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POSITIVE OPERATORS ON BANACH LATTICES

M.Sc. THESIS

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ABSTRACT

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This work deals with positive (nonnegative) operators, particularly the positivity conditions for a matrix operator. What we are trying to obtain is to find some criteria for an operator defined on a Banach space to be positive with respect to some basis in regard to have a partial answer to *invariant subspace problem.*

Keywords : Invariant subspace, Banach lattice, Positive operator, Change of basis.

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ÖZET

BANACH ÖRGÜLERİ ÜZERİNDE POZİTİF OPERATÖRLER

ÖZLEM ÇAVUŞOĞLU

Eldeki çalışmada pozitif (negatif-olmayan) operatörlerle; özel olarak bir matris operatörünün pozitif olma koşulları ile ilgilenilmektedir. Burada yapılmaya çalışılan bir Banach uzayı üzerinde tanımlanan bir operatörün bir baza göre pozitif olabilmesi için gerekli kriterler elde ederek *invaryant alt-uzay problemi*'ne kısmi bir yanıt bulabilmektir.

Anahtar Kelimeler : İnvaryant alt-uzay, Banach örgüsü, pozitif operatör, baz değişimi.

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Chapter 1

Introduction

The study of operators has become an important research field in analysis after the appearance of Stefan Banach's monograph on linear operators between normed spaces, which concentrates on topological properties of the spaces and operators. Operators preserving the order structures were first studied by F.Riesz, H. Freudenthal, and L.V. Kantorovich starting from 1930's. However, it took until 1960's to realize that topological and order structures are related much as it has been considered and should be studied together.

Starting from pre-1980 period, much has been done on the subject from all points of view by various mathematicians, related to the theory of positive operators on Banach lattices. Of these, the following one will be our main concern:

- *When a given quasinilpotent operator on a Banach space with a Schauder basis (in particular, on a Hilbert space) can be made positive with respect to some basis?*

To the best of our knowledge, this problem is unsolved even for finite dimensional spaces. However, it seems, the question arises from the so called Invariant Subspace Problem. The main idea of this problem is to find a partial answer to the invariant subspace problem for Banach spaces with a basis.

In Chapter 2, we shall give a brief introduction to the invariant subspace problem, for positive operators on Banach spaces, and we shall introduce a problem posed by Y.A.Abramovich, C.D Aliprantis and O.Burkinshaw in [2], with several reformulations.

Chapter 3 is the core of our study, in which we have tried to distinguish between the two different notions of positivity: *positivity with respect to the cone* generated by a basis and *positive-definiteness*. We shall start with the basic idea of a change of basis, and then try to relate our problem to the notion of *positive bases* and *basis preserving matrices*. Secondly, we introduce the notion of positive definiteness and try to find the conditions under which the definitions of positivity coincide. Relevant consequences and facts are also examined.

In Chapter 4 we introduce some relevant facts arising from the spectral theory of positive operators in connection with the work done by R. Drnovšek [6] and X.D. Zhang [14], [15].

Chapter 2

Positive Operators on Banach Spaces

In this chapter the fundamental concepts related to the subject will be given. We begin with the invariant subspace problem, from which our problem itself stems. Some basic notions on positive operators and Banach lattices will be given and the main problem of the thesis along with the invariant subspace problem will be introduced. For the unexplained terminology we refer to [4] and [11].

2.1 The Invariant Subspace Problem

The invariant subspace problem is the following question:

- *Does there exist a non-trivial closed T-invariant subspace $V \subseteq X$ for the continuous linear operator T on X?*

This question is a natural consequence of the theory of eigenvectors in finite- dimensional spaces. Let λ be an eigenvalue of an operator T, then corresponding to the given eigenvalue λ, an element x for which $Tx = \lambda x$ holds is called an eigenvector . The set $N_\lambda = \{x \in X : Tx = \lambda x\}$ is called the eigenspace corresponding to the eigenvalue λ. If x is an eigenvector, it is obvious that $V = \{\lambda x : \lambda \in \mathbb{C}\}$ is T-invariant. Since there exist operators with no eigenvalues in infinite dimensional spaces, some other concepts are needed, and the concept of non-trivial invariant subspace is the most natural one.

If we let X be a finite dimensional complex Banach space of dimension greater than one, and the operator T is not a multiple of the identity operator I, then N_λ is a non-trivial closed T-hyperinvariant subspace. When $T \neq \lambda I$, for each $x \in N_\lambda$ and S in the commutant[1] of T, we have $TSx = STx = S(\lambda x) = \lambda Sx$, so that $Sx \in N_\lambda$. This subspace is clearly closed and non-trivial . Hence, every non-zero operator T on a finite dimensional complex Banach space X of dimension greater than one, has a non-trivial closed hyperinvariant subspace.

The invariant subspace problem is considered only when X is an infinite-dimensional seperable Banach space. In the opposite case, i.e, if X is non-separable, then for a fixed point $0 \neq x \in X$, the subspace

$$V_x = \overline{\text{span}}\{x, Tx, T^2x, ...\}$$

is a non-trivial closed T-invariant subspace.

For the hyperinvariant subspaces, a similar concept occurs: let us take all operators commuting with T instead of taking the iterates of T, and define,

$$G_x = \overline{\text{span}}\{Ax : A \text{ commutes with } T\}.$$

Then G_x is a non-trivial closed T-hyperinvariant subspace.

The first result about the existence of an invariant subspace is proved in 1935 by J. von Neumann. He proved that every compact operator on a Hilbert space has a non-trivial closed invariant subspace. The most powerful contribution to invariant subspace problem came in 1973 from V.I. Lomonosov. He introduced an elegant technique to solve some hard problems in the theory that is connected with compact operators. He proved that a non-zero compact operator on a Banach space has hyperinvariant subspaces.

