $Cov(X_i, X_j) = \rho_{ij}\sigma_i\sigma_j$. The parameters of the model; means, variances and the correlations can be easily constructed from the historical data.

$$X_i \sim N(\mu_i, \sigma_i^2) \text{ , } i = 1, \ldots, d \tag{2.9}$$

$$X_p \approx \sum_{i=1}^{d} w_i X_i \;\Rightarrow\; X_p \sim N(\mu_p, \sigma_p^2) \text{ , } \sum_{i=1}^{d} w_i = 1 \tag{2.10}$$

$$\mu_p = \sum_{i=1}^{d} w_i \mu_i \;= w^T \mu \text{ , } \sigma_p^2 = \sum_{i=1}^{d}\sum_{j=1}^{d} w_i w_j \rho_{ij}\sigma_i\sigma_j = w^T \Sigma w \tag{2.11}$$

$$\mu^T = (\mu_1, \ldots, \mu_d) \text{ , } \Sigma = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1d}\sigma_1\sigma_d \\ \vdots & \ddots & \vdots \\ \rho_{d1}\sigma_d\sigma_1 & \cdots & \sigma_d^2 \end{pmatrix} \tag{2.12}$$

According to this approach, the portfolio return can be represented as one-dimensional normal random variable. If the VaR horizon is Δt, the portfolio returns will follow normal distribution with mean $\mu_p \Delta t$ and variance $\sigma_p^2 \Delta t$. Then VaR_α for

Δt can be easily calculated by:

$$VaR_\alpha = P_0(1 - e^{(\mu_p \Delta t + z_{1-\alpha}\sigma_p\sqrt{\Delta t})}) \tag{2.13}$$

where $z_{1-\alpha}$ is the $1-\alpha$ quantile of the standard normal distribution. For example if $P_0 = 1$, $\mu_p = 0$, $\sigma_p = 0.01$ and $\Delta t = 10$ days, then $VaR_{0.99} = 1 - e^{0\times 10 - 2.32\times 0.01\times\sqrt{10}} \simeq 0.0707$, which means that the portfolio loss for 10 days will be less than or equal to 0.0707 with 99 per cent confidence.

2.3.2. Exact Multinormal Model

If the asset logreturns follow a multinormal distribution, the exact portfolio logreturn is:

$$X_p = \log\left(\sum_{i=1}^{d} w_i e^{X_i}\right) \neq \sum_{i=1}^{d} w_i X_i \tag{2.14}$$

No closed form solution for the exact VaR of a portfolio is available for this model. Thus the risk of the portfolio can be calculated only by simulation. Below an algorithm for calculating the return of a d-dimensional portfolio is given. The algorithm consists of the following steps [16]:

1. Generate d-dimensional vector $Z = (Z_1, \ldots, Z_d)$ of d independent standart normal random variates.
2. Find the Cholesky decomposition L of the correlation matrix R such that $R = LL^T$ where L is a lower triangular matrix.
3. Set $X = LZ$

By this algorithm, standard multivariate normal random variates with correlation matrix R can be generated and then the portfolio return can be calculated as:

$$X_p = log(w^T e^{\mu\Delta t + \sigma X\sqrt{\Delta t}}) \tag{2.15}$$

where μ and σ are d-dimensional vectors of the means and the standard deviations of asset returns.

From the empirical results found in this study, the exact multinormal portfolio logreturn was found to be an upper bound for the approximate multinormal portfolio logreturn. Therefore the approximate risk is an upper bound for the exact risk. Thus we have the theorem: (We did not find it in the literature)

Theorem 2.3.1 (Upper bound for the approximate portfolio logreturn) *The exact portfolio logreturn is an upper bound for the approximate portfolio logreturn. Therefore the approximate risk of the portfolio is an upper bound for the exact risk of the portfolio.*

Proof:
Let $w = (w_1, \ldots, w_d)$ be the weight vector of assets in the portfolio and $x = (x_1, \ldots, x_d)$ be the vector of asset logreturns at the end of the risk period. Then at the end of the risk period, the approximate portfolio logreturn will be:

$$f_a(w) = w^T x = w_1 x_1 + w_2 x_2 + \ldots + w_d x_d \tag{2.16}$$

and the exact portfolio logreturn will be:

$$f_e(w) = \log(w^T e^x) = \log(w_1 e^{x_1} + w_2 e^{x_2} + \ldots + w_d e^{x_d}) \tag{2.17}$$

where $w_d = 1 - \sum_{i=1}^{d-1} w_i$. Therefore a d-dimensional portfolio logreturn can be expressed as a $d-1$ dimensional function in terms of the weights:

$$f_a(w) = w^T x = w_1 x_1 + w_2 x_2 + \ldots + (1 - \sum_{i=1}^{d-1} w_i) x_d \tag{2.18}$$

$$f_e(w) = \log(w^T e^x) = \log(w_1 e^{x_1} + w_2 e^{x_2} + \ldots + (1 - \sum_{i=1}^{d-1} w_i) e^{x_d}) \tag{2.19}$$

If all the x_i's are equal ($x_i = x, \forall i$), then $f_a = f_e = x$, and if w_i equals 1 for asset i, then $f_a = f_e = x_i$.

f_a is a hyperplane defined in $[0, 1]^{d-1}$. For the exact returns to be an upper bound for the approximate returns, and thus the approximate risk to be an upper bound for the exact risk, f_e must be equal to f_a at the corner points of the hyperplane and for all other points f_e must be greater than f_a. Since f_a is a linear function of w, f_e must be a concave function of w to satisfy these properties. The necessary and sufficient condition for f_e to be a concave function is that the Hessian matrix of f_e must be negative semi-definite. It is not difficult to show (see Appendix A) that we can write:

$$\nabla^2(f_e) = -\frac{1}{a} A \tag{2.20}$$

where

$$a = \left(w_1 e^{x_1} + w_2 e^{x_2} + \ldots + (1 - \sum_{i=1}^{d-1} w_i) e^{x_d} \right)^2 \tag{2.21}$$

and A is a positive semi-definite matrix. Since a is always non-negative, it is clear that $\nabla^2(f_e)$ is negative semi-definite. $\square$

The multinormal model is widely used in risk estimation and portfolio management because of its nice properties. It is flexible, very easy to use and the speed of calculation is very fast. However there are important drawbacks of this method:

- It is well-known that the asset returns are not normally distributed. They have fat tails and high kurtosis,
- The dependence between the assets is assumed to be linear. But there are non-linear dependencies between the asset returns,
- It does not consider the possibility of extreme joint co-movement of the asset

returns.

2.3.3. Historical Simulation

Historical simulation is a non-parametric method since it does not require any assumption about the distributions of asset returns. In this method, historical returns are used for the estimation of future returns. For example if the risk of a portfolio will be calculated for t days, t realizations from the historical data are selected randomly for each of the assets and used for the next t days' returns. By repeating this several times, a distribution for the profits and the losses of the portfolio is obtained, and the corresponding risk measure is calculated.

This method is easy to implement and it allows for non-normal returns. Risk estimates of the historical simulation are expected to be more accurate than the multinormal model because of having more extremes in the tails. However a typical problem with this approach is that there are not enough data in the tails. Thus the extreme events are possibly underestimated since we restrict ourselves to use only the observed data. The process also assumes that the returns are i.i.d. but there might be serial dependence between the asset returns. Also the data size to be used is an issue. Large samples increase the accuracy of the risk estimates but also increase the probability of using irrelevant data since the further we go into the past for the data, the less relevant this information may be to today's market because of the possible changes in the dependence between the assets.

2.3.4. GARCH Process

In this section, we mainly follow the book of Holton [13].

The multinormal model assumes cross-correlation (cross-covariance) that is the correlation between the asset returns in the portfolio. But another type of correlation might also exist within the returns of the same asset, i.e. the return of an asset at a current time might be affected from the past returns of the same asset. According to this approach, today's return is dependent on the past returns.

To model this situation, Engle [17] proposed *Autoregressive Conditional Heteroskedasticity* (ARCH) model. This model assumes that today's variance depends on the last q errors (returns for financial assets). An ARCH(q) process W has the following conditional distribution:

$$W^t \sim N(0, {}^{t|t-1}\sigma^2) \tag{2.22}$$

$${}^{t|t-1}\sigma^2 = \beta_0 + \sum_{i=1}^{q} \beta_{1i}{}^{t-i}W^2 \tag{2.23}$$

Bollerslev [18] extended this ARCH model by allowing ${}^{t|t-1}\sigma^2$ to also depend on its last p values. This model is called *Generalized Autoregressive Conditional Heteroskedasticity* (GARCH) model. A GARCH(p,q) process W has the following conditional distribution:

$$W^t \sim N(0, {}^{t|t-1}\sigma^2) \tag{2.24}$$

$${}^{t|t-1}\sigma^2 = \beta_0 + \sum_{i=1}^{q} \beta_{1i}{}^{t-i}W^2 + \sum_{j=1}^{p} \beta_{2j}{}^{t-j|t-j-1}\sigma^2 \tag{2.25}$$

In finance, ARCH(1) and GARCH(1,1) models are commonly used. But these models are valid for univariate time series, that is GARCH(1,1) can be used to model the serial dependence for only the return distributions of individual assets. To model the autocorrelation of the individual assets as well as the cross-correlation between them, Bollerslev proposed a multidimensional extension of the univariate GARCH model. This model comprises d univariate GARCH processes ${}^tW_i$, related to one another with a constant conditional correlation matrix ρ. This model is called *Constant*

Conditional Correlation GARCH(CCC-GARCH) and has the following form:

$$W^t \sim {}^{t-1}N_d(0, {}^{t|t-1}\Sigma) \tag{2.26}$$

$${}^{t|t-1}\Sigma = {}^{t|t-1}\sigma\rho{}^{t|t-1}\sigma \tag{2.27}$$

where ρ is the correlation matrix of the asset returns and,

$${}^{t|t-1}\sigma^2 = \begin{pmatrix} \sqrt{{}^{t|t-1}\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \sqrt{{}^{t|t-1}\sigma_2^2} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{{}^{t|t-1}\sigma_d^2} \end{pmatrix} \tag{2.28}$$

with conditional variances ${}^{t|t-1}\sigma_k^2$ modeled with univariate GARCH(p,q) processes:

$${}^{t|t-1}\sigma_k^2 = \beta_{k0} + \sum_{i=1}^{q} \beta_{k1i}{}^{t-i}W_k^2 + \sum_{j=1}^{p} \beta_{k2j}{}^{t-j|t-j-1}\sigma_k^2, \ k = 1, \dots d \tag{2.29}$$

The parameters of the models can be estimated by maximum likelihood.

The idea of GARCH is that it tries to estimate today's volatility by making connections to the past returns and past volatilities. Thus it produces a "memory in volatility" and shows that the volatility exhibits long-range dependence. It explains the extreme movements by the changes in volatility. As a consequence in many of the practical applications the historical data of the assets returns are treated as the "raw" data and the return series of individual assets are "filtered" using univariate GARCH(1,1) processes. Then the filtered data are treated as i.i.d. The multivariate models are built upon these filtered data and after estimating the future returns in terms of the filtered data, these estimated returns are defiltered by the same GARCH processes and the risk estimation is performed.

GARCH models only explain the conditional volatility and not account for the unexpected market risks. But the market risk comprises many sources and the system is very complex to discover. Especially in the last decade, the national and global political and economical developments have caused many shocks in the stock markets. For these reasons Harold and Jianping [19] worked on the effects of the political developments to the stock returns in Hong-Kong stock market. In their work, they used a components-jump volatility filter. Their filter consists of two parts the first of which is a fundamental ARCH derivative filter of volatility to capture the long-term volatility. The second part is a Poisson process which accounts for the extreme returns. The model identifies the dates with jump, or surprises, return movements and associates them with the political news announcements thus allowing them to quantify the return and volatility effects of political events. They found that extreme return jumps in the market were closely associated with political news, and that the impact of these news was asymmetric, that is with bad news having a greater volatility effect relative to good news. At the return level, they also found that the largest market movements in Hong Kong were often associated with major political news.

As it can be seen the extreme movements of the returns might be caused generally by the national political or other economical developments, but not only by the long-term volatility. Especially in the last few years, stock markets have highly been affected by the global news. Thus a shock in one of the world markets also causes extreme losses in other markets. Also for 1 day risk estimation, the only thing GARCH does is to estimate the current volatility and use the classical normal model. But there is no guarantee that there will not be a global or national news that will cause a shock in the stock markets even in one day.

To conclude, using GARCH filters seems a reasonable choice at the first glance since the assumption of i.i.d data is not very realistic. But the filtering is performed for the individual assets and when the copula model is introduced in Chapter 3, it will be seen that wrong filtering would destroy the true dependence structure between the assets. Because of all these reasons, in this work the filtering was not performed for the raw data. Nevertheless CCC-GARCH was used as an alternative method for estimating portfolio risk.

3. COPULA

In this section the copula methodology will be explained. The role of the copula of several random variables is precisely to offer a complete and unique description of the dependence structure, excluding all the information on the marginal distribution of the random variables. The theoretical parts of this chapter mainly follow [20] and [21].

3.1. Introduction

According to the classical normality assumption, the return distribution of a portfolio can fully be described by the mean vector and the covariance matrix of the asset returns. Therefore the dependence between the asset returns is described by the linear correlation coefficient. But it is well-known that the classical linear correlation coefficient is not an adequate measure of dependence between financial assets because there are nonlinear dependencies between the assets.

Before giving the definition of the copula, the definitions of different dependence measures will be informative to better understand the relation between the dependence and copulas.

3.2. Dependence Measures

The notion of independence of random variables is easy to define. Two random variables X and Y are independent if and only if:

$$\mathrm{P}(X \leq x \ and \ Y \leq y) = \mathrm{P}(X \leq x)\mathrm{P}(Y \leq y) \tag{3.1}$$

or equivalently

$$\mathrm{P}(X \leq x|Y) = \mathrm{P}(X \leq x) \tag{3.2}$$

In other words, two random variables are independent if the knowledge of an information about one of them does not bring any new insight about the other one.

If two random variables are not independent, they are dependent. Two random variables X and Y are *mutually completely dependent* if the knowledge of X implies the knowledge of Y. This mean that there is a one-to-one mapping f such that $Y = f(X)$, almost everywhere. The mapping f is either strictly increasing or strictly decreasing. In the first case, X and Y are said to be *comotonic*, in the other case they are said to be *countermonotonic* [20].

There exist different methods to measure the dependence between random variables. In the following sections, these methods will be explained.

3.2.1. Linear Dependence

Linear dependence is measured by the linear correlation coefficient, also called *the Pearson's product-moment coefficient.* Given two random variables X and Y, the linear correlation coefficient is defined as:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X).Var(Y)}} \tag{3.3}$$

provided that $Var(X)$ and $Var(Y)$ exist. $Cov(X,Y)$ is the covariance of X and Y.

$\rho(X,Y)$ is called "linear" correlation coefficient because its knowledge is equivalent to that of the coefficient β of the linear regression $Y = \beta X + \epsilon$, where ϵ is the residual which is linearly uncorrelated to X. The relation between ρ and β is:

$$\rho = \beta\sqrt{Var(X)/Var(Y)} \tag{3.4}$$

The properties of the linear correlation coefficient can be summarized as:

- It varies betweeen -1 and 1; $-1 \leq \rho(X,Y) \leq 1$,
- The linear correlation coefficient is invariant under a monotonic affine change of variables of the form:

$$\left.\begin{array}{l} X' = aX + b \\ Y' = cY + d \end{array}\right\} \Rightarrow \rho(X',Y') = sign(ac)\rho(X,Y)$$

The use of the linear correlation coefficient has some advantages and shortcomings [21]. The advantages are:

- It is often straightforward to calculate the correlation coefficient,
- Correlation and covariance are easy to manipulate under linear operations,
- It is a natural measure of dependence for multivariate normal distributions and, more generally, for multivariate spherical and elliptical distributions.

The shortcomings are:

- The variances of X and Y must be finite or the correlation coefficient cannot be defined. This is not ideal for a dependence measure and might cause problems when working with heavy-tailed distributions.
- If the linear correlation between two random variables is zero, it does not mean that they are independent.
  For example given a random variable w uniformly distributed in $[0, 2\pi]$, for the couple of random variables $(U,V) = (cosw, sinw)$, $\rho(U,V) = 0$ although they are not independent [20].
- The linear correlation coefficient has the serious deficiency that it is not invariant under nonlinear monotonic transforms. As a consequence, it does not give the true dependence between random variables.

Since the financial asset returns are far from the normal distribution and have heavy tails, the use of the linear correlation coefficient to describe the dependence between assets will have the above drawbacks. It is therefore necessary to look for other measures of dependence.

3.2.2. Rank Correlation

A fundamental question for financial risk management is: "Do the prices of two or more assets tend to rise or fall together?". A natural way to find an answer to this question is to compare the probability that they rise (or fall) together with the probability that one of the two assets rises (respectively falls) while the other one falls (respectively rises) [20]. The first claim is called *concordance*, i.e. they move in the same direction without regarding up or down, the second claim is called *discordance*, which is the opposite of concordance. Rank correlation deals with measuring the concordance of random variables. The two most popular rank correlation measures are *Kendall's tau* and *Spearman's rho*.

<u>3.2.2.1. Kendall's Tau.</u> Let us consider a random sample of n observations from a continuous random vector (X, Y). There are $\binom{n}{2}$ distinct pairs (X_i, Y_i) and (X_j, Y_j) of observations in the sample, and each pair is either concordant or discordant. Then an estimate of Kendall's tau rank correlation coefficient for the sample is given by:

$$\hat{\tau} = \binom{n}{2}^{-1} \sum_{i<j} sign\left[(X_i - X_j)\,(Y_i - Y_j)\right] \tag{3.5}$$

For a pair of two independent realizations (X_1, Y_1) and (X_2, Y_2) of the same random variables (X, Y), the mathematical formulation of Kendall's tau is:

$$\tau = \mathrm{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathrm{P}[(X_1 - X_2)(Y_1 - Y_2) < 0] \tag{3.6}$$

The left term on the right-hand-side gives the probability of concordance while the right term on the right-hand-side represents the probability of discordance. For continuous random variables, Kendall's tau can be rewritten as:

$$\tau = 2\text{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - 1 \tag{3.7}$$

From this equation, it can be seen that Kendall's tau varies between -1 and 1. The lower bound is reached if and only if the variables (X, Y) are countermonotonic, while the upper bound is attained if and only if (X, Y) are comotonic.

Given any monotonic mappings G_X and G_Y not necessarily linear, one has

$$\begin{aligned} X_1 \geq X_2 \quad &\Leftrightarrow \quad G_X(X_1) \geq G_X(X_2), \\ Y_1 \geq Y_2 \quad &\Leftrightarrow \quad G_Y(Y_1) \geq G_Y(Y_2). \end{aligned} \tag{3.8}$$

which means that the ranking of the pairs (X_i, Y_i) does not change under monotonic transformations. Therefore Kendall's tau is invariant under monotonic transformation of the marginal distributions. As a consequence, it only depends on the copula of (X, Y) [20].

<u>3.2.2.2. Spearman's rho.</u> Spearman's rho is defined as the difference between the probability of concordance and the probability of discordance for the pairs of random variables (X_1, Y_1) and (X_2, Y_3), where the pairs (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) are three independent realizations drawn from the same distribution.

Let $\{(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\}$ denote a random sample of n observations from a continuous random vector (X, Y). The sample estimator of Spearman's rho $\rho_s(X, Y)$ is defined as:

$$\hat{\rho}_s(X, Y) = \frac{12}{n(n^2 - 1)} \sum_{i=1}^{n} \left(rank(X_i) - \frac{n+1}{2} \right) \left(rank(Y_i) - \frac{n+1}{2} \right) \tag{3.9}$$

For continuous random variables, Spearman's rho can be rewritten as:

$$\rho_s = 3\mathrm{P}[(X_1 - X_2)(Y_1 - Y_3) > 0] - \mathrm{P}[(X_1 - X_2)(Y_1 - Y_3) < 0] \tag{3.10}$$

In fact Spearman's rho is related to the linear correlation of the rank. Considering two random variables X and Y with marginal distributions F_X and F_Y, Spearman's rho equals:

$$\rho_s = \rho(F_X(X), F_Y(Y)) = \frac{Cov(F_X(X), F_Y(Y))}{\sqrt{Var(F_X(X))Var(F_Y(Y))}} \tag{3.11}$$

Kruskal [22] showed that Kendall's tau and Spearman's rho have the following relation:

$$\begin{array}{cccc} \frac{3\tau-1}{2} & \leq \rho_s \leq & -\frac{\tau^2-2\tau-1}{2}, & \tau \geq 0 \\ \frac{\tau^2+2\tau-1}{2} & \leq \rho_s \leq & \frac{3\tau+1}{2}, & \tau \leq 0 \end{array} \tag{3.12}$$

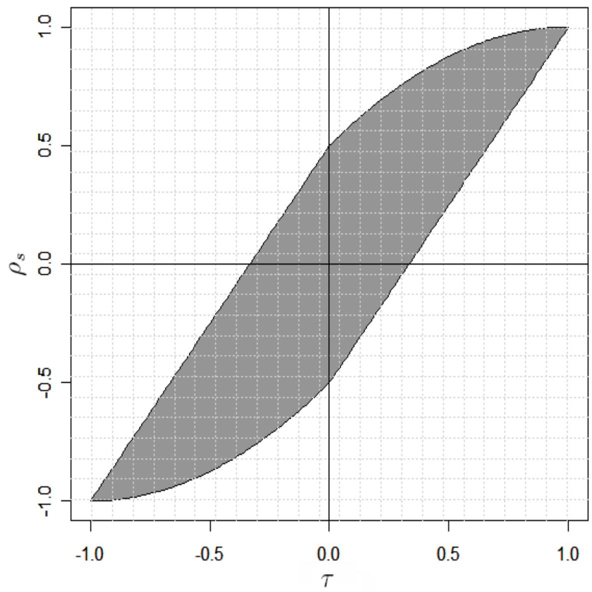

Figure 3.1. The relation between Kendal's tau and Spearman's rho

To summarize the two rank correlation coefficients, Kendall's tau and Spearman's rho enjoy the same set of properties:

1. They are defined for any pair of continuous random variables (X, Y),
2. They are symmetric,

$$\rho_s(X, Y) = \rho_s(Y, X) \tag{3.13}$$

$$\tau(X, Y) = \tau(Y, X) \tag{3.14}$$

3. They range in $[-1, 1]$,

$$-1 \leq \rho_s(X, Y), \tau(X, Y) \leq 1 \tag{3.15}$$

4. They equal zero if and only if X and Y are independent.
5. If the pair of random variables (X_1, X_2) is more dependent than the pair (Y_1, Y_2) in the following sense:

$$C_X(u, v) \geq C_Y(u, v), \forall u, v \in [0, 1], \tag{3.16}$$

then the same ranking holds for any of these two measures.
6. If (X, Y) is comotonic, then $\rho_s(X, Y) = \tau(X, Y) = 1$, if they are countermonotonic, then $\rho_s(X, Y) = \tau(X, Y) = -1$

Any measure of dependence fulfilling these six properties is called a *concordance measure* [20].

3.2.3. Tail Dependence

Tail dependence is a weakened form of positive quadrant dependence to focus on local dependences. For instance, one could wish to focus on the lower left tails of the asset return distributions to find out whether the joint losses appear to occur more likely together than one could expect from statistically independent losses. By definition, the upper tail dependence coefficient is:

$$\lambda_U = \lim_{u \to 1^-} P\left(X > F_X^{-1}(u) | Y > F_Y^{-1}(u)\right) \tag{3.17}$$

In other words, given that Y is very large (at some probability level u), the probability that X is very large at the same probability level u defines asymptotically the upper tail dependence coefficient λ_U. λ_U can also be interpreted in terms of VaR. Indeed, the quantiles $F_X^{-1}(u)$ and $F_Y^{-1}(u)$ are nothing but the VaR of portfolios X and Y at confidence level u if the losses are regarded as positive. Thus the coefficient λ_U provides the probability that X exceeds the VaR at level u, assuming that Y has exceeded the VaR at the same probability level, when this level goes to one [20].

Tail dependence is a copula property which will be introduced within this chapter, i.e. it is independent of the marginals of X and Y. Let C be the copula of the variables X and Y. If their bivariate copula has the limit:

$$\lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u} = \lim_{u \to 1^-} 2 - \frac{\log C(u,u)}{\log u} = \lambda_U \tag{3.18}$$

then C has an upper tail dependence coefficient λ_U. In a similar way, the lower tail dependence can be defined as:

$$\lambda_L = \lim_{u \to 0^+} P\left(X < F_X^{-1}(u) | Y < F_Y^{-1}(u)\right) \tag{3.19}$$

and if C has the limit:

$$\lim_{u \to 0^+} \frac{C(u,u)}{u} = \lambda_L \tag{3.20}$$

then C has a lower tail dependence λ_L [21].

3.3. Definition of the Copula

A function $C : [0,1]^n \rightarrow [0,1]$ is an *n-copula* (n-dimensional copula) if it enjoys the following properties:

- $\forall u \in [0,1], C(1,\ldots,1,u,1,\ldots,1) = u$,
- $\forall u_i \in [0,1], C(u_1,\ldots,u_n) = 0$ if at least one of the u_i's equals zero,
- C is grounded and n-increasing, i.e., the C-volume of every box whose vertices lie in $[0,1]^n$ is positive.

For the 2-dimensional case, these properties become:

- $C(u,1) = u$ and $C(1,v) = v, \forall u, v \in [0,1]$,
- $C(u,0) = C(0,v) = 0, \forall u, v \in [0,1]$,
- $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0,$ for $u_1 \leq u_2$ and $\mathrm{v}_1 \leq v_2$.

The following theorem is known as *Sklar's Theorem.* It is perhaps the most important result regarding copulas and is used in essentially all applications of copulas.

Theorem 3.3.1 (Sklar's Theorem)
Given an n-dimensional distribution function F with marginals $F_1, F_2, \ldots, F_n$, there exist an n-copula $C : [0,1]^n \rightarrow [0,1]$ such that:

$$F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n)) \tag{3.21}$$

If $F_1, F_2, \ldots, F_n$ are all continuous then C is unique; otherwise C is uniquely determined on $RanF_1 \times \ldots \times RanF_n$, where $RanF_i$ is the range of the marginal i. Conversely, if C is an n-copula and $F_1, F_2, \ldots, F_n$ are distribution functions, then the function F defined above is an n-dimensional distribution function with marginals $F_1, F_2, \ldots, F_n$.

Indeed given a multivariate distribution function F with marginals $F_1, F_2, \ldots, F_n$, for any $(u_1, u_2, \ldots, u_n)$ in $[0,1]^n$:

$$C\left(u_1, \ldots, u_n\right) = F\left(F_1^{-1}(u_1), \ldots, F_1^{-1}(u_n)\right) \tag{3.22}$$

is an n-copula.

Theorem 3.3.2 (Invariance Theorem)
If $X_1, X_2, \ldots, X_n$ has copula C, then $Y_1 = h_1(X_1), \ldots, Y_n = h_n(X_n)$ has the same copula C, if Y_i is an increasing function of X_i.

$$C(F_1(x_1), \ldots, F_n(x_n)) = C(h_1(F_1(x_1)), \ldots, h_n(F_n(x_n))) \tag{3.23}$$

By the invariance theorem, it can be seen that the copula is *not* affected by non-linear transformations of the random variables.

From the above definitions, it can be seen that a copula is nothing but a multivariate distribution function supported in $[0,1]^n$ with uniform marginals. From Sklar's Theorem we see that for continuous multivariate distribution functions, the univariate marginals and the multivariate dependence structure can be separated, and the dependence structure can be represented by a copula. Thus copulas are very useful models for representing multivariate distributions with arbitrary marginals. One can model the marginal distributions of a multivariate distribution and find a copula to capture the dependence between the marginals.

$$F(x_1, \ldots, x_n) = \mathrm{P}\left\{X_1 \leq x_1, \ldots, X_n \leq x_n\right\} = C(F_1(x_1), \ldots, F_n(x_n)) \tag{3.24}$$

Once we find a copula for a multivariate distribution, we can switch between the copula and the multivariate distribution environments. Thus working with copulas can be easier than working with multivariate distribution functions. For example if we want to simulate from a multivariate distribution, we can shift to the copula environment and simulate from the copula, then we find the corresponding random variates by transforming them back to the multivariate environment.

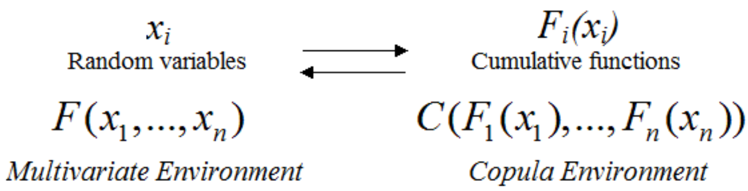

Figure 3.2. The multivariate and copula environments

Due to the property that copulas are n-increasing, an upper and a lower bound can be found for any copulas. Choosing $u_2 = v_2 = 1$ in 2-increasing inequalities, any bivariate copula satisfies:

$$C(u, v) \geq u + v - 1 \tag{3.25}$$

and since a copula must be non-negative, a lower bound is obtained for any bivariate copulas:

$$C(u, v) \geq \max(u + v - 1, 0) \tag{3.26}$$

Similarly, choosing alternatively $(u_1 = 0, v_2 = 1)$ and $(u_2 = 1, v_1 = 0)$, an upper bound is obtained for any bivariate copula:

$$C(u, v) \leq \min(u, v) \tag{3.27}$$

Proposition 3.3.1 (Fréchet-Hoeffding Upper and Lower Bounds)
Given an n-copula C, for all $u_1, \ldots, u_n \in [0, 1]$, the following relation holds for any

copula:

$$\max(u_1 + \ldots + u_n - n + 1, 0) \leq C(u_1, \ldots, u_n) \leq \min(u_1, \ldots, u_n) \tag{3.28}$$

3.3.1. Copula Families

There are mainly two families of copulas used for financial applications:

- Elliptical Copulas
- Archimedean Copulas

<u>3.3.1.1. Elliptical Copulas.</u> Elliptical copulas are derived from multivariate elliptical distributions. The two most important copulas of this family are the normal (Gaussian) copula and the t (Student's) copula. By construction, these two copulas are close to each other in their central parts and become closer in the tails when the number of degrees of freedom of the t-copula increases [20].

The normal copula is an elliptical copula derived from the multivariate normal distribution. Let Φ denote the standard normal distribution function and $\Phi_{\rho,n}$ the n-dimensional standard normal distribution function with correlation matrix ρ. Then, the n-dimensional normal copula with correlation ρ is:

$$C_{\rho,n}(u_1, \ldots, u_n) = \Phi_{\rho,n}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)) \tag{3.29}$$

whose density can be written as:

$$c_{\rho,n}(u_1, \ldots, u_n) = \frac{\partial\, C_{\rho,n}(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n} \tag{3.30}$$

and reads:

$$c_{\rho,n}(u_1,\dots,u_n)=\frac{1}{\sqrt{|\rho|}}\exp\left(-0.5y^T(u)(\rho^{-1}-I)y(u)\right) \tag{3.31}$$

where $y^T(u) = (\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_n))$, I is the identity matrix and $|.|$ is the determinant. Thus a normal copula is completely determined by its correlation matrix ρ [20].

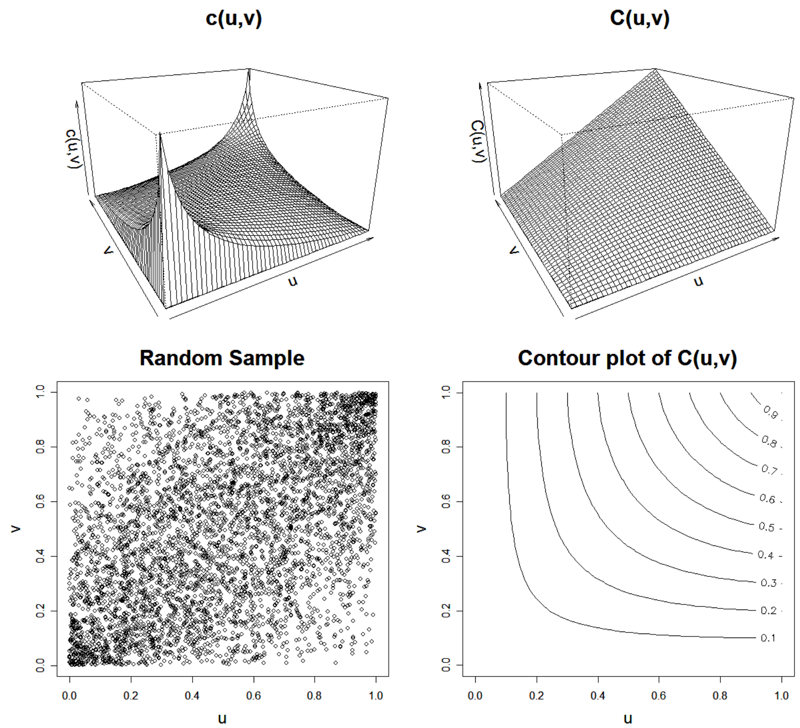

Figure 3.3. The normal copula with $\rho = 0.5$

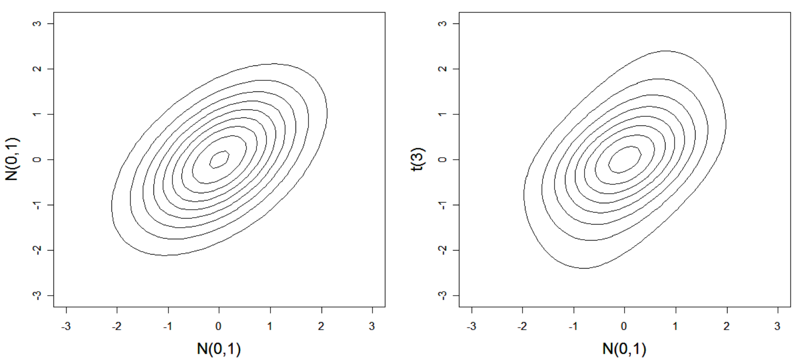

Figure 3.4. MVD's with the normal copula ($\rho = 0.5$) and marginals N(0,1) and t(3)

If X has the stochastic representation:

$$X = \mu + \frac{\sqrt{\upsilon}}{\sqrt{\chi_\upsilon^2}} Z \tag{3.32}$$

where $\mu \in \mathbb{R}^n$, $Z \sim N_n(0, \Sigma)$ is an n-dimensional vector of independent standard normal random variables and χ_υ^2 is a chi-square distribution with υ degrees of freedom, then X has a multivariate t-distribution with mean μ and covariance matrix $\frac{\upsilon}{\upsilon-2}\Sigma$ for $\upsilon > 2$.

The t-copula is an elliptical copula derived from the multivariate t distribution. Let T_υ be the standard t-distribution function with υ degrees of freedom and $T_{n,\rho,\upsilon}$ be the multivariate t distribution function with υ degrees of freedom and shape matrix ρ. Then the corresponding t-copula is:

$$C_{n,\rho,\upsilon}(u_1, \ldots, u_n) = T_{n,\rho,\upsilon}({T_\upsilon}^{-1}(u_1), \ldots, {T_\upsilon}^{-1}(u_n)) \tag{3.33}$$

The density of the t-copula is:

$$c_{n,\rho,\upsilon}(u_1, \ldots, u_n) = \frac{1}{\sqrt{|\rho|}} \frac{\Gamma\left(\frac{\upsilon+n}{2}\right)\left[\Gamma\left(\frac{\upsilon}{2}\right)\right]^{n-1}}{\left[\Gamma\left(\frac{\upsilon+1}{2}\right)\right]^n} \frac{\prod_{k=1}^{n}\left(1+\frac{y_k^2}{\upsilon}\right)^{\frac{\upsilon+1}{2}}}{\left(1+\frac{\mathbf{y}'\rho^{-1}\mathbf{y}}{\upsilon}\right)^{\frac{\upsilon+n}{2}}} \tag{3.34}$$

where $y^T(u) = (T_\upsilon^{-1}(u_1), \ldots, T_\upsilon^{-1}(u_n))$.

Since the t-distribution tends to the normal distribution when υ goes to infinity, the t-copula also tends to the normal copula as $\upsilon \to +\infty$ [20].

$$\upsilon \to +\infty \Rightarrow \sup_{u \in [0,1]^n} |C_{n,\rho,\upsilon}(u) - C_{\rho,n}(u)| \to 0 \tag{3.35}$$

The description of the t-copula relies on two parameters: the correlation matrix

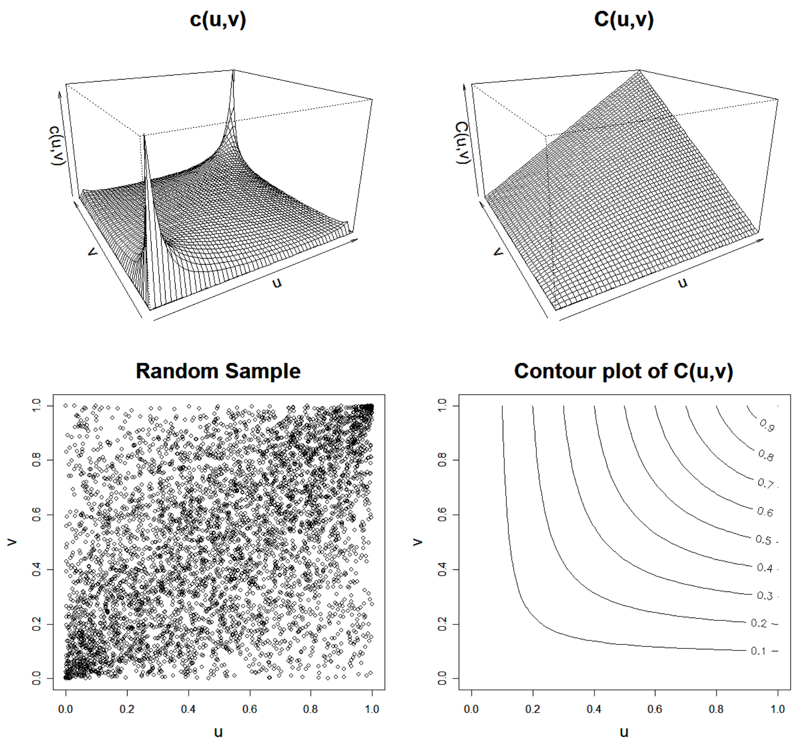

Figure 3.5. The t-copula with $\rho = 0.5$ and $\upsilon = 3$

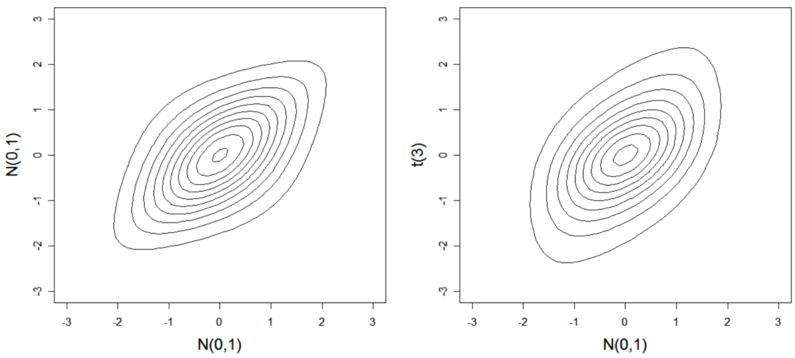

Figure 3.6. MVD's with the t-copula ($\rho = 0.5$, $\upsilon = 3$) and marginals N(0,1) and t(3)

ρ as for the normal copula, and in addition the number of degrees of freedom υ. An accurate estimation of the parameter υ is rather difficult and this can have an important impact on the estimated value of the shape matrix. As a consequence, the t-copula may be more difficult to calibrate than the normal copula [20].

3.3.1.2. Archimedean Copulas. This copula family includes a very large number of copulas enjoying a certain number of interesting properties. They allow for a great variety of different dependence structures. In contrast to elliptical copulas, all commonly encountered Archimedean copulas have closed form expressions [20].

Definition 3.3.1 Archimedean Copula
An Archimedean copula has the following form:

$$C(u_1, \ldots, u_n) = \varphi^{-1}(\varphi(u_1) + \ldots + \varphi(u_n)) \tag{3.36}$$

for all $0 \leq u_1, \ldots, u_n \leq 1$ *and where* φ *is a function called the generator, satisfying the following properties:*

1. $\varphi : [0, 1] \to [0, \infty]$,
2. $\varphi(0) = \infty, \varphi(1) = 0$,
3. *for all* $t \in (0, 1)$, $\varphi'(t) < 0$, *i.e.* φ *is decreasing,*
4. *for all* $t \in (0, 1)$, $\varphi''(t) \geq 0$, *i.e.* φ *is convex.*

It is not necessary for $\varphi(0)$ to be infinite for φ to generate a copula. When $\varphi(0)$ is finite, the Archimedean copula generated by φ is given by a pseudo-inverse of φ:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0) \\ 0, & t \geq \varphi(0) \end{cases} \tag{3.37}$$

For Archimedean copulas, the complexity of the dependence structure between n variables, usually described by an n-dimensional function, is reduced and embedded into the function of a single variable, the generator φ [20].

Among the large number of copulas in the Archimedean family, Clayton, Gumbel, Frank and Ali-Mikhail-Haq copulas are best known. For these four copulas, their parameter ranges and the generators are given in Table 3.1.

