# LOOP ZERO FORCING ON GRAPHS

**Dissertation**

**Mohamoud Ahmed HUSSEIN**

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**Eskişehir**
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## FINAL APPROVAL FOR THESIS

This thesis titled LOOP ZERO FORCING ON GRAPHS has been prepared and submitted by Mohamoud Ahmed HUSSEIN in partial fulfillment of the requirements in "Eskişehir Technical University Directive on Graduate Education and Examination" for the Degree of PhD in Mathematics Department has been examined and approved on 18/08/2023.

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**ABSTRACT**

LOOP ZERO FORCING ON GRAPHS

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The loop zero forcing number of a graph is based on the loop color change rule: If a white vertex $v$ is the only white neighbor of a black vertex $u$, then change the color of $v$ to black. If the neighbors of a white vertex $v$ are all black, then change the color of $v$ to black. A minimum loop zero forcing set is a set of black vertices that changes the entire graph to black and has a minimum cardinality. The loop zero forcing number of a graph is the cardinality of a minimum loop zero forcing set of the graph. The loop propagation time is the least number of rounds; it takes for a minimum loop zero forcing set to change the whole graph to black. A failed loop zero forcing number is the largest set that fails to change the entire graph to black. The loop throttling number minimizes the sum of a loop zero forcing set and its loop propagation time.

In this work, we establish various properties of the loop zero forcing number, loop propagation time, loop throttling number, and failed loop zero forcing number of graphs. Graphs giving the extreme values of these concepts are also characterized.

**Keywords:** Loop zero forcing number, Loop propagation time, Loop throttling number, Failed loop zero forcing number

# ÖZET

## ÇİZGELERDE DÖNGÜ SIFIR ZORLAMA

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Bir çizgenin döngü sıfır zorlama sayısı, döngü renk değiştirme kuralına dayanır. Bu kuralda eğer beyaz renkli $v$ köşe noktası, $u$ siyah köşe noktasının tek beyaz komşusu ise, $v$ köşe noktasının rengi siyah olarak değiştirilir. Ayrıca, beyaz bir $v$ köşe noktasının tüm komşuları siyah ise, $v$ köşe noktasının rengi siyah olarak değiştirilir. Minimum döngü sıfır zorlama kümesi, çizgenin tüm köşe noktalarını siyah renge dönüştüren minimum sayıda elemana sahip siyah renkteki köşe noktaların kümesidir. Bir çizgenin döngü sıfır zorlama sayısı, minimum döngü sıfır zorlama kümesinin eleman sayısıdır. Döngü yayılım süresi, çizgenin tüm köşe noktalarını siyaha dönüştürmek için minimum döngü sıfır zorlama kümelerine karşılık gelen tur sayılarının minimumudur. Başarısız döngü sıfır zorlama sayısı, çizgenin tüm köşe noktalarını siyah renge dönüştüremeyen en geniş köşe noktalar kümesidir. Döngü kısma sayısı ise bir döngü sıfır zorlama kümesinin eleman sayısı ile döngü yayılma süresinin toplamını minimize eder.

Bu çalışmada, çizgenin döngü sıfır zorlama sayısı, döngü yayılım süresi, döngü kısma sayısı ve başarısız döngü sıfır zorlama sayısının çeşitli özellikleri incelenmiştir. Ayrıca, bu kavramların ekstrem değerlerini veren çizgeler karakterize edilmiştir.

**Anahtar Sözcükler :** Döngü sıfır zorlama sayısı, Döngü yayılım süresi,
Döngü kısma sayısı, Başarısız döngü sıfır zorlama sayısı

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Mohamoud Ahmed HUSSEIN

18/08/2023

# STATEMENT OF COMPLIANCE WITH ETHICAL PRINCIPLES AND RULES

I hereby truthfully declare that this thesis is an original work prepared by me; that I have behaved in accordance with the scientific ethical principles and rules throughout the stages of preparation, data collection, analysis and presentation of my work; that I have cited the sources of all the data and information that could be obtained within the scope of this study, and included these sources in the references section; and that this study has been scanned for plagiarism with "scientific plagiarism detection program" used by Eskişehir Technical University, and that "it does not have any plagiarism" whatsoever. I also declare that, if a case contrary to my declaration is detected in my work at any time, I hereby express my consent to all the ethical and legal consequences that are involved.

Mohamoud Ahmed HUSSEIN

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## GLOSSARY OF SYMBOLS AND ABBREVIATIONS

| | | |
|---|---|---|
| $V$ | : | Vertex set |
| $E$ | : | Edge set |
| $\delta(G)$ | : | Minimum degree of a graph $G$ |
| $\Delta(G)$ | : | Maximum degree of a graph $G$ |
| $d(u,v)$ | : | Distance between vertices $u$ and $v$ |
| $N(u)$ | : | (Open) neighborhood of a vertex $u$ |
| $N[u]$ | : | Closed neighborhood of a vertex |
| $Z(G)$ | : | Standard zero forcing number of a graph $G$ |
| $Z_\ell(G)$ | : | Loop zero forcing number of a graph $G$ |
| $Z_+(G)$ | : | PSD zero forcing number of a graph $G$ |
| $Z_-(G)$ | : | Skew zero forcing number of a graph $G$ |
| $pt(G)$ | : | Standard propagation time number of a graph $G$ |
| $pt_\ell(G)$ | : | Loop propagation time number of a graph $G$ |
| $pt_+(G)$ | : | PSD propagation time number of a graph $G$ |
| $pt_-(G)$ | : | Skew propagation time number of a graph $G$ |
| $th(G)$ | : | Standard throttling number of a graph $G$ |
| $th_\ell(G)$ | : | Loop throttling number of a graph $G$ |
| $th_+(G)$ | : | PSD throttling number of a graph $G$ |
| $th_-(G)$ | : | Skew throttling number of a graph $G$ |
| $F(G)$ | : | Standard failed zero number of a graph $G$ |
| $F_\ell(G)$ | : | Loop failed zero number of a graph $G$ |
| $F_-(G)$ | : | Skew failed zero number of a graph $G$ |

# 1. INTRODUCTION

## 1.1. Graph Theory Basics

A graph $G$ is considered simple if it adheres to the rule of having neither loops nor multiple edges. A loop is defined as an edge connecting a vertex to itself. The interaction between two vertices $u$ and $v$ in the vertex set $V(G)$, where $\{u, v\}$ is an element of the edge set $E(G)$, is termed an edge and can be symbolized as $uv$. In a simple graph, $u$ and $v$ are distinct if the edge $uv$ is present. On the other hand, a graph $G = (V, E)$ that accommodates loops is termed a loop graph. It's important to note that while a loop graph can possess loops, it's not obligatory for it to include them. The count of elements in the vertex set (and respectively, the edge set) of graph $G$ is referred to as the order (and respectively, the size) of $G$.

The (open) neighborhood $N(v)$ of a vertex $v$ in $V(G)$ encompasses vertices $u$ such that $uv$ is an edge in $E(G)$. The (closed) neighborhood $N[v]$ includes $N(v)$ as well as the vertex $v$. Two distinct vertices $u$ and $v$ are classified as adjacent if there exists an edge $uv$ linking them. The degree of a vertex $v$, denoted as $\deg(v)$, is the count of vertices adjacent to it. The maximum degree in $G$, marked as $\Delta(G)$, signifies the degree of the vertex with the greatest degree. Conversely, the minimum degree in $G$, indicated as $\delta(G)$, represents the degree of the vertex with the smallest degree. A vertex with a degree of one is termed a pendant, and a vertex connected to a pendant is known as a support vertex. If a support vertex has only one non-pendant neighbor, it is referred to as an end-support vertex.

Throughout the rest of this thesis, graphs are assumed to be simple, unless explicitly stated otherwise. If we have a graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$, then $G$ represents a path on $n$ vertices, denoted as $P_n$. On the other hand, if $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, then $G$ represents a cycle on $n$ vertices, denoted as $C_n$.

A graph $G$ is considered connected if for any two vertices $x$ and $y$ belonging to $V$, there exists a path that connects $x$ and $y$. A tree is a special type of connected graph that is acyclic, meaning it does not contain any cycles.