Until the middle of the 1970's, the invariant subspace problem was phrased more stronger then it is mentioned above: it asked whether *every* continuous linear operator on a (separable) Banach space has a non-trivial invariant subspace. In 1976 P. Enflo solved this question negatively by constructing an example of a continuous operator on a Banach space without a non-trivial closed invariant subspace. This

[1]The commutant of a continuous operator $T : X \to X$ on a Banach space is the set of all continuous operators that commute with T.

counterexample confined the invariant subspace problem for operators on Banach spaces to the search for various classes of operators for which it can be guarantied the existence of an invariant subspace.

Among the various questions in the theory of invariant subspaces, in this thesis we handle the question that asks the conditions for a change of basis under which a nonpositive operator can be made positive on a Banach space with a basis. It actually asks: *When a given quasinilpotent operator on a Banach space with a Schauder basis(in particular, on a Hilbert space) can be made positive with respect to some basis?*

To the best of our knowledge, this problem is unsolved for even finite dimensional spaces.

2.2 Positive Operators on Banach Spaces with Bases

Recall that any set equipped with an order relation $\geq$ (i.e., $\geq$ is reflexive, antisymmetric and transitive) is called a partially ordered set. A real vector space X with an order relation is called an ordered vector space whenever the following extra conditions are satisfied:

- If $x \geq y$, then $x + z \geq y + z$ for each $z \in X$.

- If $x \geq y$, then $\alpha x \geq \alpha y$ for each scalar $\alpha \geq 0$.

Note that almost all concepts defined for real ordered vector spaces carry over to complex ordered vector spaces with natural modifications. Due to this reason, unless otherwise stated, the real ordered vector spaces will be considered.

An element x in an ordered vector space X is called positive whenever $x \geq 0$ holds.

Definition 2.2.1. An operator $T : X \to Y$ between two ordered vector spaces is said to be *positive* if $Tx \geq 0$ for each $x \geq 0$.

For a positive operator T, it follows that $Ty \leq Tx$ whenever $y \leq x$ holds. The notation $T \geq S$ means $T - S \geq 0$, or equivalently $Tx \geq Sx$ for each $x \geq 0$.

An ordered vector space E is called a *Riesz space* (or a vector lattice) if for each pair of elements $x, y \in E$, the set $\{x, y\}$ has an infimum and a supremum in E. Using the standart notation, we shall write

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y := \inf\{x, y\}.$$

A seminorm p on a Riesz space is said to be a *lattice seminorm* whenever $|x| \leq |y|$ implies $p(x) \leq p(y)$. A Riesz space equipped with a lattice norm is referred to as a normed Riesz space. If a normed Riesz space is also norm complete (i.e., a Banach space), then it is called a Banach lattice.

2.2.1 Spaces with a Schauder Basis

Definition 2.2.2. A sequence $\{x_n\}$ in a Banach space X is called a *Schauder basis* (or simply a basis) of X if for every $x \in X$ there exists a uniquely determined sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$, where the series assumed to converge in the norm.

Associated with the basis the standart sequence of *coefficient functionals* $f_n \quad (n = 1, 2, ...)$ is defined by

$$f_n(x) = \alpha_n \; for \; x = \sum_{i=1}^{\infty} \alpha_i x_i \in X.$$

Obviously each f_n is a linear functional on X, and, as is well-known each of $f_n \in X'$ (topological dual) for each n, is continuous (Cf. [3], [2]).

If we fix a basis for a Banach space X, every operator $T : X \to X$ can be identified with an infinite matrix $[t_{ij}]$. On the other hand, we can also say that an infinite matrix $[t_{ij}]$ define an operator on X. However, an operator corresponding to a matrix need not be bounded. It is important to note that an operator $[t_{ij}]$ is positive if and only if for each pair (i, j), $t_{ij} \geq 0$ holds. For *quasinilpotent* operators acting on a Banach

Space with a basis, there are some results on the existence of invariant subspaces obtained so far.

Definition 2.2.3. A continuous operator $T : X \to X$ on a Banach space with spectral radius $r(T) = \lim_{n\to\infty}(\| T^n \|)^{\frac{1}{n}} = 0$ is said to be *quasinilpotent*.

The above mentioned concepts lead to the following results obtained by various mathematicians and are collected in the survey on invariant subspace problem by Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw [3]:

Theorem 2.2.4. Let $T : X \to X$ be a continuous positive operator on a Banach space with a basis. If T commutes with a non-zero positive operator that is quasinilpotent at a non-zero positive vector, then T has a non-trivial closed invariant subspace.

Proof. Cf. [3, Thm 7.1, p.31]. □

Corollary 2.2.5. Let X be a Banach space with a basis. If $T : X \to X$ is a continuous quasinilpotent positive operator, then T has a non-trivial closed invariant subspace.

Theorem 2.2.6. Let X be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on X that is quasinilpotent at a non-zero positive vector. If for a double sequence $\{w_{ij}\}$ of complex numbers, the weighted matrix $B = [w_{ij}a_{ij}]$ defines a continuous operator B on X, then the operator B has a non-trivial closed invariant subspace.

Proof. Cf. [3, Thm 7.3, p.32]. □

Corollary 2.2.7. Let X be a Banach space with a basis. Assume that a positive matrix $A = [a_{ij}]$ defines a continuous operator on X which is quasinilpotent at a non-zero positive vector. If a continuous operator $T : X \to X$ defined by a matrix $T = [t_{ij}]$ satisfies $t_{ij} = 0$ whenever $a_{ij} = 0$, then T has a non-trivial closed invariant subspace.