Table 3.1. Parameter ranges and generators of archimedean copulas

Copula	Parameter Range	$\varphi(\theta)$
Clayton	$[-1,\infty)\backslash\{0\}$	$\frac{t^{-\theta}-1}{\theta}$
Gumbel	$[1,\infty)$	$(-\ln t)^{\theta}$
Frank	$(-\infty,\infty)\backslash\{0\}$	$-\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$
Ali-Mikhail-Haq	$[-1,1)$	$\ln\left(\frac{1-\theta(1-t)}{t}\right)$

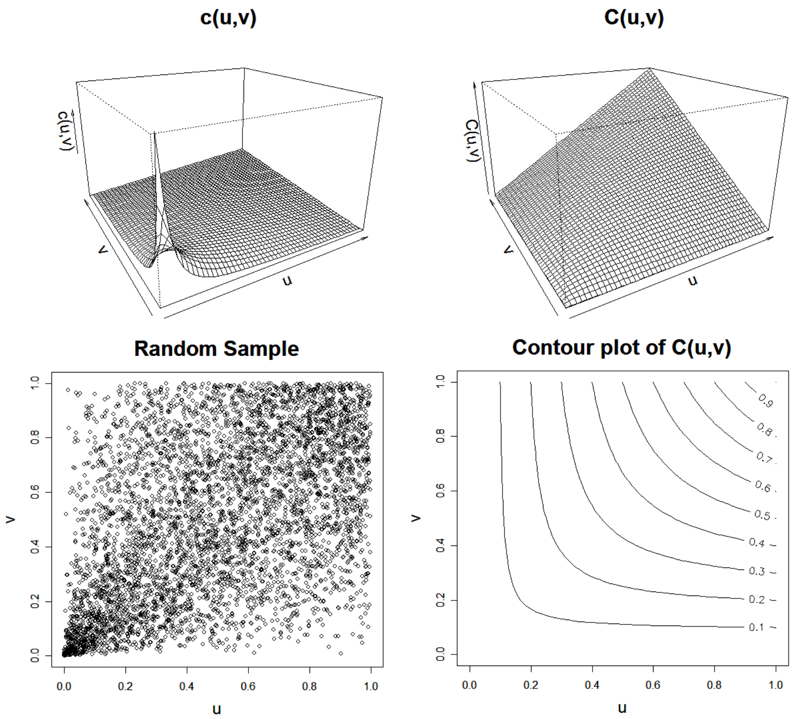

Figure 3.7. Clayton copula with $\theta = 1$

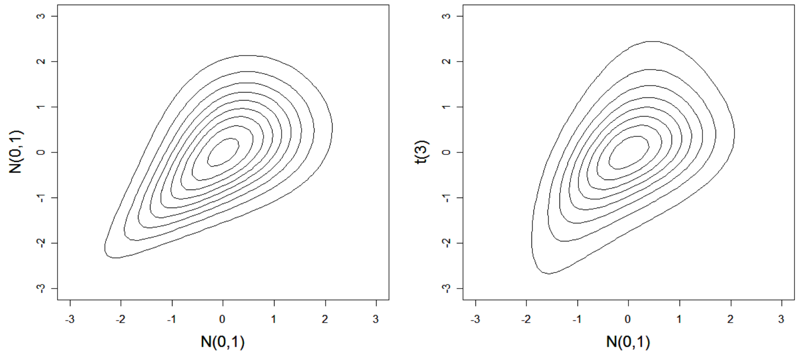

Figure 3.8. MVD's with Clayton copula ($\theta = 1$) and marginals N(0,1) and t(3)

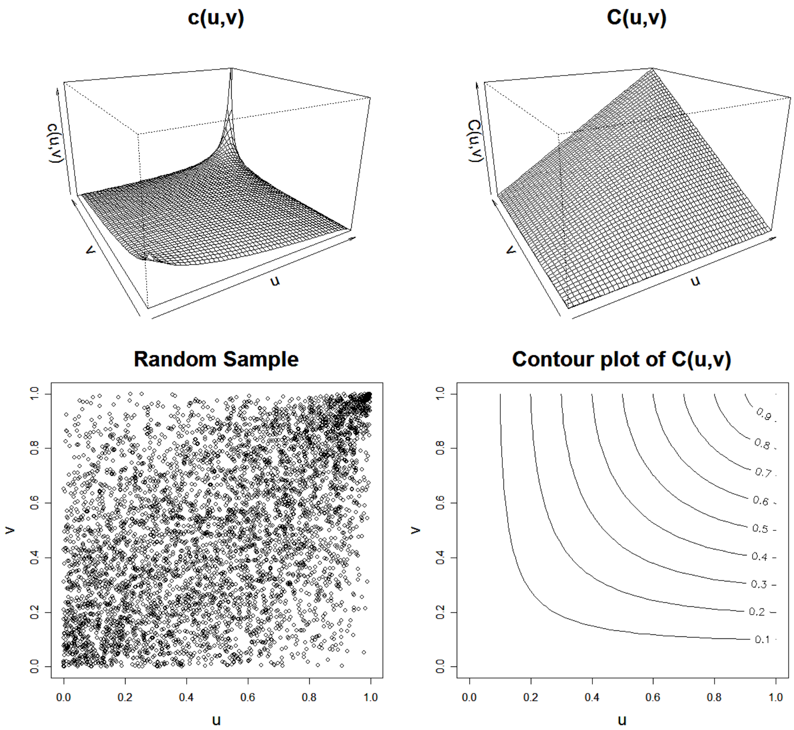

Figure 3.9. Gumbel copula with $\theta = 1.5$

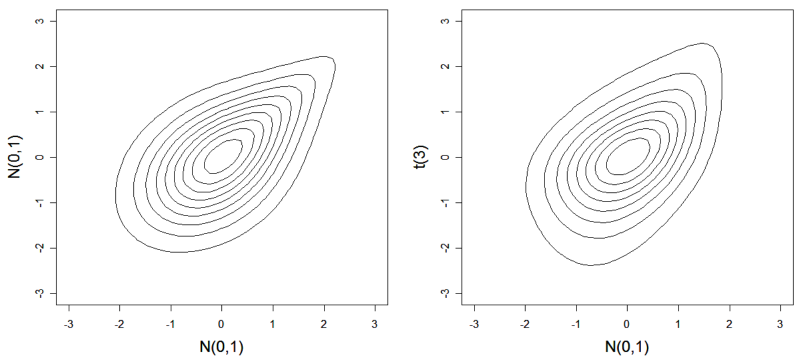

Figure 3.10. MVD's with Gumbel copula ($\theta = 1.5$) with marginals N(0,1) and t(3)

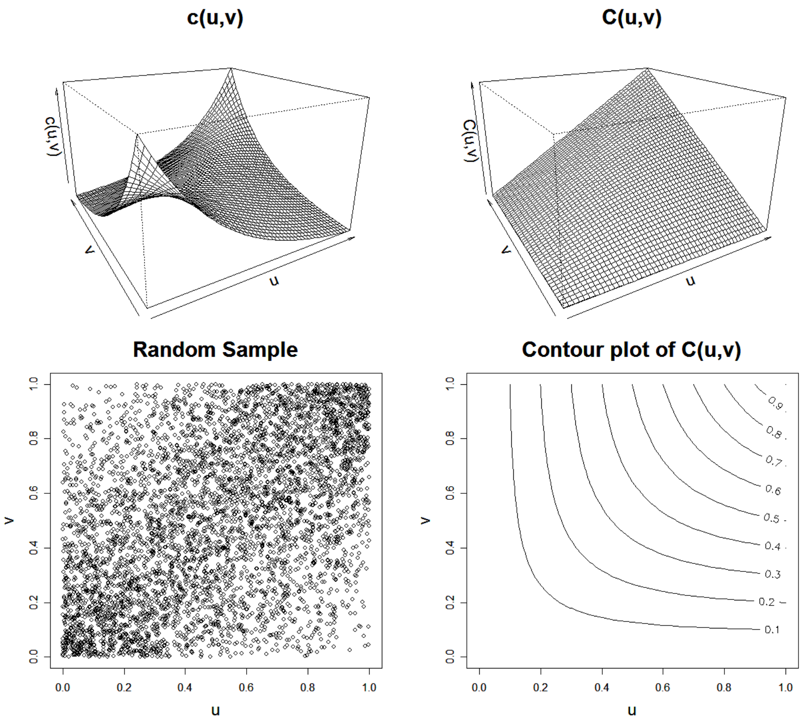

Figure 3.11. Frank copula with $\theta = 3$

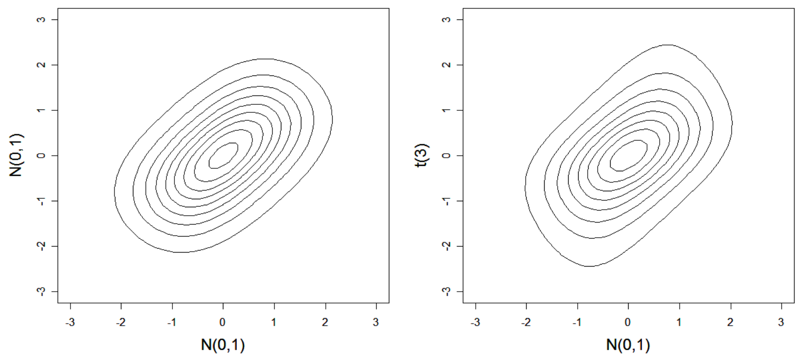

Figure 3.12. MVD's with Frank copula ($\theta = 3$) and marginals N(0,1) and t(3)

3.3.2. Dependence Structure of Copulas

Alternative dependence measures to the classical correlation coefficient have been explained in the previous sections. These were the rank correlation coefficients and two of them were the Spearman's rho and the Kendall's tau. Rank correlation coefficients

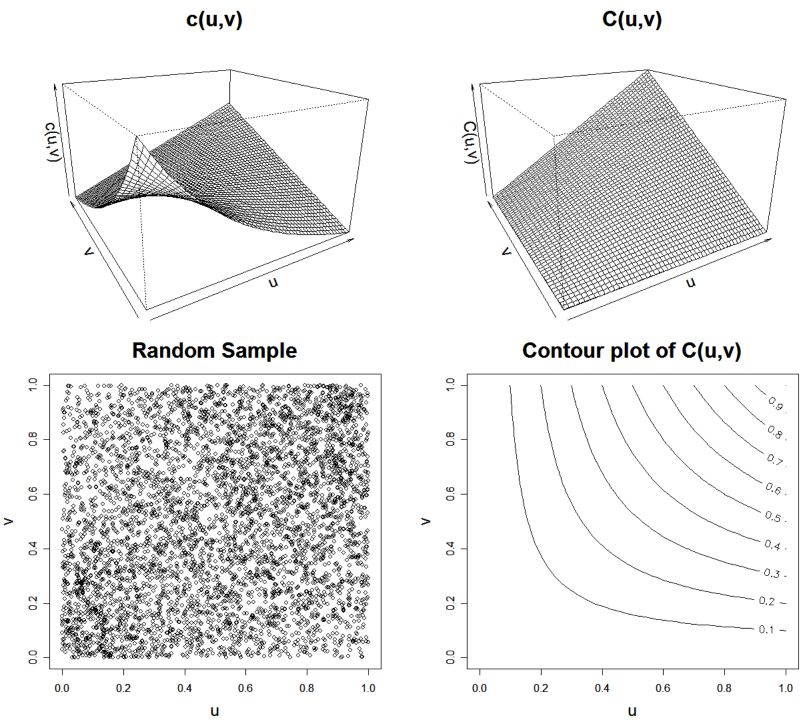

Figure 3.13. Ali-Mikhail-Haq copula with $\theta = 0.5$

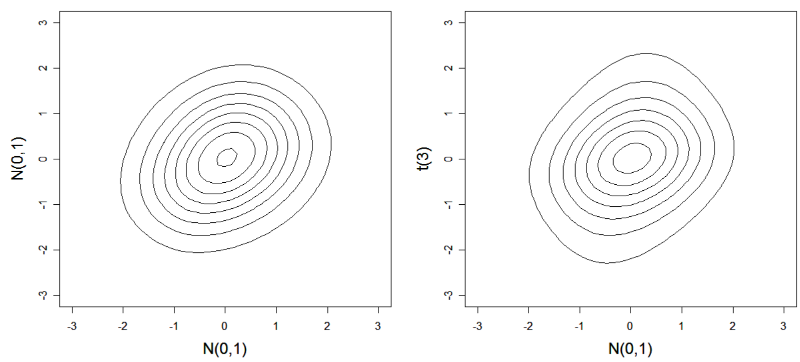

Figure 3.14. MVD's with Ali-Mikhail-Haq copula ($\theta = 0.5$) and marginals N(0,1) and t(3)

of two random variables X and Y can be expressed in terms of their copula C:

$$\tau(C) = 4 \iint_{[0,1]^2} C(u,v) dC(u,v) - 1 \tag{3.38}$$

$$\rho_s(C) = 12 \iint_{[0,1]^2} C(u,v) du dv - 3 \tag{3.39}$$

$\tau = \frac{2}{\pi} \arcsin \rho$ and $\rho_s = \frac{6}{\pi} \arcsin \rho$ holds for any pair of random variables whose dependence structure is given by an elliptical copula. The parameter ρ denotes the linear correlation coefficient, when it exists, of the elliptical distribution associated with the considered elliptical copula [20].

The generator of an Archimedean copula fully embodies the properties of dependence. As a consequence, Kendall's tau, Spearman's rho and tail dependence coefficients can be expressed in terms of its generator [23]. For a bivariate Archimedean copulas having generator φ, Kendall's tau is given by:

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt \tag{3.40}$$

For the four bivariate Archimedean family of copulas, the rank correlation coefficients are given in terms of their parameters in Table 3.2.

Table 3.2. Rank correlation coefficients of archimedean copulas

Copula	τ	ρ_S
Clayton	$\frac{\theta}{\theta+2}$	complicated
Gumbel	$\frac{\theta-1}{\theta}$	no closed form
Frank	$1 - \frac{4}{\theta}[1 - D_1(\theta)]$	$1 - \frac{12}{\theta}[D_1(\theta) - D_2(\theta)]$
Ali-Mikhail-Haq	$\frac{3\theta-2}{\theta}$	complicated

$D_k(x)$ denotes the "Debye" function: $D_k(x) = \frac{k}{x} \int_0^x \frac{t^k}{(e^t-1)} dt$

Any Archimedean copula, with a strict generator φ, ($\varphi(0) = \infty, \varphi^{[-1]} = \varphi^{-1}$), has a coefficient of upper tail dependence given by:

$$\lambda_U = 2 - 2 \lim_{t \to 0} \frac{\varphi^{-1\prime}(2t)}{\varphi^{-1\prime}(t)} \tag{3.41}$$

As a consequence, if $\varphi^{-1\prime}(0) > -\infty$, the coefficient of the upper tail dependence is identically zero. For an Archimedean copula to present tail dependence, it is necessary that $\lim_{t \to 0} \varphi^{-1\prime}(t) = -\infty$. Similarly, the coefficient of lower tail dependence is:

$$\lambda_L = 2 \lim_{t \to \infty} \frac{\varphi^{-1\prime}(2t)}{\varphi^{-1\prime}(t)} \tag{3.42}$$

so that $\varphi^{-1\prime}(\infty)$ must be equal to zero in order for an Archimedean copula to have a non-zero lower tail dependence [20]. In Table 3.3, the coefficients of tail dependences of the mentioned Archimedean copulas are given. It can be seen that Clayton copula has no upper tail dependence, while Gumbel copula has no lower tail dependence. Frank and Ali-Mikhail-Haq copula have none of them.

Table 3.3. Tail dependence coefficients of archimedean copulas

Copula	λ_U	λ_L
Clayton	0	$2^{-1/\theta}$
Gumbel	$2 - 2^{1/\theta}$	0
Frank	0	0
Ali-Mikhail-Haq	0	0

If (X, Y) has a normal copula with correlation coefficient ρ, the tail dependence is zero for all $\rho \in [-1, 1)$. In contrast, if (X, Y) follows the t-copula, the tail dependence coefficient is:

$$\lambda = 2\bar{T}_{v+1}\left(\sqrt{v+1}\sqrt{\frac{1-\rho}{1+\rho}}\right) \tag{3.43}$$

where $\bar{T}$ denotes the survival function. The coefficient is greater than zero for all $\rho > -1$

and it is symmetric in both tails [21]. This means that although the correlation of the t-copula is zero, it still contains tail dependences.

3.4. Estimation of Copula Parameters

Parameter estimation of a given copula is also called *calibration.* In this section some of the most popular techniques which appeared in the statistical literature and have common use in modeling financial and economic variables will be summarized.

3.4.1. Parametric Estimation

There are two basic parametric methods for copula fitting: *Maximum Likelihood Method* and *Inference Functions for Margins Method.*

3.4.1.1. Maximum Likelihood Method. Let F be a multivariate distribution function with continuous marginals F_i and copula C. The density of the joint distribution function F is given by:

$$f(x_1, \dots, x_n) = c(F_1(x_1), \dots, F_n(x_n)) \prod_{i=1}^{n} f_i(x_i) \tag{3.44}$$

where f_i is the density function of marginal F_i and c is the density of the copula given by:

$$c(u_1, \dots, u_n) = \frac{\partial C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} \tag{3.45}$$

Assume that we have an n-dimensional dataset of length T; $X = \{(x_1^t, \dots, x_n^t)\}_{t=1}^{T}$. Let $\delta = (\beta_1, \dots, \beta_n, \alpha)$ be the vector of all the parameters to estimate, where β_i is the vector of the parameters of marginal distribution F_i, and α is the vector of the copula parameters. Then the log-likelihood function can be written as:

$$l(\delta) = \sum_{t=1}^{T} \ln c(F_1(x_1^t;\beta_1),\ldots,F_n(x_n^t;\beta_n);\alpha) + \sum_{t=1}^{T}\sum_{i=1}^{n} \ln(f_i(x_i^t;\beta_i)) \tag{3.46}$$

The Maximum Likelihood (ML) estimator $\hat{\delta}$ of the parameter vector δ is the one, which maximizes the log-likelihood function above:

$$\hat{\delta} = \arg\max_{\delta} l(\delta) \tag{3.47}$$

For this procedure to work properly, the choice of the marginal distributions is crucial. It is thus appropriate to model each marginal distribution and perform a first ML estimation of their corresponding parameters. Then, together with the choice of a suitable copula, these preliminary estimates of the parameters of the marginal distributions provide useful starting points to globally maximize the log-likelihood [20].

<u>3.4.1.2. Inference Functions for Margins.</u> The problem with the ML method is that it is computationally intensive in the case of high dimensions as it requires to jointly estimate the parameters of the marginals and the parameters of the copula [20]. Also the dependency parameter of the copula function may be a convoluted expression of the marginal parameters. Therefore, an analytical expression of the gradient of the likelihood function might not exist. Only numerical gradients may be computable, implying a slowing down of the numerical procedure [24].

In the Inference Functions for Margins (IFM) method, the parameters are not estimated jointly but the parameters of the marginal distributions are estimated separately from the parameters of the copula. In other words, the estimation process is divided into the following two steps:

1. Estimating the parameters of the marginal distributions using ML;
2. Estimating the parameters of the copula using the estimated parameters in the

first step.

For the first step, the log-likelihood functions for the marginals are:

$$l_i(\beta_i) = \sum_{t=1}^{T} \ln(f_i(x_i^t; \beta_i)), \ \ i = 1, \dots n \tag{3.48}$$

The ML estimator $\hat{\beta}_i$ of the parameter vector β_i for marginal i is the one which maximizes $l_i(\beta_i)$:

$$\hat{\beta}_i = \arg\max_{\beta_i} l(\beta_i) \tag{3.49}$$

For the second step the log-likelihood function of the copula is:

$$l(\alpha) = \sum_{t=1}^{T} \ln c(F_1(x_1^t; \hat{\beta}_1), \dots, F_n(x_n^t; \hat{\beta}_n); \alpha) \tag{3.50}$$

The ML estimator $\hat{\alpha}$ of the parameter vector α for the copula is the one which maximizes $l(\alpha)$:

$$\hat{\alpha} = \arg\max_{\alpha} l(\alpha) \tag{3.51}$$

While asymptotically less efficient than the ML method, this approach has the obvious advantage of reducing the dimensionality of the optimization problem [20]. Also in practice, one has to deal with samples of different lengths. The accuracy of the IFM method is much better than the ML method, when the size of the intersection of the marginal samples is small [25].

3.4.2. Semiparametric Estimation

In parametric methods, it is assumed that the true model belongs to a given family of multivariate distribution, i.e. a family of copula and families of univariate marginal distributions. Such a modeling requires a very accurate knowledge of the true distributions and can lead to bad estimations of the copula parameters if the marginals are misspecified [20]. Thus, especially when in doubt concerning the univariate marginal distributions, a semiparametric approach may be preferable. Here the parametric representation is done only for the copula. No assumption is made about the marginal distributions.

The semiparametric method explained here is called *Canonical Maximum Likelihood* (CML) method. In this method, the marginal distributions are not modeled as specific distributions and empirical distributions are used for the marginals. This method is very similar to IFM method except for the first step. In CML, the estimation step is divided into the following two steps:

1. Transforming the dataset $X = \{(x_1^t, \ldots, x_n^t)\}_{t=1}^T$ into uniform variates $\hat{U} = \{(\hat{u}_1^t, \ldots, \hat{u}_n^t)\}_{t=1}^T$ using empirical distributions.
2. Estimating the parameters of the copula using the estimated uniforms of the first step.

For the first step, the dataset is transformed into uniforms by:

$$\hat{u}_i^k = \hat{F}_i(x_i^k) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}_{\{x_i^t \leqslant x_i^k\}}, \; k = 1, \ldots, T; \; i = 1, \ldots, n \tag{3.52}$$

For the second step the log-likelihood function is:

$$l(\alpha) = \sum_{t=1}^{T} \ln c(\hat{u}_1^t, \ldots, \hat{u}_n^t; \alpha) \tag{3.53}$$

The ML estimator $\hat{\alpha}$ of the parameter vector α for the copula is the one which maximizes $l(\alpha)$:

$$\hat{\alpha} = \arg\max_{\alpha} l(\alpha) \tag{3.54}$$

3.4.3. Nonparametric Estimation

This very first copula estimation method dates back to the work by Deheuvels in 1979. Considering n-dimensional random vector $X = (X_1, \ldots, X_n)$ whose copula is C and given a sample of size T, $X = \{(x_1^t, \ldots, x_n^t)\}_{t=1}^T$, a natural idea is to estimate the empirical distribution function of F of X as:

$$\hat{F}(x) = \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}_{\{x_1^t \leqslant x_1, \ldots, x_n^t \leqslant x_n\}} \tag{3.55}$$

and the empirical marginal distribution functions of X_i's as:

$$\hat{F}_i(x_i) = \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}_{\{x_i^t \leqslant x_i\}} \tag{3.56}$$

Then Sklar's theorem can be applied to obtain a nonparametric estimation of copula C. However, even if the marginals of F are continuous, their empirical counterparts are not. Thus the estimated copula is not unique. However, a nonparametric estimator of C can be obtained defined at the discrete points $\left(\frac{i_1}{T}, \ldots, \frac{i_n}{T}\right),\ i_k \in \{1, 2, \ldots, T\}$ [20]. By inverting the empirical marginal distributions one can obtain an empirical copula:

$$\hat{C} = \left(\frac{i_1}{T}, \ldots, \frac{i_n}{T}\right) = \frac{1}{T} \sum_{t=1}^{T} 1_{\{x_1^t \leqslant x_1(i_1;T), \ldots, x_n^t \leqslant x_n(i_n;T)\}} \tag{3.57}$$

where $x_i(k;T)$ denotes the k^{th} order statistics of the sample $(x_i^1, \ldots, x_i^T)$.

The empirical distribution function $\hat{F}$ converges, almost surely, uniformly to the underlying distribution function F from which the sample is drawn, as the sample size

T goes to infinity. This property still holds for the nonparametric estimator defined by the empirical copula [20].

$$\sup_{u \in [0,1]^n} \left| \hat{C}(u) - C(u) \right| \overset{T \to \infty}{\to} 0 \tag{3.58}$$

Similarly, the empirical density $\hat{c}$ of the empirical copula $\hat{C}$ can be estimated by:

$$\hat{c} = \left(\frac{i_1}{T}, \ldots, \frac{i_n}{T} \right) = \begin{cases} \frac{1}{T}, & \text{if } \{x_1(i_1;T), \ldots, x_n(i_n;T)\} \text{ belongs to the sample} \\ 0, & \text{otherwise} \end{cases} \tag{3.59}$$

3.5. Copula Selection

In this section, the selection procedure of the copula which best fits to the dataset will be explained. In general we assume that we have a finite subset of copulas $\tilde{C} = \{C_k\}_{1 \leqslant k \leqslant K}$, $\tilde{C} \subset C$.

The first thing to be done is to consider the log-likelihood and AKAIKE Information Criterion (AIC) values if a parametric estimation was done. The higher the log-likelihood value and the smaller the AIC value, the better the copula fits to the dataset. AIC value is calculated as in [26]:

$$\text{AIC} = \text{- } 2 \ \times \ \text{log-likelihood value} + 2 \ \times \ \text{number of estimated parameters} \tag{3.60}$$

Secondly, If $\tilde{C}$ corresponds to the Archimedean family, it is sufficient to identify the generator. Genest and Rivest [27] have developed an empirical method to identify the copula in the Archimedean case. If C is an Archimedean copula with generator φ, the function K is:

$$K = u - \frac{\varphi(u)}{\varphi'(u)} = \mathrm{P}\left\{C(U_1, \ldots, U_n) \leqslant u\right\} \tag{3.61}$$

A nonparametric estimate of K is given by:

$$\hat{K}(u) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}_{[\vartheta_i \leqslant u]} \tag{3.62}$$

where

$$\vartheta_i = \frac{1}{T-1}\sum_{t=1}^{T} \mathbf{1}_{[x_1^t < x_1^i, \ldots, x_n^t < x_n^i]} \tag{3.63}$$

The idea is to select an Archimedean copula by fitting $\hat{K}(u)$. Frees and Valdez [28] propose to use a Q-Q Plot of K and $\hat{K}$. Since $K(u)$ is a distribution, one could also define the best copula among $\tilde{C}$ as the copula which gives the minimum distance in L^2, between $K(u)$ and $\hat{K}(u)$ [29].

$$d_2(\hat{K}, K) = \int_0^1 [K(u) - \hat{K}(u)]^2 du \tag{3.64}$$

In general, If $\tilde{C}$ corresponds to the general copula families, L^p distance between the empirical copula and theoretical copula can be used for copula selection. We can choose the copula C_k in the subset, which has the minimum of this distance for example in L^2 [29]:

$$d_2(\hat{C}, C_k) = \left(\sum_{t_1=1}^{T} \cdots \sum_{t_n=1}^{T} \left[\hat{C}\left(\frac{t_1}{T}, \ldots \frac{t_n}{T}\right) - C_k\left(\frac{t_1}{T}, \ldots \frac{t_n}{T}\right) \right]^2 \right)^{1/2} \tag{3.65}$$

This distance may also be used to estimate the vector of parameters $\omega \in \alpha$ of a given copula $C(u; \omega)$:

$$\hat{\omega} = \arg\min_{\omega \in \alpha} \left(\sum \left[\hat{C}(u) - C(u;\omega) \right]^2 \right)^{1/2} \tag{3.66}$$

To conclude, selection of a copula among a subset of copulas can be done considering the following criterions:

- Log-likelihood values,
- AIC values,
- The distance between the empirical and the theoretical copulas.

3.6. Simulation from Copulas

An important practical application of copulas consists in the simulation of random variables with prescribed marginals and various dependence structures in order to perform Monte-Carlo studies. Sklar's theorem shows that the generation of n random variables $X_1, \ldots, X_n$ with marginals $F_1, \ldots, F_n$ and copula C can be performed as follows [20]:

1. Generate n random variables $u_1, \ldots, u_n$ with uniform marginals and copula C,
2. Apply the inversion method for each u_i in order to generate each x_i:

$$x_i = F_i^{-1}(u_i) \tag{3.67}$$

where F_i^{-1} denotes the inverse of F_i, that is the quantile of F_i.

Therefore, the main difficulty in generating n random variables following the joint distribution $F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$ lies in the generation of n auxiliary random variables with uniform marginals and dependence structure given by the copula C [20]. In the next section two algorithms are given for generating random variables from copulas. The first one is specific to elliptical copulas while the second one can be applied to a wide range of copulas.

3.6.1. Simulation from Elliptical Copulas

Simulation of random variables whose dependence structure is given by an elliptical copula is particularly simple. The simulation from elliptical copulas is equivalent to the problem of simulation from elliptically distributed random variables [20].

Simulation of an n-dimensional random vector following an n-dimensional normal copula with correlation matrix ρ can easily be performed by the following algorithm:

1. Generate n independent standard normal random variables $z = (z_1, \ldots, z_n)$,
2. Find the Cholesky decomposition L of the correlation matrix ρ such that, $\rho = LL^T$, where L is a lower triangular matrix,
3. Set $y = Lz$,
4. Evaluate $x_i = \Phi(y_i)$, $i = 1, \ldots, n$, where Φ denotes the univariate standard normal distribution function.

To generate an n-dimensional random vector drawn from a t-copula with υ degrees of freedom and shape matrix ρ, the following algorithm can be performed:

1. Generate n independent standard normal random variables $z = (z_1, \ldots, z_n)$,
2. Find the Cholesky decomposition L of the correlation matrix ρ such that, $\rho = LL^T$, where L is a lower triangular matrix,
3. Set $y = Lz$,
4. Generate a random variable s, independent of z and following a χ^2 distribution with υ degrees of freedom,
5. Set $t = \frac{\sqrt{\upsilon}}{\sqrt{s}} y$,
6. Evaluate $x_i = T_\upsilon(t_i),$, $i = 1, \ldots, n$, where T_υ denotes the univariate standard t-distribution function with υ degrees of freedom.

To generate an n-dimensional random vector drawn from a more complicated elliptical copula, it is useful to say that any centered and elliptically distributed random vector X admits the following stochastic representation [20]:

$$X = RN \tag{3.68}$$

where N is a vector of centered normal random variables with covariance matrix Σ, and R is a positive random variable independent of N.

3.6.2. Simulation from General Copulas

The second general method is *Conditional Distribution Method* which is based on the fact that:

$$\begin{aligned} \mathrm{P}[U_1 \leqslant u_1, \ldots, U_n \leqslant u_n] = \; & \mathrm{P}[U_n \leqslant u_n | U_1 = u_1, \ldots, U_{n-1} = u_{n-1}] \\ & \times \mathrm{P}[U_1 = u_1, \ldots, U_{n-1} = u_{n-1}] \end{aligned} \tag{3.69}$$

which gives

$$\begin{aligned} \mathrm{P}[U_1 \leqslant u_1, \ldots, U_n \leqslant u_n] = \; & \mathrm{P}[U_n \leqslant u_n | U_1 = u_1, \ldots, U_{n-1} = u_{n-1}] \\ & \times \mathrm{P}[U_{n-1} \leqslant u_{n-1} | U_1 = u_1, \ldots, U_{n-2} = u_{n-2}] \\ & \vdots \\ & \times \mathrm{P}[U_2 \leqslant u_2 | U_1 = u_1] \times \mathrm{P}[U_1 \leqslant u_1] \end{aligned} \tag{3.70}$$

Therefore applying this reasoning to an n-copula C and denoting by C_k the copula of the k first variables yields:

$$C\left(u_1, \ldots, u_n\right) = C_n\left(u_n | u_1, \ldots, u_{n-1}\right) \ldots C_2\left(u_2 | u_1\right) \underbrace{C_1\left(u_1\right)}_{=u_1} \tag{3.71}$$

where

$$C_k(u_k | u_1, \ldots, u_{k-1}) = \frac{\partial_{u_1} \ldots \partial_{u_{k-1}} C_k(u_1, \ldots, u_k)}{\partial_{u_1} \ldots \partial_{u_{k-1}} C_{k-1}(u_1, \ldots, u_{k-1})} \tag{3.72}$$

As a consequence, to simulate n random variables with copula C, the following procedure can be applied:

- Generate n uniform and independent random variables $\upsilon_1, \ldots, \upsilon_n$,
- Set $u_1 = \upsilon_1$,
- Set $u_2 = C_2^{-1}(\upsilon_2 | u_1)$,
- $\vdots$
- Set $u_n = C_n^{-1}(\upsilon_n | u_1, \ldots, u_{n-1})$

This algorithm is particularly efficient in the case of Archimedean copulas. But for large dimensions, the inversion of the conditional copulas can become intractable. Thus simulating from elliptical copulas is much easier than from Archimedean copulas [20].

3.7. Empirical Results for Copula Fitting

As an application, different copulas were fitted to a dataset of stock prices. The fitting was performed for stock portfolios of two, three, four, five and 10 stocks. The stocks are traded in New York Stock Exchange (NYSE) and the data include the adjusted daily closing prices of 15 stocks between 16/09/2002-18/09/2007. Therefore the data consists of five years' (1261 data points) daily adjusted closing prices. The stocks were selected from different sectors and industries as much as possible, so as to minimize the correlation between them and to see how the dependences act between the stocks, even between quite unrelated ones. The selected stocks and their industries are given in Table 3.4.

The daily logreturns were calculated from the daily adjusted closing prices. The logreturns can simply be summed to find the future price of a stock. This cannot be done for arithmetic returns as mentioned before. After transformation we had 1260

Table 3.4. Stocks from NYSE

Symbol	Company Name	Sector	Industry
BP	BP plc	Basic Material	Major Integrated Oil & Gas
UNP	Union Pacific Co.	Services	Railroads
GM	General Motors Co.	Consumer Goods	Auto Manufacturers - Major
PG	Procter & Gamble Co.	Consumer Goods	Personal Products
MOT	Motorola Inc.	Technology	Communication Equipment
MMM	3M Company	Conglomerates	Conglomerates
JNJ	Johnson & Johnson	Healthcare	Drug Manufacturers - Major
IBM	International Business Machines Corp.	Technology	Diversified Computer Systems
DIS	Walt Disney Co.	Services	Entertainment - Diversified
MCD	McDonald's Corp.	Services	Restaurants
DD	EI DuPont de Nemours & Co.	Basic Materials	Agricultural Chemicals
CAT	Caterpillar Inc.	Industrial Goods	Farm & Construction Machinery
DAI	Daimler AG	Consumer Goods	Auto Manufacturers - Major
HON	Honeywell International Inc.	Industrial Goods	Aerospace/Defense Products & Services
T	AT&T Inc.	Technology	Telecom Services - Domestic

data points.

$$x_i^t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}),\ \ i = 1, \ldots, 15;\ \ , t = 1, \ldots, 1260 \qquad (3.73)$$

The estimated correlation matrix of the logreturns is given in Table 3.5, rounded to the third digit. It can be seen that dependence exists among the stock logreturns although they belong to different industries. The maximum linear correlation is 0.530 between DAI and HON. The minimum is 0.123 between PG and MOT.

A set of copulas was fitted to the joint return distributions of portfolios. The fitting was done by IFM method. Because of the reasons mentioned before, no filtering was performed to the data.

CML method was not performed since the empirical distributions are used for the marginals in CML. However it is well-known that the empirical distributions do not have enough data in the tails. Nevertheless we performed both CML and IFM

Table 3.5. Correlation matrix of the stock returns

	BP	UNP	GM	PG	MOT	MMM	JNJ	IBM	DIS	MCD	DD	CAT	DAI	HON	T
BP	1.000	0.376	0.257	0.226	0.216	0.307	0.283	0.285	0.297	0.187	0.390	0.380	0.454	0.354	0.272
UNP	0.376	1.000	0.346	0.306	0.298	0.381	0.262	0.362	0.366	0.277	0.530	0.453	0.454	0.446	0.301
GM	0.257	0.346	1.000	0.201	0.256	0.287	0.186	0.341	0.342	0.282	0.384	0.339	0.513	0.406	0.306
PG	0.226	0.306	0.201	1.000	0.123	0.347	0.366	0.269	0.307	0.274	0.341	0.290	0.321	0.308	0.267
MOT	0.216	0.298	0.256	0.123	1.000	0.275	0.166	0.417	0.396	0.280	0.338	0.312	0.370	0.353	0.313
MMM	0.307	0.381	0.287	0.347	0.275	1.000	0.320	0.393	0.377	0.302	0.490	0.403	0.433	0.408	0.276
JNJ	0.283	0.262	0.186	0.366	0.166	0.320	1.000	0.314	0.319	0.215	0.346	0.278	0.335	0.268	0.283
IBM	0.285	0.362	0.341	0.269	0.417	0.393	0.314	1.000	0.473	0.304	0.473	0.397	0.482	0.516	0.406
DIS	0.297	0.366	0.342	0.307	0.396	0.377	0.319	0.473	1.000	0.325	0.463	0.416	0.483	0.479	0.379
MCD	0.187	0.277	0.282	0.274	0.280	0.302	0.215	0.304	0.325	1.000	0.334	0.277	0.351	0.319	0.248
DD	0.390	0.530	0.384	0.341	0.338	0.490	0.346	0.473	0.463	0.334	1.000	0.455	0.488	0.521	0.380
CAT	0.380	0.453	0.339	0.290	0.312	0.403	0.278	0.397	0.416	0.277	0.455	1.000	0.482	0.472	0.313
DAI	0.454	0.454	0.513	0.321	0.370	0.433	0.335	0.482	0.483	0.351	0.488	0.482	1.000	0.530	0.441
HON	0.354	0.446	0.406	0.308	0.353	0.408	0.268	0.516	0.479	0.319	0.521	0.472	0.530	1.000	0.380
T	0.272	0.301	0.306	0.267	0.313	0.276	0.283	0.406	0.379	0.248	0.380	0.313	0.441	0.380	1.000

as a pilot study for a portfolio of five stocks. The risk results were not so different. Thus we concluded that this was because of the continuity of the empirical distributions. However we suppose that a parametric approach is superior to a non-parametric approach.

In IFM method, the first step is fitting the marginal distributions. It is known that financial data are far from the normal distribution. They are fat tailed and have high kurtosis. However, to see whether these problems also exist in our dataset, it was considered as a model to the marginals. In Table 3.6, the parameters of the fitted normal distributions to the asset returns were given with the log-likelihood values.

The histograms were plotted for each of the stock returns and the fitted normal models were compared. Since Q-Q plots are very useful tools to see the tails of the empirical and theoretical quantiles, Q-Q plots were also investigated. The histograms with the fitted normal lines, and Q-Q plots are given in Appendix B.1. For a final investigation, χ^2 tests were also performed to test the normality.

Table 3.6. Fitted normal distributions to the stocks returns

Stock	**Mean**	**SD**	**Log-likelihood**
BP	0.000515	0.01288	3,695.328
UNP	0.000602	0.01388	3,601.814
GM	0.000011	0.02297	2,966.869
PG	0.000388	0.00921	4,118.245
MOT	0.000474	0.02379	2,922.678
MMM	0.000425	0.01175	3,811.356
JNJ	0.000200	0.01026	3,982.154
IBM	0.000419	0.01336	3,649.664
DIS	0.000660	0.01607	3,417.077
MCD	0.000811	0.01587	3,432.823
DD	0.000301	0.01307	3,677.692
CAT	0.001141	0.01696	3,349.448
DAI	0.000807	0.01800	3,273.946
HON	0.000761	0.01610	3,415.030
T	0.000598	0.01523	3,484.373

Table 3.7. χ^2 test results for the fitted normal distributions

Stock	χ^2	**df**	**p-value**
BP	116.191	99	0.114
UNP	163.810	99	4.50e-05
GM	222.698	99	1.62e-11
PG	148.889	99	8.86e-04
MOT	217.778	99	6.56e-11
MMM	185.873	99	2.94e-07
JNJ	223.175	99	1.42e-11
IBM	236.191	99	3.18e-13
DIS	185.714	99	3.05e-07
MCD	180.635	99	1.04e-06
DD	131.587	99	1.59e-02
CAT	134.286	99	1.06e-02
DAI	171.429	99	8.69e-06
HON	155.556	99	2.48e-04
T	245.873	99	1.72e-14

As it can be easily seen from the histograms and the fitted normal distributions, the return distributions are far from normal. Especially in the Q-Q plots, the heavy tails can be easily observed. Also in χ^2 tests, all the fitted normal models were rejected at five per cent level except BP. This is because BP has less outliers than the other stocks. But in the Q-Q plot, the non-normality and heavy tails of BP can also be easily seen.

Since the normal distribution was found to model the asset returns inadequately especially in the tails, as expected before, other distributions were considered for modeling the asset returns. *t-distribution* is one of the candidates for the marginals since it has fatter tails which can capture the extreme returns. The other candidate is the *Generalized Hyperbolic Distribution* (GHD). In the following a short description of the t-distribution will be given. GHD will be explained with more details.

A random variable which follows a t-distribution with υ degrees of freedom has the stochastic representation:

$$t_\upsilon = \frac{\sqrt{\upsilon}}{\sqrt{s}} Z \tag{3.74}$$

where Z is a random variable following the standard normal distribution and s has a χ^2 distribution with υ degrees of freedom. The pdf of the t-distribution with υ degrees of freedom is:

$$f(x) = \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\upsilon\pi}\,\Gamma\left(\frac{\upsilon}{2}\right)} \left(1 + \frac{x^2}{\upsilon}\right)^{-\left(\frac{\upsilon+1}{2}\right)} \tag{3.75}$$

Generalized Hyperbolic Distribution (GHD) was introduced by Barndorff-Nielsen in 1977 [30] in his work, where he studied the mass size distribution of wind blown sand particles. The facts about GHD in this section follow [12].

GHD is a mean-variance mixture of a normal distribution with the *Generalized Inverse Gaussian* (GIG) distribution. It allows representation of the skewness and kurtosis, and its tails tend to be heavier than those of the normal. We can use the same formulations for the univariate GHD simply by setting $d = 1$ and $\Sigma = \sigma^2$.

Definition 3.7.1 Generalized Hyperbolic Distribution
The random variable X is said to have a multivariate GHD if:

$$X = \mu + W\gamma + \sqrt{W}AZ \tag{3.76}$$

where

1. $Z \sim N_k(0, I_k)$
2. $A \in \mathbb{R}^{d\times k}$ *is a matrix,*
3. μ *and* γ *are parameter vectors in* $\mathbb{R}^d$
4. $W \geq 0$ *is a scalar-valued random variable which is independent of Z and follows a GIG distribution.*

The density of the GIG distribution is:

$$f(w;\lambda,\chi,\psi) = \left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}} \frac{w^{\lambda-1}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \exp\left\{-\frac{1}{2}\left(\frac{\chi}{w} + \psi w\right)\right\} \tag{3.77}$$

with parameters satisfying:

$$\begin{aligned} &\chi > 0, \psi \geqslant 0, \quad if \quad \lambda < 0 \\ &\chi > 0, \psi > 0, \quad if \quad \lambda = 0 \\ &\chi \geqslant 0, \psi > 0, \quad if \quad \lambda > 0 \end{aligned} \tag{3.78}$$

If X follows a GIG distribution, it can be written $X \sim N^{-}(\lambda, \chi, \psi)$.

From the definition of GHD, it can be seen that:

$$X|W \sim N_d(\mu + W\gamma, W\Sigma) \tag{3.79}$$

where $\Sigma = AA^T$. This is also why it is called normal mean-variance mixture distribution. From the mixture definition, we can obtain the expected value and variance of X as:

$$E(X) = \mu + E(W)\gamma \tag{3.80}$$

$$Var(X) = E(W)\Sigma + Var(W)\gamma\gamma' \tag{3.81}$$

when the mixture variable W has finite variance $Var(W)$. If the mixing variable is $W \sim N^{-}(\lambda, \chi, \psi)$, then the joint density of the d-dimensional GHD in the non-singular case (Σ has rank d) is given by:

$$f(x;\lambda,\chi,\psi,\mu,\Sigma,\gamma) = c \times \frac{K_{\lambda-\frac{d}{2}}\left(\sqrt{(\chi + (x-\mu)'\Sigma^{-1}(x-\mu))\,(\psi + \gamma'\Sigma^{-1}\gamma)}\right) e^{(x-\mu)'\Sigma^{-1}\gamma}}{\left(\sqrt{(\chi + (x-\mu)'\Sigma^{-1}(x-\mu))\,(\psi + \gamma'\Sigma^{-1}\gamma)}\right)^{\frac{d}{2}-\lambda}} \tag{3.82}$$

where c is the normalizing constant:

$$c = \frac{\left(\sqrt{\chi\psi}\right)^{-\lambda}\psi^{\lambda}\left(\psi + \gamma'\Sigma^{-1}\gamma\right)^{\frac{d}{2}-\lambda}}{(2\pi)^{\frac{d}{2}}\,|\Sigma|^{\frac{1}{2}}\,K_{\lambda}\left(\sqrt{\chi\psi}\right)} \tag{3.83}$$

and $|.|$ denotes the determinant. $K_{\lambda}(.)$ is a modified Bessel function of the third kind.

The parametrization of GHD density can be done in several ways. One of the mostly used representation is the above $(\lambda, \chi, \psi, \mu, \Sigma, \gamma)$ parametrization. However it has a drawback of an identification problem. Indeed $GHD_d(\lambda, \chi, \psi, \mu, \Sigma, \gamma)$ and $GHD_d(\lambda, \chi/k, k\psi, \mu, k\Sigma, k\gamma)$ are identical for any $k > 0$. This problem causes a lot of trouble in the calibration, where the redundant information causes the algorithm to be unstable [31]. Thus, the following change of variables is a suitable way for overcoming this problem:

$$\bar{\alpha} = \sqrt{\chi\psi} \tag{3.84}$$

$$\psi = \bar{\alpha}\frac{K_{\lambda+1}(\bar{\alpha})}{K_{\lambda}(\bar{\alpha})} \text{ and } \chi = \frac{\bar{\alpha}^2}{\psi} = \bar{\alpha}\frac{K_{\lambda}(\bar{\alpha})}{K_{\lambda+1}(\bar{\alpha})} \tag{3.85}$$

The GHD distribution contains several special cases which are known under special names:

1. **Hyperbolic Distribution (HD):** If $\lambda = 1$, we get the multivariate GHD whose univariate marginals are one-dimensional hyperbolic distributions. If $\lambda = (d+1)/2$, we get the d-dimensional hyperbolic distribution, however its marginals are not hyperbolic distributions any more.
2. **Normal Inverse Gaussian Distribution (NIG):** If $\lambda = -1/2$, then the distribution is known as Normal Inverse Gaussian (NIG).
3. **Variance Gamma Distribution (VG):** If $\lambda > 0$ and $\chi = 0$ (or alternatively $\lambda > 0$ and $\bar{\alpha} = 0$), then we get a limiting case known as the Variance Gamma (VG) distribution.
4. **Skewed t-distribution:** If $\lambda = -\upsilon/2, \chi = \upsilon$ and $\psi = 0$ (or alternatively $\bar{\alpha} = 0$ and $\lambda < -2$), we get a limiting case which is called the skewed t-distribution.