The subgraph obtained from graph $G$ by removing all vertices in set $S$ and all edges incident with these vertices is represented as $G - S$. In the special case where $S$ contains

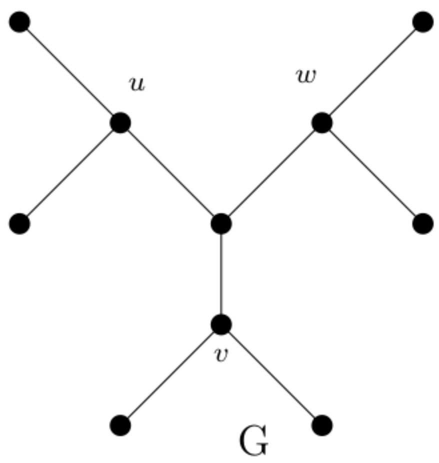



**Figure 1.1.** *$G$ is a caterpillar with $u$, $v$ and $w$ support vertices*

only one vertex $v$, we denote the subgraph as $G-\{v\}$ or simply $G-v$. A star is a specific type of graph, denoted as $K_{1,k}$, where $k \geq 1$ is a tree structure with the possibility of having at most one non-pendant vertex. A double star is a tree with exactly two adjacent end-support vertices. Caterpillar is a tree such that the resulting graph is a path after removing all pendants. A spider is a tree with an identified vertex, called the center, of degree greater than two such that the graph obtained by removing the center is a disjoint union of paths called legs. The length of a leg is the number of vertices in that leg. A balanced spider has legs of the same length. We use $T_{p,q}$ to denote the balanced spider with $p$ legs, each of length $q$. Two distinct vertices $u$ and $v$ in a graph $G$ are called twins if $N[u] = N[v]$, and independent twins if $N(u) = N(v)$. A set $W$ of three vertices in a graph $G$ is called a triplet if every vertex $x \in N(W) \backslash W$ has at least two neighbors in $W$. A triplet is dependent if $G[W]$ is connected, and independent if $G[W]$ is disconnected. The distance between two vertices $u$ and $v$, denoted by $d(u, v)$, is the length of the shortest path connecting them. The distance from a set $U$ to a vertex $v \in V(G)$ is $d(U, v) = \min_{u \in U} d(u, v)$. The eccentricity of a set $U$ of vertices in a graph $G$ is defined by $ecc(U) = \max_{v \in V(G)} d(U, v)$.

## 1.2. Literature Review

Standard zero forcing is based on the standard color change rule: for a simple graph $G$ with some initially black colored vertices, a white vertex $u$ is colored black if $u$ is the only white neighbor of a black vertex $v$. We call such a black vertex $v$ a forcing vertex or in other words, we say $v$ forces $u$, and write $v \to u$. A (standard) zero forcing set is an initial set $S \subseteq V(G)$ of black vertices such that the final coloring of $S$ is the vertex set of $G$. A (standard) zero forcing number of a graph $G$, denoted $Z(G)$, is the cardinality of a minimum (standard) zero forcing set of $G$.

For the simple graph $G$ in Figure 1.2, an initial set $S = \{a, b, e\}$ is colored black. Since vertex $c$ is the only white neighbor of $b$, $b$ forces $c$. Similarly, $a$ forces $d$. Figure 1.2 shows the complete process.

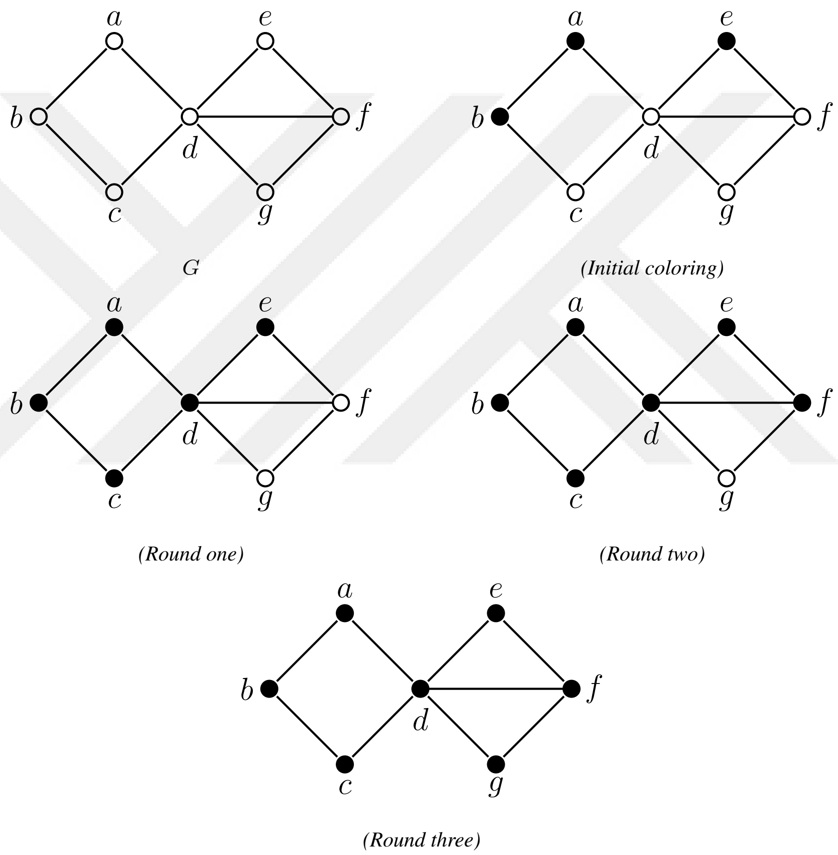



**Figure 1.2.** *An example of a standard zero forcing process*

The standard zero forcing number is as an upper bound for the maximum nullity. Let $S_n(\mathcal{R})$ denote the set of real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathcal{R})$, the graph of $A$, denoted $\mathcal{G}$, is the graph with vertices $\{1, 2, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\}$. The maximum nullity of $G$ is

$$M(G) = \max\{null(A) : A \in S_n(\mathcal{R}) \text{and } \mathcal{G} = G\},$$

and minimum rank of $G$ is

$$mr(G) = \min\{null(A) : A \in S_n(\mathcal{R}) \text{ and } \mathcal{G} = G\}.$$

**Theorem 1.1** ([3])**.** *For any graph G, $M(G) \leq Z(G)$.*

Various properties of the zero forcing number of graphs were studied in the literature. Before we state some of these properties, we give some definitions.

The path cover number of a graph $G$, denoted $P(G)$, is the minimum number of vertex disjoint induced paths that cover all the vertices of $G$. A planar graph $G$ is a graph on two parallel paths if $P(G) = 2$. A unicyclic graph is a connected graph containing exactly one cycle. A cactus is a graph where each edge is in at most one cycle.

**Theorem 1.2** ([3])**.** *For any graph G,*

$$\delta(G) \leq Z(G).$$

**Proposition 1.3** ([6])**.** *For any graph G, G has at least two minimum zero forcing sets.*

**Theorem 1.4** ([21])**.** *For any graph G,*

$$P(G) \leq Z(G).$$

The known family of graphs that satisfy $Z(G) = P(G)$ are trees [21], unicyclic graphs [13] and cactus graphs [31]. However, the difference between the two parameters could be arbitrarily large, for example

$$P(K_n) = \left\lceil \frac{n}{2} \right\rceil < n - 1 = Z(K_n).$$

The next theorem characterizes graphs with extreme standard zero forcing number.

**Theorem 1.5** ([3, 17, 32])**.** *Let G be a graph. Then*

1. $Z(G) = 1$ *if and only if G is a path graph.*

2. $Z(G) = 2$ *if and only if G is a graph on two parallel paths.*

3. $Z(G) = |G| - 1$ *if and only if G is a complete graph.*

4. $Z(G) = |G|$ *if and only if G is complement of a complete graph.*

The behavior of the zero forcing number under vertex and edge removal were also determined.

**Theorem 1.6** ([13, 29]). *Let $G$ be a graph, $v \in V(G)$ and $e \in E(G)$. Then*

$$|Z(G-v) - Z(G)| = 1 \textit{ and } |Z(G-e) - Z(G)| = 1.$$

*All bounds are tight.*

Positive semidefinite zero (PSD) forcing uses the positive semidefinite color change rule: let $S$ be the set of black vertices and let $W_1, W_2, \ldots, W_k$ be the set of vertices of the components $G - S$. If $v \in W_i$ is the only white neighbor of $u \in S$ in $G[W_i \cup S]$, then change the color of $v$ to black. The positive semidefinite zero forcing, denoted $Z_+(G)$, is the cardinality of a minimum PSD zero forcing set of $G$.

The positive semidefinite zero forcing number is an upper bound for positive semidefinite maximum nullity. Let $S_n(\mathcal{R})$ denote the set of real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathcal{R})$, the graph of $A$, denoted $\mathcal{G}$, is the graph with vertices $\{1, 2, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\}$. The maximum positive semidefinite nullity of $G$ is

$$M_+(G) = \max\{null(A) : A \in S_n(\mathcal{R}) \text{ is positive semidefinite and } \mathcal{G} = G\},$$

and minimum positive semidefinite rank of $G$ is

$$mr_+(G) = \min\{null(A) : A \in S_n(\mathcal{R}) \text{ is positive semidefinite and } \mathcal{G} = G\}.$$

**Theorem 1.7** ([6]). *For any graph $G$, $M_+(G) \leq Z_+(G)$.*

Some properties of the positive semidefinite zero forcing number of graphs were also studied in literature. For a non-trivial connected graph $G$, $\delta(G) \leq Z_+(G)$ [23]. Every graph $G$ have at least two minimum PSD zero forcing sets [14, 28]. For any graph $G$, $T(G) \leq Z(G)$ [5] and $T(G) = Z(G)$ if $G$ is a partial 2-tree [5, 15], where $T(G)$ is the minimum number of disjoint trees that cover the vertices of $G$ .

Graphs with extreme positive semidefinite zero forcing number were also characterized in [14]. $Z_+(G) = 1$ if and only if $G$ is a tree. $Z_+(G) = 2$ if and only if either $G$ is disconnected of two tree components or $G$ is connected such that exactly one block of $G$ has a cycle and $G$ does not a $K_4$ or $T_3$ minor. $Z_+(G) = |G| - 1$ if and only if $G = K_r \cup (|G| - r)$ with $r \geq 2$. $Z_+(G) = |G|$ if and only if $G$ is complement of complete graph.

