T.R.

GEBZE TECHNICAL UNIVERSITY

GRADUATE SCHOOL OF NATURAL AND APPLIDED SCIENCES

MONOTONE ITERATIVE TECHNIQUES FOR SET VALUED DIFFERENTIAL EQUATIONS IN METRIC SPACES

BATOUL BALLOUT

A THESIS SUBMITTED FOR THE DEGREE OF

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GEBZE TEKNİK ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

METRİK UZAYLARDA KÜME DİFERANSİYEL DENKLEMLER İÇİN MONOTON İTERASYON TEKNİKLER

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…………….. tarih ve …….. sayılı kararı.

SUMMARY

The preliminary materials providing the requisite tools and substantial background to study the set differential equations in metric spaces were collected since the metric space concerned consists of all nonempty compact convex sets in finite dimensional space. The requisite theorems and definitions in set differential equations are given to study the monotone iterative technique for set differential equations in metric spaces utilizing the method of upper and lower solutions. It is well known that the method of upper and lower solutions with the monotone iterative technique offers abstract as well as deductive existence result in a closed set that is generated by upper and lower solutions. The natural question is whether it is possible to extend the monotone method when the given function is the difference of two or three functions. So that we can obtain some known results as special cases and some new results. The answer is positive, this offers a new look into the monotone method results developed so far and also combine all the results in a single set up. The results took into account here are so public that they include several special cases of interest, and this leads to the possibility of having four or eight types of upper and lower solutions. In this thesis we consider coupled lower and upper solutions were considered and two sequences which converge to coupled minimal and maximal solutions respectively were developed.

Keywords: Set Valued Differential Equation, Monotone Iterative Technique, Upper and Lower Solutions.

ÖZET

Metrik uzaylarda verilmiş küme diferansiyel denklemleri incelemek için öncelikle gerekli araçları ve önemli koşulları sağlayan bazı öncelikli materyalleri veriyoruz, çünkü ilgili metrik uzay sonlu boyutlu uzayda tüm boşolmayan kompakt konveks kümelerden oluşur, bu yüzden küme diferansiyel denklemlerdeki gerekli teoremler ve tanımlara ihtiyacımız var. Metrik uzaylarda küme diferansiyel denklemler için monoton iterasyon tekniğin incelenmesi, karşılaştırma sonuçlarının, alt ve üst çözüm yöntemlerinin kullanılması, monoton iterasyon yöntemin üst ve alt çözüm yöntemlerinin bir sonucu olarak ortayı çıktığı bilinmektedir. Üst ve alt çözümlerin oluşturduğu kapalı ve sınırlı bir kümede verilen sürekli fonksiyonlar yani doğal denklemler, bu verilen fonksiyon iki veya üç farklı fonksiyonun kombinasyonu olarak verildiğinde bu monoton iterasyon tekniğinin bu tip denklemlere uygulanıp uygulanamayacağı ilğinç bir açık problem ola gelmiştir. Bu problem çözüldüğünde böylece özel durumlar olarak bilinen bazı sonuçları ve bazı yeni sonuçlar elde edebiliriz. Bu problemin çözümü mümkün olup ve cevabı da olumludur. Bu fikir monotone iterasyon yöntemine yeni bir bakış kazandırıyor ve yeni bir perspektif sunmuştur. Şimdiye kadar elde edilen sonuçları genelleştirip ve literatürde ki tek bir fonkosiyon için tüm sonuçları elde ettik. Elde edilen tüm sonuçlar burada göz önünde bulundurulduğunda çok özel ilgi alanları içerdiğini ve bu sayede dört ve sekiz tip üst ve alt çözümleri kullanılmıştır. Bu tezde, eşleşmiş alt ve üst çözümleri ele almakta ve eşleşmiş minimum ve maksimum çözümlere düzgün yakınsayan monoton fonksiyon dizi çiftleri elde edilmiştir.

Anahtar kelimeler: Küme Differansiyel Denklemler, Monotone Iterasyon Teknikler, Alt ve Üst Çözümler.

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LIST of ABBREVIATIONS and ACRONYMS

ABBREVIATIONS and ACRONYMS	DESCRIPTIONS
D_H	: Hukuhara Derivative
SDE	: Set Differential Equations
IVP	: Initial Value Problem
MQMP	: Mixed Quasi Monotone Property

1. INTRODUCTION

Multivalued differential equations (now known as set differential equations (SDEs) generated by multivalued differential inclusions) have been introduced in a semi-linear metric space, consisting of all nonempty, compact, convex subsets of an initial finite or infinite dimensional space[1]. The basic existence and uniqueness results of such (SDEs) have been investigated and their solutions have compact, convex values. Also, these generated (SDEs) have been employed as a material to prove the existence of solutions in a united method, of multivalued differential inclusions [1, 2, 3, 4]. The third chapter is devoted to the basic theory of set differential equations (SDEs). We begin with the formulation of the initial value problem of (SDEs) in the metric space $(K_C(R^n), D)$. Utilizing the properties of the Hausdorff metric $D[\cdot,\cdot]$ and employing the known theory of differential and integral inequalities, we establish a variety of comparison results, that are required for later discussion[1, 5]. The monotone iterative technique is considered for SDE in the fourth chapter, employing the method of upper and lower solutions. The results considered are so general that they contain several special cases of interest [1, 6, 7, 8, 9, 10]. In the fifth chapter, comparison results with initial time difference are given, we will study the monotone iterative technique for one (single), two and three functions with initial time difference [11, 12, 13].

2. PRELIMINARIES

Definition 2.1: We will define three spaces of non-empty subsets of R^n*, namely,*

- $K_C(R^n)$ *consisting of all nonempty compact convex subsets of* R^n*,*
- $K(R^n)$ *consisting of all nonempty compact subsets of* R^n*,*
- $C(R^n)$ *consisting of all nonempty closed subsets of* R^n*,*

recall that a nonempty subset A *of* R^n*, is convex if for all* $a_1, a_2 \in A$ *and all* $\lambda \in [0 , 1]$*, the point:*

$$a = \lambda a_1 + (1 - \lambda) a_2 \qquad (2.1)$$

belongs to A*. For any nonempty subset* A *of* R^n*, we denote by* coA *Its convex hull. That is the totality of points* a *of the form (2.1) or, equivalently, the smallest Convex subset containing* A*, clearly:*

$$A \subseteq coA = co(coA) \qquad (2.2)$$

with $A = coA$ *if* A *is convex.*

Let A and B be two nonempty subsets of R^n. And let $\lambda \in R$, we define the following Makowski addition and scalar multiplication by:

$$A + B = \{a + b, \qquad a \in A, b \in B\} \qquad (2.3)$$

and

$$\lambda A = \{\lambda a \, , a \in A\} \qquad (2.4)$$

then we have some useful known examples in literature about the convex sets as follows:

Example 2.1: An interval $[a , b] \subset R$ *is a convex set. To see this let* $c , d \in [a , b]$ *and assume, without loss of generality that* $c < d$*, let* $\lambda \in (0\ 1)$*, then*

$$\begin{aligned} a \le (1 - \lambda)c + \lambda c &< \\ (1 - \lambda)c + \lambda d &< \\ (1 - \lambda)d + \lambda d = d \le b. \end{aligned} \qquad (2.5)$$

Example 2.2: A disk with center $(0,0)$ *and raduis c is a convex subset of* R^2*. We can easily show that by using the usual distance formula in* R^2*, namely:*

$$\|x-y\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \tag{2.6}$$

and the triangle inequality:

$$\|u+v\| \leq \|u\| + \|v\| \tag{2.7}$$

if the disc's center is the origin and its radius c, then x *belong to the disc if and only if* $\|x\|^2 \leq c^2$*. Let* u *and* v *belong to the disc. And let* $0 \leq \lambda \leq 1$*. Then:*

$$\|\lambda u + (1-\lambda)v\|^2 = \lambda^2\|u\|^2 + 2\lambda(1-\lambda)\|u\|\|v\| + (1-\lambda)^2\|v\|^2 \tag{2.8}$$

since, $\|u\|^2 \leq c^2$ *and* $\|v\|^2 \leq c^2$

$$\|\lambda u + (1-\lambda)v\|^2 \leq \lambda^2c^2 + 2\lambda(1-\lambda)c^2 + (1-\lambda)^2c^2 \tag{2.9}$$

So, we arrive at:

$$\|\lambda u + (1-\lambda)v\|^2 \leq c^2. \tag{2.10}$$

Example 2.3: In R^n *the set* $H \coloneqq \{x \in R^n : a_1\, x_1 + a_2x_2 + \cdots + a_nx_n = c\}$ *is a convex set. For any choice of constants* a_i *in* R^n*. It is defining equation is a generalization of the usual equation of a plane in* R^3*, namely the equation:*

$$ax + by + cz + d = 0 \tag{2.11}$$

We will show that H is a convex set. To do that Let $x^1, x^2 \in H$ *and define* $z \in R^3$ *by*

$$Z = \sum_{i=1}^{n} a_i\,[(1-\lambda)x_i^1 + \lambda x_i^2] = \sum_{i=1}^{n}[(1-\lambda)a_ix_i^1 + \lambda a_ix_i^2]$$

$$= (1-\lambda)\sum_{i=1}^{n} a_i\, x_i^1 + \lambda\sum_{i=1}^{n} a_i\, x_i^2 = (1-\lambda)c + \lambda c = c \tag{2.12}$$

Hence $Z \in H$.

Example 2.4: As a generalization of the previous example, let A be an $m \times n$ matrix, $b \in R^m$ and let $S \coloneqq \{x \in R^n : Ax = b\}$ (the set S is just the set of all solutions of the linear equation $Ax = b$) then the set S is a convex subset of R^n. Let $x^1, x^2 \in S$ then:

$$A\big((1-\lambda)x^1 + \lambda x^2\big) = (1-\lambda)A\,x^1 + \lambda A\,x^2 \qquad (2.13)$$
$$= (1-\lambda)b + \lambda b = b.$$

Note 2.1: There are always two. So, called trivial examples. These are the empty set ϕ, and the entire space R^n, note also that a singleton $\{x\}$ is convex, in this latter case as in the case of empty set, the definition is satisfied.

Then we have the following propositions:

Propositions: The spaces $C(R^n), K(R^n)$ and $K_C(R^n)$ are closed under the operations of additions and scalar multiplication. In fact, the following properties hold:

- $A + \theta = \theta + A = A$, *where* $\theta \in R^n$ *is the zero element of* R^n
- $(A + B) + C = A + (B + C)$
- $A + C = B + C \Rightarrow A = B$
- $1.A = A$
- $\lambda(A + B) = \lambda A + \lambda B$
- $(\lambda + \mu)A = \lambda A + \mu A$

where $A, B, C \in K_C(R^n), \lambda, \mu \in R_+$.

The Hausdorff metric:

Let x be a point in R^n and A a nonempty subset of R^n the distance $d(x, A)$ from x to A is defined by:

$$d(x, A) = \inf\{\|x - a\| : a \in A\} \qquad (2.14)$$

thus $d(x, A) = d(x, \bar{A}) \geq 0$ and $d(x, A) = 0$, if and only if $x \in \bar{A}$, where $\bar{A}$ is the closure of $A \subseteq R^n$. We shall call the subset:

$$S_\varepsilon(A) = \{x \in R^n : d(x, A) < \varepsilon\} \tag{2.15}$$

an ε_neighborhood of A, and its closure is the subset:

$$\bar{S}_\varepsilon(A) = \{x \in R^n : d(x, A) \leq \varepsilon\} \tag{2.16}$$

we shall denote by:

$$\bar{S}_1^n = \bar{S}_1\,(\theta) \tag{2.17}$$

which is clearly a compact subset of R^n. Note also that:

$$\bar{S}_\varepsilon(A) = A + \varepsilon \bar{S}_1^n \tag{2.18}$$

for any $\varepsilon > 0$ and any nonempty subset A of R^n. We shall for convenience sometimes write $S(A, \varepsilon)$ and $\bar{S}_\varepsilon(A)$. Now let A and B be nonempty subsets of R^n. We define the Hausdorff separation of B from A by:

$$d_H(B, A) = sup\{d(b, A) : b \in B\} \tag{2.19}$$

or, equivalently:

$$d_H(B, A) = inf\{\varepsilon > 0 \ : B \subseteq A + \varepsilon \bar{S}_1^n\} \tag{2.20}$$

we have $d_H(B, A) \geq 0$ with $d_H(B, A) = 0$ iff $B \subseteq \bar{A}$, also the triangle inequality:

$$d_H(B, A) \leq d_H(B, C) + d_H(C, A) \tag{2.21}$$

holds, for all nonempty subsets A, B and C of R^n, in general however:

$$d_H(A, B) \neq d_H(B, A) \tag{2.22}$$

we define the Hausdorff distance between nonempty subsets A and B of R^n by:

$$D(A, B) = max\{d_H(A, B), d_H(B, A)\} \tag{2.23}$$

which is symmetric in A and B. Consequently:

- $D(A, B) \geq 0$ with $D(A, B) = 0$ iff $\bar{A} = \bar{B}$
- $D(A, B) = D(B, A)$
- $D(A, B) \leq D(A, C) + D(C, B)$

for any nonempty subsets A, B and C of R^n.

If we restrict our attention to nonempty closed subsets of R^n, we find that the Hausdorff distance is a metric known as the Hausdorff metric. Thus $(C(R^n), D)$ is a metric space.

And so, we have the following properties:

Proposition 2.1: $(C(R^n), D)$ *is a complete separable metric space in which* $K(R^n)$ *and* $K_C(R^n)$ *are closed subsets. Hence* $(K(R^n), D)$ *and* $(K_C(R^n), D)$ *are also complete separable metric spaces.*

We need the following result which deals with the law of cancellation to proceed further.

Lemma 2.1: Let A *and* $B \in K_C(R^n)$ *and* $C \in C(R^n)$ *then, If* $A + C \subseteq B + C$ *then* $A \subseteq B$.

Proof 2.1: Let a *be any element of* A*, we need to show that* $a \in B$*. For any* $c_1 \in C$*, we have* $a + c_1 \in B + C$*, that is there exist* $b_1 \in B$ *and* $c_2 \in C$ *with* $a + c_1 = b_1 + c_2$ *for the same reason,* $b_2 \in B$ *and* $c_3 \in C$*, with* $a + c_2 = b_2 + c_3$ *it will be exist. Iterate the steps indefinitely and sum the first* n *of the equations we get:*

$$na + \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} b_i + \sum_{i=2}^{n+1} c_i \qquad (2.24)$$

which implies:

$$na + c_1 + c_2 + \cdots + c_n = \left(\sum_{i=1}^{n} b_i\right) + c_2 + c_3 + \cdots + c_n + c_{n+1} \qquad (2.25)$$

$$na + c_1 = \left(\sum_{i=1}^{n} b_i\right) + c_{n+1} \qquad (2.26)$$

then:

$$a = \frac{1}{n}\left(\sum_{i=1}^{n} b_i\right) + \frac{c_{n+1}}{n} - \frac{c_1}{n} \qquad (2.27)$$

we will set:

$$x_n = \frac{1}{n}\sum_{i=1}^{n} b_i \quad (2.28)$$

consequently,

$$a = x_n + \frac{c_{n+1}}{n} - \frac{c_1}{n} \quad (2.29)$$

we observe that $x_n \in B$. *For all* n, *as* $n \to \infty$

$$\frac{c_{n+1}}{n} - \frac{c_1}{n} \to 0 \quad (2.30)$$

this shows that x_n *converges to* a. *And since* B *is compact* $a \in B$.

Proposition 2.2: If $A, B \in K_c(R^n)$ *and* $C \in K(R^n\ )$ *then:*

$$D(A + C, B + C) = D(A, B) \quad (2.31)$$

Proof 2.2: Let $\lambda \geq 0$ *and* S *denote the closed unit sphere of the space. Consider the following inequalities:*

i) $A + \lambda S \supset B$
ii) $B + \lambda S \supset A$
iii) $A + C + \lambda S \supset B + C$
iv) $B + C + \lambda S \supset A + C$

Put $d_1 = D(A, B)$ *and* $d_2 = D(A + C, B + C)$, *then* d_1 *is the infimum of the positive numbers* λ *for which (i) and (ii) hold. Similarly,* d_2 *is the infimum of the positive numbers* λ *for which (iii) and (iv) hold. Since (iii) and (iv) follow from (i) and (ii) by adding* C *we have* $d_1 \geq d_2$. *conversely since by Lemma (2.1) cancelling* C *is allowed in (iii) and (iv). We obtain* $d_1 \leq d_2$, *which proves the proposition.*

Proposition 2.3: If $A, B \in K(R^n)$:

- $D(coA\ , co\ B) \leq D(A, B)$ (2.32)
- *If* $A, A', B, B' \in K_C(R^n)$ *then:*

$$D(A + A', B + B') \leq D(A, B) + D(A', B') \quad (2.33)$$

$$D(A - A', B - B') \leq D(A, B) + D(A', B') \quad (2.34)$$

provided the difference $A - A', B - B'$ *exist.*

- *furthermore with* $\beta = max\{\lambda , \mu\}$, *we have:*

$$D(\lambda A, \mu B) \leq \beta D(A, B) + |\lambda - \mu|[D(A, \theta) + D(B, \theta)] \qquad (2.35)$$

and if $A - B$ *exist:*

$$D(\lambda A, \lambda B) = \lambda D(A - B, \theta) \qquad (2.36)$$

- $D(tA, tB\ ) = tD(A, B)$, *for all* $t \geq 0$. *(2.37)*

In general, $A + (-A) \neq \{\theta\}$, this is illustrated by the following example.

Example 2.5: Let $B = [0, 5]$, *so that* $(-1)B = [-5, 0]$, *and therefore*

$$B + (-1)B = [0,5] + [-5,0] = [-5, 5]$$

thus, adding (-1) times a set does not constitute a natural operation of subtraction.

Definition 2.2: For a fixed A *and* B *in* $K_C(R^n)$, *if there exists an element* $C \in K_C(R^n)$ *such that* $A = B + C$, *then we say that the Hukuhara difference of* A *and* B *exists and is denoted by* $A - B$.

When the Hukuhara difference exists, it is unique. This follows from this property:

$$A + C = B + C \ \Rightarrow A = B \qquad (2.38)$$

suppose for a fixed A and B in $K_C(R^n)$ there exists an element $C \in K_C(R^n)$, such that:

$$A = B + C \qquad (2.39)$$

and for an element $G \in K_C(R^n)$ such that:

$$A = B + G \qquad (2.40)$$

then according to the property (2.38)

$$B + C = \ B + G \Rightarrow C = G. \qquad (2.41)$$

The following example explains the Definition (2.2).

Example 2.6: We get from the previous example that:

$$[-5,5]-[-5,0]=[0,1] \textit{ and } [-5,5]-[0,5]=[-5,0]$$

note that the Hukuhara difference $A-B$ is different from the set:

$$A+(-B)=\{a+(-b);\quad a\in A\,,b\in B\}$$

from the previous example, we can set:

$$B=[0,5]\qquad A=[-5,5]$$

then we can note that:

$$A-B=[-5,5]-[0,5]=[-5,0]$$

$$A+(-B)=[-5,5]+[-5,0]=[-5,5].$$

The next proposition provides the necessary and sufficient condition for the existence of the Hukuhara difference $A-B$.

Proposition 2.4: Let $A,B\in K_C(R^n)$ for the difference $A-B$ to exist. It is necessary and sufficient to have the following condition. If $a\in\delta A$, there exists at least a point c such that:

$$a\in B+c\subset A \tag{2.42}$$

Proof 2.4: Necessity: suppose the difference $A-B$ exists. Let $C=A-B$, then. $A=B+C$. If $a\in\delta A$, then

$$a\in B+C \tag{2.43}$$

that is

$$a=b+c\ \ ,b\in B\,,c\in C \tag{2.44}$$

also, if $z\in B$ then:

$$z+c\in A \tag{2.45}$$

and therefore

$$a\in B+c\subset A \tag{2.46}$$

Sufficiency:

Suppose $a \in B + c \subset A$ *holds, consider the set* $C = \{x\ : B + x \subseteq A\}$ *clearly* C *is compact and we have* $B + C \subseteq A$. *Now if* d *and* $d' \in C$, *then we have* $B + d\ \subseteq A$ *and* $B + d' \subseteq A$ *from which we obtain:*

$$(1-\lambda)(B+d) + \lambda(B+d') \subset A \quad \textit{for} \quad 0 \le \lambda \le 1 \tag{2.47}$$

that since A *is convex. We can write the L.H.S as* $B + z$ *where* $z = (1-\lambda)d + d'$ *hence* $z \in C$ *and* C *is convex.*

Now let $u \in A$. *A straight line through* u *meets* δA *at two points* $a\ , a'$. *By hypothesis there exist elements* $d\ , d' \in C$. *Such that:*

$$a \in B + d \ \ \textit{and}\ a' \in B + d' \tag{2.48}$$

we can write:

$$u = (1-\lambda)a + \lambda a' \ \ \textit{with} \ \ 0 < \lambda < 1 \tag{2.49}$$

then:

$$u \in B + z \tag{2.50}$$

where $z = (1-\lambda)d + \lambda d' \in C$, *hence* $A \subseteq B + C$, *thus* $A = B + C$. *And the proof is complete.* ■

We note that an indispensable condition for the Hukuhara difference $A - B$ to exist is that some translate of B is a subset of A, however, in general the Hukuhara difference need not exist as is seen from the next example.

Example 2.7: $\{0\} - [0,2]$ *does not exist, since no translate of* $[0,2]$ *can ever belong to the singleton set* $\{0\}$.

Definition 2.3: Let I *be an interval of real numbers. Let* U *a multifunction:*

$$U: I \longrightarrow K_C(R^n) \tag{2.51}$$

be given, U *is Hukuhara differentiable at a point* $t_0 \in I$, *if there exists* $D_H U(t_0) \in K_C(R^n)$ *such that the limits:*

$$lim_{\Delta t \to 0^+} \frac{U(t_0 + \Delta t) - U(t_0)}{\Delta t} \tag{2.52}$$

and

$$lim_{\Delta t \to 0^+} \frac{U(t_0) - U(t_0 - \Delta t)}{\Delta t} \tag{2.53}$$

both exist and are equal to $D_H U(t_0)$.

clearly, implicit in the definition of $D_H U(t_0)$ is the existence of the differences $U(t_0) - U(t_0 - \Delta t)$ and $U(t_0 + \Delta t) - U(t_0)$, for all $\Delta t > 0$ sufficiently small.

Definition 2.4: We shall say that $F: [0,1] \to K_c(\mathbb{R}^n)$ is integrally bounded on $[0,1]$ if there exists an integrable function $g: [0,1] \to \mathbb{R}$ such that

$$\|F(t)\| \leq g(t), \qquad \forall t \in [0,1]. \tag{2.54}$$

Theorem 2.1: $F: T \to K_c(\mathbb{R}^n)$ is measurable if and only if there exists a sequence $\{f_i\}$ of measurable selectors of F such that:

$$F(t) = \overline{\bigcup \{f_\iota(t): \iota = 1,2, \cdots\}}, \text{ for each } t \in T \tag{2.55}$$

Theorem 2.2: Let $F: [0,1] \longrightarrow K_C(R^n)$ be measurable and integrally bounded. Then $A: [0,1] \longrightarrow K_C(R^n)$ defined by:

$$A(t) = \int_0^t F(s)ds \tag{2.56}$$

for all $t \in [0,1]$ is Hukuhara differentiable for almost all $t_0 \in (0,1)$, with the Hukuhara derivative $D_H A(t_0) = F(t_0)$.

Definition 2.5: We consider mappings F from a domain T in R^n into the metric space $(K_c(R^n)\ ,D)$. Thus, $F: T \to K_c(R^n)$ or equivalently

$$F(t) \in K_c(R^n), \textit{for all } t \in T \qquad (2.57)$$

we shall call such a mapping F is a (compact convex) set valued mapping from T to R^n.

Definition 2.6: We shall say that a set valued mapping F satisfying (2.57) is continuous at $t_0 \in T$ if $\forall \epsilon > 0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$, such that

$$D[F(t), F(t_0)] < \varepsilon, \qquad \forall t \in T \ \textit{with} \ \| t - t_0 \| < \delta \qquad (2.58)$$

alternatively, we can write (2.7) in terms of the convergence of sequences, that is

$$\lim_{t_n \to t_0} D[F(t_n), F(t_0)] = 0, \qquad \forall t_n \in T \ \textit{with}\ t_n \to t_0 \in T \qquad (2.59)$$

using the Hausdorff separation d_H and neighborhoods, we see that (2.7) is combination of

$$d_H(F(t), F(t_0)) < \varepsilon \qquad (2.60)$$

that is

$$F(t) \subset S_\epsilon\big(F(t_0)\big) \equiv F(t_0) + \epsilon S_1^n \qquad (2.61)$$

and

$$d_H(F(t_0), F(t)) < \varepsilon \qquad (2.62)$$

that is

$$F(t_0) \subset S_\epsilon\big(F(t)\big) \equiv F(t) + \epsilon S_1^n \qquad (2.63)$$

For all $t \in T$ with $\| t - t_0 \| < \delta$. As before, $S_1^n = \{x \in \mathbb{R}^n : \| x \| < 1\}$ is the open unit ball in $\mathbb{R}^n$. If the mapping F satisfies (2.60), (2.61), then we say that is upper semicontinuous at t_0, or that is lower semicontinuous at t_0, if it satisfies (2.62), (2.63). Thus F is continuous at t_0 if and only if it is both lower semicontinuous and upper semicontinuous at t_0.

3. BASIC THEORY

3.1. Basic Definitions and Theorems:

Let us consider the initial value problem (IVP) for the set differential equation:

$$D_H U = F(t,U), \qquad U(t_0) = U_0 \in K_C(R^n), \qquad t_0 \geq 0 \qquad (3.1.1)$$

where $F \in C[R_+ \times K_C(R^n), K_C(R^n)]$ and $D_H U$ is the Hukuhara derivative of U.The mapping $U \in C^1[J, K_C(R^n)]$ where $J = [t_0, t_0 + a]$, $a > 0$, is said to be a solution of (3.1.1) on J, if it satisfies (3.1.1) on J. Since $U(t)$ is continuously differentiable, we have:

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s) ds, \qquad t \in J \qquad (3.1.2)$$

we therefore associate with the IVP (3.1.1), the following integral equation:

$$U(t) = U_0 + \int_{t_0}^{t} F\big(s, U(s)\big) ds\,, \qquad t \in J \qquad (3.1.3)$$

where the integral in (3.1.3) is the Hukuhara integral observe also that $U(t)$ is a solution of (3.1.1) if and only if it satisfies (3.1.3) on J. Since F is continuously differentiable in R_+, then any solution of (3.1.1) is also a solution of (3.1.3) and conversely.

Proof:

Any solution of the differential equation (3.1.1) converts it into an identity in U i.e.: $D_H U = F(t,U)$, an integration of this equality with $U(t_0) = U_0$ gives (3.1.3) conversely, if $U(t)$ is any solution of (3.1.3) then the substitution $t = t_0$ in (3.1.3) gives $U(t_0) = U_0$, further, since $F(t,U)$ is continuous, by differentiating (3.1.3) we find that $D_H U = F(t,U)$.■

Definition 3.1.1: Let $r(t)$ be the solution of (3.1.1) on J, then $r(t)$ is said to be a maximal solution of (3.1.1) if for every solution $U(t)$ of (3.1.1) on J. The inequality:

$$U(t) \leq r(t), \quad t \in J \qquad (3.1.4)$$

holds.

Definition 3.1.2: Let $\rho(t)$ be a solution of (3.1.1) on J, then $\rho(t)$ is said to be a minimal solution of (3.1.1). If for every solution $U(t)$ of (3.1.1) on J. The inequality:

$$U(t) \geq \rho(t), \quad t \in J \tag{3.1.5}$$

holds.

Definition 3.1.3: A function $W(t) \in C^1[R_+, K_C(R^n)]$, is said to be upper solution of (3.1.1) if:

$$D_H W \geq F(t, W), \qquad W(t_0) \geq U_0 \quad on \ J \tag{3.1.6}$$

holds.

Definition 3.1.4: A function $V(t) \in C^1[R_+, K_C(R^n)]$, is said to be lower solution of (3.1.1) if:

$$D_H V \leq F(t, V), \qquad V(t_0) \leq U_0 \quad on \ J \tag{3.1.7}$$

holds.