An important property of GHD is that, they are closed under linear transformations. If $X \sim GHD_d(\lambda, \chi, \psi, \mu, \Sigma, \gamma)$ and $Y = BX + b$ where $B \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$, then $Y \sim GHD_k(\lambda, \chi, \psi, B\mu + b, B\Sigma B', B\gamma)$ which means that the linear transformations of GHD still remain in the GHD class. This seems a useful property for portfolio management. If we set $B = w^T = (w_1, w_2, \ldots, w_d)$ and $b = 0$, then the portfolio $y = w^T X$ is a one-dimensional GHD:

$$y \sim GHD_1(\lambda, \chi, \psi, w^T\mu, w^T\Sigma w, w^T\gamma) \tag{3.86}$$

Although this property shows at the first glance the multivariate GHD as being a nice model for the multivariate asset returns, each marginal return distribution would have the same parameters λ, χ and ψ in this multivariate model. Thus the only different parameters would be the location, shape and skewness parameters μ, σ and γ for the marginals but not the kurtosis parameters. However the marginals should be allowed to have different kurtosis parameters since it is an important characteristic for the asset returns. Consequently, GHD was considered as a model only for the univariate asset returns.

As it was stated before, since the normal distribution was found to model the asset returns inadequately, the t-distribution and the univariate GHD were considered as models for the individual asset returns. The parameters of the two distributions were calibrated by maximizing the log-likelihood functions. In Table 3.8, the log-likelihood values are given for the three fitted distributions; normal, t and GHD, for the 15 stock returns.

Table 3.8. Log-likelihood values for normal, t and GHD

Stock	**Normal** ($n_{par} = 2$)	**t** ($n_{par} = 3$)	**GHD** ($n_{par} = 5$)	**Which to use?**
BP	3,695.328	3,719.487	**3,719.542**	t
UNP	3,601.814	3,650.921	**3,658.346**	GHD
GM	2,966.869	3,083.574	**3,086.960**	GHD
PG	4,118.245	4,152.403	**4,152.959**	t
MOT	2,922.678	3,083.017	**3,083.021**	t
MMM	3,811.356	3,916.721	**3,916.883**	t
JNJ	3,982.154	4,058.852	**4,063.086**	GHD
IBM	3,649.664	3,810.663	**3,811.292**	t
DIS	3,417.077	3,515.138	**3,515.374**	t
MCD	3,432.823	3,546.553	**3,546.580**	t
DD	3,677.692	3,741.153	**3,741.256**	t
CAT	3,349.448	3,421.904	**3,422.331**	t
DAI	3,273.946	3,325.086	**3,328.472**	GHD
HON	3,415.030	3,485.854	**3,486.730**	t
T	3,484.373	3,596.471	**3,599.156**	GHD

As it can be seen from Table 3.8, the t-distribution and the GHD modeled the marginals much better than the normal distribution. Although the GHD has the highest log-likelihood values of all the stocks, the t-distribution values are very close to GHD values for 10 stocks. For these stocks, t-distribution will be used since it is easy to simulate from the t-distribution. However, for the five stocks (UNP, GM, JNJ, DAI, T), the difference between the t and the GHD is relatively high. Thus for these five stocks, GHD will be used to model the asset returns. In Table 3.9 and Table 3.10, the estimated parameters of the asset return distributions are given.

Table 3.9. Parameters of the fitted t-distributions for stock returns

Stock	**Location**	**Scale**	**df**
BP	0.000541	0.0109	7.05
PG	0.000313	0.0075	5.84
MOT	0.000512	0.0155	3.51
MMM	0.000345	0.0082	3.94
IBM	0.000153	0.0085	3.28
DIS	0.000530	0.0113	3.86
MCD	0.000850	0.0111	4.00
DD	0.000229	0.0101	5.03
CAT	0.001326	0.0131	5.26
HON	0.000522	0.0118	4.23

Table 3.10. Parameters of the fitted GHD for stock returns

Stock	λ	$\bar{\alpha}$	μ	σ	γ
UNP	0.380139	0.989883	-0.002142	0.013706	0.002745
GM	-0.398183	0.577251	-0.001230	0.022872	0.001240
JNJ	-0.271792	0.724601	-0.000307	0.010315	0.000507
DAI	0.270337	0.949899	-0.000205	0.018014	0.001011
T	-0.617913	0.574358	0.000069	0.015250	0.000531

The histograms of the asset returns with the fitted t and GHD lines, and Q-Q plots are given in Appendix B.2. It can be easily seen from the histograms that the t-distribution and the GHD can capture the high kurtosis and fat tails. In Table 3.11, the results of χ^2 tests are given. According to the table, all the fitted distributions are acceptable at five per cent level except T. However from the histogram and fitted GHD line, the fitted GHD for T seems a suitable distribution. Also in Q-Q plot, GHD can capture the extremes of T. Therefore we accepted the fitted GHD for T.

Table 3.11. χ^2 test results for the fitted t and GHD distributions

Stock	χ^2	**df**	**p-value**
BP	100.793	99	0.430
UNP	81.111	99	0.904
GM	79.523	99	0.924
PG	112.698	99	0.163
MOT	104.285	99	0.338
MMM	100.634	99	0.435
JNJ	92.539	99	0.663
IBM	112.063	99	0.174
DIS	93.333	99	0.641
MCD	103.968	99	0.346
DD	81.904	99	0.893
CAT	79.841	99	0.921
DAI	94.761	99	0.601
HON	99.523	99	0.466
T	130.793	99	0.017

Q-Q plots indicate that the fitted t and GHD models are capable to explain the extreme returns in the tails. Also χ^2 tests show that the fitted distributions are acceptable at five per cent level. Thus the t-distribution and the GHD were concluded to be adequate models for the marginal asset return distributions.

After fitting the marginal distributions, the observations in the dataset were transformed into the uniform variates by CDF tranformation. Since we do not have a closed form expression for the CDF of GHD, the transformation is done with numerical integration. Therefore we used the "ghyp" package of the statistical software R [8] and its built-in functions for this operation. With this transformation the first step of the IFM method is completed.

After the completion of the first step of the IFM method, a set of copulas was fitted to the transformed data. The log-likelihood functions of the copulas were maximized for the portfolios consisting of two, three, four, five and 10 stocks. There were 20 portfolios of two stocks, 10 portfolios of three stocks, four portfolios of four stocks, three portfolios of five stocks and two portfolios of 10 stocks where the portfolios were constructed with arbitrarily chosen stocks.

The maximization of the copula log-likelihood for a portfolio consisting of three stocks is explained below to be an example for the fitting procedure of the t-copula. Let ρ be the shape matrix and υ be the degress of freedom of the t-copula. ρ is initialized as the sample's Spearman's rho and υ is initialized arbitrarily, i.e $\upsilon = 5$. The elements of ρ are:

$$\rho = \begin{vmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{vmatrix} \tag{3.87}$$

where $\rho_{ij} = \rho_{ji}$. Then the log-likelihood function of the t-copula with ρ and υ is:

$$l(\rho, \upsilon) = \sum_{t=1}^{T} \ln c(u_1^t, \dots, u_n^t; \rho, \upsilon) \tag{3.88}$$

where $c(u_1^t, \dots, u_n^t; \rho, \upsilon)$ is the density of the t-copula given by $c_{n,\rho,\upsilon}(u_1^t, \dots, u_n^t)$ with the expression in Equation (3.34). We used the "copula" package of the statistical software R [9, 32] and its built-in functions for the log-likelihood maximizations.

As the dimension increases, the expression of the pdf of the Archimedean copulas become more complex and thus the pdf is not available due to intensive computing involved in symbolically differentiating the CDF. Therefore the fitting for the

Archimedean copulas could not be performed for the portfolios of 10 stocks. The results for two Elliptical and four Archimedean copulas are given in Appendix C.

The results show that the t-copula with the t and GHD marginals seems to be a proper copula to model the dependencies between the stocks. It is the best fitting copula according to the log-likelihood and AIC values for all the portfolios and it is the best according to L^2 distances for almost all the portfolios. For some of the portfolios of two stocks, the differences between different models seem to be small. For example, for the pair UNP-JNJ, there is not a big difference between the normal, t and Frank copula according to three criterions. L^2 distance is minimal for the Frank copula for this stock pair. But the t-copula still has the highest log-likelihood and smallest AIC values. However these exceptions are very rare and the t-copula is always the best fitting model according to the log-likelihood and AIC values for all the portfolios consisting of two, three, four, five and 10 stocks.

To conclude, according to the empirical results the t-copula has a considerable difference from the other copulas for modeling portfolios. Thus for multivariate financial returns, especially for portfolios of more than two assets, the t-copula with the t and GHD marginals seems to be an adequate model to capture the dependencies and explain the extreme returns of assets.

3.8. Problems with Turkish Data

We also wanted to consider data of İstanbul Stock Exchange (ISE) and see which dependences exist for Turkish stocks. With this aim five stocks were selected which were included in the ISE National 100 index between 21/11/2002 and 04/12/2007 so as to have 1261 data points for closing prices. The daily logreturns were calculated from these daily closing prices. After the transformation we had 1260 data points of the logreturns for each stock. The selected stocks and their industries were given in Table 3.12.

Table 3.12. Stocks from ISE

Symbol	**Company Name**	**Industry**
ARCLK	Arçelik A.Ş.	Metals
SAHOL	Sabancı Holding A.Ş.	Holdings
TCELL	Türkcell	Technology
THYAO	Türk Hava Yolları	Services
TUPRS	Tüpraş	Chemicals

The estimated correlation matrix shows that the maximum linear correlation is 0.597 between ARCLK and SAHOL. The minimum is 0.057 between THYAO and TUPRS.

Table 3.13. Correlation matrix of the stock returns for ISE

	ARCLK	SAHOL	TCELL	THYAO	TUPRS
ARCLK	1.000	0.597	0.191	0.094	0.127
SAHOL	0.597	1.000	0.244	0.116	0.203
TCELL	0.191	0.244	1.000	0.074	0.093
THYAO	0.094	0.116	0.074	1.000	0.057
TUPRS	0.127	0.203	0.093	0.057	1.000

The histogram of the returns of ARCLK is given in Figure 3.15. The situation is interesting because the distributions do not show continuity. The discreteness shows itself especially around zero. When we increase the number of bins of the histograms, the picture became clearer. There are many zeros in the middle of the histograms and also the positive and the negative returns start a bit away from zero. This indicates a discreteness problem for Turkish stocks. Since the problem seems largest around zero, the number of zeros in Turkish and New York data (for the first five stocks) is given in Table 3.14.

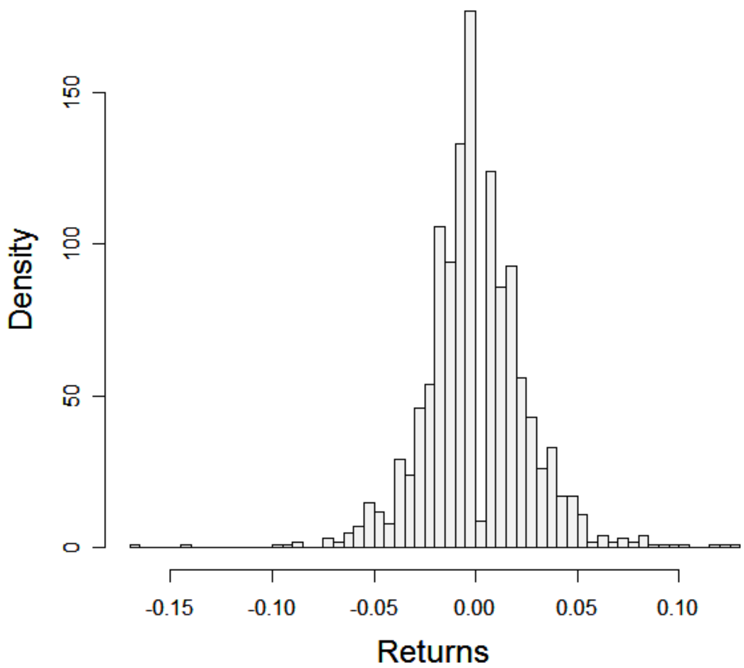

Figure 3.15. The histogram for the logreturns of ARCLK

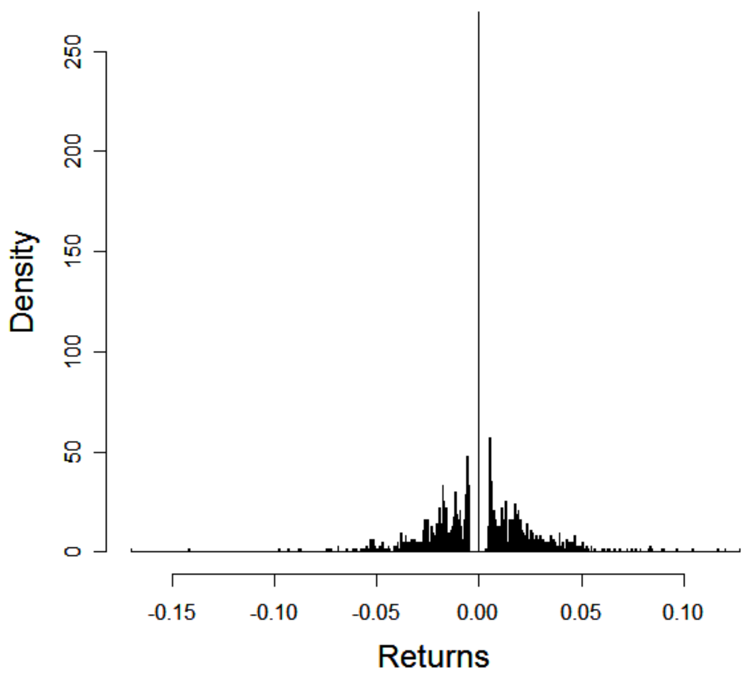

Figure 3.16. The histogram for the logreturns of ARCLK with 500 bins

Table 3.14. Zeros in NYSE and ISE

Stock	**Number of Zeros**	**Length of the data**
BP	10	1260
UNP	14	1260
GM	6	1260
PG	12	1260
MOT	22	1260
ARCLK	171	1260
SAHOL	167	1260
TCELL	165	1260
THYAO	194	1260
TUPRS	205	1260

Table 3.14 explains the reason of the discreteness. The number of zeros in Turkish data is nearly 10 times larger than the New York data and thus Turkish data are much more discrete. The reason that leads to many zeros in Turkish data is that, in ISE the prices are rounded and there is a lower bound for the price movements to increase or decrease. For example in NYSE, the smallest price movement of a stock is \$0.01. This does not change from stock to stock whatever its price is, i.e. the smallest price movement for a \$60 per share stock is \$0.01 and it is the same for a \$0.50 per share stock. Thus the stocks in NYSE show more continuity. But the situation is different in ISE.

In ISE, the smallest price movements occur according to the price interval of the stocks. The minimum movements of the stocks according to the price intervals are given in Table 3.15

According to this rule, a 3.5YTL stock can move at least 0.02YTL up or down. But a 12YTL stock can move at least 0.1YTL up or down. Because of these movement restrictions the prices are rounded and this causes the discreteness of the data.

Table 3.15. Price movements in ISE

Price Interval	**Movement**
0.00-2.50	0.01
2.50-5.00	0.02
5.00-10.00	0.05
10.00-25.00	0.1
25.00-50.00	0.25
50.00-100.00	0.5
100.00-	1

In Figure 3.17, the empirical copula is given for a stock pair, ARCLK and SAHOL. Since there are many zeros, the empirical cumulative distribution suddenly increases at zero and the copula also becomes discrete. This situation is the same for the other stock pairs.

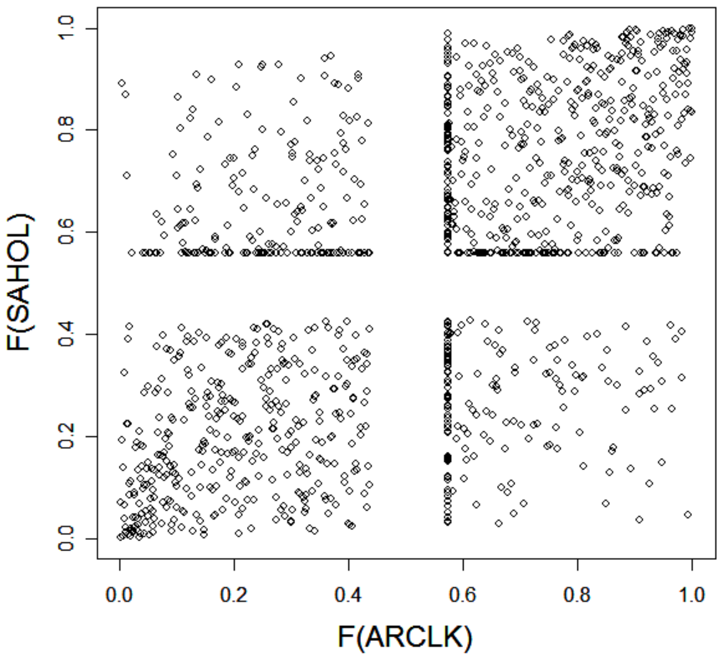

Figure 3.17. Empirical copula of ARCLK and SAHOL

To be able to fit copula to the Turkish data, this discreteness issue has to be fixed. Some noise can be added to the data to make them more continuous. But it would possibly destroy the underlying dependence. Thus we concluded that Turkish data are not proper for copula fitting because of the marginal fitting problems. Other approaches like *Extreme Value Theory* (EVT) might be considered for assessing the tail risks. In her Ph.D. thesis, Ünal [33] worked with calculating VaR and ES for Turkish stocks using EVT.

4. MODERN PORTFOLIO RISK SIMULATION

The classical risk calculation methods assume normality of returns or use the historical data of returns. The main classical model for risk calculation is the approximate multinormal model and it is still widely used for the risk calculation of financial assets because of its nice properties. In fact multinormal model is an adequate model for the risks at lower levels, i.e. $VaR_{0.95}$. But for more extreme risks like $VaR_{0.99}$, it is not a good model because of its thin tails. Copulas, introduced in Chapter 3, are better models for risk estimation because of the ability of combining arbitrary marginals with a dependence structure to represent the multivariate distributions. Thus we have a wide range of distributions for the marginals and very different structures for the dependence between them. So we are able to model the individual asset returns with heavy-tailed distributions to obtain more realistic models for the asset returns. From the empirical results found in this study, the t-copula with the t and GHD marginals seems to be suitable model for the risk calculation for NYSE stocks. Thus we will analyze this model in detail and call it shortly the "t-t copula model".

The aim of this chapter is to compare the results of different risk calculation methods for the single stocks and stock portfolios. For the single stocks, the risks were calculated by:

- Normal Model
- Historical Simulation
- GARCH process

For the portfolios, the risk calculations were performed by:

- Approximate Multinormal Model
- Exact Multinormal Model (Monte Carlo simulation)
- Historical Simulation
- GARCH process

- t copula with t and GHD marginals (t-t copula)

4.1. Empirical Results for Different Methods

The risk calculation was performed for five individual stocks and three portfolios consisting of five stocks. The considered risk measures are VaR and ES at 99 per cent level and the time horizons for the risk calculations are one, five, 10 and 22 days. 95 per cent level risk results were not given because 95 per cent means one out of 20 days, which means that our risk estimations will fail more than once a month. But this does not seem a reasonable time horizon for the extremes, i.e. the extremes should be seen more rarely. Also as it was stated before, the normal model is a suitable model for the risk estimations at 95 per cent level.

4.1.1. The Risks of Single Stocks

For the single stocks, the risk calculation was performed for BP, UNP, GM, PG and MOT. The risk estimates of the single stocks are given with their standard errors in Appendix D.1. The "%" column is the relative risk of the corresponding methods to the risk of the approximate normal model. This enables a fast comparison between the methods. The risks were calculated by Monte-Carlo simulations with 100 repetitions for each model. The sample size within each repetition is 1000.

For one day, the risk estimates of the historical simulation method are higher than the risk of the normal model for all the stocks. Thus historical simulation is able to capture the extreme risks better than the normal model as stated before. But for GARCH, the risk estimates are smaller than the normal model risks for PG and MOT. This is because GARCH uses the current variance, i.e. conditional variance, but normal model uses the sample variance. PG and MOT have lower variances than their sample variances at the current time. In Table 4.1 the sample and current standard deviations of 5 stocks are given.

Table 4.1. Sample and GARCH standard deviations of five stocks

	BP	**UNP**	**GM**	**PG**	**MOT**
Sample sd	0.01288	0.01388	0.02297	0.00921	0.02379
GARCH sd	0.01312	0.02178	0.03220	0.00882	0.01644

For multidays, the risk estimates of the historical simulation tend to decrease compared to the normal risk. This is a validation of that the logreturns are not i.i.d. normal. Otherwise the sum of the i.i.d normal variates should have still been normal. The empirical distributions have higher kurtosis than the fitted normal distributions and they do not have enough data in the tails although they have more extremes than the fitted normal models. Thus the risk estimates decrease compared to the normal model estimates as the time horizon increases.

For the GARCH model, the situation is different. The GARCH model estimates the risks assuming normality given the current variance. So if the stock is in high volatility period, the classical normal model, which uses the sample variance, underestimates the risks. But if the stock is in low volatility period, in this case the classical normal model overestimates the risks. Thus for multi-days, since it is a dynamic model, GARCH risks change from stock to stock according to the trend of the volatility. The estimated daily standard deviations of five stocks are given through Figure 4.1-Figure 4.5. The red lines are the samples' standard deviations.

To sum up, we can say from these results that in general the normal model underestimates the risk for one day. But as the time horizon of the risk measures increases, the normal model tends to overestimate the risk.

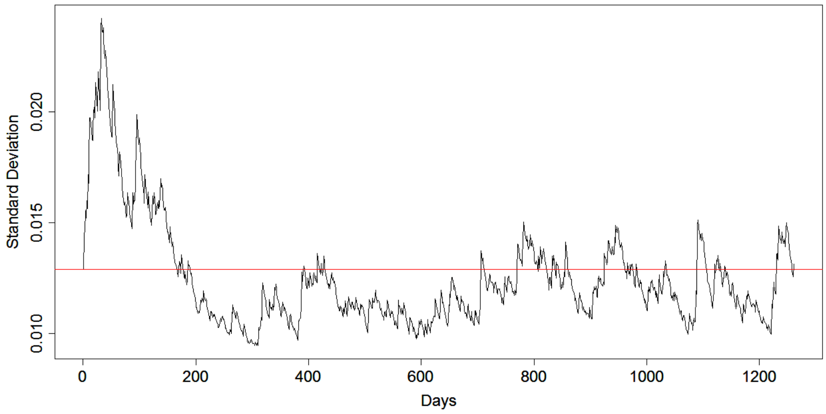

Figure 4.1. Daily standard deviations of BP

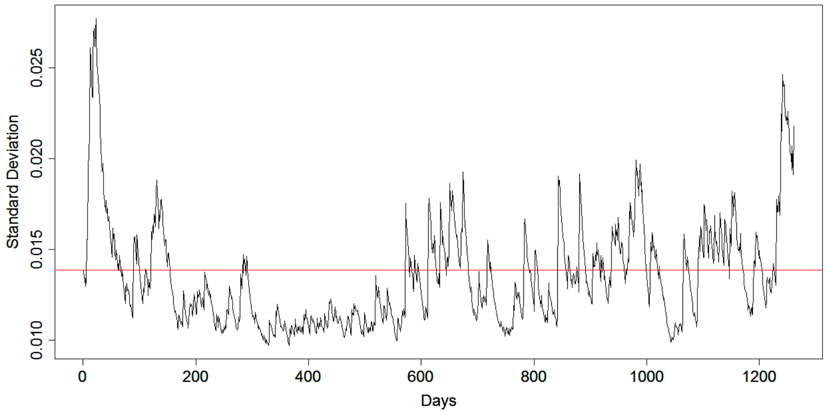

Figure 4.2. Daily standard deviations of UNP

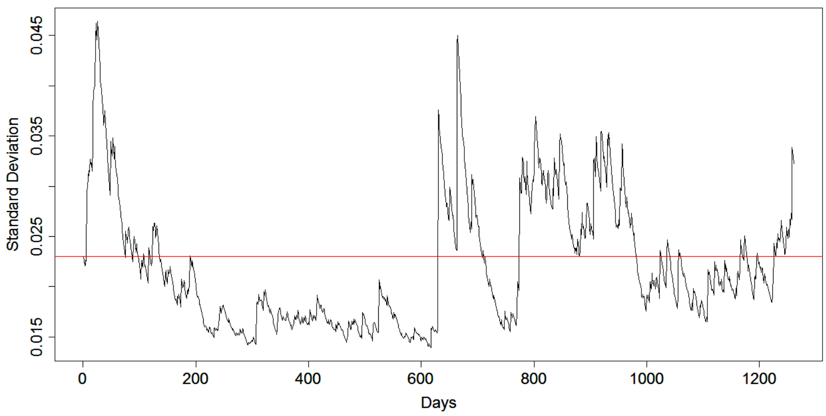

Figure 4.3. Daily standard deviations of GM

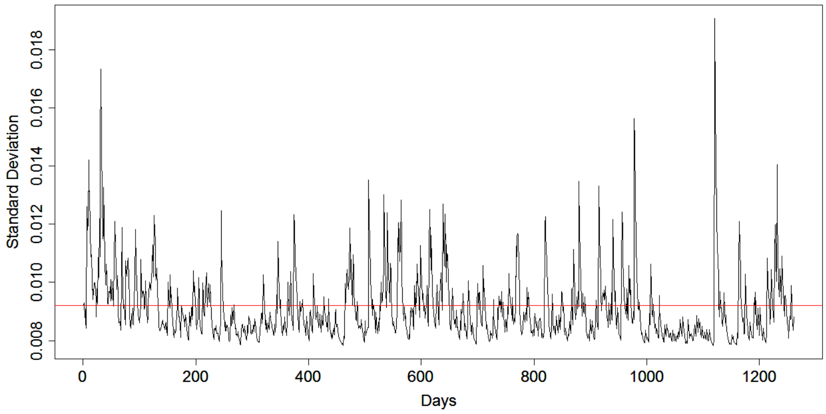

Figure 4.4. Daily standard deviations of PG

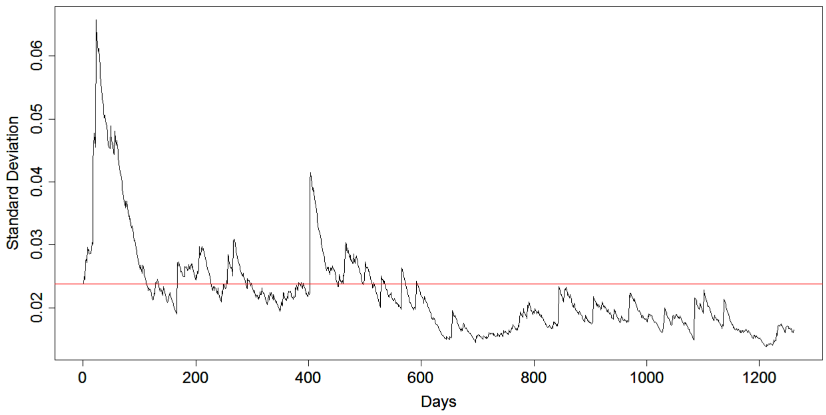

Figure 4.5. Daily standard deviations of MOT

4.1.2. The Risk of Portfolios

Three portfolios consisting of five stocks were used for risk calculation. The portfolios are the same portfolios which were used in copula fitting examples. These are:

1. BP-GM-MOT-IBM-HON
2. GM-JNJ-MCD-DD-DAI
3. UNP-PG-MMM-JNJ-DIS

Three sub-portfolios, which include the same stocks with different weights, were constructed from each of the three portfolios so as to obtain *conservative, balanced* and *risky portfolios* of the same stocks. To determine the risk levels of the sub-portfolios, the approximate portfolio variances were used. As mentioned before, the variance of the portfolio according to the approximate multinormal model is:

$$\sigma_p^2 = w^T \Sigma w \tag{4.1}$$

where w is the weight vector of the assets and Σ is the covariance matrix of the asset returns. In Table 4.2 the specifications of the portfolios, i.e. the weights of the stocks and the approximate portfolio variances, are given.

The risk estimates of different methods for three portfolios are given with their standard errors in Appendix D.2. The "%" column is the relative risk of the corresponding method to the risk of the approximate multinormal model. The risks were calculated by Monte-Carlo simulations with 100 repetitions for each method. The sample size for each repetition was 1000.

<u>4.1.2.1. The Risk for One Day.</u> For one day VaR, all the alternative methods have higher risks than the approximate normal model for all three portfolios except the exact simulation method. The exact simulation risks must be less than the approximate risks

Table 4.2. Portfolio specifications

	Sub-portfolio	BP	GM	MOT	IBM	HON	σ_p^2
	Conservative	0,40	0,15	0,05	0,20	0,20	0.000120
First	**Balanced**	0,20	0,20	0,20	0,20	0,20	0.000151
portfolio	**Risky**	0,10	0,20	0,40	0,10	0,20	0.000213
		GM	**JNJ**	**MCD**	**DD**	**DAI**	σ_p^2
	Conservative	0,05	0,40	0,20	0,20	0,15	0.000090
Second	**Balanced**	0,20	0,20	0,20	0,20	0,20	0.000127
portfolio	**Risky**	0,40	0,10	0,20	0,10	0,20	0.000193
		UNP	**PG**	**MMM**	**JNJ**	**DIS**	σ_p^2
	Conservative	0,15	0,40	0,20	0,20	0,05	0.000059
Third	**Balanced**	0,20	0,20	0,20	0,20	0,20	0.000071
portfolio	**Risky**	0,20	0,10	0,20	0,10	0,40	0.000100

since the approximate risk is an upper bound for the exact risk. Also for the risky sub-portfolio of the first portfolio, GARCH risk estimate is less than the risk estimate of the approximate normal model. This is because of that the current variance of MOT is less than its sample variance and its weights is 0.40 for this portfolio. So the portfolio is dominated by MOT and GARCH estimate became smaller than the approximate normal risk.

For the conservative and balanced sub-portfolios of the first portfolio, the highest risks are given by historical simulation. This might seem inconsistent at the first glance since it was said that historical data lack of extremes in the tails. But the stocks of these portfolios have relatively more data in their tails and they dominate these portfolios. For example 40 per cent of the conservative sub-portfolio of the first portfolio is BP, 20 per cent of it is HON and five percent of it is MOT. The empirical quantiles of these stocks are smaller than the quantiles of the fitted models. So the historical simulation results are higher than the t-t copula results. However the difference is not very large.

Table 4.3. Quantiles of empirical and fitted distributions for the stocks of the first portfolio

	BP	**GM**	**MOT**	**IBM**	**HON**
1% Empirical Quantile	-0.033475	-0.060796	-0.068059	-0.034562	-0.046229
Fitted Distribution	t	GHD	t	t	t
1% Quantile of Fitted Distribution	-0.032072	-0.062123	-0.062311	-0.035928	-0.042392

Table 4.4. Sample and GARCH standard deviations for the stocks of the first portfolio

	BP	**GM**	**MOT**	**IBM**	**HON**
Sample sd	0.01288	0.02297	0.02379	0.01336	0.0161
GARCH sd	0.01312	0.03220	0.01644	0.01264	0.01546

For the second portfolio, the highest risks are given by GARCH in the balanced and risky sub-portfolios. This is again because of the high volatility trend of the stocks. But contrasts to the first portfolio, t-t copula risks are higher than the historical simulation risks for all three sub-portfolios. In this case the empirical data are deficient in the tails and not able to capture the extremes but the fitted marginal models are able to do so. As it can be seen from Table 4.5, the fitted distributions have quite similar quantiles. Thus the risk estimates of the t-t copula and historical simulation are close to each other which can be seen as a validation of that t-t copula reflects the true dependence between the financial assets.

Table 4.5. Quantiles of empirical and fitted distributions for the stocks of the second portfolio

	GM	**JNJ**	**MCD**	**DD**	**DAI**
1% Empirical Quantiles	-0.060796	-0.027357	-0.040821	-0.034656	-0.046626
Fitted Distribution	GHD	GHD	t	t	GHD
1% Quantile of Fitted Distribution	-0.062123	-0.027536	-0.040741	-0.033672	-0.046408

Table 4.6. Sample and GARCH standard deviations for the stocks of the second portfolio

	GM	**JNJ**	**MCD**	**DD**	**DAI**
Sample sd	0.02297	0.01026	0.01587	0.01307	0.01800
GARCH sd	0.03220	0.00806	0.01715	0.01602	0.02075

For the third portfolio, the highest risks are given by GARCH model for all three sub-portfolios because of the high volatility trends. Again t-t copula risks are higher than the historical simulation risks. This again indicates that t-t copula model is able to capture the extremes more than the historical simulation. However the difference between the two models is not very large which again indicates that t-t copula model can reflect the true dependence between the asset returns.

Table 4.7. Quantiles of empirical and fitted distributions for the stocks of the third portfolio

	UNP	**PG**	**MMM**	**JNJ**	**DIS**
1% Empirical Quantiles	-0.035964	-0.023498	-0.028558	-0.027357	-0.041328
Fitted Distribution	GHD	t	t	GHD	t
1% Quantile of Fitted Distribution	-0.033134	-0.023477	-0.030640	-0.027536	-0.042671

Table 4.8. Sample and GARCH standard deviations for the stocks of the third portfolio

	UNP	**PG**	**MMM**	**JNJ**	**DIS**
Sample sd	0.01388	0.00921	0.01175	0.01026	0.01607
GARCH sd	0.02178	0.00882	0.01713	0.00806	0.02114

For one day ES, all the alternative methods have higher risks than the approximate normal model for all three portfolios except the exact simulation method and the risky sub-portfolio of the first portfolio because of the same reasons with one day $VaR_{0.99}$.

For all sub-portfolios of the first portfolio, the highest risks are given by historic simulation. This might be expected because the same situation occurred for one day $VaR_{0.99}$. This is a result of that the stocks of the first portfolio have relatively more data in their tails. Also the differences between the risk estimates of the two methods are relatively higher than the differences of $VaR_{0.99}$. This is because of that the ES is related with the returns lower than the VaR. VaR results are close to each other since the empirical and theoretical quantiles are similar. But since the empirical distributions have some outliers, historical simulation give higher ES estimates than the t-t copula.

For the second portfolio, the highest risks are given by historical simulation in the conservative sub-portfolio, by t-t copula in the balanced sub-portfolio and by GARCH in the risky sub-portfolio. We do not regard GARCH results since we know that it depends on the current volatility trends. In this case the t-t copula risks are higher than the historical simulation results for the balanced and risky sub-portfolios. They are very close to each other for the conservative sub-portfolio. This situation was also occurred in $VaR_{0.99}$ since the empirical and theoretical quantiles are quite similar.

For the third portfolio, the highest risks are given by the t-t copula for the conservative sub-portfolio and by GARCH model for the balanced and risky sub-portfolios because of the high volatility trends. Again t-t copula risks are higher than the historical simulation risks as in the case of $VaR_{0.99}$, which again indicates that t-t copula model is able to capture the extremes than the historical simulation.

To sum up the results for one day, all the alternative models give higher risks than the approximate multinormal model (except exact simulation). GARCH model generally give the highest risks especially for VaR. But it estimates the risk to be conditionally normal and thus it does not seem a proper model for one day. The t-t copula model gives higher or similar risks compared to historical simulation. Also the differences between the estimates of the two models are not so large. This indicates that the t-t copula model is able to capture the extreme returns and it reflects the true dependence between the assets since the results are relatively closer to each other than the other models and the true dependence is embedded into the historical data. Thus the t-t copula model is concluded to be a close to real model and is promising for one day risk estimation.

4.1.2.2. The Risk for Multidays. For multidays, GARCH gives the highest risks for the second and third portfolios for both risk measures. For the first portfolio, either historical simulation or GARCH gives the highest risks. However higher risks do not mean the true risks. GARCH is a useful tool for modeling long term volatility in memory but it assumes conditional multinormality for portfolios. Thus the dependence structure between the assets is again considered to be linear and this is known to be incorrect. Thus we focus on the results of the historical simulation and the t-t copula.

For the first portfolio, historical simulation results are generally higher than the t-t copula results for both risk measures. This situation was also occurred for one day $VaR_{0.99}$. The reason is that the empirical quantiles are smaller than the theoretical quantiles and there are some outliers in the empirical data. This reason caused the historical simulation estimates to be higher than the t-t copula estimates. For the second and third portfolios, t-t copula results are always higher than the historical simulation results for both risk measures. However the difference is not very large especially in the second portfolio. This is because of that for the stocks of the second and third portfolios, the empirical and theoretical quantiles are quite similar.

An important detail for the t-t copula is that, the selection of the marginal asset return distributions is very crucial. The more the univariate marginal models capture the extremes, the more accurate is the risk estimate. For example in the first portfolio, some of the marginal distributions are not able to capture the extreme outliers. Thus the historical simulation results are higher than the t-t copula results. Therefore the care has to be taken for modeling the marginals.

To sum up the results for multi-days, disregarding GARCH, the t-t copula gave higher risk estimates than the historical simulation and approximate multinormal model for almost all the portfolios (and sub-portfolios). This is an indication of explaining the extreme events. But as for single stocks, as the time horizon increases, all the risk estimates of all the models, except GARCH, tend to decrease compared to the approximate multinormal model. This again indicates that approximate multinormal model underestimates the risk for short time horizons. But as the time horizon increases, it moves towards to overestimate it.

4.1.3. Sensitivity Analysis

We performed three sensitivity analyses to see how the simulation output is sensitive to the input parameters. As mentioned before as the degrees of freedom (df) increases, the t-copula approaches to the normal copula and the t-distribution approaches to the normal distribution. Also the df complicates the calibration of the t-copula. Therefore we performed two sensitivity analyses for the df of the t-copula of the third portfolio and the df of PG which is included in the third portfolio. Also we performed a third sensitivity analysis for the correlation coefficient of the t-copula between two stocks.

The df of the t-copula of the third portfolio is 11.46 and it is 5.84 for PG. The correlation coefficient for PG-MMM is 0.394. We added $\pm 2SE$ to their estimated values and calculated the portfolio risk for the conservative weights. The results can be seen in Figure 4.6 where the sample size is 1000 with 100 replications (100,000 simulations). The dots are the risk estimates and the lines are the confidence intervals for the risk estimates. The red dots and lines represent the risks found by the original parameters.

One cay say from Figure 4.6 that the df of the t-copula does not have a significant effect on the results and the risk is not so sensitive to df since it is relatively high and the sample is relatively small. Its effect also seems unstable. However the trend of the risk shows that the risk estimate increases as the df decreases as it should be expected. But the situation is different for PG. As the df of PG increases, the risk estimate decreases significantly. Thus the risk estimate is more sensitive to the df of PG than the df of the t-copula. The correlation coefficient also has an effect on the risk but it is not as much as the effect of the df of PG.

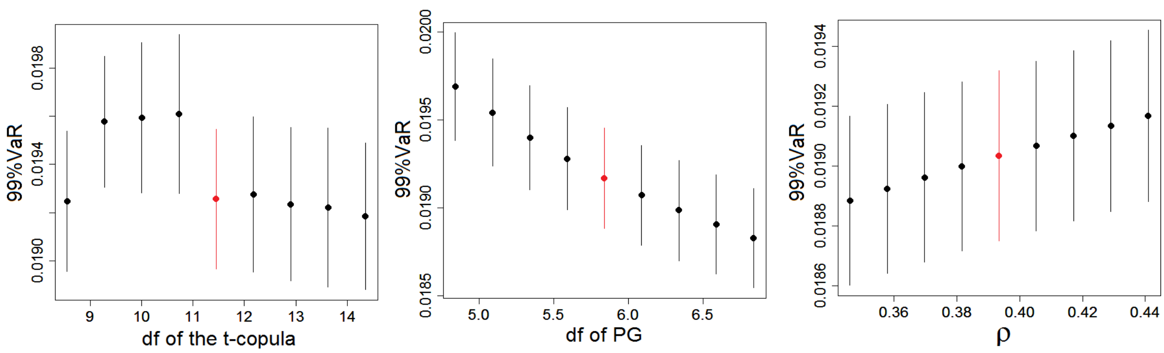

Figure 4.6. Sensitivity of the risk to simulation parameters ($n = 1,000,000$)

The same analyses were performed by increasing the number of replications to 1000 with the same sample size (1,000,000 simulations). The results are given in Figure 4.7. The effect of the df of the t-copula is more clear and the trend shows itself more clearly since the sample is relatively larger. However the confidence intervals of the risks estimates are still large to conclude on the significant effect of the df of the t-copula for the risk estimate. Again the sensitivity of the risk estimate to the df of PG is very significant. The correlation coefficient also has relatively significant effect on the risk estimate but it has still large confidence intervals.

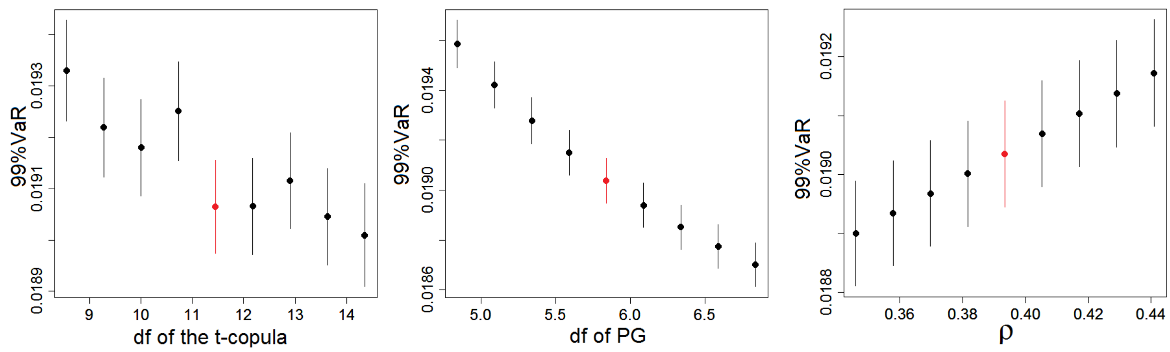

Figure 4.7. Sensitivity of the risk to simulation parameters ($n = 100,000$)

From the results of the sensitivity analyses, we can conclude that the estimated risk is more sensitive to the marginal distributions rather than the copula especially for small samples. We know from the copula fitting examples that the t-copula is the best fitting copula to the asset returns especially in higher dimensions. Thus, to estimate accurate risks, we should estimate the parameters of the t-copula accurately and to achieve this, we should select the true marginal distributions and estimate their parameters accurately. Therefore the selection of the marginal distributions for the asset returns is very crucial for the risk calculation and it has a relatively strong effect on the estimated risks.

Since the t-copula showed that it is a nice tool to model the dependencies between financial assets, and since the t-distribution and GHD showed that they are very good models for the univariate asset return distributions, t-t copula is concluded to be a "close to real model" for the risk calculation of portfolios. Therefore the portfolio risk calculation by the t-t copula model will be explained in detail in the next section.

