CUKUROVA UNIVERSITY

INSTITUTE OF NATURAL AND APPLIED SCIENCES

PhD THESIS

Harmonic Analysis Approach Method to Some Problems in Additive Combinatorics Theory

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ABSTRACT

In this thesis, we develop better formulas for the distribution of quadratic residues in certain subsets of finite fields. For this purpose, we consider the sets $S_p = \{t^2 : t \in \mathbb{F}_p\}$ and $C_p = \{t^3 : t \in \mathbb{F}_p\}$, and we use the results on Gauss and Kummer sums. We prove that for any integer $k \geq 3$ and for an odd prime number p, the number of k-term arithmetic progressions in S_p is given by

$$\frac{p^2}{2^k} + R,$$

where

$$|R| \leq \left(\frac{k-2}{4} - \frac{k-2}{2^{k-1}}\right) \cdot p^{\frac{3}{2}} + c_k \cdot p$$

and c_k is a computable constant depending only on k. The proof also uses finite Fourier analysis and certain types of Weil estimates. Also, we obtain some formulas that give the exact number of arithmetic progressions of length ℓ in the set S_p when $\ell \in \{3,4,5\}$ and p is an odd prime number. For $\ell = 4,5$, our formulas are based on the number of points on certain elliptic curves, and the error term is best possible due to the Sato-Tate conjecture.

Keywords: Arithmetic progressions, Arithmetic geometry, Weil estimates, Sato-Tate conjecture.

ÇUKUROVA ÜNİVERSİTESİ
FEN BİLİMLERİ ENSTİTÜSÜ

DOKTORA TEZİ

Toplamsal Kombinatorik Teorisinden Gelen Harmonik Analiz Problemleri

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Sadık EYİDOĞAN

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ÖZ

Bu tezde, sonlu cisimlerin belirli alt kümelerindeki aritmetik dizilerin sayısını bulmaya odaklandık. Bu amaç doğrultusunda, sonlu cisimlerin karesel ve küpsel sayıların oluşturduğu $S_p = \{t^2 : t \in \mathbb{F}_p\}$ ve $C_p = \{t^3 : t \in \mathbb{F}_p\}$ kümelerini ele alıp Gauss ve Kummer toplamları üzerine var olan sonuçları kullandık. Tezimizin esas bulgusu olan $k \geq 3$ tam sayısı ve p tek asal sayısı için S_p'deki k uzunluklu aritmetik dizilerin sayısının, hata terimi

$$|R| \leq \left(\frac{k-2}{4} - \frac{k-2}{2^{k-1}}\right) \cdot p^{\frac{3}{2}} + c_k \cdot p$$

olacak şekilde

$$\frac{p^2}{2^k} + R$$

ile verildiğini ispat ettik. Buradaki c_k değeri, sadece k değerine bağlı hesaplanabilir bir sabittir. İspatımız içeriği sonlu Fourier analizi ve birlirli tipteki Weil sanılarından oluşmaktadır. Ayrıca tezimizde sırasıyla $\ell = 3,4,5$ için S_p'deki ℓ uzunluklu aşikar olmayan aritmetik dizilerin sayısını tam olarak hesaplayan formüller elde edildi. $\ell = 4,5$ durumunda elde edilen formüller içerisinde eliptik eğriler mevcuttur. $\ell = 4,5$ için formüller eliptik eğriler içerdiğinden hata terimi mevcuttur ve bu hata teriminin en iyisi olduğu Sato-Tate sanısını kullanılarak gösterildi. Son bölümde, sonlu cisimlerde S_p kümesindeki 3 uzunluklu aritmetik dizilerin sayını bulma problemini, küpsel elemanların oluşturduğu C_p kümesinde ele alınmıştır.

Anahtar Kelimeler: Aritmetik diziler, Aritmetik geometri, Weil sanıları, Sato-Tate sanısı.

GENİŞLETİLMİŞ ÖZET

$\mathbb{F}_p$ sonlu bir cisim olmak üzere, $\mathbb{F}_p$ üzerindeki *karesel sayıların* oluşturduğu $S_p = \{t^2 : t \in \mathbb{F}_p\}$ kümesi tanımlansın. Tarihsel sürece baktığımızda sonlu cisimler üzerindeki karesel sayıların dağılımı farklı bir çok matematikçi tarafından çalışılmıştır. Geçmiş 100 yıldan fazla süreçte $k \geq 1$ ve $p > k$ tek asal sayısı için $\mathbb{F}_p^\times = \mathbb{F}_p \setminus \{0\}$'de kaç tane davranışı önceden belirlenen karesel veya karesel olmayan $(a, a+1, \ldots, a+k-1)$ k'lısının mevcut olduğu en iyi şekilde saymak istenmiştir. Daha iyi ifade edilecek olunursa, seçilen k tane $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$ işaretleri için

$$N_p(\varepsilon_1, \ldots, \varepsilon_k) = \left| \left\{ a \in \mathbb{F}_p^\times : \left(\frac{a+i-1}{p}\right) = \varepsilon_i \text{ for } i = 1, \ldots, k \right\} \right|$$

değerini hesaplamak asıl amaçtır. Buradaki $\left(\frac{\cdot}{p}\right)$ gösterimi Legendre sembolü olup $\mathbb{F}_p$'deki bir elemanın karesel olup olmağına karar verir. Ayrıca $(a, a+1, \ldots, a+k-1)$ k'lısına, k uzunluklu karesel yol deseni denir.

1896 yılında Aladov yayınladığı (Aladov, 1896) çalışmasıyla $\mathbb{F}_p^\times$'deki 2 uzunluklu karesel tüm yol desenlerin sayısını ve 3 uzunluklu karesel bazı yol desenlerin sayısını belirlemiştir. 1930'ların başında ise Davenport, (Davenport, 1931 ve 1933) çalışmalarında $k \geq 4$ için bu problemi ele almıştır. İlerleyen süreçte Kazt'ın (Chapter 9, Katz, 1988) çalışmasında verilen bir k uzunluklu karesel yol deseni ve $p > k$ tek asal sayısı için

$$\left| N_p(\varepsilon_1, \ldots, \varepsilon_k) - \frac{p}{2^k} \right| < (k-1)\sqrt{p} + \frac{k}{2}$$

eşitsizliğinin sağlandığı gösterilmiştir. Üstelik karesel yol desenin ardışık olma şartını da ortadan kaldırarak keyfi farklardan oluşan desenler için de hesaplanmıştır:

$p > k$ asal sayı ve $c_1, \ldots, c_k \in \mathbb{F}_p$ de farklı değerler olmak üzere

$$\left\{ a \in \mathbb{F}_p^\times : \left(\frac{a+c_i}{p}\right) = \varepsilon_i \text{ for } i = 1, \ldots, k \right\}$$

kümesinin eleman sayısı N_p ile gösterelim. Bu durumda,

$$\left| N_p - \frac{p}{2^k} \right| < (k-1)\sqrt{p} + \frac{k}{2}$$

eşitsizliği sağlanır.

Her bir i değeri için $\varepsilon_i = 1$ olarak alınırsa, yukarıdaki sonuçtan S_p'deki k uzunluklu aritmetik dizi $(k-APs)$'lerin sayının hata terimi

$$|H| \leq (k-1) \cdot p^{\frac{3}{2}} + O_k(p)$$

olacak şekilde

$$\frac{p^2}{2^k} + H$$

ile verildiği sonucuna ulaşılır. Bu tezde, ilk olarak S_p'deki k uzunluklu aritmetik dizilerin sayısını daha iyi hata terimi ile hesaplayan ana teoremimizi ispatlayacağız. Şimdi bu teoremimizi ifade edelim:

$k \geq 4$ tam sayısı ve $p > 3$ asal sayısı için S_p'deki k uzunluklu aritmetik dizilerin sayısı, hata terimi

$$|R| \leq \left(\frac{k-2}{4} - \frac{k-2}{2^{k-1}} \right) \cdot p^{\frac{3}{2}} + c_k \cdot p$$

olacak şekilde

$$\frac{p^2}{2^k} + R$$

ile verilir.

Ayrıca tezimizde sırasıyla $k = 3,4,5$ için S_p'deki k uzunluklu aşikar olmayan aritmetik dizilerin sayısını tam olarak hesaplayan formüller elde edildi. $k = 4,5$ durumunda elde edilen formüller, içerisinde eliptik eğriler barındırır. $k = 4,5$ için formüller eliptik eğriler içerdiğinden hata terimi mevcuttur ve bu hata teriminin en iyisi olduğu Sato-Tate sanısını kullanılarak gösterildi. Formüllerimizin bazılarında eliptik eğrileri mevcut olduğundan, eliptik eğrilerin çalışılmasının önemli bir yönünün, eğri üzerindeki noktaların etkili bir şekilde sayılmasına dair yöntemler geliştirmek olduğunu belirtmek gerekir.

Son bölümde, $\mathbb{F}_p$'deki karesel elemanların oluşturduğu kümedeki 3 uzunluklu aritmetik dizilerin sayını bulma problemini, küpsel elemanların oluşturduğu

$$C_p = \{t^3 : t \in \mathbb{F}_p\}$$

kümesinde ele alınmıştır. S_p'deki 3 uzunluklu aritmetik dizilerin sayısını belirlemede Gauss toplamları kullanılırken C_p'deki 3 uzunluklu aritmetik dizilerin sayısını belirlemede Kummer toplamları kullanılarak formül elde edildi.

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SYMBOLS AND ABBREVIATIONS

k-APs	:	Arithmetic progressions of length k
$\mathbb{F}_p$	:	The finite field with p elements
S_p	:	$S_p = \{t^2 : t \in \mathbb{F}_p\}$
C_p	:	$C_p = \{t^3 : t \in \mathbb{F}_p\}$
$O(\,)$	:	Big O notation
$e_N(x)$	:	$e^{2\pi ix/N}$
$\hat{f}$	:	Fourier transform of f
$\mathbb{Z}_N$	:	Group of integers modulo N
$\left(\frac{\cdot}{p}\right)$	:	Legendre symbol
$\#E(\mathbb{F}_p)$	:	The order of the elliptic curve E over the finite field $\mathbb{F}_p$
φ	:	Euler's phi function

1. INTRODUCTION

In 1927, van der Waerden (van der Waerden, 1927) proved a celebrated theorem regarding the existence of arithmetic progressions in any partition of the positive integers with finitely many classes. This is one of the fundamental results of Ramsey theory, and this theorem has been strengthened in many different directions. In 1936, a strengthening of van der Waerden's theorem was conjectured by Erdős and Turán (Erdős and Turán, 1936), which states that any subset of positive integers with a positive upper density contains arbitrarily long arithmetic progressions. For a subset A of positive integers, its upper density is defined as

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1,\ldots,N\}|}{N}.$$

In 1953, this conjecture was confirmed by Roth (Roth, 1953) for arithmetic progressions of length three. Actually, his proof shows not only the conjecture is true for arithmetic progressions of length three, but it also provides an explicit upper bound for the largest size of a subset of $\{1,\ldots,N\}$ with no non-trivial arithmetic progressions of length three (which is denoted by $r_3(N)$). In 1969, Szemerédi (Szemerédi, 1965) extended the aforementioned result to arithmetic progressions of length four, and then in 1975 he developed his combinatorial method to resolve the conjecture for arbitrarily long arithmetic progressions, see (Szemerédi, 1975). The affirmative answer to Erdős and Turán's conjecture is now known as Szemerédi's theorem, which is one of the cornerstones of additive combinatorics. There is also a finitary version of Szemerédi's theorem which is equivalent to Szemerédi's theorem itself. Let $\varepsilon > 0$, and let k be a positive integer. Then, there is some $N(\varepsilon,k)$ such that if $n \geq N(\varepsilon,k)$, then any subset of $\{1,2,\ldots,n\}$ with at least εn elements contains a k-term arithmetic progression. The smallest such $N(\varepsilon,k)$ is called the Szemerédi number denoted by $S(\varepsilon,k)$.

A second proof of Szemerédi's theorem was given by Furstenberg (Furstenberg, 1977) using ergodic theory in 1977. Furstenberg's proof was a major breakthrough in terms of both his techniques, which gave rise to many natural generalizations of the theorem, for example the density version of the Hales-Jewett theorem (Furstenberg, 1991) and the polynomial Szemerédi theorem (Bergelson and Leibman, 1996). Despite their depths and impacts, the proofs of Szemerédi and Furstenberg fail to give upper bounds for $r_k(N)$ (which is the largest size of a subset of $\{1,\ldots,N\}$ with no non-trivial k-term arithmetic progressions),

since Szemerédi's proof applies van der Waerden's theorem and Furstenberg's proof uses the axiom of choice.

Gowers developed new Fourier analytic methods to reprove Szemerédi's theorem for arithmetic progressions of length four (Gowers, 1998) in 1998, and arbitrarily long arithmetic progressions (Gowers, 2001) in 2001. In fact, he gave not only a proof of the full Szemerédi's theorem but also a quantitative bound for $r_k(N)$.

As well as in the integers, Szemerédi-type problems have been extensively studied in subsets of finite fields. While much work has been done on the problem of whether subsets of finite fields contain arithmetic progressions, in this study we concentrate on how many arithmetic progressions we have in certain subsets of finite fields. Here, we consider the set $S_p = \{t^2 : t \in \mathbb{F}_p\}$ and we obtain the exact asymptotic for the number of k-term arithmetic progressions in this set. Our approach relies on the estimation of character sums, which has been a recurrent topic in number theory. A typical exponential and character sum is of the form

$$T_1 = \sum_{(x_1,\ldots,x_n)\in\mathbb{F}_p^n} \psi(q(x_1,\ldots,x_n))$$

and

$$T_2 = \sum_{(x_1,\ldots,x_n)\in\mathbb{F}_p^n} \chi(q(x_1,\ldots,x_n)),$$

where $q(x_1,\ldots,x_n) \in \mathbb{F}_p[x_1,\ldots,x_n]$ of degree d, $\psi(x)$ is a non-trivial additive character and $\chi(x)$ is a non-trivial multiplicative character on the finite field $\mathbb{F}_p$. The expectation is the estimate

$$|T_i| \leq cp^{n/2}, \tag{1.1}$$

where c is a constant depending on n and the degree d of the polynomial $q(x_1,\ldots,x_n)$, and this is sort of a randomness. The above estimation corresponds to the Riemann hypothesis in finite fields. The estimation (1.1) was first achieved by Hasse (Hasse, 1936) for single-variable smooth cubics and then generalized by Weil (Weil, 1948). For each odd prime number p and for each non-linear polynomial $f \in \mathbb{Z}[X]$, we denote the Weil sum by

$$s(f,p) = \sum_{x\in\mathbb{F}_p} e_p(f(x)),$$

where $e_p(x) = e^{2\pi ix/p}$. In 1948, Weil proved as a consequence of his work (Weil, 1948) in algebraic geometry that if p is an odd prime number and $f \in \mathbb{Z}[X]$ is a non-linear polynomial with $f \notin p\mathbb{Z}[X]$, then we have

$$|s(f,p)| \leq (\deg f - 1) \cdot \sqrt{p}.$$

The higher dimensional version for the estimation of the exponential sum T_1 was obtained in the seminal works of Deligne (Deligne, 1974 and 1980) where he proved the Riemann hypothesis for finite fields that was also conjectured by Weil. More precisely, Deligne (Deligne, 1974) proved that if p does not divide d and if the homogeneous part q_d with degree d of q defines a smooth hypersurface in $\mathbb{P}^{n-1}$, then the expected estimation for T_1 holds with $c = (d-1)^n$. Later on, Katz (Katz, 2002) proved the multiplicative version of Deligne's result and obtained an estimation for the sum T_2. In this article, our algebraic sets that we encounter are highly singular and this is why we need singular character sum estimations. An estimation of this type was proved by Rojas-León (Rojas-León, 2005), extending the work of Katz. In a very recent work, Rojas-León (Rojas-León, 2022) deduced an estimation for multi-variable multiplicative character sums, which extends the well-known estimates for both classical Jacobi sums and one-variable polynomial multiplicative character sums. The result of Rojas-León (Rojas-León, 2022) will be crucial to prove our first theorem of this paper, and we obtain an asymptotic for the number of k-term arithmetic progressions (k-APs) in S_p with a better error term. Moreover, our error term is sharp and best possible when $k \in \{4,5\}$, owing to the celebrated Sato-Tate conjecture (a theorem now), see (Barnet-Lamb et al., 2011, Clozel et al., 2008, Harris et al., 2010, Taylor, 2008). Observe that our estimate in the next theorem is reminiscent of the Riemann hypothesis in the sense of finite fields.

Theorem 1.1 Let $k \geq 4$ be a positive integer and $p > 3$ be a prime number. The number of k-APs in S_p is given by

$$\frac{p^2}{2^k} + R,$$

where

$$|R| \leq \left(\frac{k-2}{4} - \frac{k-2}{2^{k-1}}\right) \cdot p^{\frac{3}{2}} + c_k \cdot p$$

and c_k is an explicit computable constant depending only on k.