Definition 3.1.5: A family of functions $F = \{f_\alpha(t)\}_{\alpha \in A}$ defined on real interval J is said to be equicontinuous on J if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ independent of $f_\alpha \in F$ and also of t_1, t_2 in J in such that:

$$|f_\alpha(t_1) - f_\alpha(t_2)| < \varepsilon \quad whenever \qquad |t_1 - t_2| < \delta \tag{3.1.8}$$

Lemma (Ascoli's Lemma) 3.1.1: Let $F = \{f_\alpha(t)\}_{\alpha \in A}$ be a family of functions which is uniformly bounded and equicontinuous on an interval J. Then every sequence of functions $\{f_n(t)\}$ in F contains a subsequence $\{f_{n_k}(t)\}$, k = 0, 1, 2, ... which is uniformly convergent on every compact sub-interval of J.

Lemma (Grownwall's Lemma) 3.1.2: Assume that $f, g \in C[R_+ \times R^n, R^n]$, non-negative functions, for $t \geq t_0$, and $c > 0$, be a constant. Then the inequality:

$$f(t) \leq c + \int_{t_0}^{t} f(s)g(s)ds, \qquad t \geq t_0 \tag{3.1.9}$$

implies the inequality:

$$f(t) \leq c\, exp\left[\int_{t_0}^{t} g(s)ds\right] \tag{3.1.10}$$

Proof 3.1.2: We have:

$$f(t) \leq c + \int_{t_0}^{t} g(s)f(s)ds, \qquad c > 0 \tag{3.1.11}$$

then we can write:

$$\frac{f(t)g(t)}{c + \int_{t_0}^{t} f(s)g(s)ds} \leq g(t) \tag{3.1.12}$$

Let

$$r(t) = c + \int_{t_0}^{t} g(s)f(s)ds, \;\; t \geq t_0 \tag{3.1.13}$$

then:

$$r'(t) = f(t)g(t), \;\; r(t_0) = c \tag{3.1.14}$$

$$r'(t) = f(t)g(t) \leq r(t)g(t) \tag{3.1.15}$$

which leads to:

$$\int_{t_0}^{t} \frac{r'(s)}{r(s)} ds \leq \int_{t_0}^{t} g(s)ds \tag{3.1.16}$$

$$ln[r(t)] - ln[r(t_0)] \leq \int_{t_0}^{t} g(s)ds \tag{3.1.17}$$

$$ln[r(t)] - ln[c] \leq \int_{t_0}^{t} g(s)ds \tag{3.1.18}$$

$$ln\left[\frac{r(t)}{c}\right] \leq \int_{t_0}^{t} g(s)ds \tag{3.1.19}$$

$$\frac{r(t)}{c} \leq exp\left[\int_{t_0}^{t} g(s)ds\right] \tag{3.1.29}$$

$$r(t) \leq c\, exp\left[\int_{t_0}^{t} g(s)ds\right] \tag{3.1.30}$$

then we arrive at: $f(t) \leq r(t) \leq c\, exp\left[\int_{t_0}^{t} g(s)ds\right]$*. And the proof is complete.* ■

Remark:

In the previous lemma if we have $c = 0$ *then,* $f(t) \equiv 0$.

Proof:

For any $t \geq t_0$, *we define* $r(t)$ *as follows:*

$$r(t) = \int_{t_0}^{t} f(s)g(s)ds, \tag{3.1.30}$$

so that $r(t_0) \equiv 0$. *And so:*

$$r'(t) = f(t)g(t) \tag{3.1.31}$$

thus, we have: $f(t) \leq r(t)$ *then:*

$$r'(t) = f(t)g(t) \leq r(t)g(t) \tag{3.1.32}$$

$$r(t) - \big(r(t)g(t)\big) \leq 0 \tag{3.1.33}$$

multiplying both sides of this inequality by:

$$exp\left(-\int_{t_0}^{t} g(s)ds\right) \tag{3.1.34}$$

we arrive at:

$$\left(exp\left(-\int_{t_0}^{t} g(s)ds\right) r(s)\right)' \leq 0 \tag{3.1.35}$$

thus:

$exp\left(-\int_{t_0}^{t} g(s)ds\right) r(s)$ *is nonincreasing. Since* $r(t_0) = 0$, *it follows that:*

$$r(t) \leq 0$$

and hence:

$$f(t) \leq r(t) \leq 0 \tag{3.1.36}$$

however, since the function $f(t)$ *is nonnegative, we find that* $f(t) \equiv 0$.

Definition 3.1.6: Let $F \in C[R_+ \times K_C(R^n), K_C(R^n)]$ *is said to satisfy the uniform lipschitz condition in* R_+ *if:*

$$D\left[F\big(t,U(t)\big),F\big(t,V(t)\big)\right] \leq LD[U(t),V(t)] \tag{3.1.37}$$

for all $t \in R_+$ and $U,V \in K_C(R^n)$. Having the same t and the nonnegative constant L is called the Lipchitz constant.

Theorem (Lipschitz uniqueness theorem) 3.1.3: Assume that $F \in C[R_+ \times K_C(R^n), K_C(R^n)]$, *and satisfies the uniform lipschitz condition*

$$D\left[F\big(t,U(t)\big),F\big(t,V(t)\big)\right] \leq LD[U(t),V(t)] \tag{3.1.38}$$

for all $t \in R_+$ and $U,V \in K_C(R^n)$, then the IVP (3.1.1) has a unique solution in R_+

Proof 3.1.3: Suppose that $V(t)$ and $U(t)$ are two solutions of (3.1.1) on J, where $V,U \in C^1[J,K_C(R^n)]$ then the equivelant Volterra integral equations are:

$$V(t) = V(t_0) + \int_{t_0}^{t} F\big(s,V(s)\big)ds \tag{3.1.39}$$

$$U(t) = U(t_0) + \int_{t_0}^{t} F\big(s,U(s)\big)ds \tag{3.1.40}$$

since $V(t_0) = U(t_0) = U_0$ and by using Hausdorff metric properties and (3.1.7) we obtain:

$$\begin{aligned} D[U(t),V(t)] &= D\left[U_0 + \int_{t_0}^{t} F\big(s,U(s)\big)ds\ ,U_0 + \int_{t_0}^{t} F\big(s,V(s)\big)ds\right] \\ &= D\left[\int_{t_0}^{t} F\big(s,U(s)\big)ds\ ,\int_{t_0}^{t} F\big(s,V(s)\big)ds\right] \\ &= D\left[\int_{t_0}^{t} \left[F\big(s,U(s)\big),F\big(s,V(s)\big)\right]ds\right] \\ &\leq \int_{t_0}^{t} D[F(s,U(s),F(s,V(s))]\,ds \\ &\leq L\int_{t_0}^{t} D[U(s),V(s)]ds \end{aligned} \tag{3.1.41}$$

now by applying Grownwall's lemma, where $c = 0$. We arrive at:

$$D[V(t),U(t)] = 0 \tag{3.1.42}$$

as a consequently:

$$V(t) = U(t). \tag{3.1.43}$$

And the proof is complete.■

Theorem 3.1.4: Assume that:

i) $V, W \in C^1[R_+, K_C(R^n)]$, *and* $F \in C[R_+ \times K_C(R^n), K_C(R^n)]$, $F(t,X)$ *is monotone nondecreasing in* X *for each* $t \in R_+$*:*

$$D_H V \leq F(t,V), \qquad D_H W \geq F(t,W), \qquad t \in R_+ \tag{3.1.44}$$

ii) *For any* $X, Y \in K_C(R^n)$ *such that* $X \geq Y$, $t \in R_+$*:*

$$F(t,X) \leq F(t,Y) + L(X-Y) \quad \textit{for some} \quad L > 0 \tag{3.1.45}$$

then:

$$V(t_0) \leq W(t_0) \ \textit{implies that} \ V(t) \leq W(t), \ t \geq t_0$$

Proof 3.1.4: Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) > 0$, *and define* $\breve{W}(t) = W(t) + \varepsilon e^{2Lt}$. *Since* $V(t_0) \leq W(t_0) < \breve{W}(t_0)$, *it is enough to prove that:*

$$V(t) < \breve{W}(t), \qquad t \geq t_0 \tag{3.1.45}$$

to arrive to our conclusion, in view of the fact $\varepsilon > 0$. *Is arbitrary, let* $t_1 > 0$ *be the supremum of all positive numbers* $\delta > 0$, *such that:*

$$V(t_0) < \breve{W}(t_0) \ \textit{implies} \ V(t) < \breve{W}(t) \ \textit{on} \ [t_0, \delta].$$

It is clear that $t_1 > t_0$, *and* $V(t_1) \leq \breve{W}(t_1)$, *this follows, using the nondecreasing nature of* F *and condition (3.1.45) that:*

$$\begin{aligned} D_H V(t_1) &\leq F\big(t_1, V(t_1)\big) \leq F\left(t_1, \breve{W}(t_1)\right) \\ &\leq F\big(t_1, W(t_1)\big) + L\big(\breve{W} - W\big) \\ &\leq D_H W(t_1) + L\varepsilon e^{2Lt_1} \\ &< D_H W(t_1) + 2L\varepsilon e^{2Lt_1} = D_H \breve{W}(t_1) \end{aligned} \tag{3.1.46}$$

consequently, it follows that there exists an $\eta > 0$ *satisfying:*

$$V(t) - \breve{W}(t) > V(t_1) - \breve{W}(t_1), \qquad t_1 - \eta < t < t_1 \tag{3.1.47}$$

this implies that $t_1 > t_0$ *cannot be the supremum in view of the continuity of the functions involved and therefore the relation*

$$V(t) < \breve{W}(t), \qquad t \geq t_0 \tag{3.1.48}$$

is true, and then we can write

$$V(t) < \breve{W}(t) = W(t) + \varepsilon e^{2Lt} \tag{3.1.49}$$

and then:

$$lim_{\varepsilon\to 0} V(t) \leq lim_{\varepsilon\to 0}[W(t) + \varepsilon e^{2Lt}] \tag{3.1.50}$$

$$V(t) \leq W(t). \tag{3.1.51}$$

And the proof is complete.■

Theorem 3.1.5: Let $F \in C[J \times K_C(R^n), K_C(R^n)]$ *and* $D\left[F\big(t, U(t)\big), \theta\right] \leq M$, *then there exists a solution of the IVP (3.1.1) on* J *.*

Existence via Upper and Lower Solutions:

If we know the existence of lower and upper solutions V, W such that, $V \leq W$, we can prove the existence of solutions in closed set:

$$\Omega = \{(t, U) \in J \times K_C(R^n), \qquad t_0 \leq t \leq t_0 + a, \qquad V(t) \leq U(t) \leq W(t)\}$$

this is what we will prove in the next theorem.

Theorem 3.1.6: Let $J = [t_0, t_0 + a], V, W \in C^1[J, K_C(R^n)]$, *be lower and upper solutions of (3.1.1) Such that* $V(t) \leq W(t)$ *on* J, *and* $F \in C[\Omega, K_C(R^n)]$, *then, there exists a solution* $U(t)$ *of (3.1.1), Such that:*

$$V(t) \leq U(t) \leq W(t) \tag{3.1.52}$$

Proof 3.1.6: Let $P: J \times K_C(R^n) \to K_C(R^n)$ *defined by:*

$$P(t, U) = max\{V(t), min\{U(t), W(t)\}\} \tag{3.1.53}$$

then $F(t, P(t, U(t))$ *defines a continuous extension of* F *to* $J \times K_C(R^n)$ *which is also bounded since* F *is bounded on* Ω*. Therefore, the solution of (3.1.1) Exists on* J*. For* $\varepsilon > 0$ *consider,*

$$W_\varepsilon(t) = W(t) + \varepsilon(1 + t) \quad (3.1.54)$$

$$V_\varepsilon(t) = V(t) - \varepsilon(1 + t) \quad (3.1.55)$$

clearly,

$$W_\varepsilon(t_0) = W(t_0) + \varepsilon(1 + t_0) = U_0 + \varepsilon(1 + t_0) \quad (3.1.56)$$

$$V_\varepsilon(t_0) = V(t_0) - \varepsilon(1 + t_0) = U_0 - \varepsilon(1 + t_0) \quad (3.1.57)$$

as a result, $V_\varepsilon(t_0) < U_0 < W_\varepsilon(t_0)$*, we wish to show that* $V_\varepsilon(t) < U(t) < W_\varepsilon(t)$ *on* J*. Suppose that* $t_1 \in (t_0, t_0 + a)$ *is such,* $V_\varepsilon(t) < U(t) < W_\varepsilon(t)$ *on* $[t_0,\ t_1)$*, and*

$$U(t_1) = W_\varepsilon(t_1) \quad (3.1.58)$$

then

$$U(t_1) = W_\varepsilon(t_1) = W(t_1) + \varepsilon(1 + t_1) \quad (3.1.59)$$

this means

$$U(t_1) > W(t_1) \quad (3.1.60)$$

and so:

$$P(t_1, U(t_1)) = max[V(t_1), min[U(t_1), W(t_1)]] = max[V(t_1), W(t_1)] = W(t_1)$$

also

$$V(t_1) \leq P(t_1, U(t_1)) \leq W(t_1) \quad (3.1.61)$$

hence

$$D_H W(t_1) \geq F(t_1, W(t_1)) \geq F(t_1, P(t_1, U(t_1)) = D_H U(t_1)$$

$$D_H W(t_1) \geq D_H U(t_1) \quad (3.1.71)$$

we have:

$$W(t_1) = W_\varepsilon(t_1) - \varepsilon(1 + t_1) \quad (3.1.72)$$

$$D_H W(t_1) = D_H W_\varepsilon(t_1) - \varepsilon \quad (3.1.73)$$

$$D_H W_\varepsilon(t_1) = D_H W(t_1) + \varepsilon \quad (3.1.74)$$

then we can arrive at the relation

$$D_H W_\varepsilon(t_1) > D_H W(t_1) \qquad (3.1.75)$$

from (3.1.71) and (3.1.75) we obtain

$$D_H W_\varepsilon(t_1) > D_H U(t_1) \qquad (3.1.76)$$

this contradicts

$$W_\varepsilon(t) > U(t) \text{ for } t \in [t_0,\ t_1) \qquad (3.1.77)$$

consequently

$$V_\varepsilon(t) < U(t) < W_\varepsilon(t) \text{ on } J \qquad (3.1.78)$$

letting $\varepsilon \to \infty$ we get, $V(t) < U(t) < W(t)$ on J. Completing the proof.■

Theorem (Peano uniqueness theorem) 3.1.7: Let $F \in C[J \times K_c(R^n), K_c(R^n)]$, and $F(t,U)$ is nonincreasing in U for each fixed t in $J = [t_0, t_0 + a\ ]$. then, the initial value problem (3.1.1) has at most one solution in $[t_0, t_0 + a\ ]$.

Proof 3.1.7: Suppose $U(t)$ and $V(t)$ are two solutions of (3.1.1) which differ somewhere in $[t_0, t_0 + a\ ]$. we assume that $V(t) > U(t)$ for $t_1 < t < t_1 + \varepsilon \leq t_0 + a$, and $V(t) = U(t)$ for $t_0 < t \leq t_1$. Thus, for all $t \in (t_1, t_1 + \varepsilon]$, since the function F is nonincreasing we have $F(t, U(t)) \geq F\big(t, V(t)\big)$, and hence $D_H U(t) \geq D_H V(t)$. This implies that the function $\varphi(t) = V(t) - U(t)$ is nonincreasing. Further, since $\varphi(t_1) = 0$, we have $\varphi(t) \leq 0$ in $[t_1, t_1 + \varepsilon]$. This contradiction proves that $V(t) = U(t)$ in $[t_0, t_0 + a\ ]$.■

Lemma 3.1.3: Suppose that $F\big(t, U(t)\big)$ is nonincreasing in U, then:

i) There exist lower and upper solutions V_0, W_0 of IVP,

$$D_H U = F\big(t, U(t)\big), \qquad U(t_0) = U_0 \qquad (3.1.79)$$

such that $V_0 \leq W_0$ on $J = [t_0, t_0 + a]$.

ii) There exists a unique solution U of IVP on J, such that $V_0 \leq U(t) \leq W_0$.

Proof 3.1.3: Let $V_0(t) = -R_0 + \varphi(t)$, $W_0(t) = R_0 + \varphi(t)$, where $\varphi(t)$ is the solution of:

$$D_H\varphi(t) = F(t,0), \qquad \varphi(t_0) = U_0 \tag{3.1.80}$$

choose $R_0 > 0$ *sufficiently large so that:*

$$V_0 \le 0 \le W_0$$

since F *is non-increasing this implies that:*

$$D_H V_0 = D_H\varphi(t) = F(t,0) \le F(t,V_0) \tag{3.1.81}$$

$$D_H W_0 = D_H\varphi(t) = F(t,0) \ge F(t,W_0) \tag{3.1.82}$$

and we can write

$$V_0(t_0) = -R_0 + \varphi(t_0) = -R_0 + U_0 \le U_0 \tag{3.1.83}$$

$$W_0(t_0) = R_0 + \varphi(t_0) = R_0 + U_0 \ge U_0 \tag{3.1.84}$$

$D_H V_0 \le F(t,V_0)$, $V(t_0) \le U_0$, *which means that* V_0 *is lower solution*

$D_H W_0 \le F(t,W_0)$, $W(t_0) \ge U_0$, *which means that* W_0 *is upper solution*

now, according to the theorem (existence via upper and lower solutions) there exists a solution $U(t)$ *of IVP, such that* $V_0 \le U(t) \le W_0$ *on* J, *and since* F *is non-increasing, uniqueness is obvious. And the proof is complete.*■

3.2 Dini Derivatives and Comparison Principles

Definition 3.2.1: We adopt the following notation for Dini derivatives:

$$D^+U(t) = lim_{h\to 0^+} sup \frac{1}{h}[U(t+h) - U(t)], \tag{3.2.1}$$

$$D_+U(t) = lim_{h\to 0^+} inf \frac{1}{h}[U(t+h) - U(t)], \tag{3.2.2}$$

$$D^-U(t) = lim_{h\to 0^-} sup \frac{1}{h}[U(t+h) - U(t)], \tag{3.2.3}$$

$$D_-U(t) = lim_{h\to 0^-} inf \frac{1}{h}[U(t+h) - U(t)], \tag{3.2.4}$$

where $U \in C[J,R]$. *When* $D^+U(t) = D_+U(t)$, *the right derivative will be denoted by* $U'_+(t)$. *Similarly* $U'_-(t)$ *denotes the left derivative. We have the following results concerning the Dini derivative.*

Theorem 3.2.1: Let $m \in C[J,R_+]$, *and* $g \in C[J \times R_+, R_+]$ *and:*

$$D_m^+(t) \le g\big(t,m(t)\big), \qquad t \in J \tag{3.2.5}$$

then $m(t_0) \leq w_0$ *implies:*

$$m(t) \leq r(t), \qquad t \in J \tag{3.2.6}$$

where, $r(t)$ *is the maximal solution of the scalar differential equation.*

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0 \tag{3.2.7}$$

existing on J, *where* $J = [t_0, t_0 + a]$.

Proof 3.2.1: Let G *be such that:*

$$G(t, w) = \begin{cases} g(t, w(t)) & w \geq m(t) \\ g(t, m(t)) & w < m(t) \end{cases} \tag{3.2.8}$$

let $w(t)$ *be the solution of the scalar differential equation (3.2.7), Suppose that* $w(t) < m(t)$ *for some* t, *then there exists a* $t_1 > t_0$ *such that:*

$$w(t_1) \leq m(t_1) \tag{3.2.9}$$

and

$$w'(t_1) < D^+ m(t_1) \tag{3.2.10}$$

and so:

$$D^+ m(t_1) \leq g(t_1, m(t_1)) = G(t, w(t_1)) = w'(t_1) \tag{3.2.11}$$

which is a contradiction with:

$$D^+ m(t_1) > w'(t_1) \tag{3.2.12}$$

it therefore, follows that:

$$w(t) \geq m(t) \tag{3.2.13}$$

which implies that $w(t)$ *is the solution of the (3.2.7) in view of the definition of* $G(t, w)$. *since* $r(t)$ *is the maximal solution of (3.2.7), we have* $w(t) \leq r(t)$ *from which it follows that*

$$m(t) \leq r(t), \ t \in J. \tag{3.2.14}$$

And the proof is complete. ■

Theorem 3.2.2: Let $g \in C[J \times R_+, R_+]$ *and that* $g(t,w)$ *is monotone nondecreasing in* w *for each* t*, let* $m \in C[J, R_+], t \in J$ *and* $m(t_0) \leq w_0$*, and:*

$$m(t) \leq m(t_0) + \int_{t_0}^{t} g\big(s, m(s)\big)ds \quad t \in J \tag{3.2.15}$$

then:

$$m(t) \leq r(t), \qquad t \in J \tag{3.2.16}$$

where, $r(t)$ *is the maximal solution of (3.2.7) existing on* J*.*

Proof 3.2.2: Define

$$v(t) = m(t_0) + \int_{t_0}^{t} g\big(s, m(s)\big)ds \tag{3.2.17}$$

so that:

$$m(t) \leq v(t) \tag{3.2.18}$$

and:

$$v'(t) = g\big(t, m(t)\big) \tag{3.2.19}$$

since g *is monotone, using (3.2.18) we obtain the differential inequality:*

$$v'(t) = g\big(t, m(t)\big) \leq g\big(t, v(t)\big), \qquad t \in J \tag{3.2.20}$$

from the application of Theorem (3.2.1) we obtain:

$$v(t) \leq r(t), \qquad t \in J \tag{3.2.21}$$

proving the theorem. ■

Theorem 3.2.3: Assume that $F \in C[J \times K_c(R^n), K_C(R^n)]$ *and* $t \in J$*,* $U, V \in K_C(R^n)$

$$D[F(t,U), F(t,V)] \leq g(t, D[U,V]) \tag{3.2.22}$$

where $g \in C[J \times R_+, R_+]$ *and* $g(t,w)$ *is monotone nondecreasing in* w *for each* $t \in J$*. Suppose further that the maximal solution* $r(t, t_0, w_0)$ *of the scalar differential* *equation (3.2.7) exists on J. Then if $U(t)$, $V(t)$ are any two solutions through (t_0, U_0), (t_0, V_0) respectively on J. It follows that:*

$$D[U(t), V(t)] \leq r(t, t_0, w_0), \qquad t \in J \tag{3.2.23}$$

provided that $[U_0, V_0] \leq w_0$.

Proof 3.2.3: Set $m(t) = D[U(t), V(t)]$, so that $m(t_0) = D[U_0, V_0] \leq w_0$ then. In view of the properties of the metric D, we get:

$$\begin{aligned} m(t) &= D\left[U_0 + \int_{t_0}^{t} F\big(s, U(s)\big)ds, V_0 + \int_{t_0}^{t} F\big(s, V(s)\big)ds\right] \\ &\leq D\left[U_0 + \int_{t_0}^{t} F\big(s, U(s)\big)ds, U_0 + \int_{t_0}^{t} F\big(s, V(s)\big)ds\right] + \\ &\quad + D\left[U_0 + \int_{t_0}^{t} F\big(s, V(s)\big)ds, V_0 + \int_{t_0}^{t} F\big(s, V(s)\big)ds\right] \\ &= D\left[\int_{t_0}^{t} F\big(s, U(s)\big)ds, \int_{t_0}^{t} F\big(s, V(s)\big)ds\right] + D[U_0, V_0] \end{aligned} \tag{3.2.24}$$

now using the properties of integral and (3.2.22) we observe that:

$$\begin{aligned} m(t) &\leq m(t_0) + \int_{t_0}^{t} \Big[D\big[F\big(s, U(s)\big), F(s, V(s)\big]\Big] ds \\ &\leq m(t_0) + \int_{t_0}^{t} g(s, D[U(s), V(s)])\ ds \\ &= m(t_0) + \int_{t_0}^{t} g(s, D[U(s), V(s)]) \quad t \in J \end{aligned} \tag{3.2.25}$$

now applying Theorem (3.2.2), we conclude that:

$$m(t) \leq r(t, t_0, w_0), \ t \in J. \tag{3.2.26}$$

And the proof is complete.■

Remark:

If we employ the theory of differential inequalities instead of integral inequalities, we can dispense with the monotone character of $g(t,w)$ assumed in Theorem (3.2.3). This is the content of the next comparison principle.

Theorem 3.2.4: Let the assumptions of Theorem (3.2.3) hold except the nondecreasing property of $g(t,w)$ in w, then the conclusion (3.2.23) is valid.

Proof 3.2.4: For small $h > 0$, the Hukuhara differences $U(t+h) - U(t)$, $V(t+h) - V(t)$ exist. And we have for $t \in J$. Set $m(t) = D[U(t), V(t)]$ and then:

$$m(t+h) - m(t) = D[\,U(t+h)\,, V(t+h)\,] - D[U(t)\,, V(t)] \qquad (3.2.27)$$

using Hausdorff metric properties, we get:

$$\begin{aligned} D[\,U(t+h)\,, V(t+h)\,] &\le D\big[U(t+h), U(t) + hF\big(t, U(t)\big)\big] \\ &+ D\big[U(t) + hF\big(t, U(t)\big), V(t+h)\big] \end{aligned} \qquad (3.2.28)$$

and:

$$\begin{aligned} &D\big[U(t) + hF\big(t, U(t)\big), V(t+h)\big] \\ &\qquad \le D[V(t) + hF(t, V(t), V(t+h)] \\ &\qquad + D\big[U(t) + hF\big(t, U(t)\big), V(t) + hF(t, V(t))\big] \end{aligned} \qquad (3.2.29)$$

also, we observe that:

$$\begin{aligned} &D\big[U(t) + hF\big(t, U(t)\big), V(t) + hF\big(t, V(t)\big)\big] \\ &\qquad \le D\big[U(t) + hF\big(t, U(t)\big), U(t) + hF\big(t, V(t)\big)\big] \\ &\qquad + D\big[U(t) + hF\big(t, V(t)\big), V(t) + hF\big(t, V(t)\big)\big] \\ &\qquad = D\big[hF\big(t, U(t)\big), hF\big(t, V(t)\big)\big] + D[U(t), V(t)] \end{aligned} \qquad (3.2.30)$$

hence, it follows that:

$$
\begin{aligned}
m(t+h)-m(t) = D[\,U(t+h)\,,V(t+h)\ \,] &- D[U(t)\,,V(t)] \\
&\leq\ D\big[U(t+h),U(t)+hF\big(t,U(t)\big)\big] \\
&+\ D[V(t)+hF(t,V(t),V(t+h)] \\
&+\ D\big[hF\big(t,U(t)\big),hF\big(t,V(t)\big)\big]+\ D[U(t),V(t)] \\
&- D[U(t),V(t)] \qquad \textit{(3.2.31)}
\end{aligned}
$$

and then we can write the estimate:

$$
\begin{aligned}
\frac{m(t+h)-m(t)}{h}\ \leq \\
&\leq\ \frac{1}{h}D\big[U(t+h),U(t)+hF\big(t,U(t)\big)\big] \\
&+\frac{1}{h}D[V(t)+hF(t,V(t),V(t+h)] \\
&+\ \frac{1}{h}D\big[hF\big(t,U(t)\big),hF\big(t,V(t)\big)\big] \qquad \textit{(3.2.32)}
\end{aligned}
$$

and consequently, in view of the properties of D *and the fact* $U(t),V(t)$ *are solutions of (3.1.1), we find that:*

$$
\begin{aligned}
D_m^+(t) = lim_{h\to 0^+}sup\frac{1}{h}&[m(t+h)-m(t)] \\
&\leq lim_{h\to 0^+}supD\left[\frac{U(t+h)-U(t)}{h},F\big(t,U(t)\big)\right] \\
&+lim_{h\to 0^+}supD\left[F(t,V(t),\frac{V(t+h)-V(t)}{h}\right] \\
&+D\big[F\big(t,U(t)\big),F\big(t,V(t)\big)\big] \qquad \textit{(3.2.33)}
\end{aligned}
$$

here, we have used the fact that:

$$
\begin{aligned}
D\big[U(t+h),U(t)+hF\big(t,U(t)\big)\big] \\
&= D\big[U(t)+Z(t,h),U(t)+hF\big(t,U(t)\big)\big] \\
&= D\big[Z(t,h)+U(t),U(t)+hF\big(t,U(t)\big)\big] \\
&= D\big[Z(t,h),hF\big(t,U(t)\big)\big] \\
&= D[U(t+h)-U(t),hF(t,U(t))] \qquad \textit{(3.2.34)}
\end{aligned}
$$

where, $Z(t,h)=U(t+h)+U(t)$*. And so, the conclusion (3.2.23) follows from Theorem (3.2.1).*■

The next comparison result provides an estimate under weaker assumptions:

Theorem 3.2.5: Assume that $F \in C[J \times K_C(R^n), K_C(R^n)\ ]$ *and:*

$$lim_{h\to 0^+} sup \frac{1}{h}\Big[D\big[U + hF\big(t, U(t)\big), V + hF(t, V)\big] - D[U(t), V(t)]\Big]$$
$$\leq g(t, D[U(t), V(t)]), \qquad t \in J \tag{3.2.35}$$

where, $U(t), V(t) \in K_C(R^n), g \in C[J \times R_+, R_+]$*, the maximal solution* $r(t, t_0, w_0)$ *of (3.2.7) exists on* J*, then the conclusion (3.2.23) is valid.*

Proof 3.2.5: Proceeding as in the proof of Theorem (3.2.4) we see that:

$$\begin{aligned} m(t+h) - m(t) &= D[\ U(t+h)\ , V(t+h)\ \ ] - D[U(t)\ , V(t)] \\ &\leq\ D\big[U(t+h), U(t) + hF\big(t, U(t)\big)\big] \\ &+\ D[V(t) + hF(t, V(t), V(t+h)] \\ &+\ D\big[U(t) + hF\big(t, U(t)\big), V(t) + hF\big(t, V(t)\big)\big] \\ &- D[U(t), V(t)] \end{aligned} \tag{3.2.36}$$

and so:

$$D_m^+(t) = lim_{h\to 0^+} sup \frac{1}{h}[m(t+h) - m(t)] \leq$$
$$lim_{h\to 0^+} sup \tfrac{1}{h}\Big[D\big[U + hF\big(t, U(t)\big), V + hF(t, V)\big] - D[U(t), V(t)]\Big]$$
$$+ lim_{h\to 0^+} supD\left[\frac{U(t+h) - U(t)}{h}, F\big(t, U(t)\big)\right]$$
$$+\ lim_{h\to 0^+} supD\left[F\big(t, V(t)\big), \frac{V(t+h) - V(t)}{h}\right] \leq g(t, D[U(t), V(t)]$$
$$= g\big(t, m(t)\big)\ , t \in J \tag{3.2.37}$$

the conclusion follows as before by Theorem (3.2.1). And so, the proof is complete. ■

Remark:

We wish to remark that in Theorem (3.2.5), $g(t, w)$ *need not be nonnegative and therefore the estimate in Theorem (3.2.5) would be finer than the estimate in Theorem (3.2.3) and (3.2.4).*

Corollary 3.2.1: Assume that $F \in C[J \times K_C(R^n), K_C(R^n)\ ]$ *and either:*

- $D[F(t,U),\theta] \le g(t, D[U,\theta])$ *or*
- $lim_{h \to 0^+} sup \frac{1}{h}\Big[D\big[U + hF\big(t,U(t)\big),\theta\big] - D[U(t),\theta]\Big] \le g(t, D[U(t),\theta])\ ; t \in J$

where $g \in C[J \times R_+, R_+]$, *then if* $D[U_0,\theta] \le w_0$ *, we have:*

$$D[U(t),\theta] \le r(t,t_0,w_0), \qquad t \in J \tag{3.2.38}$$

where $r(t,t_0,w_0)$ *is the maximal solution of (3.2.7) on* J.