4.2. Estimating the Risk of t-t Copula Models

A detailed explanation of the portfolio risk calculation by the t-t copula model is given in this section.

4.2.1. Naive Simulation

In this study, the t-t copula risk calculations were performed using the algorithm which is given in pseudo code below. The inputs of the algorithm are:

- $\beta = (\beta_1, \ldots, \beta_d)$: the parameter vector for d marginal distributions $F_1, \ldots, F_d$ (either t or GHD),
- ρ and υ: the parameters for the t-copula,
- w: the weight vector of the stocks,
- n_{outer}: the number of replications of simulation,
- n_{inner}: the sample size for each simulation,
- n_{days}: time horizon for risk calculation.

The outputs of the algorithm are the means and standard errors of the estimated risk measures ($VaR_{0.99}$ and $ES_{0.99}$).

INPUT: $\beta = (\beta_1, \ldots, \beta_d)$, ρ and df, w, n_{outer}, n_{inner} and n_{days}
for $i = 1$ to n_{outer} **do**
  for $j = 1$ to n_{inner} **do**
    Set $x^{(t)} \leftarrow 0$ for $t = 1, \ldots, d$
    for $k = 1$ to n_{days} **do**
      Generate a d-dimensional t-copula u with parameters ρ and df
      Set $x^{(t)} \leftarrow x^{(t)} + F_t^{-1}(u^{(t)}, \beta_t)$ for $t = 1, \ldots d$, where $(.)^{(t)}$ is the t^{th} entry of $(.)$ and F_t^{-1} is the inverse of the t^{th} marginal.
    end for
    $r^{(j)} = 1 - \sum_{t=1}^{d} w^{(t)} e^{x^{(t)}}$
  end for
  Sort r, set $VaR_{0.99}^{(i)} \leftarrow r^{(n_{inner} \times 0.99)}$ and set $ES_{0.99}^{(i)} \leftarrow \frac{\sum_{j>(n_{inner} \times 0.99)} r^{(j)}}{n_{inner} \times 0.01}$
end for
OUTPUT: The means and the standard errors of $VaR_{0.99}$ and $ES_{0.99}$

Figure 4.8. Naive simulation algorithm for the t-t copula

This naive algorithm was implemented in R and used for portfolio risk calculation by the t-t copula method. To reduce the error of the estimated risks, variance reduction techniques (VRT) were also applied to the naive algorithm.

4.2.2. Variance Reduction

The theoretical part of this section mainly follows [16].

A simulation can be regarded as a function which inputs some values and outputs one or more values. If there is randomness in the simulation, the output will be different in each repetition of the simulation. Therefore there will be some noise in the output. Less noise in the output means more accurate results. VRT try to decrease the noise of the output of simulations. There are several VRT to reduce the noise of simulation outputs. In this study, variance reduction was performed for the t-t copula model using Antithetic Variates (AV) and Importance Sampling (IS) methods.

4.2.2.1. Antithetic Variates Method. In the AV method, the variance of a simulation output is decreased using negatively correlated input variables. Suppose that we would like to estimate $\mu = E[Y]$ where $Y = q(X)$ is the output of a simulation experiment and the function q describes the operation of a single simulation. Suppose that we have generated two samples, Y_1 and Y_2. Then an unbiased estimate of μ is given by:

$$\hat{\mu} = \frac{Y_1 + Y_2}{2} \tag{4.2}$$

and

$$V(\hat{\mu}) = \frac{V(Y_1) + V(Y_2) + 2Cov(Y_1, Y_2)}{2} \tag{4.3}$$

The variance of $\hat{\mu}$ can be reduced if we can arrange Y_1 and Y_2 so that $Cov(Y_1, Y_2) < 0$. Y_i is the output of the i^{th} simulation run and thus Y_i is a function of the input random variables $X_i = (X_{i1}, \ldots, X_{in})$. We cannot directly control the simulation of Y_i but we can directly control the simulation of the input vector X_i for the i^{th} simulation. For most simulations, the outputs are monotonically dependent on the input variables. Thus to obtain negatively correlated outputs for Y_1, we should use negatively correlated inputs for X_1.

We can obtain negatively correlated input variables X according to its type:

- If X is $U(0, 1)$, then $X_2 = 1 - X_1$
- If X is symmetric around 0, then $X_2 = -X_1$
- For general X with CDF $F(x)$ and inverse $F^{-1}(u)$, we generate a uniform random variate $U(0, 1)$ and use $X_1 = F^{-1}(U)$ and $X_2 = F^{-1}(1 - U)$.

To make a fair comparison between the naive and AV simulations, we should simulate $n/2$ correlated pairs (Y_1, Y_2) for AV. Therefore in total we have n simulations for the naive and n simulations for the AV.

In the t-t copula method, we can control the simulation of the t-copula. Thus AV was performed in the t-copula generation step. As mentioned before, a d-dimensional t-copula is generated as:

$$x = T_v(t) \tag{4.4}$$

where

$$t = \frac{\sqrt{v}}{\sqrt{s}} y, \;\; y \sim N(0, \Sigma), \;\; s \sim \chi_v^2 \tag{4.5}$$

Therefore the calculated risk by the t-t copula simulation is a function of the input variables X, standard multinormal variates, and χ^2 variates. Thus we can control either these two variates or both of them. Here we used AV for the multinormal variates since they are symmetric around zero. So we simply took $y_2 = -y_1$. We also kept the same variates s for χ^2 in both simulations. This resulted for the new multivariate t variates as the negative of the first ones, i.e. $t_2 = -t_1$. Since the t-copula is generated by taking the CDF of the multivariate t distribution, the t-copula for AV uses $1 - u$, if the copula was u in the first simulation since the standard t-distribution is symmetric around zero.

The results are given in Table 4.9 for the balanced sub-portfolio of the first portfolio for one day $VaR_{0.99}$ and $ES_{0.99}$. For the naive simulation, the sample size is 1000 with 100 repetitions. For AV, the sample size is 500 with 100 repetitions since we should halve the sample size for AV.

From the results, one can say that no variance reduction was reached. However the CPU time of the AV simulation is only half of the naive simulation. This technique

Table 4.9. Naive simulation and AV results

	$VaR_{0.99}$		$ES_{0.99}$		
Method	**Value**	**SE**	**Value**	**SE**	**CPU time (sec)**
Naive	0.02648	0.00024	0.03436	0.00036	3.39
AV	0.02653	0.00025	0.03466	0.00044	1.67

did not reduce the variance but did the running time. Therefore we can double the sample size and obtain a reduction in the variances of the risk estimates with similar CPU times. The results are given in Table 4.10. In this case the variances are reduced and the variance reduction factors (VRF) are :

$$VRF_{VaR} = \left(\frac{0.00024}{0.00015}\right)^2 \approx 2.56$$
$$VRF_{ES} = \left(\frac{0.00036}{0.00026}\right)^2 \approx 1.91$$

Table 4.10. Naive simulation and AV results with doubled sample size

	99% VaR		**99% ES**		
Method	**Value**	**SE**	**Value**	**SE**	**CPU time (sec)**
Naive	0.02648	0.00024	0.03436	0.00036	3.39
AV	0.02680	0.00015	0.03480	0.00026	3.18

4.2.2.2. Importance Sampling Method. Importance Sampling (IS) is a variance reduction technique especially useful for rare event simulations. We can simply represent a simulation as the following integral form:

$$\mu = E(Y) = E_f[q(X)] = \int q(x)f(x)dx \qquad (4.6)$$

where Y denotes the single output of a simulation, x is the d-dimensional vector of input variables of the simulation, the function q describes the output of a single simulation run and $f(x)$ is the joint density function of all input variables. A single repetition of the naive simulation algorithm for the evaluation of the integral above consists of the two steps:

- Generate the input variables vector X
- Calculate $Y = q(X)$

The integral above can be also represented for IS as:

$$\mu = E(Y) = E_g[q(X)f(x)] = \int q(X)\frac{f(x)}{g(x)}g(x)dx = \int q(X)w(x)g(x)dx \tag{4.7}$$

where $g(x)$ is called the *IS density* and the correction factor $w(x)$ is called *weight* in the simulation and *likelihood ratio* in the statistical literature. A single repetition of the IS algorithm consists of the following steps:

- Generate the input variable X with density g(x).
- Calculate $Y = q(X)\frac{f(x)}{g(x)}$

IS density $g(x)$ should be selected such that it is possible to generate random variates with density $g(x)$ and the variance of the estimator is small. It can be shown that the variance is minimized for:

$$g(x) = \frac{|q(x)f(x)|}{\int |q(x)f(x)|dx} \tag{4.8}$$

but in practice the denominator is unknown. Also the estimate of IS is unbiased and has a bounded variance if the IS density $g(x)$ has higher tails than $q(x)f(x)$. So the selection should be done regarding two principles:

- The IS density $g(x)$ should mimic the behavior of $|q(X)f(x)|$,
- The IS density $g(x)$ must have higher tails than $|q(X)f(x)|$.

In practice the selection of the IS density is a trial and error process guided by these two general rules.

For the d-dimensional case, the above integral becomes a d-dimensional integration. $f(x)$ becomes the product of the d marginal densities if they are i.i.d.

$$f(x) = \prod_{i=1}^{d} f_i(x_i) \tag{4.9}$$

In such a case the IS density is also the product of the marginal IS densities. Thus the weight becomes:

$$w(x) = \prod_{i=1}^{d} \frac{f_i(x_i)}{g_i(x_i)} \tag{4.10}$$

In higher dimensions, i.e. $d \geq 5$, even a small difference between f and g may lead to very small and very large weight values and thus to instable results. Thus care has to be taken in selecting the parameters of g for higher dimensions.

For our case, IS is performed for calculating $ES_{0.99}$ for the portfolio since it is the highest risk. The IS is used for the standard normal variates and the χ^2 variates in the t-copula generation step. For the standard normal variates, the IS density is selected by shifting the mean zero to a negative value and increasing the variance so as to have higher tails than the original standard normal densities. The IS density for the normal variates is in the general form:

$$g(x) \sim N(\mu_{is}, \sigma_{is}) \tag{4.11}$$

where μ_{is} and σ_{is} was zero and one in the naive simulation. For the χ^2 variates, the IS density is selected by decreasing the degrees of freedom to increase the probability of the rare events. Thus the IS density for the χ^2 variates is:

$$g(x) \sim \chi^2(df_{is}) \tag{4.12}$$

where df_{is} was the degrees of freedom of the t-copula in the naive simulation. After trying several values for the parameters of the normal and χ^2 IS densities, the results were founded to be the best which are given in Table 4.11.

Table 4.11. Naive simulation and IS results

	$ES_{0.99}$	
Method	**Value**	**SE**
Naive	0.03436	0.00036
IS with normals (μ_{is}=-0.25 ,σ_{is}=1.42)	0.03431	0.00026
IS with χ^2 (df_{is}=7.5)	0.03449	0.00029

The obtained VRF's are:

$$VRF_{Normal} = \left(\frac{0.00036}{0.00026}\right)^2 \approx 1.91$$
$$VRF_{\chi^2} = \left(\frac{0.00036}{0.00028}\right)^2 \approx 1.65$$

5. PORTFOLIO OPTIMIZATION

In this section the classical model for portfolio selection will be explained and how to construct optimal portfolios using the classical model will be illustrated. Then the copula based portfolio optimization will be introduced.

5.1. Classical Model for Portfolio Optimization

What we call "classical model" is in fact the model used in Modern Portfolio Theory (MPT). MPT proposes how rational investors will use diversification to optimize their portfolios and how a risky asset should be priced. In the classical model, the portfolio return is assumed to be the weighted sum of the individual returns of the portfolio assets. Since the return of an asset is a random variable, the return of the portfolio is also a random variable with an expected value and a variance.

Markowitz [1] set up a quantitative framework for portfolio selection. In this framework it is assumed that asset returns follow a multivariate normal distribution. This means that the return of a portfolio can be completely described by an expected return and a variance. This framework is called the mean-variance approach. In the mean-variance approach of Markowitz, the aim of the investor is to minimize the risk at a given expected return level, or maximize the expected return at a given risk level. Markowitz explains this situation in his famous paper as:

"...the rule that the investor does (or should) maximize discounted expected, or anticipated, returns is rejected both as a hypothesis to explain, and as a maximum to guide investment behavior. He considers the rule that the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing. There is a rule which implies both that the investor should diversify and that he should maximize expected return. This rule is a special case of the "expected returns-variance of returns" rule. It assumes that there is a portfolio which gives both maximum expected return and minimum variance, and it commends this portfolio to the investor"

[1].

5.1.1. Mathematical Model of the Problem

In this section the portfolio optimization problem will be defined as a mathematical model. Let the parameters and variables of the problem be:

- d: the number of available assets available
- μ_i: the expected return of asset i, $i = 1, \ldots, d$
- σ_{ij}: the covariance between assets i and j, $i = 1, \ldots, d$ and $j = 1, \ldots, d$
- r: the desired expected return
- w_i: the weight of asset i in the portfolio, $i = 1, \ldots, d$

Then according to the mean-variance model of Markowitz, we get the following mathematical problem which is referred to as P1:

P1:

$$\min \quad \sum_{i=1}^{d} \sum_{j=1}^{d} w_i w_j \sigma_{ij} \tag{5.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{d} w_i \mu_i = r \tag{5.2}$$

$$\sum_{i=1}^{d} w_i = 1 \tag{5.3}$$

$$0 \leqslant w_i \leqslant 1, \qquad i = 1, \ldots, d \tag{5.4}$$

We can write Equation (5.1) in vector-matrix notation as $w^T \Sigma w$, where Σ is the covariance matrix of asset returns.

The objective of the problem is to minimize the risk of the portfolio which is measured as the variance. Equation (5.2) ensures the expected return of the portfolio to be r and Equation (5.3) ensures the sum of the weights to be equal to one. Equation

(5.4) implies that short-selling is not allowed. Therefore all the currency will be invested into the portfolio.

P1 is a convex *quadratic programming* (QP) problem which can easily be solved by computationally effective algorithms [34]. It is sometimes referred as *unconstrained portfolio optimization problem* since there are no additional constraints about the decision variables. The problem can be extended by adding the cardinality constraints by which one can restrict the minimum and maximum weights for each asset and the number of assets in the portfolio. Then the problem becomes a *cardinality constrained* problem as:

$$\min \quad \sum_{i=1}^{d}\sum_{j=1}^{d} w_i w_j \sigma_{ij} \tag{5.5}$$

$$\text{s.t.} \quad \sum_{i=1}^{d} w_i \mu_i = r \tag{5.6}$$

$$\sum_{i=1}^{d} w_i = 1 \tag{5.7}$$

$$\sum_{i=1}^{d} z_i = K \tag{5.8}$$

$$\varepsilon_i z_i \leqslant w_i \leqslant \delta_i z_i, \quad i = 1, \ldots, d \tag{5.9}$$

$$z_i \in \{0, 1\}, \qquad i = 1, \ldots, d \tag{5.10}$$

where ε_i and δ_i represent respectively the minimum and maximum weight of asset i in the portfolio. z_i is a 0-1 decision variable which takes zero if asset i is not included in the portfolio and takes one if it is included. K is the number of assets to be included in the portfolio. In this case the constrained problem is a *Mixed Integer Quadratic Programming* (MIQP) problem which is a harder problem to solve than the classical QP. But in this study we will deal only with the unconstrained case.

5.1.2. Different Representations of the Mathematical Model

The problem in mean-variance framework can also be represented by different models. In the first alternative, the constraint about the expected return is added to the objective function and it is deleted from the constraint set. This model represents the trade-off between the risk and the return [35].

P2:

$$\begin{aligned} \min \quad & \lambda \left[\sum_{i=1}^{d} \sum_{j=1}^{d} w_i w_j \sigma_{ij} \right] - (1-\lambda) \sum_{i=1}^{d} w_i \mu_i && (5.11) \\ \text{s.t.} \quad & \sum_{i=1}^{d} w_i = 1 && (5.12) \\ & 0 \leqslant w_i \leqslant 1, \qquad i = 1, \ldots, d && (5.13) \end{aligned}$$

This optimization problem is referred to as P2. In this model, $\lambda = 0$ represents maximizing the expected return, irrespective of the risk involved, and the optimal solution will involve just the single asset with the highest return. Conversely, $\lambda = 1$ represents minimizing the risk, irrespective of the return involved, and the optimal solution will typically involve a number of assets. Values of λ satisfying $0 < \lambda < 1$ represent an explicit trade-off between the risk and the return, generating solutions between the two extremes $\lambda = 0$ and $\lambda = 1$ [34].

The second alternative model is the opposite of the mean-variance model. In the mean-variance model, the risk is minimized for a given level of expected return. However in this second alternative model, the expected return is maximized for a given level of risk, therefore the expected return term is defined as the objective function and the risk term is added into the constraint set. The objective can be also defined as minimizing the negative expected return which is equivalent to maximizing the positive expected return. This problem is referred to as P3:

P3:

$$
\begin{aligned}
\min \qquad & -\sum_{i=1}^{d} w_i \mu_i && (5.14)\\
\text{s.t.} \qquad & \sum_{i=1}^{d} w_i = 1 && (5.15)\\
& \sum_{i=1}^{d}\sum_{j=1}^{d} w_i w_j \sigma_{ij} = q && (5.16)\\
& 0 \leqslant w_i \leqslant 1, \qquad i = 1, \ldots, d && (5.17)
\end{aligned}
$$

5.1.3. Efficient Frontier

Every possible asset combination can be plotted in risk-return space, and the collection of all such possible portfolios defines a region in this space. The line along the upper edge of this region is known as the *efficient frontier*. Combinations along this line represent the portfolios which have the lowest risk for a given level of expected return. The efficient frontier can be obtained by resolving P1 for varying values of r, P2 for varying values of λ or P3 for varying values of q. In other words, the three problem formulations lead to the same efficient frontiers when varying the parameters r, λ and q respectively. Thus a portfolio that is efficient for one of the three problem formulations will be also efficient for the other two formulations [36].

Since it is known that the variance is not an adequate risk measure, the mean-variance framework can be transformed into a mean-risk framework by defining the objective function as VaR or ES in P1.

5.1.4. Minimum Risk Portfolio

The minimum risk portfolio (MRP) is the one which gives the minimum risk on the efficient frontier without considering the expected return. The portfolios which have smaller returns than the return of the MRP can be disregarded since there is

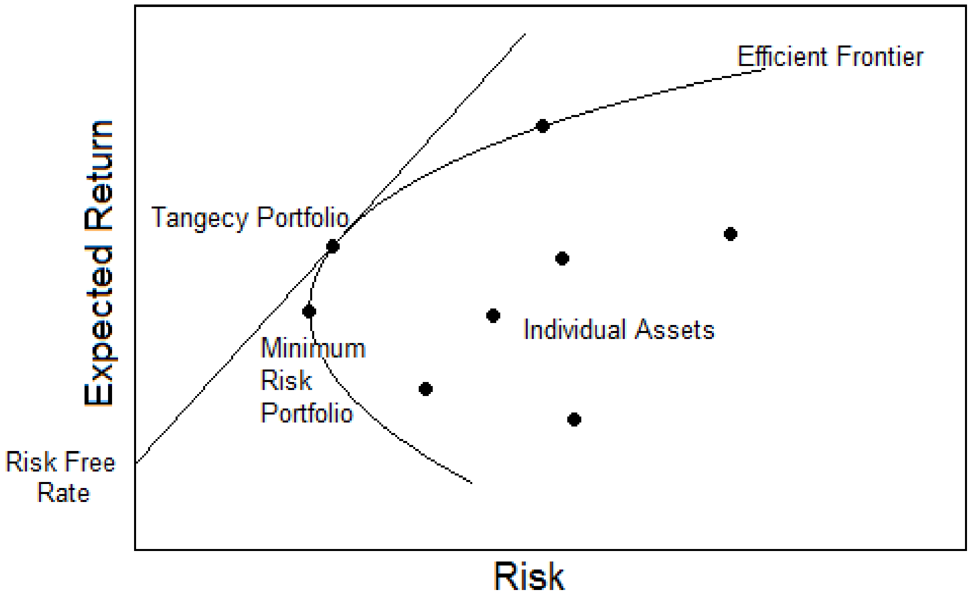

Figure 5.1. Efficient frontier

always a portfolio with higher return and less risk.

5.1.5. Sharpe Ratio

If there is a benchmark asset such as the risk-free asset, the Sharpe ratio represents a measure of the amount of additional return (above the benchmark asset) a portfolio provides compared to its risk. For an asset i, the Sharp ratio is defined as:

$$S_i = \frac{E(r_i) - r_f}{\sigma_i} \tag{5.18}$$

where $E(r_i)$ and σ_i is the expected return and the standard deviation of asset i and r_f is the return of the risk-free asset which is assumed to be constant. $E(r_i) - r_f$ is called the *risk premium* or *equity premium*. It is the expected excess return of asset i over the risk-free return. The Sharpe ratio is used to characterize how well the return of an asset compensates the investor for the risk taken. When comparing two assets, each with the same expected return $E(r)$, against the same benchmark return r_f, the asset with the higher Sharpe ratio gives more return per the risk taken.

5.1.6. Tangency Portfolio

The portfolio with the highest Sharpe ratio on the efficient frontier is known as the *tangency portfolio* (TP). The combination of the risk-free asset and the TP produces a larger return that is above the efficient frontier. Under certain assumptions, the TP must consists of all assets available to investors and each asset must be held in proportion to its market value relative to the total market value of all assets. Therefore the TP is often called the *market portfolio* [35].

5.1.7. Capital Market Line

When the market portfolio is combined with the risk-free asset, the result is the Capital Market Line (CML). All points along the CML have higher expected returns than the portfolios on the efficient frontier. The slope of the CML determines the additional expected return needed to compensate for a unit change in risk. The expression for the CML is:

$$E(r_p) = r_f + \left(\frac{E(r_m) - r_f}{\sigma_m} \right) \sigma_p \tag{5.19}$$

where r_p and σ_p is the expected return and the standard deviation of the portfolio which consists of the market portfolio and the risk-free asset. In other words, the CML says that the expected return of a portfolio is equal to the risk-free asset plus a risk premium, where the risk premium is equal to the market price of risk times the quantity of risk for the portfolio [35].

5.2. Application of the Classical Model to NYSE Data

As an application, the classical unconstrained mean-variance model (P1) was applied to NYSE data to find efficient portfolios, i.e. the optimal portfolios for given levels of expected return. With this aim, five stocks were selected arbitrarily from NYSE data as candidates to construct a portfolio. The time horizon was assumed to be one day and no cardinality constraints were used since the aim of the study is not to solve a complicated portfolio optimization problem with Markowitz approach. The aim is to compare the result of the classical Markowitz approach with the results of the copula based optimization which will be explained in this chapter.

The parameters of the problem were estimated from the sample statistics. Thus the expected returns were taken as the sample means of the asset returns, and the covariances between the assets were taken as the sample covariances. In Table 5.1 and Table 5.2, the estimated expected returns and covariances of the stocks are given.

Table 5.1. Expected returns of five stocks

	BP	**UNP**	**GM**	**PG**	**MOT**
$E(r)$	0.000515	0.000603	0.000012	0.000388	0.000474

Table 5.2. Covariances of five stocks

	BP	**UNP**	**GM**	**PG**	**MOT**
BP	0.000166	0.000067	0.000076	0.000027	0.000066
UNP	0.000067	0.000193	0.000110	0.000039	0.000098
GM	0.000076	0.000110	0.000528	0.000043	0.000140
PG	0.000027	0.000039	0.000043	0.000085	0.000027
MOT	0.000066	0.000098	0.000140	0.000027	0.000566

The QP was solved by using the quadratic program solver package “quadprog” of the statistical software R [10]. The efficient frontier was obtained by resolving P1 for varying values of the expected portfolio return. The expected portfolio return was varied sequentially between the minimum and maximum expected returns of the individual assets since the portfolio return can take the values between these two extremes. By this method 100 efficient portfolios were obtained. The risks (variances) and the expected returns of these 100 portfolios were plotted and the efficient frontier was obtained which is given in Figure 5.2. This frontier includes only the efficient portfolios after the MRP since the portfolios which have smaller expected returns than the return of the MRP can be disregarded. Using the same efficient portfolios, another efficient frontier was obtained by taking the risk as $VaR_{0.99}$ since it is known that the variance is not an adequate risk measure as mentioned before. This new frontier was used to determine the MRP and the TP.

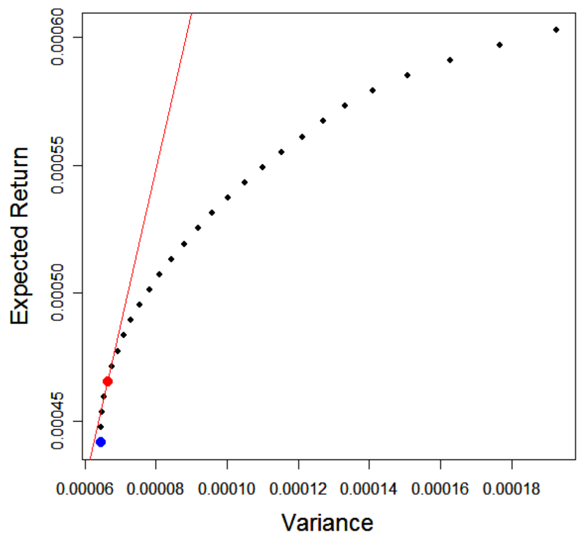

Figure 5.2. Mean-variance efficient frontier of five stocks

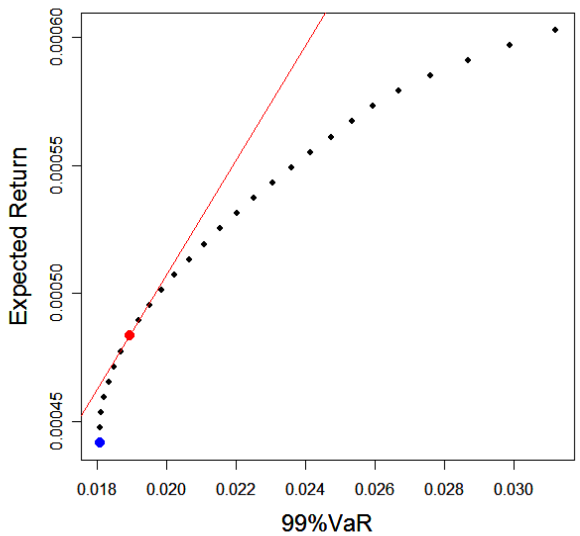

Figure 5.3. Mean-$VaR_{0.99}$ efficient frontier of five stocks

The risk-free asset was taken as the 13-week US treasury bill. The 13-week return data were transformed to daily logreturns. Then the last year's (252 data points) average daily logreturn was used as the risk-free return. After transformations, the daily risk-free return was found to be around 0.00006.

In the two frontiers, the blue dots indicate the MRP's and the red dots indicate the TP's. The red lines are the CML's. In Table 5.3, the weights of the assets in these two portfolios (MRP and TP) are given rounded to the third digit when the risk measure is taken as $VaR_{0.99}$.

Table 5.3. The weights of five stocks in MRP and TP

	BP	**UNP**	**GM**	**PG**	**MOT**	**E(r)**	**99% VaR**
MRP	0.227	0.108	0.005	0.622	0.038	0.000441	0.01807
TP	0.267	0.276	0	0.434	0.023	0.000483	0.01893

The weights of the two portfolios are given as bar charts in Figure 5.4.

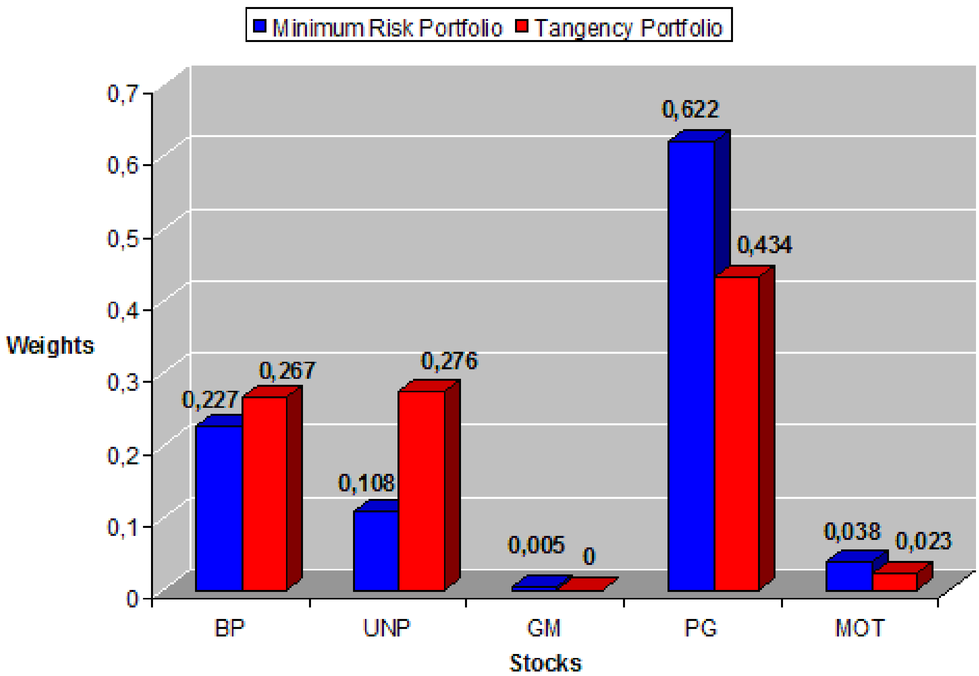

Figure 5.4. The weights of five stocks in MRP and TP

If we do not take into account the normality assumption, the main disadvantage of the Markowitz model is that, the number of parameters of the problem exponentially increases if the asset universe increases. For example if there are n candidate assets for constructing a portfolio, one needs to estimate n standard deviations for the individual assets and $n(n-1)/2$ correlations between the assets which makes $n(n+1)/2$ parameters in total [37]. Thus several practical models were developed which are based on the Markowitz model. These models are called the General Equilibrium Models (GET) and the most important one is the Capital Asset Pricing Model (CAPM).

5.3. CAPM

In this section the CAPM will be explained with the main ideas. CAPM is used in finance to determine a theoretically appropriate required rate of return of an asset, if that asset is to be added to an already well-diversified portfolio. It has some assumptions about the market conditions and the investors' preferences. The reader is referred to [35] and [37] for the detailed information of CAPM and derivation of the formulas given in this section.

CAPM separates the risk of individual assets as the *systematic* and *unsystematic risk*. The systematic risk (or the market risk) is common for all the assets in the market and it cannot be diversified away. It is the risk caused by socioeconomical and political events. The only risk that can be diversified away is the unsystematic risk which is the risk associated with individual assets.

CAPM uses a benchmark asset for estimating systematic risks of individual assets. This benchmark asset is called the market portfolio. The market portfolio is assumed to include all possible risky investments each of which is weighted with its market value. In practice, the benchmark asset is generally taken as a market index such as NYSE composite or S&P500. The systematic risk of an asset is represented by β which is a measure of the sensitivity of individual assets to the market conditions. β of the market portfolio is defined as one. Assets with β less than one have less systematic risks where the assets with β higher than one are more risky assets. The β coefficient of an asset is calculated as:

$$\beta_i = \frac{Cov(r_i, r_m)}{Var(r_m)} \tag{5.20}$$

where r_i is the random variable of the returns of asset i and r_m is the random variable of the returns of the market portfolio. A risk-averse investor who makes decisions based on the mean-variance framework should construct an efficient portfolio using a combination of the market portfolio and the risk-free asset. All these efficient portfolios lie on the CML. According to CAPM, the expected return of an individual asset is calculated by:

$$E(r_i) = r_f + \beta_i \left(E(r_m) - r_f\right) \tag{5.21}$$

where r_f is the return of the risk-free asset and it is assumed to be a constant. This equation states that the expected return of an asset is a linear function of its index of systematic risk measured by β.

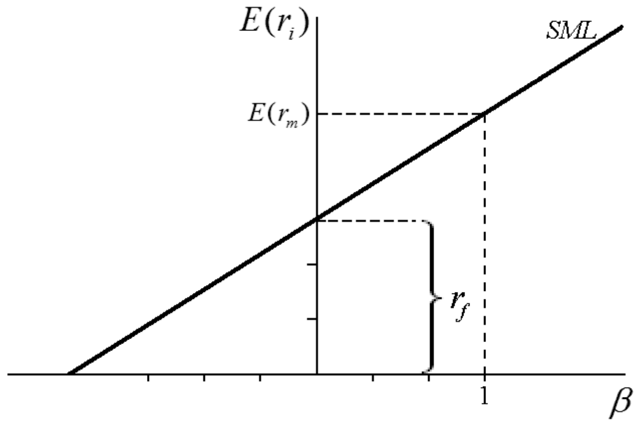

Figure 5.5. Security market line

Equation (5.21) is called the Security Market Line (SML). The expected return of individual assets will lie on the SML but not on the CML. Only the market portfolio will lie on both of these lines. The empirical analogue for Equation (5.21) is:

$$r_i = r_f + \beta_i \left(r_m - r_f\right) + \varepsilon_i \tag{5.22}$$

where ϵ_i is the error term independent of r_m. This equation is called the Security Characteristic Line (SCL).

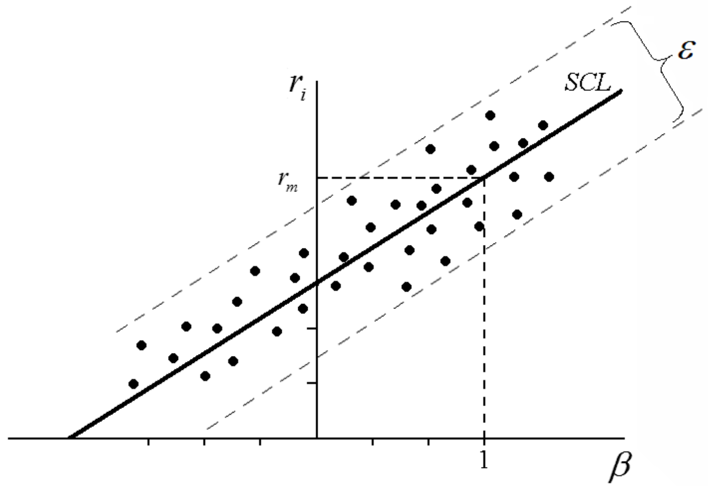

Figure 5.6. Security characteristic line

The main drawback of CAPM is the normality assumption of asset returns. However if we remember that the asset universe consists of hundreds of stocks, CAPM is a nice model for pre-determining the candidate stocks for constructing portfolios, i.e. stocks with lower β's and higher returns. Thus in this study CAPM will be used as a predetermination tool and the stocks which are selected by CAPM will be used for portfolio optimization.

5.4. Application of CAPM to NYSE Data

CAPM was applied to NYSE data and the resulting SML is given in Figure 5.7. The candidate stocks were selected as the stocks which have relatively higher returns and smaller systematic risks ($\beta \leq 1$) according to SML. Therefore BP, UNP, PG, MMM, MCD and T were selected as the candidate stocks to construct the optimal portfolios. In Table 5.4 and Table 5.5, the estimated expected returns and covariances of these six stocks are given.

Table 5.4. Expected returns of six stocks

	BP	**UNP**	**PG**	**MMM**	**MCD**	**T**
$E(r)$	0.000515	0.000603	0.000388	0.000426	0.000812	0.000599

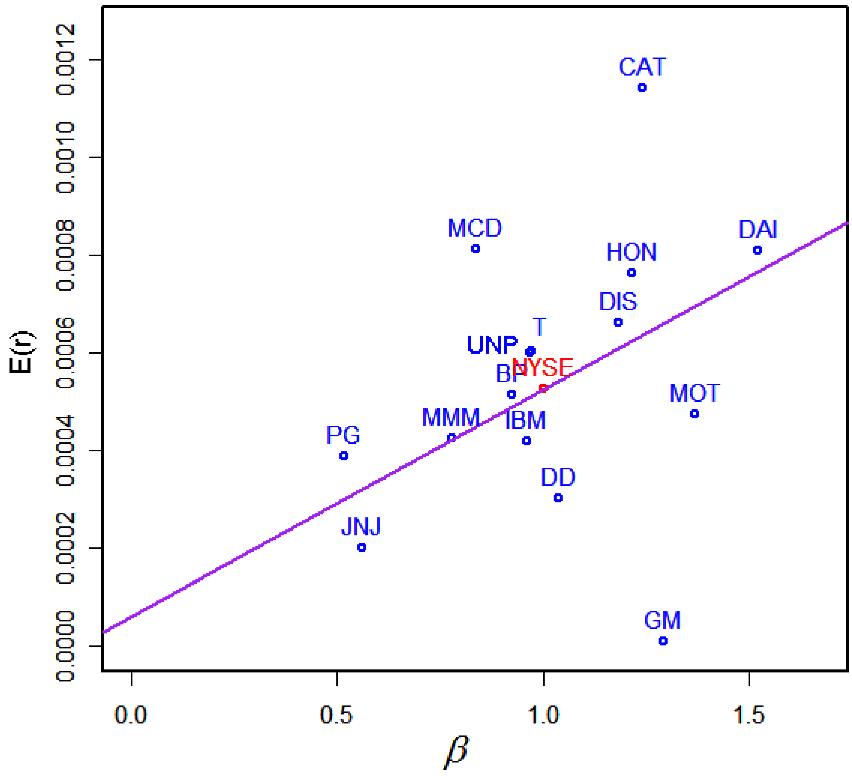

Figure 5.7. The security characteristic line of NYSE stocks

Table 5.5. Covariances of six stocks

	BP	**UNP**	**PG**	**MMM**	**MCD**	**T**
BP	0.000166	0.000067	0.000027	0.000046	0.000038	0.000053
UNP	0.000067	0.000193	0.000039	0.000062	0.000061	0.000064
PG	0.000027	0.000039	0.000085	0.000038	0.000040	0.000037
MMM	0.000046	0.000062	0.000038	0.000138	0.000056	0.000049
MCD	0.000038	0.000061	0.000040	0.000056	0.000252	0.000060
T	0.000053	0.000064	0.000037	0.000049	0.000060	0.000232

Before applying P1 for the six stocks selected by CAPM, we made some rearrangements on it. Firstly the expected portfolio return constraint is changed from equality to inequality since the solution will be the same in both cases. Because if we increase the expected portfolio return after the MRP, the risk of the portfolio will increase since the efficient frontier is a non-decreasing curve. Thus the expected portfolio return constraint will take its lower bound at the optimal solution if it is an inequality constraint. Also the upper bounds ($w_i \leq 1$) for the weights are removed from the model since we have two constraints satisfying the upper bound restrictions, those are the weights must be non-negative and they must sum to one. Thus a weight can be at most one (and the others must be zero in this case). By these rearrangements, we obtain the following model P1′ which is a simpler model than P1 to work with:

P1′:

$$\min \quad \sum_{i=1}^{d}\sum_{j=1}^{d} w_i w_j \sigma_{ij} \tag{5.23}$$

$$\text{s.t.} \quad \sum_{i=1}^{d} w_i \mu_i \geqslant r \tag{5.24}$$

$$\sum_{i=1}^{d} w_i = 1 \tag{5.25}$$

$$w_i \geqslant 0, \qquad i = 1, \ldots, d \tag{5.26}$$

The efficient frontier was obtained by resolving P1′ for varying values of the expected portfolio return. Since we do not regard the portfolios which have smaller expected returns than the return of the MRP, we first found the MRP and then varied the expected portfolio return between the return of the MRP and the maximum expected portfolio return, by increasing the expected return by 0.00001 at each step. In Figure 5.8, the efficient frontier is given where the risk measure is taken as $VaR_{0.99}$. The blue dot indicates the MRP and the red dot indicates the TP. The red line is the CML. In Table 5.6, the specifications of these two portfolios are given. The weights are rounded to the third digit. The weights of the two portfolios are also given as bar charts in Figure 5.9.

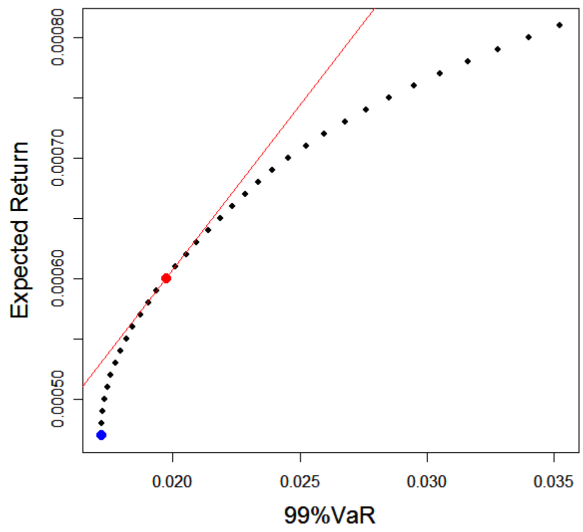

Figure 5.8. Mean-$VaR_{0.99}$ efficient frontier of six stocks

Table 5.6. The weights of six stocks in MRP and TP

	BP	**UNP**	**PG**	**MMM**	**MCD**	**T**	**E(r)**	$VaR_{0.99}$
MRP	0.172	0.051	0.479	0.161	0.066	0.071	0.000470	0.01720
TP	0.179	0.155	0.212	0.020	0.298	0.136	0.000600	0.01974

Since the TP will change as the risk-free return changes, a lower and an upper bound were found for the sensitivity of the current solution to the risk-free return. The current TP will be the same if the risk-free return remains between 0.000053 and 0.000083. Both bounds can be seen in Figure 5.10.

To sum up the results, the first portfolio consists of five arbitrarily selected stocks and the second portfolio consist of six stocks which were selected according to CAPM. The MRP of the second portfolio has a higher return and smaller risk than the MRP of the first one. Thus CAPM gave a better MRP. The expected return of the TP of the second portfolio is 0.000600 and its $VaR_{0.99}$ is 0.01974. These values are 0.000483 and 0.018930 for the fisrt portfolio. We have relatively higher return and higher risk in the TP of CAPM. But the risk is nearly 0.030 if we expect the same return from the first portfolio. This shows that the risk-return trade-off is more efficient in CAPM. It is

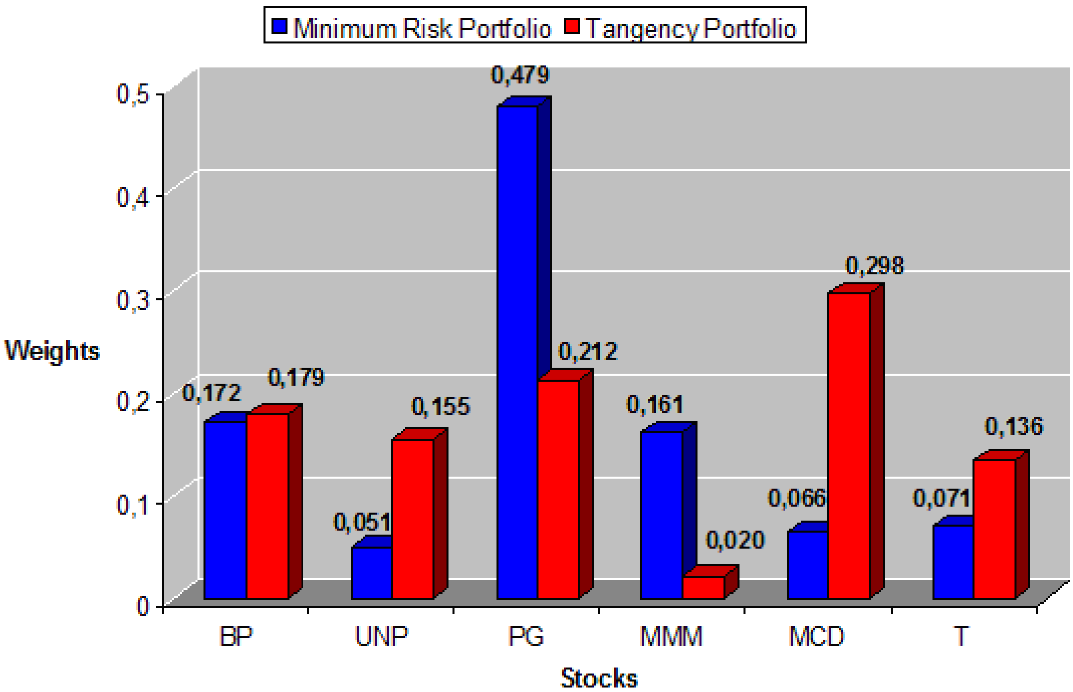

Figure 5.9. The weights of six stocks in MRP and TP

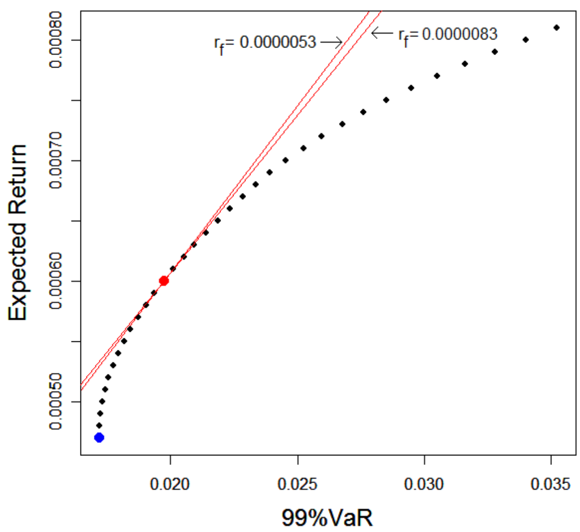

Figure 5.10. Lower and upper bounds for the risk-free return

needless to say that as we take more candidate stocks, we can construct more efficient risk-return portfolios using CAPM. However we will use these six stocks in the copula based portfolio optimization which will be explained in the next section.