2.2.2 The Case of Non-positive Operators

Let us consider a quasinilpotent operator on a Banach space with a basis. Suppose that the operator is not positive with respect to this basis, which implies that our aforementioned invariant subspace theorems do not apply. However, if we can change the basis, in such a way that the operator becomes positive with respect to the new basis, then the operator would have a non-trivial closed invariant subspace. As a simple example that illustrates this point, consider the operator

$$T = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

on $\mathbb{R}^2$ with the standart basis $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$. If we introduce the new basis $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1 - \mathbf{e}_2$, then it can be seen that in this basis the operator T has the following matrix representation

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

As is seen, with this new basis the matrix of the same operator T become positive. Then a natural question follows:

- *When a given quasinilpotent operator on a Banach space with a Schauder basis (in particular, on a Hilbert space) can be made positive with respect to some basis?*

This problem is unsolved even for finite-dimensional spaces. Once this problem is solved, then the results of the previous section can be easily applied to this type of operators. Hence it may be possible to the have a partial answer to invariant subspace problem for non-positive operators.

Chapter 3

Concepts of Positivity

It is well known that there exist various concepts of positivity. On Hilbert Spaces and on general Banach spaces the notion of positivity is defined differently; the definition of positivity we introduced in Chapter 2 is not compatible with the standart positivity concept in Hilbert spaces. As an instance, there are at least two different notions of positivity. In this chapter we shall define positivity for Banach spaces and Hilbert spaces and try to relate them to our problem.

3.1 Positivity with respect to a Cone

The set of all positive elements of X is denoted by X^+. X^+ is referred to as the positive cone, or simply the cone, of X.

Definition 3.1.1. A subset C of a vector space X is said to be a *cone* whenever $C + C \subseteq C$, $\alpha C \subseteq C$ for each $\alpha \geq 0$, and $C \cap (-C) = \{0\}$.

Every cone C determines a partial order $\geq$ on X by letting $y \geq x$, if and only if $y - x \in C$. In fact $C = \{x \in X : x \geq 0\}$, and the elements of C are referred to as *positive vectors*. A (partially) *ordered vector space* (X, C) is a vector space X equipped with a cone C.

Definition 3.1.2. An operator $T : X \to X$ on an ordered vector space (X, C) is said to be *positive* if $Tx \geq 0$ for each $x \geq 0$, where the order $\geq$ is induced by C.

Equivalently, we have the following definition as well:

Definition 3.1.3. An operator $T : X \to X$ on a Banach space with a basis $\{x_n\}$ is said to be positive (with respect to the basis $\{x_n\}$) if $T(C) \subseteq C$, where C is the cone generated by $\{x_n\}$.

Every basis $\{x_n\}$, then, gives rise to a closed cone C defined by

$$C = \{x = \sum_{n=1}^{\infty} \alpha_n x_n : \alpha_n \geq 0 \text{ for each n=1,2,... } \}.$$

The cone C will be referred to as the cone generated by the basis $\{x_n\}$ and it is easily observed that each coefficient functional f_n is automatically positive with respect to the cone generated by the basis $\{x_n\}$. Hence the notion of positivity is strongly related to the basis selected. Once a basis is specified, then all notions of positivity are understood with respect to the cone generated by this basis. We stressed in the previous chapter that if we fix a basis for a Banach space X, every operator $T : X \to X$ can be identified with an infinite matrix $[t_{ij}]$, and an infinite matrix $[t_{ij}]$ defines an operator on X. Recall that an operator $[t_{ij}]$ is now positive if and only if for each pair (i, j), $t_{ij} \geq 0$ holds.

Let us first observe what corresponds to a change of basis in a finite-dimensional vector space. Let $\dim X = n$, and $\alpha = \{\alpha_1, ..., \alpha_n\}$ and $\beta = \{\beta_1, ..., \beta_n\}$ be two bases in X. Let $x \in X$, and $x = \sum_i \xi_i \alpha_i = \sum_i \eta_i \beta_i$.

Let the linear transformation A be defined by $A\alpha_i = \beta_i$ for $i = 1, ..., n$,i.e.,

$$A(\sum_i \xi_i \alpha_i) = \sum_i \xi_i \beta_i$$

.

Let $[a_{ij}]$ be the matrix of A in the basis α, i.e., $\beta_j = A\alpha_j = \sum_i a_{ij}\alpha_i$. Since $\sum_i \xi_i \beta_i = 0$ implies $\xi_1 = \xi_2 = ... = \xi_n = 0$, it can be observed that A is invertible. Since

$$\sum_i \eta_i \beta_i = \sum_j \eta_i A\alpha_j = \sum_j \eta_j \sum_i a_{ij}\alpha_i \quad (3.1)$$

$$= \sum_i (\sum_j a_{ij}\eta_j)\alpha_i, \quad (3.2)$$

we obtain

$$\xi_i = \sum_j a_{ij}\eta_j.$$

The invertible linear transformation A is reduced to the transformation of bases. We then have

$$T\alpha_j = \sum_i t_{ij}\alpha_i$$

and

$$T\beta_j = \sum_i s_{ij}\beta_i.$$

Using the linear transformation A defined above, we can write

$$\begin{aligned} T\beta_j &= TA\alpha_j = T(\sum_k a_{kj}\alpha_k) && (3.3)\\ &= \sum_k a_{kj}T\alpha_k = \sum_k a_{kj}\sum_i t_{ik}\alpha_i && (3.4)\\ &= \sum_i(\sum_k t_{ik}a_{kj}\alpha_i) && (3.5) \end{aligned}$$

and

$$\begin{aligned} \sum_k s_{kj}\beta_k &= \sum_k s_{kj}A\alpha_k && (3.6)\\ &= \sum_k s_{kj}\sum_i a_{ik}\alpha_i && (3.7)\\ &= \sum_i(\sum_k a_{ik}s_{kj}\alpha_i). && (3.8) \end{aligned}$$