When we look at the historical process of the distribution of quadratic residues or counting quadratic residue patterns in finite fields, it is seen that it has been widely handled by different mathematicians. Over the past 100 years, for $k \geq 1$ and an odd prime $p > k$, it

has been desirable to count how many k-tuples of consecutive numbers $a, a+1, \ldots, a+k-1$ in $\mathbb{F}_p^\times$ have predetermined quadratic residue or nonresidue behavior. For a choice of k signs $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$, set

$$N_p(\varepsilon_1, \ldots, \varepsilon_k) = \left| \left\{ a \in \mathbb{F}_p^\times : \left(\frac{a+i-1}{p}\right) = \varepsilon_i \text{ for } i = 1, \ldots, k \right\} \right|,$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. In 1896, Aladov (Aladov, 1896) counted each quadratic residue patterns of length 2, and some quadratic residue patterns of length 3 in $\mathbb{F}_p^\times$. In the 1930s, Davenport (Davenport, 1931 and 1933) considered this counting problem for $k \geq 4$. It was shown (Chapter 9, Katz, 1988) that for k signs $\varepsilon_1, \ldots, \varepsilon_k \in \{\pm 1\}$, and an odd prime $p > k$,

$$\left| N_p(\varepsilon_1, \ldots, \varepsilon_k) - \frac{p}{2^k} \right| < (k-1)\sqrt{p} + \frac{k}{2}.$$

Moreover, quadratic residue patterns with gaps that are not necessarily consecutive was also counted: if $p > k$ and $c_1, \ldots, c_k$ are distinct in $\mathbb{F}_p$, the set

$$\left\{ a \in \mathbb{F}_p^\times : \left(\frac{a+c_i}{p}\right) = \varepsilon_i \text{ for } i = 1, \ldots, k \right\}$$

has a size N_p, and it satisfies

$$\left| N_p - \frac{p}{2^k} \right| < (k-1)\sqrt{p} + \frac{k}{2}.$$

See also (Conrad, 2018). When we fix $\varepsilon_i = 1$ for each i, this yields that the number of k-APs in S_p is given by

$$\frac{p^2}{2^k} + H, \tag{1.2}$$

where

$$|H| \leq (k-1) \cdot p^{\frac{3}{2}} + O_k(p).$$

In this case, it is seen that the estimate in Theorem 1.1 for the number of k-APs in S_p has a better error term than that of (1.2).

In our following result, we obtain the formulas which give the exact number of non-trivial arithmetic progressions of length 3 in the set S_p. Note that "non-trivial" means that the common difference of the arithmetic progression is not zero.

Proposition 1.2 Let p be an odd prime number. The number of non-trivial 3-APs in S_p is given by the following table:

The formula	**The prime number** p
$\frac{1}{8}(p+3)(p-1)$	$p \equiv 1 \pmod 8$
$\frac{1}{8}(p-3)(p-1)$	$p \equiv 3 \pmod 8$
$\frac{1}{8}(p-1)(p-1)$	$p \equiv 5 \pmod 8$
$\frac{1}{8}(p+1)(p-1)$	$p \equiv 7 \pmod 8$

A formula that determines the number of non-trivial 4-APs in S_p can be given in the following result, and it depends on the number of points on the elliptic curve

$$E : y^2 = x(x+3)(x+4).$$

Since we will use elliptic curves in some of our formulas, it would be necessary to point out that an important aspect of the study of elliptic curves is devising effective ways of counting points on the curve. There are several approaches to do so, and the algorithms devised have been proved to be useful tools in the study of various fields, see (Lercier and Morain, 1995, Schoof, 1985 and 1995).

We also note that the error term in the following theorem is sharp.

Theorem 1.3 Let $p > 3$ be a prime number. The number of non-trivial 4-APs in S_p is given by the following formula:

$$\begin{aligned}\frac{(p+1)^4}{16p^2} + \frac{(p-1)(5p+1)}{16p^2} - \frac{p+1}{2} \\ + \frac{p-1}{16} \cdot \left(\frac{-1}{p}\right) \cdot \left(2 \cdot \left(\frac{-6}{p}\right) + 4 \cdot \left(\frac{-2}{p}\right) + 2 \cdot \left(\frac{2}{p}\right) + 2 \cdot \left(\frac{-3}{p}\right) + 2\right) \\ + \frac{1}{16} \cdot \left(\frac{-1}{p}\right) \cdot (p-1)\left(\#E(\mathbb{F}_p) - p - 1\right),\end{aligned}$$

where the elliptic curve E over $\mathbb{F}_p$ is defined by

$$E : y^2 = x(x+3)(x+4),$$

and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Moreover, the number of non-trivial 4-APs in S_p is given by

$$\frac{p^2}{16} + R_p,$$

where

$$|R_p| \leq \frac{1}{8} \cdot p^{\frac{3}{2}} + O(p),$$

and the error term R_p and the above coefficient $\frac{1}{8}$ are best possible in the sense that $O(p^{\frac{3}{2}})$ cannot be replaced by a smaller function of p, and $\frac{1}{8}$ cannot be replaced by a smaller constant.

The resulting formula for the number of 5-APs in S_p is quite long and involves more elliptic curves. The exact formula can be found in its proof.

Theorem 1.4 Let $p > 3$ be a prime number. There exist explicitly computable polynomials $f \in \mathbb{Z}[X]$, $g \in \mathbb{Z}[X_1, X_2, X_3, X_4]$ and $h_i \in \mathbb{Z}[X_1, X_2]$ with $\deg f = 3$, $\deg_{X_1} g = 3$ and $\deg_{X_1} h_i = 2$ for $i \in \{1,2,3\}$ such that the number of 5-APs in S_p is given by

$$\frac{(p+1)^5}{32p^3} + \frac{f(p)}{32p^3} + \frac{g\left(p, \left(\frac{-1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right)\right)}{32p^2} + \sum_{i=1}^{3} \frac{h_i\left(p, \left(\frac{-1}{p}\right)\right)}{32p} \left(\#E_i(\mathbb{F}_p) - p - 1\right),$$

where the elliptic curve E_i over $\mathbb{F}_p$ is defined by

$$E_1 : y^2 = x(x+3)(x+4),$$
$$E_2 : y^2 = x(x+4)(x+6),$$
$$E_3 : y^2 = x(x+8)(x+9),$$

and $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Moreover, the number of 5-APs in S_p is given by

$$\frac{p^2}{32} + O(p^{\frac{3}{2}}),$$

and the error term is best possible in the sense that $O(p^{\frac{3}{2}})$ cannot be replaced by a smaller function of p.

Can we take our question above a step further, and give a formula that calculates the number of 3-APs in the set of cubes

$$C_p = \{t^3 : t \in \mathbb{F}_p\}$$

in $\mathbb{F}_p$? In addition to our results which make use of Gauss sums for the number of non-trivial 3-APs in S_p, we give the following result using Kummer sums for the number of non-trivial 3-APs in C_p.

Theorem 1.5 Let p be a prime number with $p \equiv 1 \pmod 3$. Let Q_p denote the number of non-trivial 3-APs in C_p. Then,

$$Q_p = \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(pc_p + 4p + 8) - \frac{p+2}{3},$$

where $c_p \in \mathbb{Z}$ with $c_p = O(\sqrt{p})$ is a computable constant which depends on p. If p is of the form $u^2 + 27v^2$ for some integers u and v with $u \equiv 2 \pmod 3$, then

$$Q_p = \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(2up + 12p + 8) - \frac{p+2}{3}.$$

In the following table, using SageMath (SageMath, 2018), we give the calculations of the formulas, we obtained in our theorems above, for some certain values. Note that if $p \not\equiv 1 \pmod 3$, then $C_p = \mathbb{F}_p$ and so $Q_p = p(p-1)$.

Table 1.1 The number of non-trivial 3 and 4 -APs in S_p and C_p for prime numbers p between 20 and 50.

$20 < p < 50$	#3-APs in S_p	#4-APs in S_p	#3-APs in C_p
23	66	44	23×22
29	98	28	29×28
31	120	30	50
37	162	54	60
41	220	120	41×40
43	210	84	70
47	276	138	47×46

2. PRELIMINARIES

In this paper, we make use of Fourier analysis on finite abelian groups. In particular, our main tool will be the Fourier transform of functions which are defined on the finite cyclic group $\mathbb{Z}_N$. Throughout this note, $e_N : \mathbb{Z}_N \to \mathbb{C}$ is defined as $e_N(x) = e^{2\pi ix/N}$ for any $x \in \mathbb{Z}_N$. This function has the following well-known property, which is known as *orthogonality*:

$$\sum_{m\in\mathbb{Z}_N} e_N(mu) = \begin{cases} 0 & \text{if } u \neq 0, \\ N & \text{if } u = 0. \end{cases} \tag{2.1}$$

Given a function $f : \mathbb{Z}_N \to \mathbb{C}$, its *Fourier transform* $\widehat{f}$ at $m \in \mathbb{Z}_N$ is defined by

$$\widehat{f}(m) = N^{-1} \sum_{x\in\mathbb{Z}_N} e_N(-xm) f(x). \tag{2.2}$$

Basically, the Fourier transform of $f : \mathbb{Z}_N \to \mathbb{C}$ is another function that is defined as the average of the values $f(x)$ multiplied by the corresponding roots of unity, namely $e_N(-xm)$, $x \in \mathbb{Z}_N$. It has numerous useful properties. Among them, the one we require is the inversion formula. The *inversion formula* states that with the above definition of the Fourier transform, we can recover f from its Fourier coefficients via the formula

$$f(x) = \sum_{m\in\mathbb{Z}_N} e_N(xm)\widehat{f}(m). \tag{2.3}$$

This basic feature of the Fourier transform appears frequently in the proof of our results. Surely, there are much more practical properties of the Fourier transform. For more detailed information about Fourier analysis on $\mathbb{Z}_N$, one can consult (Stein and Shakarchi, 2003).

In the following definition, we describe an arithmetic progression in $\mathbb{Z}$.

Definition 2.1 (Arithmetic Progressions) An arithmetic progression of length k (k-AP) in $\mathbb{Z}$ is a sequence of k-integers such that each difference between two consecutive terms is the same constant.

We say that a set $A \subseteq \mathbb{Z}$ contains arbitrarily long arithmetic progressions if for any $k \in \mathbb{N}$, there is a non-trivial k-AP in A. There are some distinctions between arithmetic progressions in $\mathbb{Z}$ and $\mathbb{Z}_N$. We define an arithmetic progression in $\mathbb{Z}_N$ in the following way.

Definition 2.2 A k-term arithmetic progression in $\mathbb{Z}_N$, $x_0, x_2, ..., x_{k-1}$, is a sequence of integers satisfying

$$2x_i \equiv x_{i-1} + x_{i+1} \pmod{N},$$

for all $i = 1, ..., k-2$.

The disadvantage is that arithmetic progressions in $\mathbb{Z}_N$ are not necessarily arithmetic progressions in $\mathbb{Z}$ (they might "wrap around"). For instance, in $\mathbb{Z}_{102}$, $\{65,100,33\}$ is a 3-term arithmetic progression but not in $\mathbb{Z}$. Nevertheless, there is a relation between lengths of arithmetic progressions in $\mathbb{Z}$ and $\mathbb{Z}_N$. The following proposition was stated by Bourgain without proof in (Bourgain, 1990). Thus, we felt the need to prove this proposition.

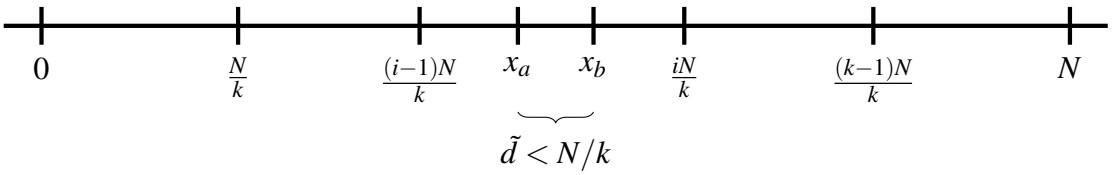
Proposition 2.3 If there exists a non-trivial $\left(2k^2-2k+1\right)$-AP in $\mathbb{Z}_N$, then there is a non-trivial arithmetic progression in $\mathbb{Z}$ of length k contained in this given arithmetic progression in $\mathbb{Z}_N$.

Proof: Let $x_1, x_2, \ldots, x_{2k^2-2k+1}$ be a non-trivial $\left(2k^2-2k+1\right)$-AP in $\mathbb{Z}_N$. Now, we divide the interval $[0,N)$ into k disjoint parts

$$[0,N) = \bigcup_{i=1}^{k} \left[\frac{(i-1)N}{k}, \frac{iN}{k}\right).$$

By the pigeonhole principle, there exist an interval $\left[\frac{(i-1)N}{k}, \frac{iN}{k}\right)$ and $a, b \in \{1, \ldots, k+1\}$ with $a < b$ and $i \in \{1, \ldots, k\}$ such that

$$x_a, x_b \in \left[\frac{(i-1)N}{k}, \frac{iN}{k}\right) \text{ and } x_a \neq x_b :$$



Note that after choosing the appropriate representation of the elements of the arithmetic progression on classes modulo N, we consider the elements as integers.

Hence, the set $\left\{x_a, x_b, x_{a+2(b-a)}, \ldots, x_{a+(2k-2)(b-a)}\right\}$ has the following properties:

- $|x_{a+(i-1)(b-a)} - x_{a+i(b-a)}| = \tilde{d}$ for $i \in \{1, 2, \ldots, 2k-2\}$,
- $a + (2k-2)\cdot(b-a) \leq 2k^2 - 2k + 1$ for $a, b \in \{1, \ldots, k+1\}$.

It means that $\{x_{a+i(b-a)}\}_{i=0}^{2k-2}$ is an arithmetic progression on one of the intervals $(-2N,N)$ and $[0,3N)$. Now, without loss of generality, we assume that $\{x_{a+i(b-a)}\}_{i=0}^{2k-2}$ is a $(2k-1)$-AP on $[0,3N)$. If $\{x_{a+i(b-a)}\}_{i=0}^{2k-2} \cap [2N,3N) \neq \emptyset$, then there exists an arithmetic progression of length at least k on the intervals $[N,2N)$ since $\tilde{d} < N/k$. For the other case, if $\{x_{a+i(b-a)}\}_{i=0}^{2k-2} \cap [2N,3N) = \emptyset$, by the pigeonhole principle and as $\tilde{d} < N/k$, there exists an arithmetic progression of length at least k on one of the intervals $[0,N)$ and $[N,2N)$. Thus, we conclude that there exists a non-trivial k-AP in $\mathbb{Z}$ obtained from the k-AP on one of the intervals $[0,N)$ and $[N,2N)$. □

The proposition above provides a way to connect $\mathbb{Z}_N$-progressions to $\mathbb{Z}$-progressions. In particular, finding a $\left(2k^2-k+1\right)$-AP in $\mathbb{Z}_N$ gives rise to the existence of a k-AP in $\{1,...,N\}$. In the further parts of this note, we prove that there are long APs in some special subsets of $\mathbb{Z}_N$. Hence, if we lift those sets up to $\mathbb{Z}$, that is to say, see them as a subset of $\mathbb{Z}$, then we obtain $\mathbb{Z}$-APs.

The characteristic function $A(x)$ of a set $A \subseteq \mathbb{Z}_N$ is defined as

$$A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

We will often use the following lemma, which determines the number of k-APs in the set $A \subseteq \mathbb{Z}_N$.

Lemma 2.4 Let A be any subset of $\mathbb{Z}_N$ and $k \geq 3$. Then, the number of k-APs in the set A is

$$\frac{N^2|A|^k}{N^k} + H,$$

where

$$\begin{aligned} H =& N^2 \sum_{(x_1,x_2,\ldots,x_{k-2})\neq 0} \widehat{A}(x_1)\cdots\widehat{A}(x_{k-2})\widehat{A}(x_1+2x_2+\cdots+(k-2)x_{k-2}) \\ & \cdot\widehat{A}(-2x_1-3x_2-\cdots-(k-1)x_{k-2}). \end{aligned}$$

Proof: Let A be any subset of $\mathbb{Z}_N$ and $k \geq 3$. Define

$$Q_N(t) = |\{(y_1,y_2,\ldots,y_k) \in A^k : y_{i+1}-y_i = t \text{ for } i \in \{1,\ldots,k-1\}\}|$$

as the number of k-term arithmetic progressions in A with common difference t.
So, the number of k-APs in A is equal to

$$\sum_{t\in\mathbb{Z}_N} Q_N(t) = \sum_{y_1\in\mathbb{Z}_N}\sum_{t\in\mathbb{Z}_N} A(y_1)A(y_1+t)\cdots A(y_1+(k-1)t). \tag{2.4}$$

If we use the Fourier inversion formula (2.3) for each of the terms in Equation (2.4), namely for

$$A(y_1+t), A(y_1+2t), \ldots, A(y_1+(k-1)t),$$

we obtain the following sums which depend on the Fourier coefficients of A:

$$\begin{aligned}\sum_{t\in\mathbb{Z}_N} Q_N(t) &= \sum_{y_1,t\in\mathbb{Z}_N} A(y_1) \sum_{x_0\in\mathbb{Z}_N} e_N(x_0(y_1+t))\widehat{A}(x_0)\cdots \sum_{x_{k-2}\in\mathbb{Z}_N} e_N(x_{k-2}(y_1+(k-1)t))\widehat{A}(x_{k-2}) \\ &= \sum_{(y_1,x_0,x_1,\ldots,x_{k-2})} A(y_1)\widehat{A}(x_0)\cdots\widehat{A}(x_{k-2})e_N(y_1(x_0+x_1+\cdots+x_{k-2})) \\ &\quad \cdot \sum_{t\in\mathbb{Z}_N} e_N(t(x_0+2x_1+\cdots+(k-1)x_{k-2})).\end{aligned}$$

By orthogonality, we have

$$\sum_{t\in\mathbb{Z}_N} e_N(t(x_0+2x_1+\cdots+(k-1)x_{k-2})) = \begin{cases} N & \text{if } x_0+2x_1+\cdots+(k-1)x_{k-2}=0, \\ 0 & \text{otherwise.}\end{cases}$$

From this orthogonality relation, we obtain that

$$\begin{aligned}\sum_{t\in\mathbb{Z}_N} Q_N(t) = N \sum_{(y_1,x_1,x_2,\ldots,x_{k-2})} & A(y_1)\widehat{A}(x_1)\cdots\widehat{A}(x_{k-2})\widehat{A}(-2x_1-\cdots-(k-1)x_{k-2}) \\ & \cdot e_N(y_1(-x_1-2x_2-\cdots-(k-2)x_{k-2})).\end{aligned}$$

Then, one can conclude that

$$\begin{aligned}\sum_{t\in\mathbb{Z}_N} Q_N(t) = N^2 \sum_{x_1,x_2,\ldots,x_{k-2}} & \widehat{A}(x_1)\cdots\widehat{A}(x_{k-2})\widehat{A}(x_1+2x_2+3x_3+\cdots+(k-2)x_{k-2}) \\ & \cdot\widehat{A}(-2x_1-3x_2-\cdots-(k-1)x_{k-2}).\end{aligned}$$

We denote this again by

$$\sum_{t\in\mathbb{Z}_N} Q_N(t) = \frac{N^2|A|^k}{N^k} + H$$

where

$$\begin{aligned}H = N^2 \sum_{(x_1,x_2,\ldots,x_{k-2})\neq 0} & \widehat{A}(x_1)\cdots\widehat{A}(x_{k-2})\widehat{A}(x_1+2x_2+\cdots+(k-2)x_{k-2}) \\ & \cdot\widehat{A}(-2x_1-3x_2-\cdots-(k-1)x_{k-2}),\end{aligned}$$

and the proof is complete. □

Definition 2.5 Let p be an odd prime number. An integer a which is not divisible by p is said to be a quadratic residue modulo p if it is congruent to a perfect square modulo p and is a quadratic nonresidue modulo p otherwise. The Legendre symbol is a function of a and p and it is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p, \\ 0 & \text{if } p \mid a. \end{cases}$$

Below we list three important properties of the Legendre symbol which will be frequently used.