Corollary 3.2.2: The function $g(t,w) = \lambda(t)w, \lambda(t) \ge 0$, *and continuous is admissible in Theorem (3.2.3) to give:*

$$m(t) \le m(t_0) + \int_{t_0}^{t} \lambda(s)m(s)ds\ ; t \in J \tag{3.2.39}$$

then the Grownwall's inequality implies:

$$m(t) \le m(t_0) exp\left[\int_{t_0}^{t} \lambda(s)ds\ \right], \qquad t \in J \tag{3.2.40}$$

wich shows that (3.2.23) reduces to:

$$D[U(t),V(t)] \le D[U_0\ , V_0] exp\left[\int_{t_0}^{t} \lambda(s)ds\right], \qquad t \in J \tag{3.2.41}$$

Corollary 3.2.3: The function $g(t,w) = -\lambda(t)w$, $\lambda(t) \ge 0$, *is also admissible in Theorem (3.2.5) to give:*

$$D[U(t),V(t)] \le D[U_0\ , V_0] exp\left[-\int_{t_0}^{t} \lambda(s)ds\right], \qquad t \in J \tag{3.2.42}$$

if $\lambda(t) > 0$, *we find that:*

$$D[U(t),V(t)] \le D[U_0\ , V_0] e^{-\lambda(t-t_0)}, \qquad t \in J \tag{3.2.43}$$

if $J = [t_0\ , \infty]$, *we see that:*

$$lim_{t \to \infty} D[U(t),V(t)] = 0 \tag{3.2.44}$$

showing the advantage of Theorem (3.2.5).

3.3 Local Existence and Uniqueness

We shall begin by proving the existence and uniqueness result under assumptions more general than the Lipchitz type condition, which exhibits the idea of the comparison principles.

Theorem 3.3.1: Assume that:

i) $F \in C[R_0, K_C(R^n)]$ *and* $D[F(t,U(t)),\theta] \leq M_0$ *where:*

$R_0 = J \times B(U_0,b)$, $B(U_0,b) = [U \in K_C(R^n) : D[U,U_0] \leq b]$ *on* R_0.

ii) $g \in C[J \times [0,2b], R_+]$, $G(t,w) \leq M_1$ *on* $J \times [0,2b]$, $G(t,0) \equiv 0$ *and* $G(t,w)$ *is nondecreasing in* w *for each* $t \in J$*. And* $w(t) \equiv 0$ *is the only solution of:*

$$w' = G(t,w), \qquad w(t_0) = 0 \quad on\ J \tag{3.3.1}$$

iii) $D[F(t,U),F(t,V)] \leq g(t,D[U,V])$ *on* R_0.

Then the successive approximation defined by:

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} F(s,U_n(s))ds, \qquad n = 0,1,2,\cdots \tag{3.3.2}$$

exists on $J_0 = [t_0, t+\eta]$, *where* $\eta = min\left\{a, \frac{b}{M}\right\}$, $M = max\{M_0, M_1\}$ *as continuous function and converge uniformly to the unique solution* $U(t)$ *of the IVP (3.1.1) on* J_0.

Proof 3.3.1: using the properties of Hausdorff metric, we get by induction:

$$\begin{aligned} D[U_{n+1}, U_0] &= D\left[U_0 + \int_{t_0}^{t} F(s,U_n(s))ds\ , U_0\right] \\ &= D\left[\int_{t_0}^{t} F(s,U(s))ds\ ,\theta\right] \\ &\leq \int_{t_0}^{t} D[F(s,U_n(s)),\theta\ ] \leq M_0\,(t-t_0) \leq M_0 a \leq b \end{aligned} \tag{3.3.3}$$

and as a consequently, the successive approximations $\{U_n(t)\}$ *are well defined on* J_0*. we shall next define the successive approximations of (3.1.1) as follows:*

$$w_0 = M(t - t_0)$$

$$w_{n+1}(t) = \int_{t_0}^{t} g\big(s, w_n(s)\big)ds\,, t \in J_0 \quad, n = 0, 1, 2, \cdots \tag{3.3.4}$$

an easy induction, in view of the monotone character of $g(t,w)$ in w, proves that $\{w_n(t)\}$ are well defined and:

$$0 \le w_{n+1}(t) \le w_n(t), \qquad t \in J_0 \tag{3.3.5}$$

since $|w'_n(t)| \le g\big(t, w_{n-1}(t)\big) \le M_1$, clearly the sequence $\{w_n\}$ is nonincreasing and uniformly bounded, lets prove that it is equicontinuous:

$$\forall \varepsilon > 0, \qquad \exists \delta = \delta(\varepsilon), \qquad \forall n \in N, \qquad \forall\, t_1, t_2 \in J_0, \qquad t_2 > t_1$$

$$|t_1 - t_2| \le a = \delta = \frac{\varepsilon}{M}$$

$$\begin{aligned}
|w_n(t_1) - w_n(t_2)| &= \left|\int_{t_0}^{t_1} g\big(s, w_{n-1}(s)\big)ds - \int_{t_0}^{t_2} g\big(s, w_{n-1}(s)\big)ds\right| \\
&= \left|\int_{t_0}^{t_2} g\big(s, w_{n-1}(s)\big)ds - \int_{t_0}^{t_1} g\big(s, w_{n-1}(s)\big)ds\right| \\
&= \left|\int_{t_0}^{t_2} g\big(s, w_{n-1}(s)\big)ds + \int_{t_1}^{t_0} g\big(s, w_{n-1}(s)\big)ds\right| \\
&= \left|\int_{t_1}^{t_2} g\big(s, w_{n-1}(s)\big)ds\right| \le (t_2 - t_1)M_1 \le aM = \varepsilon
\end{aligned} \tag{3.3.6}$$

we conclude by Ascoli-arzela theorem that $\{w_n(t)\}$ has a subsequence $\{w_{n_k}(t)\}$ that converges uniformly to $w(t)$, and monotonicity of the sequence $\{w_n(t)\}$ shows that:

$$lim_{n\to\infty}\, w_n(t) = w(t) \tag{3.3.7}$$

uniformly on J_0.We have $w' = g(t,w)$ and then:

$$\int_{t_0}^{t} w'(s)ds = \int_{t_0}^{t} g\big(s, w(s)\big)\, ds \tag{3.3.8}$$

$$w(t) = w_0 + \int_{t_0}^{t} g\big(s, w(s)\big)\, ds \tag{3.3.9}$$

and since $w(t)$ satisfies the equivalent Volterra integral equation of (3.3.1) then it is a solution of (3.3.1), then by condition (ii)

$$w(t) \equiv 0 \;\; on\, J_0$$

we observe that:

$$D[U_1(t), U_0(t)] \leq \int_{t_0}^{t} D[F(s, U_0(s))] \leq M(t - t_0) \equiv w_0(t) \qquad (3.3.10)$$

assume that for some $K > 1$, *we have:*

$$D[U_k(t), U_{k-1}(t)] \leq w_{k-1}(t) \quad on\, J_0 \qquad (3.3.11)$$

Since

$$D[U_{k+1}(t), U_k(t)] \leq \int_{t_0}^{t} D\big[F\big(s, U_k(s)\big), F(s, U_{k-1}(s))\big]ds \qquad (3.3.11)$$

using condition (iii) and the monotone character of $g(t, w)$, *we get:*

$$D[U_{k+1}(t), U_k(t)] \leq \int_{t_0}^{t} g(s, D[U_k(s), U_{k-1}(s)])ds \leq$$

$$\leq \int_{t_0}^{t} g\big(s, w_{k-1}(s)\big)ds = w_k(t) \qquad (3.3.12)$$

thus, by induction, the estimate:

$$D[U_{n+1}(t), U_n(t)] \leq w_n(t), \qquad t \in J_0 \qquad (3.3.13)$$

is true for all n.

Letting $u(t) = D[U_{n+1}(t), U_n(t)]$, $t \in J_0$, *the proof of Theorem (3.2.4) shows that:*

$$D^+u(t) \leq g(t, D[U_n(t), U_{n-1}(t)]) \leq g\big(t, w_{n-1}(t)\big) \ ; t \in J_0 \qquad (3.3.14)$$

now let $n \leq m$, *setting* $v(t) = D[U_n(t), U_m(t)]$ *we obtain from (3.2.4):*

$$\begin{aligned} D^+v(t) \leq D[D_H U_n(t), D_H U_m(t)] &= D\big[F\big(t, U_{n-1}(t)\big), F(t, U_{m-1}(t)\big] \\ &\leq D\big[F\big(t, U_n(t)\big), F\big(t, U_{n-1}(t)\big)\big] \\ &+ D[F(t, U_n(t), F(t, U_m(t)] \\ &+ D\big[F\big(t, U_m(t)\big), F\big(t, U_{m-1}(t)\big)\big] \\ &\leq g\big(t, w_{n-1}(t)\big) + g\big(t, w_{m-1}(t)\big) \\ &+ g(t, D[U_n(t), U_m(t)]) \\ &\leq g\big(t, v(t)\big) + 2g\big(t, w_{n-1}(t)\big), \quad t \in J_0 \end{aligned} \qquad (3.3.15)$$

here we have used the argument of the proof of the Theorem (3.2.4) the monotone character of $g(t,w)$ *and the fact that* $w_{m-1} \leq w_{n-1}$ *since* $n \leq m$ *and* $w_n(t)$ *is a decreasing sequence. The comparison Theorem (3.2.1) yields the estimate:*

$$v(t) \leq r_n(t), \qquad t \in J_0 \tag{3.3.16}$$

where, $r_n(t)$ *is the maximal solution of the equation:*

$$r_n' = g\big(t, r_n(t)\big) + 2g\big(t, w_{n-1}(t)\big), \qquad r_n(t_0) = 0 \tag{3.3.17}$$

for each n, *and since as* $n \to \infty$, $2g\big(t, w_{n-1}(t)\big) \to 0$ *uniformly on* J_0, *it follows by lemma that* $r_n(t) \to 0$ *as* $n \to \infty$ *uniformly, on* J_0. *This implies from (3.3.1) and the definition of* $v(t)$ *that* $U_n(t)$ *converges uniformly to* $U(t)$, *according to Cauchy's criterion and clearly* $U(t)$ *is a solution of (3.1.1). To show uniqueness, let* $V(t)$ *be another solution of (3.1.1), on* J_0 *Then setting* $m(t) = D[U(t), V(t)]$ *and noting that* $m(t_0) = 0$, *we get, as before* $D^+m(t) \leq g\big(t, m(t)\big)$ $t \in J_0$, *and* $m(t) \leq r(t, t_0, 0), t \in J_0$, *by Theorem (3.2.3). By assumption* $r(t, t_0, 0) \equiv 0$, *we get* $U(t) \equiv V(t)$ *on* J_0. *Proving the theorem.* ■

We shall discuss, in the next result, the continuous dependence of solutions with initial values. we need the following lemma before we proceed.

Lemma 3.3.1: Let $F \in C[J \times K_C(R^n), K_C(R^n)]$ *and let:*

$$G(t,r) = max\big\{D\big[F\big(t, U(t)\big), \theta\big] : D[U(t), U_0(t)] \leq r\big\} \tag{3.3.18}$$

assume that $r^*(t, t_0, 0)$ *is the maximal solution of*

$$w' = G(t, w), \qquad w(t_0) = 0 \ \text{ on } J \tag{3.3.19}$$

let $U(t) = U(t, t_0, U_0)$ *be the solution of (3.1.1) then:*

$$D[U(t), U_0] \leq r^*(t, t_0, 0) \tag{3.3.20}$$

Proof 3.3.1: Define $m(t) = D[U(t), U_0]$, $t \in J$. *Then Theorem (3.2.4) shows that:*

$$D^+m(t) \leq D[D_H U(t), \theta] = D\big[F\big(t, U(t)\big), \theta\big]$$

$$\leq max_{D[U,U_0]\leq m(t)} D\big[F\big(t, U(t)\big), \theta\big] = G\big(t, m(t)\big) \tag{3.3.21}$$

this implies by Theorem (3.2.1) that:

$$D[U(t), U_0] \leq r^*(t, t_0, 0), \qquad t \in J \tag{3.3.22}$$

proving the lemma. ■

Theorem 3.3.2: Under the same assumptions of Theorem (3.3.1), Assume further that the solution $w(t, t_0, w_0)$ of (3.3.1) through every point (t_0, w_0) are continuous with respect to (t_0, w_0). Then the solutions $U(t) = U(t, t_0, U_0)$ of (3.1.1) are continuous relative to (t_0, U_0).

Proof 3.3.2: Let $U(t) = U(t, t_0, U_0)$ and $V(t) = V(t, t_0, V_0)$ be two solutions of (3.1.1). then defining $m(t) = D[U(t), V(t)]$, we get from Theorem (3.2.3) the estimate:

$$D[U(t), V(t)] \leq r(t, t_0, D[U_0, V_0]\,), \qquad t \in J \tag{3.3.23}$$

since

$$lim_{U_0 \to V_0} D[U(t), V(t)] \leq \ lim_{U_0 \to V_0} r(t, t_0, d[U_0, V_0]) \tag{3.3.24}$$

uniformly on J and by hypothesis $r(t, t_0, 0) \equiv 0$, it follows that:

$$0 \leq lim_{U_0 \to V_0} D[U(t), V(t)] \leq 0 \tag{3.3.25}$$

and then we can write:

$$lim_{U_0 \to V_0} U(t, t_0, U_0) = V(t, t_0, V_0) \tag{3.3.26}$$

uniformly, and hence the continuity of $U(t, t_0, U_0)$ relative to U_0 is valid. To prove the continuity relative to t_0, we let $U(t) = U(t, t_0, U_0)$ and $V(t) = V(t, \tau_0, U_0)$ be two solutions of (3.1.1). And let $\tau_0 > t_0$, as before setting:

$m(t_0) = d[U(t_0), V(t_0)]$, we obtain from lemma (3.3.1) that:

$$m(\tau_0) \leq r^*(\tau_0, t_0, 0) \tag{3.3.27}$$

and consequently, by Theorem (3.2.3) we arrive at:

$$m(t) \leq \check{r}(t), \qquad t \geq \tau_0 \tag{3.3.28}$$

where $\check{r}(t) = \check{r}\big(t, \tau_0, r^*(\tau_0, t_0, 0)\big)$ *is the maximal solution of (3.3.1) through* $(\tau_0, r^*(\tau_0, t_0, 0))$, *since* $r^*(t_0, t_0, 0) \equiv 0$, *we have :*

$$lim_{\tau_0 \to t_0} \check{r}\big(t, \tau_0, r^*(\tau_0, t_0, 0)\big) = \check{r}(t, t_0, 0) \tag{3.3.29}$$

uniformly on J, *by hypothesis* $\check{r}(t, t_0, 0) = 0$, *which proves the continuity of* $U(t, t_0, U_0)$ *with respect to* t_0 *and so the proof is complete.*■

3.4 Global existence:

We consider the set differential equation:

$$D_H U(t) = F\big(t, U(t)\big), \qquad U(t_0) = U_0 \in K_C(R^n) \tag{3.4.1}$$

where $F \in C[R_+ \times K_C(R^n), K_C(R^n)]$, in this section, we shall investigate the existence of solutions for $t \geq t_0$, assuming local existence, we shall prove the following global existence result.

Theorem 3.4.1: Assume that $F \in C[R_+ \times K_C(R^n), K_C(R^n)]$ *and:*

$$D[F(t, U), \theta\,] \leq G(t, D[U, \theta]), \qquad (t, U) \in R_+ \times K_C(R^n) \tag{3.4.2}$$

where $g \in C[R_+^2, R_+]$, $g(t, w)$ *is nondecreasing in* w *for each* $t \in R_+$ *and the maximal solution* $r(t, t_0, w_0)$ *of (3.3.1) exists on* $[t_0, \infty)$ *suppose further that* F *is smooth enough to guarantee local existence of solutions of (3.4.1), for any* $(t_0, U_0) \in R_+ \times K_C(R^n)$, *then the largest interval of existence of any solution* $U(t, t_0, U_0)$ *of (3.4.1) such that* $D[U_0, \theta] \leq U_0$ *is* $[t_0, \infty)$.

Proof 3.4.1: Let $U(t) = U(t, t_0, U_0)$ *be any solution of (3.4.1) with* $[U_0, \theta] \leq w_0$, *which exists on* $[t_0, \beta]$,$t_0 < \beta < \infty$ *and the value of* β *cannot be increased.*

Define:

$$m(t) = \; D[U_0, \theta] \tag{3.4.3}$$

then the Corollary (3.2.1) shows that:

$$m(t) \leq r(t, t_0, D[U\ ,\theta]), \qquad t_0 \leq t \leq \beta \tag{3.4.4}$$

for any t_1, t_2 such that $t_0 < t_1 < t_2 < \beta$, we have:

$$D[U(t_1), U(t_2)] = D\left[U_0 + \int_{t_0}^{t_1} F\big(s, U(s)\big)ds\ , U_0 + \int_{t_0}^{t_2} F\big(s, U(s)\big)ds\right]$$

$$= D\left[\theta, -\int_{t_0}^{t_1} F\big(s, U(s)\big)ds + \int_{t_0}^{t_2} F\big(s, U(s)\big)ds\right]$$

$$= D\left[\int_{t_1}^{t_2} F\big(s, U(s)\big)ds\ , \theta\right]$$

$$\leq \int_{t_1}^{t_2} D\big[F\big(s, U(s)\big), \theta\big]ds \leq \int_{t_1}^{t_2} g(s, D[U(s), \theta])ds \tag{3.4.5}$$

the relation (3.4.4) and the nondecreasing nature of $g(t, w)$ yields:

$$D[U(t_1), U(t_2)] \leq \int_{t_1}^{t_2} g(s, r(t, t_0, w_0)ds$$

$$= r(t_2, t_0, w_0) - r(t_1, t_0, w_0) \tag{3.4.6}$$

since $lim_{t\to\beta}\, r(t, t_0, w_0)$ exists and finite by hypothesis, taking the limit as $t_1, t_2 \to \beta$, and using the Cauchy criterion for convergence, it follows from (3.4.6) that $lim_{t\to\beta}\, U(t, t_0, U_0)$ exists. We define:

$$U(\beta, t_0, U_0) = lim_{t\to\beta^-}\, U(t, t_0, U_0) \tag{3.4.7}$$

and consider the initial value problem

$$D_H U(t) = F\big(t, U(t)\big), \quad U(\beta) = U(\beta, t_0, U_0). \tag{3.4.8}$$

by the assumed local existence, we see that $U(t, t_0, U_0)$ can be continued beyond β, contradicting our assumption that β cannot be continued, hence every solution $U(t, t_0, U_0)$ of (3.4.1) such that $D[U_0, \theta] \leq w_0$ exists globally on $[t_0, \infty)$, and the proof is complete. ■

4. Monotone Iterative Technique

4.1. Monotone Iterative Technique for Single Function

Theorem 4.1.1: Let $F \in C[J \times K_C(R^n), K_C(R^n)]$, *and* $V, W \in C^1[J, K_C(R^n)]$ *be lower and upper solutions of*

$$D_H U = F(t, U), \qquad U(0) = U_0 \in K_C(R^n) \tag{4.1.1}$$

such that $V \leq W$ *on* J. *Where* $J = [0, T]$ *Suppose furthermore that:*

$$F(t, X) - F(t, Y) \geq -M(X - Y) \tag{4.1.2}$$

for $V \leq Y \leq X \leq W$ *and* $M \geq 0$. *Then, there exist monotone sequences* $\{W_n\}, \{V_n\}$ *in* $K_C(R^n)$ *such that:*

$$V_n \to \rho(t), \qquad W_n \to R(t) \;\; in \;\; K_C(R^n)$$

as $n \to \infty$, *and* (ρ, R) *are coupled minimal and maximal solutions of (4.1.1) respectively.*

Proof 4.1.1: We set $V_0 = V$, $W_0 = W$, *for any* $\eta \in C^1[J, K_C(R^n)]$ *such that* $V_0 \leq \eta \leq W_0$ *and consider the linear differential equation:*

$$D_H U = F(t, \eta) - M(U - \eta), \qquad U(0) = U_0 \tag{4.1.3}$$

from (4.1.2) we obtain:

$$F(t, X) \geq F(t, Y) - M(X - Y). \tag{4.1.4}$$

Choose $Y = \eta$ *and* $X = U$,

$$D_H U = F(t, U) \geq F(t, \eta) - M(U - \eta) \tag{4.1.5}$$

it is clear that for every such η, *there exists a unique solution* U *of the equation (4.1.3) on* J. *Define a mapping* A *by* $A\eta = U$, *this mapping will be used to define sequences* $\{W_n\}$, $\{V_n\}$, *lets prove that:*

i) $V_0 \leq AV_0$

ii) *A is monotone operator on the segment:*

$$[V_0, W_0] = \{U \in C^1[J, K_C(R^n)],\ V_0 \leq U \leq W_0; t \in J\}$$

to prove (i), set $AV_0 = V_1$ where V_1 is the unique solution of (4.1.3) with $\eta = V_0$ in (4.1.3) set $\eta = V_0, U = V_1$

$$D_H V_1 = F(t, V_0) - M(V_1 - V_0), \qquad V_1(0) = U_0 \tag{4.1.6}$$

setting: $\varphi = V_0 - V_1$, we see that:

$$D_H\varphi = D_H V_0 - D_H V_1 = D_H V_0 - F(t, V_0) + M(V_1 - V_0) \tag{4.1.7}$$

so, we can obtain:

$$D_H\varphi \leq F(t, V_0) - F(t, V_0) + M(V_1 - V_0) = M(V_1 - V_0) = -M\varphi \tag{4.1.8}$$

and we have $\varphi(0) = V_0(0) - V_1(0) \leq U_0 - U_0 = 0$, since $V_0(0) \leq U_0$, we obtain the differential inequality:

$$D_H\varphi \leq -M\varphi, \qquad \varphi(0) \leq 0 \tag{4.1.9}$$

and so, we obtain that $V_0 \leq AV_0$. Similarly, we can prove $W_0 \geq AW_0$. To prove (ii), let $\eta_1, \eta_2 \in [V_0, W_0]$ such that $\eta_1 \leq \eta_2$, suppose that: $U_1 = A\eta_1, U_2 = A\eta_2$. We shall show that $A\eta_1 \leq A\eta_2$, set $\varphi = U_1 - U_2 \Rightarrow D_H\varphi = D_H U_1 - D_H U_2$ from (4.1.3) we can write:

$$U_1 = A\eta_1 \implies D_H U_1 = F(t, \eta_1) - M(U_1 - \eta_1), \qquad U_1(0) = U_0 \tag{4.1.10}$$
$$U_2 = A\eta_2 \implies D_H U_2 = F(t, \eta_2) - M(U_2 - \eta_2), \qquad U_2(0) = U_0 \tag{4.1.11}$$

$$\begin{aligned} D_H\varphi &= D_H U_1 - D_H U_2 = \\ &= F(t, \eta_1) - M(U_1 - \eta_1) - F(t, \eta_2) + M(U_2 - \eta_2) \\ &\leq M(\eta_2 - \eta_1) - M(U_1 - \eta_1) + M(U_2 - \eta_2) \end{aligned}$$
$$D_H\varphi \leq -M(U_1 - U_2) = -M\varphi \tag{4.1.12}$$

With $\varphi(0) = U_2(0) - U_1(0) = U_0 - U_0 = 0$, we obtain the differential inequality:

$$D_H\varphi \leq -M\varphi, \qquad \varphi(0) = 0 \tag{4.1.13}$$

we arrive at $\varphi \leq 0 \Rightarrow U_1 \leq U_2 \Rightarrow A\eta_1 \leq A\eta_2$ whenever $\eta_1 \leq \eta_2$, this complete the proof of (ii). We can now define the sequences V_n, W_n as follows:

$$V_n = AV_{n-1}, \qquad W_n = AW_{n-1} \tag{4.1.14}$$

and we conclude from the previous arguments that:

$$V_0 \le AV_0 = V_1 \le AV_1 = V_2 \le \cdots \le V_n \le \cdots \le W_n = AW_{n-1} \le W_{n-1}$$
$$= AW_{n-2} \le W_{n-2} \le \cdots \le W_2 \le W_1 \le W_0. \quad \text{(4.1.15)}$$

on J. It's easy to show that ρ, R are solutions of the (4.1.1) on J, since $\{W_n\}, \{V_n\}$ are solutions of (4.1.1) where

$$D_H V_n = F(t, V_{n-1}) - M(V_n - V_{n-1}), \qquad V_n(0) = U_0 \quad \text{(4.1.16)}$$
$$D_H W_n = F(t, W_{n-1}) - M(W_n - W_{n-1}), \qquad W_n(0) = U_0 \quad \text{(4.1.17)}$$

by integrating both sides with respect to t on the interval $[0, T]$,

$$\int_0^t D_H V_n ds = \int_0^t F(s, V_{n-1})\, ds \ - M \int_0^t (V_n(s) - V_{n-1}(s))ds \quad \text{(4.1.18)}$$

$$V_n(t) - V_n(0) = \int_0^t F(s, V_{n-1})\, ds \ - M \int_0^t (V_n(s) - V_{n-1}(s))ds \quad \text{(4.1.19)}$$

$$V_n(t) = U_0 + \int_0^t F(s, V_{n-1})\, ds \ - M \int_0^t (V_n(s) - V_{n-1}(s))ds \quad \text{(4.1.20)}$$

taking the limit as $n \to \infty$

$$\rho(t) = U_0 + \int_0^t F(s, \rho(s))\, ds \quad \text{(4.1.21)}$$

if $\rho(t)$ is the solution of equivalently Volterra integral equation, it is also the solution of the corresponding IVP. In the same way we can prove that R is a solution of the IVP. To prove that ρ, R are respectively minimal and maximal solutions of (4.1.1), we must show that if U any solution of (4.1.1) such that $V_0 \le U \le W_0$ then, $V_0 \le \rho \le U \le R \le W_0$, to do this, suppose that for some n

$$V_n \le U \le W_n \ \ on\, J \quad \text{(4.1.22)}$$

and set $\varphi = V_{n+1} - U$ so that:

$$D_H\varphi = D_H V_{n+1} - D_H U =$$
$$= F(t, V_n) - M(V_{n+1} - V_n) - F(t, U) \le$$
$$\le M(U - V_n) - M(V_{n+1} - V_n) = -M\varphi \quad \text{(4.1.23)}$$

with $\varphi(0) = 0$, *using the comparison results we arrive at* $\varphi \leq 0$, $U \geq V_{n+1}$, *in the same way, we can show that* $U \geq W_{n+1}$, *and as a result* $V_{n+1} \leq U \leq W_{n+1}$ *on* J, *since* $V_0 \leq U \leq W_0$, *on* J *and this prove by mathematical induction that :*

$$V_n \leq U \leq W_n \text{ on } J, \quad \text{for all } n$$

taking the limit as $n \to \infty$, *we conclude that* $\rho \leq U \leq R$, *and the proof is complete.*■

Corollary 4.1.1: In addition to the assumptions of Theorem (4.1.1), we assume:

$$F(t, X) - F(t, Y) \leq M(X - Y) \tag{4.1.24}$$

where $V(t) \leq Y \leq X \leq W(t)$, *and* $M > 0$, *then we have unique solution of (4.1.1) such that* $R = U = \rho$.