The behavior of the positive semidefinite zero forcing number under vertex and edge removal were also determined. Let $G$ be a graph, $v \in V(G)$ and $e \in E(G)$, then $|Z_+(G) - Z_+(G-v)| = 1$ and $|Z_+(G-e) - Z_+(G)| = 1$, and all bounds are tight [14].

Skew zero forcing uses the skew zero forcing color change rule: if exactly of one neighbor $v$ of $u$ is white, then change the color of $v$ to black. The skew zero forcing number of a graph $G$, denoted $Z_-(G)$, is the cardinality of a minimum skew zero forcing set of $G$. The set of skew-symmetric matrices described by a graph $G$ is

$$\mathcal{S}^-(G) = \{A : A^T = -A, \mathcal{G}(A) = G\}.$$

The minimum skew rank of a graph $G$ is defined as

$$mr^-(G) = \min\{rank(A) : A \in \mathcal{S}^-(G)\},$$

and the maximum skew nullity of $G$ is defined as

$$M^-(G) = \max\{null(A) : A \in \mathcal{S}^-(G)\}.$$

**Theorem 1.8** ([20, 25]). *For any graph G, $M^-(G) \leq Z_-(G)$.*

Graphs having the highest possible skew forcing numbers and lowest possible skew forcing number have been determined (See [20, 24, 25, 27]).

The standard propagation time for standard zero forcing was defined in [22] and [11] as follows.

**Definition 1.9.** Let $G$ be a graph, and $S$ a standard zero forcing set for $G$. Define $S^0 = S$. For $t \geq 0$, let $S^{t+1}$ is the set of vertices $v$ for which there exists a black vertex $u$ in $G$ such that $v$ is the only white neighbor of $u$ not in $\cup_{r=0}^{t} S^r$.

**Definition 1.10.** The standard propagation time of a standard zero forcing set $S$ in a graph $G$, denoted $pt(G, S)$, is the minimum $t_0$ such that $V(G) = \cup_{t=0}^{t_0} S^t$.

**Definition 1.11.** The minimum standard propagation time of $G$ is

$$pt(G) = \min\{pt(G, S) : S \text{ is a minimum standard zero forcing set of } G\}.$$

**Definition 1.12.** The maximum standard propagation time of $G$ is

$$PT(G) = \max\{pt(G, S) : S \text{ is a minimum standard zero forcing set of } G\}.$$

A subset $S$ of vertices of a graph $G$ is an efficient standard zero forcing set for a graph $G$ if $pt(G, S) = pt(G)$ and $S$ is a minimum standard zero forcing set of $G$.

A lower bound on $pt(G)$ for a graph $G$ was stated in [22] as

$$pt(G) \geq \frac{|G| - Z(G)}{Z(G)}.$$

Since if $S$ is an efficient standard zero forcing set, at most $|S|$ can be forced in each round.

For any graph $G$, $0 \leq pt(G) \leq PT(G) \leq |G| - 1$. Graphs with extreme standard propagation time were characterized in [22].

**Proposition 1.13** ([3, 32])**.** *Let G be a graph. Then*

1. $pt(G) = 0$ *if and only if G is the complement of a complete graph.*
2. $pt(G) = |G| - 1$ *if and only if G is a path graph.*

**Proposition 1.14** ([22])**.** *Let G be a graph. Any two of the following conditions imply the third.*

1. $|G| = 2Z(G)$.
2. $pt(G) = 1$.
3. *G is a matched-sum graph.*

The difference $PT(G) - pt(G)$ of a graph $G$ is called the propagation time discrepancy, and denoted as $pd(G)$. The standard propagation time interval of a graph $G$ is defined as

$$[pt(G), PT(G)] = [pt(G), pt(G) + 1, \ldots, PT(G) - 1, PT(G)].$$

A graph $G$ is said to possess a full standard propagation time interval if each integer within the standard propagation time interval corresponds to the standard propagation time of at least one minimum standard zero forcing set of $G$. However, research has demonstrated in [22] that this property does not hold true for all graphs, as some graphs do not have a full standard propagation time interval.

The positive semidefinite propagation time is defined in a similar manner as the standard propagation time but adheres to the positive semidefinite zero forcing definition.

**Definition 1.15.** Let $G$ be a graph and $S$ a positive semidefinite zero forcing set. Let $S^0 = S$. For $t \geq 1$, let $S^t$ is the set of black vertices obtained by connecting $S^{t-1}$ to each of the connected components of $G[V \backslash S^{t-1}]$ and carrying out the positive zero forcing process.

Alternative definitions include the minimum positive semidefinite propagation time of a graph $G$, denoted as $pt_+(G)$. This indicates the smallest value of $pt_+(G, S)$ when considering all minimum positive semidefinite zero forcing sets $S$ of $G$. Likewise, the maximum positive semidefinite propagation time of $G$, denoted as $PT_+(G)$, is the largest value of $pt_+(G, S)$ among all minimum positive semidefinite zero forcing sets $S$ of $G$.

Moreover, for any given graph $G$, a minimum positive semidefinite zero forcing set $S$ is deemed efficient when the value of $pt_+(G, S)$ is equal to $pt_+(G)$. In other words, an efficient set $S$ attains the minimum positive semidefinite propagation time for the graph $G$.

The following are known results regarding graphs with extreme positive semidefinite propagation time.

**Proposition 1.16** ([33])**.** *Let $G$ be a graph.*

1. $pt_+(G) = 0$ *if and only if $G$ is the complement of a complete graph.*
2. *If $uv \in E(G)$, then $pt_+(G) = 1$ if and only if there exists a minimum positive semidefinite zero forcing set $S$ of $G$ such that $N(S) = V(G)$ and $S$ has full forcing.*
3. $pt_+(G) = |G| - 2$ *if and only if $G$ is one of $P_3, P_4, C_3$ or $P_2 \cup P_1$.*
4. $pt_+(G) = |G| - 1$ *if and only if $G = K_2$.*

The skew propagation time was introduced in [27]. The minimum and maximum skew propagation times can be defined similarly to minimum and maximum standard propagation times, but following the skew zero forcing. The next results summarize some results about extreme propagation time.

**Proposition 1.17** ([27])**.** *Let $G$ be a graph.*

1. $pt_-(G) = 0$ *if and only if $G$ is the complement of a complete graph.*
2. *If there exists a minimum skew forcing set $S$ such that $G[V \backslash S] = sK_2$, then $pt_-(G) = 1$.*

The idea of throttling was introduced to tackle the challenge of minimizing both the size of a zero forcing set and its propagation time. The concept of throttling emerged from a question Richard Brualdi posed to Michael Young during a presentation on zero forcing and propagation time at the 2011 International Linear Algebra Society Conference in Braunschweig, Germany. Subsequently, Butler and Young explored the notion of sum throttling for standard zero forcing in their work [13], inspired by this question. Since then, researchers have extensively investigated sum throttling, delving into various parameters, including standard zero forcing, positive semidefinite zero forcing, and their minor monotone floors, power domination, and Cops and Robbers (as discussed in [27, Chapter 10] for a comprehensive survey).

Now, we will introduce the concept of sum throttling for the three zero forcing procedures under consideration: standard zero forcing, positive semidefinite zero forcing, and skew zero forcing. The standard throttling number of a set $S$ in a graph $G$ is defined as $th(G, S) = |S| + pt(G, S)$. Similarly, the standard $k$-throttling number is denoted as $th(G, k) = k + pt(G, k)$, where $k$ is an integer parameter. Finally, the standard throttling number of $G$ of order $n$ is represented by

$$th(G) = \min_{S \subseteq V(G)} th(G, S) = \min_{Z(G) \leq k \leq n} th(G, k).$$

**Theorem 1.18** ([10])**.** *Let G be a graph. Then*

1. $th(G) = 1$ *if and only if* $|G| = 1$*.*

2. $th(G) = 2$ *if and only if* $|G| = 2$*.*

3. $th(G) = 3$ *if and only if* $|G| = 3$ *or G is one of* $2K_2$*,* $P_4$ *or* $C_4$*.*

4. $th(G) = |G|$ *if and only if G does not contain an induced* $P_4$*,* $C_4$*, or bowtie graph.*

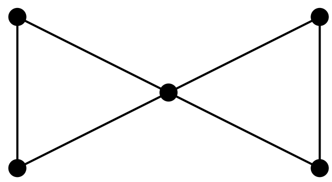


**Figure 1.3.** *Bowtie*

The positive semidefinite throttling number of $S$ in $G$ is $th_+(G, S) = |S| + pt_+(G, S)$, and the positive semidefinite $k$-throttling number is $th_+(G, k) = k + pt_+(G, k)$. The positive semidefinite throttling number of $G$ of order $n$ is

$$th_+(G) = \min_{S \subseteq V(G)} th_+(G,S) = \min_{Z_+(G) \leq k \leq n} th_+(G,k).$$