Proposition 2.6 (Pandey, 2018) For integers b and c with $p \nmid b$,

$$\sum_{\ell=0}^{p-1} \left(\frac{b\ell + c}{p}\right) = 0.$$

Proposition 2.7 (Pandey, 2018) Let a, b and c be integers, and let p be an odd prime. Then

$$\sum_{\ell=0}^{p-1} \left(\frac{a\ell^2 + b\ell + c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } p \nmid \left(b^2 - 4ac\right), \\ \left(\frac{a}{p}\right)(p-1) & \text{if } p \mid \left(b^2 - 4ac\right). \end{cases}$$

Remark 2.8 Let k be an odd positive integer and $m \geq 1$. Let $a_{ij} \in \mathbb{F}_p$ for $1 \leq i \leq k$ and $1 \leq j \leq m$. Then

$$\sum_{x_1, x_2, \ldots, x_m \in \mathbb{F}_p} \left(\frac{a_{11}x_1 + \cdots + a_{1m}x_m}{p}\right) \cdots \left(\frac{a_{k1}x_1 + \cdots + a_{km}x_m}{p}\right) = 0.$$

This equation is quickly obtained by defining new variables $y_j = ax_j$ for a chosen $a \in \mathbb{F}_p$ with $\left(\frac{a}{p}\right) = -1$. This property will be used frequently without being specified in the following sections.

In 1924, Artin estimated the correctness of the following theorem on elliptic curves. However, Artin was not able to prove his estimate. In 1933, Hasse proved the estimate of Artin. Then, Weil generalized the result of Hasse, as we mentioned in the previous section. The following two theorems will play an important role in finding the number of arithmetic progressions of length 4 and 5 in S_p.

Theorem 2.9 (Hasse) (Theorem 4.2, Washington, 2008) Let E be an elliptic curve over the finite field $\mathbb{F}_p$. Then, the order of $E(\mathbb{F}_p)$ satisfies

$$|p + 1 - \#E(\mathbb{F}_p)| \leq 2\sqrt{p}.$$

Theorem 2.10 (Theorem 4.14, Washington, 2008) Let E be an elliptic curve defined by $y^2 = x^3 + Ax + B$ over the finite field $\mathbb{F}_p$ where p is an odd prime. Then,

$$\#E(\mathbb{F}_p) = p + 1 + \sum_{x \in \mathbb{F}_p} \left(\frac{x^3 + Ax + B}{p} \right).$$

Around 1960, independently, Mikio Sato and John Tate inquired about the distribution of numbers $\frac{\#E(\mathbb{F}_p) - p - 1}{\sqrt{p}}$ within the interval $[-2, 2]$ as p approaches to infinity. The question of whether these numbers are uniformly distributed within the interval is the first one that comes to mind. In other words, is it true that for any interval $[a, b] \subseteq [2, 2]$, we have

$$\lim_{N \to \infty} \frac{|\{p \leq N : (\#E(\mathbb{F}_p) - p - 1)/\sqrt{p} \in [a, b]\}|}{|\{p \leq N\}|} = b - a?$$

This question marks the origin of the Sato-Tate conjecture (see Murty and Murty, 2009). The available numerical data appeared to indicate a different outcome. To be more specific, Sato and Tate were inclined to predict the following statement for elliptic curves without complex multiplication. If we write

$$(\#E(\mathbb{F}_p) - p - 1)/\sqrt{p} = 2\cos\theta_p, \;\; 0 \leq \theta_p \leq \pi$$

and $[\alpha, \beta] \subseteq [0, \pi]$, then, their conjecture says

$$\lim_{N \to \infty} \frac{|\{p \leq N : \theta_p \in [\alpha, \beta]\}|}{|\{p \leq N\}|} = \frac{2}{\pi} \int_\alpha^\beta \sin^2\theta \, d\theta = \frac{\beta - \alpha}{\pi} - \frac{1}{2\pi}(\sin 2\beta - \sin 2\alpha).$$

The original conjecture and its generalization to all totally real fields was proved by Laurent Clozel, Michael Harris, Nicholas Shepherd-Barron, and Richard Taylor under mild assumptions in 2008 (Clozel et al, 2008), and completed by Thomas Barnet-Lamb, David Geraghty, Harris, and Taylor in 2011 (Barnet-Lamb et al, 2011).

In this thesis, we will use a corollary of Chebotarev–Sato–Tate theorem to show that the formulas that give the number of non-trivial 4 and 5-APs in S_p have the error term which is best possible. Since Barnet-Lamb et al proved Sato-Tate conjecture by removing mild assumptions, we can also remove this assumption from Murty's result (Corollary 2, Murty and Murty, 2009). Thus, we can give the following theorem.

Theorem 2.11 Let E be an elliptic curve defined over the rational numbers without complex multiplication. Let q be a natural number and a an integer with $(a, q) = 1$. For $0 \leq \alpha \leq \beta \leq \pi$, the density of primes p for which $\theta_p \in [\alpha, \beta]$ and $p \equiv a \mod q$ is

$$\frac{2}{\pi\varphi(q)} \int_\alpha^\beta \sin^2\theta \, d\theta.$$

Let $k \geq 6$ be a positive integer. When estimating the number of arithmetic progressions of length k in S_p, we need a conclusion that yields more than Hasse's theorem. In 2022, Rojas-León proved an estimate for multi-variable multiplicative character sums over affine subspaces of $\mathbb{A}_k^n$, which generalizes the well-known estimates for both classical Jacobi sums and one-variable polynomial multiplicative character sums (Rojas-León, 2022). The following theorem proved by Rojas-León is of fundamental importance in finding the number of arithmetic progressions of length k in S_p with a better error term.

Theorem 2.12 (Corollary 2, Rojas-León, 2022) Let $k = \mathbb{F}_q$ be a finite field, with $q = p^a$ a prime power. Let $\chi_1, \ldots, \chi_n : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be n non-trivial multiplicative characters. Let $L_1, \ldots, L_n : \mathbb{A}_k^d \to \mathbb{A}_k^1$ be affine linear forms, with $L_i(t) = a_{i,1}t_1 + \cdots + a_{i,d}t_d + b_i$, and let $V_i \subseteq \mathbb{A}_k^d$ be the hyperplane defined by $L_i(t) = 0$. Suppose that the affine map $\mathbb{A}_k^d \to \mathbb{A}_k^n$ defined by the L_i is injective (that is, that the matrix (a_{ij}) has rank d), and that for every $I \subseteq \{1, \ldots, n\}$ with $|I| \leq d+1$ we have $\dim(\cap_{i \in I} V_i) \leq d - |I|$. Then, we have the estimate

$$\left| \sum_{t \in k^d} \chi_1(L_1(t)) \cdots \chi_n(L_n(t)) \right| \leq D_L \cdot q^{d/2},$$

where

$$D_L := (-1)^d + \sum_{j=1}^{d} (-1)^{d+j} a_j,$$

and a_j is the number of subsets $I \subseteq \{1, \ldots, n\}$ with $|I| = j$ such that $\cap_{i \in I} V_i \neq \emptyset$.

Observe that the algebraic sets occurring in the previous theorem are highly singular, so one cannot apply the results of (Katz, 2002) and (Rojas-León, 2005) immediately. Although the character sum estimates are in the realm of analytic number theory, the technique behind them is the use of ℓ-adic cohomology and Grothendieck's trace formula, see also the works of Deligne (Deligne, 1977a and 1977b).

Given that p and q are two distinct odd primes, suppose we know whether q is a quadratic residue of p or not. The natural question is as follows: will p be a quadratic residue of q? One of Gauss' favorite theorems, which is the law of quadratic reciprocity answers this question. The law of quadratic reciprocity is a very deep theorem with over two hundred fifty proofs.

Proposition 2.13 (Law of Quadratic Reciprocity) (Berndt et al., 1998) Let p and q be two distinct odd prime numbers. Then,

$$\left(\frac{p}{q}\right) \cdot \left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$$

holds.

For an odd prime number p, an integer a and $k \in \mathbb{Z}_{>0}$, a general Gauss sum is defined as

$$G_k(a,p) = \sum_{m=0}^{p-1} e_p(am^k). \tag{2.5}$$

When $k = 1$ and $p \nmid a$, as mentioned before, the sum of all p-th roots of unity, which is a geometric sum and can be easily evaluated to be zero. When $k \geq 2$, the task of determining the sum then becomes considerably more difficult. In fact, even for the initial case $k = 2$, it took Gauss several years to accomplish this. In late May of 1801, Gauss conjectured that

$$G_2(1,p) = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod 4, \\ i\sqrt{p} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{2.6}$$

On August 30, 1805, Gauss wrote in his diary that he devoted some time to this problem every week for more than four years before he was able to prove his conjecture on the signs of these sums (Berndt and Evans, 1981). The sum $G_2(a,p)$ introduced by Gauss in 1801 is now called the quadratic Gauss sum.

Theorem 2.14 (Theorem 1.5.2, Berndt et al., 1998) Let a be an integer not divisible by a prime $p > 2$. Then

$$G_2(a,p) = \sum_{m=0}^{p-1} e_p(am^2) = \left(\frac{a}{p}\right) G_2(1,p) = \begin{cases} \left(\frac{a}{p}\right)\sqrt{p} & \text{if } p \equiv 1 \pmod 4, \\ i\left(\frac{a}{p}\right)\sqrt{p} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

We also recall Weil's theorem from the introduction.

Theorem 2.15 (Weil, 1948) Let p be an odd prime number. Let $f \in \mathbb{Z}[X]$ be a non-linear polynomial such that $f \notin p\mathbb{Z}[X]$. We denote the Weil sum by

$$s(f,p) = \sum_{x \in \mathbb{F}_p} e_p\left(f(x)\right),$$

where $e_p(x) = e^{2\pi i x/p}$. Then, we have

$$|s(f,p)| \leq (\deg f - 1) \cdot \sqrt{p}.$$

3. PROOF OF THEOREM 1.1

Let $p > k$ be an odd prime number. Let

$$Q_p(t) = |\{(x_1,\dots,x_k) \in S_p^k \mid x_{i+1} - x_i = t \text{ for } i \in \{1,\dots,k-1\}\}| \tag{3.1}$$

denote the number of k-term arithmetic progressions in S_p with common difference t. By Lemma 2.4, the number of k-APs in S_p is equal to

$$\frac{p^2|S_p|^k}{p^k} + R,$$

where

$$R = p^2 \sum_{(x_1,x_2,\dots,x_{k-2})\neq 0} \widehat{S_p}(x_1)\cdots\widehat{S_p}(x_{k-2})\widehat{S_p}(x_1+2x_2+\cdots+(k-2)x_{k-2})$$
$$\cdot \widehat{S_p}(-2x_1-3x_2-\cdots-(k-1)x_{k-2}).$$

Note that when $0 \neq m \in \mathbb{F}_p$,

$$\begin{aligned}
\widehat{S_p}(m) &= \frac{1}{p}\sum_{x\in\mathbb{F}_p} e_p(-mx)S_p(x) \\
&= \frac{1}{p}\sum_{x\in S_p} e_p(-mx) \\
&= \frac{1}{2p}\left(\sum_{t\in\mathbb{F}_p} e_p(-mt^2)+1\right) \\
&= \frac{1}{2p}\left(\varepsilon\left(\frac{-m}{p}\right)\sqrt{p}+1\right) \\
&= \frac{1}{2p}+\frac{1}{2\sqrt{p}}\varepsilon\left(\frac{-m}{p}\right),
\end{aligned}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4 \\ i & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and $\widehat{S_p}(0) = \dfrac{|S_p|}{p} = \dfrac{p+1}{2p}$ in case $m = 0$.

Next, we find the upper bound mentioned for the error term R. By the expression of R and the Fourier transform of S_p which were given above, it is sufficient to find an upper bound for the following expressions in forms A, B, C and D since they are the largest ones:

$$A = p^2 \cdot \left(\frac{p+1}{2p}\right)^{k-2-m} \cdot \frac{1}{\left(2\sqrt{p}\right)^{m+2}} \sum_{x_{a_1},\ldots,x_{a_m}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_m}}{p}\right) \left(\frac{-a_1 x_{a_1} - \cdots - a_m x_{a_m}}{p}\right) \tag{3.2}$$
$$\cdot \left(\frac{(a_1+1)\,x_{a_1} + \cdots + (a_m+1)\,x_{a_m}}{p}\right),$$

$$B = p^2 \cdot \left(\frac{p+1}{2p}\right)^{k-1-m} \cdot \frac{1}{\left(2\sqrt{p}\right)^{m+1}} \sum_{\substack{x_{a_1},\ldots,x_{a_m} \\ (a_1+1)x_{a_1}+\cdots+(a_m+1)x_{a_m}=0}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_m}}{p}\right) \tag{3.3}$$
$$\cdot \left(\frac{-a_1 x_{a_1} - \cdots - a_m x_{a_m}}{p}\right),$$

$$C = p^2 \cdot \left(\frac{p+1}{2p}\right)^{k-1-m} \cdot \frac{1}{\left(2\sqrt{p}\right)^{m+1}} \sum_{\substack{x_{a_1},\ldots,x_{a_m} \\ -a_1 x_{a_1}-\cdots-a_m x_{a_m}=0}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_m}}{p}\right) \tag{3.4}$$
$$\cdot \left(\frac{(a_1+1)\,x_{a_1} + \cdots + (a_m+1)\,x_{a_m}}{p}\right),$$

$$D = p^2 \cdot \left(\frac{p+1}{2p}\right)^{k-m} \cdot \frac{1}{\left(2\sqrt{p}\right)^{m}} \sum_{\substack{x_{a_1},\ldots,x_{a_m} \\ -a_1 x_{a_1}-\cdots-a_m x_{a_m}=0 \\ (a_1+1)x_{a_1}+\cdots+(a_m+1)x_{a_m}=0}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_m}}{p}\right), \tag{3.5}$$

where $a_i \in \{1,2,\ldots,k-2\}$ such that $a_i \neq a_j$ when $i \neq j$, and $m \in \{2,\ldots,k-2\}$. Now, we observe that in case of $m = 1$, the equations from (3.2) to (3.5) are equal to zero. By the properties of the Legendre symbol, we get

$$\sum_{x_{a_1}} \left(\frac{-x_{a_1}}{p}\right) \left(\frac{-a_1 x_{a_1}}{p}\right) \left(\frac{(a_1+1)\,x_{a_1}}{p}\right) = \left(\frac{a_1\,(a_1+1)}{p}\right) \sum_{x_{a_1}} \left(\frac{x_{a_1}}{p}\right).$$

Then, it follows from orthogonality that

$$\left(\frac{a_1\,(a_1+1)}{p}\right) \sum_{x_{a_1}} \left(\frac{x_{a_1}}{p}\right) = 0.$$

Thus, equation (3.2) is equal to zero. As $a_1 \in \{1,2,\ldots,k-2\}$, we also have

$$\sum_{\substack{x_{a_1} \\ (a_1+1)x_{a_1}=0}} \left(\frac{-x_{a_1}}{p}\right) \left(\frac{-a_1 x_{a_1}}{p}\right) = 0.$$

Similarly, the other equations are shown to be equal to zero.