Proof 4.1.1: If we set $\varphi = R - \rho$, *then*

$$D_H\varphi = D_H R - D_H\rho = F(t, R) - F(t, \rho) \leq M(R - \rho),$$

which gives

$$D_H\varphi \leq M\varphi, \quad \varphi(0) = 0 \tag{4.1.25}$$

hence, we get $\varphi(t) \leq 0$ *on* J *which implies that* $R \leq \rho$, *we have already that* $\rho \leq R$, *so we obtain that, so we obtain that* $R = U = \rho$, *is the unique solution of (4.1.1). And the proof is complete.*■

4.2. Monotone Iterative Technique for Sum Two Functions:

To develop the monotone iterative technique, we shall consider the following set differential equation:

$$D_H U = F(t, U) + G(t, U), \qquad U(0) = U_0 \in K_C(R^n) \tag{4.2.1}$$

where $F, G \in C[J \times K_C(R^n),\ K_C(R^n)]$, and $J = [0, T]$.

Definition 4.2.1: Let $V, W \in C^1[R_+, K_C(R^n)]$, *then* V, W *are said to be:*

i) *Natural lower and upper solutions of (4.2.1) if:*

$$D_H V \leq F(t, V) + G(t, V),\ D_H W \geq F(t, W) + G(t, W),\ t \in J \qquad (4.2.2)$$

ii) Coupled lower and upper solutions of type I of (4.2.1) if:

$$D_H V \leq F(t, V) + G(t, W),\ D_H W \geq F(t, W) + G(t, V),\ t \in J \qquad (4.2.3)$$

iii) Coupled lower and upper solutions of type II of (4.2.1) if:

$$D_H V \leq F(t, W) + G(t, V),\ D_H W \geq F(t, V) + G(t, W),\ t \in J \qquad (4.2.4)$$

iv) Coupled lower and upper solutions of type III of (4.2.1) if:

$$D_H V \leq F(t, W) + G(t, W),\ D_H W \geq F(t, V) + G(t, V),\ t \in J \qquad (4.2.5)$$

we observe that whenever we have $V(t) \leq W(t), t \in J$.

Theorem 4.2.1: Assume that:

1. *$V, W \in C^1[J, K_C(R^n)]$ are coupled lower and upper solutions of type I relative to (4.2.1) with $V(t) \leq W(t), \quad t \in J$.*
2. *$F, G \in C[J \times K_C(R^n), K_C(R^n)]$, $F(t, X)$ is nondecreasing in X, and $G(t, Y)$ is nonincreasing in Y, for each $t \in J$.*
3. *F, G map bounded sets into bounded sets in $K_C(R^n)$.*

Then there exist monotone sequences $\{W_n\}, \{V_n\}$ in $K_C(R^n)$ such that:

$$V_n \to \rho(t), \qquad W_n \to R(t) \ \text{in} \ K_C(R^n)$$

and (ρ, R) are coupled minimal and maximal solutions of (4.2.1) respectively, that is they satisfy:

$$D_H \rho(t) = F(t, \rho) + G(t, R), \qquad \rho(0) = U_0 \ \ on\, J \qquad (4.2.6)$$

$$D_H R(t) = F(t, R) + G(t, \rho), \qquad R(0) = U_0 \ \ on\, J \qquad (4.2.7)$$

Proof:

For each $n \geq 0$, define the unique solutions $V_{n+1}(t), W_{n+1}(t)$ by:

$$D_H V_{n+1}(t) = F(t, V_n) + G(t, W_n), \quad V_{n+1}(0) = U_0 \ \ on\, J \qquad (4.2.8)$$

$$D_H W_{n+1}(t) = F(t, W_n) + G(t, V_n), \quad W_{n+1}(0) = U_0 \ \ on\, J \qquad (4.2.9)$$

where $V(0) \leq U_0 \leq W(0)$, we set $V_0 = V, W_0 = W$, our aim to prove:

$$V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_2 \leq W_1 \leq W_0 \qquad (4.2.10)$$

we have using the fact that $V_0 \le W_0$*, and the nondecreasing character of* F*:*

$$D_H V_0(t) \le F(t, V_0) + G(t, W_0) \tag{4.2.11}$$

and, we have:

$$D_H V_1(t) = F(t, V_0) + G(t, W_0) \tag{4.2.12}$$

we can obtain that:

$$D_H V_0(t) \ \le D_H V_1(t) \tag{4.2.13}$$

consequently, according to Theorem (3.1.4) we arrive at $V_0(t) \le V_1(t)$*. A similar argument shows that* $W_1(t) \le W_0(t)$*. We next prove that* $V_1 \le W_1$ *on* J*. For this purpose, consider:*

$$D_H V_1(t) = F(t, V_0) + G(t, W_0) \tag{4.2.14}$$

$$D_H W_1(t) = F(t, W_0) + G(t, V_0) \tag{4.2.15}$$

And

$$V_1(0) = W_1(0) = U_0$$

since $V_0(t) \le W_0(t)$*, then:*

$$F(t, V_0) \le F(t, W_0), \qquad F(t, X) \text{ is monotone nondecreasing in } X.$$

$$G(t, V_0) \ge G(t, W_0), \qquad G(t, Y) \text{ is monotone nonincreasing in } Y.$$

so, we obtain:

$$D_H V_1(t) \le F(t, W_0) + G(t, W_0) \text{ on } J \tag{4.2.16}$$

$$D_H W_1(t) \ge F(t, W_0) + G(t, W_0) \text{ on } J \tag{4.2.17}$$

consequently, we arrive at:

$$D_H V_1(t) \le D_H W_1(t) \tag{4.2.18}$$

and then according to Theorem (3.1.4), we arrive at:

$$V_1(t) \le W_1(t) \text{ on } J \tag{4.2.19}$$

and as a result, we obtain:

$$V_0 \le V_1 \le W_1 \le W_0 \tag{4.2.20}$$

assume that for some $j > 1$*, we have:*

$$V_{j-1} \le V_j \le W_j \le W_{j-1} \quad on\ J \tag{4.2.21}$$

then we show that:

$$V_j \le V_{j+1} \le W_{j+1} \le W_j \quad on\ J \tag{4.2.22}$$

to do this, we have

$$D_H V_j(t) = F(t, V_{j-1}) + G(t, W_{j-1}), \quad V_j(0) = U_0 \ \ on\ J \tag{4.2.23}$$

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, W_j), \quad V_{j+1}(0) = U_0 \ \ on\ J \tag{4.2.24}$$

so, we can write:

$$D_H V_j(t) = F(t, V_{j-1}) + G(t, W_{j-1}) \le F(t, V_j) + G(t, W_j) = D_H V_{j+1}(t) \tag{4.2.25}$$

consequently, $V_j(t) \le V_{j+1}(t)$ *on* J*, in the same way we arrive at* $W_{j+1} \le W_j$ *on* J*. Next, we show that* $V_{j+1} \le W_{j+1}$*,* $t \in J$*, we have:*

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, W_j), \quad V_{j+1}(0) = U_0 \ \ on\ J \tag{4.2.26}$$

$$D_H W_{j+1}(t) = F(t, W_j) + G(t, V_j), \quad W_{j+1}(0) = U_0 \ \ on\ J \tag{4.2.27}$$

then we can write:

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, W_j) \le F(t, W_j) + G(t, W_j) \tag{4.2.28}$$

$$D_H W_{j+1}(t) = F(t, W_j) + G(t, V_j) \ge F(t, W_j) + G(t, W_j) \tag{4.2.29}$$

and as a result:

$$V_{j+1}(t) \le W_{j+1}(t) \quad on\ J \tag{4.2.30}$$

hence (4.2.22) follows and consequently, by induction (4.2.10) is valid for all n*. Clearly sequences* $\{W_n\}, \{V_n\}$ *are uniformly bounded on. To Show that they are equicontinuous, consider for any* $t_1 < t_2$ *where* $t_1, t_2 \in J$*.*

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon), \quad \forall n \in N, \quad |t_2 - t_1| < T = \delta = \frac{\varepsilon}{M}$$

$D[V_n(t_2), V_n(t_1)] =$

$$= D\left[U_0 + \int_0^{t_2}\{F(s, V_{n-1}(s)) + G(s, W_{n-1}(s))\}ds, U_0 + \int_0^{t_1}\{F(s, V_{n-1}(s)) + G(s, W_{n-1}(s))\}ds\right]$$

$$\leq \int_{t_1}^{t_2} D[\{F(s, V_{n-1}(s)) + G(s, W_{n-1}(s))\}, \theta]ds$$

$$\leq M|t_2 - t_1| < MT = \varepsilon \qquad (4.2.31)$$

we used here the properties of integral and the metric D, together with the fact that $F + G$ are bounded since $\{W_n\}, \{V_n\}$ are uniformly bounded, hence $\{V_n\}$ is equicontinuous on J, Ascoli's theorem gives a subsequence $\{V_{n_k}\}$ which converges uniformly to $\rho(t) \in K_C(R^n)$, and since $\{V_n\}$ is monotone nondecreasing sequence, the entire sequence $\{V_n\}$ converges uniformly to $\rho(t)$ on J. By the same way we can show that the sequence $\{W_n\}$ converges uniformly to $R(t)$ on J, it therefore follows, using the integral representation of (4.2.8) and (4.2.9) that $\rho(t)$ and $R(t)$ satisfy:

$$D_H\rho(t) = F(t, \rho) + G(t, R), \qquad \rho(0) = U_0 \ \ on\ J \qquad (4.2.32)$$

$$D_H R(t) = F(t, R) + G(t, \rho), \qquad R(0) = U_0 \ \ on\ J \qquad (4.2.33)$$

and that

$$V_0 \leq \rho \leq R \leq W_0 \qquad (4.2.34)$$

we next claim that $(\rho, R)$are coupled minimal and maximal solutions of (4.2.1), that is, if $U(t)$ is any solution of (4.2.1) such that:

$$V_0 \leq U(t) \leq W_0 \qquad (4.2.35)$$

then

$$V_0 \leq \rho \leq U(t) \leq R \leq W_0 \qquad t \in J \qquad (4.2.36)$$

suppose that for some n,

$$V_n \leq U(t) \leq W_n \qquad t \in J \qquad (4.2.37)$$

then we have using the monotone nature of F and G and (4.2.37):

$$D_H U = F(t, U) + G(t, U) \geq F(t, V_n) + G(t, W_n), \qquad U(0) = U_0 \qquad (4.2.38)$$

$$D_H V_{n+1} = F(t, V_n) + G(t, W_n), \qquad V_{n+1}(0) = U_0 \qquad (4.2.39)$$

consequently, we arrive at:

$$D_H U \geq D_H V_{n+1} \qquad (4.2.40)$$

according to Theorem (3.1.4), we arrive at:

$$V_{n+1} \leq U \quad \text{on } J \qquad (4.2.41)$$

in the same way,

$$W_{n+1} \geq U \quad \text{on } J \qquad (4.2.42)$$

hence by induction the relation (4.2.37) is true for all $n \geq 1$, *taking the limit* $n \to \infty$, *we obtain (4.2.36) proving the claim. The proof is complete.*■

Corollary 4.2.1: If, in addition to the assumptions of Theorem (4.2.1), F *and* G *satisfy whenever* $X \geq Y, X, Y \in K_C(R^n)$

$$F(t, X) \leq F(t, Y) + N_1(X - Y) \qquad (4.2.43)$$

$$G(t, X) + N_2(X - Y) \geq G(t, Y) \qquad (4.2.44)$$

where $N_1, N_2 > 0$ *then* $\rho = R = U$ *is the unique solution of (4.2.1).*

Proof 4.2.1: Since $\rho \leq R$, *and then* $R = \rho + m$ *or* $m = R - \rho$, *now*

$$\begin{aligned} D_H\rho + D_H m = D_H R &= F(t, R) + G(t, \rho) \\ &\leq F(t, \rho) + N_1(R - \rho) + G(t, R) + N_2(R - \rho) \\ &= D_H\rho + (N_1 + N_2)m \qquad (4.2.45) \end{aligned}$$

which means,

$$D_H m \leq (N_1 + N_2)m, \qquad m(0) = 0 \qquad (4.2.46)$$

which leads by using the comparison results to $R \leq \rho$ *on* J, *proving the uniqueness of* $\rho = R = U$. *And the proof is complete.*■

Theorem 4.2.2: Assume that (i) and (ii) of the Theorem (4.2.1) hold, then for any solution of (4.2.1) $U(t)$ *with* $V_0 \leq U(t) \leq W_0$ *on* J, *we have the iterates* $\{V_n\}, \{W_n\}$ *satisfying*

$$V_0 \leq V_2 \leq \cdots \leq V_{2n} \leq U(t) \leq V_{2n+1} \leq \cdots \leq V_3 \leq V_1 \ on\, J \tag{4.2.47}$$
$$W_1 \leq W_3 \leq \cdots \leq W_{2n+1} \leq U(t) \leq W_{2n} \leq \cdots \leq W_2 \leq W_0 \ on\, J \tag{4.2.48}$$

provided that $V_0 \leq V_2, W_2 \leq W_0$ *on* J, *where the iterative schemes are given by:*

$$D_H V_{n+1} = F(t, W_n) + G(t, V_n), \qquad V_{n+1}(0) = U_0 \tag{4.2.49}$$
$$D_H W_{n+1} = F(t, V_n) + G(t, W_n), \qquad W_{n+1}(0) = U_0 \tag{4.2.50}$$

furthermore, the monotone sequences $\{V_{2n}\}, \{V_{2n+1}\}, \{W_{2n}\}, \{W_{2n+1}\}$ *in* $K_C(R^n)$ *converge to* ρ, R, R^*, ρ^* *in* $K_C(R^n)$ *respectively and verify*

$$D_H R = F(t, R^*) + G(t, \rho), \qquad R(0) = U_0 \tag{4.2.51}$$
$$D_H \rho = F(t, \rho^*) + G(t, R), \qquad \rho(0) = U_0 \tag{4.2.52}$$
$$D_H R^* = F(t, \rho) + G(t, R^*), \qquad R^*(0) = U_0 \tag{4.2.53}$$
$$D_H \rho^* = F(t, R) + G(t, \rho^*), \qquad \rho^*(0) = U_0 \ on\, J \tag{4.2.54}$$

Proof 4.2.2: We shall first show that V_0, W_0 *are coupled lower and upper solutions* V_0, W_0 *of the Type* II *of (4.2.1) exist on* J, *satisfying* $V_0 \leq W_0$ *on* J, *for this purpose, consider the IVP*

$$D_H Z = F(t, \theta) + G(t, \theta), \qquad Z(0) = U_0 \tag{4.2.55}$$

let $Z(t)$ *be the solution of (4.2.55) which exists on* J, *define* V_0, W_0 *by*

$$R_0 + V_0 = Z \ and \ W_0 = Z + R_0$$

where the positive vector $R_0 = (R_{01}, R_{02}, \ldots, R_{0n})$ *is chosen sufficiently large so that we have* $V_0 \leq \theta \leq W_0$ *on* J, *after that using the monotone character of* F *and* G, *we can write*

$$D_H V_0 = D_H Z = F(t, \theta) + G(t, \theta) \leq F(t, W_0) + G(t, V_0), \tag{4.2.56}$$

With $V_0(0) = Z(0) - R_0 \leq Z(0) = U_0$. *In the same way,* $D_H W_0 \geq F(t, V_0) + G(t, W_0)$, $W_0(0) \geq U_0$. *And as a result,* V_0, W_0 *are the coupled lower and upper solutions of type* II *of (4.2.1).*

Let $U(t)$ be any solution of (4.2.1) such that $V_0 \leq U(t) \leq W_0$ on J, we shall show that

$$V_0 \leq V_2 \leq U \leq V_3 \leq V_1$$
$$W_1 \leq W_3 \leq U \leq W_2 \leq W_0 \qquad (4.2.57)$$

on J, since $U(t)$ is a solution of (4.2.1), we have, using the monotone character of F and G, and the fact $V_0 \leq U(t) \leq W_0$,

$$D_H U = F(t,U) + G(t,U) \leq F(t,W_0) + G(t,V_0), \qquad U(0) = U_0 \qquad (4.2.58)$$

and V_1 satisfies

$$D_H V_1 = F(t,W_0) + G(t,V_0), \quad V_1(0) = U_0 \text{ on } J \qquad (4.2.60)$$

which yields that $U(t) \leq V_1$ on J, in the same way, $W_1 \leq U(t)$. After that, we show that $V_2 \leq U(t)$ on J, note that

$$D_H V_2 = F(t,W_1) + G(t,V_1), \qquad V_2(0) = U_0 \qquad (4.2.59)$$

and then because of the monotonicity of F and G, we get

$$D_H U = F(t,U) + G(t,U) \geq F(t,W_1) + G(t,V_1), U(0) = U_0 \text{ on } J \qquad (4.2.60)$$

we can write:

$$D_H V_2 \leq D_H U \qquad (4.2.61)$$

consequently, according to Theorem (3.1.4), we arrive at $V_2 \leq U(t)$, on J. A similar argument shows that $U \leq W_2$ on J, next we find utilizing the assumption $V_0 \leq V_2$, $W_2 \leq W_0$ on J and the monotonicity of F and G, we have:

$$D_H V_1 = F(t,W_0) + G(t,V_0), \quad V_1(0) = U_0 \text{ on } J \qquad (4.2.62)$$
$$D_H V_3 = F(t,W_2) + G(t,V_2) \leq F(t,W_0) + G(t,V_0), \qquad (4.2.63)$$

With $V_3(0) = U_0$ on J, we arrive at:

$$D_H V_3 \leq D_H V_1 \qquad (4.2.64)$$

which follows that $V_3(t) \leq V_1(t)$ on J, in the same way one can show that $W_1 \leq W_3$ on J, also, employing similar reasoning, one can prove that $U(t) \leq V_3$ and $W_3 \leq U(t)$ on J, proving the relations (4.2.27). Now assuming for some $n > 2$, the inequalities

$$V_{2n-4} \leq V_{2n-2} \leq U \leq V_{2n-1} \leq V_{2n-3},$$
$$W_{2n-3} \leq W_{2n-1} \leq U \leq W_{2n-2} \leq W_{2n-4}, \text{ on } J \qquad (4.2.65)$$

hold, it can be shown, employing similar arguments that

$$V_{2n-2} \leq V_{2n} \leq U \leq V_{2n+1} \leq V_{2n-1},$$
$$W_{2n-1} \leq W_{2n+1} \leq U \leq W_{2n} \leq W_{2n-2}, \text{ on } J \qquad (4.2.66)$$

thus, by induction (4.2.47) and (4.2.48) are valid for all $n = 0, 1, 2, \cdots,$ *Since* $\{V_n\}, \{W_n\} \in K_C(R^n)$ *for some* n, *employing a similar reasoning as in (4.2.1) we conclude:*

$$lim_{n\to\infty} V_{2n} = \rho, \quad lim_{n\to\infty} V_{2n+1} = R$$
$$lim_{n\to\infty} W_{n+1} = \rho^*, \quad lim_{n\to\infty} W_{2n} = R^* \qquad (4.2.67)$$

exist, in $K_C(R^n)$, *uniformly on* J, *it therefore follows by suitable use of the integral representation (4.2.49) and (4.2.50) that* $\rho,\ R,\ R^*, \rho^*$ *satisfy that corresponding set differential equation in (4.2.1) on* J, *also from (4.2.47) and (4.2.48), we obtain:*

$$\rho \leq U \leq R,\ \rho^* \leq U \leq R^* \text{on } J. \qquad (4.2.68)$$

The proof therefore is complete.■

Corollary 4.2.2: Under the assumptions of Theorem (4.2.2), if F *and* G *satisfy the assumptions of Corollary (4.2.1) then,* $\rho = \rho^* = R = R^* = U$, *is the unique solution of (4.2.1)*

Proof 4.2.2: Let $q_1 + \rho = R$, $q_2 + \rho^* = R^*$ *where* $q_1, q_2 \geq 0$ *on* J, *since* $\rho \leq R$ *and* $\rho^* \leq R^*$ *on* J, *then using the assumptions, we can write:*

$$D_H(q_1 + q_2) \leq (N_1 + N_2)(q_1 + q_2),\ q_1(0) + q_2(0) = 0 \text{ on } J \qquad (4.2.69)$$

So, using Theorem (3.1.4) implies that $q_1 + q_2 \leq 0$ *on* J, *and as a result, we get* $\rho = R = U$ *and* $\rho^* = R^* = U$ *on* J. *And this proves the corollary.*■

4.3 Monotone Iterative Technique with the other Types of Coupled Lower and Upper Solutions

We used in the Theorem (4.2.1) the first type of coupled lower and upper solutions, now we can study monotone iterative technique for the other types, by making some changes.

Theorem 4.3.1: Suppose that:

i) $V, W \in C^1[J, K_C(R^n)]$ *are coupled lower and upper solutions of natural type relative to (4.2.1) with* $V(t) \leq W(t), t \in J$

ii) $F, G \in C[J \times K_C(R^n), K_C(R^n)]$, $F(t, X)$ *is nondecreasing in* X, *and* $G(t, Y)$ *is nondecreasing in* Y, *for each* $t \in J$.

iii) F, G *map bounded sets into bounded sets in* $K_C(R^n)$.

Then we have two monotone sequences $\{W_n\}, \{V_n\}$ *in* $K_C(R^n)$ *such that:*

$$V_n \to \rho(t), \qquad W_n \to R(t) \ \ in \ \ K_C(R^n)$$

and (ρ, R) *are coupled minimal and maximal solutions of (4.2.1) respectively, and they are satisfying*

$$D_H \rho(t) = F(t, \rho) + G(t, \rho), \qquad \rho(0) = U_0 \ \ on\, J \qquad (4.3.1)$$

$$D_H R(t) = F(t, R) + G(t, R), \qquad R(0) = U_0 \ \ on\, J \qquad (4.3.2)$$

Proof 4.3.1: For each $n \geq 0$, *define the unique solutions* $V_{n+1}(t), W_{n+1}(t)$ *by the relations:*

$$D_H V_{n+1}(t) = F(t, V_n) + G(t, V_n), \qquad V_{n+1}(0) = U_0 \ \ on\, J \qquad (4.3.3)$$

$$D_H W_{n+1}(t) = F(t, W_n) + G(t, W_n), \quad W_{n+1}(0) = U_0 \ \ on\, J \qquad (4.3.4)$$

where $V(0) \leq U_0 \leq W(0)$, *we set* $V_0 = V, W_0 = W$, *our aim to prove:*

$$V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_2 \leq W_1 \leq W_0 \qquad (4.3.5)$$

we have using the fact that $V_0 \leq W_0$ *and the nondecreasing character of* F *:*

$$D_H V_0(t) \leq F(t, V_0) + G(t, V_0) \qquad (4.3.6)$$

and we can write from (4.3.3) by substituting $n = 0$

$$D_H V_1(t) = F(t, V_0) + G(t, V_0) \tag{4.3.7}$$

we can obtain that:

$$D_H V_0(t) \le D_H V_1(t) \tag{4.3.8}$$

consequently, according to Theorem (3.1.4) we arrive at $V_0(t) \le V_1(t)$. *A similar argument shows that* $W_1(t) \le W_0(t)$. *We next prove that* $V_1 \le W_1$ *on* J. *For this purpose, consider:*

$$D_H V_1(t) = F(t, V_0) + G(t, V_0) \tag{4.3.9}$$

$$D_H W_1(t) = F(t, W_0) + G(t, W_0) \tag{4.3.10}$$

And $V_1(0) = W_1(0) = U_0$. *since* $V_0(t) \le W_0(t)$ *then:*

$$F(t, V_0) \le F(t, W_0), \qquad F(t, X) \text{ is monotone nondecreasing in } X.$$

$$G(t, V_0) \le G(t, W_0), \qquad G(t, Y) \text{ is monotone nondecreasing in } Y.$$

so, we obtain:

$$D_H V_1(t) \le F(t, W_0) + G(t, W_0) \text{ on } J \tag{4.3.11}$$

$$D_H W_1(t) = F(t, W_0) + G(t, W_0) \text{ on } J \tag{4.3.12}$$

consequently, we arrive at:

$$D_H V_1(t) \le D_H W_1(t) \tag{4.3.13}$$

consequently, according to Theorem (3.1.4)

$$V_1(t) \le W_1(t) \text{ on } J \tag{4.3.14}$$

and as a result, we obtain:

$$V_0 \le V_1 \le W_1 \le W_0 \tag{4.3.15}$$

assume that for some $j > 1$, *we have:*

$$V_{j-1} \le V_j \le W_j \le W_{j-1} \quad \text{ on } J \tag{4.3.16}$$

then we show that:

$$V_j \le V_{j+1} \le W_{j+1} \le W_j \quad \text{ on } J \tag{4.3.17}$$

to do this, consider:

$$D_H V_j(t) = F(t, V_{j-1}) + G(t, V_{j-1}), \quad V_j(0) = U_0 \quad on\ J \tag{4.3.18}$$

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, V_j), \quad V_{j+1}(0) = U_0 \quad on\ J \tag{4.3.19}$$

so, we can write:

$$D_H V_j(t) = F(t, V_{j-1}) + G(t, V_{j-1}) \leq F(t, V_j) + G(t, V_j) = D_H V_{j+1}(t) \tag{4.3.20}$$

consequently, $V_j(t) \leq V_{j+1}(t)$ *on* J*, similarly we can get* $W_{j+1} \leq W_j$ *on* J*. Next, we show that* $V_{j+1} \leq W_{j+1}$*,* $t \in J$*, we have:*

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, V_j), \quad V_{j+1}(0) = U_0 \quad on\ J \tag{4.3.21}$$

$$D_H W_{j+1}(t) = F(t, W_j) + G(t, W_j), \quad W_{j+1}(0) = U_0 \quad on\ J \tag{4.3.22}$$

then we can write:

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, V_j) \leq F(t, W_j) + G(t, W_j) \tag{4.3.23}$$

$$D_H W_{j+1}(t) = F(t, W_j) + G(t, W_j) \tag{4.3.24}$$

and as a result:

$$D_H V_{j+1}(t) \leq D_H W_{j+1}(t) \quad on\ J \tag{4.3.25}$$

consequently, utilizing Theorem (3.1.4) we arrive at:

$$V_{j+1}(t) \leq W_{j+1}(t) \quad on\ J \tag{4.3.26}$$

hence the relation (4.3.17) follows, and consequently by induction the relation (4.3.5) is valid for all n*. Clearly sequences* $\{W_n\}, \{V_n\}$ *are uniformly bounded on. To Show that they are equicontinuous, consider for any* $t_1 < t_2$ *where* $t_1, t_2 \in J$