5.5. Copula Based Portfolio Optimization

The classical portfolio optimization problem assumes that the portfolio return is a multivariate normal random variable. Thus the risk of the portfolio, which is the objective function of the optimization problem, is a deterministic function of the problem parameters. However it is a well-known fact that the multivariate normal model is not an adequate model for portfolio risk calculation and the t-t copula model which was introduced in Chapter 4 is a more realistic model. Therefore one can easily say that the optimal portfolios which can be found by solving P1′ are *not* the optimal portfolios any more. However we can find the true optimal portfolios by changing the objective function of P1′ by the risk function of the t-t copula model and changing the approximate expected portfolio return constraint by the exact expected portfolio return constraint. P1′ is a deterministic optimization problem since the objective function and the constraints have closed form expressions and include no randomness. But in the copula model, we can find the risk of the portfolio only by "simulation". Thus the new problem becomes a *stochastic optimization problem*. We can write the stochastic optimization problem by changing the objective term and the expected portfolio return constraint of P1′ as:

SP1:

$$\min \quad E\left(F(w_1, \ldots, w_d)\right) \tag{5.27}$$

$$\text{s.t.} \quad log\left(\sum_{i=1}^{d} w_i e^{\mu_i}\right) \geqslant r \tag{5.28}$$

$$\sum_{i=1}^{d} w_i = 1 \tag{5.29}$$

$$w_i \geqslant 0, \qquad i = 1, \ldots, d \tag{5.30}$$

where F is the stochastic response function of the t-t copula model whose value can be found by simulation. We will refer to this problem as SP1. If we represent the weight vector as w, then the stochastic response can be written as:

$$F(w) = f(w) + \varepsilon(w) \tag{5.31}$$

where $f(w)$ is the deterministic function of $E(F(w))$ and $\varepsilon(w)$ is a stochastic function (or the noise) with $E(\varepsilon(w)) = 0$ for all w. Then the optimization problem is to minimize $f(w)$ subject to the same constraint set. f is called the *objective function* and F the *response function.* Typically f is not known explicitly, and the optimization method must work with F. For stochastic simulation optimizations, the response function is computed from the output quantities of one or many replications of the simulation [38].

What makes simulation based optimization more difficult than the ordinary deterministic optimization is its stochastic nature. Banks et al. [39] explain this situation as:

"Even when there is no uncertainty, optimization can be very difficult if the number of design variables is large, the problem contains a diverse collection of design variable types, and little is known about the structure of the performance function. Optimization via simulation adds an additional complication because the performance of a particular design cannot be evaluated exactly, but instead must be estimated. Because we have estimates, it may not be possible to conclusively determine if one design is better than another, frustrating optimization algorithms that try to move in improving directions. In principle, one can eliminate this complication by making so many replications, or such long runs, at each design point that the performance estimate has essentially no variance. In practice, this could mean that very few alternative designs will be explored due to the time required to simulate each one".

There are several methods for optimizing non-linear functions that cannot be solved analytically. These methods are necessarily iterative in nature and the user must supply some starting values or initial guesses for the parameters. Many of these methods use first-order or even second-order derivatives to determine a search direction to improve the value of the objective function. However these techniques are totally deterministic in nature and when applied to problems affected by noise, whether it be error in measurement or uncertainty in prediction, they are either unable to reach an optimum at all or they may reach a false optimum. Distinct from derivative-based search methods is a class of techniques known as *direct-search methods.* In contrast to other optimization techniques which require derivative of the objective function to determine a search direction, a direct-search method relies solely on the value of the objective function on a set of points. Direct-search methods are used for both deterministic and stochastic applications. They are effective techniques in deterministic applications especially when derivatives are unavailable or are computationally intensive. Also they are robust with respect to small perturbations in the function's value; and therefore, they are used often in applications where noise is present [40].

For the copula based portfolio optimization, which is a stochastic optimization problem, we used the Nelder-Mead Simplex Search (NMSS) algorithm. NMSS is a direct-search method and was developed in 1965 by Nelder and Mead [41] for unconstrained optimization of deterministic functions. However, it has been applied frequently to the optimization of stochastic simulation models [40]. NMSS algorithm begins with the function's value on a set of $n+1$ points in the parameter space if we have n decision variables in our problem. This set of points in the parameter space defines a polytope in R^n which has $n+1$ vertices and is called a *simplex.* The algorithm proceeds through a sequence of operations on the simplex to direct it presumably towards a local optimum. Assuming that the problem is a minimization, the algorithm does this by replacing the worst vertex (the vertex which has the highest objective value) in the simplex with a new point that has a lower objective value through one of the following operations: *reflection, expansion* or *contraction.* If all of these operations fail to find a new point to replace the worst point in the simplex, then the entire simplex shrinks towards the vertex with the lowest objective value [38].

In the following the NMSS algorithm is given from [38] and [40] for minimization problems. We implemented this algorithm in R to use it for the copula based portfolio optimization problem.

Step 1. Initialization: For a function of n parameters, choose $n+1$ extreme points to form an initial n-dimensional simplex. For a simplex of size a, each vertex of the initial simplex is found by:

$$x_i = x_0 + pe_i + \sum_{k=1(k\neq i)}^{n} qe_k,\ \ i = 1,\ldots,n \tag{5.32}$$

where x_0 is the initial starting values of n parameters, e_i are the unit base vectors and,

$$p = \frac{a}{n\sqrt{2}}\left(\sqrt{n+1} + n - 1\right) \tag{5.33}$$

$$q = \frac{a}{n\sqrt{2}}\left(\sqrt{n+1} - 1\right) \tag{5.34}$$

Evaluate the response function $F(x_i)$ at each point (vertex) x_i of the simplex for $i = 1, 2, \ldots, n+1$.

Step 2. New Iteration: At the start of each iteration, identify the vertices where the highest, second highest, and lowest response function values occur. Let x_{worst}, x_{sworst} and x_{best} respectively denote these points, and let F_{worst}, F_{sworst} and F_{best} respectively represent the corresponding observed function values.

Step 3. Reflection: Find x_{cent}, the centroid of all points other than x_{worst}:

$$x_{cent} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{5.35}$$

where $x_1, \ldots, x_n$ are the vertices which have smaller objective values than F_{worst}. Generate a new point x_{ref} by reflecting x_{worst} through x_{cent}. Reflection is carried out according to the following equation, where α is the refection coefficient ($\alpha > 0$):

$$x_{ref} = (1+\alpha)x_{cent} - \alpha x_{worst} \tag{5.36}$$

Step 4.a. Accepting Reflection: If $F_{best} \leqslant F_{ref} \leqslant F_{sworst}$ then replace x_{ref} by x_{worst} and Go To Step 6.

Step 4.b. Expansion: If $F_{ref} < F_{best}$ then the reflection is expanded, with the hope that more improvement will result by extending the search in the same direction. The expansion point is calculated by the following equation, where the expansion coefficient is γ ($\gamma > 1$):

$$x_{exp} = \gamma x_{ref} + (1 - \gamma) x_{cent} \tag{5.37}$$

If $F_{\exp} < F_{best}$, then x_{exp} replaces x_{worst} in the simplex; otherwise, the expansion is rejected and x_{ref} replaces x_{worst}. Go To Step 6.

Step 4.c. Contraction: If $F_{ref} > F_{sworst}$, then the simplex contracts. If $F_{ref} \leqslant F_{worst}$, then replace x_{ref} by x_{worst} before contraction. The contraction point is calculated by the following equation, where the contraction coefficient is β ($0 < \beta < 1$):

$$x_{cont} = \beta x_{worst} + (1 - \beta) x_{cent} \tag{5.38}$$

If $F_{cont} \leqslant F_{worst}$, then accept the contraction and replace x_{cont} by x_{worst} and Go To Step 6. Otherwise (If $F_{cont} > F_{worst}$) the contraction is failed and Go To Step 5.

Step 5. Shrink: The entire simplex shrinks by a factor δ ($0 < \delta < 1$), retaining only x_{best}. The shrink is performed by replacing each vertex of the simplex by (except x_{best}):

$$x_i \leftarrow \delta x_i + (1 - \delta) x_{best} \tag{5.39}$$

Evaluate the response function $F(x_i)$ at each generated new vertex x_i of the simplex. Go To Step 6.

Step 6. Stopping Criterion: If the stopping criterions are satisfied then STOP and output x_{best} and F_{best}, otherwise Go To Step 2.

Nelder and Mead [41] used $\alpha = 1$, $\gamma = 2$, $\beta = 0.5$ and $\delta = 0.5$. For the stopping criterion, they computed the standard deviation of the (deterministic) objective function values over all $n+1$ extreme points, and they stopped when the standard deviation S_f dropped below 10^{-8}, where:

$$S_f = \left(\sum_{i=1}^{n+1} \left(f(x_i) - \bar{f} \right)^2 \Big/ (n+1) \right)^{1/2} \tag{5.40}$$

and

$$\bar{f} = \frac{\sum_{i=1}^{n+1} f(x_i)}{n+1} \tag{5.41}$$

In Figure 5.11, the reflection, expansion, contraction and shrink steps of NMSS algorithm is illustrated. The original simplex is shown with a dashed line [42].

NMSS algorithm is used for unconstrained optimization problems but the copula based portfolio optimization problem is a constrained problem. However, we can transform this constrained problem into an unconstrained problem by relaxing the constraints and penalizing them in the objective term. But with this representation of the problem (SP1), the equality constraint is hard to satisfy within NMSS algorithm since the weights might possibly sum over one or below one. Therefore we would possibly penalize this equality constraint in each step and obtain no optimal (not necessarily global) solution with this representation of the problem. Thus we transformed SP1 into a problem with inequality constraints which are easier to satisfy when we relax them. SP1 is a d-dimensional problem and the equality constraint allows us to reduce the dimension of the problem to $d-1$. Because if we know the values of $d-1$ decision variables, we can find the value of the d^{th} one by $1 - \sum_{i=1}^{d-1} w_i$. Therefore if we replace

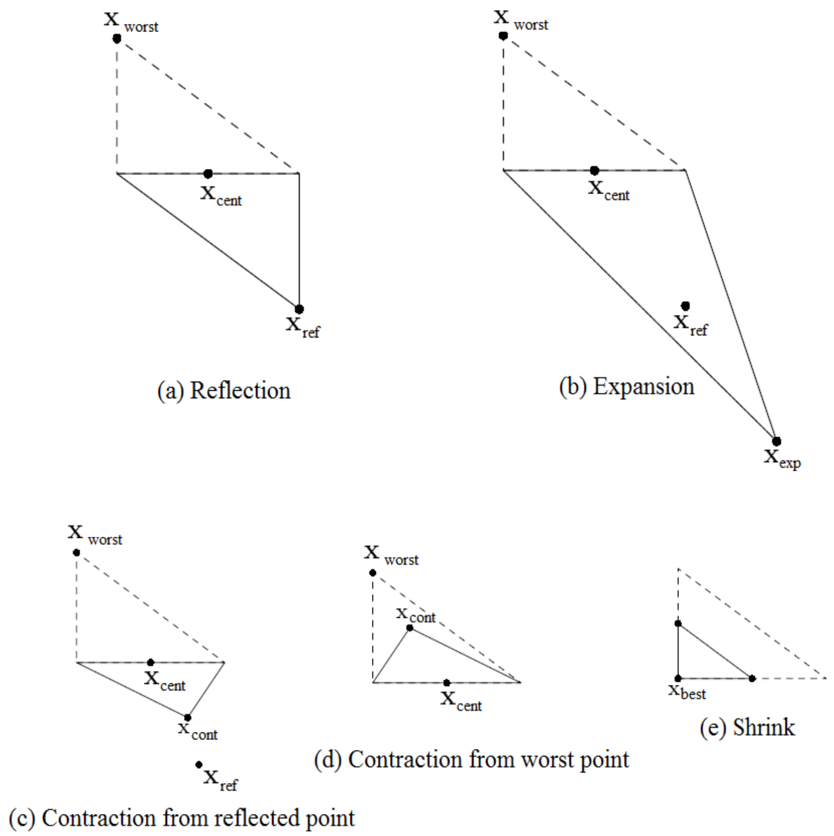

Figure 5.11. Steps of Nelder-Mead simplex search algorithm

w_d by $1 - \sum_{i=1}^{d-1} w_i$ in SP1, we will transform SP1 into SP2, which is a $d-1$-dimensional problem, and obtain the following mathematical model with simple calculus:

SP2:

$$\begin{aligned} \min \quad & E\left(F(w_1, \ldots, w_{d-1})\right) && (5.42) \\ \text{s.t.} \quad & \sum_{i=1}^{d-1} w_i \left(e^{\mu_i} - e^{\mu_d}\right) \geqslant e^r - e^{\mu_d} && (5.43) \\ & -\sum_{i=1}^{d-1} w_i \geqslant -1 && (5.44) \\ & w_i \geqslant 0, \qquad i = 1, \ldots, d-1 && (5.45) \end{aligned}$$

Now we can relax the constraints and penalize them in the objective term. If we represent the left-hand side of the j^{th} constraint as $g_j(w)$ and right-hand side of it as c_j in SP2, then we will have SP3 as:

SP3:

$$\begin{aligned} \min \quad & \mathrm{E}\left(F(w_1, \ldots, w_{d-1})\right) - \sum_{j=1}^{d+1} \lambda_j \min\left(0, g_j(w) - c_j\right) && (5.46) \\ \text{s.t.} \quad & \lambda_j > 0 \ , \ j = 1, \ldots d+1 && (5.47) \end{aligned}$$

If the constraint $g_j(w) \geq c_j$ is not satisfied in SP3, then $\min\left(0, g_j(w) - c_j\right)$ will become negative and the positive penalty λ_j will increase the objective value of the problem.

After transformations we have the final mathematical model of the problem which is an unconstrained stochastic optimization problem. Now we can apply NMSS algorithm for this problem (SP3).

In this study, we used $\alpha = 1$, $\gamma = 2$ and $\beta = 0.5$ as Nelder and Mead [41] suggested. $\delta = 0.9$ is recommended by Barton and Ivey [43] for optimization of noisy functions. Therefore we used $\delta = 0.9$ and found in a few pilot runs that it generally gives better results than $\delta = 0.5$. Also we did not use the standard deviation for the stopping criterion. We stopped the algorithm if the difference between the objective values of the worst and the best vertices of the simplex is smaller than a predetermined value which we take as 10^{-8}. By this stopping criterion, we wanted to guarantee that our solution in the final simplex would be sensitive to a particular digit. We used $\lambda_j = 1$ for all j as our objective value will be probably at level 10^{-2}. Since essentially no theoretical results have been proved explicitly for the global convergence of NMSS algorithm despite its widespread use [42], we used multi-start for the optimizations, that is we run the same optimization with different initial starting values and selected the best solution between them.

5.5.1. Empirical Results

We estimate the objective function in SP3 by the t-t copula method which is introduced in Chapter 4. Thus firstly we fitted the t-copula to the data of the six stocks. The copula parameters are given in Table 5.7.

Table 5.7. The copula parameters for six stocks ($\upsilon = 12.40$)

	BP	**UNP**	**PG**	**MMM**	**MCD**	**T**
BP		0.367	0.227	0.329	0.213	0.284
UNP			0.330	0.417	0.304	0.328
PG				0.395	0.319	0.328
MMM					0.345	0.322
MCD						0.284
T						

When NMSS algorithm produces a new vertex in the simplex, we have a new weight vector of the portfolio assets. We input this vector into t-t copula algorithm and get the risk estimate which is $\mathrm{E}\left(F(w_1, \ldots, w_{d-1})\right)$ in SP3. This is one evaluation of the objective function and lasts nearly 6.5 seconds (for $n_{outer} = 200$ and $n_{inner} = 1,000$). The lines two-eight of the t-t copula algorithm, which is given in Chapter 4, is not affected by the input weight vector and the main part of the running time is expected to occur between these two lines since this part of the algorithm consists of random number generations. So if we perform the operations thorough line two-eight at the beginning of the optimization, we obtain the random numbers for the asset returns before the optimization and we do not need to perform the same operations at each function evaluation if we store them into the computer memory. Thus we can use the same returns when the weight vector is changed by NMSS. In fact we use common random numbers (CRN) for the asset returns by this method. The only thing left is to calculate the losses given the weights and the CRN, and estimating the risk. When we revised the t-t copula algorithm for CRN, the CPU time reduces to about 0.05 second from 6.5 seconds which means that we speeded up the function evaluation more than 100 times.

The Markowitz efficient frontier is given in Figure 5.8 for the six stocks. If we calculate the risks of these efficient portfolios by the t-t copula method and plot them in the risk-return space by their exact portfolio returns, we obtain Figure 5.12. The black dots are the efficient portfolios found by P1′ and the gray dots are their copula risk-returns.

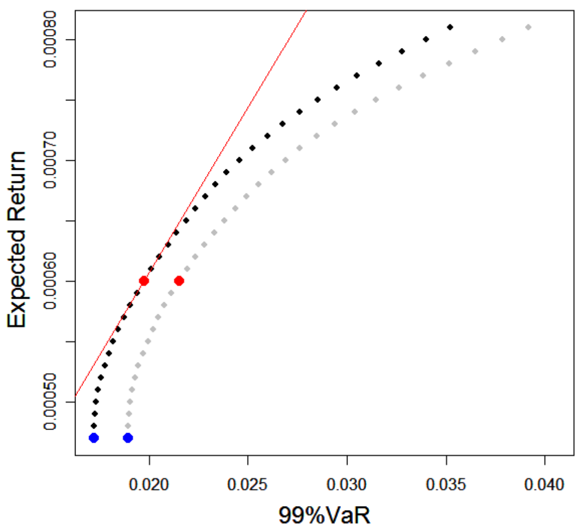

Figure 5.12. Efficient frontier for six stocks with copula risk and Markowitz weights

As it can be seen from Figure 5.12, the copula frontier is relatively far from the normal frontier. However this copula frontier is not the “true” efficient frontier of the copula method since we used the weights which we found by solving P1$'$. In fact by this plot we just wanted to see how the copula risk-return structure of the efficient portfolios found by the classical model is different than the original one. Therefore by solving SP3 with NMSS, we can find efficient portfolios between these two frontiers which will form the true copula frontier. We performed the copula based optimization as follows:

- We stored the CRN into the memory using the t-t copula algorithm ($n_{outer} = 200$ and $n_{inner} = 1,000$),
- We performed a grid search to determine the initial starting points for multi-start,
- We found the MRP by taking the expected portfolio return as zero and solving SP3,
- We varied the expected portfolio return between 0.00050 and 0.00063 and solved SP3 to obtain 15 efficient portfolios in total.

We performed the grid search by dividing the five-dimensional hypercube into equal intervals of length 0.1 and calculating the objective function at their feasible intersection points. For example a two-dimensional grid search for our problem is illustrated in Figure 5.13 where each point represents a feasible solution. By this method, given a level of expected return, we can find the feasible points which satisfy this return and then select some of them to form the initial starting point. For example for a specified level of expected return, we selected the first three points which satisfy the corresponding return and have the smallest risks, and then find the central point of them as the initial starting point. For the other restarts, we used the first five, 10 and 20 points, and for the fifth restart we used the weights found by the classical Markowitz model at that return level.

The CPU time for the optimization was about 20 minutes to produce 15 efficient portfolios. The efficient frontier which was found by solving SP3 is added to Figure 5.12 and given in Figure 5.14. The corresponding part of the frontier is enlarged to see the difference more precisely in Figure 5.15

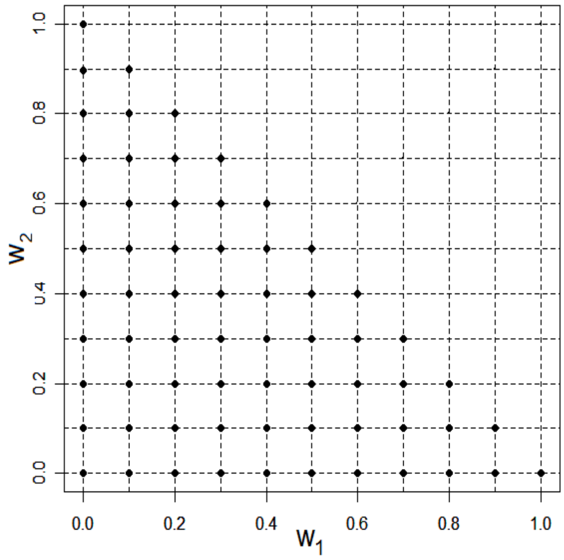

Figure 5.13. Grid search for two-dimensional portfolio

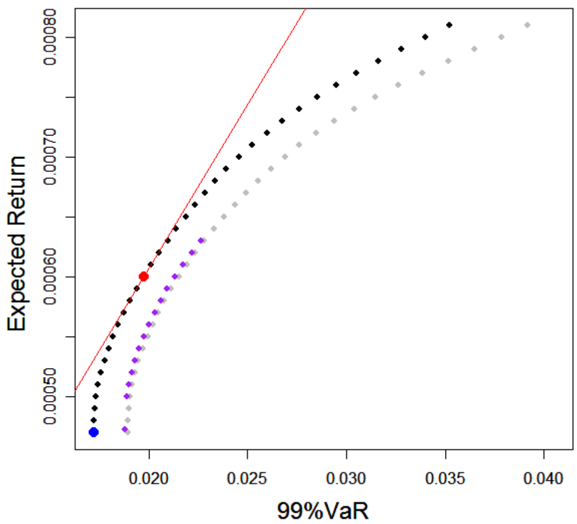

Figure 5.14. Efficient frontier by copula based optimization

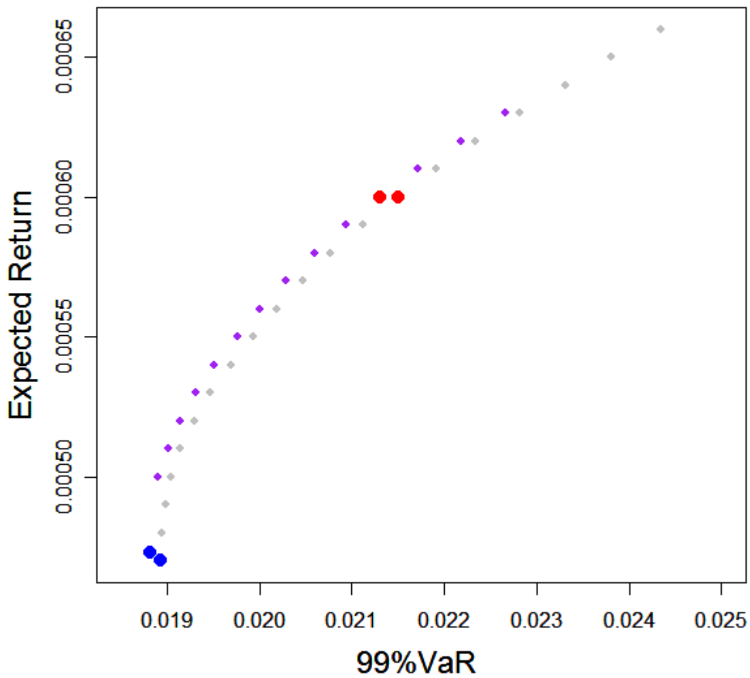

Figure 5.15. Enlarged efficient frontier by copula based optimization

The optimization results are given in Table 5.8 with the specifications of the MRP and the TP. The weights are rounded to the third digit. The weights of the MRP's for both frontiers are given as bar charts in Figure 5.16. The weights of the TP's are also given in Figure 5.17.

Table 5.8. Specifications of MRP and TP for the copula based optimization

	BP	**UNP**	**PG**	**MMM**	**MCD**	**T**	**E(r)**	$VaR_{0.99}$
MRP	0.193	0.083	0.472	0.142	0.066	0.044	0.000473	0.01881
TP	0.189	0.227	0.197	0.005	0.275	0.107	0.000600	0.02131

From the figures we can say that the difference between the weights are not negligible. Thus the copula based optimization has changed the appearance of the efficient portfolios. In Table 5.9 the risk and the expected return levels of both frontiers are given to see how SP3 gave better results than the copula frontier with the classical Markowitz weights. The first row is the MRP's and the twelfth row is the TP's. From the table one can say that the average improvement is nearly one per cent.

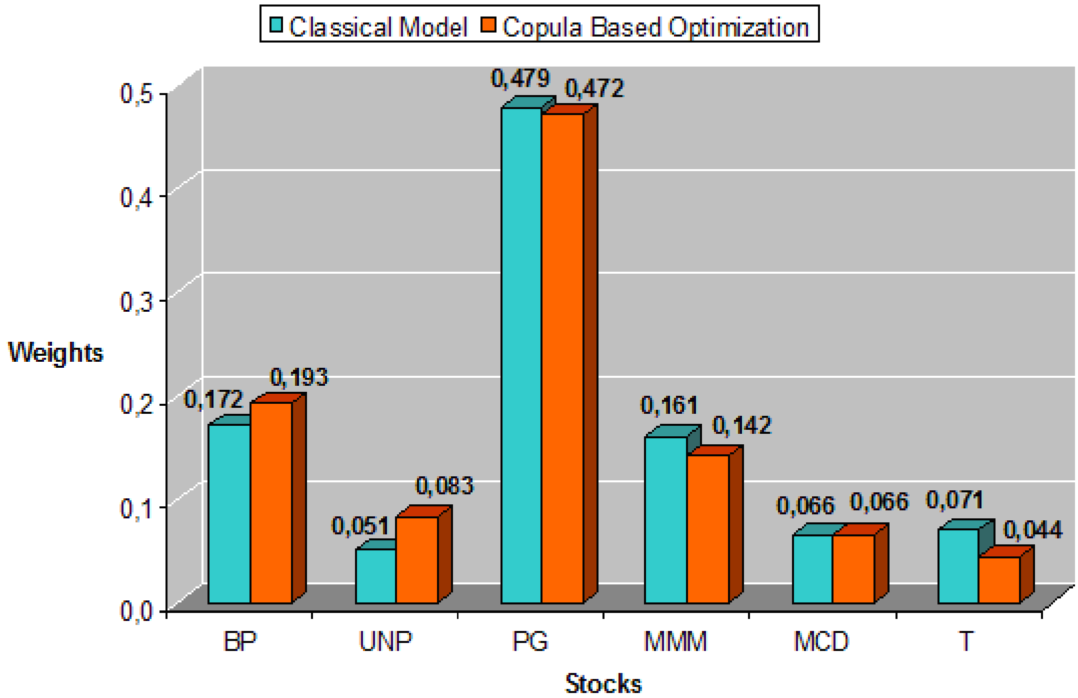

Figure 5.16. Weights of MRP's in the classical model and the copula based optimization

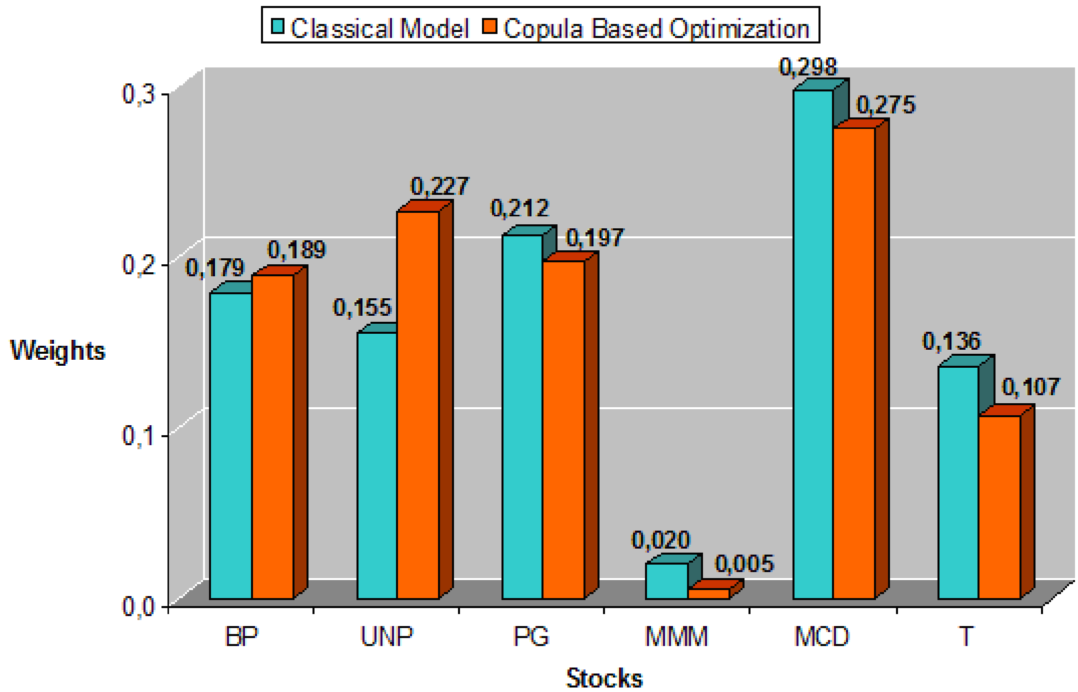

Figure 5.17. Weights of TP's in the classical model and the copula based optimization

Table 5.9. Risk and return levels of the classical model and the copula based optimization

	$E(r)$ **for P**$1'$	$VaR_{0.99}$	$E(r)$ **for SP3**	$VaR_{0.99}$	**% improvement**
1	0.000470	0.01892	0.000473	0.01881	0.006107
2	0.000500	0.01904	0.000500	0.01890	0.007404
3	0.000510	0.01913	0.000510	0.01900	0.006748
4	0.000520	0.01929	0.000520	0.01913	0.008216
5	0.000530	0.01946	0.000530	0.01931	0.007872
6	0.000540	0.01969	0.000540	0.01950	0.009412
7	0.000550	0.01993	0.000550	0.01976	0.008581
8	0.000560	0.02018	0.000560	0.02000	0.009055
9	0.000570	0.02046	0.000570	0.02028	0.008679
10	0.000580	0.02076	0.000580	0.02059	0.008087
11	0.000590	0.02111	0.000590	0.02094	0.008186
12	0.000600	0.02151	0.000600	0.02130	0.009388
13	0.000610	0.02191	0.000610	0.02171	0.008926
14	0.000620	0.02233	0.000620	0.02217	0.007074
15	0.000630	0.02281	0.000630	0.02265	0.006965

5.5.2. Accuracy of the Results

From the empirical results, we can conclude that we can obtain optimal portfolios by stochastic optimization with the t-t copula model. However we produced a sample of asset returns and used this sample to optimize the portfolios. Therefore our optimization problem became a semi-stochastic (or semi-deterministic) optimization because of the deterministic sample. Thus it is needless to say that as we produce another sample, the optimal portfolios will change. For this reason we produced nine more samples for the asset returns and ran the optimization algorithm for these samples to obtain 10 different efficient copula frontiers. In Figure 5.18, these 10 frontiers are plotted where each group corresponds to a return level. The dots in the lower-left part are the MRP's and the dashed lines represent the risk-return intervals.

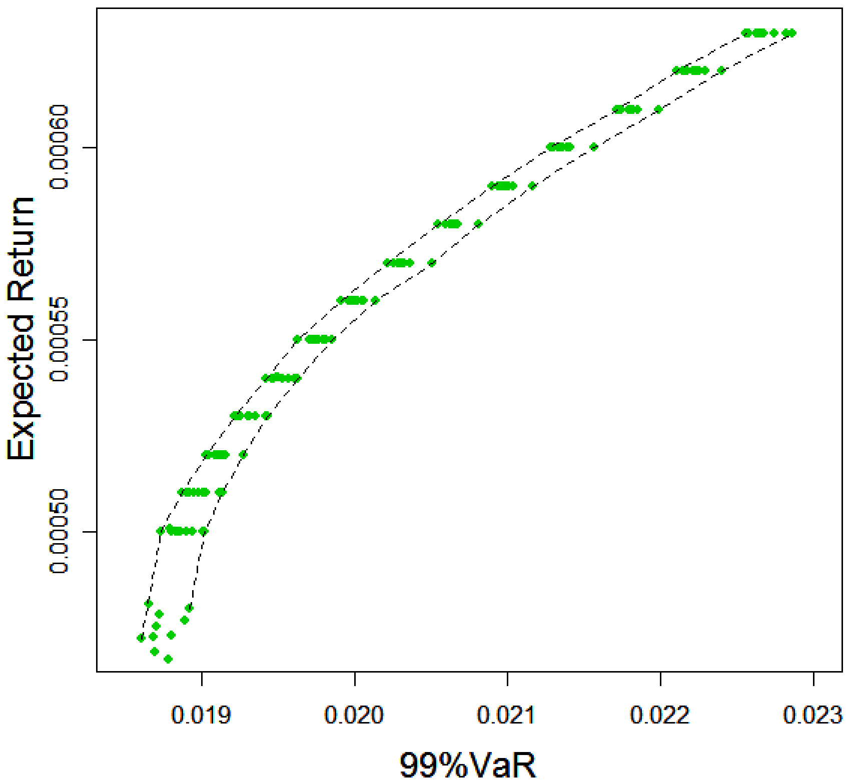

Figure 5.18. Copula frontiers obtained by 10 different samples

In Figure 5.19 and Figure 5.20, the weights found by 10 optimizations are given for the MRP's and the TP's. The dashed lines are the weights of the individual stocks found from the classical Markowitz model and the black lines are the means of the weights of the individual stocks found from 10 optimizations.

From Figure 5.19, we can say that the weights of BP and UNP are greater than the Markowitz weights in the copula model and the difference for UNP is relatively higher. However MMM, MCD and T have smaller weights in the copula model except MCD in two cases. The weights of PG are approximately the same in both methods.

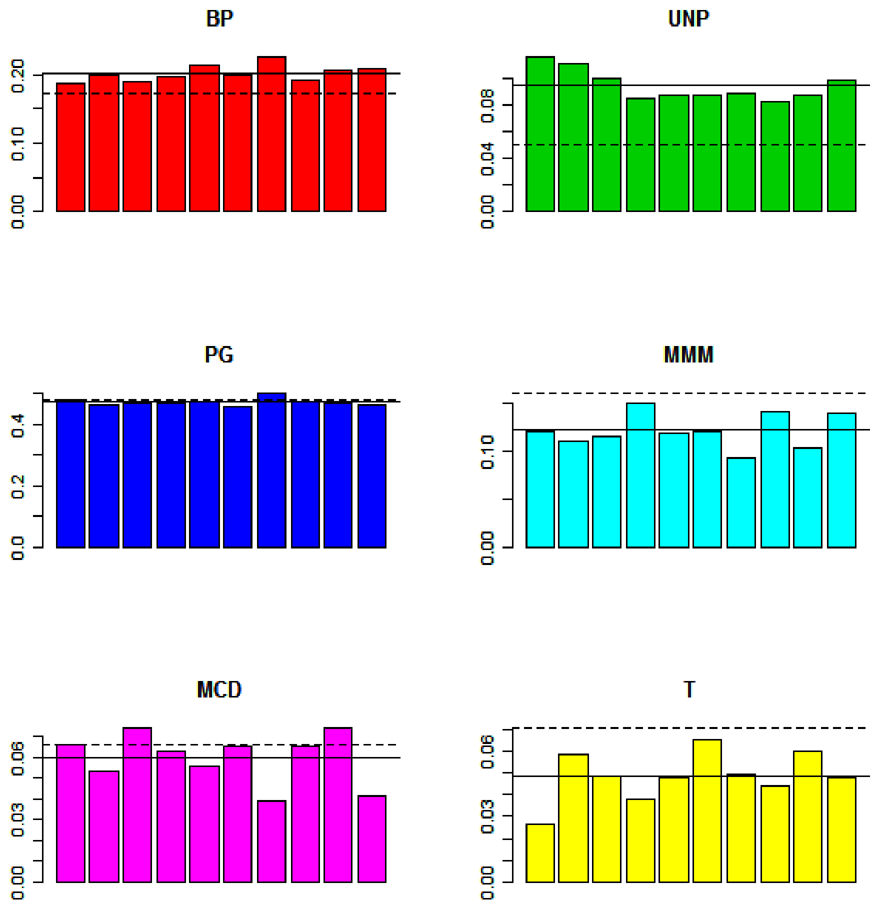

Figure 5.19. MRP weights of six stocks found by 10 optimizations

In Figure 5.20, the weights of BP and UNP are again greater than the Markowitz weights and the difference for UNP is again relatively higher. The weights of PG, MMM, MCD an T are smaller than the Markowitz model except a few cases for PG and MMM.

Since we used relatively small samples for the optimizations, the standard errors of risk estimates of the optimal portfolios are relatively high and thus the real risk of an optimal portfolio will be different than its estimation. In Table 5.10, the mean and

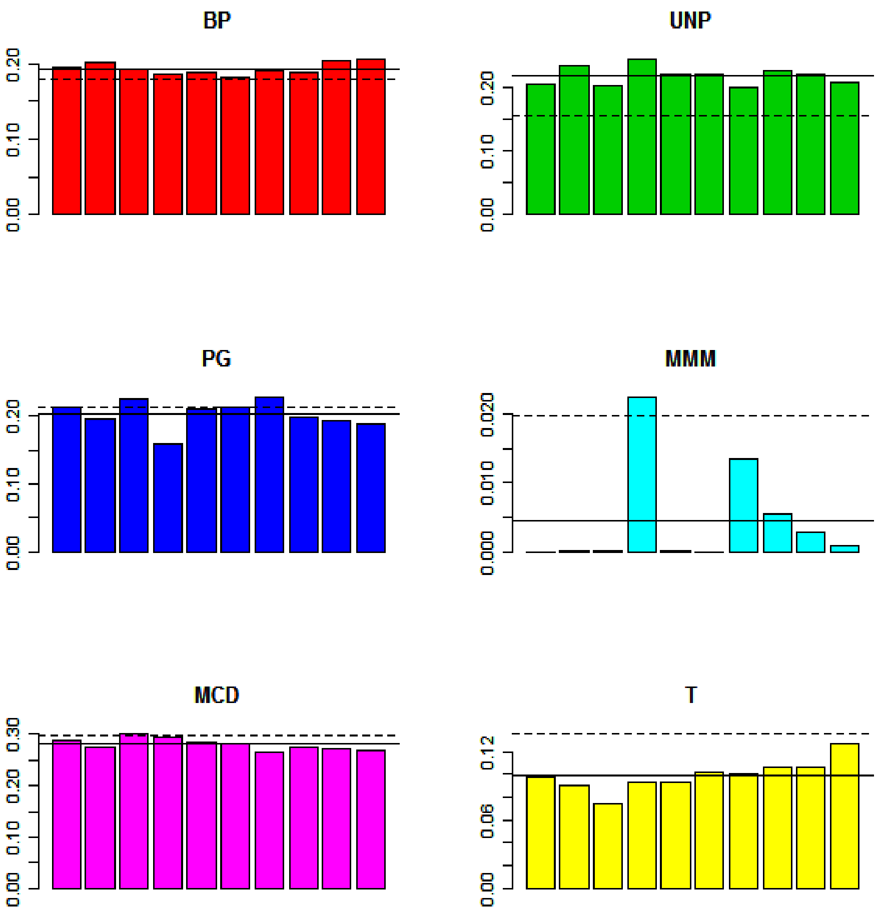

Figure 5.20. TP weights of six stocks found by 10 optimizations

the standard errors of the risk estimates of the optimized portfolios for one efficient frontier are given. According to the table, one can say that the risk of the efficient portfolios can deviate nearly one per cent from its mean ($\mu \pm 2$SE) if we use a sample of size 200,000 (n_{outer} = 200, n_{inner}=1,000). Therefore we produced a large sample of size 32,000,000 and calculated the risks of the portfolios which we found by the 10 optimizations. We regard these risks as the "real risks" since the sample is relatively larger than the samples used for the optimizations. The standard errors of the real risk estimates of the optimal portfolios will reduce nearly 13 times with this larger sample.

Table 5.10. Risk estimates and their standard errors of the optimal portfolios

Return level	$VaR_{0.99}$	**SE**
0	0.01881	0.000095
0.00050	0.01890	0.000091
0.00051	0.01900	0.000091
0.00052	0.01913	0.000092
0.00053	0.01931	0.000093
0.00054	0.01950	0.000095
0.00055	0.01976	0.000095
0.00056	0.02000	0.000097
0.00057	0.02028	0.000101
0.00058	0.02059	0.000102
0.00059	0.02094	0.000105
0.00060	0.02130	0.000112
0.00061	0.02171	0.000115
0.00062	0.02217	0.000116
0.00063	0.02265	0.000119

In Figure 5.21, the real risk-returns for the optimized portfolios are added to Figure 5.18 to see how the real risks of the optimized portfolios are. The black dots are the real risk-returns of the optimized portfolios and the red dots are the real risk-returns of portfolios which are obtained by classical multinormal model. The yellow dots are the optimized portfolios (with sample size 200,000) which have the minimum real risks at each return level. The real risks of the classical model seems to be higher than the real risks of the optimized portfolios and the portfolios which have the minimum real risks at each return level generally gave smaller risks in the optimizations.