We first find the upper bound for (3.2). Since we cannot apply Theorem 2.12 immediately to the sum, we bring the expressions to the appropriate forms. Using change of variables, namely $x_{a_i} = x_{a_m} x_{a_i}$ for $i \in \{1,\dots,m-1\}$, and by the properties of the Legendre symbol, A above becomes

$$\frac{p^2 \cdot (p+1)^{k-2-m}}{2^k \cdot p^{k-2-m} \cdot p^{\frac{m+2}{2}}} \cdot p \sum_{x_{a_1},\dots,x_{a_{m-1}}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_{m-1}}}{p}\right) \tag{3.6}$$
$$\cdot \left(\frac{-a_1 x_{a_1} - \cdots - a_{m-1} x_{a_{m-1}} - a_m}{p}\right) \left(\frac{(a_1+1)x_{a_1} + \cdots + (a_{m-1}+1)x_{a_{m-1}} + a_m + 1}{p}\right).$$

Now, we calculate (3.6) with the help of Theorem 2.12.

Take affine linear forms $L_1,\dots,L_{m+1} : \mathbb{A}_k^{m-1} \to \mathbb{A}_k^1$ as

$$\begin{aligned}
L_1(x) &= -1 \cdot x_{a_1} \\
L_2(x) &= -1 \cdot x_{a_2} \\
&\vdots \\
L_{m-1}(x) &= -1 \cdot x_{a_{m-1}} \\
L_m(x) &= -a_1 \cdot x_{a_1} + \cdots + -a_{m-1} \cdot x_{a_{m-1}} - a_m \\
L_{m+1}(x) &= (a_1+1) \cdot x_{a_1} + \cdots + (a_{m-1}+1) \cdot x_{a_{m-1}} + (a_m+1).
\end{aligned}$$

The affine map defined by L_i is injective since the matrix (a_{ij}) has rank $m-1$. Now, let $V_i \subseteq \mathbb{A}_k^{m-1}$ be the hyperplane defined by $L_i(x) = 0$ for each $i \in \{1,\dots,m+1\}$, and $I \subset \{1,\dots,m+1\}$ be a subset with $|I| \le m$. When $|I| = m$,

$$\bigcap_{i \in I} V_i = \emptyset,$$

that is to say $\dim(\cap_{i\in I} V_i) = -1$. When $|I| \le m-1$,

$$\dim\left(\bigcap_{i \in I} V_i\right) \le m - 1 - |I|$$

holds since the intersections of the hyperplanes V_i do not coincide with themselves. Hence, by applying Theorem 2.12, we get the following inequality

$$|A| \le \frac{1}{2^k} \cdot D_L \cdot p^{\frac{3}{2}} + O_k(p^{\frac{1}{2}}), \tag{3.7}$$

where

$$D_L := (-1)^{m-1} + \sum_{j=1}^{m-1} (-1)^{m-1+j} c_j$$

and c_j is the number of subsets $I \subset \{1,\dots,m+1\}$ with $|I| = j$ such that

$$\bigcap_{i\in I} V_i \neq \emptyset.$$

Notice that

$$c_j = \binom{m+1}{j}$$

when $j \in \{1,...,m-1\}$. Using the well-known identity

$$\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} = 0,$$

the value D_L becomes

$$(-1)^{m-1} + \sum_{j=1}^{m-1} (-1)^{m-1+j} \binom{m+1}{j} = (-1)^{m-1} + (-1)^m (1 + (-1)^m (m+1) + (-1)^{m+1}) = m. \tag{3.8}$$

Therefore, (3.7) and the previous equality yield that

$$|A| \leq \frac{1}{2^k} \cdot m \cdot p^{\frac{3}{2}} + O_k(p^{\frac{1}{2}}).$$

Now, we rewrite the other forms and bring them to the form A. Note that the properties of the Legendre symbol and change of variables will be used again. If we arrange the indices on (3.3) and (3.4) using

$$(a_1+1)x_{a_1} + \cdots + (a_m+1)x_{a_m} = 0 \text{ and } -a_1 x_{a_1} - \cdots - a_m x_{a_m} = 0,$$

we get the following sums:

$$\frac{p^2 \cdot (p+1)^{k-1-m}}{2^k \cdot p^{k-1-m} \cdot p^{\frac{m+1}{2}}} \sum_{x_{a_1},\dots,x_{a_{m-1}}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_{m-1}}}{p}\right) \tag{3.9}$$
$$\cdot \left(\frac{(a_1+1)x_{a_1} + \cdots + (a_{m-1}+1)x_{a_{m-1}}}{p}\right) \left(\frac{(a_m - a_1)x_{a_1} + \cdots + (a_m - a_{m-1})x_{a_{m-1}}}{p}\right),$$

$$\frac{p^2 \cdot (p+1)^{k-1-m}}{2^k \cdot p^{k-1-m} \cdot p^{\frac{m+1}{2}}} \sum_{x_{a_1},\dots,x_{a_{m-1}}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_{m-1}}}{p}\right) \tag{3.10}$$
$$\cdot \left(\frac{a_1 x_{a_1} + \cdots + a_{m-1}x_{a_{m-1}}}{p}\right) \left(\frac{(a_m - a_1)x_{a_1} + \cdots + (a_m - a_{m-1})x_{a_{m-1}}}{p}\right).$$

Similarly, if the same method as in form A is applied for (3.9) and (3.10), we get

$$|B| \leq \frac{m}{2^k} \cdot p^{\frac{3}{2}} + O_k(p^{\frac{1}{2}}), \tag{3.11}$$

$$|C| \leq \frac{m}{2^k} \cdot p^{\frac{3}{2}} + O_k(p^{\frac{1}{2}}). \tag{3.12}$$

If we first use $-a_1 x_{a_1} - \cdots - a_m x_{a_m} = 0$ to rewrite the indices of the form D, we get the following sum in order to find the upper bound for (3.5):

$$\frac{p^2 \cdot (p+1)^{k-m}}{2^k \cdot p^{k-m} \cdot p^{\frac{m}{2}}} \sum_{\substack{x_{a_1},\ldots,x_{a_{m-1}} \\ (1-a_m^{-1}a_1)x_{a_1}+\cdots+(1-a_m^{-1}a_{m-1})x_{a_{m-1}}=0}} \left(\frac{-x_{a_1}}{p}\right) \cdots \left(\frac{-x_{a_{m-1}}}{p}\right) \cdot \left(\frac{a_1 x_{a_1} + \cdots + a_{m-1} x_{a_{m-1}}}{p}\right). \tag{3.13}$$

Then, again if the same method is used as in forms B and C, we deduce that

$$|D| \leq \frac{m}{2^k} \cdot p^{\frac{3}{2}} + O_k(p^{\frac{1}{2}}). \tag{3.14}$$

There can be at most 2^{k-2} expressions for (3.2), (3.3), (3.4) and (3.5) in the error term R. We also know that half of these expressions are zero by Remark 2.8. Thus, we deduce the upper bound for R as

$$|R| \leq 2 \cdot \sum_{m=2}^{k-2} \binom{k-2}{m} \cdot \frac{m}{2^k} \cdot p^{\frac{3}{2}} + O_k(p) = \left(\frac{k-2}{4} - \frac{k-2}{2^{k-1}}\right) \cdot p^{\frac{3}{2}} + O_k(p)$$

using the equality

$$\sum_{m=2}^{k-2} \binom{k-2}{m} \cdot m = \sum_{m=1}^{k-2} \binom{k-2}{m} \cdot m - (k-2) = (k-2) \cdot 2^{k-3} - (k-2).$$

Moreover, if the sums for the error term contributing to p are determined and calculated, as in our proof, the constant c_k can be found explicitly. □

An Application of Theorem 1.1 Let $\{S_{p_i}\}_{i \in \mathbb{N}}$ be a sequence of sets such that $p_1 = 5$ and p_i is a prime number. This time, we consider S_{p_i} as a subset of $\{1, \ldots, p_i\} \subset \mathbb{N}$. Let us make an assumption for now. Assume that when $i > j \geq 1$,

$$S_{p_i} \cap \{1, 2, \ldots, p_j\} = S_{p_j}.$$

Let us define

$$A = \bigcup_{i \geq 1} S_{p_i}.$$

Claim: The set A contains arbitrarily long arithmetic progressions.

Proof: Let $k \geq 3$ be a positive integer. By Theorem 1.1, for a sufficiently large prime number p_i, the set S_{p_i} contains non-trivial arithmetic progressions of length $2k^2 - 2k + 1$ modulo p_i. It follows from Proposition 2.3 that S_{p_i} contains non-trivial arithmetic progressions of length k in $\mathbb{Z}$. Hence, A contains arbitrarily long arithmetic progressions. (Thus, we proved the claim without using Szemerédi's theorem.)

Now, let us prove the above assumption, namely the existence of such sequences.

Proposition 3.1 Let q be a prime number such that $q \equiv 5 \pmod 8$ and S_q be the set of quadratic residues modulo q, that is $S_q = \{k \in \{1,2,\ldots,q\} : x^2 \equiv k \pmod q \text{ for some } x\}$. Then, there exists a prime number $p > q$ such that $p \equiv 5 \pmod 8$ with

$$S_p \cap \{1,2,\ldots,q\} = S_q.$$

Proof: Let q be a prime number such that $q \equiv 5 \pmod 8$. Now, let us divide the primes in $\{1,2,\ldots,q\}$ into two sets according to be quadratic or quadratic nonresidue modulo q. Let $p_1,\ldots,p_k \in \{1,2,\ldots,q\}$ be the list of primes where $\left(\frac{p_i}{q}\right) = 1$ and $2 = q_1,\ldots,q_r \in \{1,2,\ldots,q\}$ be the list of primes where $\left(\frac{q_i}{q}\right) = -1$.

For $i \in \{2,\ldots,r\}$, choose $a_i \in \{1,\ldots,q_i\}$ such that $\left(\frac{a_i}{q_i}\right) = -1$. Consider the following congruences:

$$\begin{aligned} X &\equiv 5 \pmod 8, \\ X &\equiv 1 \pmod{p_i}, \text{for each } i \in \{1,2,\ldots,k\}, \\ X &\equiv a_i \pmod{q_i}, \text{for each } i \in \{2,3,\ldots,r\}. \end{aligned}$$

By Chinese Remainder theorem, the solution set is an arithmetic progression

$$(a + n \cdot 8p_1 \cdots p_k q_2 \cdots q_r)_n,$$

where $0 \leq a < 8p_1 \cdots p_k q_2 \cdots q_r$. Moreover, $\gcd(a, 8p_1 \cdots p_k q_2 \cdots q_r) = 1$. Recall Dirichlet's theorem on arithmetic progressions (Theorem 7.9, Apostol, 1976), which states that if a and ℓ are relatively prime positive integers, then there are infinitely many primes of the form $a + n\ell$ with $n \in \mathbb{N}$. By Dirichlet's theorem, the above arithmetic progression contains a prime number, say $p > q$. Combining the law of quadratic reciprocity and $p \equiv 5 \pmod 8$, we obtain that for any odd prime s,

$$\left(\frac{p}{s}\right) \cdot \left(\frac{s}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{s-1}{2}} = 1.$$

Thus, $\left(\frac{p}{s}\right) = 1$ if and only if $\left(\frac{s}{p}\right) = 1$. As $\left(\frac{p}{p_i}\right) = \left(\frac{1}{p_i}\right) = 1$ for each $i \in \{1,2,\ldots,k\}$, we get $\left(\frac{p_i}{p}\right) = 1$. Similarly, as $\left(\frac{p}{q_i}\right) = \left(\frac{a_i}{q_i}\right) = -1$ for each $i \in \{2,3,\ldots,r\}$, we have $\left(\frac{q_i}{p}\right) = -1$. Also, as $p \equiv 5 \pmod 8$, we obtain that $\left(\frac{2}{p}\right) = -1$. □

4. SOME APPLICATIONS OF QUADRATIC GAUSS SUMS

4.1. Proof of Proposition 1.2

In this section, we prove Proposition 1.2 using quadratic Gauss sums.

Proof of Proposition 1.2 Let p be an odd prime. Let

$$Q_p(t) = |\{(x,y,z) \in S_p^3 \mid y-x = z-y = t\}| \tag{4.1}$$

denote the number of 3-term arithmetic progressions in S_p with common difference t.

By Lemma 2.4, we have that

$$\sum_{t\in\mathbb{F}_p} Q_p(t) = \frac{p^2|S_p|^3}{p^3} + p^2 \sum_{\ell\neq 0} \widehat{S}_p(\ell)\widehat{S}_p(\ell)\widehat{S}_p(-2\ell). \tag{4.2}$$

Recall that when $0 \neq m \in \mathbb{F}_p$,

$$\widehat{S}_p(m) = \frac{1}{2p} + \frac{1}{2\sqrt{p}}\varepsilon\left(\frac{-m}{p}\right),$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Using $\sum_{\ell\neq 0}\left(\frac{\ell}{p}\right) = 0$, we conclude that

$$\begin{aligned} p^2 \sum_{\ell\neq 0} \widehat{S}_p(\ell)\widehat{S}_p(\ell)\widehat{S}_p(-2\ell) &= p^2 \sum_{\ell\neq 0} \left(\frac{1}{2p}\left(1+\varepsilon\left(\frac{-\ell}{p}\right)\sqrt{p}\right)\right)^2 \left(\frac{1}{2p}\left(1+\varepsilon\left(\frac{2\ell}{p}\right)\sqrt{p}\right)\right) \\ &= \frac{1}{8p}\sum_{\ell\neq 0}\left(2\varepsilon^2 p\left(\frac{-2}{p}\right) + \varepsilon^2 p + 1\right). \end{aligned}$$

Note that

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ -1 & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{4.3}$$

and

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } 7 \pmod 8, \\ -1 & \text{if } p \equiv 3 \text{ or } 5 \pmod 8. \end{cases} \tag{4.4}$$

When we assume $p \equiv 1 \pmod 8$, it follows from equations (4.3) and (4.4) that

$$p^2 \sum_{\ell\neq 0} \widehat{S}_p(\ell)\widehat{S}_p(\ell)\widehat{S}_p(-2\ell) = \frac{1}{8p}(3p+1)(p-1).$$

Hence, we find that the number of non-trivial 3-APs in S_p is

$$\sum_{t\in\mathbb{Z}_p} Q_N(t) - |S_p| = \left(p^2 \left(\frac{p+1}{2p} \right)^3 + \frac{(p-1)(3p+1)}{8p} \right) - \left(\frac{p+1}{2} \right)$$
$$= \frac{(p-1)(p+3)}{8}.$$

In addition, if the processes are done by considering, respectively, conditions $p \equiv 3 \pmod 8$, $p \equiv 5 \pmod 8$ and $p \equiv 7 \pmod 8$ in the same way, we obtain the following formulas

$$\frac{(p-1)(p-3)}{8}, \ \frac{(p-1)(p-1)}{8} \ \text{ and } \ \frac{(p-1)(p+1)}{8}.$$

□

The above proposition actually gives the number of non-trivial solutions of the Diophantine congruence $x^2 + y^2 \equiv 2z^2 \pmod p$. Now, we consider this situation from another perspective. When we look at the significant developments on arithmetic progressions in recent years, it can be seen that the Diophantine equation $x^n + y^n = 2z^n$ has no non-trivial primitive solutions in $\mathbb{Z}_{>0}$ when $n \geq 3$, and this was proved by Darmon and Merel (Darmon and Merel, 1997). An integer solution (x,y,z) is called primitive if $\gcd(x,y,z) = 1$. In contrast to the Darmon-Merel Theorem, we will observe that for $n \geq 3$, the congruence $x^n + y^n \equiv 2z^n \pmod p$ has a non-trivial solution when p is a sufficiently large prime number. In order to get this observation, it is enough to show that there exist non-trivial arithmetic progressions of length 3 in $\Omega_p^n = \left\{ x^n : \ x \in \mathbb{F}_p \right\}$, and this can be achieved by van der Waerden's theorem (van der Waerden, 1927).

Remark 4.1 For $n \geq 3$, $\Omega_p^n = \left\{ x^n : \ x \in \mathbb{F}_p \right\}$ contains non-trivial arithmetic progressions of length 3 when p is a sufficiently large prime number.

Proof: Let $m = \left| \mathbb{F}_p^\times / \left(\mathbb{F}_p^\times \right)^n \right|$. Note that $1 \leq m \leq n$. Let $\{g_1, g_2, \ldots, g_m\}$ be a set of representatives of $\mathbb{F}_p^\times / \left(\mathbb{F}_p^\times \right)^n$, in other words

$$\mathbb{F}_p^\times = \bigsqcup_{i=1}^{m} g_i \left(\mathbb{F}_p^\times \right)^n.$$

Define a coloring ψ of $\{1,2,\ldots,p-1\}$ by $m-$many colors as follows. For each $a \in \{1,2,\ldots,p-1\}$, there is a unique g_i such that $a \in g_i \left(\mathbb{F}_p^\times \right)^n$. Set $\psi(a) = i$. By van der Waerden's theorem (van der Waerden, 1927), if p is large enough, there are distinct elements $x,y,z \in \mathbb{F}_p^\times$ such that

$$x + y = 2z \text{ and } \psi(x) = \psi(y) = \psi(z) = i.$$

As we can write $x = g_i x_1^n,\ y = g_i y_1^n,\ z = g_i z_1^n$, we obtain nonzero distinct elements $x_1^n, y_1^n, z_1^n \in \left(\mathbb{F}_p^\times\right)^n = \Omega_p^n \setminus \{0\}$ such that $x_1^n + y_1^n = 2z_1^n$.

4.2. Proof of Theorem 1.3

First, let us calculate the sums specified in the following lemma, which we will need in the proofs of Theorem 1.3 and Theorem 1.4.