**Theorem 1.19** ([9]). *Let $G$ be a graph. Then*

1. $th_+(G) = 1$ *if and only if $G$ is an isolated vertex.*
2. $th_+(G) = 2$ *if and only if $G$ is a star or two isolated vertices.*
3. $th_+(G) = 3$ *if and only if $G$ is one of the following*
   (a) *$G$ is disconnected, and*
      i. *$G$ is three isolated vertices*
      ii. *$G$ has two components, each component is a star or an isolated vertex, and at least one component is a star*
   (b) *$G$ is a tree with diameter three or four, or*
   (c) *$G$ is connected, and $G$ has a cycle or is a tree with $diam(G) = 5$, and there exists $u, v \in V(G)$ such that*
      i. $N(u) \cup N(v) = V(G)$
      ii. $\deg(w) \leq 2$ *for all $w \notin \{u, v\}$, and*
      iii. *if $w_1, w_2 \in N(u)$ or $w_1, w_2 \in N(v)$, then $w_1$ is not adjacent to $w_2$.*
4. $th_+(G) = |G| - 1$ *if and only if $\alpha(G) = 2$ and $G$ does not have an induced $C_5$, house or double diamond subgraph (See Figure 1.4).*
5. $th_+(G) = |G|$ *if and only if $G = K_{|G|}$.*

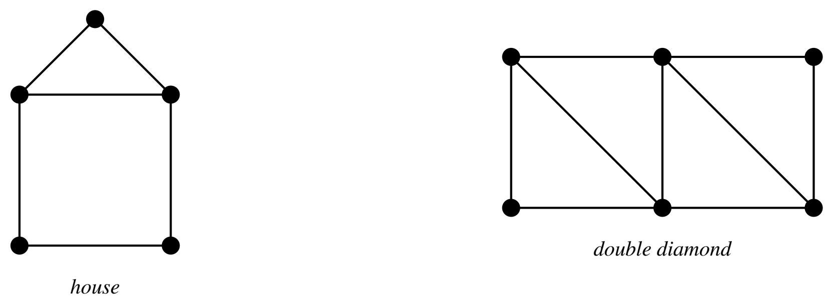



**Figure 1.4.** *House and double diamond graphs*

The skew throttling number of $S$ in $G$ is $th_-(G,S) = |S| + pt_-(G,S)$, and the skew $k$-throttling number is $th_-(G,k) = k + pt_-(G,k)$. The skew throttling number of $G$ of order $n$ is

$$th_{-}(G) = \min_{S\subseteq V(G)} th_{-}(G,S) = \min_{Z_{-}(G)\leq k\leq n} th_{-}(G,k).$$

For $s, t \geq 0$, define $H(s,t)$ by $V(H(s,t)) = \{b\} \cup \{x_i, y_i : i = 1, 2, \ldots, s\} \cup \{z_i, w_i : i = 1, 2, \ldots, t\}$ and $E(H(s,t)) = \{bx_i, x_iy_i : i = 1, 2, \ldots, s\} \cup \{bz_i, bw_i : i = 1, 2, \ldots, t\}$.

**Theorem 1.20** ([12]). *Let G be a graph. Then*

1. $th_{-}(G) = 1$ *if and only if* $G = rK_2$ *for* $r \geq 1$ *or G is an isolated vertex.*
2. $th_{-}(G) = 2$ *if and only if G is one of* $2K_1$, $H(s,t) \cup rK_2$ *with* $r + s + t \geq 1$ *or* $(\tilde{G} \circ K_1) \cup rK_2$ *where each component of* $\tilde{G}$ *has an edge.*
3. $th_{-}(G) = 3$ *if and only if G is cograph, does not have an induced* $2K_2$*, and has at least one edge.*
4. $th_{-}(G) = |G|$ *if and only if* $G = nK_1$.

A subset $S \subseteq V(G)$ of a graph $G$ is considered a failed zero forcing set if the final coloring of $S$ does not force all the vertices in the graph (i.e., $pt(G,S) = \infty$). The standard failed zero forcing number of a graph $G$, denoted as $F(G)$, refers to the maximum number of vertices in a failed zero forcing set within $G$. The concept of standard failed zero forcing number was introduced and investigated in [18]. Since the vertex set of a graph $G$ forms a zero forcing set, it follows that $F(G)$ is at most $|G| - 1$. Additionally, if $S$ is a minimum zero forcing set of $G$ and $v \in S$, then the set $S - \{v\}$ is a failed zero forcing set, implying that $F(G)$ is greater than or equal to $Z(G) - 1$. Notably, the failed zero forcing number is strictly lower than the zero forcing number only in the case of complete graphs and the complement of complete graphs. Furthermore, graphs with exceptionally high failed zero forcing numbers were also characterized in relevant studies.

**Proposition 1.21** ([18, 19]). *Let G be a graph.*

1. $F(G) = 0$ *if and only if G is an isolated vertex or* $K_2$.
2. $F(G) = 1$ *if and only if G is a pair of isolated vertices,* $K_3$, $P_3$ *or* $P_4$.
3. $F(G) = 2$ *if and only if G is one of the graphs shown in Figure 1.5.*
4. $F(G) = |G| - 2$ *if and only if G has dependent or independent twin.*

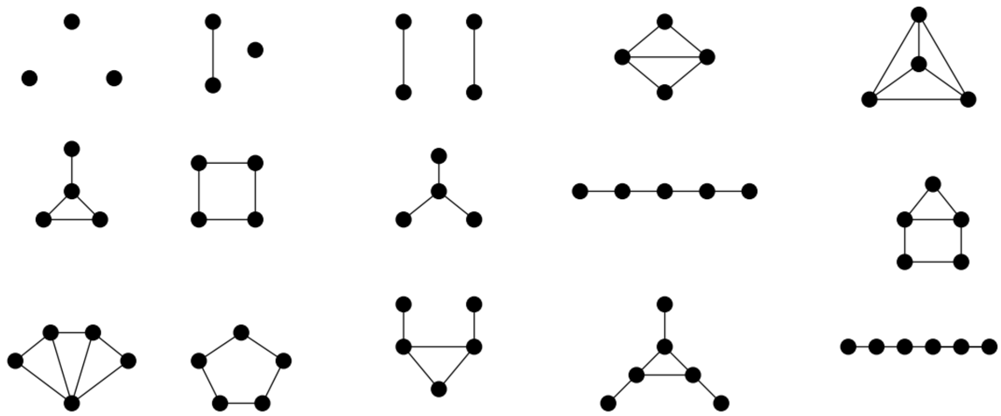


**Figure 1.5.** *The* $15$ *graphs with* $F(G) = 2$ *in Proposition 1.21*

5. $F(G) = |G| - 1$ *if and only if* $G = \overline{K_{|G|}}$.

The failed skew zero forcing number parameter denoted as $F_{-}(G)$, refers to the largest size of an initial set of vertices in a graph G which will not cause all vertices in the graph to become black-colored eventually under the skew color change rule. This parameter was first introduced in reference [2] and further explored in he failed skew zero forcing number was introduced in [2] and studied in [26] .

Graphs with extremely high and low failed skew zero forcing numbers were characterized. Before presenting these characterizations, we need to introduce some definitions. A graph $G$ is classified as a doubly extended bouquet-dipole if it comprises two vertices, $u$ and $v$, each belonging to a nonempty set of odd cycles. All other vertices on these cycles have a degree of two. Additionally, $u$ and $v$ are connected by a path of even order that alternates between single even order paths, where all internal vertices have a degree of two, and multiple even order paths, also with internal vertices having a degree of two. An $n$-blocking refers to a subgraph of $P_{2n+1}$, where the external vertices have degrees higher than two, and the internal vertices have a degree of two. Specifically, when $n = 1$, this subgraph is referred to as a 1-blocking.

**Theorem 1.22** ([2, 26])**.** *Let* $G$ *be a graph.*

1. $F_{-}(G) = 0$ *if and only if* $G$ *is one of the following graphs.*

   *(a) An odd cycle, or a nonempty set of odd cycles whose intersection is a single vertex.*

   *(b) An doubly extended bouquet-dipole.*

2. *$F_{-}(G) = 1$ if and only if G is $K_4$ or $P_3$ or G has all five of the following properties.*

   *(a) Exactly 1 disjoint 1-blocking, no more than 2 non-disjoint 1-blockings that share a vertex of degree 3, and no other $n$-blocking.*

   *(b) All degree 2 vertices are in a 1-blocking, an even path with external vertices with higher degree than 2, and even multiple path, or an odd cycle with exactly one vertex of degree higher than 2.*

   *(c) There are no pendant even cycles on G and no vertices of degree 1.*

   *(d) Outside of the 1-blocking, no vertex has more than one neighbor of degree 3.*

   *(e) All but one of the exterior vertices of the 1-blockings can have one neighbor that is not in an odd cycle or adjacent to another exterior of a 1-blocking through an even path.*

3. *$F_{-}(G) = |G| - 2$ if and only if G has a dependent twin and no isolated vertices.*

4. *$F_{-}(G) = |G| - 1$ if and only if G contains an isolated vertex.*

This thesis comprises four chapters. Within Chapter 2, we delve into the concepts of loop zero forcing and loop propagation time, subjecting them to analysis. In Section 2.1, we establish definitions and make observations pertaining to loop zero forcing. The subsequent Section 2.2 elucidates various properties of loop zero forcing sets. Moving on to Section 2.3, we delve into the examination of loop propagation time, wherein we define its characteristics and explore graphs characterized by high and low propagation times. Section 2.4 is dedicated to the exploration of the loop zero forcing number and propagation time as applied to specific families of graphs.