Comparing the results obtained above, we obtain

$$\sum_k a_{ik}s_{kj} = \sum_k t_{ik}a_{kj}.$$

Using the matrix multiplication, we may write the above equation as

$$[A][T]_\beta = [T]_\alpha[A]$$

, where $[T]_\beta$ corresponds to the operator T with respect to the basis β, $[T]_\alpha$ corresponds to the operator T with respect to the basis α, $[A]$ is the matrix corresponding to A. This is equivalent to

$$[T]_\beta = [A]^{-1}[T]_\alpha[A], \tag{3.9}$$

We should also remember that if such an equality holds for two matrices $[T]_\alpha$ and $[T]_\beta$, these matrices are called similar. The important thing to remember here is that the spectral concepts (i.e., the spectrum, the spectral radius, the multiplicities of the eigenvalues, etc.) remain invariant for similar matrices. Consequently, the quasinilpotence of an operator on a finite dimensional space is preserved after a change of basis.

It is a standart fact from linear algebra that if the characteristic polynomial of a matrix corresponding to an operator splits into linear factors, then there exist a basis in which the matrix of the operator is in Jordan canonical form. Since the Jordan canonical form of a matrix is similar to the matrix itself, it is easy to state the following:

Let T be an operator on a Banach space with a Schauder basis. If the spectrum of the operator consists only of nonnegative elements, then it can be made positive with respect to some basis.

We shall also mention another fact of special interest before we close off with finite-dimensionality.

Definition 3.1.4. The *Gramian* of a finite or infinite sequence $\{f_n\}$ of vectors, is the matrix whose (i, j) entry is the inner product $\langle f_i, f_j \rangle$.

Proposition 3.1.5. Every positive matrix is Gramian and every Gramian is positive.

Proof. Let G be a Gramian matrix. By definition, G is a symmetric matrix, therefore all the eigenvalues are real. Hence, $\det G$ is just product of the eigenvalues. It is easy to show that $\det G \geq 0$ (for details, see [13], Thm.5.67, p.191). It follows then, all the eigenvalues are positive; thus, the Jordan canonical form is a positive matrix. Further discussion can be found in [8], [13]. □

Definition 3.1.6. A Hilbert matrix is a matrix whose (i, j) entry is $\frac{1}{i+j+1}$, where $i, j \in \mathbb{N}$. The norm of the Hilbert matrix is equal to π and its matrix is a Hilbert matrix.

Proposition 3.1.7. There exists an operator A on a seperable infinite dimensional Hilbert space with $\| A \| \leq \pi$

Proof. Cf. [8, p.201]. □

Corollary 3.1.8. Hilbert matrix is positive.

Proof. Cf. [8, p.201]. □

Another related concept is the notion of "positive basis", which makes it possible to give some partial answers to our main question. The theorems of the remaining part of this subsection are from [12].

Definition 3.1.9. Let X be a Banach space. A basis x_n of X is said to be *positive* if for every linear isometry $T : X \to X$ with

$$T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i \quad (j = 1, 2, ...), \tag{3.10}$$

there exist another linear isometry $T_+ : X \to X$ satisfying

$$T_+(x_j) = \sum_{i=1}^{\infty} |a_{ij}| x_i \quad (j = 1, 2, ...). \tag{3.11}$$

Proposition 3.1.10. A basis $\{x_n\}$ of X is *positive* if and only if for every linear isometry $T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i (j = 1, 2, ...)$ and any scalars $\alpha_1, ..., \alpha_n$ the series

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} |a_{ij}| \alpha_j \right) x_i \tag{3.12}$$

converges and

$$\left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} |a_{ij}| \alpha_j \right) x_i \right\| = \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} a_{ij} \alpha_j \right) x_i \right\|. \tag{3.13}$$

Proof. ($\Rightarrow$) Let$\{x_n\}$ be positive. Then,

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} |a_{ij}| \alpha_j \right) x_i \right\| &= \left\| T_+ \left(\sum_{j=1}^{n} \alpha_j x_j \right) \right\| = \left\| T \left(\sum_{j=1}^{n} \alpha_j x_j \right) \right\| \\ &= \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} a_{ij} \alpha_j \right) x_i \right\|. \end{aligned}$$

($\Leftarrow$) Assume conversely that 3.13 is satisfied and $T : X \to X$ is a linear isometry with $T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i (j = 1, 2, ...)$. Taking $\alpha_1 = ... = \alpha_{k-1} = \alpha_{k+1} = ...\alpha_n = 0, \alpha_k = 1$, it follows that $\sum_{i=1}^{\infty} |a_{ij}| x_i (j = 1, 2, ...)$ converges. Let $T_+ = \sum_{i=1}^{\infty} |a_{ij}| x_i (j = 1, 2, ...)$ and extend T_+ by linearity to the (dense) linear subspace of X spanned by $\{x_n\}$. Since T is a linear isometry, for any scalars $\alpha_1, ..., \alpha_n$ we get

$$\begin{aligned}\left\| T_+ \left(\sum_{j=1}^{n} \alpha_j x_j \right) \right\| &= \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} |a_{ij}| \alpha_j \right) x_i \right\| = \left\| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} a_{ij} \alpha_j \right) x_i \right\| \\ &= \left\| T \left(\sum_{j=1}^{n} \alpha_j x_j \right) \right\| = \left\| \sum_{j=1}^{n} \alpha_j x_j \right\|,\end{aligned}$$

and it follows that T_+ is a linear isometry on a dense subspace of X. Hence it can be extended to a linear isometry $T_+ : X \to X$. □

Now let us give some examples of Banach spaces with positive bases.