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon), \quad \forall n \in N, \quad |t_2 - t_1| < T = \delta = \frac{\varepsilon}{M}$$

$$D[V_n(t_2), V_n(t_1)] =$$

$$= D\left[U_0 + \int_0^{t_2} \{F(s, V_{n-1}(s)) + G(s, V_{n-1}(s))\}ds, U_0 + \int_0^{t_1} \{F(s, V_{n-1}(s)) + G(s, V_{n-1}(s))\}ds\right]$$

$$\leq \int_{t_1}^{t_2} D[\{F(s, V_{n-1}(s)) + G(s, V_{n-1}(s))\}, \theta]ds$$

$$\leq M|t_2 - t_1| < MT = \varepsilon \qquad (4.3.27)$$

here we utilized the properties of integral and the metric D, together with the fact that $F + G$ are bounded since $\{W_n\}, \{V_n\}$ are uniformly bounded, hence $\{V_n\}$ is equicontinuous on J, the corresponding Ascoli's theorem gives a subsequence $\{V_{nk}\}$ which converges uniformly to $\rho(t) \in K_C(R^n)$, and since $\{V_n\}$ is monotone nondecreasing sequence, the entire sequence $\{V_n\}$ converges uniformly to $\rho(t)$ on J. The same arguments apply to the sequence $\{W_n\}$ and $W_n \to R$ uniformly on J, it therefore follows, using the integral representation of (4.3.3) and (4.3.4) that $\rho(t)$ and $R(t)$ satisfy:

$$D_H\rho(t) = F(t, \rho) + G(t, \rho), \qquad \rho(0) = U_0 \quad on\ J \qquad (4.3.28)$$

$$D_H R(t) = F(t, R) + G(t, R), \qquad R(0) = U_0 \quad on\ J \qquad (4.3.29)$$

and that

$$V_0 \leq \rho \leq R \leq W_0 \qquad (4.3.30)$$

we next claim that (ρ, R) are coupled minimal and maximal solutions of (4.2.1), that is, if $U(t)$ is any solution of (4.2.1) such that:

$$V_0 \leq U(t) \leq W_0 \qquad (4.3.31)$$

then

$$V_0 \leq \rho \leq U(t) \leq R \leq W_0 \qquad t \in J \qquad (4.3.32)$$

suppose that for some n,

$$V_n \leq U(t) \leq W_n, \qquad t \in J \qquad (4.3.33)$$

then we have using the monotone nature of F and G and (4.3.33):

$$D_H U = F(t,U) + G(t,U) \geq F(t,V_n) + G(t,V_n), \qquad U(0) = U_0 \tag{4.3.34}$$

$$D_H V_{n+1} = F(t,V_n) + G(t,V_n), \quad V_{n+1}(0) = U_0 \tag{4.3.35}$$

hence

$$D_H U \geq D_H V_{n+1} \tag{4.3.36}$$

consequently, utilizing Theorem (3.1.4) we arrive at:

$$V_{n+1} \leq U \quad on\ J \tag{4.3.37}$$

in the same way we can show that,

$$W_{n+1} \geq U \quad on\ J \tag{4.3.38}$$

hence by induction the relation (4.3.33) is true for all $n \geq 1$, *taking the limit* $n \to \infty$, *we get (4.3.32) proving the claim. The proof is complete.*■

Corollary 4.3.1: If, in addition to the assumptions of Theorem (4.3.1), F *and* G *satisfy whenever* $X \geq Y,\ X, Y \in K_C(R^n)$.

$$F(t,X) \leq F(t,Y) + N_1(X - Y) \tag{4.3.39}$$

$$G(t,X) \leq G(t,Y) + N_2(X - Y) \tag{4.3.40}$$

where $N_1, N_2 > 0$ *then* $\rho = R = U$ *is the unique solution of (4.2.1).*

Proof 4.3.1: Since $\rho < R$, *and then* $R = \rho + m$ *or* $m = R - \rho$, *now*

$$\begin{aligned} D_H\rho + D_H m = D_H R &= F(t,R) + G(t,R) \\ &\leq F(t,\rho) + N_1(R - \rho) + G(t,\rho) + N_2(R - \rho) \\ &= D_H\rho + (N_1 + N_2)m \end{aligned} \tag{4.3.41}$$

which means,

$$D_H m \leq (N_1 + N_2)m, \qquad m(0) = 0 \tag{4.3.42}$$

using comparison results leads to $R \leq \rho\ \ on\ J$, *proving the uniqueness of* $\rho = R = U$. *Complete the proof.*■

Theorem 4.3.2: Assume that:

i) $V, W \in C^1[J, K_C(R^n)]$ *are coupled lower and upper solutions of type II relative to (4.2.1) with* $V(t) \le W(t), t \in J$

ii) $F, G \in C[J \times K_C(R^n), K_C(R^n)], F(t,X)$ *is nonincreasing in* X, *and* $G(t,Y)$ *is nondecreasing in* Y, *for each* $t \in J$.

iii) $F\ , G$ *map bounded sets into bounded sets in* $K_C(R^n)$.

Then there exist monotone sequences $\{W_n\}, \{V_n\}$ *in* $K_C(R^n)$ *such that:*

$$V_n \to \rho(t), \qquad W_n \to R(t) \ \ in \ \ K_C(R^n)$$

and (ρ, R) *are coupled minimal and maximal solutions of (4.2.1) respectively, that is they satisfy:*

$$D_H\rho(t) = F(t,R) + G(t,\rho), \qquad \rho(0) = U_0 \ \ on\, J \tag{4.3.43}$$

$$D_H R(t) = F(t,\rho) + G(t,R), \qquad R(0) = U_0 \ \ on\, J \tag{4.3.44}$$

Proof 4.3.2: For each $n \ge 0$, *define the unique solutions* $V_{n+1}(t), W_{n+1}(t)$ *by:*

$$D_H V_{n+1}(t) = F(t,W_n) + G(t,V_n), \qquad V_{n+1}(0) = U_0 \ \ on\, J \tag{4.3.45}$$

$$D_H W_{n+1}(t) = F(t,V_n) + G(t,W_n), \qquad W_{n+1}(0) = U_0 \ \ on\, J \tag{4.3.46}$$

where $V(0) \le U_0 \le W(0)$, *we set* $V_0 = V, W_0 = W$. *our aim to prove:*

$$V_0 \le V_1 \le V_2 \le \cdots \le V_n \le W_n \le \cdots \le W_2 \le W_1 \le W_0. \tag{4.3.47}$$

we have using the fact that: $V_0 \le W_0$ *and the nondecreasing character of:*

$$D_H V_0(t) \le F(t,W_0) + G(t,V_0) \tag{4.3.48}$$

and, we have:

$$D_H V_1(t) = F(t,W_0) + G(t,V_0) \tag{4.3.50}$$

we can obtain that:

$$D_H V_0(t) \ \le D_H V_1(t) \tag{4.3.51}$$

consequently, using Theorem (3.1.4), we arrive at $V_0(t) \ \le V_1(t)$. *A similar argument shows that* $W_1(t) \le W_0(t)$,*we next prove that* $V_1 \le W_1$ *on* J. *For this purpose, consider:*

$$D_H V_1(t) = F(t, W_0) + G(t, V_0) \qquad (4.3.52)$$

$$D_H W_1(t) = F(t, V_0) + G(t, W_0) \qquad (4.3.53)$$

With $V_1(0) = W_1(0) = U_0$. *since* $V_0(t) \le W_0(t)$ *then:*

$$F(t, W_0) \le F(t, V_0), \qquad F(t, X) \text{ is monotone nonincreasing in } X.$$

$$G(t, V_0) \le G(t, W_0), \qquad G(t, Y) \text{ is monotone nondecreasing in } Y.$$

so, we obtain:

$$D_H V_1(t) \le F(t, V_0) + G(t, W_0) \text{ on } J \qquad (4.3.55)$$

$$D_H W_1(t) = F(t, V_0) + G(t, W_0) \text{ on } J \qquad (4.3.56)$$

consequently, we arrive at:

$$D_H V_1(t) \le D_H W_1(t) \qquad (4.3.57)$$

consequently,

$$V_1(t) \le W_1(t) \text{ on } J \qquad (4.3.58)$$

and as a result, we obtain:

$$V_0 \le V_1 \le W_1 \le W_0 \qquad (4.3.59)$$

assume that for some $j > 1$, *we have:*

$$V_{j-1} \le V_j \le W_j \le W_{j-1} \quad \text{on } J \qquad (4.3.60)$$

then we show that:

$$V_j \le V_{j+1} \le W_{j+1} \le W_j \quad \text{on } J \qquad (4.3.61)$$

to do this, consider:

$$D_H V_j(t) = F\left(t, W_{j-1}\right) + G\left(t, V_{j-1}\right), \qquad V_j(0) = U_0 \ \text{ on } J \qquad (4.3.62)$$

$$D_H V_{j+1}(t) = F\left(t, W_j\right) + G\left(t, V_j\right), \qquad V_{j+1}(0) = U_0 \ \text{ on } J \qquad (4.3.63)$$

so, we can write:

$$D_H V_j(t) = F\left(t, W_{j-1}\right) + G\left(t, V_{j-1}\right) \le F\left(t, W_j\right) + G\left(t, V_j\right) = D_H V_{j+1}(t) \qquad (4.3.64)$$

consequently, $V_j(t) \le V_{j+1}(t)$ *on* J*, similarly we can get* $W_{j+1} \le W_j$ *on* J*. Next, we show that* $V_{j+1} \le W_{j+1}$*;* $t \in J$ *we have:*

$$D_H V_{j+1}(t) = F(t, W_j) + G(t, V_j), \qquad V_{j+1}(0) = U_0 \ \ on\, J \qquad (4.3.65)$$

$$D_H W_{j+1}(t) = F(t, V_j) + G(t, W_j), \qquad W_{j+1}(0) = U_0 \ \ on\, J \qquad (4.3.66)$$

then we can write:

$$D_H V_{j+1}(t) = F(t, W_j) + G(t, V_j) \le F(t, V_j) + G(t, W_j)$$
$$D_H W_{j+1}(t) = F(t, V_j) + G(t, W_j) \qquad (4.3.67)$$

and as a result:

$$V_{j+1}(t) \le W_{j+1}(t) \qquad on\, J \qquad (4.3.68)$$

hence (4.3.61) follows and consequently by induction (4.3.47) is valid for all n*. Clearly sequences* $\{W_n\}, \{V_n\}$ *are uniformly bounded on* J*. To show that they are equicontinuous, consider for any* $t_1 < t_2$ *where* $t_1, t_2 \in J$

$$\forall \varepsilon > 0, \qquad \exists \delta = \delta(\varepsilon), \qquad \forall n \in N, \quad |t_2 - t_1| < T = \delta = \frac{\varepsilon}{M}$$

$$D[V_n(t_2), V_n(t_1)] =$$
$$= D\left[U_0 + \int_0^{t_2} \{F(s, W_{n-1}(s)) + G(s, V_{n-1}(s))\} ds, U_0 + \int_0^{t_1} \{F(s, W_{n-1}(s)) + G(s, V_{n-1}(s))\} ds\right]$$
$$\le \int_{t_1}^{t_2} D[\{F(s, W_{n-1}(s)) + G(s, V_{n-1}(s))\}, \theta] ds$$
$$\le M|t_2 - t_1| < MT = \varepsilon \qquad (4.3.69)$$

here we utilized the properties of integral and the metric D*, together with the fact that* $F + G$ *are bounded since* $\{W_n\}, \{V_n\}$ *are uniformly bounded, hence* $\{V_n\}$ *is equicontinuous on* J*, the corresponding Ascoli's theorem gives a subsequence* $\{V_{n_k}\}$ *which converges uniformly to* $\rho(t) \in K_C(R^n)$*, and since* $\{V_n\}$ *is monotone nondecreasing sequence, the entire sequence* $\{V_n\}$ *converges uniformly to* $\rho(t)$ *on* J*. Similar arguments apply to the sequence* $\{W_n\}$ *and* $W_n \to R$ *uniformly on* J*, it* *therefore follows, using the integral representation of (4.3.45) and (4.3.46) that* $\rho(t)$ *and* $R(t)$ *satisfy:*

$$D_H\rho(t) = F(t,R) + G(t,\rho), \qquad \rho(0) = U_0 \ \ on\, J \tag{4.3.70}$$

$$D_H R(t) = F(t,\rho) + G(t,R), \qquad R(0) = U_0 \ \ on\, J \tag{4.3.71}$$

and that

$$V_0 \leq \rho \leq R \leq W_0 \tag{4.3.72}$$

we next claim that (ρ, R)*are coupled minimal and maximal solutions of (4.2.1), that is, if* $U(t)$ *is any solution of (4.2.1) such that:*

$$V_0 \leq U(t) \leq W_0 \tag{4.3.73}$$

then

$$V_0 \leq \rho \leq U(t) \leq R \leq W_0 \qquad t \in J \tag{4.3.74}$$

suppose that for some n*,*

$$V_n \leq U(t) \leq W_n \qquad t \in J \tag{4.3.75}$$

then we have using the monotone nature of F *and* G *and (4.3.75):*

$$D_H U = F(t,U) + G(t,U) \geq F(t,W_n) + G(t,V_n), \qquad U(0) = U_0 \tag{4.3.76}$$

$$D_H V_{n+1} = F(t,W_n) + G(t,V_n), \qquad V_{n+1}(0) = U_0 \tag{4.3.77}$$

hence,

$$D_H U \geq D_H V_{n+1} \tag{4.3.78}$$

consequently, according to Theorem (3.1.4) we arrive at:

$$V_{n+1} \leq U \quad \ on\ \ J \tag{4.3.79}$$

similarly,

$$W_{n+1} \geq U \quad \ on\ \ J \tag{4.3.80}$$

hence by induction the relation (4.3.75) is true for all $n \geq 1$*, taking the limit* $n \to \infty$ *we get (4.3.74) proving the claim. The proof is complete.*■

Corollary 4.3.2: If, in addition to the assumptions of Theorem (4.3.2) F *and* G *satisfy whenever* $X \geq Y, X, Y \in K_C(R^n)$

$$F(t,X)+N_1(X-Y)\geq F(t,Y) \quad (4.3.81)$$

$$G(t,X)\leq G(t,Y)+N_2(X-Y) \quad (4.3.82)$$

where $N_1\ ,N_2>0$, *then* $\rho=R=U$ *is the unique solution of (4.2.1).*

Proof 4.3.2: Since $\rho<R$, *and then* $R=\rho+m$ *or* $m=R-\rho$, *now*

$$\begin{aligned} D_H\rho+\ D_Hm=D_HR &= F(t,\rho)+G(t,R) \\ &\leq F(t,R)+N_1(R-\rho)+G(t\ ,\rho)+N_2(R-\rho) \\ &= D_H\rho+(N_1+N_2)m \quad (4.3.83) \end{aligned}$$

which means,

$$D_Hm\leq(N_1+N_2)m, \qquad m(0)=0 \quad (4.3.84)$$

utilizing Theorem (3.1.4) we arrive at $R\leq\rho\ \ on\ \ J$, *proving the uniqueness of* $\rho=R=U$, *completeing the proof.*■

Theorem 4.3.3: Assume that:

i) $V,W\in C^1[J,K_C(R^n)]$ *are coupled lower and upper solutions of Type III relative to (4.2.1) with* $V(t)\leq W(t),\ \ t\in J$

ii) $F,G\in C[J\times K_C(R^n),K_C(R^n)]$, $F(t,X)$ *is nonincreasing in* X, *and* $G(t,Y)$ *is nonincreasing in* Y, *for each* $t\in J$.

iii) F,G *map bounded sets into bounded sets in* $K_C(R^n)$.

Then there exist monotone sequences $\{W_n\},\{V_n\}$ *in* $K_C(R^n)$ *such that:*

$$V_n\to\rho(t), \qquad W_n\to R(t)\ \ in\ \ K_C(R^n)$$

and (ρ,R) *are coupled minimal and maximal solutions of (4.2.1) respectively, that is they satisfy:*

$$D_H\rho(t)=F(t,R)+G(t,R), \qquad \rho(0)=U_0\ \ on\ J \quad (4.3.85)$$

$$D_HR(t)=F(t,\rho)+G(t,\rho), \qquad R(0)=U_0\ \ on\ J \quad (4.3.86)$$

Proof 4.3.3: For each $n\geq 0$, *define the unique solutions* $V_{n+1}(t),W_{n+1}(t)$ *by:*

$$D_HV_{n+1}(t)=F(t,W_n)+G(t,W_n), \qquad V_{n+1}(0)=U_0\ \ on\ J \quad (4.3.87)$$

$$D_HW_{n+1}(t)=F(t,V_n)+G(t,V_n), \qquad W_{n+1}(0)=U_0\ \ on\ J \quad (4.3.88)$$

where $V(0) \leq U_0 \leq W(0)$, *we set* $V_0 = V$, $W_0 = W$. *our aim to prove:*

$$V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_2 \leq W_1 \leq W_0. \tag{4.3.89}$$

we have using the fact that: $V_0 \leq W_0$ *and the nondecreasing character of* F*:*

$$D_H V_0(t) \leq F(t, W_0) + G(t, W_0) \tag{4.3.90}$$

and, we have:

$$D_H V_1(t) = F(t, W_0) + G(t, W_0) \tag{4.3.91}$$

we can obtain that:

$$D_H V_0(t) \leq D_H V_1(t) \tag{4.3.92}$$

consequently, utilizing Theorem (3.1.4) we arrive at $V_0(t) \leq V_1(t)$. *A similar argument shows that* $W_1(t) \leq W_0(t)$.*We next prove that* $V_1 \leq W_1$ *on* J. *For this purpose, consider:*

$$D_H V_1(t) = F(t, W_0) + G(t, W_0) \tag{4.3.93}$$

$$D_H W_1(t) = F(t, V_0) + G(t, V_0) \tag{4.3.94}$$

With $V_1(0) = W_1(0) = U_0$. *since* $V_0(t) \leq W_0(t)$ *then:*

$$F(t, V_0) \leq F(t, W_0), \qquad F(t, X) \text{ is monotone nonincreasing in } X.$$

$$G(t, V_0) \leq G(t, W_0), \qquad G(t, Y) \text{ is monotone nonincreasing in } Y.$$

so, we obtain:

$$D_H V_1(t) = F(t, W_0) + G(t, W_0) \leq F(t, V_0) + G(t, V_0) \text{ on } J \tag{4.3.95}$$

$$D_H W_1(t) = F(t, V_0) + G(t, V_0) \text{ on } J \tag{4.3.96}$$

hence,

$$D_H V_1(t) \leq D_H W_1(t) \tag{4.3.97}$$

consequently, according to Theorem (3.1.4) we arrive at:

$$V_1(t) \leq W_1(t) \text{ on } J \tag{4.3.98}$$

and as a result, we obtain:

$$V_0 \leq V_1 \leq W_1 \leq W_0 \tag{4.3.99}$$

assume that for some $j > 1$, *we have:*

$$V_{j-1} \leq V_j \leq W_j \leq W_{j-1} \quad on\ J \tag{4.3.100}$$

then we show that:

$$V_j \leq V_{j+1} \leq W_{j+1} \leq W_j \quad on\ J \tag{4.3.101}$$

to do this, consider:

$$D_H V_j(t) = F(t, W_{j-1}) + G(t, W_{j-1}), \qquad V_j(0) = U_0 \ \ on\ J \tag{4.3.102}$$

$$D_H V_{j+1}(t) = F(t, W_j) + G(t, W_j), \qquad V_{j+1}(0) = U_0 \ \ on\ J \tag{4.3.103}$$

so, we can write:

$$D_H V_j(t) = F(t, W_{j-1}) + G(t, W_{j-1}) \leq$$
$$\leq F(t, W_j) + G(t, W_j) = D_H V_{j+1}(t) \tag{4.3.104}$$

consequently, $V_j(t) \leq V_{j+1}(t)$ *on* J*, in the same way we can obtain* $W_{j+1} \leq W_j$ *on* J*. Next, we show that* $V_{j+1} \leq W_{j+1}$ *,* $t \in J$*, then we have:*

$$D_H V_{j+1}(t) = F(t, W_j) + G(t, W_j), \qquad V_{j+1}(0) = U_0 \ \ on\ J \tag{4.3.105}$$

$$D_H W_{j+1}(t) = F(t, V_j) + G(t, V_j), \qquad W_{j+1}(0) = U_0 \ \ on\ J \tag{4.3.106}$$

then we can write:

$$D_H V_{j+1}(t) = F(t, W_j) + G(t, W_j) \leq F(t, V_j) + G(t, V_j) \tag{4.3.107}$$

$$D_H W_{j+1}(t) = F(t, V_j) + G(t, V_j) \tag{4.3.108}$$

and as a result:

$$V_{j+1}(t) \leq W_{j+1}(t) \quad on\ J \tag{4.3.109}$$

consequently (4.3.101) follows and by induction (4.3.89) is valid for all n*. Clearly sequences* $\{W_n\}, \{V_n\}$ *are uniformly bounded on* J*. To show that they are equicontinuous, consider for any* $t_1 < t_2$ *where* $t_1, t_2 \in J$

$$\forall \varepsilon > 0, \qquad \exists \delta = \delta(\varepsilon), \qquad \forall n \in N, \qquad |t_2 - t_1| < T = \delta = \frac{\varepsilon}{M}$$

$D[V_n(t_2), V_n(t_1)] =$

$$= D\left[U_0 + \int_0^{t_2} \{F(s, W_{n-1}(s)) + G(s, W_{n-1}(s))\} ds, U_0 + \int_0^{t_1} \{F(s, W_{n-1}(s)) + G(s, W_{n-1}(s))\} ds\right]$$

$$\leq \int_{t_1}^{t_2} D[\{F(s, W_{n-1}(s)) + G(s, W_{n-1}(s))\}, \theta] ds$$

$$\leq M|t_2 - t_1| < MT = \varepsilon \qquad (4.3.110)$$

here we utilized the properties of integral and the metric D, together with the fact that $F + G$ are bounded since $\{W_n\}, \{V_n\}$ are uniformly bounded, hence $\{V_n\}$ is equicontinuous on J, the corresponding Ascoli's Theorem gives a subsequence $\{V_{n_k}\}$ which converges uniformly to $\rho(t) \in K_C(R^n)$, and since $\{V_n\}$ is monotone nondecreasing sequence of functions, the entire sequence $\{V_n\}$ converges uniformly to $\rho(t)$ on J. Similar arguments apply to the sequence $\{W_n\}$ and $W_n \rightarrow R$ uniformly on J, it therefore follows, using the integral representation of (4.3.87) and (4.3.88) that $\rho(t)$ and $R(t)$ satisfy:

$$D_H \rho(t) = F(t, R) + G(t, R), \qquad \rho(0) = U_0 \;\; on\, J \qquad (4.3.111)$$

$$D_H R(t) = F(t, \rho) + G(t, \rho), \qquad R(0) = U_0 \;\; on\, J \qquad (4.3.112)$$

and that

$$V_0 \leq \rho \leq R \leq W_0. \qquad (4.3.113)$$

we next claim that (ρ, R) are coupled minimal and maximal solutions of (4.2.1), that is, if $U(t)$ is any solution of (4.2.1) such that:

$$V_0 \leq U(t) \leq W_0 \qquad (4.3.114)$$

then,

$$V_0 \leq \rho \leq U(t) \leq R \leq W_0, \quad t \in J \qquad (4.3.115)$$

suppose that for some n,

$$V_n \leq U(t) \leq W_n, \; t \in J \qquad (4.3.116)$$

then, we have using the monotone nature of F and G and (4.3.116):

$$D_H U = F(t,U) + G(t,U) \geq F(t,W_n) + G(t,W_n),\ \ U(0) = U_0 \quad (4.3.117)$$

$$D_H V_{n+1} = F(t,W_n) + G(t,W_n), \qquad V_{n+1}(0) = U_0 \quad (4.3.118)$$

Hence,

$$D_H V_{n+1} \leq D_H U \quad (4.3.119)$$

consequently, utilizing the Theorem (3.1.4) we arrive at:

$$V_{n+1} \leq U \quad on\ J \quad (4.3.120)$$

similarly,

$$W_{n+1} \geq U \quad on\ J. \quad (4.3.121)$$

hence by induction the relation (4.3.116) is true for all $n \geq 1$, *taking the limit* $n \rightarrow \infty$, *we get (4.3.115) proving the claim. The proof is complete.* ■

Corollary 4.3.3: If, in addition to the assumptions of Theorem (4.3.3) F *and* G *satisfy whenever* $X \geq Y, X, Y \in K_C(R^n)$

$$F(t,X) + N_1(X - Y) \geq F(t,Y) \quad (4.3.122)$$

$$G(t,X) + N_2(X - Y) \geq G(t,Y) \quad (4.3.123)$$

where $N_1, N_2 > 0$, *then* $\rho = R = U$ *is the unique solution of (4.2.1).*

Proof 4.3.3: Since $\rho < R$, *and then* $R = \rho + m$ *or* $m = R - \rho$, *now*

$$\begin{aligned} D_H\rho + D_H m = D_H R &= F(t,\rho) + G(t,\rho) \\ &\leq F(t,R) + N_1(R-\rho) + G(t\,,R) + N_2(R-\rho) \\ &= D_H\rho + (N_1 + N_2)m \end{aligned} \quad (4.3.124)$$

which means,

$$D_H m \leq (N_1 + N_2)m, \qquad m(0) = 0 \quad (4.3.125)$$

which leads to $R \leq \rho$ *on* J, *proving the uniqueness of* $\rho = R = U$. *Completes the proof.* ■

4.4. Monotone Iterative Technique for Sum of Three Functions:

To develop the monotone iterative technique and arrive at a generalization, we shall consider the IVP:

$$D_H U = F(t,U) + G(t,U) + H(t,U), \qquad U(t_0) = U_0 \in K_C(R^n) \qquad (4.4.1)$$

where F, G and $H \in C[J \times K_C(R^n), K_C(R^n)]$, $U \in C^1[J, K_C(R^n)]$.

we need the following definitions which various possible notions of lower and upper solutions relative to (4.4.1).

Definition 4.4.1: Let $V, W \in C^1[R_+, K_C(R^n)]$, then V, W are said to be:

i) Natural lower and upper solutions of (4.4.1) if:

$$D_H V \le F(t,V) + G(t,V) + H(t,V),$$
$$D_H W \ge F(t,W) + G(t,W) + H(t,W),\ t \in J \qquad (4.4.2)$$

ii) Coupled lower and upper solutions of type I of (4.4.1) if:

$$D_H V \le F(t,V) + G(t,W) + H(t,W),$$
$$D_H W \ge F(t,W) + G(t,V) + H(t,V),\ t \in J \qquad (4.4.3)$$

iii) Coupled lower and upper solutions of type II of (4.4.1) if:

$$D_H V \le F(t,W) + G(t,V) + H(t,W),$$
$$D_H W \ge F(t,V) + G(t,W) + H(t,V),\ t \in J \qquad (4.4.4)$$

iv) Coupled lower and upper solutions of type III of (4.4.1) if:

$$D_H V \le F(t,W) + G(t,W) + H(t,V),$$
$$D_H W \ge F(t,V) + G(t,V) + H(t,W),\ t \in J \qquad (4.4.5)$$

v) Coupled lower and upper solutions of type IV of (4.4.1) if:

$$D_H V \le F(t,V) + G(t,V) + H(t,W)$$
$$D_H W \ge F(t,W) + G(t,W) + H(t,V),\ t \in J \qquad (4.4.6)$$

vi) *Coupled lower and upper solutions of type* V *of (4.4.1) if:*

$$D_H V \leq F(t,V) + G(t,W) + H(t,V)$$
$$D_H W \geq F(t,W) + G(t,V) + H(t,W), \ t \in J \qquad (4.4.7)$$

vii) *Coupled lower and upper solutions of type* VI *of (4.4.1) if:*

$$D_H V \leq F(t,W) + G(t,V) + H(t,V)$$
$$D_H W \geq F(t,V) + G(t,W) + H(t,W), \ t \in J \qquad (4.4.8)$$

viii) *Coupled lower and upper solutions of type* VII *of (4.4.1) if:*

$$D_H V \leq F(t,W) + G(t,W) + H(t,W),$$
$$D_H W \geq F(t,V) + G(t,V) + H(t,V), \ t \in J \qquad (4.4.9)$$

we observe that whenever we have $V(t) \leq W(t), t \in J$.