Lemma 4.2 Let $p > 3$ be a prime number and for nonzero $m \in \mathbb{F}_p$,

$$\widehat{S}_p(m) = \frac{1}{2p} + \frac{1}{2\sqrt{p}}\varepsilon\left(\frac{-m}{p}\right),$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Then, for $a, b, c, d \in \mathbb{F}_p$ with $abcd(ad-bc) \neq 0$, we have

(I) $$\sum_{m \neq 0} \widehat{S}_p(m)\widehat{S}_p(am)\widehat{S}_p(bm) = \left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{a}{p}\right) + \left(\frac{b}{p}\right) + \left(\frac{ab}{p}\right)\right)\right)(p-1).$$

(II) $$\sum_{\substack{cm_1+dm_2 \neq 0 \\ am_1+bm_2=0 \\ m_2 \neq 0,\ m_1 \neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(cm_1+dm_2)$$
$$= \left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-ab}{p}\right) + \left(\frac{b(bc-ad)}{p}\right) + \left(\frac{-a(bc-ad)}{p}\right)\right)\right)(p-1).$$

(III) $$\sum_{\substack{cm_1+dm_2 \neq 0 \\ am_1+bm_2 \neq 0 \\ m_2 \neq 0,\ m_1 \neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(am_1+bm_2)\widehat{S}_p(cm_1+dm_2)$$
$$= \frac{(p-1)(p-3)}{16p^4} + \frac{1}{16p^2}(p-1)\left(\frac{-1}{p}\right)(\#E(\mathbb{F}_p) - p - 1)$$
$$- \frac{1}{16p^3}\varepsilon^2\left(\left(\frac{a}{p}\right) + \left(\frac{b}{p}\right) + \left(\frac{c}{p}\right) + \left(\frac{d}{p}\right)\right)(p-1)$$
$$- \frac{1}{16p^3}\varepsilon^2\left(\left(\frac{a\cdot(ad-bc)}{p}\right) + \left(\frac{d\cdot(ad-bc)}{p}\right)\right)(p-1)$$
$$- \frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-b\cdot(ad-bc)}{p}\right) + \left(\frac{-c\cdot(ad-bc)}{p}\right)\right)(p-1)$$
$$- \frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-ab}{p}\right) + \left(\frac{-cd}{p}\right) + \left(\frac{ac}{p}\right) + \left(\frac{bd}{p}\right)\right)(p-1)$$

where the elliptic curve E over $\mathbb{F}_p$ is defined by

$$E : y^2 = x(x-bc)(x-ad).$$

Proof: ***(I).*** As $ab \neq 0$, we see that

$$\sum_{m\neq 0} \widehat{S}_p(m)\widehat{S}_p(am)\widehat{S}_p(bm)$$
$$= \sum_{m\neq 0} \left(\frac{1}{2p} + \frac{1}{2\sqrt{p}}\varepsilon\left(\frac{-m}{p}\right)\right)\left(\frac{1}{2p} + \frac{1}{2\sqrt{p}}\varepsilon\left(\frac{-am}{p}\right)\right)\left(\frac{1}{2p} + \frac{1}{2\sqrt{p}}\varepsilon\left(\frac{-bm}{p}\right)\right).$$

Since we know that the sum of product of terms containing an odd number of Legendre symbols is zero, we just need to calculate the product of terms containing an even number of Legendre symbols. Thus, we obtain that

$$\sum_{m\neq 0} \widehat{S}_p(m)\widehat{S}_p(am)\widehat{S}_p(bm) = \sum_{m\neq 0}\left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{ab}{p}\right) + \left(\frac{a}{p}\right) + \left(\frac{b}{p}\right)\right)\right)$$
$$= \left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{ab}{p}\right) + \left(\frac{a}{p}\right) + \left(\frac{b}{p}\right)\right)\right)(p-1).$$

(II). Since $am_1 + bm_2 = 0$, we have

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2=0\\ m_2\neq 0,\ m_1\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(cm_1+dm_2) = \sum_{m_1\neq 0} \widehat{S}_p(m_1)\widehat{S}_p(-b^{-1}am_1)\widehat{S}_p((c-db^{-1}a)m_1).$$

Then, by *(I)* the desired equality is obtained.

(III). Now, let us calculate the last sum:

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(am_1+bm_2)\widehat{S}_p(cm_1+dm_2) = T(1)+T(2)+T(3)+T(4),$$

where

$$T(1) = \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.5}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-am_1-bm_2}{p}\right) \tag{4.6}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.7}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_2}{p}\right)\left(\frac{-am_1-bm_2}{p}\right) \tag{4.8}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-am_1-bm_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.9}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right), \tag{4.10}$$

$$T(2) = \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^4} \tag{4.11}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^2}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-am_1-bm_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right), \tag{4.12}$$

$$T(3) = \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{7/2}}\varepsilon\left(\frac{-m_1}{p}\right) \tag{4.13}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{7/2}}\varepsilon\left(\frac{-m_2}{p}\right) \tag{4.14}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{7/2}}\varepsilon\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.15}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{7/2}}\varepsilon\left(\frac{-am_1-bm_2}{p}\right), \tag{4.16}$$

$$T(4) = \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{5/2}}\varepsilon^3\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.17}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{5/2}}\varepsilon^3\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-am_1-bm_2}{p}\right) \tag{4.18}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{5/2}}\varepsilon^3\left(\frac{-m_1}{p}\right)\left(\frac{-am_1-bm_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.19}$$

$$+ \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^{5/2}}\varepsilon^3\left(\frac{-m_2}{p}\right)\left(\frac{-am_1-bm_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right). \tag{4.20}$$

We start by calculating (4.5). First, we edit the index of the sum by the inclusion-exclusion principle:

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) = \sum_{m_2\neq 0, m_1\neq 0} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.21}$$

$$- \sum_{\substack{cm_1+dm_2=0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right)$$

$$- \sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2=0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right).$$

Then, by Proposition 2.7 or by a change of variable, we obtain that

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) \tag{4.22}$$

$$= -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{c}{p}\right)+\left(\frac{-b\cdot(ad-bc)}{p}\right)\right)(p-1).$$

By arranging the coefficients in equation (4.22), we calculate (4.6), (4.7) and (4.8), respectively:

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-am_1-bm_2}{p}\right) = -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{a}{p}\right)+\left(\frac{d\cdot(ad-bc)}{p}\right)\right)(p-1),$$

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) = -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{d}{p}\right)+\left(\frac{a\cdot(ad-bc)}{p}\right)\right)(p-1),$$

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_2}{p}\right)\left(\frac{-am_1-bm_2}{p}\right) = -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{b}{p}\right)+\left(\frac{-c\cdot(ad-bc)}{p}\right)\right)(p-1).$$

If the same method in (4.21) is applied for (4.9) and (4.10), then we deduce that

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-am_1-bm_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right) = -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{ac}{p}\right)+\left(\frac{bd}{p}\right)\right)(p-1),$$

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^3}\varepsilon^2\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right) = -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-ab}{p}\right)+\left(\frac{-cd}{p}\right)\right)(p-1).$$

Also, it is quickly obtained that

$$\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^4} = \frac{(p-1)(p-3)}{16p^4}.$$

Now, we compute (4.12) with the help of Theorem 2.10. Observe that

$$\begin{aligned}&\sum_{\substack{cm_1+dm_2\neq 0\\ am_1+bm_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \frac{1}{16p^2}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-am_1-bm_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right)\\ &= \sum_{m_2\neq 0, m_1\neq 0} \frac{1}{16p^2}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-am_1-bm_2}{p}\right)\left(\frac{-cm_1-dm_2}{p}\right). \qquad (4.23)\end{aligned}$$

First, let us make (4.23) convenient to use Theorem 2.10. The change of variable

$$m_2 = m\cdot m_1$$

is bijective, so we have

$$\frac{1}{16p^2}\sum_{m_1\neq 0,m\neq 0}\left(\frac{-m_1}{p}\right)\left(\frac{-m\cdot m_1}{p}\right)\left(\frac{-am_1-bm\cdot m_1}{p}\right)\left(\frac{-cm_1-dm\cdot m_1}{p}\right)$$
$$=\frac{1}{16p^2}(p-1)\left(\frac{-1}{p}\right)\sum_{m\neq 0}\left(\frac{-m}{p}\right)\left(\frac{-bm-a}{p}\right)\left(\frac{-dm-c}{p}\right).$$

Next, we deal with the sum

$$\sum_{m\neq 0}\left(\frac{-m}{p}\right)\left(\frac{-bm-a}{p}\right)\left(\frac{-dm-c}{p}\right). \tag{4.24}$$

Consider the curve $y^2=-x(-bx-a)(-dx-c)=-bdx^3-(bc+ad)x^2-acx$, which can be rewritten as $(-bdy)^2=(-bdx)^3-(bc+ad)(-bdx)^2+acbd(-bdx)$. By replacing $-bdx$ with x and $-bdy$ with y, we arrive at the elliptic curve

$$E: y^2=x^3-(bc+ad)x^2+acbdx=x(x-bc)(x-ad). \tag{4.25}$$

Hence, by Theorem 2.10, we obtain that

$$\sum_{m\in\mathbb{F}_p}\left(\frac{-m(bm+a)(dm+c)}{p}\right)=\#E(\mathbb{F}_p)-p-1 \tag{4.26}$$

where the elliptic curve E over $\mathbb{F}_p$ is defined as above (4.25).

By Remark 2.8, we compute that each sum in $T(3)$ and $T(4)$ is zero. Thus, we conclude that

$$\begin{aligned}T(1)+T(2)+T(3)+T(4)=&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{c}{p}\right)+\left(\frac{-b\cdot(ad-bc)}{p}\right)\right)(p-1)\\&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{a}{p}\right)+\left(\frac{d\cdot(ad-bc)}{p}\right)\right)(p-1)\\&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{d}{p}\right)+\left(\frac{a\cdot(ad-bc)}{p}\right)\right)(p-1)\\&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{b}{p}\right)+\left(\frac{-c\cdot(ad-bc)}{p}\right)\right)(p-1)\\&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{ac}{p}\right)+\left(\frac{bd}{p}\right)\right)(p-1)\\&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-ab}{p}\right)+\left(\frac{-cd}{p}\right)\right)(p-1)\\&+\frac{(p-1)(p-3)}{16p^4}\\&+\frac{1}{16p^2}(p-1)\left(\frac{-1}{p}\right)(\#E(\mathbb{F}_p)-p-1).\end{aligned}$$

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 Let $p>3$ be a prime number. Let

$$Q_p(t)=|\{(x,y,z,v)\in S_p^4 \mid y-x=z-y=v-z=t\}| \tag{4.27}$$

denote the number of 4-term arithmetic progressions in S_p with common difference t.

By Lemma 2.4, the number of 4-APs in S_p is equal to

$$\sum_{t\in\mathbb{F}_p} Q_p(t)=\frac{|S_p|^4}{p^2}+p^2\sum_{(m_1,m_2)\neq(0,0)}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2)\widehat{S}_p(m_1+2m_2). \tag{4.28}$$

Note that the following system of equations

$$\begin{aligned}-2x_1-3x_2&=0\\ x_1+2x_2&=0\end{aligned}$$

has a unique solution since we have

$$\begin{vmatrix}-2 & -3\\ 1 & 2\end{vmatrix}=-1,$$

which is invertible in $\mathbb{F}_p$. Denote the last splitted term by

$$\begin{aligned}H&=p^2\sum_{(m_1,m_2)\neq(0,0)}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2)\widehat{S}_p(m_1+2m_2)\\
&=p|S_p|\sum_{m_2\neq0,\ m_1=0}\widehat{S}_p(m_2)\widehat{S}_p(-3m_2)\widehat{S}_p(2m_2)\\
&+p|S_p|\sum_{m_1\neq0,\ m_2=0}\widehat{S}_p(m_1)\widehat{S}_p(-2m_1)\widehat{S}_p(m_1)\\
&+p|S_p|\sum_{\substack{m_1+2m_2\neq0\\ -2m_1-3m_2=0\\ m_2\neq0,\ m_1\neq0}}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_1+2m_2)\\
&+p|S_p|\sum_{\substack{-2m_1-3m_2\neq0\\ m_1+2m_2=0\\ m_2\neq0,\ m_1\neq0}}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2)\\
&+p^2\sum_{\substack{-2m_1-3m_2\neq0\\ m_1+2m_2\neq0\\ m_2\neq0,\ m_1\neq0}}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2)\widehat{S}_p(m_1+2m_2).\end{aligned}$$

By Lemma 4.2, we compute the five sums mentioned above respectively:

$$
\begin{aligned}
T_1 &= \sum_{m_2\neq 0,\ m_1=0} \widehat{S}_p(m_2)\widehat{S}_p(-3m_2)\widehat{S}_p(2m_2) \\
&= \left(\frac{1}{8p^3}+\frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-6}{p}\right)+\left(\frac{2}{p}\right)+\left(\frac{-3}{p}\right)\right)\right)(p-1),
\end{aligned}
$$

$$
\begin{aligned}
T_2 &= \sum_{m_1\neq 0,\ m_2=0} \widehat{S}_p(m_1)\widehat{S}_p(-2m_1)\widehat{S}_p(m_1) \\
&= \left(\frac{1}{8p^3}+\frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-2}{p}\right)+\left(\frac{1}{p}\right)+\left(\frac{-2}{p}\right)\right)\right)(p-1),
\end{aligned}
$$

$$
\begin{aligned}
T_3 &= \sum_{\substack{m_1+2m_2\neq 0\\ -2m_1-3m_2=0\\ m_2\neq 0,\ m_1\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_1+2m_2) \\
&= \left(\frac{1}{8p^3}+\frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-6}{p}\right)+\left(\frac{-3}{p}\right)+\left(\frac{2}{p}\right)\right)\right)(p-1).
\end{aligned}
$$

$$
\begin{aligned}
T_4 &= \sum_{\substack{-2m_1-3m_2\neq 0\\ m_1+2m_2=0\\ m_2\neq 0,\ m_1\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2) \\
&= \left(\frac{1}{8p^3}+\frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-2}{p}\right)+\left(\frac{-2}{p}\right)+\left(\frac{1}{p}\right)\right)\right)(p-1).
\end{aligned}
$$

Now, let us calculate the last sum by (III) of Lemma 4.2:

$$
\begin{aligned}
T_5 &= \sum_{\substack{-2m_1-3m_2\neq 0\\ m_1+2m_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2)\widehat{S}_p(m_1+2m_2) \\
&= \frac{(p-1)(p-3)}{16p^4}+\frac{1}{16p^2}(p-1)\left(\frac{-1}{p}\right)(\#E(\mathbb{F}_p)-p-1) \\
&\quad -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{1}{p}\right)+\left(\frac{2}{p}\right)+\left(\frac{-2}{p}\right)+\left(\frac{-3}{p}\right)\right)(p-1) \\
&\quad -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{1}{p}\right)+\left(\frac{-3}{p}\right)\right)(p-1) \\
&\quad -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-2}{p}\right)+\left(\frac{2}{p}\right)\right)(p-1) \\
&\quad -\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-2}{p}\right)+\left(\frac{-6}{p}\right)+\left(\frac{-2}{p}\right)+\left(\frac{-6}{p}\right)\right)(p-1)
\end{aligned}
$$

where the elliptic curve E over $\mathbb{F}_p$ is defined by

$$E : y^2 = x(x+4)(x+3).$$

Thus, we conclude that the number of non-trivial 4-APs in S_p is given by the following formula:

$$\begin{aligned}
\sum_{t\in\mathbb{F}_p} Q_p(t) - |S_p| = \frac{|S_p|^4}{p^2} \\
&+ p|S_p|\left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-6}{p}\right) + \left(\frac{2}{p}\right) + \left(\frac{-3}{p}\right)\right)\right)(p-1) \\
&+ p|S_p|\left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-2}{p}\right) + \left(\frac{1}{p}\right) + \left(\frac{-2}{p}\right)\right)\right)(p-1) \\
&+ p|S_p|\left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-6}{p}\right) + \left(\frac{-3}{p}\right) + \left(\frac{2}{p}\right)\right)\right)(p-1) \\
&+ p|S_p|\left(\frac{1}{8p^3} + \frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-2}{p}\right) + \left(\frac{-2}{p}\right) + \left(\frac{1}{p}\right)\right)\right)(p-1) \\
&+ \frac{(p-1)(p-3)}{16p^2} + \frac{1}{16}(p-1)\left(\frac{-1}{p}\right)(\#E(\mathbb{F}_p) - p - 1) \\
&- \frac{1}{16p}\varepsilon^2\left(\left(\frac{1}{p}\right) + \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right) + \left(\frac{-3}{p}\right)\right)(p-1) \\
&- \frac{1}{16p}\varepsilon^2\left(\left(\frac{1}{p}\right) + \left(\frac{-3}{p}\right)\right)(p-1) \\
&- \frac{1}{16p}\varepsilon^2\left(\left(\frac{-2}{p}\right) + \left(\frac{2}{p}\right)\right)(p-1) \\
&- \frac{1}{16p}\varepsilon^2\left(\left(\frac{-2}{p}\right) + \left(\frac{-6}{p}\right) + \left(\frac{-2}{p}\right) + \left(\frac{-6}{p}\right)\right)(p-1) \\
&- |S_p|.
\end{aligned}$$

When the above equation is rearranged, the desired formula is deduced.