Lemma 3.1.11. Let $\{x_n\}$ be the natural basis of c_0. A continuous linear mapping $T : c_0 \to c_0$, with $T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i (j = 1, 2, ...)$ is an isometry if and only if

$$\sup_{1 \le i < \infty} |a_{ij}| = 1 \ (j = 1, 2, ...), \tag{3.14}$$

$$\sum_{j=1}^{\infty} |a_{ij}| \le 1 \ (i = 1, 2, ...). \tag{3.15}$$

Proof. Cf. [12]. □

Proposition 3.1.12. The natural basis of c_0 is positive.

Proof. If we pass from $\{a_{ij}\}$ to $\{|a_{ij}|\}$, the conditions 3.14 and 3.15 remain invariant. From the previous lemma the assertion follows. □

Lemma 3.1.13. Let $\{x_n\}$ be the natural basis of l^p where $1 \le p \ne 2$. If the mapping $T : l^p \to l^p$, with $T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i \ (j = 1, 2, ...)$, is a linear isometry, then

$$a_{ij} a_{im} = 0 \ (j = 1, 2, ...; j \ne m). \tag{3.16}$$

Proof. Cf. [12]. □

Proposition 3.1.14. Let $1 \le p \ne 2$. Then the natural basis of l^p is positive.

Proof. Let $\{x_n\}$ be the natural basis of l^p, $T : l^p \to l^p$ be an isometry with $T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i$ $(j = 1, 2, ...)$ and $\alpha_1, ..., \alpha_n$ be arbitrary scalars. Then by the lemma above, for each i at most one of $a_{i1}, ..., a_{in}$ is different than O. Hence it follows that $|\sum_{i=1}^{n} |a_{ij}|\alpha_j| = |\sum_{i=1}^{n} a_{ij}\alpha_j|$. Hence

$$\begin{aligned}\left\|\sum_{i=1}^{\infty}\left(\sum_{j=1}^{n}|a_{ij}|\alpha_j\right)x_i\right\| &= \left(\sum_{i=1}^{\infty}\left|\sum_{j=1}^{n}|a_{ij}|\alpha_j\right|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty}\left|\sum_{i=1}^{n}a_{ij}\alpha_j\right|^p\right)^{\frac{1}{p}} \\ &= \left\|\sum_{i=1}^{\infty}\left(\sum_{i=1}^{n}a_{ij}\alpha_j\right)x_i\right\|.\end{aligned}$$

Thus by proposition 3.1.6, $\{x_n\}$ is a positive basis of X. □

Lemma 3.1.15. Let $\mathscr{H}$ be a (finite or infinite dimensional) Hilbert space and $\{x_n\}$ be an orthonormal basis of $\mathscr{H}$. Then $\{x_n\}$ is not positive.

Proof. Let us define

$$y_1 = \frac{x_1 + x_2}{\|x_1 + x_2\|}, \quad y_2 = \frac{x_1 - x_2}{\|x_1 - x_2\|}, \quad y_n = x_n \quad (n = 3, 4, ...). \tag{3.17}$$

Then $\{y_n\}$ is an othonormal basis of $\mathscr{H}$, hence there is a linear isometry $T : \mathscr{H} \to \mathscr{H}$ such that

$$T(x_n) = y_n, \quad (n = 1, 2, ...). \tag{3.18}$$

Then

$$\begin{aligned}\|T_+(x_1 + x_2)\| &= \left\|\frac{x_1 + x_2}{\|x_1 + x_2\|} + \frac{x_1 + x_2}{\|x_1 - x_2\|}\right\| \\ &= \|x_1 + x_2\|\left\|\frac{1}{\|x_1 + x_2\|} + \frac{1}{\|x_1 - x_2\|}\right\| \\ &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 2 > \sqrt{2} = \|x_1 + x_2\|\end{aligned}$$

which means T_+ is not an isometry, i.e., the basis is not positive. □

The following proposition shows that the solution of the problem of existence of a positive basis in finite or infinite dimensional Banach spaces is not always affirmative.

Proposition 3.1.16. If $\mathscr{H}$ is a (finite or infinite dimensional) Hilbert space, then $\mathscr{H}$ has no positive basis.

Proof. Let $\{x_n\}$ be a positive basis of $\mathscr{H}$, and assume without loss of generality that $\{x_n\}$ is normalized. Let

$$x = \frac{x_i - x_j}{\|x_i - x_j\|} \tag{3.19}$$

for arbitrary indices i, j. Consider x^* and x as first elements of two different orthonormal bases. Then there should exist a linear isometry such that $T(x*) = x$. Since $\{x_n\}$ is a positive basis,

$$\left\| \frac{x_i + x_j}{\|x_i - x_j\|} \right\| = \|T_+(x^*)\| = \|T(x^*)\| = \left\| \frac{x_i - x_j}{\|x_i - x_j\|} \right\|,$$

Hence $\langle x_i, x_j \rangle = 0$. Since i, j are choosen arbitrarily, it follows that $\{x_n\}$ is orthonormal, contradicting the previous lemma. □

By definition, the positivity of bases of a space X depends very much on the properties of linear isometries. If the only isometries of a Banach space X onto itself are $T = I_X$ or $T = -I_X$, then obviously every basis of X is positive. And the answer to the question of existence of non-positive bases in finite-dimensional Banach spaces is negative [12]. In the case of infinite dimensional Banach spaces the existence of non-positive bases is not known so far [12].

Example 3.1.17. Let $T : c_0 \to c_0$ be a linear isometry with $T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i \quad (j = 1, 2, ...)$, where $\{x_n\}$ is a basis of c_0. Then there exists a basis with respect to which T is positive.

Example 3.1.18. Let $T : l^p \to l^p$, with $1 \leq p \neq 2$, be a linear isometry. Let $T(x_j) = \sum_{i=1}^{\infty} a_{ij} x_i \quad (j = 1, 2, ...)$, where $\{x_n\}$ is a basis of l^p. Then there exists a basis with respect to which T is positive.