In the subsequent Chapter 3, we shift our focus to the study of failed loop zero forcing. Section 3.1 introduces definitions and outlines observations relevant to failed loop zero forcing. Within Section 3.2, we delve into the characterization of graphs with high and low failed loop zero forcing numbers. The exploration continues in Section 3.3, where we scrutinize the failed loop zero forcing number within specific families of graphs.

Chapter 4 marks our introduction and investigation of throttling loop zero forcing. The foundational definitions and observations for loop throttling are laid out in Section 4.1. Within Section 4.2, we apply loop throttling to trees and present our findings. Moving forward in Section 4.3, we characterize graphs exhibiting high and low failed loop throttling numbers. Lastly, in Section 4.4, we turn our attention to the study of loop throttling numbers in the context of particular families of graphs.

# 2. LOOP ZERO FORCING

## 2.1. Loop Zero Forcing Set

Zero forcing is a graph coloring process wherein each vertex is initially assigned either a black or white color, aiming to transform the entire vertex set to black through the repeated application of color change rules. These color change rules are specific conditions that allow a vertex to change the color of a neighboring white vertex to black. When a color change rule is applied to a vertex $u$, resulting in the color change of a white vertex $v$, we express this as $u \rightarrow v$ (possible $u = v$).

Given an initial set $S$ of black-colored vertices from the graph $G$, the final coloring of $S$ is the set of vertices where no further color changes are possible. If the final coloring of a zero forcing set $S$ corresponds to the vertex set of the graph $G$, then $S$ is referred to as a zero forcing set of $G$. Furthermore, $S$ is considered a minimum zero forcing set if it has the smallest number of vertices among all zero forcing sets.

The zero forcing number of a graph $G$ represents the cardinality of a minimum zero forcing set $S$ of $G$. The propagation time of a zero forcing set refers to the number of steps needed to color the entire graph using the zero forcing process. The propagation time of a graph $G$ is the minimum propagation time among all minimum zero forcing sets of $G$.

The color change rule employed in the zero forcing process varies across different variants. In the standard zero forcing, the standard color change rule is utilized, which involves changing the color of a white vertex $v$ to black only if $v$ is the sole white neighbor of a black vertex $u$. The standard zero forcing number, denoted as $Z(G)$, was introduced and analyzed in [3] and [7, 21], respectively. Additionally, the standard propagation time $pt(G)$ was investigated in [22]. Notably, the zero forcing number of any graph $G$ provides an upper limit on the maximum nullity of the family of symmetric matrices having the off-diagonal nonzero pattern described by the vertices of $G$ [3].

On the other hand, positive semidefinite zero forcing uses the positive semidefinite color change rule, which operates as follows: Given a set $S$ of black vertices and $W_1, W_2, \ldots, W_k$ as the vertex sets of components $G - S$, if $v \in W_i$ is the only white neighbor of $u \in S$ in $G[W_i \cup S]$, then the color of $v$ is changed to black. The positive semidefinite zero forcing number, denoted as $Z_+(G)$, was introduced and studied in [6] and [14], respectively. Moreover, the positive semidefinite propagation time, $pt_+(G)$, was analyzed in [33]. $Z_+(G)$ provides an upper bound on the maximum nullity of the family of positive semidefinite matrices corresponding to graph $G$ [6].

Let $\hat{G}$ be a loop graph. The underlying simple graph of $\hat{G}$ is the graph $G$ obtained from $\hat{G}$ by deleting all loops. The set of symmetric matrices described by a loop graph $\hat{G}$ is

$$\mathcal{S}(\hat{G}) = \{A = a_{ij} \in \mathcal{S}_n(\mathbb{R}) : a_{ij} \neq 0 \text{ if and only if } \{i, j\} \in E(\hat{G})\}.$$

The maximum nullity of $\hat{G}$ is

$$M(\hat{G}) = \max\{null(A) : A \in \mathcal{S}(\hat{G})\}.$$

The standard color change rule for a loop graph $\hat{G}$, denoted as $CCR\text{-}Z(\hat{G})$, is defined as follows: If there is exactly one neighbor $v$ of a vertex $u$ that is white, then change the color of $v$ to black. The zero forcing number of a loop graph, $Z(\hat{G})$, is determined by finding the minimum zero forcing set using the color change rule $CCR\text{-}Z(\hat{G})$.

The enhanced zero forcing number of a graph $G$, denoted as $\hat{Z}(G)$, is defined as the maximum value of $Z(\hat{G})$ taken over all loop graphs $\hat{G}$ such that the underlying simple graph of $\hat{G}$ is $G$. In other words, $Z(\hat{G})$ is the largest zero forcing number among all loop graphs whose underlying simple graph is $G$.

The next two theorems show that $M(G)$ of any graph $G$ can be at most $Z(G)$.

**Theorem 2.1** ([3])**.** *For any loop graph $\hat{G}$, $M(\hat{G}) \leq Z(\hat{G})$.*

**Theorem 2.2** ([4])**.** *For any graph $G$, $M(G) \leq \hat{Z}(G) \leq Z(G)$.*

Define the loop zero forcing number of a graph $G$, denoted by $Z_\ell(G)$, to be the zero forcing number of the loop graph $\hat{G}$ with loops at every positive degree vertex whose underlying simple graph is $G$. The color change rule for the loop zero forcing number of simple graphs can be stated as follows: if a white vertex $v$ is the only white of black vertex $u$, then $u$ forces $v$. If the neighbors of a white vertex $u$ are all black, then $u$ forces itself.

The following theorem gives the relationship among the three zero forcing numbers discussed above.

**Theorem 2.3** ([4])**.** *For any graph $G$, $Z_+(G) \leq Z_\ell(G) \leq Z(G)$.*

## 2.2. Properties of Loop Zero Forcing Set

Let $S$ be a loop zero forcing set of a graph $G$. A chronological list of forces of $S$ is a list of forces performed to achieve the final coloring of $S$ in the order they are executed. If a vertex $v$ can force itself and it is the only white of one of its neighbors $u$, then record the force $u \to v$. A forcing chain for a chronological list of forces is a sequence of vertices $(u_1, u_2, \ldots, u_k)$ such that $u_i \to u_{i+r}$ for some integer $r$ and $u_{i+s} \to u_{i+s}$ for $s = 1, 2, \ldots, r-1$ or $u_i$ does not force any vertex and $u_{i+p} \to u_{i+p}$ for $p = 1, 2, \ldots, q$ where $i + q = k$. If a vertex forces itself, then include it in only one forcing chain containing one of its neighbors. A forcing chain is maximal if it is not proper subsequence of any other forcing chain. A forcing chain can have only one vertex $u$ (i.e., $u \in S$ and $u$ does not perform any force). Every forcing chain of $S$ starts with a vertex in $S$, so the cardinality of $S$ and the number of forcing chains are equal.

**Proposition 2.4.** *Let $G$ be a connected graph. Then $G$ has at least two minimum loop zero forcing sets.*

*Proof.* Suppose $S'$ is a minimum loop zero forcing set of a graph $G$. If $u \in S'$ forces $v \in V(G)\backslash S'$ in round one, then let $S = (S'\backslash\{u\}) \cup \{v\}$. Then $S$ is a loop zero forcing set such that $|S| = |S'|$. If $v \in V(G)\backslash S'$ forces $v$ at round one, let $S = (S'\backslash\{u\}) \cup \{v\}$ where $u \in S'$ is a neighbor of $v$, then $S$ is a loop zero forcing set such that $|S'| = |S|$. □

**Proposition 2.5.** *Let $G$ be a graph of order $n \geq 3$. Then there exists a minimum loop zero forcing set containing no pendant vertices.*

*Proof.* Let $S'$ be a minimum loop zero forcing set of a graph $G$, and let $v$ be a support vertex and $u$ a pendant neighbor of $v$. Suppose $u \in S'$. Let $S = (S'\backslash\{u\}) \cup \{v\}$. Then $S$ is a loop zero forcing set such that $|S| = |S'|$. □

**Proposition 2.6.** *Let $G$ be a graph, $S$ be a minimum loop zero forcing set of $G$. For $v \in S$, we have $|N(v) \cap V(G)\backslash S| \geq 1$.*

*Proof.* Let $S$ be a minimum loop zero forcing set of a graph $G$ and $v \in S$. Suppose $N(v) \subseteq S$. Let $S' = S\backslash\{v\}$. Then $S'$ is a loop zero forcing set of $G$ such that $|S| > |S'|$ contradicting to the assumption that $S$ is minimum loop zero forcing set. Hence, $|N(v) \cap V(G)\backslash S| \geq 1$. □