Next, we prove the second part of the theorem by applying the Sato-Tate conjecture, which is a theorem now. Recall by Hasse's theorem that

$$|p+1-\#E(\mathbb{F}_p)| \leq 2\sqrt{p}.$$

Write

$$2\cos\theta_p = \frac{\#E(\mathbb{F}_p) - p - 1}{\sqrt{p}},$$

where $\theta_p \in [0,\pi]$. By SageMath (SageMath, 2018), our elliptic curve

$$E : y^2 = x(x+3)(x+4)$$

has no complex multiplication and its j-invariant is $35152/9 \notin \mathbb{Z}$. Then by the Sato-Tate conjecture (Barnet-Lamb et al., 2011, Harris et al., 2010), we know that

$$\lim_{N\to\infty} \frac{|\{p \leq N : 0 \leq \theta_p \leq \alpha\}|}{|\{p \leq N\}|} = \frac{2}{\pi}\int_0^\alpha \sin^2\theta \, d\theta = \frac{1}{\pi}(\alpha - \sin\alpha \cdot \cos\alpha)$$

for any $\alpha \in [0,\pi]$. Let $\varepsilon > 0$ be given. Then, choosing $\alpha \in (0,\pi]$ sufficiently small with respect to ε and assembling the first part of the theorem, the Hasse bound and the previous consequence of the Sato-Tate conjecture, the number of non-trivial 4-APs in S_p is given by

$$\frac{p^2}{16} + R_p,$$

where

$$\left(\frac{1}{8} - \varepsilon\right) p^{\frac{3}{2}} \leq |R_p| \leq \left(\frac{1}{8} + \varepsilon\right) p^{\frac{3}{2}}$$

holds for infinitely many prime numbers p. Hence, the error term $O(p^{\frac{3}{2}})$ and the constant $\frac{1}{8}$ are both best possible. □

5. PROOF OF THEOREM 1.4

We will make use of the following two lemmas in the proof Theorem 1.4.

Lemma 5.1 Let $p>3$ be a prime number and for nonzero $m\in\mathbb{F}_p$,

$$\widehat{S}_p(m)=\frac{1}{2p}+\frac{1}{2\sqrt{p}}\varepsilon\left(\frac{-m}{p}\right),$$

where

$$\varepsilon=\begin{cases}1 & \text{if } p\equiv 1 \pmod 4,\\ i & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Then, for $a,b,c,d,e\in\mathbb{F}_p$ with

$$abcd(ad-bc)\neq 0 \text{ and } e(a-cd)(b-ce)\neq 0,$$

respectively, we have

$$\begin{aligned}
(I')\ &\sum_{m_1\neq 0,\ m_2\neq 0}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(am_1+bm_2)\widehat{S}_p(cm_1+dm_2)\\
&=\left(\frac{1}{8p^3}+\frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-ab}{p}\right)+\left(\frac{b(bc-ad)}{p}\right)+\left(\frac{-a(bc-ad)}{p}\right)\right)\right)\frac{p^2-1}{2p}\\
&+\left(\frac{1}{8p^3}+\frac{1}{8p^2}\varepsilon^2\left(\left(\frac{-cd}{p}\right)+\left(\frac{-d(bc-ad)}{p}\right)+\left(\frac{c(bc-ad)}{p}\right)\right)\right)\frac{p^2-1}{2p}\\
&+\frac{(p-1)(p-3)}{16p^4}+\frac{1}{16p^2}(p-1)\left(\frac{-1}{p}\right)(\#E(\mathbb{F}_p)-p-1)\\
&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{a}{p}\right)+\left(\frac{b}{p}\right)+\left(\frac{c}{p}\right)+\left(\frac{d}{p}\right)\right)(p-1)\\
&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{a\cdot(ad-bc)}{p}\right)+\left(\frac{d\cdot(ad-bc)}{p}\right)\right)(p-1)\\
&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-b\cdot(ad-bc)}{p}\right)+\left(\frac{-c\cdot(ad-bc)}{p}\right)\right)(p-1)\\
&-\frac{1}{16p^3}\varepsilon^2\left(\left(\frac{-ab}{p}\right)+\left(\frac{-cd}{p}\right)+\left(\frac{ac}{p}\right)+\left(\frac{bd}{p}\right)\right)(p-1)
\end{aligned}$$

where the elliptic curve E over $\mathbb{F}_p$ is defined by

$$E: y^2=x(x-bc)(x-ad).$$

$$\begin{aligned}
(II')\ &\sum_{\substack{am_1+bm_2+cm_3\neq 0\\ dm_1+em_2+m_3=0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(am_1+bm_2+cm_3)\\
&=\sum_{\substack{(a-cd)m_1+(b-ce)m_2\neq 0\\ -dm_1-em_2\neq 0\\ m_2\neq 0,\ m_1\neq 0}}\widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-dm_1-em_2)\widehat{S}_p((a-cd)m_1+(b-ce)m_2).
\end{aligned}$$

Proof: (I'). Combining (II) and (III) in Lemma 4.2, we obtain (I').

(II'). Using $dm_1+em_2+m_3=0$, we get (II').

Lemma 5.2 (I'') For $a,b,c,\beta,\gamma\in\mathbb{F}_p$ with $abc\beta\gamma(c-a\gamma)\neq 0$, we have

$$\sum_{\substack{am_1+bm_2+cm_3\neq 0\\ m_1+\beta m_2+\gamma m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)=\left(\left(\frac{-ab}{p}\right)+\left(\frac{-\beta}{p}\right)+\left(\frac{-(\beta c-\gamma b)(c-a\gamma)}{p}\right)\right)(p-1).$$

(II'') For $a,b,c,\alpha,\beta,\gamma\in\mathbb{F}_p$ with $abc\alpha\beta\gamma\neq 0$, we have

$$\sum_{\substack{\alpha m_1+\beta m_2+\gamma m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}}\left(\frac{-m_2}{p}\right)\left(\frac{am_1+bm_2+cm_3}{p}\right)$$
$$=\left(\left(\frac{-b}{p}\right)+\left(\frac{-\alpha(b\alpha-a\beta)}{p}\right)+\left(\frac{-\gamma(b\gamma-c\beta)}{p}\right)\right)(p-1).$$

Proof: (I'') By the inclusion-exclusion principle and the properties of the Legendre symbol,

$$\sum_{\substack{am_1+bm_2+cm_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)-\sum_{\substack{am_1+bm_2+cm_3\neq 0\\ m_1+\beta m_2+\gamma m_3=0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right) \tag{5.1}$$
$$=\sum_{m_1\neq 0,m_2\neq 0,m_3\neq 0}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)-\sum_{\substack{am_1+bm_2+cm_3=0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)$$
$$-\sum_{\substack{(b-a\beta)m_2+(c-a\gamma)m_3\neq 0\\ m_2\neq 0,m_3\neq 0}}\left(\frac{\beta m_2+\gamma m_3}{p}\right)\left(\frac{-m_2}{p}\right)$$
$$=-\sum_{m_2\neq 0,m_3\neq 0}\left(\frac{a^{-1}bm_2+a^{-1}cm_3}{p}\right)\left(\frac{-m_2}{p}\right)-\sum_{m_2\neq 0,m_3\neq 0}\left(\frac{\beta m_2+\gamma m_3}{p}\right)\left(\frac{-m_2}{p}\right)$$
$$+\sum_{\substack{(b-a\beta)m_2+(c-a\gamma)m_3=0\\ m_2\neq 0,m_3\neq 0}}\left(\frac{\beta m_2+\gamma m_3}{p}\right)\left(\frac{-m_2}{p}\right)$$
$$=-\sum_{m_2\neq 0,m_3\neq 0}\left(\frac{bm_2+cm_3}{p}\right)\left(\frac{-am_2}{p}\right)-\sum_{m_2\neq 0,m_3\neq 0}\left(\frac{\beta m_2+\gamma m_3}{p}\right)\left(\frac{-m_2}{p}\right)$$
$$+\sum_{m_2\neq 0}\left(\frac{(\beta c-\gamma b)m_2}{p}\right)\left(\frac{-(c-a\gamma)m_2}{p}\right).$$

It follows from Proposition 2.7 that

$$(5.1)=\sum_{m_3\neq 0}\left(\frac{-ab}{p}\right)+\sum_{m_3\neq 0}\left(\frac{-\beta}{p}\right)+\sum_{m_2\neq 0}\left(\frac{-(\beta c-\gamma b)(c-a\gamma)}{p}\right).$$

Then, this yields (I'').

(II'') By the inclusion-exclusion principle,

$$\begin{aligned}
&\sum_{\substack{\alpha m_1+\beta m_2+\gamma m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \left(\frac{-m_2}{p}\right)\left(\frac{am_1+bm_2+cm_3}{p}\right)\\
&= \sum_{m_1\neq 0,m_2\neq 0,m_3\neq 0} \left(\frac{-m_2}{p}\right)\left(\frac{am_1+bm_2+cm_3}{p}\right) - \sum_{\substack{\alpha m_1+\beta m_2+\gamma m_3=0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \left(\frac{-m_2}{p}\right)\left(\frac{am_1+bm_2+cm_3}{p}\right)\\
&= \sum_{m_2\neq 0,m_3\neq 0} \left(\frac{-m_2}{p}\right)\sum_{m_1}\left(\frac{am_1+bm_2+cm_3}{p}\right) - \sum_{m_2\neq 0,m_3\neq 0} \left(\frac{-m_2}{p}\right)\left(\frac{bm_2+cm_3}{p}\right)\\
&- \sum_{\substack{-\alpha^{-1}\beta m_2-\alpha^{-1}\gamma m_3\neq 0\\ m_2\neq 0,m_3\neq 0}} \left(\frac{-m_2}{p}\right)\left(\frac{(b-a\alpha^{-1}\beta)m_2+(c-a\alpha^{-1}\gamma)m_3}{p}\right)\\
&= \sum_{m_2\neq 0,m_3\neq 0} \left(\frac{-m_2}{p}\right)\sum_{m_1}\left(\frac{am_1+bm_2+cm_3}{p}\right) - \sum_{m_2\neq 0,m_3\neq 0} \left(\frac{-m_2}{p}\right)\left(\frac{bm_2+cm_3}{p}\right) \qquad (5.2)\\
&- \sum_{m_2\neq 0,m_3\neq 0} \left(\frac{-\alpha m_2}{p}\right)\left(\frac{(b\alpha-a\beta)m_2+(c\alpha-a\gamma)m_3}{p}\right)\\
&+ \sum_{m_2\neq 0} \left(\frac{-\alpha m_2}{p}\right)\left(\frac{(b\alpha-a\beta)m_2-(c\alpha-a\gamma)\gamma^{-1}\beta m_2}{p}\right).
\end{aligned}$$

Then, it follows Proposition 2.6, Proposition 2.7 and change of variables that

$$(5.2) = \left(\frac{-b}{p}\right)(p-1) + \left(\frac{-\alpha(b\alpha-a\beta)}{p}\right)(p-1) + \left(\frac{-\gamma(b\gamma-c\beta)}{p}\right)(p-1).$$

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4 Let $p>3$ be a prime number. Let

$$Q_p(t) = |\{(x_1,x_2,x_3,x_4,x_5)\in S_p^5 \mid x_{i+1}-x_i = t \text{ for } i\in\{1,\dots,4\}\}| \qquad (5.3)$$

denote the number of 5-term arithmetic progressions in S_p with common difference t.

By Lemma 2.4, the number of 5-APs in S_p is equal to

$$\begin{aligned}
\sum_{t\in\mathbb{F}_p} Q_p(t) &= \frac{|S_p|^5}{p^3}\\
&+ p^2 \sum_{(m_1,m_2,m_3)\neq(0,0,0)} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(-2m_1-3m_2-4m_3)\widehat{S}_p(m_1+2m_2+3m_3).
\end{aligned}$$

Denote the last splitted term by

$$
\begin{aligned}
H = {} & p|S_p| \sum_{m_1\neq 0, m_2\neq 0} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2)\widehat{S}_p(m_1+2m_2) \\
& + p|S_p| \sum_{m_1\neq 0, m_3\neq 0} \widehat{S}_p(m_1)\widehat{S}_p(m_3)\widehat{S}_p(-2m_1-4m_3)\widehat{S}_p(m_1+3m_3) \\
& + p|S_p| \sum_{m_2\neq 0, m_3\neq 0} \widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(-3m_2-4m_3)\widehat{S}_p(2m_2+3m_3) \\
& + |S_p|^2 \sum_{m_3\neq 0} \widehat{S}_p(m_3)\widehat{S}_p(-4m_3)\widehat{S}_p(3m_3) \\
& + |S_p|^2 \sum_{m_2\neq 0} \widehat{S}_p(m_2)\widehat{S}_p(-3m_2)\widehat{S}_p(2m_2) \\
& + |S_p|^2 \sum_{m_1\neq 0} \widehat{S}_p(m_1)\widehat{S}_p(-2m_1)\widehat{S}_p(m_1) \\
& + p|S_p| \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3=0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(-2m_1-3m_2-4m_3) \\
& + p|S_p| \sum_{\substack{-2m_1-3m_2-4m_3=0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(m_1+2m_2+3m_3) \\
& + |S_p|^2 \sum_{\substack{-2m_1-3m_2-4m_3=0 \\ m_1+2m_2+3m_3=0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3) \\
& + p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(-2m_1-3m_2-4m_3)\widehat{S}_p(m_1+2m_2+3m_3).
\end{aligned} \tag{5.4}
$$

By (I') in Lemma 5.1, we compute the first three sums mentioned above, respectively:

$$
\begin{aligned}
& p|S_p| \sum_{m_1\neq 0, m_2\neq 0} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(-2m_1-3m_2)\widehat{S}_p(m_1+2m_2) \\
& = \frac{f_1(p)}{32p^3} + \frac{g_1\left(p, \left(\frac{-1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right)\right)}{32p^2} \\
& + \frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_1(\mathbb{F}_p) - p - 1\right),
\end{aligned} \tag{5.5}
$$

where f_1 and g_1 are two polynomials of degree 3 with respect to p, and the elliptic curve E_1 over $\mathbb{F}_p$ is defined by

$$
E_1 : y^2 = x^3 + 7x^2 + 12x = x(x+3)(x+4).
$$

$$p|S_p| \sum_{m_1\neq 0, m_3\neq 0} \widehat{S}_p(m_1)\widehat{S}_p(m_3)\widehat{S}_p(-2m_1-4m_3)\widehat{S}_p(m_1+3m_3) \tag{5.6}$$
$$= \frac{f_2(p)}{32p^3} + \frac{g_2\left(p, \left(\frac{-1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right)\right)}{32p^2}$$
$$+ \frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_2(\mathbb{F}_p)-p-1\right),$$

where f_2 and g_2 are two polynomials of degree 3 with respect to p, and the elliptic curve E_2 over $\mathbb{F}_p$ is defined by

$$E_2 : y^2 = x^3 + 10x^2 + 24x = x(x+4)(x+6).$$

$$p|S_p| \sum_{m_2\neq 0, m_3\neq 0} \widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(-3m_2-4m_3)\widehat{S}_p(2m_2+3m_3) \tag{5.7}$$
$$= \frac{f_3(p)}{32p^3} + \frac{g_3\left(p, \left(\frac{-1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right)\right)}{32p^2}$$
$$+ \frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_3(\mathbb{F}_p)-p-1\right),$$

where f_3 and g_3 are two polynomials of degree 3 with respect to p, and the elliptic curve E_3 over $\mathbb{F}_p$ is defined by

$$E_3 : y^2 = x^3 + 17x^2 + 72x = x(x+8)(x+9).$$

By (I) in Lemma 4.2, we calculate the other three following sums in (5.4), respectively.

$$|S_p|^2 \sum_{m_3\neq 0} \widehat{S}_p(m_3)\widehat{S}_p(-4m_3)\widehat{S}_p(3m_3) = \frac{f_4(p)}{32p^3} + \frac{g_4\left(p, \left(\frac{-1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right)\right)}{32p^2}, \tag{5.8}$$

where f_4 and g_4 are two polynomials of degree 3 with respect to p.

$$|S_p|^2 \sum_{m_2\neq 0} \widehat{S}_p(m_2)\widehat{S}_p(-3m_2)\widehat{S}_p(2m_2) = \frac{f_5(p)}{32p^3} + \frac{g_5\left(p, \left(\frac{-1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right)\right)}{32p^2}, \tag{5.9}$$

where f_5 and g_5 are two polynomials of degree 3 with respect to p.

$$|S_p|^2 \sum_{m_1\neq 0} \widehat{S}_p(m_1)\widehat{S}_p(-2m_1)\widehat{S}_p(m_1) = \frac{f_6(p)}{32p^3} + \frac{g_6\left(p, \left(\frac{-1}{p}\right), \left(\frac{2}{p}\right), \left(\frac{3}{p}\right)\right)}{32p^2}, \tag{5.10}$$

where f_6 and g_6 are two polynomials of degree 3 with respect to p.