Example 3.1.19. Let X be a finite dimensional Banach space and $T : X \to X$ be a linear isometry with $T(x_j) = \sum_{i=1}^{n} a_{ij} x_i \quad (j = 1, 2, ..., n)$, where $\{x_n\}$ is a basis of X. Then there exists a basis with respect to which T is positive.

Let X be a Banach space and $\xi := (\xi_n)_{n \in \mathbb{N}}$ be a sequence of non-zero vectors in X. Define a linear space of sequences of scalars as

$$A_1(\xi) := \left\{ (\alpha_n)_{n \in \mathbb{N}} \subset \mathbb{K} \mid \sum_{i=1}^{\infty} \alpha_i \xi_i \text{ converges} \right\},$$

and endow $A_1(\xi)$ with the norm

$$\|(\alpha_n)_{n\in\mathbb{N}}\|_\xi := \sup_{1\le n<\infty} \left\| \sum_{i=1}^{n} \alpha_i \xi_i \right\|.$$

It is well-known [12, Proposition 3.1] that $(A_1(\xi), \|\cdot\|_\xi)$ is a Banach space which is isomorphic to X by the mapping

$$(\alpha_n)_{n\in\mathbb{N}} \mapsto \sum_{i=1}^{\infty} \alpha_i \xi_i.$$

Let now X be a Banach space with a (Schauder) basis and $x := (x_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ be the associated sequence of coefficient functionals. Let $T : X \to X$ be a linear operator which is quasinilpotent at a non-zero vector x_0. For each $n \in \mathbb{N}$, let the sequence $y := (y_n)_{n\in\mathbb{N}}$ be defined by

$$y_n := \sum_{k=1}^{n} f_k\left(T^k x_0\right) x_0.$$

Then there exists a unique infinite matrix $a := (a_{ij})$ such that

$$y_j = \sum_{i=1}^{\infty} a_{ij} x_i \qquad (j = 1, 2, \ldots).$$

Define $v_a : A_1(y) \to A_1(x)$ for each $(\alpha_n)_{n\in\mathbb{N}} \in A_1(y)$ via

$$v_a\left((\alpha_n)_{n\in\mathbb{N}}\right) := \left(\sum_{j=1}^{\infty} a_{ij}\alpha_j \right)_{i\in\mathbb{N}}.$$

In the light of what have been seen above, one can make the following observations:

First, we can ask the following: Is v_a one-to-one and onto? If the answer is affirmative, then, by [12] a is a basis-preserving matrix, i.e., y is a basis of X.

Second, let X be a Banach space with a basis, $x := (x_n)_{n\in\mathbb{N}}$, $y := (y_n)_{n\in\mathbb{N}}$ be a sequence of vectors in X, and $T : X \to X$ be a bounded linear operator. Denote by C_x the cone generated by x. Assume that:

(i) y is a basis for X,

(ii) $Tx \in C_y$, and

(iii) $y \in C_x$.

Then T is a positive operator with respect to y.

Two natural questions then follows: With a fixed basis x for X in hand, is it then possible to substract a new basis$y$ from C_x so that $Tx \in C_y$? And if $y \in C_x$ is a basis, what is the relation between C_x and C_y?

3.2 Positive Definiteness

Let us recall that Hilbert spaces have no positive bases. So, we have not said anything about Hilbert spaces so far. Actually in the case of Hilbert spaces the notion of "positivity" of an operator is not that useful. In Hilbert spaces, more generally in inner product spaces, the notion of "positive-definiteness" enters the scene, which we shall recover below.

In Hilbert Spaces, the notion of positivity of a transformation A is defined by one of the following three equivalent conditions [9]:

- $A = B^2$ where B is self-adjoint,
- $A = C^*C$ for some C,
- A is self-adjoint and $\langle\, Ax, x\rangle \geq 0$ for all x.

It is more convenient to use the latter as the definition of positivity:

Definition 3.2.1. A linear transformation T on a Hilbert space H is said to be positive or, H -positive if T is self-adjoint and $\langle Tx, x\rangle \geq 0$ for all x in H.

To prevent a confusion, we shall call it as H -positive. As it is customary we can write $A \geq B$ whenever $A - B \geq 0$. However, difference of two transformations which are not self-adjoint shall become positive. Therefore it is more convenient to consider inequalities for self-adjoint transformations only. Since the necessary and sufficient condition for a transformation on a unitary space (i.e., complex inner product space) to be Hermitian is that $\langle Tx, x\rangle \geq 0$ be real for all x, there is no need to define the notion of "positivity" again for complex Hilbert spaces [[**?**, **?**, hal] p140].

The crucial thing here to observe is that the notion of positivity of an operator T on a Hilbert space is not related to the positivity of its matricial representation. Following example shall be nice to observe this fact:

Example 3.2.2. The matrix

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is self-adjoint and positive if we consider the natural ordering of real numbers. However, it is not H -positive: Let $x = (1, -1)$. Then $Tx = (0, 1)$ and $\langle Tx, x\rangle = -1$, which implies that T is not H-positive.

Example 3.2.3. The matrix

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is self-adjoint, but not positive. However, it is H -positive: Let $x = (x_1, x_2)$ be a positive vector. Then $Tx = (x_1, -x_2)$ and $\langle Tx, x\rangle = {x_1}^2 + {x_2}^2 \geq 0$.

Definition 3.2.4. Let E be a normed vector space and F be a normed Riesz space. A linear map $T : E \to F$ is called majorizing if for every null sequence $(x_n)_{n\in\mathbb{N}}$ in E, $(|Tx_n|)_{n\in\mathbb{N}}$ is a majorized sequence in F, i.e., there exists a positive vector y in F such that $|Tx_n| \leq y$ for all $n \in \mathbb{N}$.