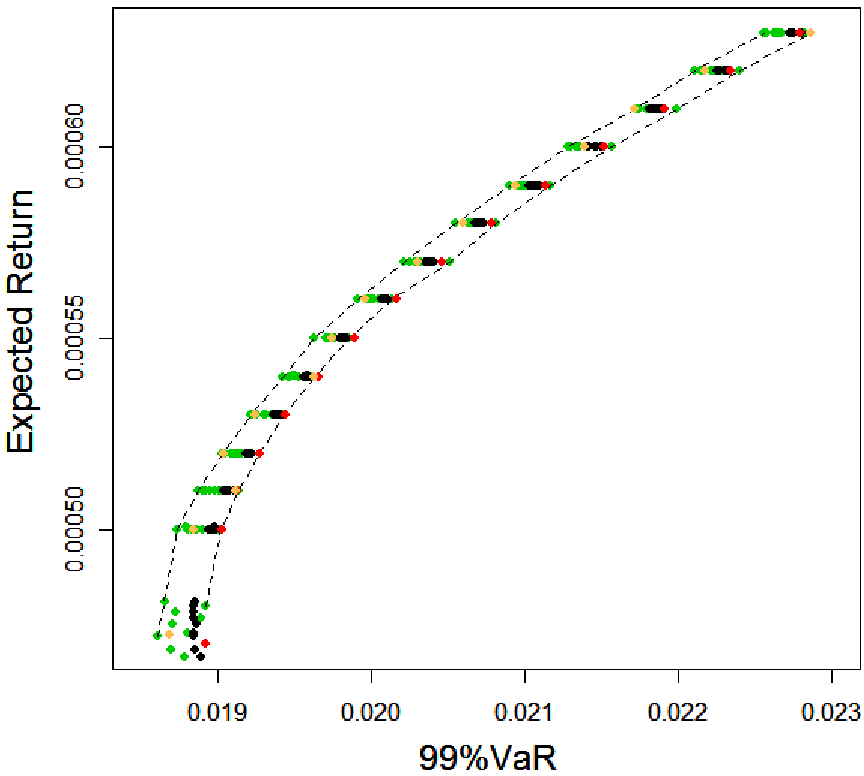

Figure 5.21. Real risks of the portfolios of 10 efficient frontiers

In Figure 5.22 the risks of the portfolios obtained by 10 optimizations are plotted against their real risks. The red dots are the real risks of the classical multinormal model. According to the plot, the real risks of the classical model are almost always greater than the real risks of the optimized portfolios which proves that our copula based stochastic optimization gave reasonable results even with a sample of only size 200,000. However the difference is still small.

We can conclude that, our copula based portfolio optimization gives smaller real risks than the real risks of the classical Markowitz model. However to obtain “close to real” optimal portfolios, we should increase the sample size and run the optimization with this sample (for example with a sample of size 10,000,000), or we should repeat the optimization several times and find the close to real frontier within these optimized portfolios.

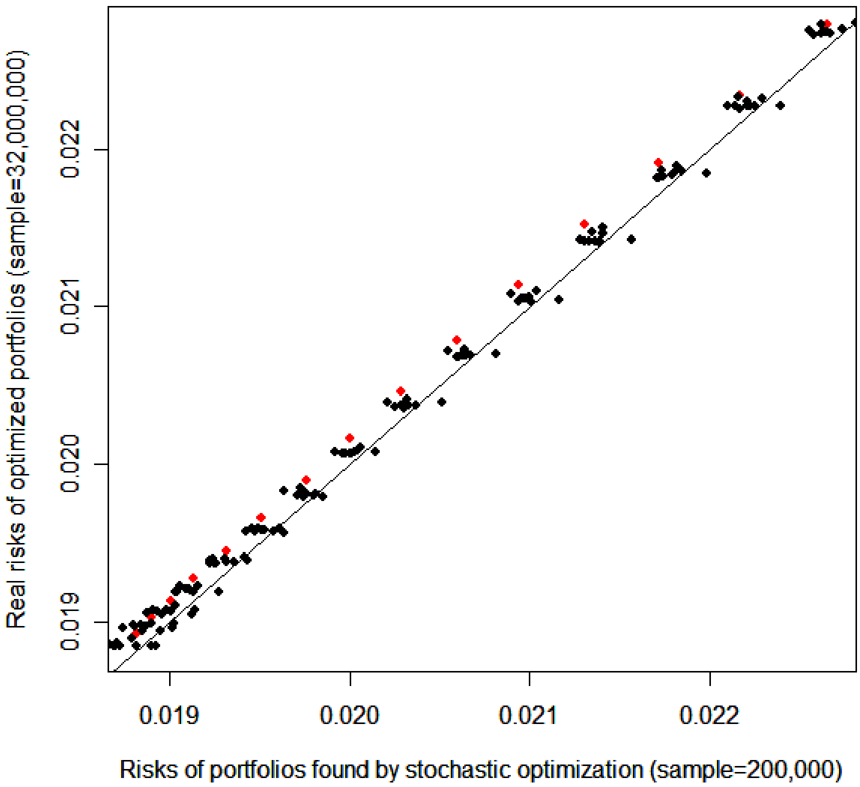

Figure 5.22. Estimated risks vs. real risks of the optimized portfolios (10 efficient frontiers)

6. CONCLUSIONS

In this study, we used the copula model to construct multivariate return distributions of stock portfolios for portfolio risk calculation, as an alternative to the classical portfolio risk calculation methods. Although the classical Markowitz model is still widely used in risk management because of its nice properties, its drawbacks are well-known by risk managers. First of all the normality assumption of asset returns is not realistic since the empirical distributions show that they have fatter tails and higher kurtosis than the normal distribution. Also it assumes linear dependence which does not regard the extreme co-movements and non-linear dependencies between assets. Historical simulation has the serious deficiency that there are not enough data in the empirical tails. GARCH models explain the extreme returns with the change in the volatility but it assumes conditional normality. These important drawbacks of the classical methods can be solved by the copula approach.

Copulas are very useful tools for modeling multivariate distributions since one can model the marginal distributions separately and then find a copula to represent the dependence structure between them. Thus, it is possible to use a wide range of univariate distributions for the marginals and several different non-linear dependence structures to relate them to each other. We fitted different copulas to our dataset consisting of stock returns from NYSE to model the return distributions of stock portfolios. We found that the t-distribution and the GHD are very nice models for the stock returns since they are able to capture high kurtosis and extreme returns in the tails. The t-copula is found to be the best fitting copula according to the log-likelihood and AIC values for arbitrarily constructed portfolios of two, three, four, five and 10 stocks . Thus we concluded that the t-copula with t and GHD marginals, which we called the “t-t copula model”, is an adequate model to represent the portfolio return distributions.

We simulated from the fitted copulas with Monte Carlo method to generate price return scenarios for portfolios and calculated the VaR and ES at 99 per cent level for different time horizons. We also compared the results of the copula method with the results of the classical portfolio risk calculation methods; the Markowitz approach, (known as variance-covariance approach, approximate multinormal model), historical simulation, Monte-Carlo simulation (with normality assumptions) and CCC-GARCH process. We found that the t-t copula outperforms the classical models. Also the results of the t-t copula and historical simulation were quite similar, which can be seen as a validation that the t-t copula can capture the true dependence between asset returns since the true dependence is embedded into the historical data. We also realized that the selection of marginal asset return distributions is very crucial since empirical quantiles were smaller than the fitted quantiles for some stocks. The sensitivity analyses also showed that the risk estimates are more sensitive to the parameters of the marginal distributions than they are to copula parameters, which emphasizes the importance of marginal distributions.

The classical Markowitz model is used in portfolio optimization problems. However the optimal portfolio found by this model would not be the true optimal portfolio since it uses an inadequate model for portfolio risk estimation. Therefore we developed a model for the copula based stochastic portfolio optimization to find the "true" optimal portfolios. We selected six candidate stocks from our dataset by CAPM and used them in the optimization. We implemented NMSS algorithm in R and optimized the portfolios by minimizing $VaR_{0.99}$ for different expected portfolio return levels. We used a semi-deterministic approach by storing the asset returns (200,000 returns) for the optimizations to speed up the optimization procedure. We repeated the optimization for 10 different return samples and calculated the risk of the optimized portfolios with a sample of 32,000,000 returns. We regarded these risks as the "real risks" since the sample is much larger than the optimization sample. We found that the real risks of the optimized portfolios were smaller than the real risks of the optimal portfolios found by the classical Markowitz model. Thus our copula based optimization gives acceptable results even with a sample of only size 200,000.

This study can be extended in several ways. The first thing to be done is to model the distributions of marginal assets with more realistic models. Although the t and GHD seem to be adequate, the tails of the return distributions are very crucial for the risk calculation. Thus, Extreme Value Theory (EVT) can be used to model the tails of the assets, whereas the center of the distributions is still modeled by the t or GHD. Also we did not filter our data by GARCH(1,1) since it could possibly destroy the true dependence. However a realistic filter with jump processes can be used to filter the univariate data as Harold and Jianping [19] suggest.

Another important extension is possible in the optimization part. We used a relatively small sample for the optimizations and repeated it 10 times to see how the optimal portfolios change according to the random samples. We used the larger sample only for calculating the “real” risks of the optimized portfolios. But the optimization could be done with a large sample that the optimization results become the true optimal results. However the NMSS algorithm, which is a direct search method, would possibly necessitate longer CPU times for this optimization and thus the need for an “exact” algorithm to solve this stochastic optimization problem arises.

APPENDIX A: PROOFS

The Hessian matrix of f_e is:

$$\nabla^2(f_e) = \begin{pmatrix} \frac{\partial^2 f}{\partial w_1^2} & \frac{\partial^2 f}{\partial w_1 \partial w_2} & \cdots & \frac{\partial^2 f}{\partial w_1 \partial w_{d-1}} \\ \frac{\partial^2 f}{\partial w_2 \partial w_1} & \frac{\partial^2 f}{\partial w_2^2} & \vdots & \frac{\partial^2 f}{\partial w_2 \partial w_{d-1}} \\ \vdots & \cdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial w_{d-1} \partial w_1} & \frac{\partial^2 f}{\partial w_{d-1} \partial w_2} & \cdots & \frac{\partial^2 f}{\partial w_{d-1}^2} \end{pmatrix} \tag{A.1}$$

When we generalize the second order partial derivatives we obtain:

$$\frac{\partial^2 f}{\partial w_i^2} = \frac{-(e^{x_i} - e^{x_d})^2}{\left(w_1 e^{x_1} + w_2 e^{x_2} + \ldots + (1 - \sum_{i=1}^{d-1} w_i) e^{x_d}\right)^2} \tag{A.2}$$

and

$$\frac{\partial^2 f}{\partial w_i \partial w_j} = \frac{-(e^{x_i} - e^{x_d})(e^{x_j} - e^{x_d})}{\left(w_1 e^{x_1} + w_2 e^{x_2} + \ldots + (1 - \sum_{i=1}^{d-1} w_i) e^{x_d}\right)^2} \tag{A.3}$$

When we rearrange the matrix, we obtain:

$$\nabla^2 (f_e) = -\frac{1}{a} A \tag{A.4}$$

where

$$a = \left(w_1 e^{x_1} + w_2 e^{x_2} + \ldots + (1 - \sum_{i=1}^{d-1} w_i) e^{x_d}\right)^2 \tag{A.5}$$

and

$$A = \begin{pmatrix} (e^{x_1} - e^{x_d})^2 & \cdots & (e^{x_1} - e^{x_d})(e^{x_{d-1}} - e^{x_d}) \\ \vdots & \ddots & \vdots \\ (e^{x_{d-1}} - e^{x_d})(e^{x_1} - e^{x_d}) & \cdots & (e^{x_{d-1}} - e^{x_d})^2 \end{pmatrix} \tag{A.6}$$

We can write $A = bb^T$ where b is a column vector:

$$b = \begin{pmatrix} e^{x_1} - e^{x_d} \\ e^{x_2} - e^{x_d} \\ \vdots \\ e^{x_{d-1}} - e^{x_d} \end{pmatrix} \tag{A.7}$$

A is positive semi-definite since $v^T A v \geq 0$ for all column vector v:

$$v^T A v = v^T (bb^T) v = (v^T b)(v^T b)^T = (v^T b)^2 \geq 0 \tag{A.8}$$

APPENDIX B: HISTOGRAMS AND Q-Q PLOTS OF STOCK RETURNS

B.1. Fitted Normal Distributions

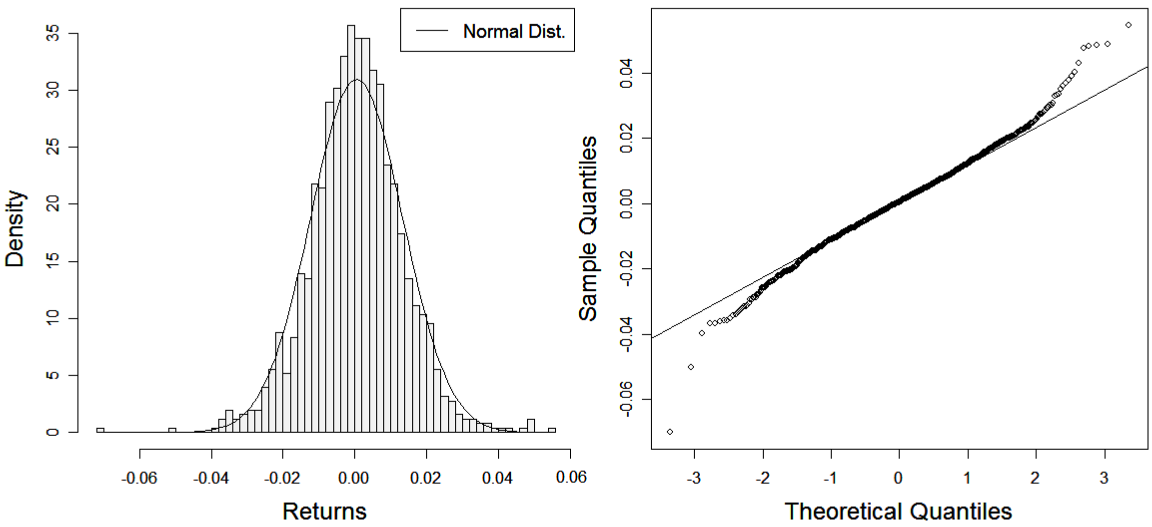

Figure B.1. Histogram and Q-Q plot for the logreturns of BP

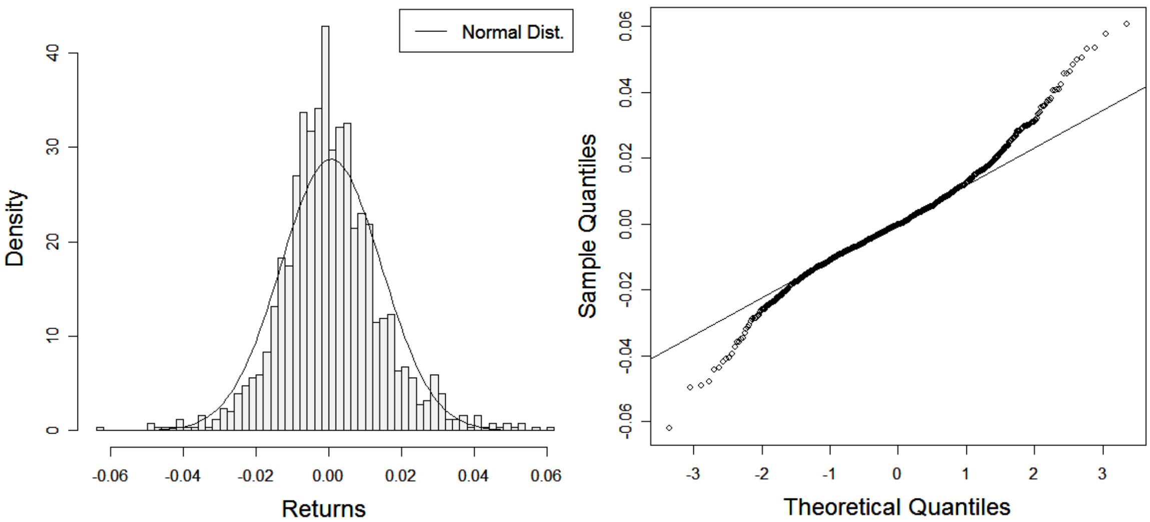

Figure B.2. Histogram and Q-Q plot for the logreturns of UNP

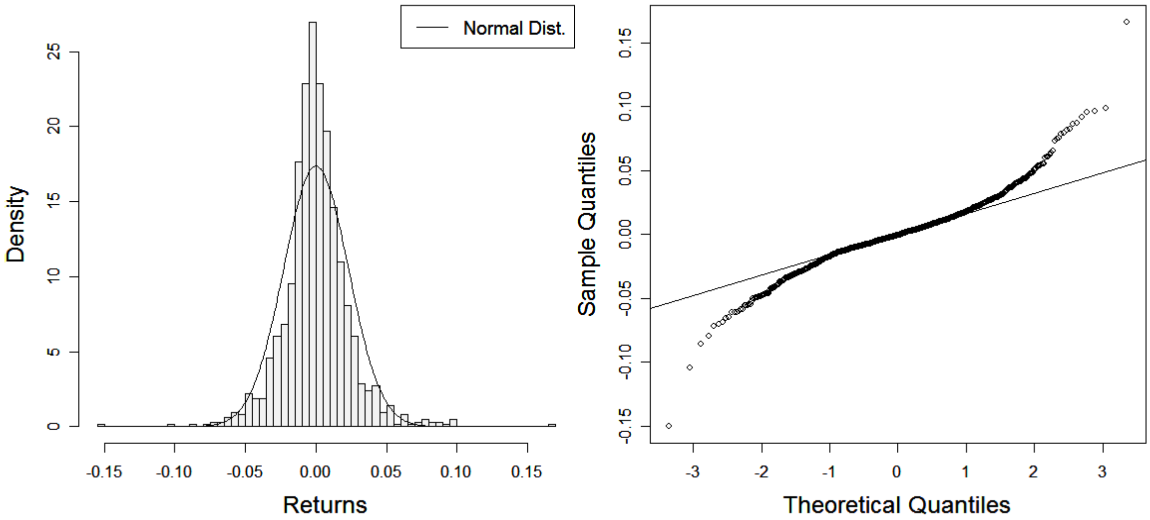

Figure B.3. Histogram and Q-Q plot for the logreturns of GM

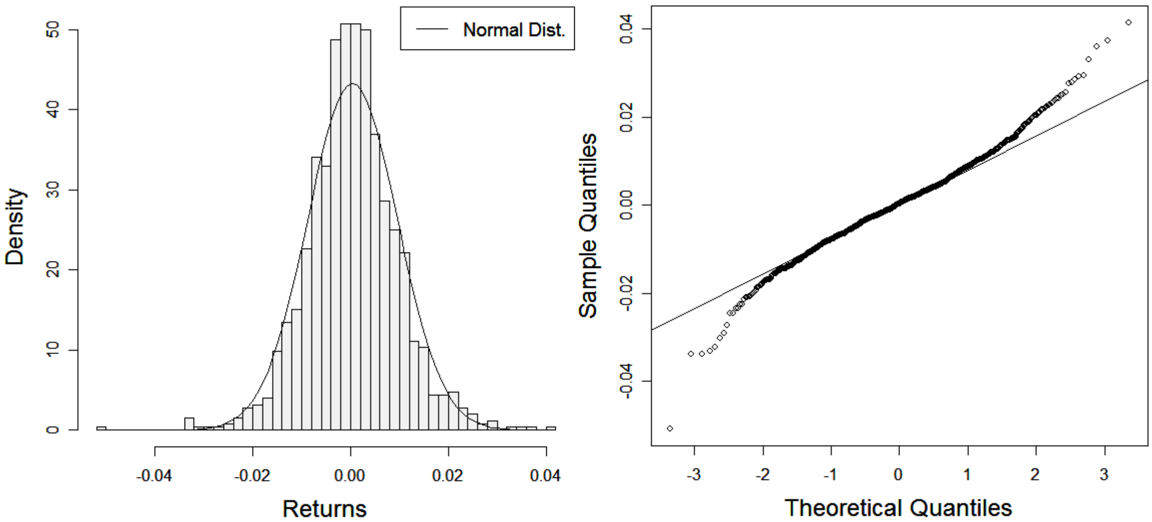

Figure B.4. Histogram and Q-Q plot for the logreturns of PG

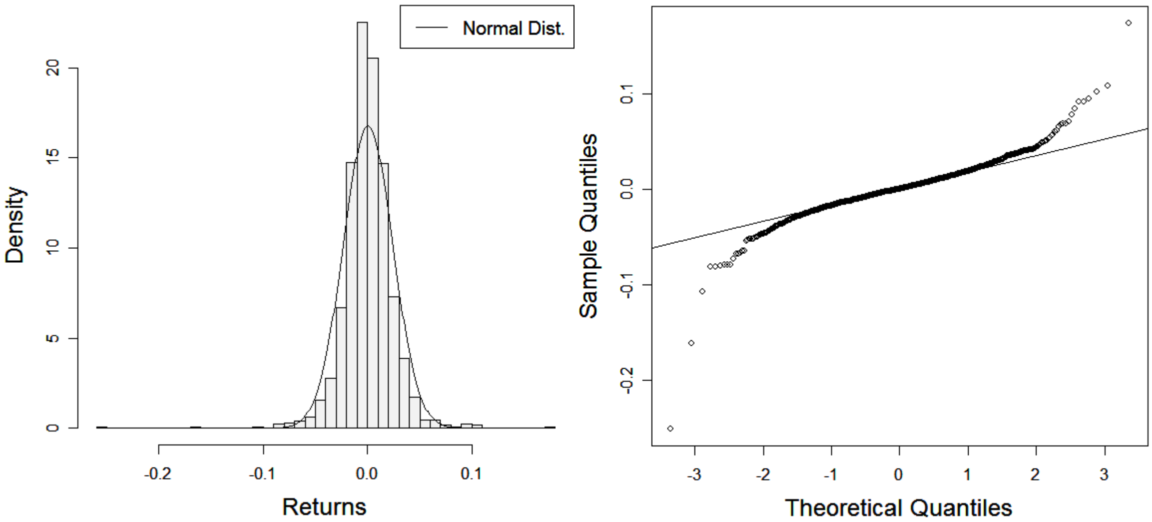

Figure B.5. Histogram and Q-Q plot for the logreturns of MOT

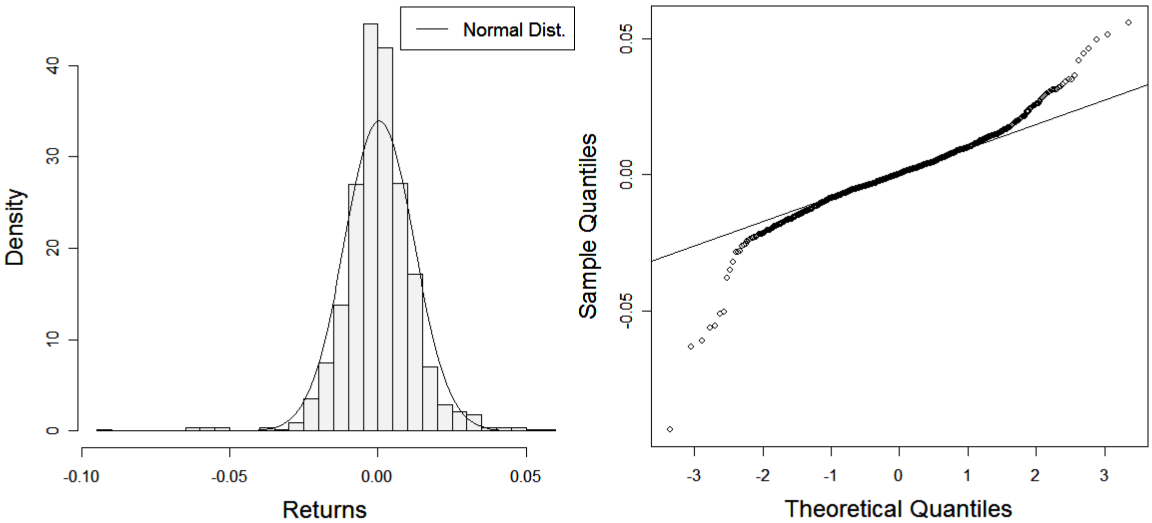

Figure B.6. Histogram and Q-Q plot for the logreturns of MMM

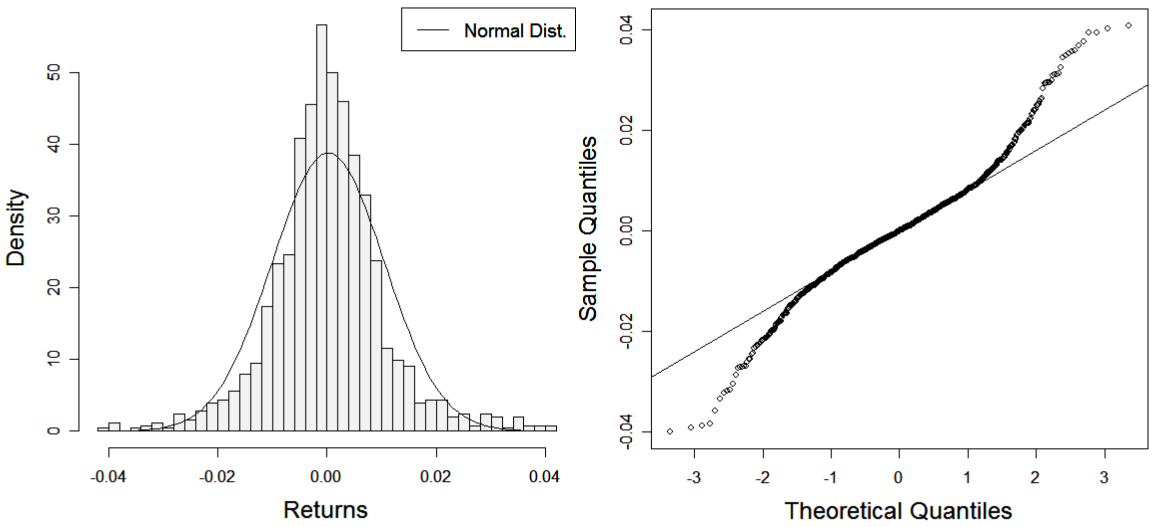

Figure B.7. Histogram and Q-Q plot for the logreturns of JNJ

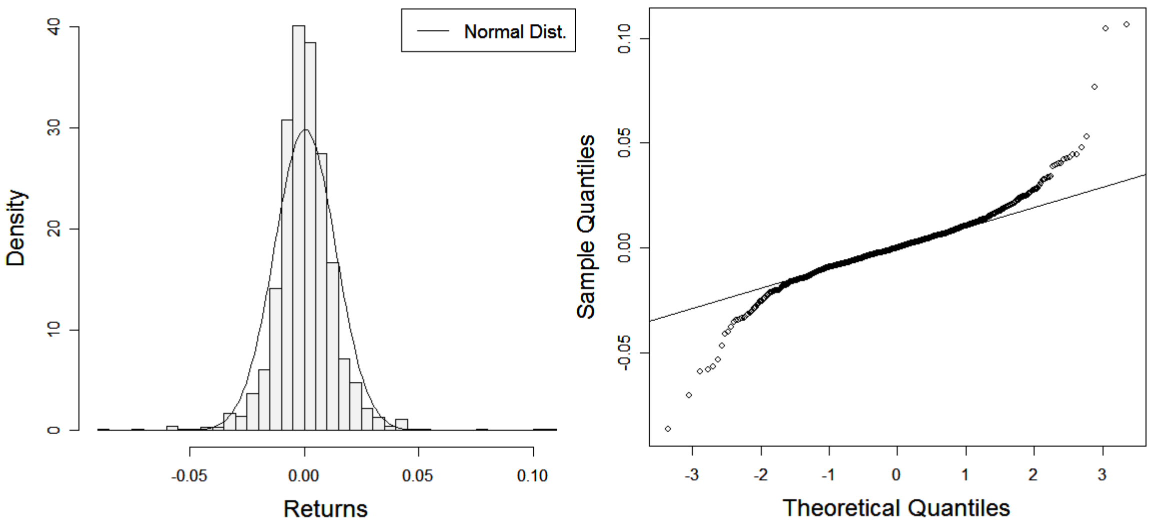

Figure B.8. Histogram and Q-Q plot for the logreturns of IBM

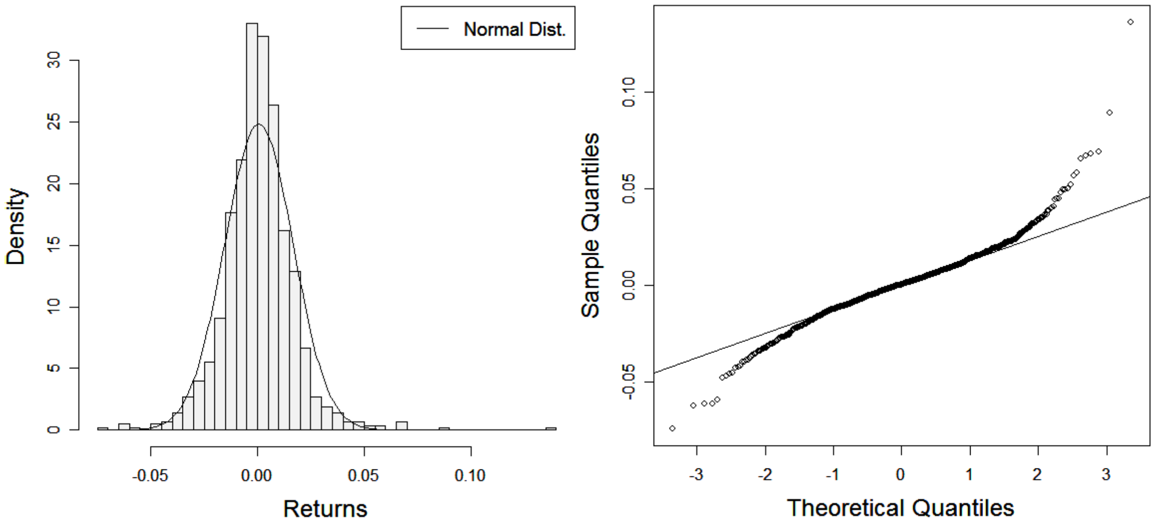

Figure B.9. Histogram and Q-Q plot for the logreturns of DIS

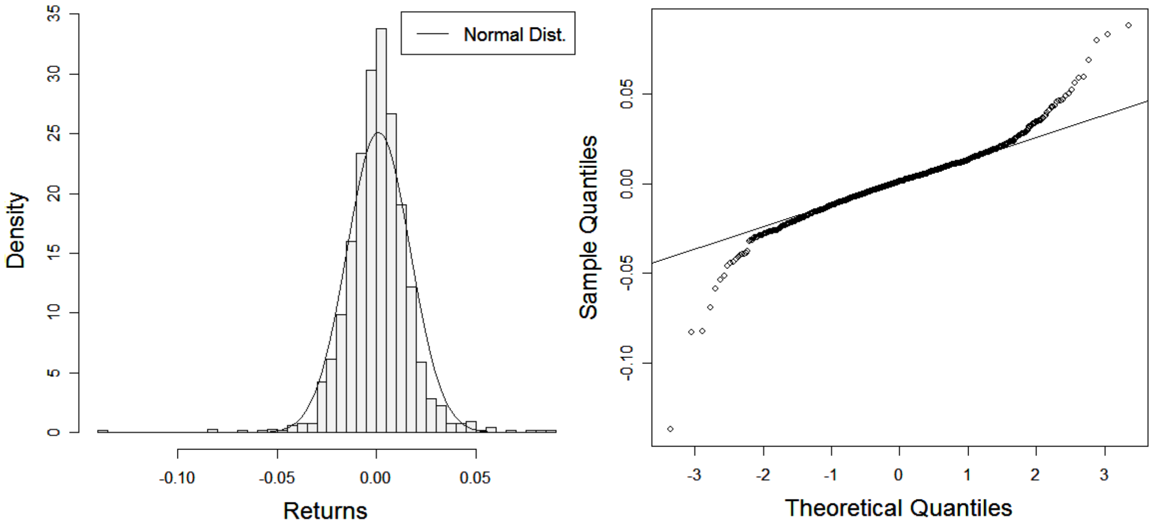

Figure B.10. Histogram and Q-Q plot for the logreturns of MCD

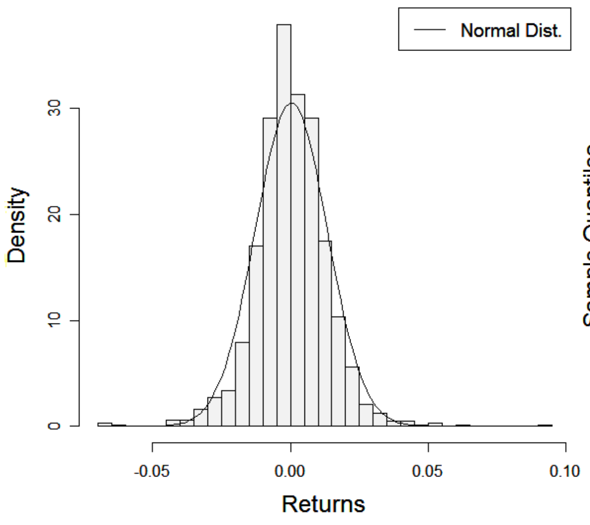

Figure B.11. Histogram and Q-Q plot for the logreturns of DD

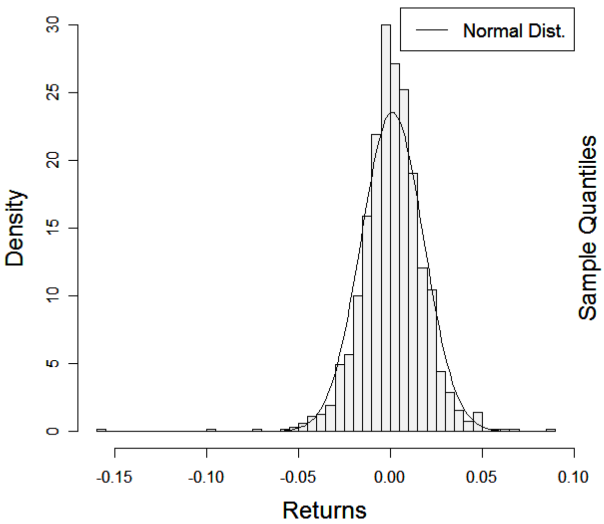

Figure B.12. Histogram and Q-Q plot for the logreturns of CAT

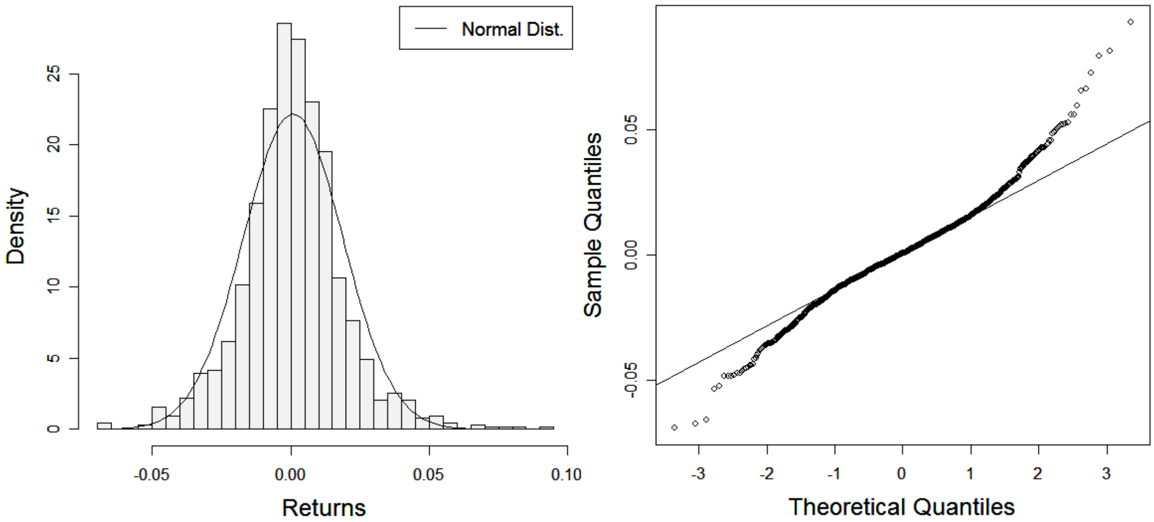

Figure B.13. Histogram and Q-Q plot for the logreturns of DAI

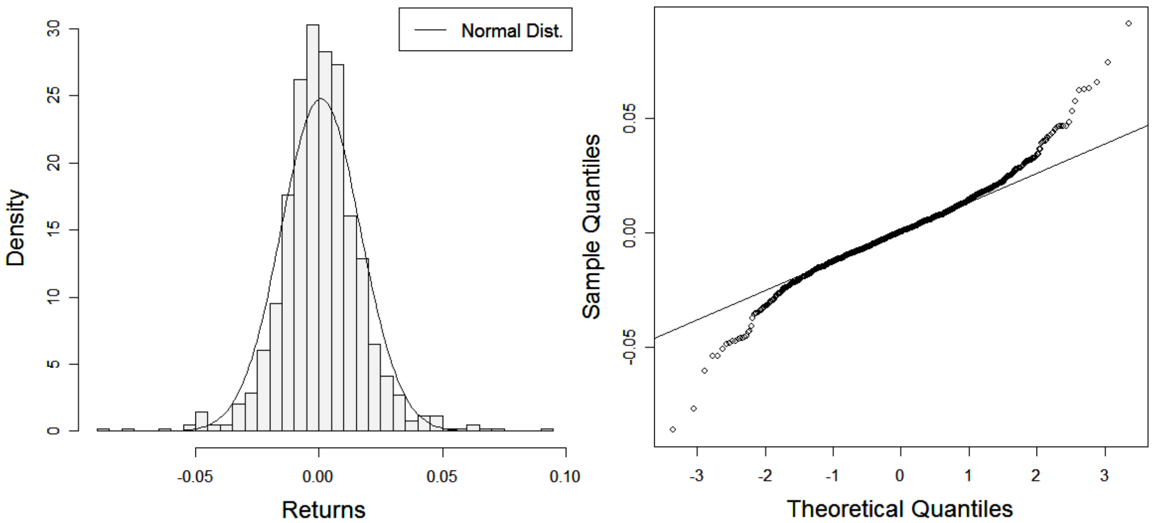

Figure B.14. Histogram and Q-Q plot for the logreturns of HON

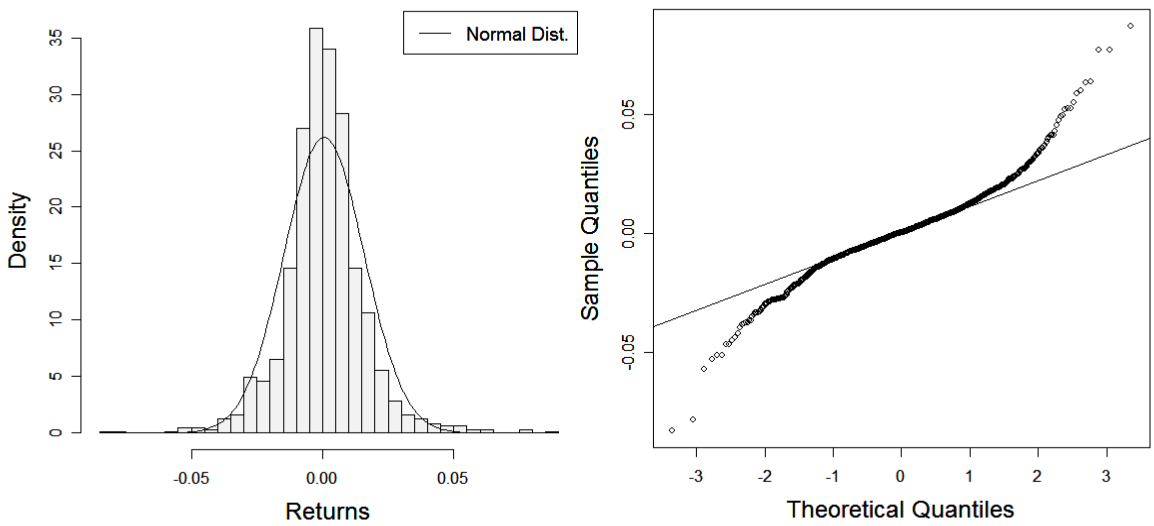

Figure B.15. Histogram and Q-Q plot for the logreturns of T

B.2. Fitted T and GHD Distributions

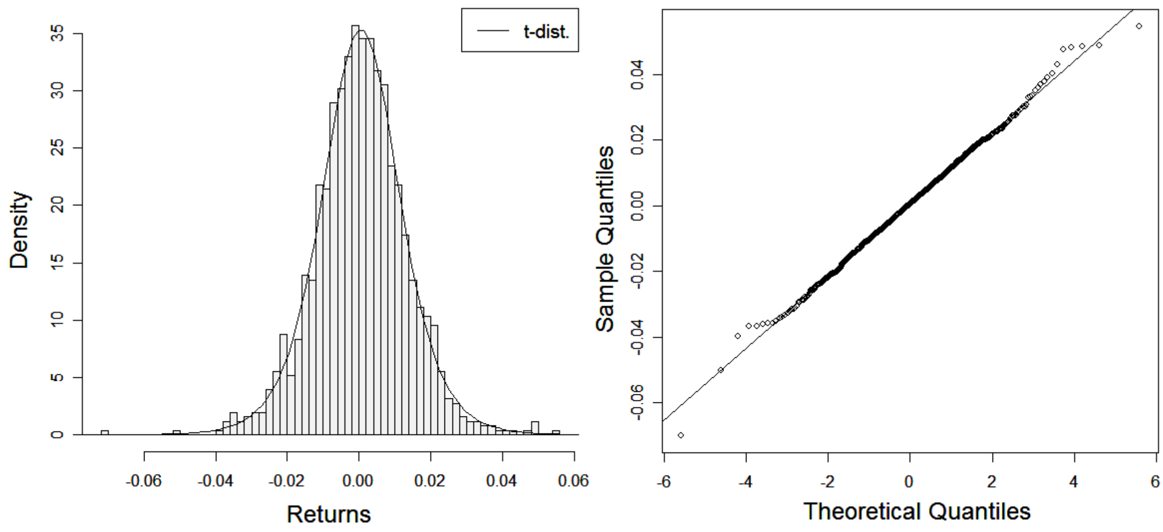

Figure B.16. Histogram and Q-Q plot for the logreturns of BP

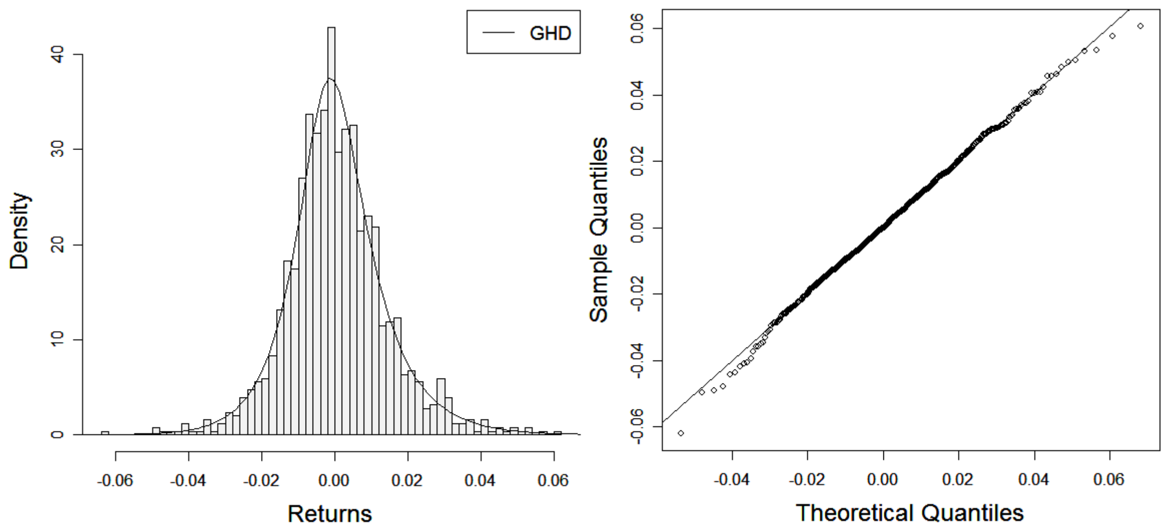

Figure B.17. Histogram and Q-Q plot for the logreturns of UNP

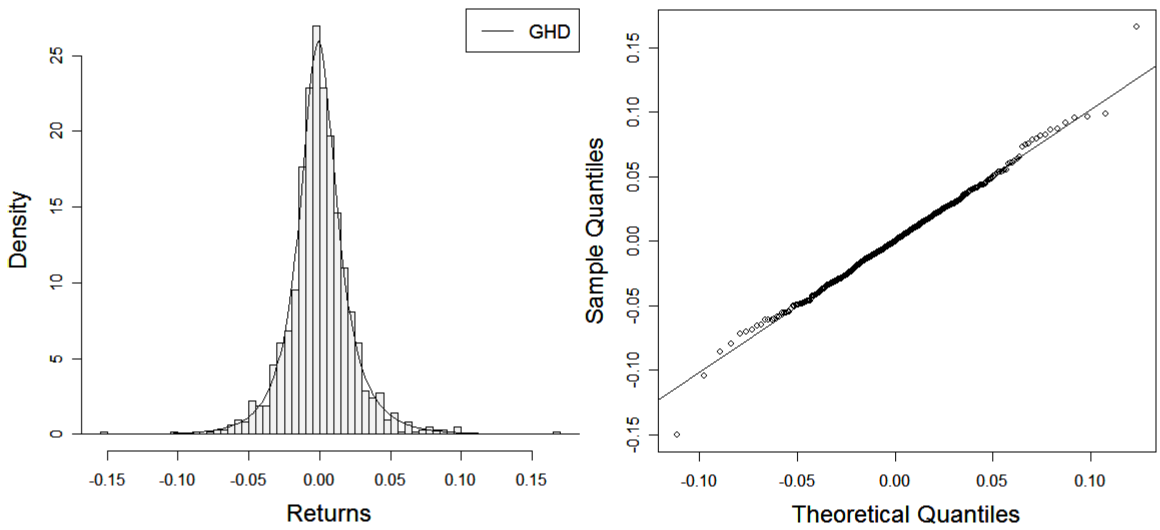

Figure B.18. Histogram and Q-Q plot for the logreturns of GM

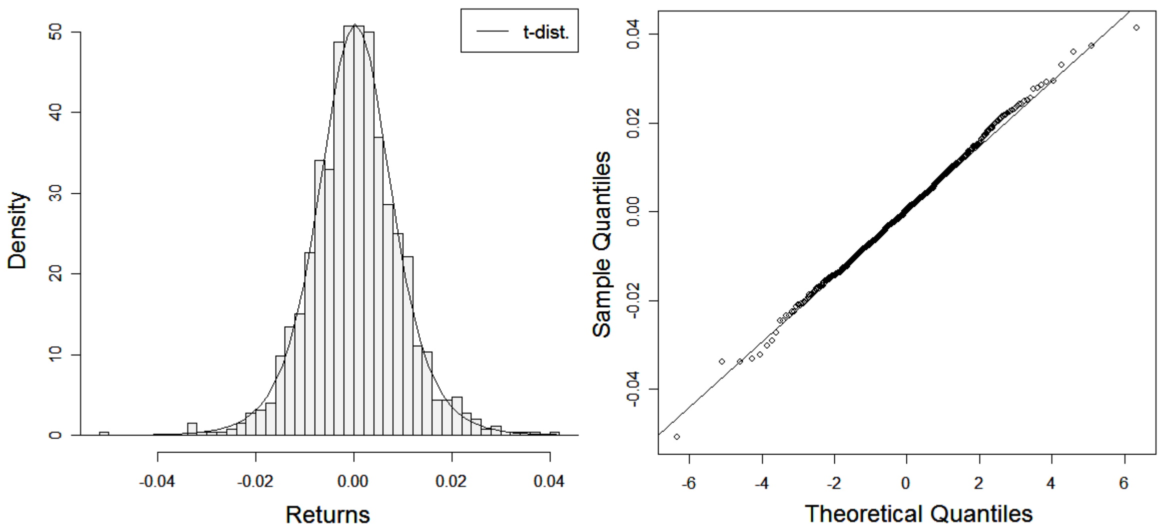

Figure B.19. Histogram and Q-Q plot for the logreturns of PG

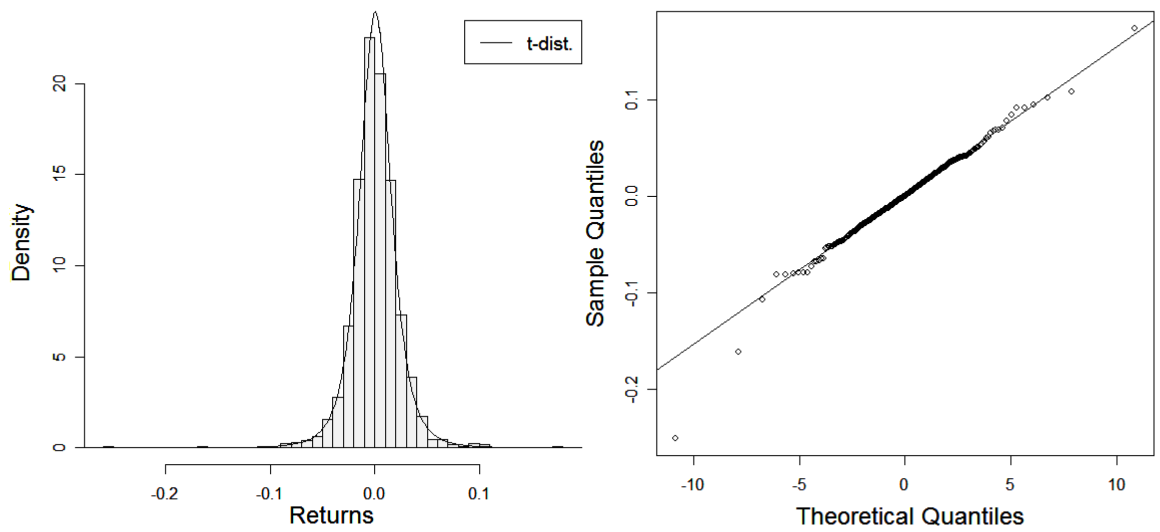

Figure B.20. Histogram and Q-Q plot for the logreturns of MOT

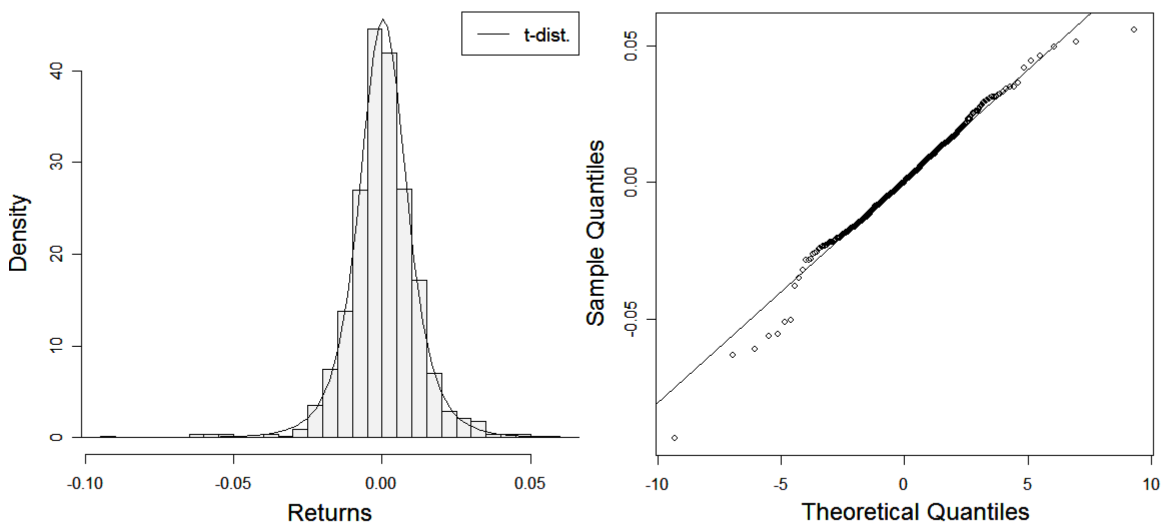

Figure B.21. Histogram and Q-Q plot for the logreturns of MMM

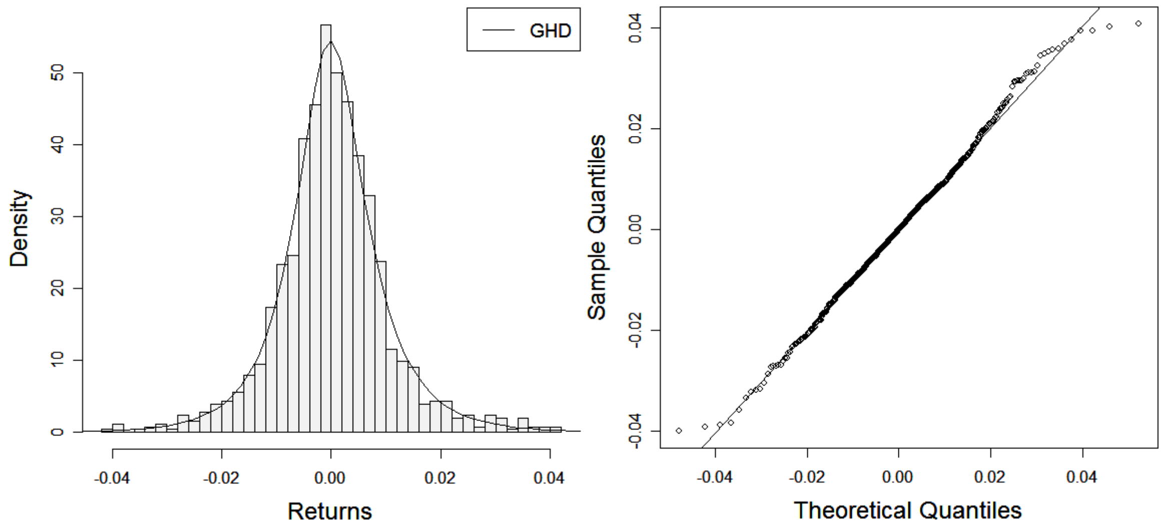

Figure B.22. Histogram and Q-Q plot for the logreturns of JNJ

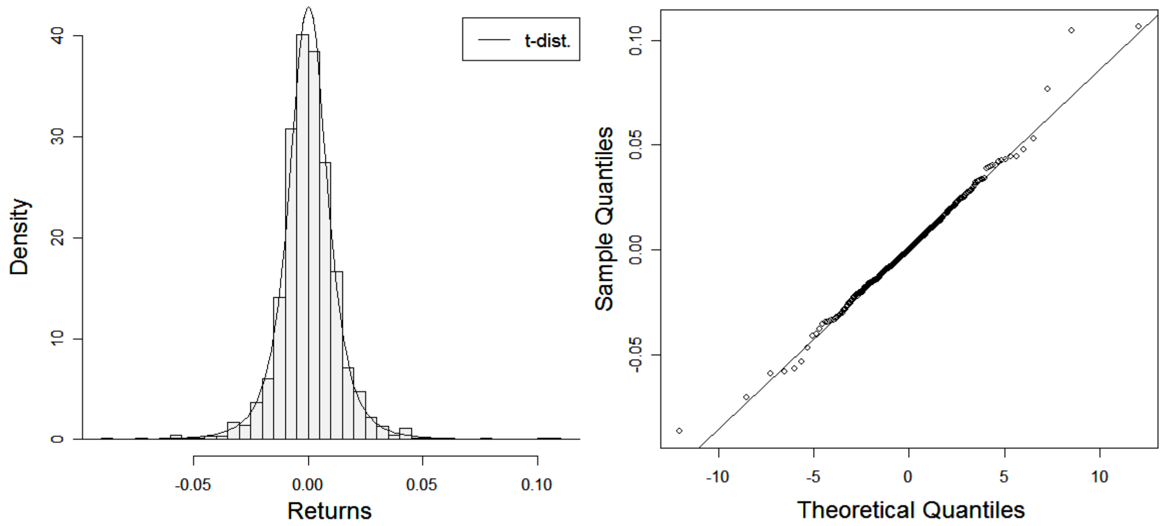

Figure B.23. Histogram and Q-Q plot for the logreturns of IBM

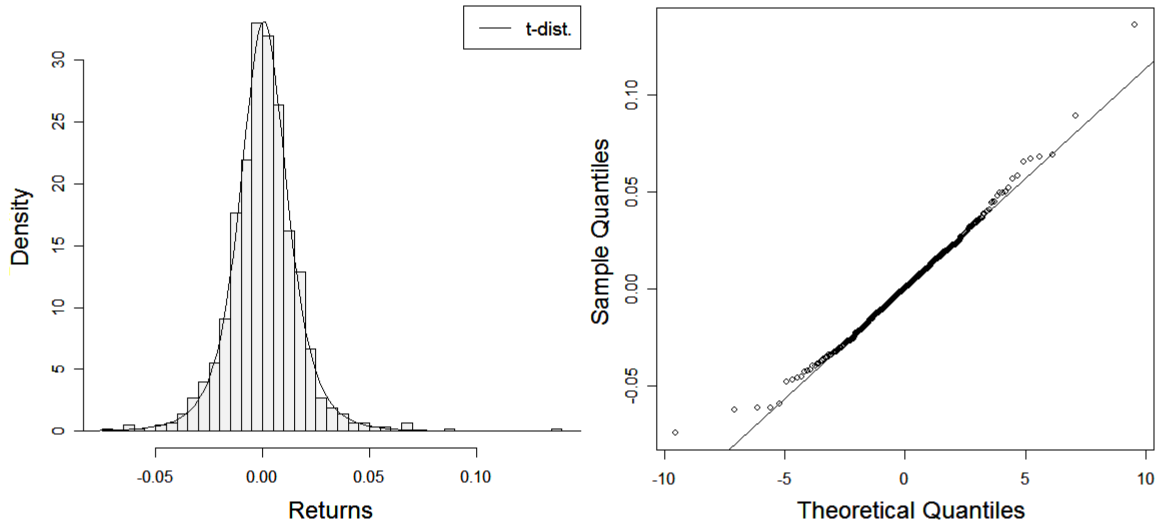

Figure B.24. Histogram and Q-Q plot for the logreturns of DIS

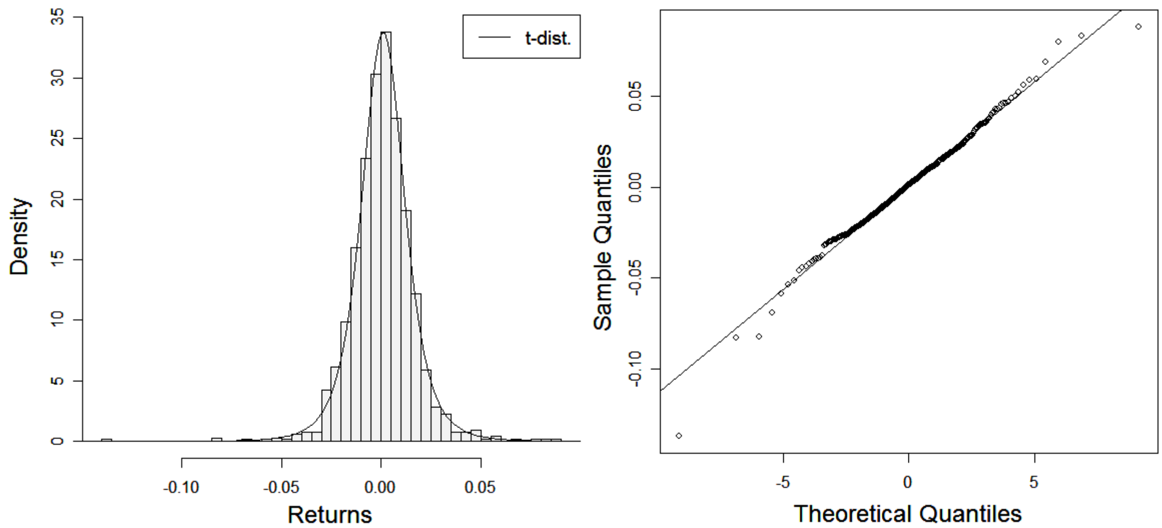

Figure B.25. Histogram and Q-Q plot for the logreturns of MCD

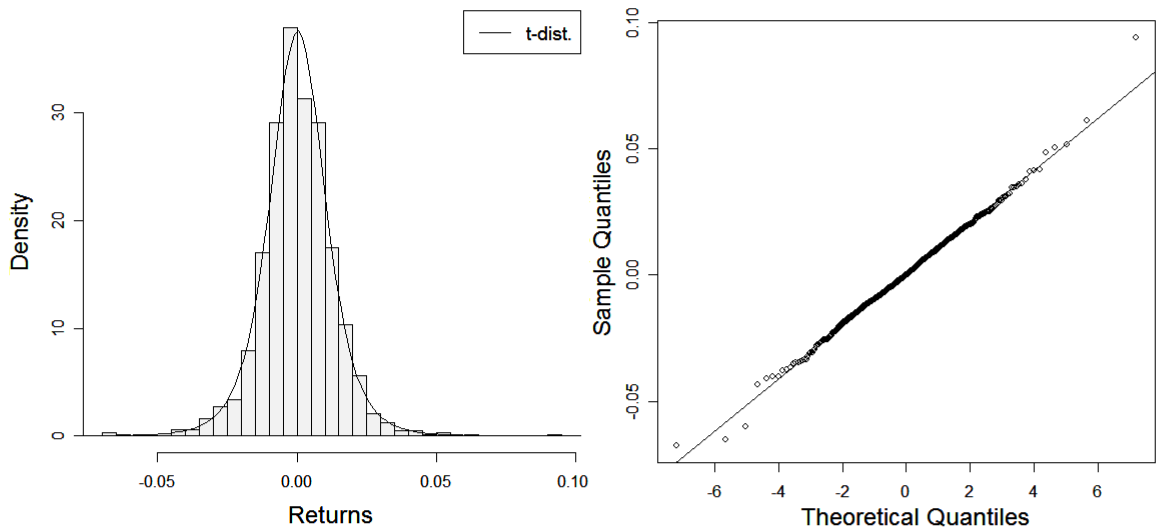

Figure B.26. Histogram and Q-Q plot for the logreturns of DD

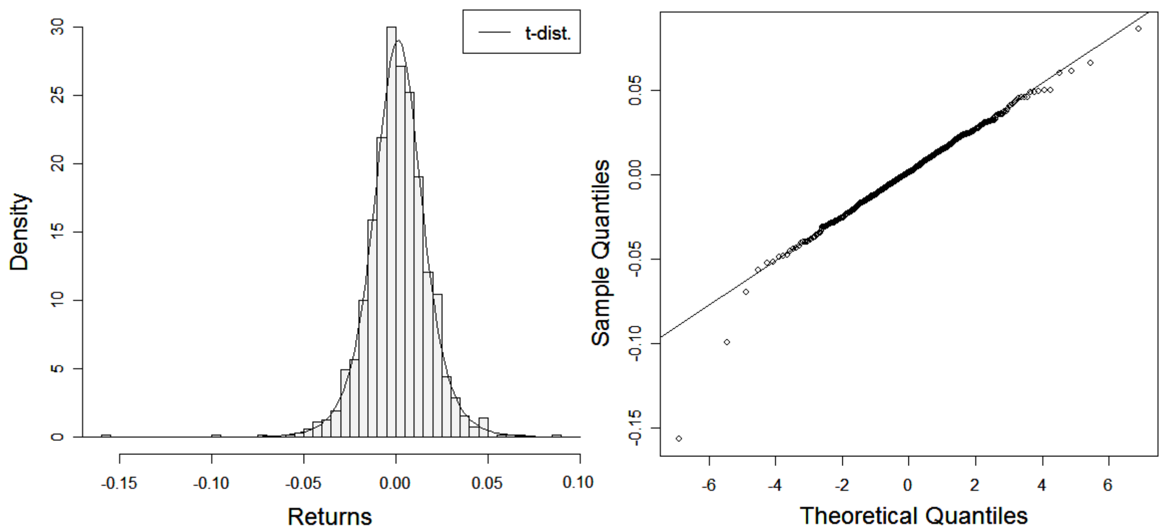

Figure B.27. Histogram and Q-Q plot for the logreturns of CAT

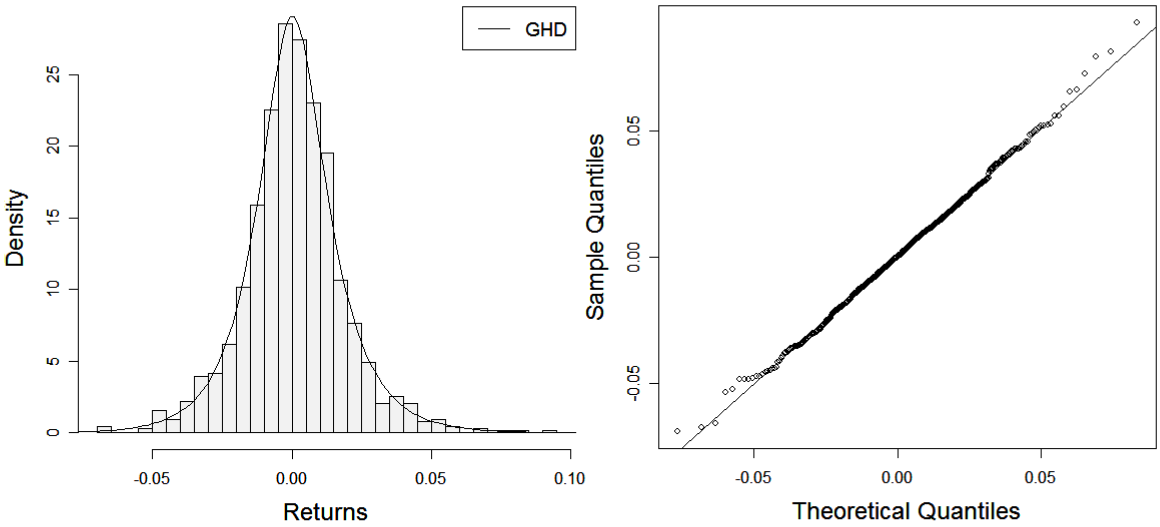

Figure B.28. Histogram and Q-Q plot for the logreturns of DAI

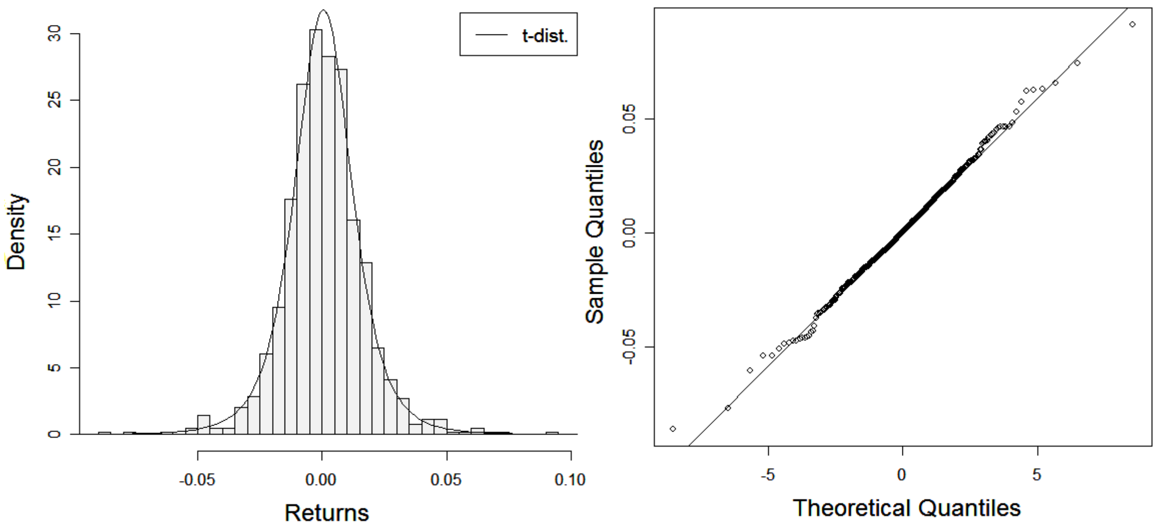

Figure B.29. Histogram and Q-Q plot for the logreturns of HON

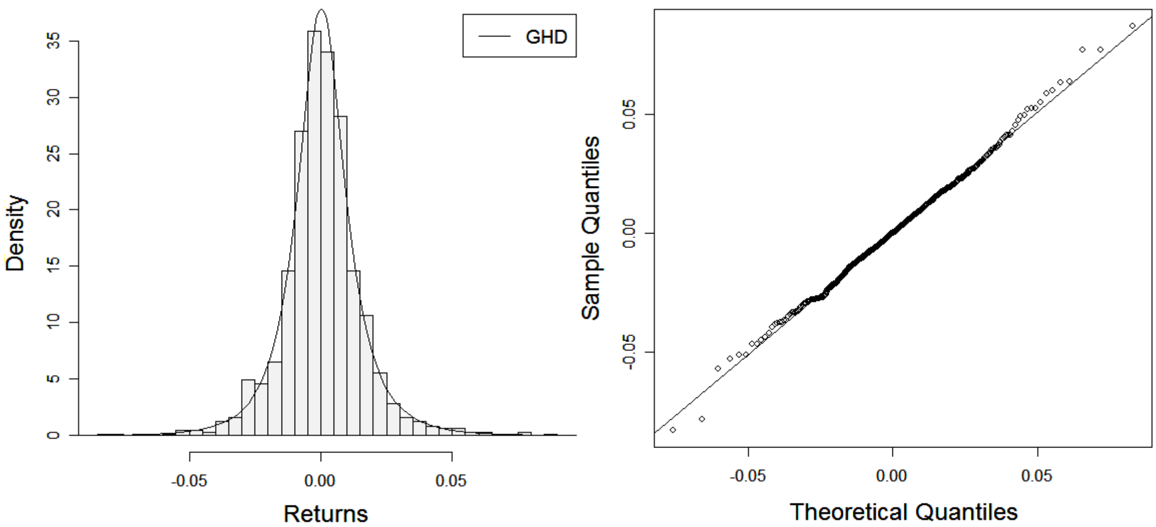

Figure B.30. Histogram and Q-Q plot for the logreturns of T

APPENDIX C: COPULA FITTING RESULTS FOR NYSE PORTFOLIOS

The portfolios which are used in copula fitting examples are referred as:

Portfolio1: GM-DD

Portfolio2: PG-DIS

Portfolio3: GM-DIS

Portfolio4: JNJ-CAT

Portfolio5: DIS-CAT

Portfolio6: UNP-HON

Portfolio7: PG-DAI

Portfolio8: UNP-JNJ

Portfolio9: MOT-T

Portfolio10: MMM-DAI

Portfolio11: GM-HON

Portfolio12: PG-IBM

Portfolio13: MOT-MMM

Portfolio14: IBM-HON

Portfolio15: HON-T

Portfolio16: JNJ-MCD

Portfolio17: JNJ-HON

Portfolio18: MCD-DD

Portfolio19: UNP-DIS

Portfolio20: BP-T

Portfolio21: BP-MCD-DAI

Portfolio22: PG-IBM-DAI

Portfolio23: BP-GM-DIS

Portfolio24: JNJ-DD-HON

Portfolio25: BP-UNP-HON

Portfolio26: PG-MMM-IBM

Portfolio27: DD-HON-T

Portfolio28: MOT-IBM-DD

Portfolio29: BP-MOT-IBM

Portfolio30: DD-CAT-HON

Portfolio31: MMM-DIS-MCD-DD

Portfolio32: BP-MOT-IBM-DD

Portfolio33: MOT-MMM-CAT-HON

Portfolio34: DIS-MCD-DAI-T

Portfolio35: GM-JNJ-MCD-DD-DAI

Portfolio36: BP-GM-MOT-IBM-HON

Portfolio37: UNP-PG-MMM-JNJ-DIS

Portfolio38: UNP-GM-PG-MOT-MMM-JNJ-IBM-MCD-CAT-T

Portfolio39: BP-GM-MOT-MMM-DIS-MCD-DD-DAI-HON-T

AMH: Ali-Mikhail-Haq copula

Llh: Log-likelihood value

Distance: L^2 distance between the empirical and theoretical copulas

C.1. Copula Fitting Results for Portfolios of Two Stocks

Table C.1. Results of copula fittings for portfolio1

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.395	0.022	106.791	0.239	−211.581
Student-t	0.400/υ=6.18	0.025/SE_{υ}=1.34	121.053	0.223	−238.106
Clayton	0.538	0.046	92.106	0.469	−182.212
Gumbel	1.322	0.029	102.242	0.366	−202.484
Frank	2.584	0.182	101.790	0.218	−201.580
AMH	0.846	0.030	101.678	0.301	−201.356

Table C.2. Results of copula fittings for portfolio2

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.322	0.024	68.960	0.214	−135.921
Student-t	0.332/υ=6.50	0.027/SE_{υ}=1.35	84.978	0.181	−165.955
Clayton	0.425	0.044	61.157	0.401	−120.314
Gumbel	1.239	0.026	64.848	0.309	−127.695
Frank	2.117	0.179	70.268	0.220	−138.536
AMH	0.742	0.040	68.206	0.254	−134.412

Table C.3. Results of copula fittings for portfolio3

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.367	0.023	91.465	0.264	−180.930
Student-t	0.376/υ=8.09	0.025/SE_{υ}=2.08	101.294	0.250	−198.587
Clayton	0.481	0.044	75.610	0.471	−149.219
Gumbel	1.286	0.027	88.964	0.380	−175.928
Frank	2.428	0.180	92.178	0.251	−182.357
AMH	0.786	0.033	88.811	0.299	−175.622

Table C.4. Results of copula fittings for portfolio4

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.288	0.025	54.576	0.190	−107.152
Student-t	0.291/υ=8.85	0.027/SE_{υ}=2.56	61.845	0.185	−119.690
Clayton	0.319	0.041	40.569	0.409	−79.139
Gumbel	1.217	0.025	58.211	0.213	−114.421
Frank	1.751	0.175	49.932	0.220	−97.864
AMH	0.646	0.047	47.407	0.262	−92.814

Table C.5. Results of copula fittings for portfolio5

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.429	0.021	128.871	0.228	−255.743
Student-t	0.437/v=5.37	0.025/SE_v=0.97	150.579	0.201	−297.158
Clayton	0.584	0.047	106.330	0.455	−210.661
Gumbel	1.373	0.030	131.748	0.382	−261.496
Frank	2.857	0.183	123.136	0.256	−244.272
AMH	0.870	0.025	121.992	0.274	−241.985

Table C.6. Results of copula fittings for portfolio6

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.470	0.020	158.304	0.226	−314.608
Student-t	0.480/v=9.11	0.022/SE_v=2.55	166.477	0.212	−328.953
Clayton	0.682	0.049	135.331	0.564	−268.661
Gumbel	1.409	0.031	142.090	0.396	−282.180
Frank	3.266	0.186	157.940	0.177	−313.880
AMH	0.916	0.020	151.257	0.407	−300.515

Table C.7. Results of copula fittings for portfolio7

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.341	0.024	77.935	0.259	−153.871
tudent-t	0.360/v=7.70	0.026/SE_v=1.80	90.219	0.215	−176.438
Clayton	0.460	0.045	68.296	0.457	−134.592
Gumbel	1.262	0.027	72.144	0.367	−142.287
Frank	2.332	0.179	85.141	0.230	−168.282
AMH	0.774	0.035	81.925	0.278	−161.849

Table C.8. Results of copula fittings for portfolio8

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.278	0.025	50.552	0.206	−99.104
Student-t	0.283/υ=19.33	0.026/SE_{υ}=10.12	52.857	0.201	−101.715
Clayton	0.321	0.041	39.495	0.405	−76.989
Gumbel	1.191	0.024	44.384	0.307	−86.767
Frank	1.748	0.173	51.046	0.189	−100.091
AMH	0.651	0.046	49.849	0.223	−97.699

Table C.9. Results of copula fittings for portfolio9

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.324	0.024	69.988	0.192	−137.975
Student-t	0.326/υ=6.79	0.027/SE_{υ}=1.58	82.217	0.186	−160.433
Clayton	0.385	0.043	54.351	0.432	−106.701
Gumbel	1.248	0.026	71.568	0.263	−141.136
Frank	2.038	0.178	65.796	0.197	−129.593
AMH	0.723	0.041	63.328	0.257	−124.656

Table C.10. Results of copula fittings for portfolio10

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.460	0.020	151.345	0.229	−300.690
Student-t	0.475/υ=6.07	0.023/SE_{υ}=1.20	169.998	0.205	−335.996
Clayton	0.644	0.049	117.032	0.563	−232.063
Gumbel	1.414	0.031	154.284	0.367	−306.569
Frank	3.183	0.185	150.820	0.229	−299.640
AMH	0.886	0.023	139.506	0.402	−277.013

Table C.11. Results of copula fittings for portfolio11

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.418	0.021	121.922	0.196	−241.843
Student-t	0.421/v=9.56	0.024/SE_v=3.00	128.304	0.188	−252.607
Clayton	0.561	0.046	100.214	0.484	−198.429
Gumbel	1.338	0.029	112.679	0.316	−223.357
Frank	2.704	0.181	113.134	0.224	−224.267
AMH	0.856	0.028	110.047	0.334	−218.095

Table C.12. Results of copula fittings for portfolio12

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.301	0.025	59.983	0.260	−117.966
Student-t	0.328/v=5.35	0.027/SE_v=0.92	82.447	0.214	−160.894
Clayton	0.409	0.045	53.954	0.395	−105.909
Gumbel	1.231	0.026	61.998	0.344	−121.996
Frank	2.070	0.179	67.014	0.231	−132.028
AMH	0.733	0.041	64.888	0.241	−127.775

Table C.13. Results of copula fittings for portfolio13

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.319	0.024	68.470	0.245	−134.941
Student-t	0.330/v=9.51	0.026/SE_v=2.88	75.720	0.225	−147.441
Clayton	0.370	0.043	48.980	0.499	−95.960
Gumbel	1.245	0.026	68.638	0.324	−135.277
Frank	2.110	0.177	71.257	0.203	−140.515
AMH	0.727	0.040	66.992	0.280	−131.985

Table C.14. Results of copula fittings for portfolio14

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.484	0.019	169.457	0.216	−336.914
Student-t	0.477/v=3.94	0.025/SE_v=0.59	202.614	0.250	−401.229
Clayton	0.755	0.051	147.311	0.376	−292.623
Gumbel	1.437	0.031	175.146	0.358	−348.291
Frank	3.144	0.185	146.419	0.275	−290.838
AMH	0.946	0.018	151.071	0.310	−300.141

Table C.15. Results of copula fittings for portfolio15

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.382	0.022	100.362	0.193	−198.724
Student-t	0.396/v=4.55	0.026/SE_v=0.69	130.706	0.174	−257.412
Clayton	0.525	0.046	85.515	0.428	−169.031
Gumbel	1.322	0.028	109.363	0.285	−216.727
Frank	2.518	0.181	96.910	0.228	−191.820
AMH	0.830	0.032	94.496	0.289	−186.992

Table C.16. Results of copula fittings for portfolio16

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.226	0.026	33.182	0.190	−64.364
Student-t	0.234/v=8.54	0.028/SE_v=2.31	41.291	0.177	−78.582
Clayton	0.268	0.040	30.020	0.315	−58.039
Gumbel	1.156	0.023	31.324	0.234	−60.648
Frank	1.421	0.174	33.383	0.216	−64.767
AMH	0.560	0.054	32.282	0.221	−62.564

Table C.17. Results of copula fittings for portfolio17

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.279	0.025	51.345	0.190	−100.689
Student-t	0.290/υ=5.78	0.028/SE_{υ}=1.09	69.584	0.175	−135.167
Clayton	0.383	0.043	52.312	0.299	−102.624
Gumbel	1.200	0.025	51.593	0.272	−101.186
Frank	1.750	0.176	49.660	0.216	−97.319
AMH	0.693	0.047	51.091	0.235	−100.182

Table C.18. Results of copula fittings for portfolio18

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.364	0.023	89.849	0.272	−177.699
Student-t	0.375/υ=9.19	0.025/SE_{υ}=2.78	96.522	0.251	−189.044
Clayton	0.453	0.045	68.117	0.521	−134.233
Gumbel	1.289	0.027	85.226	0.398	−168.452
Frank	2.442	0.180	93.423	0.217	−184.846
AMH	0.784	0.034	87.350	0.294	−172.700

Table C.19. Results of copula fittings for portfolio19

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.377	0.023	96.742	0.203	−191.484
Student-t	0.388/υ=7.53	0.025/SE_{υ}=1.80	108.310	0.183	−212.620
Clayton	0.466	0.044	73.503	0.508	−145.006
Gumbel	1.310	0.028	99.059	0.307	−196.117
Frank	2.496	0.181	96.655	0.190	−191.311
AMH	0.790	0.033	90.028	0.293	−178.056

Table C.20. Results of copula fittings for portfolio20

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	0.277	0.025	50.257	0.187	−98.514
Student-t	0.279/υ=6.92	0.028/SE_{υ}=1.55	63.350	0.191	−122.700
Clayton	0.305	0.041	36.877	0.368	−71.753
Gumbel	1.207	0.025	57.578	0.254	−113.156
Frank	1.696	0.175	46.803	0.200	−91.607
AMH	0.635	0.048	45.173	0.208	−88.347

C.2. Copula Fitting Results for Portfolios of Three Stocks

Table C.21. Results of copula fittings for portfolio21

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	232.535	0.249	−459.069
Student-t	ρ_t/υ=11.20	SE_{ρ_t}/SE_υ=2.40	246.704	0.232	−485.407
Clayton	0.403	0.027	163.124	0.636	−324.248
Gumbel	1.240	0.018	165.275	0.543	−328.551
Frank	1.938	0.115	166.488	0.402	−330.976
AMH	0.555	0.055	29.906	4.502	−57.812

Table C.22. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio21

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.204	0.447	0.365
$SE_{\rho_{norm}}$	0.026	0.021	0.023
ρ_t	0.218	0.447	0.366
SE_{ρ_t}	0.028	0.022	0.025

Table C.23. Results of copula fittings for portfolio22

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	248.938	0.246	−491.876
Student-t	ρ_t/υ=6.66	$SE_{\rho_t}/SE_{\upsilon}$=0.90	287.727	0.213	−567.455
Clayton	0.489	0.029	202.399	0.608	−402.798
Gumbel	1.285	0.019	218.224	0.536	−434.447
Frank	2.332	0.120	221.739	0.316	−441.478
AMH	0.733	0.041	64.888	4.852	−127.775

Table C.24. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio22

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.301	0.341	0.463
$SE_{\rho_{norm}}$	0.024	0.023	0.020
ρ_t	0.330	0.357	0.460
SE_{ρ_t}	0.027	0.026	0.023

Table C.25. Results of copula fittings for portfolio23

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	167.350	0.235	−328.699
Student-t	ρ_t/υ=9.28	$SE_{\rho_t}/SE_{\upsilon}$=1.69	187.818	0.215	−367.636
Clayton	0.355	0.026	130.604	0.545	−259.207
Gumbel	1.229	0.017	164.703	0.421	−327.405
Frank	1.827	0.116	147.156	0.331	−292.313
AMH	0.608	0.052	37.980	5.105	−73.960

Table C.26. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio23

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.259	0.294	0.368
$SE_{\rho_{norm}}$	0.025	0.025	0.023
ρ_t	0.251	0.295	0.374
SE_{ρ_t}	0.028	0.027	0.025

Table C.27. Results of copula fittings for portfolio24

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	SE_{norm}	301.306	0.225	−596.612
Student-t	ρ_t/υ=6.89	SE_t/SE_υ=0.95	340.914	0.199	−673.828
Clayton	0.525	0.029	231.703	0.587	−461.406
Gumbel	1.302	0.019	236.317	0.558	−470.635
Frank	2.403	0.120	234.600	0.368	−467.201
AMH	0.804	0.035	80.085	4.960	−158.170

Table C.28. ρ_{norm}, SE_{norm}, ρ_t and SE_t for portfolio24

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.347	0.279	0.533
SE_{norm}	0.023	0.025	0.018
ρ_t	0.340	0.293	0.543
SE_t	0.026	0.027	0.020

Table C.29. Results of copula fittings for portfolio25

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	SE_{norm}	286.742	0.221	−567.485
Student-t	ρ_t/υ=8.90	SE_t/SE_υ=1.57	307.861	0.197	−607.722
Clayton	0.503	0.029	227.428	0.741	−452.856
Gumbel	1.315	0.019	252.311	0.565	−502.622
Frank	2.453	0.120	245.209	0.357	−488.418
AMH	0.808	0.034	85.461	4.785	−168.922

Table C.30. ρ_{norm}, SE_{norm}, ρ_t and SE_t for portfolio25

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.373	0.361	0.470
SE_{norm}	0.023	0.023	0.020
ρ_t	0.370	0.367	0.480
SE_t	0.025	0.025	0.022

Table C.31. Results of copula fittings for portfolio26

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	SE_{norm}	240.095	0.286	−474.190
Student-t	ρ_t/υ=6.77	SE_t/SE_υ=0.88	283.581	0.225	−559.161
Clayton	0.460	0.029	187.807	0.728	−373.613
Gumbel	1.277	0.018	211.334	0.641	−420.668
Frank	2.310	0.119	221.457	0.346	−440.914
AMH	0.836	0.030	103.687	5.322	−205.374

Table C.32. ρ_{norm}, SE_{norm}, ρ_t and SE_t for portfolio26

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.378	0.300	0.420
SE_{norm}	0.022	0.024	0.021
ρ_t	0.403	0.325	0.435
SE_t	0.024	0.027	0.024

Table C.33. Results of copula fittings for portfolio27

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	SE_{norm}	340.782	0.207	−675.564
Student-t	ρ_t/v=5.55	SE_t/SE_v=0.63	398.726	0.168	−789.453
Clayton	0.566	0.030	256.685	0.767	−511.370
Gumbel	1.358	0.020	316.483	0.562	−630.965
Frank	2.689	0.122	283.847	0.357	−565.695
AMH	0.958	0.013	192.109	5.889	−382.218

Table C.34. ρ_{norm}, SE_{norm}, ρ_t and SE_t for portfolio27

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.533	0.370	0.383
SE_{norm}	0.018	0.023	0.022
ρ_t	0.539	0.380	0.401
SE_t	0.021	0.026	0.025

Table C.35. Results of copula fittings for portfolio28

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	SE_{norm}	305.470	0.242	−604.941
Student-t	ρ_t/υ=8.01	SE_t/SE_υ=1.27	333.115	0.249	−658.2310
Clayton	0.535	0.029	246.486	0.722	−490.973
Gumbel	1.325	0.019	269.630	0.624	−537.260
Frank	2.529	0.120	260.494	0.337	−518.988 0
AMH	0.883	0.026	117.871	5.196	−233.742

Table C.36. ρ_{norm}, SE_{norm}, ρ_t and SE_t for portfolio28

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.420	0.365	0.466
SE_{norm}	0.021	0.023	0.020
ρ_t	0.424	0.368	0.472
SE_t	0.024	0.025	0.022

Table C.37. Results of copula fittings for portfolio29

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	179.525	0.207	−353.049
Student-t	ρ_t/υ=8.46	SE_{ρ_t}/SE_υ=1.43	203.152	0.191	−398.303
Clayton	0.353	0.026	130.120	0.556	−258.241
Gumbel	1.211	0.017	145.168	0.492	−288.335
Frank	1.729	0.115	133.933	0.354	−265.865
AMH	0.565	0.053	33.428	5.198	−64.855

Table C.38. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio29

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.232	0.261	0.420
$SE_{\rho_{norm}}$	0.026	0.025	0.021
ρ_t	0.231	0.253	0.425
SE_{ρ_t}	0.028	0.028	0.024