Theorem 4.4.1: Assume that:

i) $V, W \in C^1[J, K_C(R^n)]$ *are coupled lower and upper solutions of type* I, *relative to (4.4.1) with* $V(t) \leq W(t); t \in J$

ii) F, G *and* $H \in C[J \times K_C(R^n), K_C(R^n)], F(t,X)$ *is nondecreasing in* X *and* $G(t,Y), H(t,Z)$ *are nonincreasing in Y and Z respectively, for each* $t \in J$.

iii) F, G *and* H *map bounded sets into bounded sets in* $K_C(R^n)$.

Then there exist monotone sequences $\{W_n\}, \{V_n\}$ *in* $K_C(R^n)$ *such that:*

$$V_n \to \rho(t), \qquad W_n \to R(t) \ \ in \ \ K_C(R^n)$$

and (ρ, R) *are coupled minimal and maximal solutions of (4.4.1) respectively, that is they satisfy:*

$$D_H \rho(t) = F(t,\rho) + G(t,R) + H(t,R), \qquad \rho(0) = U_0 \ \ on\, J \qquad (4.4.10)$$
$$D_H R(t) = F(t,R) + G(t,\rho) + H(t,\rho), \qquad R(0) = U_0 \ \ on\, J \qquad (4.4.11)$$

Proof 4.4.1: For each $n \geq 0$, *define the unique solutions* $V_{n+1}(t), W_{n+1}(t)$ *by:*

$$D_H V_{n+1}(t) = F(t,V_n) + G(t,W_n) + H(t,W_n), \qquad V_{n+1}(0) = U_0 \ \ on\, J \qquad (4.4.12)$$
$$D_H W_{n+1}(t) = F(t,W_n) + G(t,V_n) + H(t,V_n), \qquad W_{n+1}(0) = U_0 \ \ on\, J \qquad (4.4.13)$$

where $V(0) \le U_0 \le W(0)$, *we set* $V_0 = V$, $W_0 = W$, *our aim to prove:*

$$V_0 \le V_1 \le V_2 \le \cdots \le V_n \le W_n \le \cdots \le W_2 \le W_1 \le W_0 \qquad (4.4.14)$$

we have using the fact that $V_0 \le W_0$ *and the nondecreasing character of* F*:*

$$D_H V_0(t) \le F(t, V_0) + G(t, W_0) + H(t, W_0) \qquad (4.4.15)$$

and, we have from (4.4.10) for $n = 0$

$$D_H V_1(t) = F(t, V_0) + G(t, W_0) + H(t, W_0) \qquad (4.4.16)$$

Hence, we obtain:

$$D_H V_0(t) \le D_H V_1(t) \qquad (4.4.17)$$

consequently, according to Theorem (3.1.4) we obtain $V_0(t) \le V_1(t)$. *A similar argument shows that* $W_1(t) \le W_0(t)$. *We next prove that* $V_1 \le W_1$ *on* J. *For this purpose, consider:*

$$D_H V_1(t) = F(t, V_0) + G(t, W_0) + H(t, W_0) \qquad (4.4.18)$$

$$D_H W_1(t) = F(t, W_0) + G(t, V_0) + H(t, V_0) \qquad (4.4.19)$$

With $V_1(0) = W_1(0) = U_0$. *Since* $V_0(t) \le W_0(t)$ *then:*

$F(t, V_0) \le F(t, W_0)$, $F(t, X)$ *is monotone nondecreasing in* X.

$G(t, V_0) \ge G(t, W_0)$, $G(t, Y)$ *is monotone nonincreasing in* Y.

$H(t, V_0) \ge H(t, W_0)$, $H(t, Z)$ *is monotone nonincreasing in* Z.

so, we obtain:

$$D_H V_1(t) \le F(t, W_0) + G(t, W_0) + H(t, W_0) \text{ on } J \qquad (4.4.20)$$

$$D_H W_1(t) \ge F(t, W_0) + G(t, W_0) + H(t, W_0) \text{ on } J \qquad (4.4.21)$$

consequently,

$$V_1(t) \le W_1(t) \text{ on } J \qquad (4.4.22)$$

and as a result, we obtain:

$$V_0 \le V_1 \le W_1 \le W_0 \qquad (4.4.23)$$

assume that for some $j > 1$, *we have:*

$$V_{j-1} \leq V_j \leq W_j \leq W_{j-1} \quad on\ J \tag{4.4.24}$$

then, we show that:

$$V_j \leq V_{j+1} \leq W_{j+1} \leq W_j \quad on\ J \tag{4.4.25}$$

to do this, consider:

$$D_H V_j(t) = F(t, V_{j-1}) + G(t, W_{j-1}), \quad V_j(0) = U_0 \ \ on\ J \tag{4.4.26}$$

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, W_j), \quad V_{j+1}(0) = U_0 \ \ on\ J \tag{4.4.27}$$

so that we can write:

$$D_H V_j(t) = F(t, V_{j-1}) + G(t, W_{j-1}) + H(t, W_{j-1}) \leq$$
$$\leq F(t, V_j) + G(t, W_j) + H(t, W_j) = D_H V_{j+1}(t) \tag{4.4.28}$$

consequently, $V_j(t) \leq V_{j+1}(t)$ *on* J*, similarly we can get* $W_{j+1} \leq W_j$ *on* J*. Next, we show that* $V_{j+1} \leq W_{j+1}$, $t \in J$*, we have:*

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, W_j) + H(t, W_j), \quad V_{j+1}(0) = U_0 \ \ on\ J \tag{4.4.29}$$

$$D_H W_{j+1}(t) = F(t, W_j) + G(t, V_j) + H(t, V_j), \quad W_{j+1}(0) = U_0 \ \ on\ J \tag{4.4.30}$$

then, we can write:

$$D_H V_{j+1}(t) = F(t, V_j) + G(t, W_j) + H(t, W_j) \leq F(t, W_j) + G(t, W_j) + H(t, W_j)$$
$$D_H W_{j+1}(t) = F(t, W_j) + G(t, V_j) + H(t, V_j) \geq F(t, W_j) + G(t, W_j) + H(t, V_j) \tag{4.4.31}$$

and as a result:

$$V_{j+1}(t) \leq W_{j+1}(t) \quad on\ J \tag{4.4.32}$$

hence (4.4.25) follows, and consequently the relation (4.4.14) is valid for all n*, Clearly, sequences* $\{W_n\}, \{V_n\}$ *are uniformly bounded on* J*. To show that they are equicontinuous, consider for any* $t_1 < t_2$ *where* $t_1, t_2 \in J$ *:*

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon), \quad \forall n \in N, \ \ |t_2 - t_1| < T = \delta = \frac{\varepsilon}{M}$$

$D[V_n(t_2), V_n(t_1)]$

$$= D\Bigg[U_0 + \int_0^{t_2} \{F(s, V_{n-1}(s)) + G(s, W_{n-1}(s)) + H(s, W_{n-1}(s))\} ds, U_0 + \int_0^{t_1} \{F(s, V_{n-1}(s)) + G(s, W_{n-1}(s)) + H(s, W_{n-1}(s))\} ds\Bigg]$$

$$\leq \int_{t_1}^{t_2} D[\{F(s, V_{n-1}(s)) + G(s, W_{n-1}(s)) + H(s, W_{n-1}(s))\}, \theta] ds \leq M|t_2 - t_1| < MT = \varepsilon \qquad (4.4.33)$$

here we utilized the properties of integral and the metric D, together with the fact that $F + G + H$ are bounded since $\{W_n\}, \{V_n\}$ are uniformly bounded, hence $\{V_n\}$ is equicontinuous on J, the corresponding Ascoli's theorem gives a subsequence $\{V_{n_k}\}$ which converges uniformly to $\rho(t) \in K_C(R^n)$, and since $\{V_n\}$ is monotone nondecreasing sequence, the entire sequence $\{V_n\}$ converges uniformly to $\rho(t)$ on J. Similar arguments apply to the sequence $\{W_n\}$ and $W_n \to R$ uniformly on J, it therefore follows, using the integral representation of (4.4.12) and (4.4.13) that $\rho(t)$ and $R(t)$ satisfy:

$$D_H\rho(t) = F(t, \rho) + G(t, R) + H(t, R), \qquad \rho(0) = U_0 \ \ on\ J \qquad (4.4.34)$$

$$D_H R(t) = F(t, R) + G(t, \rho) + H(t, \rho), \qquad R(0) = U_0 \ \ on\ J \qquad (4.4.35)$$

and that

$$V_0 \leq \rho \leq R \leq W_0 \qquad (4.4.36)$$

we next claim that $(\rho, R)$are coupled minimal and maximal solutions of (4.4.1), that is, if $U(t)$ is any solution of (4.4.1) such that:

$$V_0 \leq U(t) \leq W_0 \qquad (4.4.37)$$

then:

$$V_0 \leq \rho \leq U(t) \leq R \leq W_0, \ t \in J \tag{4.4.38}$$

suppose that for some n*,*

$$V_n \leq U(t) \leq W_n, \quad t \in J \tag{4.4.39}$$

then we have using the monotone nature of F *and* G *and (4.4.39):*

$$D_H U = F(t,U) + G(t,U) + H(t,U) \geq$$
$$\geq F(t,V_n) + G(t,W_n) + H(t,W_n), \ \ U(0) = U_0 \tag{4.4.40}$$
$$D_H V_{n+1} = F(t,V_n) + G(t,W_n) + H(t,W_n), \qquad V_{n+1}(0) = U_0 \tag{4.4.41}$$

Hence,

$$D_H U \geq D_H V_{n+1} \tag{4.4.42}$$

consequently, according to Theorem (3.1.4) we arrive at:

$$V_{n+1} \leq U \quad \text{on } J \tag{4.4.43}$$

similarly,

$$W_{n+1} \geq U \quad \text{on } J \tag{4.4.44}$$

hence by induction the relation (4.4.39) is true for all $n \geq 1$*, taking the limit* $n \to \infty$*, we get (4.4.38) proving the claim. The proof is complete.* ■

Corollary 4.4.1: If, in addition to the assumptions of Theorem (4.4.1) F, G *and* H *satisfy whenever* $X \geq Y, X, Y \in K_C(R^n)$

$$F(t,X) \leq F(t,Y) + N_1(X - Y) \tag{4.4.45}$$
$$G(t,X) + N_2(X - Y) \geq G(t,Y) \tag{4.4.46}$$
$$H(t,X) + N_3(X - Y) \geq H(t,Y) \tag{4.4.47}$$

where $N_1, N_2, N_3 > 0$*, then* $\rho = R = U$ *is the unique solution of (4.4.1).*

Proof 4.4.1: Since $\rho \leq R$*, and then* $R = \rho + m$ *or* $m = R - \rho$*, now*

$$D_H\rho + D_H m = D_H R = F(t,R) + G(t,\rho) + H(t,\rho)$$
$$\leq F(t,\rho) + N_1(R-\rho) + G(t,R) + N_2(R-\rho)$$
$$+ H(t,R) + N_2(R-\rho) = D_H\rho + (N_1 + N_2 + N_3)m \tag{4.4.48}$$

which means,

$$D_H m \leq (N_1 + N_2 + N_3)m, \qquad m(0) = 0 \qquad (4.4.49)$$

which leads to $R \leq \rho$ *on* J*, proving the uniqueness of* $\rho = R = U$*. Completing the proof.*■

So, if the sum of two nondecreasing (nonincreasing) functions is nondecreasing (nonincreasing) function then the monotone iterative technique for three monotone functions will reduce to the monotone iterative technique for two monotone functions.

4.5. Monotone Iterative Technique for Finite Systems:

In this section we shall attempt to study the set differential system, given by

$$D_H U = F(t, U), \qquad U(0) = U_0\,, t \in J \qquad (4.5.1)$$

where $F \in C[J \times K_C(R^n)^N,\ K_C(R^n)^N]$, $U \in K_C(R^n)^N$, $K_C(R^n)^N = (K_C(R^n) \times K_C(R^n) \times \ldots \times K_C(R^n), N \text{ times })$, $U = (U_1, U_2, \cdots, U_N)$ such that for each $i, 1 \leq i \leq N$, $U_i \in K_C(R^n)$. Note also that $U_0 \in K_C(R^n)^N$.

we have the following two possibilities to measure the new variables U, U_0, F.

1. Define $D_0[U, V] = \sum_{i=1}^{N} D[U_i,\ V_i]$, where $U, V \in K_C(R^n)^N$ and employ the metric space $(K_C(R^n)^N, D_0)$.
2. Define $\widetilde{D} :\ K_C(R^n)^N \times K_C(R^n)^N \to R_+^N$ such that
$$\widetilde{D}[U, V] = (D[U_1, V_1], D[U_2, V_2], \cdots D[U_N, V_N])$$

And employ the generalized metric space $(K_C(R^n)^N, \widetilde{D})$.

Method of lower and upper solutions: finite systems.

Many of results considered so far for a single equation will now be extended to finite systems, to avoid repetition, let us agree that the subscripts i, j range over the integers 1, 2, …, n and the vectoral in equalities mean that the same inequalities hold between their corresponding components. It is well known that a certain monotone property is needed when we deal with systems of inequalities, and we shall now define this property.

Definition 4.5.1: A function $F \in C[J \times K_C(R^n)^N, \; K_C(R^n)^N]$ is said to be quasimonotone nondecreasing if, for some i such that $1 \le i \le n$, $U \le V$ and $U_i = V_i$, $F_i(t, U) \le F_i(t, V)$.

Definition 4.5.2: A function $F \in C[J \times K_C(R^n)^N, \; K_C(R^n)^N]$ is said to be quasimonotone nonincreasing if, for some i such that $1 \le i \le n$, $U \le V$ and $U_i = V_i$, $F_i(t, U) \ge F_i(t, V)$.

Theorem 4.5.1: Let $V, W \in C^1[J \times K_C(R^n)^N]$ be lower and upper solutions of (4.5.1) respectively, suppose that the function F is quasimonotone nondecreasing function, and for each i

$$F_i(t, X_1, X_2, \cdots, X_n) - F_i(t, Y_1, Y_2, \cdots, Y_n) \le L_i \sum_{i=1}^{n} (X_i - Y_i) \qquad (4.5.2)$$

whenever $X \ge Y$, then $V(0) \le W(0)$, implies that $V(t) \le W(t)$.

Proof 4.5.1: Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) > 0$, then we will define $\widetilde{W}_i(t) = W_i(t) + \varepsilon e^{(n+1)L_i t}$. Since $V_i(t_0) \le W_i(t_0) < \widetilde{W}_i(t_0)$, it is enough to prove that:

$$V_i(t) < \widetilde{W}_i(t), \qquad t \ge t_0 \qquad (4.5.3)$$

to arrive to our conclusion, in view of the fact $\varepsilon > 0$ is arbitrary small, let $t_1 > 0$ be the supremum of all positive numbers $\delta > 0$, such that:

$$V_i(t_0) < \widetilde{W}_i(t_0) \text{ implies } V_i(t) < \widetilde{W}_i(t) \text{on } [t_0, \delta].$$

Moreover $V_j(t) \ge \widetilde{W}_j(t)$, for $i \ne j$

it is clear that $V_i(t_1) = \widetilde{W}_i(t_1)$ and for $t_1 > t_0$, and $V_i(t_1) < \widetilde{W}_i(t_1)$, this follows, using the nondecreasing nature of F and condition (4.5.2) that:

$$D_H V_i(t_1) \leq F_i\big(t_1, V_1(t_1), \cdots, V_n(t_1)\big) \leq F\left(t_1, \widetilde{W}_1(t_1), \cdots, \widetilde{W}_n(t_1)\right)$$
$$\leq F\big(t_1, W_1(t_1), \cdots, W_n(t_1)\big) + L_i\big(\widetilde{W}_i - W_i\big)$$
$$\leq D_H W_i(t_1) + nL_i\varepsilon e^{(n+1)L_i t_1}$$
$$< D_H W_i(t_1) + (n+1)L_i\varepsilon e^{(n+1)L_i t_1} = D_H \widetilde{W}_i(t_1) \qquad (4.5.4)$$

consequently, it follows that there exists an $\eta > 0$ satisfying:

$$V_i(t) - \widetilde{W}_i(t) > V_i(t_1) - \widetilde{W}_i(t_1), \qquad t_1 - \eta < t < t_1 \qquad (4.5.5)$$

this implies that $t_1 > t_0$ cannot be the supremum in view of the continuity of the functions involved and therefore the relation

$$V_i(t) < \widetilde{W}_i(t), \qquad t \geq t_0 \qquad (4.5.6)$$

is true, and then we can write

$$V_i(t) < \widetilde{W}_i(t) = W_i(t) + \varepsilon e^{(n+1)L_i t} \qquad (4.5.7)$$

and then:

$$lim_{\varepsilon \to 0} V_i(t) \leq lim_{\varepsilon \to 0}\big[W_i(t) + \varepsilon e^{(n+1)L_i t}\big]$$
$$V_i(t) \leq W_i(t) \qquad (4.5.8)$$

and the proof is complete.■

Theorem 4.5.2: Under the assumption of Theorem (4.5.1), every solution of (4.5.1) such that $V(0) \leq U(0) \leq W(0)$ satisfies $V(t) \leq U(t) \leq W(t)$ on J.

Theorem 4.5.3: Let $V, W \in C^1[J, K_C(R^n)^N]$ be lower and upper solutions of (4.5.1) such that $V(t) \leq W(t)$ on J and let $F \in C[\Omega, (R^n)^N]$, where

$$\Omega = [(t, U) \in J \times (R^n)^N : V(t) \leq U(t) \leq W(t), t \in J].$$

If F is quasimonotone nondecreasing in U, then there exists a solution $U(t)$ of (4.5.1) such that $V(t) \leq U(t) \leq W(t)$ on J, provided $V(0) \leq U(0) \leq W(0)$.

In fact, the conclusion of Theorem (4.5.3) is true without demanding F to be quasimonotone nondecreasing, which is restrictive. However, in this case, we need to *strengthen the notion of upper and lower solutions of (4.5.1). We list below such lower and upper solutions as an assumption:*

for each i,

$$D_H V_i \leq F_i(t,\sigma) \text{ for all } \sigma \text{ such that } V(t) \leq \sigma \leq W(t) \text{ and } V_i(t) = \sigma_i$$
$$D_H W_i \geq F_i(t,\sigma) \text{ for all } \sigma \text{ such that } V(t) \leq \sigma \leq W(t) \text{ and } W_i(t) = \sigma_i \quad (4.5.9)$$

Theorem 4.5.4: let $V, W \in C^1[J,\ K_C(R^n)^N]$ *with* $V(t) \leq W(t)$ *on* J *satisfying (4.5.9) and let* $F \in C[\Omega, (R^n)^N]$, *then there exists a solution* $U(t)$ *of (4.5.1) such that*

$$V(t) \leq U(t) \leq W(t) \text{ on } J, \text{ provided } V(0) \leq U(0) \leq W(0).$$

Since the assumptions of Theorem 4.5.3 imply that the assumptions of Theorem (4.5.4), it is enough to prove Theorem (4.5.4).

Proof of Theorem 4.5.4.

Consider $P: J \times K_C(R^n)^N \rightarrow K_C(R^n)^N$ *defined by*

$$P_i(t,U) = max\{V_i(t), min[U_i, W_i(t)]\}, \text{ for each } i.$$

Then $F(t, P(t,U))$ *defines a continuous extension of* F *to* $J \times K_C(R^n)^N$ *which is also bounded since* F *is bounded on* Ω. *Therefore,* $D_H U = F(t, P(t,U))$ *has a solution* U *on* J *with* $U(0) = U_0$ *by theorem 3.1.5 let us show that* $V(t) \leq U(t) \leq W(t)$ *and therefore a solution of (4.5.1). For* $\varepsilon > 0$ *and* $e = (1, \ldots, 1)$, *consider* $W_\varepsilon(t) = W(t) + \varepsilon(1+t)e$ *and* $V_\varepsilon(t) = V(t) - \varepsilon(1+t)e$. *Clearly, we have* $V_\varepsilon(0) \leq U_0 \leq W_\varepsilon(0)$. *Suppose that* $t_1 \in J$ *is such that* $V_\varepsilon(t) < U(t) < W_\varepsilon(t)$ *in* $[0, t_1)$ *but* $U_j(t_1) = W_{\varepsilon j}(t_1)$. *Then, we have* $V(t_1) \leq p(t_1, U(V)) \leq W(t_1)$ *and* $P_j(t_1, U(t_1)) = W_j(t_1)$, *hence*

$$D_H W_j(t_1) \geq F_j(t_1, U(t_1)) = D_H U_j(t_1), \quad (4.5.10)$$

which implies $D_H U_j(t_1) < D_H W_{\varepsilon j}(t_1)$, *contradicting* $U_j(t) < W_{\varepsilon j}(t)$ *for* $t < t_1$. *Therefore,* $V_\varepsilon(t) < U(t) < W_\varepsilon(t)$ *in* J. *Now,* $\varepsilon \to 0$ *yields* $V(t) \leq U(t) \leq W(t)$. *And the proof is complete.* ∎

Monotone Iterative Technique for Finite Systems

In order to develop a monotone iterative method for the system (4.5.1) so as to include several possibilities, we need to begin with some new notions. For each fixed $i, 1 \leq i \leq n$ let p_i, q_i be two nonnegative integers such that $p_i + q_i = n - 1$ so that we can split the vector U into $U = (U_i, [U]_{p_i}, [U]_{q_i})$. Then, the system (4.5.1) can be written as

$$D_H U_i = F_i\big(t, U_i, [U]_{p_i}, [U]_{q_i}\big), \qquad U(0) = U_0 \tag{4.5.11}$$

Definition 4.5.3:Let $V, W \in C^1[J, K_C(R^n)^N]$. Then V, W are said to be coupled lower and upper quasi solutions of (4.5.4) if

$$\begin{aligned} D_H V_i &\leq F_i\big(t, V_i, [V]_{p_i}, [W]_{q_i}\big), \qquad V(0) \leq U_0 \\ D_H W_i &\geq F_i\big(t, W_i, [W]_{p_i}, [V]_{q_i}\big), \quad W(0) \geq U_0 \end{aligned} \tag{4.5.12}$$

Definition 4.5.4: Let $V, W \in C^1[J, K_C(R^n)^N]$. Then V, W are said to be coupled quasisolutions of (4.5.4) if

$$\begin{aligned} D_H V_i &= F_i\big(t, V_i, [V]_{p_i}, [W]_{q_i}\big), \qquad V(0) = U_0 \\ D_H W_i &= F_i\big(t, W_i, [W]_{p_i}, [V]_{q_i}\big), \quad W(0) = U_0 \end{aligned} \tag{4.5.13}$$

Definition 4.5.5: A function $F \in C[J \times K_C(R^n)^N, K_C(R^n)^N\ ]$ is said to possess a mixed quasimonotone property (mqmp for short) if for each i, $F_i\big(t, U_i, [U]_{p_i}, [U]_{q_i}\big)$ is monotone nondecreasing in $[U]_{p_i}$ and monotone nonincreasing in $[U]_{q_i}$.

Theorem 4.5.5: Let $F \in C[J \times K_C(R^n)^N, K_C(R^n)^N\ ]$ possess mixed quasimonotone property and let V_0, W_0 be coupled lower and upper quasi-solutions of system (4.5.4) such that $V_0 \leq W_0$ on J. Suppose further that

$$F_i\big(t, U_i, [U]_{p_i}, [U]_{q_i}\big) - F_i\big(t, \widetilde{U}_i, [U]_{p_i}, [U]_{q_i}\big) \geq -M_i(U_i - \widetilde{U}_i) \tag{4.5.14}$$

whenever $V_0 \leq U \leq W_0$, *and* $V_{0,i} \leq \widetilde{U}_i \leq U_i \leq W_{0,i}$ *and* $M_i \geq 0$. *Then there exist monotone sequences* $\{V_n\}$, $\{W_n\}$ *such that* $V_n \to \rho, W_n \to R$ *as* $n \to \infty$ *uniformly and monotonically to coupled minimal and maximal quasisolutions of (4.5.4) on* J, *provided* $V_0(0) \leq U_0 \leq W_0(0)$. *Further, if* U *is any solution of (4.5.11) such that* $V_0 \leq U \leq W_0$ *then* $\rho \leq U \leq R$ *on* J .

Proof 4.5.5: For any $\eta, \mu \in C[J, K_C(R^n)^N)]$ *such that* $V_0 \leq \eta,\ \mu \leq W_0$ *on* J, *we define*

$$F_i(t,U) = F_i\big(t,\ \eta_i, [\ \eta]_{p_i}, [\mu]_{q_i}\big) -\ M_i(U_i - \eta_i), \tag{4.5.15}$$

And consider the uncoupled linear differential system

$$D_H U_i = F_i(t,U) = F_i\big(t,\ \eta_i, [\ \eta]_{p_i}, [\mu]_{q_i}\big) -\ M_i(U_i - \eta_i), \quad U(0) = U_0 \tag{4.5.16}$$

Clearly, for a given η, μ *the system (4.5.15) possess a unique solution* $U(t)$ *defined on* J. *For each* $\eta, \mu \in C[J, K_C(R^n)^N]$, *such that* $V_0 \leq \eta,\ \mu \leq W_0$ *on* J, *we define the mapping A by:*

$$A[\eta, \mu] = U$$

Where U *is the unique solution of (4.5.15). This mapping defines the sequences* $\{V_n\}, \{W_n\}$. *First we prove that*

- $V_0 \leq A[V_0, W_0],\ \ W_0 \geq A[W_0, V_0]$
- *A possess the mixed quasi monotone property on the segment* $[V_0, W_0]$ *where the segment* $[V_0, W_0] = \{U \in C[J, K_C(R^n)^N]: V_0 \leq U \leq W_0\}$

to prove (i), set $A[V_0, W_0] = V_1$ *where* V_1 *is the unique solution of (4.5.15) with* $\eta = V_0,\ \mu = W_0$. *Setting* $P_i = V_{1,i} - V_{0,i}$, *then we can write like this*

$$\begin{aligned} D_H P_i = D_H V_{1,i} - D_H V_{0,i} \geq \\ \geq F_i\big(t, V_{0,i}, [V_0]_{p_i}, [W_0]_{q_i}\big) - F_i\big(t, V_{0,i}, [V_0]_{p_i}, [W_0]_{q_i}\big) \\ - M_i\big(V_{1,i} - V_{0,i}\big) = -M_i P_i \end{aligned} \tag{4.5.17}$$

and we obtain the differential inequality $P_i(t) \geq P_i(0) \geq 0$ *on* J, *then* $V_{0,i} \leq V_{1,i}$, *in the same way we can show that* $W_{0,i} \geq W_{1,i}$. *Setting* $A[W_0, V_0\ ] = W_1$, *where* W_1 *is the unique solution of (4.5.15) with* $\eta = W_0,\ \ \mu = V_0$. *Setting* $P_i = W_{1,i} - W_{0,i}$, *it easily follows that*

$$D_H P_i = D_H W_{1,i} - D_H W_{0,i}$$
$$\leq F_i\big(t, W_{0,i}, [W_0]_{p_i}, [V_0]_{q_i}\big)$$
$$- F_i\big(t, W_{0,i}, [W_0]_{p_i}, [V_0]_{q_i}\big) - M_i\big(W_{1,i} - W_{0,i}\big)$$
$$= -M_i P_i \qquad (4.5.18)$$