Definition 3.2.5. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces. A linear map $T : \mathscr{H}_1 \to \mathscr{H}_2$ is called a Hilbert-Schmidt operator if there exists an orthonormal basis $(\varphi_\alpha)_{\alpha\in I}$ of $\mathscr{H}_1$ such that $\sum_{\alpha\in I} \|T\varphi_\alpha\|^2$ is finite.

Definition 3.2.6. By a Hilbert lattice is meant a Banach lattice whose underlying Banach space is a Hilbert space.

We denote by $\mathscr{H}^+$ and $B_{\mathscr{H}}$ the positive cone and the unit ball of the Hilbert lattice $\mathscr{H}$, respectively. As usual, $\sigma(T)$ denotes the spectrum of the operator T. Lastly, the term *positive* refers to the fact $Tx \geq 0$ whenever $x \geq 0$ for a linear operator T between ordered vector spaces.

Theorem 3.2.7. Let $\mathscr{H}$ be a separable Hilbert lattice with an orthonormal system $(\varphi_j)_{j\in\mathbb{N}}$ such that $\{\varphi_j \mid j \in \mathbb{N}\} \subset \mathscr{H}^+$, and $B_{\mathscr{H}}$ be order bounded. A linear operator $T : \mathscr{H} \to \mathscr{H}$ with $\sigma(T) \subset [0,\infty)$ is positive if and only if T is a Hilbert-Schmidt operator.

Proof. Let T be a positive operator. Since T is order bounded it maps $B_{\mathscr{H}}$ into an order interval, and since T is continuous, by [11, Corollary to Proposition 3.4], it is majorizing. Therefore, by [11, Theorem 6.9], T is a Hilbert-Schmidt operator. Conversely, let T be a Hilbert-Schmidt operator. By the spectral representation, T has the form

$$T = \sum_{j\in\mathbb{N}} \lambda_j \langle \varphi_j \otimes \varphi_j \rangle$$

with $\lambda_j \geq 0$ and $\varphi_j \geq 0$ for each $j \in \mathbb{N}$, and therefore $T \geq 0$. □

Theorem 3.2.8. Let $\mathscr{H}$ be a complex Hilbert space and $T \in \mathscr{L}(\mathscr{H})$. If $\langle Th, h\rangle \geq 0$ for all $h \in \mathscr{H}$, then there exists a basis of $\mathscr{H}$ with respect to which T is positive.

Proof. Since $\langle Th, h\rangle \geq 0$ for all $h \in \mathscr{H}$, one has $\sigma(T) \subset [0,\infty)$ by [10, Proposition 18.16], and therefore the matrix representing T can be reduced to an infinite Jordan form with non-negative diagonal elements, i.e., $T \geq 0$. □

3.3 Some related facts

All positivity concepts introduced so far can be also related to the following special settings:

Let G be the general linear group $GL(n,\mathbb{R})$, and denote by P the set of $n \times n$ symmetric, positive-definite matrices topologized by the usual topology of the $\frac{n(n+1)}{2}$-dimensional Euclidean space. The group G acts on P by mapping p into gpg^t. The action is continuous and it is transitive because if $p \in P$, then by the Principal Axis Theorem [5] there is an $n \times n$ orthogonal matrix a such that a^tpa is a diagonal matrix. Then the application of a diagonal matrix will further transform p into the $n \times n$ identity matrix.

Proposition 3.3.1. Every matrix $A \in M_n(\mathbb{C})$ is a linear combination of unitary elements of $M_n(\mathbb{C})$.

Definition 3.3.2. A complex algebra $\mathscr{A}$ of continuous operators $A : \mathscr{H} \to \mathscr{H}$, where $\mathscr{H}$ is a complex Hilbert space, is called a von Neumann algebra if $\mathscr{A}$ is self-adjoint, contains the identity operator, and is weakly closed in $\mathscr{L}(\mathscr{H})$.

The *center* of an algebra $\mathscr{A}$ is the commutative subalgebra

$$c\mathscr{A} := \{C \in \mathscr{A} \mid AC = CA \text{ for every } A \in \mathscr{A}\}.$$

If $\mathscr{A}$ is an algebra of continuous operators, then its center is $\mathscr{A} \cap \mathscr{A}'$, and so if $\mathscr{A}$ is a von Neumann algebra then so is $c\mathscr{A}$., where $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$.

Definition 3.3.3. A von Neumann algebra $\mathscr{A}$ is called a factor if it is trivial, i.e., $c\mathscr{A} = \mathscr{A} \cap \mathscr{A}' = \mathbb{C}I$.

Using these facts one can ask the positivity of the linear combinations of a finite number of matrices, each of which can be made positive. Other interesting questions to ask are how a unitary matrix can be made positive and the relation between matrix units and unitary matrices and how a matrix unit/unitary matrix/self-adjoint matrix behaves under a change of basis.

It is well-known that every von Neumann algebra is generated by its unitary operators. We can ask about the structure of a von Neumann algebra consisting of operators which can be made positive. Conversely, also a suitable *positiveness* concept for a von Neumann algebra which will force its elements to be positive operators or operators which can be made positive can be asked.

Another fact to be questioned is the significance of the center of a von Neumann algebra consisting of positive operators. It is also not known, whether every von Neumann algebra of positive operators is a factor. The relation between the natural topology of a von Neumann algebra consisting of n-dimensional operators and the usual Euclidean topology of $GL(n, \mathbb{R})$ has been not studied so far.

As far as we know, these questions are not considered in our setting in any suitable way.