**Proposition 2.7.** *Let $G$ be a graph. Then $\delta(G) \leq Z_\ell(G)$.*

*Proof.* Suppose $S$ is a minimum loop zero forcing set of graph $G$. For a black vertex $v$ to force a white vertex $u$, $v$ must have $\deg(v)-1$ black neighbors. So, $\delta(G) \leq \deg(v)-1 \leq |S|$. Similarly, for a white vertex $u$ to force itself, $u$ must have $\deg(u)$ neighbors. So, $\delta(G) \leq \deg(v) \leq |S|$. □

**Theorem 2.8.** *Let G be a graph. Then* $Z_\ell(G) = 1$ *if and only if G is a caterpillar.*

*Proof.* Consider a caterpillar graph $G$, where $v$ is an end-support vertex belonging to the vertex set $V(G)$. It is evident that $v$ acts as a loop zero forcing set, implying $Z_\ell(G) = 1$. Conversely, let $G$ be a graph such that $Z_\ell(G) = 1$, and let $\{v\}$ represent the loop zero forcing set. There exists at most one neighbor $w_1$ of $v$ that is not a pendant vertex. By forcing all pendant vertices adjacent to $v$, then $w_1$ can be forced next round. At this stage, $w_1$ is situated on the only black vertex that has white neighbors. Furthermore, there exists precisely one vertex $w_2$, distinct from $v$, that is not a pendant. Consequently, $w_1$ allows its white pendant vertices to force themselves, followed by $w_1$ forces $w_2$. This process continues until all the vertices of $G$ become black. Therefore, $G$ is a caterpillar graph. □

**Theorem 2.9** ([4])**.** *Let G be a graph. Then* $Z_+(G) = |G|$ *if and only if* $G = \bar{K}_{|G|}$.

The following theorem is an immediate result of Theorems 2.3 and 2.9.

**Theorem 2.10.** *Let G be a graph. Then* $Z_\ell(G) = |G|$ *if and only if* $G = \bar{K}_{|G|}$.

**Definition 2.11.** The caterpillar cover number $C(G)$ of a graph $G$ is the least positive integer $k$ such that there are $k$ vertex-disjoint induced caterpillars $T_1, T_2, \ldots, T_k$ in $G$ that cover all the vertices of $G$ (i.e., $V(G) = \cup_{i=1}^{k} V(T_i)$).

**Proposition 2.12.** *For any graph G,* $C(G) \leq Z_\ell(G)$.

*Proof.* Let $G$ be a graph, $S \subseteq V(G)$ be a minimum loop zero forcing set of $G$. Every vertex in $S$ contributes one forcing chain and the vertices in a forcing chain induce a caterpillar in $G$. The maximal forcing chains are disjoint and contain all the vertices of $G$. Thus $C(G) \leq Z_\ell(G)$. □

Let $\mathcal{T}$ be the class of trees obtained from one $T_{p,2}$ and $p$ $T_{2,2}$ trees by connecting each pendant of $T_{p,2}$ to a central vertex of one $T_{2,2}$. Then $C(\mathcal{T}) = p+1$ and $Z_\ell(\mathcal{T}) = 2p-1$. For trees such kind, there exists a tree $T \in \mathcal{T}$ such that $Z_\ell(T) - C(T) = k$ for any $k \in \mathbb{N}$. See Figure 2.6 for the tree of obtained from $T_{3,2}$ and 3 $T_{2,2}$.

Fix a connected graph $G$. A vertex with at least three non-pendant neighbors is called a general vertex. A pendant vertex $u$ is called a terminal vertex of a general vertex $v$ if

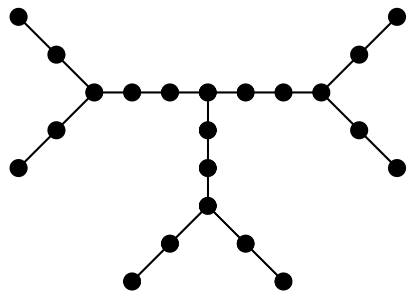


**Figure 2.6.** *$T$ is obtained from $T_{3,2}$ and 3 $T_{2,2}$*

$d(u, v) < d(u, w)$ for every other general vertex $w$, and $d(u, v) \geq 2$. A set $S \subseteq V(G)$ of terminal vertices of a general vertex $v$ such that there exists a vertex $x$ that lies on every path from a terminal vertex in $S$ to $v$ is called a batch. The batch degree of a general vertex $v$ in $G$, denoted by $bat(v)$, is the number of batches of $v$. A general vertex $v$ is an exterior general vertex ($egv$) if it has a positive batch degree. Let $bat(G)$ be the sum of batches of all general vertices of $G$, and let $eg(G)$ denote the number of exterior general vertices of $G$. A vertex is called interior vertex if it lies on a path between two general vertices. A set of at least two interior vertices is called a stick if they have the same general vertices. Let $s(G)$ be the number of sticks in a graph $G$.

**Observation 2.13.** *Let $T$ be a tree. Then $eg(T) = 0$ if and only if $T$ is a caterpillar.*

**Lemma 2.14.** *For a tree $T$ that is not a caterpillar, then $bat(T) \geq 3$.*

*Proof.* Let $T$ be a tree that is not a caterpillar. Then $T$ has at least one exterior general vertex. Assume $eg(T) = 1$. Let $v$ be the exterior general vertex of $T$. Then, by definition, $v$ have at least three non-pendant neighbors. The path from a terminal vertex of $v$ to $v$ passes through one of the non-adjacent neighbors of $v$. Therefore, $bat(T) \geq 3$. Assume $eg(T) = 2$. Let $u$ and $w$ be the exterior general vertices of $T$. Suppose $bat(u) = 1$. Then there exists a general vertex $z$ different from $w$ that lies on a path from a pendant vertex of $T$ to $u$. Either $bat(z) = 0$ or $bat(z) = 1$. But in both cases, $eg(T) \geq 3$, contradicts our assumption. Thus $bat(u) = 2$. Similarly, $bat(w) = 2$. Therefore, $bat(T) \geq 3$. Assume $eg(T) \geq 3$. Since each exterior general vertex has at least one terminal vertex, $bat(T) \geq 3$. □

**Proposition 2.15.** *Let $T$ be a tree that is not a caterpillar. Then $T$ has an exterior general vertex $v$ with $bat(v) \geq 2$.*

*Proof.* Let $T$ be a tree that is not a caterpillar. Suppose $eg(T) \leq 2$. By Lemma 2.14, $T$ has one exterior general vertex of batch degree more than one. Now suppose $eg(T) \geq 3$

and $bat(T) = eg(T)$ (i.e., every exterior general vertex has a batch degree of one). Let $u$ be an exterior general vertex such that every path between $u$ and every other exterior general vertex $v$, there exists an exterior general vertex $w$ containing the $uv$-path (possibly $v = w$). Then $u$ has two non-pendant neighbors. $\square$

**Lemma 2.16.** *Let $T$ be a tree and $S$ a minimum loop zero forcing set of $T$. An exterior general vertex $v$ is containing in $S$ if $bat(v) = 1$.*

*Proof.* Let $T$ be a tree and $S$ a minimum loop zero forcing set of $T$. Suppose $v$ is an exterior general vertex such that $v \in S$ and $bat(v) \geq 2$. Then at least one terminal pendant or terminal end-support vertex of $v$ contained in $S$. By Proposition 2.5, let $u \in S$ be a terminal end-support vertex of $v$. Since $u$ induces a caterpillar forcing chain, $u$ eventually forces $v$. Let $S' = S - \{v\}$. Then $S'$ is a loop zero forcing set of $T$ such that $|S'| < |S|$. $\square$

**Proposition 2.17.** *Let $T$ be a tree. Then there exists a minimum loop zero forcing set $S \subseteq V(T)$ that contains no exterior general vertices.*

*Proof.* Suppose $v$ is an exterior general vertex containing in a minimum loop zero forcing set $S'$ of a tree $T$. Then $bat(v) \geq 1$. Let $S = S' \backslash \{v\} \cup \{u\}$, where $u$ is a terminal end-support vertex of $v$. $S$ is a minimum loop zero forcing set of $T$. $\square$

**Lemma 2.18.** *Let $T$ be a tree. Then $eg(T) = 0$ if and only if $T$ is a caterpillar.*

*Proof.* Let $T$ be a tree. Suppose $eg(T) = 0$. Let $v \in T$ such that $|N(v)| \geq 3$. Since $v$ is not an exterior general vertex, $v$ has at most two non-pendant neighbors. Hence, $v$ is a support vertex. Thus, $T$ is a caterpillar.

Suppose $T$ is a caterpillar. Since every vertex of $T$ of degree three or more is a support vertex, $eg(T) = 0$. $\square$

**Proposition 2.19.** *Let be a tree $T$. Then $Z_\ell(T) \geq bat(T) - eg(T)$.*

*Proof.* It is obvious when $T$ is a caterpillar. Suppose $T$ is a tree that is not a caterpillar. Let $S$ be a minimum loop zero forcing set of $T$ containing neither pendants nor exterior general vertices. Let $k$ be the number of exterior general vertices of batch degree one. An exterior general vertex $v$ of $bat(v) \geq 2$ is containing in no minimum loop zero forcing set, so by Proposition 2.5, at least $bat(v) - 1$ terminal end-support vertices of $v$ contained in $S$. If $u$ and $v$ are pair of exterior general vertices of batch degree one, at least one terminal end-support vertex of $u$ or $v$ contained in $S$. So, at least $\lceil \frac{k}{2} \rceil$ terminal end-support vertices of exterior general vertices of batch degree one contained in $S$. Thus, $Z_\ell(T) \geq bat(T) - eg(T) + \lfloor \frac{k}{2} \rfloor \geq bat(T) - eg(T)$. $\square$

Next we discuss the behavior of the loop zero forcing number under vertex and edge removal.