Combining (II') in Lemma 5.1 with (III) in Lemma 4.2, we arrive at

$$\begin{aligned} & p|S_p| \sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3=0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(-2m_1-3m_2-4m_3) \\ &= \frac{f_1(p)}{32p^3} + \frac{g_1\left(p,\left(\frac{-1}{p}\right),\left(\frac{2}{p}\right),\left(\frac{3}{p}\right)\right)}{32p^2} \\ &+ \frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_1(\mathbb{F}_p)-p-1\right), \end{aligned} \tag{5.11}$$

$$\begin{aligned} & p|S_p| \sum_{\substack{-2m_1-3m_2-4m_3=0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(m_1+2m_2+3m_3) \\ &= p|S_p| \sum_{\substack{-3m_2-4m_3\neq 0\\ m_2+2m_3\neq 0\\ m_3\neq 0,\ m_2\neq 0}} \widehat{S}_p(2m_2)\widehat{S}_p(2m_3)\widehat{S}_p(-3m_2-4m_3)\widehat{S}_p(m_2+2m_3) \\ &= \frac{f_7(p)}{32p^3} + \frac{g_7\left(p,\left(\frac{-1}{p}\right),\left(\frac{2}{p}\right),\left(\frac{3}{p}\right)\right)}{32p^2} \\ &+ \frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_4(\mathbb{F}_p)-p-1\right), \end{aligned} \tag{5.12}$$

where f_7 and g_7 are two polynomials of degree 3 with respect to p, and the elliptic curve E_4 over $\mathbb{F}_p$ is defined by

$$E_4 : y^2 = x^3+40x^2+384x = x(x+16)(x+24).$$

Notice that

$$\begin{aligned} \#E_4(\mathbb{F}_p)-p-1 &= \sum_{x=0}^{p-1}\left(\frac{x(x+16)(x+24)}{p}\right) = \sum_{x=0}^{p-1}\left(\frac{4x(4x+16)(4x+24)}{p}\right) \\ &= \sum_{x=0}^{p-1}\left(\frac{x(x+4)(x+6)}{p}\right) = \#E_2(\mathbb{F}_p)-p-1. \end{aligned}$$

By (5.10), we have

$$\begin{aligned} & |S_p|^2 \sum_{\substack{-2m_1-3m_2-4m_3=0\\ m_1+2m_2+3m_3=0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3) = |S_p|^2 \sum_{m\neq 0} \widehat{S}_p(m)\widehat{S}_p(-2m)\widehat{S}_p(m) \\ &= \frac{f_6(p)}{32p^3} + \frac{g_6\left(p,\left(\frac{-1}{p}\right),\left(\frac{2}{p}\right),\left(\frac{3}{p}\right)\right)}{32p^2}. \end{aligned} \tag{5.13}$$

Now, we calculate the last sum in (5.4) using Lemma 5.2:

$$p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \widehat{S}_p(m_1)\widehat{S}_p(m_2)\widehat{S}_p(m_3)\widehat{S}_p(-2m_1-3m_2-4m_3)\widehat{S}_p(m_1+2m_2+3m_3) \tag{5.14}$$

$$= R(1)+R(2)+R(3),$$

where

$$R(1) = p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \frac{1}{32p^3}\left(\frac{-m_2}{p}\right)\left(\frac{-m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.15}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \frac{1}{32p^3}\left(\frac{-m_1}{p}\right)\left(\frac{-m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.16}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \frac{1}{32p^3}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.17}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \frac{1}{32p^3}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.18}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}} \frac{1}{32p^3}\left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right)\left(\frac{-m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right), \tag{5.19}$$

and

$$R(2) = p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^5} \tag{5.20}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.21}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_1}{p}\right)\left(\frac{-m_2}{p}\right) \tag{5.22}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_1}{p}\right)\left(\frac{-m_3}{p}\right) \tag{5.23}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_2}{p}\right)\left(\frac{-m_3}{p}\right), \tag{5.24}$$

and

$$R(3) = p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.25}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right) \tag{5.26}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_2}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.27}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_2}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right) \tag{5.28}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_1}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.29}$$

$$+ p^2 \sum_{\substack{-2m_1-3m_2-4m_3\neq 0 \\ m_1+2m_2+3m_3\neq 0 \\ m_1\neq 0, m_2\neq 0, m_3\neq 0}} \frac{1}{32p^4} \varepsilon^2 \left(\frac{-m_1}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right). \tag{5.30}$$

Let us calculate (5.15). Observe that

$$\frac{1}{32p}\sum_{\substack{-2m_1-3m_2-4m_3\neq 0\\ m_1+2m_2+3m_3\neq 0\\ m_1\neq 0,m_2\neq 0,m_3\neq 0}}\left(\frac{-m_2}{p}\right)\left(\frac{-m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right)$$
$$=\frac{1}{32p}\sum_{m_1\neq 0,m_2\neq 0,m_3\neq 0}\left(\frac{m_2m_3}{p}\right)\left(\frac{-2m_1^2-(7m_2+10m_3)\,m_1-(2m_2+3m_3)\,(3m_2+4m_3)}{p}\right). \tag{5.31}$$

We rewrite (5.31) in order to use Proposition 2.7. Since

$$b^2-4ac=(7m_2+10m_3)^2-4\cdot 2\cdot(2m_2+3m_3)\cdot(3m_2+4m_3)=(m_2+2m_3)^2,$$

equation (5.31) becomes

$$\frac{1}{32p}\sum_{\substack{m_2+2m_3=0\\ m_2\neq 0,m_3\neq 0}}\sum_{m_1\neq 0}\left(\frac{m_2m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right) \tag{5.32}$$
$$+\frac{1}{32p}\sum_{\substack{m_2+2m_3\neq 0\\ m_2\neq 0,m_3\neq 0}}\sum_{m_1\neq 0}\left(\frac{m_2m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right). \tag{5.33}$$

Now, we compute (5.32) with the help of Proposition 2.7:

$$\begin{aligned}(5.32)&=\frac{1}{32p}\sum_{\substack{m_2+2m_3=0\\ m_2\neq 0,m_3\neq 0}}\sum_{m_1}\left(\frac{m_2m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right)\\ &\quad-\frac{1}{32p}\sum_{\substack{m_2+2m_3=0\\ m_2\neq 0,m_3\neq 0}}\left(\frac{m_2m_3}{p}\right)\left(\frac{-2m_2-3m_3}{p}\right)\left(\frac{3m_2+4m_3}{p}\right)\\ &=\frac{1}{32p}\sum_{\substack{m_2+2m_3=0\\ m_2\neq 0,m_3\neq 0}}(p-1)-\frac{1}{32p}\sum_{\substack{m_2+2m_3=0\\ m_2\neq 0,m_3\neq 0}}1\\ &=\frac{1}{32p}\Big((p-1)^2-(p-1)\Big).\end{aligned}$$

By Proposition 2.7, the inclusion-exclusion principle and Equation (4.26), we get that

$$
\begin{aligned}
(5.33) &= \frac{1}{32p} \sum_{\substack{m_2+2m_3\neq 0\\ m_2\neq 0, m_3\neq 0}} \sum_{m_1} \left(\frac{m_2m_3}{p}\right)\left(\frac{-m_1-2m_2-3m_3}{p}\right)\left(\frac{2m_1+3m_2+4m_3}{p}\right)\\
&\quad - \frac{1}{32p} \sum_{\substack{m_2+2m_3\neq 0\\ m_2\neq 0, m_3\neq 0}} \left(\frac{m_2m_3}{p}\right)\left(\frac{-2m_2-3m_3}{p}\right)\left(\frac{3m_2+4m_3}{p}\right)\\
&= \frac{1}{32p} \sum_{\substack{m_2+2m_3\neq 0\\ m_2\neq 0, m_3\neq 0}} -\left(\frac{-2m_2m_3}{p}\right)\\
&\quad - \frac{1}{32p} \sum_{\substack{m_2+2m_3\neq 0\\ m_2\neq 0, m_3\neq 0}} \left(\frac{m_2m_3}{p}\right)\left(\frac{-2m_2-3m_3}{p}\right)\left(\frac{3m_2+4m_3}{p}\right)\\
&= \frac{1}{32p}\left(2(p-1) - \left(\frac{-1}{p}\right)(p-1)\left(\#E_3(\mathbb{F}_p)-p-1\right)\right).
\end{aligned}
$$

Thus, we obtain that

$$
(5.15) = \frac{1}{32p}\left(p(p-1) - \left(\frac{-1}{p}\right)(p-1)\left(\#E_3(\mathbb{F}_p)-p-1\right)\right).
$$

The other terms are quickly determined if the method when calculating (5.15) is applied. Thus, the following calculations are obtained:

$$
\begin{aligned}
(5.16) &= \frac{1}{32p}\left(\left(\frac{-6}{p}\right)p(p-1) - \left(\frac{-1}{p}\right)(p-1)\left(\#E_2(\mathbb{F}_p)-p-1\right)\right),\\
(5.17) &= \frac{1}{32p}\left(\left(\frac{6}{p}\right)p(p-1) - \left(\frac{-1}{p}\right)(p-1)\left(\#E_1(\mathbb{F}_p)-p-1\right)\right),\\
(5.18) &= \frac{1}{32p}\left(\left(\frac{6}{p}\right)p(p-1) - \left(\frac{-1}{p}\right)(p-1)\left(\#E_1(\mathbb{F}_p)-p-1\right)\right),\\
(5.19) &= \frac{1}{32p}\left(\left(\frac{-6}{p}\right)p(p-1) - \left(\frac{-1}{p}\right)(p-1)\left(\#E_2(\mathbb{F}_p)-p-1\right)\right),\\
(5.20) &= \frac{(p-1)(p^2-4p+5)}{32p^3},\\
(5.21) &= \frac{1}{32p^2}\left(\left(\frac{-6}{p}\right)+\left(\frac{-3}{p}\right)+\left(\frac{-2}{p}\right)\right)(p-1).
\end{aligned}
$$

Also, applying (I'') of Lemma 5.2 to $R(2)$ and applying (II'') of Lemma 5.2 to $R(3)$, we obtain that

$$
\begin{aligned}
(5.22) &= \frac{1}{32p^2}\left(\left(\frac{-6}{p}\right)+\left(\frac{-2}{p}\right)+\left(\frac{-2}{p}\right)\right)(p-1),\\
(5.23) &= \frac{1}{32p^2}\left(\left(\frac{-2}{p}\right)+\left(\frac{-3}{p}\right)+1\right)(p-1),\\
(5.24) &= \frac{1}{32p^2}\left(\left(\frac{-3}{p}\right)+\left(\frac{-6}{p}\right)+\left(\frac{-2}{p}\right)\right)(p-1)
\end{aligned}
$$

and

$$(5.25) = \frac{1}{32p^2}\left(\left(\frac{-1}{p}\right)+\left(\frac{2}{p}\right)+\left(\frac{2}{p}\right)\right)(p-1),$$

$$(5.26) = \frac{1}{32p^2}\left(\left(\frac{3}{p}\right)+1+\left(\frac{3}{p}\right)\right)(p-1),$$

$$(5.27) = \frac{1}{32p^2}\left(\left(\frac{-3}{p}\right)+1+\left(\frac{-3}{p}\right)\right)(p-1),$$

$$(5.28) = \frac{1}{32p^2}\left(\left(\frac{2}{p}\right)+\left(\frac{2}{p}\right)+\left(\frac{-1}{p}\right)\right)(p-1),$$

$$(5.29) = \frac{1}{32p^2}\left(\left(\frac{-2}{p}\right)+\left(\frac{-2}{p}\right)+\left(\frac{-6}{p}\right)\right)(p-1),$$

$$(5.30) = \frac{1}{32p^2}\left(1+\left(\frac{-3}{p}\right)+\left(\frac{-2}{p}\right)\right)(p-1).$$

Considering the equations from (5.15) to (5.30), we have calculated Equation (5.14). Thus, we obtain an explicit formula as expressed in the theorem.

Next, we prove the second part of the theorem by making use of a version of the Sato-Tate conjecture, and for this we refer the reader to (Barnet-Lamb et al., 2011, Harris et al., 2010) and the generalized version of (Corollary 2, Murty and Murty, 2009). By the first part of the theorem and the Hasse bound, observe that the contributions for the error term $O(p^{\frac{3}{2}})$ come from a subsum in Equations (5.5), (5.6), (5.7), (5.11) and (5.12), and they are all of the form

$$\frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_i(\mathbb{F}_p)-p-1\right)$$

for some $i \in \{1,2,3\}$, as

$$\#E_4(\mathbb{F}_p)-p-1 = \#E_2(\mathbb{F}_p)-p-1.$$

Let $V_1 = V_1(a,b)$ and $V_2 = V_2(a,b)$ be the following two elliptic curves

$$y^2 = x^3+ax^2+bx$$

and

$$y^2 = x^3-2ax^2+(a^2-4b)x$$

respectively, where $a,b \in \mathbb{Z}$ and $b(a^2-4b) \neq 0$. Then the map

$$\varphi(x,y) = \left(x+a+\frac{1}{x}, y\left(1-\frac{b}{x^2}\right)\right)$$

yields an isogeny from V_1 to V_2 with kernel $\{O,(0,0)\}$, see (pp. 110, Silverman, 1999). Thus, one infers that the elliptic curve C_1 defined by $y^2 = x(x+1)(x+4) = x^3+5x^2+4x$

and the elliptic curve C_2 defined by $y^2 = x(x-1)(x-9) = x^3 - 10x^2 + 9x$ are isogenous over $\mathbb{Q}$. Therefore, for any prime number $p > 3$, the elliptic curves C_1 and C_2 are isogenous over $\mathbb{F}_p$, and hence

$$\#C_1(\mathbb{F}_p) = \#C_2(\mathbb{F}_p)$$

by Tate's isogeny theorem (Tate, 1966).

For the rest of the proof, let $p \equiv 1 \pmod 4$ so that $\left(\frac{-1}{p}\right) = 1$. Next, we will obtain that $E_1(\mathbb{F}_p) - p - 1 = E_3(\mathbb{F}_p) - p - 1$. Now, for any prime $p > 3$

$$\begin{aligned}
\#E_1(\mathbb{F}_p) - p - 1 &= \sum_{x=0}^{p-1} \left(\frac{x(x+3)(x+4)}{p}\right) = \sum_{x=0}^{p-1} \left(\frac{-x(-x+3)(-x+4)}{p}\right) \\
&= \sum_{x=0}^{p-1} \left(\frac{x(x-3)(x-4)}{p}\right) = \sum_{x=0}^{p-1} \left(\frac{x(x+1)(x+4)}{p}\right) \\
&= \#C_1(\mathbb{F}_p) - p - 1 = \#C_2(\mathbb{F}_p) - p - 1 \\
&= \sum_{x=0}^{p-1} \left(\frac{x(x-1)(x-9)}{p}\right) = \sum_{x=0}^{p-1} \left(\frac{x(x+8)(x+9)}{p}\right) \\
&= \#E_3(\mathbb{F}_p) - p - 1.
\end{aligned}$$

In other words, the correspondig subsums of (5.5) and (5.7) coming from the elliptic curves are equal. Recall the elliptic curve

$$E_1 : y^2 = x(x+3)(x+4)$$

has no complex multiplication and its j-invariant is $35152/9 \notin \mathbb{Z}$. As in the previous theorem, set

$$2\cos\theta_p = \frac{\#E_1(\mathbb{F}_p) - p - 1}{\sqrt{p}},$$

where $\theta_p \in [0, \pi]$. Then by the proof of the Sato-Tate conjecture (Barnet-Lamb et al., 2011, Harris et al., 2010), one immediately gets that

$$\lim_{N\to\infty} \frac{|\{p \le N : 0 \le \theta_p \le \pi/6\}|}{|\{p \le N : p \equiv 1 \pmod 4\}|} = \frac{2}{\varphi(4)\pi} \int_0^{\pi/6} \sin^2\theta \, d\theta = \frac{1}{48}(2\pi - 3\sqrt{3}) \approx 0.022646,$$

and $\cos(\pi/6) = \sqrt{3}/2$. Express the number of 5-APs in S_p as

$$\frac{p^2}{32} + R_p,$$

where $R_p = O(p^{\frac{3}{2}})$. By the consequence of the Sato-Tate conjecture, for infinitely many primes $p \equiv 1 \pmod 4$, Equations (5.5), (5.7) and (5.11) will bring an error term T_p and

$$T_p \geq \frac{1}{32p}\left(p^2-1\right) 6\cos(\pi/6)\sqrt{p} = \frac{1}{32p}\left(p^2-1\right) 3\sqrt{3}\sqrt{p}. \tag{5.34}$$

The subsums in (5.6) and (5.12), namely

$$\frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_2(\mathbb{F}_p)-p-1\right)$$

and

$$\frac{1}{32p}\left(p^2-1\right)\left(\frac{-1}{p}\right)\left(\#E_4(\mathbb{F}_p)-p-1\right)$$

can cancel out at most

$$\frac{1}{32p}\left(p^2-1\right) 4\sqrt{p}$$

of the term in (5.34), by Hasse's estimate. Hence, there are two positive absolute constants c_1 and c_2 such that the inequality

$$c_1 p^{\frac{3}{2}} \leq |R_p| \leq c_2 p^{\frac{3}{2}}$$

holds for infinitely many prime numbers p. This yields that the error term $O(p^{\frac{3}{2}})$ is best possible. □

6. KUMMER SUMS AND 3-APs

Let p be an odd prime with $p \equiv 1 \pmod 3$. Then, one has that $|C_p| = 1 + \frac{p-1}{3} = \frac{p+2}{3}$ where $C_p = \{t^3 : t \in \mathbb{F}_p\}$. Let

$$K(p) = \sum_{x=0}^{p-1} e_p(x^3)$$

be the Kummer sum. The Kummer sum is related to a cubic Gauss sum which we define next. Let g be a primitive root modulo p and $w = e^{2\pi i/3}$. Define the multiplicative cubic character

$$\chi : \mathbb{F}_p \to \{0, 1, w, w^2\}$$

as follows: $\chi(0) = 0$ and $\chi(g) = w$. Thus, $\chi(g^m) = w^r$, where r is the remainder when m is divided by 3. Note also that χ extends to $\mathbb{N}$ as a Dirichlet character. Let

$$\tau_p = \sum_{x=0}^{p-1} \chi(x) e_p(x)$$

be the cubic Gauss sum. One can observe that

$$\overline{\tau_p} = \sum_{x=0}^{p-1} \overline{\chi}(x) e_p(x)$$

as $\chi(-1) = 1$. We have that τ_p has norm $\sqrt{p}$, from (Chapter 3, Davenport, 1980). Thus, $\tau_p = \sqrt{p} e^{i\theta_p}$ and $\overline{\tau_p} = \sqrt{p} e^{-i\theta_p}$ for some angle θ_p. Unlike the quadratic Gauss sum, there is no specific formula for θ_p and in fact there is an equidistribution result by Heath-Brown and Patterson in (Heath-Brown and Patterson, 1979) as p varies, and this refuted Kummer's guess.