Table C.39. Results of copula fittings for portfolio30

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	446.161	0.233	−886.322
Student-t	ρ_t/υ=6.58	$SE_{\rho_t}/SE_{\upsilon}$=0.87	493.048	0.200	−978.097
Clayton	0.688	0.032	345.201	0.916	−688.402
Gumbel	1.446	0.022	421.342	0.640	−840.684
Frank	3.287	0.125	404.577	0.346	−807.155
AMH	0.908	0.020	148.245	4.445	−294.490

Table C.40. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio30

	ρ_{12}	ρ_{13}	ρ_{23}
ρ_{norm}	0.474	0.532	0.499
$SE_{\rho_{norm}}$	0.020	0.018	0.019
ρ_t	0.483	0.543	0.517
SE_{ρ_t}	0.022	0.020	0.021

C.3. Copula Fitting Results for Portfolios of Four Stocks

Table C.41. Results of copula fittings for portfolio31

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	516.441	0.283	−1020.883
Student-t	ρ_t/υ=9.72	SE_{ρ_t}/SE_υ=1.33	554.284	0.265	−1094.567
Clayton	0.477	0.022	377.194	1.006	−752.388
Gumbel	1.309	0.015	426.679	0.848	−851.359
Frank	2.420	0.094	431.284	0.545	−860.568
AMH	0.844	0.029	103.875	6.946	−205.750

Table C.42. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio31

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}
ρ_{norm}	0.396	0.338	0.515	0.357	0.466	0.363
$SE_{\rho_{norm}}$	0.022	0.023	0.018	0.023	0.020	0.023
ρ_t	0.410	0.347	0.529	0.356	0.487	0.378
SE_{ρ_t}	0.023	0.025	0.020	0.025	0.021	0.024

Table C.43. Results of copula fittings for portfolio32

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	412.324	0.234	−812.649
Student-t	ρ_t/υ=10.17	SE_{ρ_t}/SE_υ=1.53	442.822	0.228	−871.643
layton	0.399	0.021	290.067	0.775	−578.134
Gumbel	1.246	0.014	313.962	0.707	−625.924
Frank	1.955	0.093	296.770	0.509	−591.540
AMH	0.565	0.053	33.428	6.824	−64.855

Table C.44. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio32

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}
ρ_{norm}	0.232	0.261	0.374	0.420	0.365	0.466
$SE_{\rho_{norm}}$	0.026	0.025	0.023	0.021	0.023	0.020
ρ_t	0.230	0.257	0.359	0.425	0.371	0.469
SE_{ρ_t}	0.028	0.027	0.025	0.023	0.024	0.022

Table C.45. Results of copula fittings for portfolio33

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	505.256	0.291	−998.513
Student-t	ρ_t/υ=7.23	SE_{ρ_t}/SE_υ=0.77	575.807	0.238	−1137.614
Clayton	0.464	0.022	360.706	1.080	−719.411
Gumbel	1.305	0.015	423.650	0.877	−845.300
Frank	2.393	0.095	418.370	0.578	−834.740
AMH	0.727	0.040	66.992	6.791	−131.985

Table C.46. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio33

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}
ρ_{norm}	0.319	0.352	0.356	0.455	0.431	0.499
$SE_{\rho_{norm}}$	0.024	0.023	0.023	0.020	0.021	0.019
ρ_t	0.325	0.364	0.371	0.481	0.458	0.516
SE_{ρ_t}	0.026	0.025	0.025	0.022	0.022	0.021

Table C.47. Results of copula fittings for portfolio34

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	452.174	0.214	−892.348
Student-t	ρ_t/v=7.06	SE_{ρ_t}/SE_v=0.78	512.777	0.190	−1011.554
Clayton	0.474	0.022	364.083	0.816	−726.166
Gumbel	1.290	0.015	384.362	0.604	−766.724
Frank	2.225	0.095	366.147	0.447	−730.293
AMH	0.780	0.035	83.648	7.352	−165.297

Table C.48. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio34

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}
ρ_{norm}	0.357	0.475	0.373	0.365	0.272	0.438
$SE_{\rho_{norm}}$	0.023	0.020	0.023	0.023	0.025	0.021
ρ_t	0.353	0.477	0.397	0.364	0.286	0.450
SE_{ρ_t}	0.025	0.022	0.025	0.025	0.027	0.023

C.4. Copula Fitting Results for Portfolios of Five Stocks

Table C.49. Results of copula fitting for portfolio35

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	635.296	0.222	−1250.592
Student-t	ρ_t/υ=10.36	$SE_{\rho_t}/SE_{\upsilon}$=1.25	684.252	0.235	−1346.504
Clayton	0.421	0.017	476.414	0.809	−950.827
Gumbel	1.268	0.012	484.994	0.708	−967.987
Frank	2.095	0.080	493.687	0.485	−985.373
AMH	0.584	0.052	34.564	7.996	−67.127

Table C.50. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio35

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_{23}	ρ_{24}	ρ_{25}	ρ_{34}	ρ_{35}	ρ_{45}
ρ_{norm}	0.213	0.312	0.395	0.524	0.226	0.347	0.339	0.364	0.365	0.477
$SE_{\rho_{norm}}$	0.026	0.024	0.022	0.018	0.026	0.023	0.023	0.023	0.023	0.020
ρ_t	0.221	0.315	0.406	0.521	0.237	0.354	0.357	0.377	0.368	0.488
SE_{ρ_t}	0.028	0.026	0.023	0.020	0.027	0.025	0.025	0.024	0.024	0.021

Table C.51. Results of copula fitting for portfolio36

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	569.225	0.252	−1118.451
Student-t	ρ_t/υ=10.06	SE_{ρ_t}/SE_υ=1.20	620.435	0.257	−1218.869
Clayton	0.379	0.017	407.728	0.778	−813.456
Gumbel	1.235	0.012	423.936	0.761	−845.872
Frank	1.852	0.079	403.924	0.581	−805.848
AMH	0.608	0.052	37.980	8.457	−73.960

Table C.52. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio36

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_{23}	ρ_{24}	ρ_{25}	ρ_{34}	ρ_{35}	ρ_{45}
ρ_{norm}	0.258	0.232	0.260	0.361	0.261	0.339	0.418	0.420	0.357	0.484
$SE_{\rho_{norm}}$	0.025	0.026	0.025	0.023	0.025	0.023	0.021	0.021	0.023	0.019
ρ_t	0.253	0.229	0.258	0.363	0.260	0.340	0.416	0.423	0.379	0.500
SE_{ρ_t}	0.027	0.027	0.027	0.025	0.027	0.025	0.023	0.023	0.024	0.021

Table C.53. Results of copula fitting for portfolio37

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	571.783	0.211	−1123.566
Student-t	ρ_t/υ=11.46	SE_{ρ_t}/SE_υ=1.45	619.028	0.200	−1216.055
Clayton	0.393	0.017	436.706	0.963	−871.412
Gumbel	1.259	0.012	479.625	0.794	−957.250
Frank	2.055	0.079	487.564	0.548	−973.127
AMH	0.742	0.039	69.617	8.438	−137.234

Table C.54. ρ_{norm}, $SE_{\rho_{norm}}$, ρ_t and SE_{ρ_t} for portfolio37

	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{15}	ρ_{23}	ρ_{24}	ρ_{25}	ρ_{34}	ρ_{35}	ρ_{45}
ρ_{norm}	0.322	0.409	0.277	0.376	0.379	0.397	0.321	0.354	0.396	0.316
$SE_{\rho_{norm}}$	0.024	0.021	0.025	0.022	0.022	0.022	0.024	0.023	0.022	0.024
ρ_t	0.325	0.417	0.284	0.383	0.394	0.411	0.336	0.373	0.406	0.320
SE_{ρ_t}	0.025	0.023	0.026	0.024	0.024	0.023	0.025	0.024	0.023	0.026

C.5. Copula Fitting Results for Portfolios of 10 Stocks

Table C.55. Results of copula fittings for portfolio38

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	1518.279	1.128	−2946.559
Student-t	ρ_t/υ=14.69	$SE_{\rho_t}/SE_{\upsilon}$=1.26	1632.388	1.078	−3172.776

ρ_{norm} = (0.344, 0.322, 0.304, 0.409, 0.277, 0.371, 0.302, 0.470, 0.313, 0.225, 0.261, 0.326, 0.212, 0.339, 0.312, 0.369, 0.326, 0.165, 0.378, 0.397, 0.300, 0.305, 0.312, 0.302 , 0.319, 0.187, 0.419, 0.291, 0.352, 0.324, 0.354, 0.420, 0.338, 0.455, 0.302, 0.332, 0.225, 0.287, 0.295, 0.325, 0.413, 0.387, 0.312, 0.272, 0.327)

$SE_{\rho_{norm}}$ = (0.023, 0.024, 0.024, 0.021, 0.025, 0.022, 0.024, 0.020, 0.024, 0.026, 0.025, 0.023 , 0.026, 0.023, 0.024, 0.023, 0.024, 0.026, 0.022, 0.022, 0.024, 0.024, 0.024, 0.024 , 0.024, 0.026, 0.021, 0.024, 0.023, 0.024, 0.023, 0.021, 0.023, 0.020, 0.024, 0.024 , 0.026, 0.024, 0.024, 0.024, 0.021, 0.022, 0.024, 0.025, 0.023)

ρ_t = (0.338, 0.329, 0.310, 0.418, 0.281, 0.379, 0.297, 0.475, 0.329, 0.235, 0.259, 0.326 , 0.215, 0.341, 0.306, 0.368, 0.337, 0.198, 0.398, 0.408, 0.323, 0.320, 0.329, 0.330 , 0.331, 0.204, 0.426, 0.287, 0.361, 0.331, 0.369, 0.437, 0.345, 0.476, 0.333, 0.343 , 0.233, 0.305, 0.312, 0.337, 0.426, 0.399, 0.324, 0.278, 0.350

SE_{ρ_t} = (0.024, 0.024, 0.025, 0.022, 0.026, 0.023, 0.025, 0.021, 0.025, 0.026, 0.026, 0.025 , 0.027, 0.025, 0.025, 0.024, 0.025, 0.027, 0.023, 0.023, 0.025, 0.025, 0.025, 0.025 , 0.025, 0.028, 0.023, 0.026, 0.024, 0.025, 0.024, 0.022, 0.024, 0.021, 0.025, 0.025 , 0.027, 0.025, 0.026, 0.025, 0.022, 0.023, 0.025, 0.026, 0.025)

Table C.56. Results of copula fittings for portfolio39

Copula	Parameter(s)	SE	Llh	Distance	AIC
Normal	ρ_{norm}	$SE_{\rho_{norm}}$	1939.179	0.509	−3788.358
Student-t	ρ_t/υ=12.36	SE_{ρ_t}/SE_υ=0.97	2081.991	0.496	−4071.983

ρ_{norm} = (0.259, 0.232, 0.328, 0.293, 0.203, 0.374, 0.446, 0.361, 0.276, 0.261, 0.326, 0.367 0.312, 0.394, 0.524, 0.418, 0.326, 0.319, 0.397, 0.291, 0.364, 0.377, 0.357, 0.324 0.396, 0.338, 0.515, 0.460, 0.431, 0.302, 0.357, 0.466, 0.474, 0.473, 0.372, 0.363 0.364, 0.346, 0.272, 0.476, 0.532, 0.370, 0.535, 0.437, 0.382)

$SE_{\rho_{norm}}$ = (0.025, 0.025, 0.023, 0.024, 0.026, 0.022, 0.020, 0.023, 0.025, 0.025, 0.023, 0.022 , 0.024, 0.022, 0.018, 0.021, 0.023, 0.024, 0.022, 0.024, 0.023, 0.022, 0.023, 0.024 , 0.022, 0.023, 0.018, 0.020, 0.020, 0.024, 0.023, 0.020, 0.019, 0.019, 0.022, 0.023 , 0.023, 0.023, 0.025, 0.019, 0.018, 0.022, 0.017, 0.021, 0.022)

ρ_t = (0.249, 0.245, 0.329, 0.305, 0.212, 0.374, 0.443, 0.363, 0.288, 0.263, 0.334, 0.373 , 0.312, 0.398, 0.512, 0.416, 0.336, 0.324, 0.400, 0.294, 0.375, 0.388, 0.377, 0.327 , 0.406, 0.349, 0.529, 0.475, 0.452, 0.335, 0.358, 0.486, 0.481, 0.484, 0.394, 0.379 , 0.368, 0.363, 0.290, 0.493, 0.550, 0.394, 0.550, 0.457, 0.409)

SE_{ρ_t} = (0.026, 0.026, 0.024, 0.025, 0.027, 0.024, 0.022, 0.024, 0.026, 0.026, 0.024, 0.024 0.025, 0.023, 0.020, 0.022, 0.025, 0.025, 0.023, 0.026, 0.024, 0.023, 0.024, 0.025 0.023, 0.024, 0.019, 0.021, 0.021, 0.025, 0.024, 0.021, 0.021, 0.021, 0.024, 0.024 0.024, 0.024, 0.026, 0.020, 0.018, 0.023, 0.018, 0.022, 0.023)

APPENDIX D: T-T COPULA RISK RESULTS

D.1. Risk Results for Single Stocks

Table D.1. $VaR_{0.99}$ of BP for different time horizons

	Method	VaR	SE	%
	Normal Model	0.02904		
1 day	Historic Simulation	0.03203	0.000180	1.10
	GARCH Processes	0.03009	0.000153	1.04
	Normal Model	0.06244		
5 days	Historic Simulation	0.06397	0.000433	1.02
	GARCH Processes	0.06586	0.000329	1.05
	Normal Model	0.08577		
10 days	Historic Simulation	0.08729	0.000458	1.02
	GARCH Processes	0.09196	0.000425	1.07
	Normal Model	0.12130		
22 days	Historic Simulation	0.12068	0.000636	0.99
	GARCH Processes	0.13537	0.000726	1.12

Table D.2. $VaR_{0.99}$ of UNP for different time horizons

	Method	**VaR**	**SE**	**%**
	Normal Model	0.03120		
1 day	Historic Simulation	0.03368	0.000327	1.08
	GARCH Processes	0.04872	0.000263	1.56
	Normal Model	0.06686		
5 days	Historic Simulation	0.06636	0.000348	0.99
	GARCH Processes	0.10678	0.000621	1.60
	Normal Model	0.09163		
10 days	Historic Simulation	0.09014	0.000476	0.98
	GARCH Processes	0.14514	0.000729	1.58
	Normal Model	0.12909		
22 days	Historic Simulation	0.12568	0.000561	0.97
	GARCH Processes	0.19975	0.001049	1.55

Table D.3. $VaR_{0.99}$ of GM for different time horizons

	Method	**VaR**	**SE**	**%**
	Normal Model	0.05204		
1 day	Historic Simulation	0.05662	0.000372	1.09
	GARCH Processes	0.07215	0.000335	1.39
	Normal Model	0.11261		
5 days	Historic Simulation	0.11625	0.000696	1.03
	GARCH Processes	0.15383	0.000798	1.37
	Normal Model	0.15543		
10 days	Historic Simulation	0.15647	0.000938	1.01
	GARCH Processes	0.21002	0.001117	1.35
	Normal Model	0.22157		
22 days	Historic Simulation	0.22117	0.001027	1.00
	GARCH Processes	0.29157	0.001468	1.32

Table D.4. $VaR_{0.99}$ of PG for different time horizons

	Method	VaR	SE	%
	Normal Model	0.02083		
1 day	Historic Simulation	0.02211	0.000176	1.06
	GARCH Processes	0.02007	0.000115	0.96
	Normal Model	0.04495		
5 days	Historic Simulation	0.04606	0.000243	1.02
	GARCH Processes	0.04655	0.000237	1.04
	Normal Model	0.06190		
10 days	Historic Simulation	0.06181	0.000355	1.00
	GARCH Processes	0.06547	0.000318	1.06
	Normal Model	0.08790		
22 days	Historic Simulation	0.08706	0.000448	0.99
	GARCH Processes	0.09563	0.000553	1.09

Table D.5. $VaR_{0.99}$ of MOT for different time horizons

	Method	VaR	SE	%
	Normal Model	0,05341		
1 day	Historic Simulation	0,06102	0,000790	1,14
	GARCH Processes	0,03734	0,000195	0,70
	Normal Model	0,11434		
5 days	Historic Simulation	0,12809	0,001600	1,12
	GARCH Processes	0,08326	0,000445	0,73
	Normal Model	0,15662		
10 days	Historic Simulation	0,17740	0,001300	1,13
	GARCH Processes	0,11804	0,000577	0,75
	Normal Model	0,22061		
22 days	Historic Simulation	0,24008	0,001372	1,09
	GARCH Processes	0,17220	0,000801	0,78

Table D.6. $ES_{0.99}$ of BP for different time horizons

	Method	ES	SE	%
	Normal Model	0.03325		
1 day	Historic Simulation	0.03873	0.0003570	1.16
	GARCH Processes	0.03427	0.0001902	1.03
	Normal Model	0.07150		
5 days	Historic Simulation	0.07526	0.0005224	1.05
	GARCH Processes	0.07624	0.0004046	1.07
	Normal Model	0.09824		
10 days	Historic Simulation	0.10156	0.0006350	1.03
	GARCH Processes	0.10638	0.0005597	1.08
	Normal Model	0.13901		
22 days	Historic Simulation	0.13995	0.0008284	1.01
	GARCH Processes	0.15580	0.0008689	1.12

Table D.7. $ES_{0.99}$ of UNP for different time horizons

	Method	ES	SE	%
	Normal Model	0.03572		
1 day	Historic Simulation	0.04248	0.0003291	1.19
	GARCH Processes	0.05557	0.0003140	1.56
	Normal Model	0.07657		
5 days	Historic Simulation	0.07781	0.0004808	1.02
	GARCH Processes	0.12441	0.0007208	1.62
	Normal Model	0.10496		
10 days	Historic Simulation	0.10472	0.0005933	1.00
	GARCH Processes	0.16934	0.0009847	1.61
	Normal Model	0.14799		
22 days	Historic Simulation	0.14450	0.0006945	0.98
	GARCH Processes	0.23392	0.0014316	1.58

Table D.8. $ES_{0.99}$ of GM for different time horizons

	Method	**ES**	**SE**	**%**
1 day	Normal Model	0.05935		
	Historic Simulation	0.07608	0.0010195	1.28
	GARCH Processes	0.08189	0.0004099	1.38
5 days	Normal Model	0.12784		
	Historic Simulation	0.13989	0.0009825	1.09
	GARCH Processes	0.17671	0.0010963	1.38
10 days	Normal Model	0.17586		
	Historic Simulation	0.18094	0.0011157	1.03
	GARCH Processes	0.24118	0.0014595	1.37
22 days	Normal Model	0.24933		
	Historic Simulation	0.25137	0.0013214	1.01
	GARCH Processes	0.33392	0.0019128	1.34

Table D.9. $ES_{0.99}$ of PG for different time horizons

	Method	**ES**	**SE**	**%**
1 day	Normal Model	0,02386		
	Historic Simulation	0,02966	0,0003387	1,24
	GARCH Processes	0,02312	0,0001327	0,97
5 days	Normal Model	0,05156		
	Historic Simulation	0,05471	0,0002759	1,06
	GARCH Processes	0,05473	0,0003435	1,06
10 days	Normal Model	0,07107		
	Historic Simulation	0,07235	0,0004408	1,02
	GARCH Processes	0,07697	0,0004325	1,08
22 days	Normal Model	0,10108		
	Historic Simulation	0,10146	0,0006478	1,00
	GARCH Processes	0,11008	0,0007451	1,09

Table D.10. $ES_{0.99}$ of MOT for different time horizons

	Method	ES	SE	%
	Normal Model	0,06097		
1 day	Historic Simulation	0,09278	0,0016538	1,52
	GARCH Processes	0,04262	0,0002233	0,70
	Normal Model	0,13008		
5 days	Historic Simulation	0,17815	0,0021425	1,37
	GARCH Processes	0,09576	0,0004735	0,74
	Normal Model	0,17773		
10 days	Historic Simulation	0,22144	0,0017945	1,25
	GARCH Processes	0,13677	0,0006940	0,77
	Normal Model	0,24938		
22 days	Historic Simulation	0,28372	0,0016692	1,14
	GARCH Processes	0,19870	0,0010805	0,80

D.2. Risk Results for Portfolios

Table D.11. $VaR_{0.99}$ of BP-GM-MOT-IBM-HON for different time horizons

		Conservative			Balanced			Risky		
	Method	**VaR**	**SE**	**%**	**VaR**	**SE**	**%**	**VaR**	**SE**	**%**
1 day	Approximate Multinormal Model	0.02475			0.02776			0.03297		
	Exact Multinormal Model	0.02455	0.0001268	0.99	0.02735	0.0001250	0.99	0.03259	0.0001556	0.99
	Historical Simulation	0.02872	0.0002714	1.16	0.03051	0.0003744	1.10	0.03574	0.0003176	1.08
	CCC-GARCH	0.02631	0.0001334	1.06	0.02845	0.0001310	1.02	0.03046	0.0001662	0.92
	t-t Copula	0.02707	0.0002289	1.09	0.02973	0.0002479	1.07	0.03652	0.0003246	1.11
5 days	Approximate Multinormal Model	0.05327			0.05987			0.07110		
	Exact Multinormal Model	0.05297	0.0002827	0.99	0.05927	0.0002886	0.99	0.06940	0.0003390	0.98
	Historical Simulation	0.05499	0.0003138	1.03	0.06408	0.0004577	1.07	0.07731	0.0006973	1.09
	CCC-GARCH	0.05811	0.0002796	1.09	0.06255	0.0003590	1.04	0.06642	0.0003135	0.93
	t-t Copula	0.05410	0.0003157	1.02	0.06104	0.0003879	1.02	0.07234	0.0004469	1.02
10 days	Approximate Multinormal Model	0.07322			0.08244			0.09789		
	Exact Multinormal Model	0.07134	0.0003966	0.97	0.08010	0.0003624	0.97	0.09510	0.0004392	0.97
	Historical Simulation	0.07256	0.0004569	0.99	0.08430	0.0004497	1.02	0.10535	0.0008049	1.08
	CCC-GARCH	0.08059	0.0003474	1.10	0.08572	0.0004744	1.04	0.09186	0.0004182	0.94
	t-t Copula	0.07420	0.0004483	1.01	0.08268	0.0004410	1.00	0.09833	0.0006227	1.00
22 days	Approximate Multinormal Model	0.10366			0.11705			0.13900		
	Exact Multinormal Model	0.10111	0.0005616	0.98	0.11385	0.0005284	0.97	0.13470	0.0006767	0.97
	Historical Simulation	0.10154	0.0005623	0.98	0.11627	0.0006244	0.99	0.14092	0.0008283	1.01
	CCC-GARCH	0.11452	0.0005155	1.10	0.12195	0.0006104	1.04	0.13202	0.0006790	0.95
	t-t Copula	0.10369	0.0005468	1.00	0.11554	0.0005932	0.99	0.13745	0.0006323	0.99

Table D.12. $VaR_{0.99}$ of GM-JNJ-MCD-DD-DAI for different time horizons

		Conservative			**Balanced**			**Risky**		
	Method	**VaR**	**SE**	%	**VaR**	**SE**	%	**VaR**	**SE**	%
1 day	Approximate Multinormal Model	0.02149			0.02550			0.03149		
	Exact Multinormal Model	0.02133	0.0001076	0.99	0.02519	0.0001301	0.99	0.03082	0.0001641	0.98
	Historical Simulation	0.02347	0.0001787	1.09	0.02718	0.0002717	1.07	0.03427	0.0003327	1.09
	CCC-GARCH	0.02289	0.0001208	1.07	0.03060	0.0001660	1.20	0.04027	0.0001895	1.28
	t-t Copula	0.02360	0.0002026	1.10	0.02828	0.0002130	1.11	0.03489	0.0002506	1.11
5 days	Approximate Multinormal Model	0.04629			0.05501			0.06807		
	Exact Multinormal Model	0.04587	0.0002273	0.99	0.05430	0.0002322	0.99	0.06719	0.0002852	0.99
	Historical Simulation	0.04735	0.0002792	1.02	0.05513	0.0003248	1.00	0.06792	0.0003942	1.00
	CCC-GARCH	0.05024	0.0002756	1.09	0.06657	0.0003533	1.21	0.08689	0.0004767	1.28
	t-t Copula	0.04780	0.0002885	1.03	0.05573	0.0003348	1.01	0.06885	0.0003708	1.01
10 days	Approximate Multinormal Model	0.06367			0.07574			0.09388		
	Exact Multinormal Model	0.06239	0.0003189	0.98	0.07358	0.0003598	0.97	0.09124	0.0004259	0.97
	Historical Simulation	0.06366	0.0003453	1.00	0.07393	0.0003987	0.98	0.09144	0.0005066	0.97
	CCC-GARCH	0.06873	0.0003220	1.08	0.09045	0.0004552	1.19	0.11867	0.0005277	1.26
	t-t Copula	0.06427	0.0003825	1.01	0.07548	0.0004357	1.00	0.09350	0.0004949	1.00
22 days	Approximate Multinormal Model	0.09020			0.10754			0.13369		
	Exact Multinormal Model	0.08804	0.0004381	0.98	0.10427	0.0004694	0.97	0.12792	0.0005841	0.96
	Historical Simulation	0.08868	0.0004912	0.98	0.10348	0.0005361	0.96	0.12817	0.0006217	0.96
	CCC-GARCH	0.09803	0.0005572	1.09	0.12662	0.0006538	1.18	0.16479	0.0009322	1.23
	t-t Copula	0.08996	0.0005538	1.00	0.10665	0.0005959	0.99	0.13054	0.0007480	0.98

Table D.13. $VaR_{0.99}$ of UNP-PG-MMM-JNJ-DIS for different time horizons

		Conservative			Balanced			Risky		
	Method	**VaR**	**SE**	%	**VaR**	**SE**	%	**VaR**	**SE**	%
1 day	Approximate Multinormal Model	0.01733			0.01899			0.02254		
	Exact Multinormal Model	0.01718	0.0000917	0.99	0.01881	0.0000905	0.99	0.02222	0.0001182	0.99
	Historical Simulation	0.01830	0.0001370	1.06	0.02010	0.0001375	1.06	0.02371	0.0002197	1.05
	CCC-GARCH	0.02050	0.0001105	1.18	0.02455	0.0001245	1.29	0.03046	0.0001374	1.35
	t-t Copula	0.01929	0.0001711	1.11	0.02076	0.0001632	1.09	0.02533	0.0001999	1.12
5 days	Approximate Multinormal Model	0.03727			0.04077			0.04831		
	Exact Multinormal Model	0.03719	0.0002052	1.00	0.04020	0.0001689	0.99	0.04768	0.0002342	0.99
	Historical Simulation	0.03726	0.0001974	1.00	0.04120	0.0002238	1.01	0.04799	0.0002752	0.99
	CCC-GARCH	0.04373	0.0002481	1.17	0.05259	0.0003212	1.29	0.06572	0.0003658	1.36
	t-t Copula	0.03913	0.0002497	1.05	0.04222	0.0002317	1.04	0.05048	0.0002974	1.04
10 days	Approximate Multinormal Model	0.05118			0.05590			0.06619		
	Exact Multinormal Model	0.05029	0.0002488	0.98	0.05478	0.0002755	0.98	0.06509	0.0002985	0.98
	Historical Simulation	0.05047	0.0002572	0.99	0.05489	0.0003141	0.98	0.06418	0.0003604	0.97
	CCC-GARCH	0.05960	0.0002722	1.16	0.07021	0.0003990	1.26	0.08866	0.0004786	1.34
	t-t Copula	0.05266	0.0003162	1.03	0.05743	0.0003348	1.03	0.06871	0.0004400	1.04
22 days	Approximate Multinormal Model	0.07228			0.07878			0.09315		
	Exact Multinormal Model	0.07111	0.0003676	0.98	0.07714	0.0003753	0.98	0.09105	0.0005423	0.98
	Historical Simulation	0.07088	0.0003531	0.98	0.07622	0.0004110	0.97	0.09012	0.0004930	0.97
	CCC-GARCH	0.08390	0.0003938	1.16	0.09792	0.0005572	1.24	0.12392	0.0007507	1.33
	t-t Copula	0.07440	0.0003457	1.03	0.07956	0.0003934	1.01	0.09568	0.0005439	1.03

Table D.14. $ES_{0.99}$ of BP-GM-MOT-IBM-HON for different time horizons

		Conservative			Balanced			Risky		
	Method	**ES**	**SE**	**%**	**ES**	**SE**	**%**	**ES**	**SE**	**%**
1 day	Approximate Multinormal Model	0.02835			0.03178			0.03771		
	Exact Multinormal Model	0.02805	0.0001367	0.99	0.03145	0.0001536	0.99	0.03723	0.0002072	0.99
	Historical Simulation	0.03679	0.0002760	1.30	0.04351	0.0005413	1.37	0.05338	0.0008887	1.42
	CCC-GARCH	0.02995	0.0001439	1.06	0.03231	0.0001630	1.02	0.03485	0.0001691	0.92
	t-t Copula	0.03611	0.0004328	1.27	0.03912	0.0004372	1.23	0.04947	0.0006379	1.31
5 days	Approximate Multinormal Model	0.06106			0.06854			0.08126		
	Exact Multinormal Model	0.06082	0.0003356	1.00	0.06825	0.0003695	1.00	0.07952	0.0004149	0.98
	Historical Simulation	0.06496	0.0004058	1.06	0.07910	0.0006455	1.15	0.10232	0.0009400	1.26
	CCC-GARCH	0.06720	0.0003848	1.10	0.07242	0.0004516	1.06	0.07652	0.0004681	0.94
	t-t Copula	0.06435	0.0004264	1.05	0.07502	0.0006890	1.09	0.08908	0.0008769	1.10
10 days	Approximate Multinormal Model	0.08400			0.09438			0.11181		
	Exact Multinormal Model	0.08218	0.0004342	0.98	0.09225	0.0004720	0.98	0.10868	0.0005253	0.97
	Historical Simulation	0.08480	0.0005589	1.01	0.09999	0.0006570	1.06	0.12904	0.0009842	1.15
	CCC-GARCH	0.09280	0.0005074	1.10	0.09953	0.0005818	1.05	0.10578	0.0005754	0.95
	t-t Copula	0.08720	0.0006260	1.04	0.09843	0.0006469	1.04	0.11887	0.0010661	1.06
22 days	Approximate Multinormal Model	0.11907			0.13403			0.15863		
	Exact Multinormal Model	0.11663	0.0006230	0.98	0.13122	0.0006912	0.98	0.15369	0.0008331	0.97
	Historical Simulation	0.11731	0.0007644	0.99	0.13635	0.0008472	1.02	0.16488	0.0009730	1.04
	CCC-GARCH	0.13202	0.0007496	1.11	0.14020	0.0007665	1.05	0.15146	0.0008369	0.95
	t-t Copula	0.12112	0.0008679	1.02	0.13477	0.0007583	1.01	0.16044	0.0008980	1.01

Table D.15. $ES_{0.99}$ of GM-JNJ-MCD-DD-DAI for different time horizons

		Conservative			Balanced			Risky		
	Method	**ES**	**SE**	**%**	**ES**	**SE**	**%**	**ES**	**SE**	**%**
1 day	Approximate Multinormal Model	0.02462			0.02920			0.03602		
	Exact Multinormal Model	0.02445	0.0001308	0.99	0.02891	0.0001391	0.99	0.03538	0.0001879	0.98
	Historical Simulation	0.03062	0.0003012	1.24	0.03476	0.0003422	1.19	0.04396	0.0003553	1.22
	CCC-GARCH	0.02613	0.0001283	1.06	0.03491	0.0001929	1.20	0.04589	0.0002379	1.27
	t-t Copula	0.03002	0.0003400	1.22	0.03609	0.0003299	1.24	0.04480	0.0003613	1.24
5 days	Approximate Multinormal Model	0.05311			0.06301			0.07779		
	Exact Multinormal Model	0.05272	0.0002668	0.99	0.06275	0.0003184	1.00	0.07663	0.0003942	0.99
	Historical Simulation	0.05604	0.0003471	1.06	0.06514	0.0004344	1.03	0.08027	0.0004964	1.03
	CCC-GARCH	0.05819	0.0003442	1.10	0.07679	0.0004632	1.22	0.10007	0.0005999	1.29
	t-t Copula	0.05724	0.0004498	1.08	0.06722	0.0005590	1.07	0.08196	0.0005711	1.05
10 days	Approximate Multinormal Model	0.07312			0.08679			0.10722		
	Exact Multinormal Model	0.07217	0.0003983	0.99	0.08480	0.0004667	0.98	0.10382	0.0004944	0.97
	Historical Simulation	0.07541	0.0004276	1.03	0.08625	0.0004787	0.99	0.10606	0.0006205	0.99
	CCC-GARCH	0.07953	0.0004371	1.09	0.10455	0.0005757	1.20	0.13713	0.0007642	1.28
	t-t Copula	0.07578	0.0005236	1.04	0.08888	0.0006178	1.02	0.10949	0.0006815	1.02
22 days	Approximate Multinormal Model	0.10378			0.12332			0.15254		
	Exact Multinormal Model	0.10177	0.0005352	0.98	0.12019	0.0006278	0.97	0.14664	0.0006794	0.96
	Historical Simulation	0.10313	0.0005955	0.99	0.11983	0.0006446	0.97	0.14582	0.0006875	0.96
	CCC-GARCH	0.11252	0.0006664	1.08	0.14452	0.0008056	1.17	0.18929	0.0010070	1.24
	t-t Copula	0.10493	0.0007044	1.01	0.12359	0.0006460	1.00	0.14950	0.0009506	0.98

Table D.16. $ES_{0.99}$ of UNP-PG-MMM-JNJ-DIS for different time horizons

		Conservative			Balanced			Risky		
	Method	**ES**	**SE**	**%**	**ES**	**SE**	**%**	**ES**	**SE**	**%**
1 day	Approximate Multinormal Model	0.01988			0.02178			0.02584		
	Exact Multinormal Model	0.01976	0.0001049	0.99	0.02165	0.0001027	0.99	0.02556	0.0001408	0.99
	Historical Simulation	0.02305	0.0002136	1.16	0.02507	0.0002386	1.15	0.03046	0.0003258	1.18
	CCC-GARCH	0.02338	0.0001113	1.18	0.02803	0.0001575	1.29	0.03487	0.0001838	1.35
	t-t Copula	0.02484	0.0002656	1.25	0.02678	0.0002506	1.23	0.03415	0.0004135	1.32
5 days	Approximate Multinormal Model	0.04283			0.04685			0.05548		
	Exact Multinormal Model	0.04275	0.0002286	1.00	0.04678	0.0002499	1.00	0.05479	0.0003314	0.99
	Historical Simulation	0.04379	0.0002425	1.02	0.04836	0.0002828	1.03	0.05627	0.0003734	1.01
	CCC-GARCH	0.05075	0.0003188	1.18	0.06143	0.0004123	1.31	0.07857	0.0005830	1.42
	t-t Copula	0.04682	0.0003161	1.09	0.05057	0.0003509	1.08	0.06187	0.0004736	1.12
10 days	Approximate Multinormal Model	0.05892			0.06435			0.07611		
	Exact Multinormal Model	0.05813	0.0003262	0.99	0.06334	0.0003537	0.98	0.07487	0.0003725	0.98
	Historical Simulation	0.05896	0.0003364	1.00	0.06429	0.0003591	1.00	0.07489	0.0004329	0.98
	CCC-GARCH	0.06915	0.0003942	1.17	0.08290	0.0005047	1.29	0.10653	0.0007608	1.40
	t-t Copula	0.06229	0.0004531	1.06	0.06868	0.0005885	1.07	0.08245	0.0006401	1.08
22 days	Approximate Multinormal Model	0.08349			0.09099			0.10741		
	Exact Multinormal Model	0.08245	0.0004439	0.99	0.08954	0.0004944	0.98	0.10478	0.0005652	0.98
	Historical Simulation	0.08238	0.0004570	0.99	0.08901	0.0005297	0.98	0.10375	0.0005912	0.97
	CCC-GARCH	0.09704	0.0005275	1.16	0.11352	0.0007362	1.25	0.14738	0.0011016	1.37
	t-t Copula	0.08705	0.0005397	1.04	0.09399	0.0005770	1.03	0.11285	0.0006966	1.05

REFERENCES

1. Markowitz, H., "Portfolio Selection", *Journal of Finance*, Vol. 7, pp. 77–91, 1952.

2. Sharpe, W. F., "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk", *The Journal of Finance*, Vol. 19, No. 3, pp. 425–442, 1964.

3. Ross, S. A., "The Arbitrage Theory of Capital Asset Pricing", *Journal of Economic Theory*, Vol. 13, No. 3, pp. 341–360, December 1976.

4. Holton, G. A., "History of Value-at-Risk: 1922-1998", *Working Paper*, EconWPA, 2002.

5. Artzner, P., F. Delbae, J. M. Eber, and D. Heath, "Coherent Measures of Risk", *Mathematical Finance*, Vol. 9, pp. 203–228, 1999.

6. Sklar, A., "Fonctions de Repartition a n dimensions et Leurs Marges", *Publications de l'Institut de Statistique de l'Universit de Paris*, Vol. 8, pp. 229–231, 1959.

7. R Development Core Team, *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, 2007, `http://www.R-project.org`, ISBN 3-900051-07-0.

8. Breymann, W. and D. Luethi, *ghyp: A package on generalized hyperbolic distributions*, 2007, r package version 1.0.0.

9. Yan, J., *copula: Multivariate Dependence with Copula*, 2007, r package version 0.5-7.

10. Weingessel, A., *quadprog: Functions to solve Quadratic Programming Problems.*, 2007, r package version 1.4-11.

11. Leydold, J. and W. Hörmann, *Runuran: R interface to the UNU.RAN random*

variate generators, 2007, `http://statistik.wu-wien.ac.at/unuran/`, r package version 0.7.

12. McNeil, A., R. Frey, and P. Embrechts, *Quantitative Risk Management: Concepts Techniques and Tools*, Princeton University Press, Princeton, 2005.

13. Holton, G. A., *Value-at-risk : theory and practice*, Academic Press, 2003.

14. Wikipedia, "Systemic Risk", `http://en.wikipedia.org/wiki/Systemic_risk`, [Online; accessed 15-May-2008].

15. Longerstaey, J., *Riskmetrics Technical Document*, JP Morgan, 1996.

16. Hörmann, W., "Monte Carlo Simulation in Finance", 2007, lecture Notes for Financial Simulation Course.

17. Engle, R., "Autoregressive Conditional Heteroskedasticity With Estimates of the Variance of U.K. Inflation", *Econometrica*, Vol. 50, pp. 987–1008, 1982.

18. Bollerslev, T., "Generalized Autoregressive Conditional Heteroskedasticity", *Journal of Econometrics*, Vol. 31, No. 3, pp. 307–327, 1986.

19. Harold, Y. K. and P. M. Jianping, "What Makes the Stock Market Jump? An Analysis of Political Risk on Hong Kong Stock Returns", *Journal of International Money and Finance*, Vol. 20, No. 7, pp. 1003–1016, 2001.

20. Malevergne, Y. and D. Sornette, *Extreme Financial Risks: From Dependence to Risk Management*, Springer, 2006.

21. Embrechts, P., A. McNeil, and D. Straumann, "Correlation and dependence in risk management: properties and pitfalls", *Risk Management: Value at Risk and Beyond*, Cambridge University Press, Cambridge, 2002.

22. Kruskal, W., "Ordinal Measures of Association", *Journal of the Americal Statistical*

Association, Vol. 53, pp. 814–861, 1958.

23. Genest, C. and J. MacKay, "The Joy of Copulas: Bivariate Distributions with Uniform Marginals", *The American Statistician*, Vol. 40, No. 4, pp. 280–283, 1986.

24. Rockinger, M. and E. Jondeau, "Conditional Dependency of Financial Series : An Application of Copulas", *Working Paper* 723, Les Cahiers de Recherche, Groupe HEC, 2001.

25. Patton, A. J., "Estimation of Multivariate Models for Time Series of Possibly Different Lengths", *Journal of Applied Econometrics*, Vol. 21, pp. 147–173, 2006.

26. Matteis, R., *Fitting Copulas to Data*, Diploma thesis, Institute of Mathematics of the University of Zurich, 2001.

27. Genest, C. and L. P. Rivest, "Statistical Inference Procedures for Bivariate Archimedean Copulas", *Journal of the American Statistical Association*, Vol. 88, No. 423, pp. 1034–1043, 1993.

28. Frees, E. W. and E. A. Valdez, "Understanding Relationships Using Copulas", *North American Actuarial Journal*, Vol. 2, No. 1, pp. 1–26, 1998.

29. Durrleman, V., A. Nikeghbali, and T. Roncalli, "Which Copula is the Right One?", *Working Paper*, Groupe de Recherche Operationnelle Credit Lyonnais, Paris, 2000.

30. Barndorff-Nielsen, O. E., "Exponentially Decreasing Distributions for the Logarithm of Particle Size", *Proceedings of the Royal Society London A*, Vol. 353, pp. 401–419, 1977.

31. Wenbo, H., *Calibration Of Multivariate Generalized Hyperbolic Distributions Using The EM Algorithm, with Applications in Risk Management, Portfolio Optimization and Portfolio Credit Risk*, Ph.d. thesis, The Florida State University College of Arts and Sciences, 2005.

32. Yan, J., "Enjoy the Joy of Copulas: With a Package copula", *Journal of Statistical Software*, Vol. 21, No. 4, pp. 1–21, 2007.

33. Ünal, G. E., *Extreme Value Approach in Analyzing Stock Returns in İstanbul Stock Exchange*, Ph.d. thesis, Boğaziçi University, 2006.

34. Chang, T. J., N. Meade, J. E. Beasley, and Y. M. Sharaiha, "Heuristics for Cardinality Constrained Portfolio Optimisation", *Computers & Operations Research*, Vol. 27, No. 13, pp. 1271–1302, 2000.

35. Fabozzi, F. J., S. Focardi, and P. N. Kolm, *Financial Modeling of the Equity Market from CAPM to Cointegration*, John Wiley and Sons, New Jersey, 2006.

36. Hellmich, M. and S. Kassberger, "Efficient Portfolio Optimization in the Multivariate Generalized Hyperbolic Framework", *Working Paper*, 2007.

37. Harrington, D. R., *Modern Portfolio Theory and the Capital Asset Pricing Model : a User's Guide*, Prentice-Hall, 1983.

38. Tomick, J. J., S. F. Arnold, and R. R. Barton, "Sample Size Selection for Improved Nelder-Mead Performance", *WSC '95: Proceedings of the 27th Conference on Winter Simulation*, pp. 341–345, IEEE Computer Society, Washington, DC, USA, 1995.

39. Banks, J., J. S. Carson, B. L. Nelson, and D. M. Nicol, *Discrete Event Systems Simulation*, Prentice-Hall, Englewood Cliffs, NJ, 2000.

40. Tomick, J. J., *On Convergence of the Nelder-Mead Simplex Algorithm for Unconstrained Stochastic Optimization*, Ph.d. thesis, Department of Statistics, The Graduate School, The Pennsylvania University, 1995.

41. Nelder, J. A. and R. Mead, "A Simplex Method for Function Minimization", *Computer Journal*, Vol. 7, pp. 308–313, 1965.

42. Lagarias, J. C., J. A. Reeds, M. H. Wright, and P. E. Wright, "Convergence Properties of the Nelder–Mead Simplex Method in Low Dimensions", *SIAM J. on Optimization*, Vol. 9, No. 1, pp. 112–147, 1998.

43. Barton, R. R. and J. S. Ivey, "Nelder-Mead Simplex Modifications for Simulation Optimization", *Management Science*, Vol. 42, No. 7, pp. 954–973, 1996.