It thus follows that $P_i(t) \leq P_i(0)e^{-M_i t} \leq 0$ on J, and hence $W_{0,i} \geq W_{1,i}$, which proves (i). To prove (ii), let $\eta_1, \eta_2, \mu \in [V_0, W_0]$ be such that $\eta_1 \leq \eta_2$. Suppose $A[\eta_1, \mu] = U_1$ and $A[\eta_2, \mu] = U_2$. then setting $P_i = U_{1,i} - U_{2,i}$, we find, using the mqmp of F and (4.5.14), that:

$$D_H P_i = F_i\big(t, \eta_{1,i}, [\eta_1]_{p_i}, [\mu]_{q_i}\big) - M_i\big(U_{1,i} - \eta_{1,i}\big) -$$
$$F_i\big(t, \eta_{2,i}, [\eta_2]_{p_i}, [\mu]_{q_i}\big) + M_i\big(U_{2,i} - \eta_{2,i}\big) \leq F_i\big(t, \eta_{1,i}, [\eta_2]_{p_i}, [\mu]_{q_i}\big) -$$
$$M_i\big(U_{1,i} - \eta_{1,i}\big) - F_i\big(t, \eta_{2,i}, [\eta_2]_{p_i}, [\mu]_{q_i}\big) + M_i\big(U_{2,i} - \eta_{2,i}\big) \leq$$
$$M_i\big(\eta_{2,i} - \eta_{1,i}\big) - M_i\big(U_{1,i} - \eta_{1,i}\big) + M_i\big(U_{2,i} - \eta_{2,i}\big) = -M_i P_i. \qquad (4.5.19)$$

also, since $P_i(0) = 0$, we get $U_{1,i} \leq U_{2,i}$, $A[\eta_1, \mu] \leq A[\eta_2, \mu]$. In the same way if $\eta, \mu_1, \mu_2 \in [V_0, W_0]$ such tha $\mu_1 \leq \mu_2$, suppose $A[\eta, \mu_1] = U_1$ and $A[\eta, \mu_2] = U_2$, then setting $P_i = U_{1,i} - U_{2,i}$, we find, using the mqmp of F and (4.5.14) that:

$$D_H P_i = F_i\big(t, \eta, [\eta]_{p_i}, [\mu_1]_{q_i}\big) - F_i\big(t, \eta, [\eta]_{p_i}, [\mu_2]_{q_i}\big) - M_i\big(U_{1,i} - \mu_{1,i}\big) +$$
$$M_i\big(U_{2,i} - \mu_{2,i}\big) \geq F_i\big(t, \eta, [\eta]_{p_i}, [\mu_2]_{q_i}\big) - F_i\big(t, \eta, [\eta]_{p_i}, [\mu_2]_{q_i}\big) -$$
$$M_i\big(U_{1,i} - \mu_{1,i}\big) + M_i\big(U_{2,i} - \mu_{2,i}\big) = -M_i\big(U_{1,i} - \eta\big) + M_i\big(U_{2,i} - \eta\big) =$$
$$-MP_i. \qquad (4.5.20)$$

and since $P_i(0) = 0$, we get $U_{1,i} \geq U_{2,i}$, $A[\eta, \mu_1] \geq A[\eta, \mu_2]$. It therefore follows that the mapping A satisfies (ii), consequently this implies $A[\eta, \mu] \leq A[\mu, \eta]$ whenever $\eta \leq \mu$ and $\eta, \mu \in [V_0, W_0]$. In view of (i) and (ii) above, we can define the sequences

$$V_n = A[V_{n-1}, W_{n-1}], \quad W_n = A[W_{n-1}, V_{n-1}] \qquad (4.5.21)$$

satisfying

$$V_0 \leq V_1 \leq \cdots \leq V_n \leq W_n \leq \cdots \leq W_1 \leq W_0 \qquad (4.5.22)$$

It is easy to prove that the sequences $\{V_n\}, \{W_n\}$ are monotone and converge uniformly and monotonically to coupled quasisolutions (ρ, R) of (4.5.11). Letting

$$\rho = lim_{n\to\infty}V_n, \quad R = lim_{n\to\infty}W_n, \tag{4.5.23}$$

we find

$$D_H\rho_i = F_i\big(t, \rho_i, [\rho]_{p_i}, [R]_{q_i}\big), \qquad V(0) = U_0 \tag{4.5.24}$$

$$D_H R_i = F_i\big(t, R_i, [R]_{p_i}, [\rho]_{q_i}\big), \qquad W(0) = U_0 \tag{4.5.25}$$

we shall show that (ρ, R) *are coupled minimal and maximal quasi solutions respectively. Let* (U_1, U_2) *be any coupled quasi solutions of (4.5.11) such that* $U_1, U_2 \in [V_0, W_0]$. *Let us assume that for some integer* $k > 0, V_{k-1} \le U_1,\ U_2 \le W_{k-1}$ *on* J, *then setting* $P_i = V_{k,i} - U_{1,i}$, *employing the mqmp property of* F *and (4.5.14) we arrive at:*

$$\begin{aligned} D_H P_i = D_H V_{k,i} - D_H U_{1,i} = \\ = F_i\big(t, V_{k-1,i}, [V_{k-1}]_{p_i}, [U_2]_{q_i}\big) - M_i\big(V_{k,i} - V_{k-1,i}\big) \\ - F_i\big(t, U_{1,i}, [U_1]_{p_i}, [U_2]_{q_i}\big) \\ \le F_i\big(t, V_{k-1,i}, [U_1]_{p_i}, [U_2]_{q_i}\big) - M_i\big(V_{k,i} - V_{k-1,i}\big) \\ - F_i\big(t, U_{1,i}, [U_1]_{p_i}, [U_2]_{q_i}\big) \\ \le M_i(U_{1,i} - V_{k-1,i}) - M_i\big(V_{k,i} - V_{k-1,i}\big) \le -M_i P_i \end{aligned} \tag{4.5.26}$$

since $P_i(0) = 0$, *this implies that* $V_k \le U_1$, *in the same way we can show that* $U_2 \le W_k$ *on* J. *it follows by induction that* $V_k \le U_1, U_2 \le W_k$ *on* J *for all* k, *since* $V_0 \le U_1,\ U_2 \le W_0$ *on* J. *hence, we have* $\rho \le U_1, U_2 \le R$ *on* J *proving* (ρ, R) *are coupled minimal and maximal quasi solutions of (4.5.11). Since any solution* U *of (4.5.11) such that* $U \in [W_0, V_0]$ *can be considered as* (U, U) *coupled quasi solutions of (4.5.11) wee also have* $\rho \le U \le R$ *on* J, *this complete the proof.* ■

5. Monotone Iterative Technique with Initial Time Difference

5.1. Comparison Theorems and Existence Results Relative to Initial Time Difference

In this section, we will give some basic comparison theorems and existence results relative to initial time difference.

Theorem 5.1.1: Suppose that:

- $V \in C^1\big[[\tau_0, \tau_0 + T], K_C(R^n)\big]$, $W \in C^1\big[[\zeta_0, \zeta_0 + T], K_C(R^n)\big]$ *and,* $F \in C[R_+ \times K_C(R^n), K_C(R^n]$, *for each* $t \in R_+$*:*

$$D_H V(t) \leq F\big(t, V(t)\big), \qquad V(\tau_0) \leq U_0$$
$$D_H W(t) \geq F\big(t, W(t)\big), \qquad W(\zeta_0) \geq U_0 \qquad (5.1.1)$$

- *For any* $X, Y \in K_C(R^n)$ *such that* $X \geq Y$, $t \in R_+$

$$F(t, X) \leq F(t, Y) + L(X - Y), \text{ for some } L > 0 \qquad (5.1.2)$$

- $\tau_0 < \zeta_0$ *and* $F(t, X)$ *is nondecreasing in* t *and* X, *then* $V(\tau_0) \leq W(\zeta_0)$ *implies:*

i) $V(t) \leq W(t + \xi), t \geq \tau_0$ *(ii)* $V(t - \xi) \leq W(t)$, $t \geq \zeta_0$ *Where,* $\xi = \zeta_0 - \tau_0$.

Proof 5.1.1: Let $W(t + \xi) = \widetilde{W}(t)$, *and let* $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) > 0$, *and define* $\breve{W}(t) = \widetilde{W}(t) + \varepsilon e^{2Lt}$. *Since* $V(t_0) \leq \widetilde{W}(t_0) < \breve{W}(t_0)$, *it is enough to prove that:*

$$V(t) < \breve{W}(t), \qquad t \geq t_0 \qquad (5.1.3)$$

to arrive to our conclusion, in view of the fact $\varepsilon > 0$ *is arbitrary small, let* $t_1 > 0$ *be the supremum of all positive numbers* $\delta > 0$, *such that:*

$$V(t_0) < \breve{W}(t_0) \text{ implies } V(t) < \breve{W}(t) \text{ on } [t_0, \delta].$$

It is clear that $t_1 > t_0$, *and* $V(t_1) \leq \breve{W}(t_1)$, *this follows, using the nondecreasing nature of* F *and condition (5.1.2) that:*

$$D_H V(t_1) \le F(t_1, V(t_1)) \le F\left(t_1, \breve{W}(t_1)\right) \le F\left(t_1, \widetilde{W}(t_1)\right) + L(\breve{W} - \widetilde{W})$$
$$\le F\left(t_1 + \xi, \widetilde{W}(t_1)\right) + L(\breve{W} - \widetilde{W}) \le D_H \widetilde{W}(t_1) + L\varepsilon e^{2Lt_1}$$
$$< D_H \widetilde{W}(t_1) + 2L\varepsilon e^{2Lt_1} = D_H \breve{W}(t_1) \qquad (5.1.4)$$

consequently, it follows that there exists an $\eta > 0$ satisfying:

$$V(t) - \breve{W}(t) > V(t_1) - \breve{W}(t_1), \qquad t_1 - \eta < t < t_1 \qquad (5.1.5)$$

this implies that $t_1 > t_0$ cannot be the supremum in view of the continuity of the functions involved and therefore the relation

$$V(t) < \breve{W}(t), \qquad t \ge t_0 \qquad (5.1.6)$$

is true, and then we can write

$$V(t) < \breve{W}(t) = \widetilde{W}(t) + \varepsilon e^{2Lt} \qquad (5.1.7)$$

making $\varepsilon \to 0$, we conclude that $V(t) \le W(t + \xi)$ $t \ge \tau_0$, which proves (i), To prove (ii), we set $\tilde{V}(t) = V(t - \xi)$, $t \ge \zeta_0$, and note that:

$$\tilde{V}(\zeta_0) = V(\zeta_0 - \xi) = V(\tau_0) \le W(\zeta_0) \qquad (5.1.8)$$

and letting: $\tilde{V}_0(t) = \tilde{V}(t) - \varepsilon e^{2Lt}$ for small $\varepsilon > 0$, and proceeding similarly, we derive the estimate:

$$V(t - \xi) \le W(t), \qquad t \ge \zeta_0 \qquad (5.1.9)$$

and the proof therefore is complete. ■

5.2. Monotone Iterative Technique for Single Function with Initial Time Difference

In order to develop the monotone iterative technique with initial time difference, we shall consider the IVP:

$$D_H U = F(t, U(t)), \quad U(\tau_0) = U_0 \qquad (5.2.1)$$

where $F \in C[J \times K_C(R^n), K_C(R^n)\ ]$, $U \in C^1[J \times K_C(R^n)]$, where $J = [\tau_0, \zeta_0 + T]$.

Theorem 5.2.1: Let $F \in C\big[[\tau_0, \zeta_0 + T] \times K_C(R^n), K_C(R^n)\big]$ *be nondecreasing function in* t, *and* $V \in C^1\big[[\tau_0, \tau_0 + T], K_C(R^n)\big], W \in C^1\big[[\zeta_0, \zeta_0 + T], K_C(R^n)\big]$ *are lower and upper solutions of the initial value problem (5.2.1) such that* $V(t) \leq W(t + \xi)$ *on* $J = [\tau_0, \zeta_0 + T]$, $\zeta_0 > \tau_0$ *, and* $\xi = \zeta_0 - \tau_0$, *Suppose further that:*

$$F(t, X) - F(t, Y) \geq -M(X - Y) \tag{5.2.2}$$

for $V(t) \leq Y \leq X \leq W(t + \xi)$ *and* $M \geq 0$, *then there exists monotone sequences* $\{V_n\}$, $\{\widetilde{W}_n\}$, *such that* $V_n \to \rho$ *and* $\widetilde{W}_n \to \tilde{R}$, *as* $n \to \infty$ *uniformly, and monotonically on* J, *and that* $\rho, \tilde{R}$ *are minimal and maximal solutions of (5.2.1) respectively, where* $\widetilde{W}(t) = W(t + \xi)$.

Proof 5.2.1: Since $\widetilde{W}(\tau_0) = W(\tau_0 + \xi) = W(\tau_0 + \zeta_0 - \tau_0) = W(\zeta_0)$, *and* $V(\tau_0) \leq U_0 \leq W(\zeta_0)$ *also* $D_H\widetilde{W}(t) = D_H W(t + \xi) \geq F\left(t + \xi, \widetilde{W}(t)\right), t \geq \tau_0$, *Let's set* $V = V_0, \widetilde{W} = \widetilde{W}_0$ *For any* $\eta \in C^1[J, K_C(R^n)]$ *such that* $V_0 \leq \eta \leq W_0$, *we consider the linear differential equation:*

$$D_H U = F(t, \eta) - M(U - \eta), \qquad U(\tau_0) = U_0 \tag{5.2.3}$$

It is clear that for every such η, *there exists a unique solution of (5.2.3) on* J. *Define a mapping* A *by* $A\eta = U$. *This mapping will be used to define the sequences* $\{V_n\}, \{\widetilde{W}_n\}$ *and let's prove that:*

i) $V_0 \leq AV_0$, $\widetilde{W}_0 \geq A\widetilde{W}_0$
ii) *A is monotone operator on the segment:*

$$\big[V_0, \widetilde{W}_0\big] = \big\{U \in C^1\big[J \times K_C(R^{n)}\big], \qquad V_0 \leq U \leq \widetilde{W}_0\, ; t \in J\big\}$$

we now prove (i), set $AV_0 = V_1$ *where* V_1 *is the unique solution of (5.2.3) with* $\eta = V_0$, *setting* $\varphi = V_0 - V_1$, *so we obtain:*

$$D_H\varphi = D_H V_0 - D_H V_1 \leq F(t, V_0) - F(t, V_0) + M(V_1 - V_0) = -M\varphi,$$
$$\varphi(\tau_0) = V_0 - U_0 \leq U_0 - U_0 = 0 \tag{5.2.4}$$

since $V_0 \leq U_0$, *This shows that:*

$$\varphi(t) \leq 0$$

hence $V_0 \leq V_1$ $on J$, or equivalently $V_0 \leq AV_0$. In the same way, we can prove that $\widetilde{W}_0 \geq A\widetilde{W}_0$. Setting $\varphi = \widetilde{W}_0 - \widetilde{W}_1$, we can write:

$$D_{\boldsymbol{H}}\varphi = D_H\widetilde{W}_0 - D_H\widetilde{W}_1 \geq F\left(t+\xi, \widetilde{W}_0\right) - F\left(t+\xi, \widetilde{W}_0\right) + M\left(\widetilde{W}_1 - \widetilde{W}_0\right)$$
$$= -M\varphi, \qquad \varphi(\tau_0) = \widetilde{W}_0 - U_0 \geq U_0 - U_0 = 0 \tag{5.2.5}$$

hence $\widetilde{W}_0 \geq \widetilde{W}_1$ $on J$, or equivalently $\widetilde{W}_0 \leq A\widetilde{W}_0$. In order to prove (ii), let $\eta_1, \eta_2 \in [V_0, \widetilde{W}_0]$ such that $\eta_1 \leq \eta_2$, assume that $U_1 = A\eta_1$, $U_2 = A\eta_2$, setting $\varphi = U_1 - U_2$ so that:

$$D_H\varphi = D_HU_1 - D_HU_2 =$$
$$= F(t,\eta_1) - M(U_1 - \eta_1) - F(t,\eta_2) + M(U_2 - \eta_2) \leq$$
$$\leq M(\eta_2 - \eta_1) + M(U_2 - \eta_2) - M(U_1 - \eta_1) = -M\varphi, \tag{5.2.6}$$

With $\varphi(\tau_0) = 0$, as before, this implies that $A\eta_1 \leq A\eta_2$ which is proving (ii). We can now define the sequences:

$$V_n = AV_{n-1}, \qquad \widetilde{W}_n = A\widetilde{W}_{n-1} \tag{5.2.7}$$

and the following conclusion is true:

$$V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_n \leq \cdots \leq \widetilde{W}_n \leq \cdots \leq \widetilde{W}_2 \leq \widetilde{W}_1 \leq \widetilde{W}_0, \quad on J \tag{5.2.8}$$

consequently

$$lim_{n\to\infty} V_n = \rho \qquad and \quad lim_{n\to\infty} \widetilde{W}_n = \tilde{R} \quad on J \tag{5.2.9}$$

It is easy to show that $\rho, \tilde{R}$ are solutions of (5.2.1) in view of the fact that $V_n, \widetilde{W}_n$ satisfy:

$$D_HV_n = F(t, V_{n-1}) - M(V_n - V_{n-1}), \qquad V_n(\tau_0) = U_0 \tag{5.2.10}$$
$$D_H\widetilde{W}_n = F\left(t+\xi, \widetilde{W}_{n-1}\right) - M\left(\widetilde{W}_n - \widetilde{W}_{n-1}\right), \qquad \widetilde{W}_n(\tau_0) = U_0 \tag{5.2.11}$$

to prove that $\rho, \tilde{R}$ are respectively minimal and maximal solutions of (5.2.1) we must show that if U is any solution of (5.2.1) such that $V_0 \leq U \leq \widetilde{W}_0$ on J, then:

$$V_0 \leq \rho \leq U \leq \tilde{R} \leq \widetilde{W}_0 \ on J \tag{5.2.12}$$

to do that, assume that for some n, $V_n \leq U \leq \widetilde{W}_n$ $on J$, and set:

$$\varphi = V_{n+1} - U \tag{5.2.13}$$

so that:

$$D_H\varphi = F(t,V_n) - M(V_{n+1} - V_n) - F(t,U) \leq$$
$$\leq M(U - V_n) - M(V_{n+1} - V_n) = -M\varphi, \qquad \varphi(\tau_0) = 0 \tag{5.2.14}$$

hence, it follows that: $V_{n+1} \leq U$ *on* J, *in the same way we can show that* $U \leq \widetilde{W}_{n+1}$ *on* J, *setting:*

$$\varphi = \widetilde{W}_{n+1} - U \tag{5.2.15}$$

hence: $V_{n+1} \leq U \leq \widetilde{W}_{n+1}$ *on* J. *This proves by induction that:*

$$V_n \leq U \leq \widetilde{W}_n, \qquad on\ J \tag{5.2.16}$$

for all n, *taking the limit as* $n \to \infty$, *we conclude that* $\rho \leq U \leq \tilde{R}$ *on* J. *And the proof is complete.* ■

5.3. Monotone Iterative Technique for the Sum of Two Functions with Initial Time Difference:

to improve the monotone iterative technique, we shall consider the following IVP:

$$D_H U = F(t,U) + G(t,U), \qquad U(\tau_0) = U_0 \in K_C(R^n) \tag{5.3.1}$$

where $F, G \in [J \times K_C(R^n), K_C(R^n)]$, and $J = [\tau_0, \zeta_0 + T]$.

The following given definition various possible notions of lower and upper solutions relative to (5.3.1) with initial time difference.

Definition 5.3.1: Let $V \in C^1\big[[\tau_0, \tau_0 + T], K_C(R^n)\big], W \in C^1\big[[\zeta_0, \zeta_0 + T], K_C(R^n)\big]$ *and* $V(t) \leq W(t + \xi) = \widetilde{W}(t), t \geq \tau_0$ *Where* $\xi = \zeta_0 - \tau_0$ *for* $\zeta_0 > \tau_0$, *then* V, W *are said to be :*

i) Natural lower and upper solutions of (5.3.1) if:

$$D_H V(t) \leq F\big(t, V(t)\big) + G\big(t, V(t)\big),$$
$$D_H \widetilde{W}(t) \geq F\left(t + \xi, \widetilde{W}(t)\right) + G\left(t + \xi, \widetilde{W}(t)\right), \quad t \in [\tau_0, \zeta_0 + T] \tag{5.3.2}$$

ii) Coupled lower and upper solutions of type I of (5.3.1) if:

$$D_H V(t) \le F(t, V(t)) + G\left(t+\xi, \widetilde{W}(t)\right),$$

$$D_H \widetilde{W}(t) \ge F\left(t+\xi, \widetilde{W}(t)\right) + G(t, V(t)), \qquad t \in [\tau_0, \zeta_0 + T] \qquad (5.3.3)$$

iii) Coupled lower and upper solutions of type II of (5.3.1) if:

$$D_H V(t) \le F\left(t+\xi, \widetilde{W}(t)\right) + G(t, V(t)),$$

$$D_H \widetilde{W}(t) \ge F(t, V(t)) + G\left(t+\xi, \widetilde{W}(t)\right), \qquad t \in [\tau_0, \zeta_0 + T] \qquad (5.3.4)$$

iv) Coupled lower and upper solutions of type III of (5.3.1) if:

$$D_H V(t) \le F\left(t+\xi, \widetilde{W}(t)\right) + G\left(t+\xi, \widetilde{W}(t)\right),$$

$$D_H \widetilde{W}(t) \ge F(t, V(t)) + G(t, V(t)), \qquad t \in [\tau_0, \zeta_0 + T] \qquad (5.3.5)$$

Theorem 5.3.1: Suppose that:

1. *Let $V \in C^1[[\tau_0, \tau_0 + T], K_C(R^n)]$, $W \in C^1[[\zeta_0, \zeta_0 + T], K_C(R^n)]$ are coupled lower and upper solutions of type I relative to (5.3.1), with $V(t) \le W(t+\xi) = \widetilde{W}(t)$ where $\xi = \zeta_0 - \tau_0$.*
2. *$F, G \in C[[\tau_0, \zeta_0 + T] \times K_C(R^n), K_C(R^n)]$, $F(t, X)$ is nondecreasing function in t and X, $G(t, Y)$ is nonincreasing in t and Y, for each $t \in [\tau_0, \zeta_0 + T]$*
3. *F, G map bounded sets into bounded sets in $K_C(R^n)$.*

Then there exist monotone sequences $\{\widetilde{W}_n\}, \{V_n\}$ in $K_C(R^n)$ such that:

$$V_n \to \rho(t), \qquad \widetilde{W}_n \to R(t) \text{ in } K_C(R^n)$$

and $(\rho, \tilde{R})$ are coupled minimal and maximal solutions of (5.3.1) respectively, that is they satisfy:

$$D_H \rho(t) = F(t, \rho) + G(t+\xi, \tilde{R}), \qquad \rho(\tau_0) = U_0 \text{ on } J \qquad (5.3.6)$$

$$D_H \tilde{R}(t) = F(t+\xi, \tilde{R}) + G(t, \rho), \qquad \tilde{R}(\tau_0) = U_0 \text{ on } J. \qquad (5.3.7)$$

Proof:

For each $n \ge 0$, define the unique solutions $V_{n+1}(t), \widetilde{W}_{n+1}(t)$ by:

$$D_H V_{n+1}(t) = F(t, V_n) + G(t+\xi, \widetilde{W}_n), \qquad V_{n+1}(\tau_0) = U_0 \text{ on } J \qquad (5.3.8)$$

$$D_H\widetilde{W}_{n+1}(t) = F\big(t+\xi,\widetilde{W}_n\big) + G(t,V_n), \qquad \widetilde{W}_{n+1}(\tau_0) = U_0 \ \ on\, J \tag{5.3.9}$$

where $V(\tau_0) \le U_0 \le \widetilde{W}(\tau_0)$, *we set* $V_0 = V, \widetilde{W}_0 = \widetilde{W}$. *Then we want to show that:*

$$V_0 \le V_1 \le V_2 \le \cdots \le V_n \le \widetilde{W}_n \le \cdots \le \widetilde{W}_2 \le \widetilde{W}_1 \le \widetilde{W}_0 \tag{5.3.10}$$

we have using the fact that $V_0 \le \widetilde{W}_0$ *and the nondecreasing character of* F*:*

$$D_H V_0(t) \le F(t,V_0) + G\big(t+\xi,\widetilde{W}_0\big) \tag{5.3.11}$$

and

$$D_H V_1(t) = F(t,V_0) + G\big(t+\xi,\widetilde{W}_0\big) \tag{5.3.12}$$

hence,

$$D_H V_0(t) \le D_H V_1(t) \tag{5.3.13}$$

consequently, according to the Theorem (3.1.4), we arrive at $V_0(t) \le V_1(t)$ *in the same method we can show that* $\widetilde{W}_1(t) \le \widetilde{W}_0(t)$. *Now, we will show* $V_1 \le \widetilde{W}_1$ *on* J *from the relations (5.3.8) and (5.3.9) with* $n = 0$

$$D_H V_1(t) = F(t,V_0) + G\big(t+\xi,\widetilde{W}_0\big)$$
$$D_H\widetilde{W}_1(t) = F\big(t+\xi,\widetilde{W}_0\big) + G(t,V_0) \tag{5.3.14}$$

With $V_1(\tau_0) = \widetilde{W}_1(\tau_0) = U_0$, *since* $V_0(t) \le \widetilde{W}_0(t)$ *then:*

$F(t,V_0) \le F\big(t+\xi,\widetilde{W}_0\big)$, $\qquad F(t,X)$ *is monotone nondecreasing in* X, t

$G(t,V_0) \ge G\big(t+\xi,\widetilde{W}_0\big)$, $\qquad G(t,Y)$ *is monotone nonincreasing in* Y, t

so, we obtain:

$$D_H V_1(t) \le F\big(t+\xi,\widetilde{W}_0\big) + G\big(t+\xi,\widetilde{W}_0\big), \quad on\, J \tag{5.3.15}$$
$$D_H\widetilde{W}_1(t) \ge F\big(t+\xi,\widetilde{W}_0\big) + G\big(t+\xi,\widetilde{W}_0\big), \quad on\, J \tag{5.3.16}$$

consequently, we arrive at:

$$D_H V_1(t) \le D_H\widetilde{W}_1(t) \tag{5.3.17}$$

by using the Theorem (3.1.4), we arrive at:

$$V_1(t) \le \widetilde{W}_1(t) \quad on \ J \tag{5.3.18}$$

and as a result, we obtain:

$$V_0 \le V_1 \le \widetilde{W}_1 \le \widetilde{W}_0 \quad on \ J \tag{5.3.19}$$

assume that for some $j > 1$, we have:

$$V_{j-1} \le V_j \le \widetilde{W}_j \le \widetilde{W}_{j-1} \quad on \ J \tag{5.3.20}$$

then we show that:

$$V_j \le V_{j+1} \le \widetilde{W}_{j+1} \le \widetilde{W}_j \quad on \ J \tag{5.3.21}$$