**Theorem 2.20.** *Let* $G$ *be a graph,* $u \in V(G)$

$$Z_\ell(G) - 1 \leq Z_\ell(G - u) \leq Z_\ell(G) + \deg(u) - 1.$$

*Proof.* Suppose $u$ is a support vertex with at least $\deg(u) - 1$ pendant neighbors. If $u$ has $\deg(u) = |G| - 1$, then $G$ is a star, and $G - u$ is a graph of only isolated vertices. Hence, $Z_\ell(G - u) \leq Z_\ell(G) + \deg(u) - 1 = \deg(u)$. If $u$ has $\deg(u) - 1$ pendant neighbors, let $v$ be a neighbor of $u$ such that $\deg(u) \geq 2$. For any minimum loop zero forcing set of $G$, either $v$ forces $u$ or $u$ forces $v$. In either case, every pendant neighbor of $u$ contributes one additional vertex to any minimum loop zero forcing set of $G - u$. Thus, $Z_\ell(G - u) \leq Z_\ell(G) + \deg(u) - 1$.

If $S$ is a minimum loop zero forcing set of a graph $G - u$, then $S \cup \{u\}$ is a loop zero forcing set of $G$. Thus, $Z_\ell(G) \leq Z_\ell(G - u) + 1$. □

**Proposition 2.21.** *Let* $G$ *be a graph. Then the following statements hold:*

1. *If* $u$ *and* $v$ *are twin, then at least one of* $u$ *or* $v$ *must be in every loop zero forcing set.*

2. *If* $u$*,* $v$ *and* $w$ *are triplet, then at least one of* $u$*,* $v$ *or* $w$ *must be in every loop zero forcing set.*

*Proof.* Let $G$ be a graph and $S$ be a minimum loop zero forcing set of $G$. Suppose $u$ and $v$ are twin in a graph $G$. Since no neighbors of $u$ or $v$ can perform a force until at least one of $u$ or $v$ is forced, at least one of $u$ or $v$ must be in $S$. Now Suppose $u$, $v$ and $w$ are triplet in a graph $G$. Let $S$ be a minimum loop zero forcing set of $G$. Since no neighbors of $u$, $v$ or $w$ can perform a force until at least one of $u$, $v$ or $w$ is forced, at least one of $u$, $v$ or $w$ must be in $S$. □

**Theorem 2.22.** *For a graph* $G$ *and an edge* $e = uv \in E(G)$*,*

$$Z_\ell(G) - 1 \leq Z_\ell(G - e) \leq Z_\ell(G) + 1.$$

*Proof.* Let $G$ be a graph, $e = uv \in E(G)$, $S$ be a minimum loop zero forcing set of $G$. Suppose $e$ is a bridge. Let $G_u$ and $G_v$ be the components of $G - e$ containing $u$ and $v$, respectively. Either $S \cap G_u$ (or $S \cap G_v$) is a loop zero forcing set of $G_u$ (or $G_v$, respectively). If $S \cap G_u$ is not a loop zero forcing set of $G_u$, then $(S \cap G_u) \cup \{u\}$ is a minimum loop zero forcing set of $G_u$. Similarly, $(S \cap G_v) \cup \{v\}$ is a minimum loop zero forcing set of $G_v$ if $S \cap G_v$ is not a loop zero forcing set of $G_v$. Thus, $Z_\ell(G-e) \leq Z_\ell(G)+1$.

If $S$ is a minimum loop zero forcing set of a graph $G-e$, then $S \cup \{v\}$ is a loop zero forcing set of $G$. Thus, $Z_\ell(G) \leq Z_\ell(G-e)+1$. □

The Nordhaus-Gaddum sum problem for minimum loop zero forcing is to establish tight lower and upper bounds on $Z_\ell(G) + Z_\ell(\overline{G})$.

**Theorem 2.23** ([14]). *Let G be a graph. Then*

1. $|G| - 2 \leq Z(G) + Z(\overline{G}) \leq 2|G| - 1$.
2. $|G| - 2 \leq Z_+(G) + Z_+(\overline{G}) \leq 2|G| - 1$.

**Theorem 2.24.** *Let G be a graph. Then*

$$|G| - 2 \leq Z_\ell(G) + Z_\ell(\overline{G}) \leq 2|G| - 1.$$

*Proof.* The proof follows from Theorem 2.23. □

## 2.3. Loop Propagation Time

Let $G$ be a graph. Start with initially black colored vertices $S \subseteq V(G)$. Let $S^{(i)}$ be the set of vertices colored black at round $i$, and let $S^{[i]}$ be the set of vertices that are colored black after round $i$. Thus $S^{[0]} = S^{(0)} = S$ and $S^{[i]} = S^{[i-1]} \cup S^{(i)}$ for $i \geq 1$. A subset $S \subseteq V(G)$ is a loop zero forcing set of $G$ if $S^{[t]} = V(G)$ for some integer $t$.

**Example 2.25.** *Consider graph G in Figure 2.7. Let $S = \{a, d, g\}$. Then*

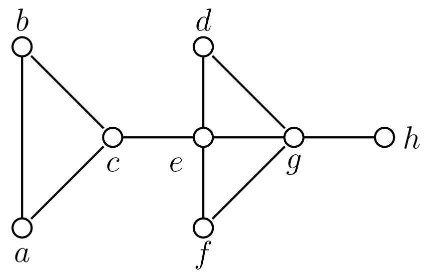



**Figure 2.7.** *graph G in Example 2.25*

$S = S^{(0)} = S^{[0]} = \{a, d, g\}$, *and*

$$\begin{aligned} S^{(1)} &= \{e, h\} & S^{[1]} &= \{a, d, e, g, h\} \\ S^{(2)} &= \{f\} & S^{[2]} &= \{a, d, e, f, g, h\} \\ S^{(3)} &= \{c\} & S^{[3]} &= \{a, c, d, e, f, g, h\} \\ S^{(4)} &= \{b\} & S^{[4]} &= \{a, b, c, d, e, f, g, h\}. \end{aligned}$$

*Therefore,* $pt_\ell(G, S) = 4$.

The loop propagation time of $S \subseteq V(G)$, denoted by $pt_\ell(G, S)$, is the least $t$ such that $S^{[t]} = V(G)$ and $pt_\ell(G, S) = \infty$ if $S$ is not a loop zero forcing set of $G$. The minimum loop propagation time of a graph $G$, denoted by $pt_\ell(G)$, is

$$pt_\ell(G) = \min\{pt_\ell(G, S) : S \text{ is a minimum loop zero forcing set of } G\}.$$

The maximum loop propagation time of a graph $G$, denoted by $PT_\ell(G)$, is

$$PT_\ell(G) = \max\{pt_\ell(G, S) : S \text{ is a minimum loop zero forcing set of } G\}.$$

For a graph $G$, two minimum loop zero forcing sets may have different loop propagation times, as Example 2.26 shows.

**Example 2.26.** *Let $G$ be the graph in Figure 2.7. It was shown in Example 2.25 that* $pt_\ell(G, S) = 4$. *Let* $J = \{a, e, g\}$. *Then*
$J = J^{(0)} = J^{[0]} = \{a, e, g\}$, *and*

$$\begin{aligned} S^{(1)} &= \{d, f, h\} & S^{[1]} &= \{a, d, e, f, g, h\} \\ S^{(2)} &= \{c\} & S^{[2]} &= \{a, c, d, e, f, g, h\} \\ S^{(3)} &= \{b\} & S^{[3]} &= \{a, b, c, d, e, f, g, h\}. \end{aligned}$$

*Therefore* $pt_\ell(G, J) = 3$.

**Definition 2.27.** Let $LZFS(G)$ be the set of all minimum loop zero forcing sets of a graph $G$. A set $S \in LZFS(G)$ is called an efficient loop zero forcing set for $G$ if $pt_\ell(G, S) = pt_\ell(G)$. Define

$$Eff_\ell(G) = \{S : S \text{ is an efficient loop zero forcing set of } G\}.$$

The following result is an immediate consequence of Proposition 2.5.

**Corollary 2.28.** *Let $G$ be a graph of order $n \geq 3$. Then there exists an efficient loop zero forcing set containing no pendant vertices.*

**Remark 2.29.** *In [22] it was shown that $\cap_{S \in Eff(G)} S = \emptyset$ is not necessarily true for standard propagation time. The same is true for loop propagation time. For example,*

$$| \cap_{S \in Eff_\ell(K_{1,n})} S| = 1.$$

**Definition 2.30.** A loop zero forcing set $S$ of a graph $G$ has full loop zero forcing if for all $i = 1, 2, \ldots, pt_\ell(G; S)$, every white neighbor of a black vertex is forced at the next round.