Chapter 4

Some spectral properties of Positive operators

A related fact to our study follows from one of the basic questions in spectral theory. It asks which additional properties should be satisfied by a bounded operator on a Banach space to be the identity operator, if the operator is *unipotent*, i.e., its spectrum consists of the number 1 only. It is already known that if the spectrum of a bounded operator T consists of the number 1 only and if it also satisfies one of the following conditions, then T is the identity opeator:

- T is a normal operator on a Hilbert space;
- T is an automorphism of a C^*-algebra;
- T is an automorphism of a commutative semisimple Banach algebra;
- T is a doubly power bounded operator on a Banach space, i.e., $\sup\{\| T^n \|: n \in \mathbb{Z}\} \leq \infty$;
- T is a lattice homomorphism on a Banach lattice.

We refer to [14] for details. C.B. Huijsmans and B. de Pagter proposed following question: Let T be a positive unipotent operator on a Banach lattice. Is it true that $T \geq I$? ([14],[15]). This question has an affirmative answer in the case of positive operators on finite dimensional Banach lattices:

Theorem 4.0.4. If T is a unipotent positive operator on a finite dimensional Banach lattice, then $T \geq I$.

Proof. Let E be an $n-$dimensional Banach lattice. Then T can be represented as a positive $n \times n$ matrix. Let $A = T - I$, then $A \geq -I$. If we consider the matrix representation af $[a_{ij}]_{n\times n}$ of A, we see that $a_{ij} \geq 0$ for $i \neq j$ and $a_{ii} \geq -1$. Since $\sigma(A^2) = 0$, the trace of A^2 is zero. But the trace of A^2 is the sum of its diagonal elements, all of which are nonnegative. On the other hand i-th diagonal element of A^2 equals to $\sum_{k=1}^{n} a_{ik}a_{ki}$, where each $a_{ik}a_{ki}$ is nonnegative. Therefore, all the diagonal elements of A^2 are zero. From this fact we obtain that $a_{ii} = 0$. Thus, we obtain that all the entries of A are nonnegative, i.e., $T - I \geq 0$. □

Although this question has been answered for the finite-dimensional case, it is still open in the general case. A partial answer to this question has been given by Xiao-Dong Zhang in [14] and [15]. Moreover, using Zhang's result above and the idea of its proof, we can slightly change the statement: If A is assumed to be an operator on a Banach lattice such that $A = T - I$ for a unipotent positive operator T, then Theorem 4.0.11 says that A is positive. Before introducing the results in general, let us give further theorems and corollaries in finite dimensional case that are obtained in the aforementioned studies.

Corollary 4.0.5. Let T be a positive operator on a finite dimensional complex Banach lattice such that $\sigma(T) \subseteq \{z : |z| = 1\}$. Then there exists a positive integer k such that $T^k \geq I$.

Proof. See [15] Corollary 4.2 □

Here we can change the setting of this corollay again by introducing an operator $A = T^k - I$ where T is a positive operator on a finite dimensional Banach lattice whose spectrum satisfies $\sigma(T) \subseteq \{z : |z| = 1\}$, and obtain that A is positive in Corollary 4.0.2 instead of $T \geq I$. If we let T be a positive contraction rather than being a positive operator, we have the following theorem:

Theorem 4.0.6. Let T be a positive contraction on a finite dimensional Banach lattice such that $\sigma(T) \subset \{z : |z| = 1\}$. Then T is an isometry.

Proof. See [15] Theorem 4.3. □

We recall that by a contraction T, it is meant that $\| T \| \leq 1$. As a consequence of Theorem 4.0.3 we have the following :

Corollary 4.0.7. If T is a positive unipotent contraction on a finite dimensional Banach lattice, then $T = I$.

Proof. See [15] Corollary 4.4. □

Theorem 4.0.8. Let T be a unipotent positive operator. If 1 is a pole of the resolvent of T, then $T \geq I$.

Proof. See [15] Theorem 5.3. □

The above theorem is proved by using the following :

Theorem 4.0.9. Let T be a unipotent positive operator on an arbitrary Banach lattice E. If there exist $0 < \alpha < 1/2$ and a constant c such that

$$\| T \|^{-n} = O(\exp(cn^{\alpha})) \text{ as } n \to \infty,$$

then $T \geq I$.

Proof. See [15] Theorem 5.1. □

Another partial answer to this question is given by Roman Drnovšek in [6]. He uses the same method as in [14] to prove similar result replacing the assumption on the growth of the negative powers of T by an assumption on the growth of the powers of the quasinilpotent operator $A = T - I$.

Let E denote a complex Dedekind complete Banach lattice, i.e., $E = \mathrm{Re}(E) \oplus i\mathrm{Re}(E)$, where $\mathrm{Re}(E)$ is a real Dedekind complete Banach lattice. $\mathcal{L}(E)$ is the set of all bounded linear operators on E. By $\mathcal{L}_r(E)$ is meant the space of all regular operators on E, i.e., those operators that can be expressed as linear combinations of positive operators. The center $\mathcal{Z}(E)$ is defined as $\mathcal{Z}(E) := \{T \in \mathcal{L}_r(E) : |T| \leq \lambda I$. The so-called *diagonal map* $\mathcal{D}$ is the associated band projection from $\mathcal{L}_r(E)$ onto $\mathcal{Z}(E)$. For further explanation, we refer to [6].

Theorem 4.0.10. Let T be a positive operator on a complex Dedekind complete Banach E, and let $A = T - I$. If $\lim_{n\to\infty} n \parallel \mathcal{D}(A^n) \parallel^{\frac{1}{n}} = 0$ and if $\mathcal{D}(T^n) \leq I$ for all $n \in \mathbb{N}$, then the operator A is positive.

Proof. See [6] Theorem 1. □

Corollary 4.0.11. Let T be a positive operator on complex Dedekind complete Banach lattice E, and let $A = T - I$. If $\lim_{n\to\infty} n \parallel A^n \parallel^{\frac{1}{n}} = 0$, then A is positive.

Proof. See [6, Corollary 1]. □

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