So, we can obtain from (5.3.8) by substituting $n = j - 1$, and $n = j$

$$D_H V_j(t) = F(t, V_{j-1}) + G(t + \xi, \widetilde{W}_{j-1}), \qquad V_j(\tau_0) = U_0 \quad on \ J \tag{5.3.22}$$

$$D_H V_{j+1}(t) = F(t, V_j) + G(t + \xi, \widetilde{W}_j), \qquad V_{j+1}(\tau_0) = U_0 \quad on \ J \tag{5.3.23}$$

and then,

$$D_H V_j(t) = F(t, V_{j-1}) + G(t + \xi, \widetilde{W}_{j-1}) \le F(t, V_j) + G(t + \xi, \widetilde{W}_j) = D_H V_{j+1}(t)$$

consequently, $V_j(t) \le V_{j+1}(t)$ on J, in the same way we can show $\widetilde{W}_{j+1} \le \widetilde{W}_j$ on J. Next, we show that $V_{j+1} \le \widetilde{W}_{j+1}$, $t \in J$ we have:

$$D_H V_{j+1}(t) = F(t, V_j) + G(t + \xi, \widetilde{W}_j), \qquad V_{j+1}(\tau_0) = U_0 \quad on \ J \tag{5.3.24}$$

$$D_H \widetilde{W}_{j+1}(t) = F(t + \xi, \widetilde{W}_j) + G(t, V_j), \qquad \widetilde{W}_{j+1}(\tau_0) = U_0 \quad on \ J \tag{5.3.25}$$

then we can write:

$$D_H V_{j+1}(t) = F(t, V_j) + G(t + \xi, \widetilde{W}_j) \le F(t + \xi, \widetilde{W}_j) + G(t + \xi, \widetilde{W}_j) \tag{5.3.26}$$

$$D_H \widetilde{W}_{j+1}(t) = F(t + \xi, \widetilde{W}_j) + G(t, V_j) \ge F(t + \xi, \widetilde{W}_j) + G(t + \xi, \widetilde{W}_j) \tag{5.3.27}$$

and as a result:

$$V_{j+1}(t) \le \widetilde{W}_{j+1}(t) \quad on \ J \tag{5.3.28}$$

hence the relation:

$$V_j \le V_{j+1} \le \widetilde{W}_{j+1} \le \widetilde{W}_j \quad on \ J \tag{5.3.29}$$

follows and consequently, by induction the relation (5.3.10) is valid for all n*. Clearly sequences* $\{\widetilde{W}_n\}, \{V_n\}$ *are uniformly bounded on* J*. Then, we will show that they are equicontinuous on* J*, consider for any* $t_1 < t_2$ *where* $t_1, t_2 \in J$

$$\forall \varepsilon > 0, \qquad \exists \delta = \delta(\varepsilon) > 0, \qquad \forall n \in N, \qquad |t_2 - t_1| < T = \delta = \frac{\varepsilon}{M}$$

$$\begin{aligned} D[V_n(t_2), V_n(t_1)] = & \\ &= D\Big[U_0 \\ &+ \int_{\tau_0}^{t_2} \{F(s, V_{n-1}(s)) + G(s + \xi, \widetilde{W}_{n-1}(s))\} ds, U_0 \\ &+ \int_{\tau_0}^{t_1} \{F(s, V_{n-1}(s)) + G(s + \xi, \widetilde{W}_{n-1}(s))\} ds\Big] \\ &\leq \int_{t_1}^{t_2} D\big[\{F(s, V_{n-1}(s)) + G(s + \xi, \widetilde{W}_{n-1}(s))\}, \theta\big] ds \\ &\leq M|t_2 - t_1| < MT = \varepsilon \end{aligned} \tag{5.3.30}$$

we used the properties of integral and the metric D*, together with the fact that* $F + G$ *are bounded since* $\{\widetilde{W}_n\}, \{V_n\}$ *are uniformly bounded, hence* $\{V_n\}$ *is equicontinuous on* J*, the corresponding Ascoli's Theorem gives a subsequence* $\{V_{n_k}\}$ *which converges uniformly to* $\rho(t) \in K_C(R^n)$*, and since* $\{V_n\}$ *is monotone nondecreasing sequence, the entire sequence* $\{V_n\}$ *converges uniformly to* $\rho(t)$ *on* J*. The same arguments apply to the sequence* $\{\widetilde{W}_n\}$ *and* $\widetilde{W}_n \to \tilde{R}$ *uniformly on* J*, it therefore follows, using the integral representation of (5.3.8) and (5.3.9) that* $\rho(t)$ *and* $\tilde{R}(t)$ *satisfy:*

$$D_H \rho(t) = F(t, \rho) + G(t + \xi, \tilde{R}), \qquad \rho(\tau_0) = U_0 \ \text{ on } J \tag{5.3.31}$$

$$D_H \tilde{R}(t) = F(t + \xi, \tilde{R}) + G(t, \rho), \qquad \tilde{R}(\tau_0) = U_0 \ \text{ on } J \tag{5.3.32}$$

and that

$$V_0 \leq \rho \leq \tilde{R} \leq \widetilde{W}_0. \tag{5.3.33}$$

we next claim that $(\rho, \tilde{R})$ *are coupled minimal and maximal solutions of (5.3.1), that is, if* $U(t)$ *is any solution of (5.3.1) such that:*

$$V_0 \leq U(t) \leq \widetilde{W}_0 \tag{5.3.34}$$

then

$$V_0 \le \rho \le U(t) \le \tilde{R} \le \widetilde{W}_0\,,\; t \in J \qquad (5.3.35)$$

suppose that for some n,

$$V_n \le U(t) \le \widetilde{W}_n,\; t \in J \qquad (5.3.36)$$

then we have using the monotone nature of F *and* G *and (5.3.36):*

$$D_H U = F(t,U) + G(t,U) \ge F(t,V_n) + G\big(t+\xi,\widetilde{W}_n\big), \qquad U(\tau_0) = U_0 \qquad (5.3.37)$$

$$D_H V_{n+1} = F(t,V_n) + G\big(t+\xi,\widetilde{W}_n\big), \qquad V_{n+1}(\tau_0) = U_0 \qquad (5.3.38)$$

hence,

$$D_H U \ge D_H V_{n+1} \qquad (5.3.39)$$

consequently, according to Theorem (3.1.4) that:

$$V_{n+1} \le U \quad on\;\; J \qquad (5.3.40)$$

in the same way,

$$\widetilde{W}_{n+1} \ge U \quad on\;\; J \qquad (5.3.41)$$

hence, by induction the relation (5.3.36) is true for all $n \ge 1$, *taking the limit as* $n \to \infty$, *we get (5.3.35) proving the claim. The proof is complete.*■

Corollary 5.3.1: If, in addition to the assumptions of Theorem (5.3.1) F *and* G *satisfy whenever* $\tilde{X} \ge Y, \tilde{X}, Y \in K_C(R^n)$

$$F\big(t+\xi,\tilde{X}\big) \le F(t,Y) + N_1(\tilde{X} - Y) \qquad (5.3.42)$$

$$G\big(t+\xi,\tilde{X}\big) + N_2(\tilde{X} - Y) \ge G(t,Y) \qquad (5.3.43)$$

where $N_1, N_2 > 0$ *then* $\rho = \tilde{R} = U$ *is the unique solution of (5.3.1).*

Proof 5.3.1: Since $\rho < \tilde{R}$, *and then* $\tilde{R} = \rho + m$ *or* $m = \tilde{R} - \rho$, *now*

$$\begin{aligned} D_H\rho + D_H m = D_H\tilde{R} &= F\big(t+\xi,\tilde{R}\big) + G(t,\rho) \\ &\le F(t,Y) + N_1\big(\tilde{X} - Y\big) + G\big(t+\xi,\tilde{X}\big) + N_2\big(\tilde{X} - Y\big) \\ &= D_H\rho + (N_1 + N_2)m \end{aligned} \qquad (5.3.44)$$

it means that,

$$D_H m \leq (N_1 + N_2)m, \qquad m(\tau_0) = 0 \tag{5.3.45}$$

using the comparison results, we can obtain that $\tilde{R} \leq \rho$ on J, proving the uniqueness of $\rho = \tilde{R} = U$. Completing the proof.■

5.4. Monotone Iterative Technique for Sum of Three Functions with Initial Time Difference

To extend the monotone iterative technique, we will take the IVP:

$$D_H U = F(t, U) + G(t, U) + H(t, U), \qquad U(\tau_0) = U_0 \in K_C(R^n) \tag{5.4.1}$$

where $F, G, H \in C[J \times K_C(R^n), K_C(R^n)]$, and $J = [\tau_0, \tau_0 + T]$.

We need the following definitions which various possible notions of lower and upper solutions relative to (5.4.1) with initial time difference.

Definition 5.4.1: Let $V \in C^1\big[[\tau_0, \tau_0 + T], K_C(R^n)\big]$, $W \in C^1\big[[\zeta_0, \zeta_0 + T], K_C(R^n)\big]$and $V(t) \leq W(t + \xi) = \widetilde{W}(t)$, $t \geq \tau_0$ Where, $\xi = \zeta_0 - \tau_0$ for $\zeta_0 > \tau_0$. Then V, W are said to be:

i) *Natural lower and upper solutions of (5.4.1) if:*

$$D_H V(t) \leq F\big(t, V(t)\big) + G\big(t, V(t)\big) + H\big(t, V(t)\big),$$
$$D_H \widetilde{W}(t) \geq F\left(t + \xi, \widetilde{W}(t)\right) + G\left(t + \xi, \widetilde{W}(t)\right) + H\left(t + \xi, \widetilde{W}(t)\right), \tag{5.4.2}$$
$$t \in [\tau_0, \zeta_0 + T]$$

ii) *Coupled lower and upper solutions of type I of (5.4.1) if:*

$$D_H V(t) \leq F\big(t, V(t)\big) + G\left(t + \xi, \widetilde{W}(t)\right) + H\left(t + \xi, \widetilde{W}(t)\right),$$
$$D_H \widetilde{W}(t) \geq F\left(t + \xi, \widetilde{W}(t)\right) + G\big(t, V(t)\big) + H\big(t, V(t)\big), \tag{5.4.3}$$
$$t \in [\tau_0, \zeta_0 + T]$$

iii) *Coupled lower and upper solutions of type II of (5.4.1) if:*

$$D_H V(t) \leq F\left(t + \xi, \widetilde{W}(t)\right) + G\big(t, V(t)\big) + H\left(t + \xi, \widetilde{W}(t)\right)$$

$$D_H\widetilde{W}(t) \geq F\big(t, V(t)\big) + G\left(t+\xi, \widetilde{W}(t)\right) + H\big(t, V(t)\big),$$
$$t \in [\tau_0, \zeta_0 + T] \tag{5.4.4}$$

iv) *Coupled lower and upper solutions of type III of (5.4.1) if:*

$$D_HV(t) \leq F\left(t+\xi, \widetilde{W}(t)\right) + G\left(t+\xi, \widetilde{W}(t)\right) + H\big(t, V(t)\big),$$
$$D_H\widetilde{W}(t) \geq F\big(t, V(t)\big) + G\big(t, V(t)\big) + H\left(t+\xi, \widetilde{W}(t)\right), \tag{5.4.5}$$
$$t \in [\tau_0, \zeta_0 + T]$$

v) *Coupled lower and upper solutions o type IV of (5.4.1) if:*

$$D_HV(t) \leq F\big(t, V(t)\big) + G\big(t, V(t)\big) + H\left(t+\xi, \widetilde{W}(t)\right),$$
$$D_H\widetilde{W}(t) \geq F\left(t+\xi, \widetilde{W}(t)\right) + G\left(t+\xi, \widetilde{W}(t)\right) + H\big(t, V(t)\big), \tag{5.4.6}$$
$$t \in [\tau_0, \zeta_0 + T]$$

vi) *Coupled lower and upper solutions of type V of (5.4.1) if:*

$$D_HV(t) \leq F\big(t, V(t)\big) + G\left(t+\xi, \widetilde{W}(t)\right) + H\big(t, V(t)\big),$$
$$D_H\widetilde{W}(t) \geq F\left(t+\xi, \widetilde{W}(t)\right) + G\big(t, V(t)\big) + H\left(t+\xi, \widetilde{W}(t)\right), \tag{5.4.7}$$
$$t \in [\tau_0, \zeta_0 + T]$$

vii) *Coupled lower and upper solutions of type VI of (5.4.1) if:*

$$D_HV(t) \leq F\left(t+\xi, \widetilde{W}(t)\right) + G\big(t, V(t)\big) + H\big(t, V(t)\big)$$
$$D_H\widetilde{W}(t) \geq F\big(t, V(t)\big) + G\left(t+\xi, \widetilde{W}(t)\right) + H\left(t+\xi, \widetilde{W}(t)\right), \tag{5.4.8}$$
$$t \in [\tau_0, \zeta_0 + T]$$

viii) *Coupled lower and upper solutions of type VII of (5.4.1) if*

$$D_HV(t) \leq F\left(t+\xi, \widetilde{W}(t)\right) + G\left(t+\xi, \widetilde{W}(t)\right) + H\left(t+\xi, \widetilde{W}(t)\right)$$
$$D_H\widetilde{W}(t) \geq F\big(t, V(t)\big) + G\big(t, V(t)\big) + H\big(t, V(t)\big),\ t \in [\tau_0, \zeta_0 + T] \tag{5.4.9}$$

Theorem 5.4.1: Assume that:

- *Let* $V \in C^1[[\tau_0, \tau_0 + T], K_C(R^n)], W \in C^1[[\zeta_0, \zeta_0 + T], K_C(R^n)]$ *are coupled lower and upper solutions of Type I relative to (5.4.1) with* $V(t) \leq W(t + \xi) = \widetilde{W}(t)$ *where* $\xi = \zeta_0 - \tau_0$.
- F, G *and* $H \in C[[\tau_0, \zeta_0 + T] \times K_C(R^n), K_C(R^n)], F(t, X)$ *is nondecreasing function in* t *and* X , $G(t, Y)$, $H(t, Z)$ *are nonincreasing functions in* Y *and* Z, *respectively and nonincreasing functions in* t.
- F, G *and* H *map bounded sets into bounded sets in* $K_C(R^n)$.

Then there exist monotone sequences $\{V_n\}$, $\{\widetilde{W}_n\}$ *in* $K_C(R^n)$ *such that:*

$$V_n \to \rho(t), \qquad \widetilde{W}_n \to R(t) \ \ in \ \ K_C(R^n)$$

and $(\rho, \tilde{R})$ *are Type I coupled minimal and maximal solutions of (5.4.1) respectively, that is they satisfy:*

$$D_H\rho(t) = F(t, \rho) + G(t + \xi, \tilde{R}) + H(t + \xi, \tilde{R}), \ \ \rho(\tau_0) = U_0 \ \ on \ J \qquad (5.4.10)$$

$$D_H\tilde{R}(t) = F(t + \xi, \tilde{R}) + G(t, \rho) + H(t, \rho), \qquad \tilde{R}(\tau_0) = U_0 \ \ on \ J \qquad (5.4.11)$$

Proof 5.4.1: For each $n \geq 0$, *define the unique solutions* $V_{n+1}(t), \widetilde{W}_{n+1}(t)$ *by:*

$$D_H V_{n+1}(t) = F(t, V_n) + G(t + \xi, \widetilde{W}_n) + H(t + \xi, \widetilde{W}_n), \qquad (5.4.12)$$
$$V_{n+1}(\tau_0) = U_0 \ on \ J$$

$$D_H \widetilde{W}_{n+1}(t) = F(t + \xi, \widetilde{W}_n) + G(t, V_n) + H(t, V_n), \qquad (5.4.13)$$
$$\widetilde{W}_{n+1}(\tau_0) = U_0 \ \ on \ J$$

where $V(\tau_0) \leq U_0 \leq \widetilde{W}(\tau_0)$, *we set* $V_0 = V, \widetilde{W}_0 = \widetilde{W}$, *our aim to prove:*

$$V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_n \leq \widetilde{W}_n \leq \cdots \leq \widetilde{W}_2 \leq \widetilde{W}_1 \leq \widetilde{W}_0 \qquad (5.4.14)$$

we have using the fact that $V_0 \leq \widetilde{W}_0$ *and the nondecreasing character of* F, G *and* H:

$$D_H V_0(t) \leq F(t, V_0) + G(t + \xi, \widetilde{W}_0) + H(t + \xi, \widetilde{W}_0) \qquad (5.4.15)$$

and, we have:

$$D_H V_1(t) = F(t, V_0) + G(t + \xi, \widetilde{W}_0) + H(t + \xi, \widetilde{W}_0) \qquad (5.4.16)$$

hence,

$$D_H V_0(t) \leq D_H V_1(t) \tag{5.4.17}$$

consequently, utilizing Theorem (3.1.4) we arrive at $V_0(t) \leq V_1(t)$. *A similar argument shows that* $\widetilde{W}_1(t) \leq \widetilde{W}_0(t)$. *We next prove that* $V_1 \leq \widetilde{W}_1$ *on* J. *To do this, consider:*

$$D_H V_1(t) = F(t, V_0) + G\left(t + \xi, \widetilde{W}_0\right) + H(t + \xi, \widetilde{W}_0) \tag{5.4.18}$$

$$D_H \widetilde{W}_1(t) = F\left(t + \xi, \widetilde{W}_0\right) + G(t, V_0) + H(t, V_0) \tag{5.4.19}$$

With $V_1(\tau_0) = \widetilde{W}_1(\tau_0) = U_0$, *since* $V_0(t) \leq \widetilde{W}_0(t)$ *then:*

$$F(t, V_0) \leq F\left(t + \xi, \widetilde{W}_0\right), \qquad F(t, X) \text{ is monotone nondecreasing in } X, t$$

$$G(t, V_0) \geq G\left(t + \xi, \widetilde{W}_0\right), \qquad G(t, Y) \text{ is monotone nonincreasing in } Y, t$$

$$H(t, V_0) \geq H\left(t + \xi, \widetilde{W}_0\right), \qquad H(t, Z) \text{ is monotone nondecreasing in } Z, t$$

so, we obtain:

$$D_H V_1(t) \leq F\left(t + \xi, \widetilde{W}_0\right) + G\left(t + \xi, \widetilde{W}_0\right) + H\left(t + \xi, \widetilde{W}_0\right) \text{ on } J \tag{5.4.20}$$

$$D_H \widetilde{W}_1(t) \geq F\left(t + \xi, \widetilde{W}_0\right) + G\left(t + \xi, \widetilde{W}_0\right) + H\left(t + \xi, \widetilde{W}_0\right) \text{ on } J \tag{5.4.21}$$

hence,

$$D_H V_1(t) \leq D_H \widetilde{W}_1(t) \tag{5.4.22}$$

consequently, by Theorem (3.1.4) we arrive at $V_1(t) \leq \widetilde{W}_1(t)$, *and as a result, we obtain:*

$$V_0 \leq V_1 \leq \widetilde{W}_1 \leq \widetilde{W}_0 \quad \text{on } J \tag{5.4.23}$$

assume that for some $j > 1$, *we have:*

$$V_{j-1} \leq V_j \leq \widetilde{W}_j \leq \widetilde{W}_{j-1} \quad \text{on } J \tag{5.4.24}$$

then, we show that:

$$V_j \leq V_{j+1} \leq \widetilde{W}_{j+1} \leq \widetilde{W}_j \quad \text{on } J \tag{5.4.25}$$

to do this, consider:

$$D_H V_j(t) = F\left(t, V_{j-1}\right) + G\left(t + \xi, \widetilde{W}_{j-1}\right) + H\left(t + \xi, \widetilde{W}_{j-1}\right), \tag{5.4.26}$$
$$V_j(\tau_0) = U_0 \quad \text{on } J$$

$$D_H V_{j+1}(t) = F(t, V_j) + G(t+\xi, \widetilde{W}_j) + H(t+\xi, \widetilde{W}_j), \quad (5.4.27)$$
$$V_{j+1}(\tau_0) = U_0 \quad on\ J$$

so, we can write:

$$D_H V_j(t) = F(t, V_{j-1}) + G(t+\xi, \widetilde{W}_{j-1}) + H(t+\xi, \widetilde{W}_{j-1})$$
$$\leq F(t, V_j) + G(t+\xi, \widetilde{W}_j) + H(t+\xi, \widetilde{W}_j) = D_H V_{j+1}(t) \quad (5.4.28)$$

consequently, $V_j(t) \leq V_{j+1}(t)$ *on* J*, similarly we can get* $\widetilde{W}_{j+1} \leq \widetilde{W}_j$ *on* J*. Next, we show that* $V_{j+1} \leq \widetilde{W}_{j+1}$, $t \in J$*, we have:*

$$D_H V_{j+1}(t) = F(t, V_j) + G(t+\xi, \widetilde{W}_j) + H(t+\xi, \widetilde{W}_j), \quad (5.4.29)$$
$$V_{j+1}(\tau_0) = U_0 \quad on\ J$$

$$D_H \widetilde{W}_{j+1}(t) = F(t+\xi, \widetilde{W}_j) + G(t, V_j) + H(t, V_j), \quad (5.4.30)$$
$$\widetilde{W}_{j+1}(\tau_0) = U_0 \quad on\ J$$

then, we can write:

$$D_H V_{j+1}(t) = F(t, V_j) + G(t+\xi, \widetilde{W}_j) + H(t+\xi, \widetilde{W}_j) \leq$$
$$\leq F(t+\xi, \widetilde{W}_j) + G(t+\xi, \widetilde{W}_j) + H(t+\xi, \widetilde{W}_j) \quad (5.4.31)$$

$$D_H \widetilde{W}_{j+1}(t) = F(t+\xi, \widetilde{W}_j) + G(t, V_j) + H(t, V_j) \geq$$
$$\geq F(t+\xi, \widetilde{W}_j) + G(t+\xi, \widetilde{W}_j) + H(t+\xi, \widetilde{W}_j) \quad (5.4.32)$$

and as a result:

$$V_{j+1}(t) \leq \widetilde{W}_{j+1}(t) \quad on\ J \quad (5.4.33)$$

hence:

$$V_j \leq V_{j+1} \leq \widetilde{W}_{j+1} \leq \widetilde{W}_j \quad on\ J \quad (5.4.34)$$

follows, and consequently by induction (5.4.14) is valid for all n*. Clearly sequences* $\{\widetilde{W}_n\}, \{V_n\}$ *are uniformly bounded on. To show that they are equicontinuous, consider for any* $t_1 < t_2$ *where* $t_1, t_2 \in J$

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon), \quad \forall n \in N, \quad |t_2 - t_1| < T = \delta = \frac{\varepsilon}{M}$$

$$D[V_n(t_2), V_n(t_1)] =$$

$$= D\left[U_0 + \int_{\tau_0}^{t_2} \left\{F\left(s, V_{n-1}(s)\right) + G\left(s+\xi, \widetilde{W}_{n-1}(s)\right)\right.\right.$$

$$\left. + H\left(s+\xi, \widetilde{W}_{n-1}(s)\right)\right\} ds,\ U_0$$

$$+ \int_{\tau_0}^{t_1} \{F(s, V_{n-1}(s)) + G\left(s+\xi, \widetilde{W}_{n-1}(s)\right)$$

$$\left. + H\left(s+\xi, \widetilde{W}_{n-1}(s)\right)\}ds\right]$$

$$\leq \int_{t_1}^{t_2} D\left[\{F(s, V_{n-1}(s)) + G\left(s+\xi, \widetilde{W}_{n-1}(s)\right)\right.$$

$$\left. + H\left(s+\xi, \widetilde{W}_{n-1}(s)\right)\}, \theta\right] ds \leq M|t_2 - t_1| < MT = \varepsilon \qquad (5.4.35)$$

Since we used the properties of integral and the metric D, together with the fact that $F + G + H$ are bounded since $\{\widetilde{W}_n\}, \{V_n\}$ are uniformly bounded, hence $\{V_n\}$ is equicontinuous on J, the corresponding Ascoli's theorem gives a subsequence $\{V_{n_k}\}$ which converges uniformly to $\rho(t) \in K_C(R^n)$, and since $\{V_n\}$ is monotone nondecreasing sequence, the entire sequence $\{V_n\}$ converges uniformly to $\rho(t)$ on J. In the same way we can show that the sequence $\{\widetilde{W}_n\}$ and $\widetilde{W}_n \to \tilde{R}$ uniformly on J, it therefore follows, using the integral representation of (5.4.12) and (5.4.13) that $\rho(t)$ and $\tilde{R}(t)$ satisfy the relations:

$$D_H\rho(t) = F(t,\rho) + G\left(t+\xi, \tilde{R}\right) + H\left(t+\xi, \tilde{R}\right),\ \rho(\tau_0) = U_0 \ \text{ on } J \qquad (5.4.36)$$

$$D_H\tilde{R}(t) = F\left(t+\xi, \tilde{R}\right) + G(t,\rho) + H(t,\rho), \qquad \tilde{R}(\tau_0) = U_0 \ \text{ on } J \qquad (5.4.37)$$

as a result,

$$V_0 \leq \rho \leq \tilde{R} \leq \widetilde{W}_0 \qquad (5.4.38)$$

we next claim that $(\rho, \tilde{R})$ are coupled minimal and maximal solutions of (5.4.1), that is, if $U(t)$ is any solution of (5.4.1) such that:

$$V_0 \leq U(t) \leq \widetilde{W}_0 \qquad (5.4.39)$$

then:

$$V_0 \leq \rho \leq U(t) \leq \tilde{R} \leq \widetilde{W}_0,\ \ t \in J \qquad (5.4.40)$$

suppose that for some n,

$$V_n \leq U(t) \leq \widetilde{W}_n, \ t \in J \tag{5.4.41}$$

then we have using the monotone nature of F, G *and* H *and (5.4.41):*

$$D_H U = F(t,U) + G(t,U) + H(t,U) \geq$$
$$\geq F(t,V_n) + G\left(t+\xi,\widetilde{W}_n\right) + H\left(t+\xi,\widetilde{W}_n\right), \qquad U(\tau_0) = U_0 \tag{4.5.42}$$

$$D_H V_{n+1} = F(t,V_n) + G\left(t+\xi,\widetilde{W}_n\right) + H\left(t+\xi,\widetilde{W}_n\right), \ V_{n+1}(\tau_0) = U_0 \tag{4.5.43}$$

hence,

$$D_H U \geq D_H V_{n+1} \tag{4.5.44}$$

consequently,

$$V_{n+1} \leq U \quad \text{on } J \tag{5.4.45}$$

similarly,

$$\widetilde{W}_{n+1} \geq U \quad \text{on } J \tag{5.4.46}$$

hence, the relation (5.4.41) is true for all $n \geq 1$ *by induction, taking the limit* $n \to \infty$ *we get (5.4.40) proving the claim. Therefore, this completes the proof of the theorem.*■

Corollary 5.4.1: If, in addition to the assumptions of Theorem (5.4.1) F *and* G *satisfy whenever* $\tilde{X} \geq Y, \tilde{X}, Y \in K_C(R^n)$

$$F\left(t+\xi,\tilde{X}\right) \leq F(t,Y) + N_1(\tilde{X} - Y) \tag{5.4.47}$$
$$G\left(t+\xi,\tilde{X}\right) + N_2(\tilde{X} - Y) \geq G(t,Y) \tag{5.4.48}$$
$$H\left(t+\xi,\tilde{X}\right) + N_3(\tilde{X} - Y) \geq G(t,Y) \tag{5.4.49}$$

where $N_1, N_3, N_2 > 0$*, then* $\rho = \tilde{R} = U$ *is the unique solution of (5.4.1).*

Proof 5.4.1: Since $\rho < \tilde{R}$*, and then* $\tilde{R} = \rho + m$ *or* $m = \tilde{R} - \rho$*, now*

$$D_H\rho + D_H m = D_H\tilde{R} = F\left(t+\xi,\tilde{R}\right) + G(t,\rho) + H(t,\rho)$$
$$\leq F(t,\rho) + N_1\left(\tilde{R}-\rho\right) + G\left(t+\xi,\tilde{R}\right) + N_2\left(\tilde{R}-\rho\right)$$
$$+ H\left(t+\xi,\tilde{R}\right) + N_2(\tilde{R}-\rho) = D_H\rho + (N_1+N_2+N_3)m \tag{5.4.50}$$

which means,

$$D_H m \leq (N_1 + N_2 + N_3)m, \qquad m(\tau_0) = 0 \tag{5.4.51}$$

which leads to $\tilde{R} \leq \rho$ *on* J*, proving the uniqueness of* $\rho = \tilde{R} = U$*. Completing the proof.* ■

6. CONCLUSION

We have studied the monotone iterative technique for set valued differential equations to generalization to study this technique. First, we study it for single function, and then we have also studied them for two and three functions which paved the way to study it for four and five functions that lead us to genralize this method the more. Hence, we have also worked on the monotonte iterative technique with initial time difference for single function, and also for two and three functions which in turn could be generalized again for finite systems in future studies.

REFERENCES

[1] Lakshmikantham V., Bhaskar T.G., Devi J.V., (2005), **“**Theory of set Differential Equations in Metric Spaces.”, Cambridge. Sci Pub.

[2] Lakshmikantham V., Leela S., Drici Z., McRae F.A., (2009), “Theory of Causal Differential Equations.”, Atlantis Press, Amsterdam – Paris.

[3] Dennis G. Zill, Mihael R. Cullen, “Differential Equations with Boundary Value Problems.” Brooks/ Cole Publishing Company. CA 93950 USA.

[4] Corduneanu, C. (2002), “ Functional Equations with Causal Operators.”,Taylor and Francis, London.

[5] McNabb A., Weir G., (1988), “Comparison theorems for causal functional differential equations.”, Proc. AMS 104, 449-452.

[6] Ladde G.S., Lakshmikantham V., Vatsala A.S., (1985), “Monotone Iterative Techniques for Nonlinear Differential Equations.” Pitman, London.

[7] Bhaskar T.G., Farzana A. McRae, (2002), “Monotone Iterative Techniques for Nonlinear Problems Involving TheDifference of Two Monotone Functions.”, Applied Mathematics and Computation 133,187-192.

[8] Drici Z., McRae F.A., Vasundhara Devi J., “Monotone iterative technique for periodic value problems with causal operators.” Nonlinear Analysis, 64(2006), 1271-1277.

[9] Yakar C., Arslan I., Cicek M., (2015), “Monotone Iterative Technique by Upper and Lower Solutions with Initial Time Difference.” Miskolc Mathematical Notes 16(1), 575-586.

[10] Yakar C., Bal B., Yakar A., (2011), “Monotone Technique in Terms of Two Monotone Functions in Finite Systems.”, Journal of Concrete and Applicable Mathematics, 233-239, ISSN: 1548-5390.

[11] Köksal S., Yakar C., (2002), “Generalized quasilinearization method with initial time difference.”, Simulation, an International Journal of Electrical, Electronic and other Physical Systems, 24(5).

[12] Yakar C., Yakar A., (2007), “An Extension of the Quasilinearization Method with Initial Time Difference.”, Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis. Watam Press. Waterloo, 14 (S2), 275-279, 1-305.

[13] Yakar C., Yakar A., (2009), “Further Generalization of Quasilinearization Method with Initial Time Difference.”, Journal of Applied Functional Analysis, 714-727.

BIOGRAPHY

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