For $x \neq 0$, note that $1 + \chi(x) + \overline{\chi}(x) = 3$ if x is in C_p and otherwise $1 + \chi(x) + \overline{\chi}(x) = 0$. This yields that

$$K(p) = \sum_{x=0}^{p-1} (1 + \chi(x) + \overline{\chi}(x)) e_p(x) = \tau_p + \overline{\tau_p} = 2\sqrt{p}\cos(\theta_p).$$

Similarly, for $a \neq 0$,

$$\begin{aligned} K(a,p) = \sum_{x=0}^{p-1} e_p(ax^3) &= \sum_{x=0}^{p-1} (1 + \chi(x) + \overline{\chi}(x)) e_p(ax) \\ &= \sum_{x=0}^{p-1} \chi(x) e_p(ax) + \sum_{x=0}^{p-1} \overline{\chi}(x) e_p(ax) \\ &= \overline{\chi}(a)\tau_p + \chi(a)\overline{\tau_p}. \end{aligned}$$

The following lemma yields the Fourier transform of the set C_p.

Lemma 6.1 For any nonzero integer m modulo p,

$$\widehat{C_p}(m) = \frac{1}{3p}(\overline{\chi}(m)\tau_p + \chi(m)\overline{\tau_p} + 2).$$

Proof: Let m be a nonzero integer modulo p. Then

$$\begin{aligned}\widehat{C_p}(m) &= \frac{1}{p}\sum_{x=0}^{p-1} e_p(-mx)C_p(x) = \frac{1}{p}\sum_{x\in C_p} e_p(-mx)\\ &= \frac{1}{p}\left(1 + \sum_{x\in C_p-\{0\}} e_p(-mx)\right) = \frac{1}{p}\left(1 + \frac{1}{3}\sum_{x=1}^{p-1} e_p(-mx^3)\right)\\ &= \frac{1}{3p}\left(2 + \sum_{x=0}^{p-1} e_p(-mx^3)\right) = \frac{1}{3p}(\overline{\chi}(m)\tau_p + \chi(m)\overline{\tau_p} + 2),\end{aligned}$$

as $\chi(m) = \chi(-m)$, and we also apply the previous observation above.

Now, we are ready to count the number of non-trivial 3-term arithmetic progressions in C_p.

Proof of Theorem 1.5 As we did in the proof of Lemma 2.4,

$$\begin{aligned}Q_p &= p^2(\widehat{C_p}(0))^3 + p^2\sum_{m=1}^{p-1}\widehat{C_p}(m)\widehat{C_p}(m)\widehat{C_p}(-2m) - \frac{p+2}{3}\\ &= \frac{(p+2)^3}{27p} + p^2\sum_{m=1}^{p-1}\widehat{C_p}(m)\widehat{C_p}(m)\widehat{C_p}(-2m) - \frac{p+2}{3}. \qquad (6.1)\end{aligned}$$

Notice that $\tau_p\overline{\tau_p} = p$, one has $\widehat{C_p}(m) = \widehat{C_p}(-m)$, $\overline{\chi}(m)\chi(m) = 1$ and $\chi(m^2) = \chi(m)^2 = \overline{\chi}(m)$ for any nonzero m modulo p. Then, using the observations above, for any nonzero m modulo p, one sees that

$$\begin{aligned}\widehat{C_p}(m)\widehat{C_p}(m)\widehat{C_p}(2m) &= \frac{1}{27p^3}(\overline{\chi}(m)\tau_p + \chi(m)\overline{\tau_p} + 2)^2(\overline{\chi}(2m)\tau_p + \chi(2m)\overline{\tau_p} + 2)\\ &= \frac{1}{27p^3}(\overline{\chi}(m^2)\tau_p^2 + \chi(m^2)\overline{\tau_p}^2 + 4 + 2p + 4\overline{\chi}(m)\tau_p\\ &\quad + 4\chi(m)\overline{\tau_p})\cdot(\overline{\chi}(2m)\tau_p + \chi(2m)\overline{\tau_p} + 2)\\ &= \frac{1}{27p^3}\Big(\overline{\chi}(2)\tau_p^3 + \overline{\chi}(m)\chi(2)\tau_p^2\overline{\tau_p} + 2\chi(m)\tau_p^2 + \chi(m)\chi(2)\overline{\tau_p}^2\tau_p + \chi(2)\overline{\tau_p}^3\\ &\quad + 2\overline{\chi}(m)\overline{\tau_p}^2 + (2p+4)\overline{\chi}(2m)\tau_p + (2p+4)\chi(2m)\overline{\tau_p} + 4p + 8 + 4\overline{\chi}(2)\chi(m)\tau_p^2\\ &\quad + 4\chi(2)p + 8\overline{\chi}(m)\tau_p + 4\overline{\chi}(2)p + 4\chi(2)\overline{\chi}(m)\overline{\tau_p}^2 + 8\chi(m)\overline{\tau_p}\Big).\end{aligned}$$

By orthogonality, for any non-principal Dirichlet character h modulo p, we have that

$$\sum_{m=1}^{p-1} h(m) = 0.$$

By the previous calculations and orthogonality of χ and $\overline{\chi}$, Equation (6.1) becomes

$$Q_p = \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(\overline{\chi}(2)\tau_p^3 + \chi(2)\overline{\tau_p}^3 + 4p + 8 + 4\chi(2)p + 4\overline{\chi}(2)p) - \frac{p+2}{3}. \quad (6.2)$$

As $\tau_p = \sqrt{p}e^{i\theta_p}$, we have the first part of the theorem. Note that $\chi(2) + \overline{\chi}(2) = 2$ if 2 is a cubic residue, and otherwise $\chi(2) + \overline{\chi}(2) = -1$. Also $\tau_p^3 + \overline{\tau_p}^3 = 2p^{3/2}\cos(3\theta_p)$. By (Chapter 3, Davenport, 1980), one has that

$$\cos(3\theta_p) = \frac{a}{2\sqrt{p}},$$

where $4p = a^2 + 27b^2$ and $a \equiv 1 \pmod{p}$. By (Chapter 3, Davenport, 1980), we know that

$$\tau_p^3 = p\sum_{t=1}^{p-1} \chi(t(1+t)) = p(A + Bw),$$

for some integers A and B. Thus, $\overline{\tau_p}^3 = p(A + Bw^2)$. Moreover $p = A^2 - AB + B^2$ and $4p = (2A - B)^2 + 3B^2$. This yields that

$$\cos(3\theta_p) = \frac{2A - B}{2\sqrt{p}}$$

and

$$\sin(3\theta_p) = \frac{B\sqrt{3}}{2\sqrt{p}}.$$

Next, we compute the sum

$$z_p = \overline{\chi}(2)\tau_p^3 + \chi(2)\overline{\tau_p}^3.$$

If $\chi(2) = 1 = \overline{\chi}(2)$, we already computed the sum. Now, suppose $\overline{\chi}(2) = w = e^{2\pi i/3}$. Then,

$$z_p = 2p^{3/2}\cos(3\theta_p + 2\pi/3) = 2p^{3/2}\left(-\frac{\cos(3\theta_p)}{2} - \frac{\sin(3\theta_p)\sqrt{3}}{2}\right) \quad (6.3)$$

$$= 2p^{3/2}\left(-\frac{2A - B}{4\sqrt{p}} - \frac{3B}{4\sqrt{p}}\right) = -p(A + B). \quad (6.4)$$

Similarly, if $\overline{\chi}(2) = w^2 = e^{-2\pi i/3}$, then

$$z_p = 2p^{3/2}\cos(3\theta_p - 2\pi/3) = 2p^{3/2}\left(-\frac{2A - B}{4\sqrt{p}} + \frac{3B}{4\sqrt{p}}\right) = -p(A - 2B). \quad (6.5)$$

By assembling (6.2), (6.3), (6.5) and the value of $\chi(2)+\overline{\chi}(2)$, which is an element of the set $\{2,-1\}$, we deduce that

$$Q_p = \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(pc_p + 4p + 8) - \frac{p+2}{3},$$

where $c_p \in \mathbb{Z}$ is a computable constant which depends on p.

Lastly, suppose that p is of the form u^2+27v^2 for some integers u and v with $u \equiv 2 \pmod 3$. Thus $4p = (2u)^2 + 27(2v)^2$. Then by (Theorem 4.15, Cox, 1986), we know that 2 is a cubic residue, in other words $\chi(2) = 1 = \overline{\chi}(2)$. Then, by (6.2) we conclude that

$$\begin{aligned} Q_p &= \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(\tau_p^3 + \overline{\tau_p}^3 + 12p + 8) - \frac{p+2}{3} \\ &= \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(2p^{3/2}\cos(3\theta_p) + 12p + 8) - \frac{p+2}{3} \\ &= \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(2p^{3/2}\frac{2u}{2\sqrt{p}} + 12p + 8) - \frac{p+2}{3} \\ &= \frac{(p+2)^3}{27p} + \frac{p-1}{27p}(2up + 12p + 8) - \frac{p+2}{3}. \end{aligned}$$

□

Example 6.2 Let $p = 31$. Then $p = u^2+27v^2$ where $u = 2$ and $v = 1$. Applying our theorem, we deduce that there are 50 many non-trivial 3-term arithmetic progressions in C_{31}. Let $p = 43$. Then $p = u^2+27v^2$ where $u = -4$ and $v = 1$. Applying our theorem, we deduce that there are 70 many non-trivial 3-term arithmetic progressions in C_{43}.

REFERENCES

Aladov, N. S. (1896). Sur la distribution des résidus quadratiques et non-quadratiques d'un nombre premier p dans la suite $1, 2, ..., p-1$. *Mat. Sb.*, 18, 61–75.

Apostol, T. M. (1976). An Introduction to Analytic Number Theory. *Springer-Verlag*, New York.

Barnet-Lamb, T., Geraghty, D., Harris, M., and Taylor, R. (2011). A Family of Calabi–Yau Varieties and Potential Automorphy II. *Publ. Res. Inst. Math. Sci.*, 47, 29–98.

Bergelson, V. and Leibman, A. (1996). Polynomial extensions of van der Waerden's and Szemerédi's theorems. *J. Amer. Math. Soc.*, 9, 725–753.

Berndt, B. and Evans, R. (1981). The determination of Gauss sums. *Bull. Amer. Math. Soc.*, 5, 107–129.

Berndt, B., Evans, R., and Williams, K. (1998). Gauss and Jacobi Sums. *Wiley and Sons*.

Bourgain, J. (1990). On arithmetic progressions in sums of sets of integers, in A Tribute to Paul Erdös, CUP.

Clozel, L., Harris, M., and Taylor, R. (2008). Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations. *Publ. Math. IHES*, 108, 1–182.

Conrad, K. (2018). Quadratic residue patterns modulo a prime. https://kconrad.math.uconn.edu/blurbs/ugradnumthy/QuadraticResiduePatterns.pdf.

Cox, D. A., (1989). Primes of the form $x^2 + ny^2$. Pure and Applied Mathematics, A Wiley Series of Texts.

Darmon, H., and Merel, L. (1997). Winding quotients and some variants of Fermat's last theorem. *J. Reine Angew. Math.*, 490, 81–100.

Davenport, H. (1980). Multiplicative Number Theory. *Springer*, New York.

Davenport, H. (1931). On the Distribution of Quadratic Residues (mod p). *J. London Math. Society*, 6, 49–54.

Davenport, H. (1933). On the Distribution of Quadratic Residues (mod p) (Second paper). *J. London Math. Society*, 8, 46–52.

Deligne, P. (1974). La Conjecture de Weil I. *Inst. Hautes Études Sci. Publ. Math.*, 43, 273–308.

Deligne, P. (1977a). Théorèmes de finitude en cohomologie l-adique, dans Cohomologie Etale, in: Séminaire de Géométrie Algébrique du Bois-Marie, SGA 4 1/2, in: Lecture Notes in Math. 569, 233–251.

Deligne, P. (1977b). Application de la formule des traces aux sommes trigonométriques, in Cohomologie Etale (SGA 4 12), *Springer Lecture Notes in Math.*, 569, 168–232.

Deligne, P. (1980). La Conjecture de Weil II. *Inst. Hautes Études Sci. Publ. Math.*, 52, 137–252.

Erdős, P., and Turán, P. (1936). On some sequences of integers. *J. Lond. Math. Soc.*, 11, 261–264.

Eyidoğan, S., Göral, H., and Kutlu, M.K. (2023). Arithmetic progressions in certain subsets of finite fields. *Finite Fields Appl.*, 91, Article ID 102264, 58 p.

Furstenberg, H. (1977). Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. d'Analyse Math.*, 31, 204–256.

Furstenberg, H., and Katznelson, Y. (1991). A density version of the Hales-Jewett theorem. *J. d'Analyse Math.*, 57, 64–119.

Gowers, W. T., (1998). A new proof of Szemerédi's Theorem for arithmetic progressions of length four. *Geom. Funct. Anal.*, 8, 529–551.

Gowers, W. T. (2001). A new proof of Szemerédi's Theorem. *Geom. Funct. Anal.*, 11, 465–588.

Harris, M., Shepherd-Barron, N., and Taylor, R. (2010). A Family of Calabi–Yau Varieties and Potential Automorphy. *Ann. Math.*, 171, 779–813.

Hasse, H. (1936). Zur Theorie der abstrakten elliptischen Funktionenkörper I. II. III.. *J. reine angew. Math.*, 175, 55–62, 69–88, 193–208.

Heath-Brown, D. R., and Patterson, S. J. (1979). The distribution of Kummer sums at prime arguments. *J. Reine Angew. Math.*, 310, 111–130.

Katz, N. M. (1988). Gauss Sums, Kloosterman Sums, and Monodromy Groups, (AM-116). *Princeton University Press*.

Katz, N. M. (2002). Estimates for nonsingular multiplicative character sums. *Int. Math. Res. Not.*, 7, 333–349.

Lercier, R., and Morain, F. (1995). Counting the number of points on elliptic curves over finite fields: strategies and performances. In: Guillou, L.C., Quisquater, JJ. (eds) Advances in Cryptology — EUROCRYPT '95. EUROCRYPT 1995. Lecture Notes in Computer Science, *Springer*, Berlin, Heidelberg.

Murty, M. R., and Murty, V. K. (2009). The Sato–Tate conjecture and generalizations in Current Trends in Science: Platinum Jubilee Special. *Indian Academy of Sciences*, 639–646.

https://mast.queensu.ca/ murty/Sato-Tate-CurrentTrends.pdf

Pandey, R. K. (2018). On certain sums with quadratic expressions involving the legendre symbol. *Journal of Integer Sequences* , 21, 1–10.

Rojas-León, A. (2005). Estimates for singular multiplicative character sums. *Int. Math. Res. Not.*, 20, 1221–1234.

Rojas-León, A. (2022). On a generalization of Jacobi sums. *Finite Fields and Their Applications*, 77, 1–15.

Roth, K. (1953). On certain sets of integers. *J. London Math Soc.*, 28, 104–109.

SageMath, the Sage Mathematics Software System (Version 8.3), (2018). The Sage Developers, http://www.sagemath.org.

Schoof, R. (1985). Elliptic curves over finite fields and the computation of square roots mod *p*. *Math. Comp.*, 44 No. 170 483–494.

R. Schoof, R. (1995). Counting points on elliptic curves over finite fields. *Journal de théorie des nombres de Bordeaux*, 7, 219–254.

Silverman, J. H. (1999). Advanced topics in the arithmetic of elliptic curves. *Springer-Verlag*, Graduate Texts in Mathematics.

Stein, E. M., and Shakarchi, R. (2003). Fourier Analysis: An Introduction. *Princeton University Press*.

Szemerédi, E. (1969). On sets of integers containing no 4 elements in arithmetic progression. *Acta Math. Acad. Sci. Hungar.*, 20, 89–104.

édi, E. (1975). On sets of integers containing no k elements in arithmetic progression. *Acta Arith.*, 27, 199–245.

Tate, J. (1966). Endomorphisms of abelian varieties over finite fields. *Invent. Math.*, 2, 134–144.

Taylor, R. (2008). Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations II. *Publications Mathématiques de l'IHES*, 108, 183–239.

van der Waerden, B. L. (1927). Beweis einer Baudetschen Vermutung. *Nieuw Arch. Wisk.*, 15, 212–216.

Washington, L. T. (2008). Elliptic Curves: Number Theory and Cryptography. *Chapman and Hall/CRC.*

Weil, A. (1948). On Some Exponential Sums. *Proceedings of the National Academy of Sciences of the United States of America*, 34, 204–207.

CURRICULUM VITAE

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