The graph in Figure 2.8, the loop zero forcing set $S_1 = \{a, b\}$ has full loop zero forcing while $S_2 = \{c, d\}$ does not have full loop zero forcing.

**Remark 2.31.** *The following statements hold for loop zero propagation time of graphs.*

1. *If G is disconnected graph with connected components $G_1, G_2, \ldots, G_k$, then*

   $$pt_\ell(G) = \max\{pt_\ell(G_i) \text{ for } i = 1, 2, \ldots, k\}$$

   *and*

   $$PT_\ell(G) = \max\{PT_\ell(G_i) \text{ for } i = 1, 2, \ldots, k\}.$$

2. *If G is a graph and $S$ is a loop zero forcing set of G. Then $pt_\ell(G) \leq |G| - |S|$, because in each round, at least one force must be performed. Therefore $pt_\ell(G) \leq |G| - Z_\ell(G)$.*

3. *If G is a graph, and $S_1$ and $S_2$ are isomorphic loop zero forcing sets of G. Then $pt_\ell(G; S_1) = pt_\ell(G; S_2)$. However, non-isomorphic loop zero forcing sets may also have equal loop propagation times. Figure 2.8 shows two non-isomorphic loop zero forcing sets with the same loop propagation times.*

4. *A loop zero forcing set $S$ of a graph G has a full loop zero forcing if $pt_\ell(G, S) = 1$ or $S$ is a standard zero forcing set of G with full standard zero forcing.*

5. *If G is a graph and $S$ is a loop zero forcing set of G. Then $ecc(S) \leq pt_\ell(G, S)$, because in each round, at most one distance can be advanced. Furthermore, if $S$ has full loop zero forcing, then $pt_\ell(G, S) = ecc(S)$.*

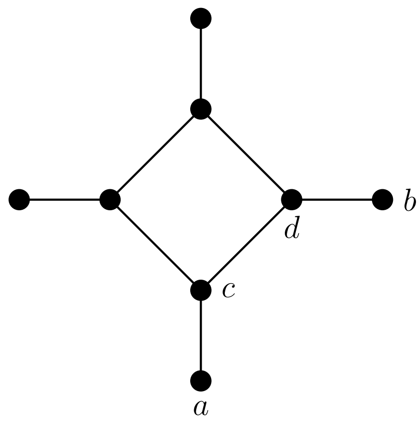


**Figure 2.8.** $pt_\ell(G, \{a, b\}) = pt_\ell(G, \{c, d\}) = 3$

In general, the $pt_\ell(G)$ and $pt(G)$ of a graph $G$ are not comparable even when $Z_\ell(G) < Z(G)$. Consider the star graph $K_{1,n}$ ($n \geq 3$). Then $Z(K_{1,n}) = n-1$ and $pt(K_{1,n}) = 2$ but $Z_\ell(K_{1,n}) = 1$ and $pt_\ell(K_{1,n}) = 1$. On the other hand, for the double star $K(n, m)$ where $n \geq 1, m \geq 2$, $Z(K(n, m)) = n + m - 2$ and $pt(K(n, m)) = 2$ but $Z_\ell(K(n, m)) = 1$ and $pt_\ell(K(n, m)) = 3$.

Since every standard zero forcing set $S$ in a graph $G$ is also a loop zero forcing in $G$, $pt_\ell(G, S) \leq pt(G, S)$. Then the following statement holds when $Z(G) = Z_\ell(G)$.

**Proposition 2.32.** *Let $G$ be a graph. If $Z_\ell(G) = Z(G)$, then $pt_\ell(G) \leq pt(G)$.*

The bound in 2.32 is tight. Examples of graphs $G$ for which $Z(G) = Z_\ell(G)$ and $pt_\ell(G) = pt(G)$ include paths, cycles and complete graphs. Consider the full house graph $H$ of order five, shown in Figure 2.9. Then $Z(H) = Z_\ell(H) = 3$ but $Z_\ell(G) = 1 < 2 = Z(G)$.

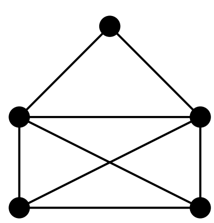


**Figure 2.9.** *house graph $H$*

**Definition 2.33.** The loop propagation time interval of a graph $G$ is defined as

$$[pt_\ell(G), PT_\ell(G)] = \{pt_\ell(G), pt_\ell(G) + 1, \ldots, PT_\ell(G) - 1, PT_\ell(G)\}.$$

**Definition 2.34.** A graph $G$ has a full loop propagation time interval if every integer in the loop propagation time interval is achievable by some minimum loop zero forcing set.

**Definition 2.35.** The loop propagation time discrepancy of a graph $G$ is defined as

$$pd_\ell(G) = PT_\ell(G) - pt_\ell(G).$$

**Remark 2.36.** *Let $G$ be the graph shown in Figure 2.10. Then $pt_\ell(G) = 4$ and $PT_\ell(G) = 6$. There is no minimum loop zero forcing set $S$ with $pt_\ell(G, S) = 5$. So, $G$ have no full loop propagation time interval.*

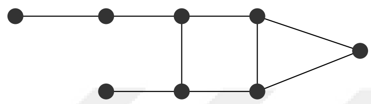


**Figure 2.10.** *Graph G in Remark 2.36*

For a graph $G$, $u \in V(G)$ and $e \in E(G)$, there is no constant bound on the difference between the $pt_\ell(G)$ and $pt_\ell(G-u)$ or $pt_\ell(G-e)$. Deleting a central vertex or an edge incident to the central vertex of a path graph reduces the loop propagation time by half. Deleting a vertex or an edge of a cycle doubles the loop propagation time.

The Nordhaus-Gaddum sum problem for loop propagation time is to establish tight lower and upper bounds on $pt_\ell(G) + pt_\ell(\overline{G})$.

**Proposition 2.37.** *Let $G$ be a graph. Then*

$$1 \leq pt_\ell(G) + pt_\ell(\overline{G}) \leq |G| + 2$$

*and both bounds are tight.*

*Proof.* If $G$ does not have an edge, then $\overline{G}$ has an edge and $pt_\ell(\overline{G}) \geq 1$. Since $pt_\ell(G) = 0$, $pt_\ell(G) + pt_\ell(\overline{G}) \geq 1$. On the other hand, by Remark 2.31, $pt_\ell(G) + pt_\ell(\overline{G}) \leq (|G| - Z_\ell(G)) + (|G| - Z_\ell(\overline{G}))$. By Theorem 2.24, $Z_\ell(G) + Z_\ell(\overline{G}) \geq |G| - 2$. Thus $pt_\ell(G) + pt_\ell(\overline{G}) \leq |G| + 2$. □

The lower bound is realized by complete graphs, and the upper bound is realized by the path of four vertices.

Graphs with extreme loop propagation times will be characterized. Note that the minimum loop propagation time of a graph $G$ is bounded by $0 \leq pt_\ell(G) \leq |G| - 1$.

**Lemma 2.38.** *Let $G$ be a graph. If $v \in V(G)$ is a support vertex with $k$ pendant neighbors, then $pt_\ell(G) \leq |G| - k$.*

*Proof.* Let $G$ be a graph and let $v \in V(G)$ be a support vertex with $k$ pendant neighbors. By Corollary 2.28, $G$ have an efficient minimum loop zero forcing set containing no pendants of $v$. Every pendant neighbor of $v$ is forced at the same round, slowing down the minimum loop propagation time by $k - 1$. Thus $pt_\ell(G) \leq |G| - k$. □

**Lemma 2.39.** *Let $G$ be a graph. If $u$ and $v$ are support vertices, each have $k$ pendant neighbors, then $pt_\ell(G) \leq |G| - 2k + 1$.*

*Proof.* Since every support vertex with $k$ neighbors slows down the loop propagation time by $k - 1$, $pt_\ell(G) \leq |G| - 2k + 1$. □

**Theorem 2.40.** *Let $G$ be a graph. Then $pt_\ell(G) = |G| - 1$ if and only if $G$ is a caterpillar in which every support vertex has exactly one pendant neighbor.*

*Proof.* Suppose $pt_\ell(G) = |G| - 1$. Then $Z_\ell(G) = 1$, so $G$ is a caterpillar. By Lemma 2.38, a support vertex of $G$ have at most one pendant neighbor.

Now, suppose $G$ is a caterpillar such that every support vertex have exactly one pendant neighbor. By Theorem 2.8, $Z_\ell(G) = 1$. In each round, either a non-pendant neighbor forces another vertex, or a pendant neighbor forces itself. Therefore $pt_\ell(G) = |G| - 1$. □

**Definition 2.41.** A graph $G$ is a graph on two parallel caterpillars if $V(G)$ can be partitioned into disjoint subsets $U_1$ and $U_2$ so that the induced subgraphs $T_i = G[U_i]$, where $i = 1, 2$ are caterpillars, furthermore $G$ can be drawn on the plane with parallel caterpillars of $T_1$ and $T_2$, ensuring that the edges between the two caterpillars (drawn as line segments, not curves) do not cross, and two pendant vertices are not adjacent.

**Definition 2.42.** Let $G$ be a graph on two parallel caterpillars $T_1$ and $T_2$. If $v \in V(G)$, then $T(v)$ denotes the parallel caterpillar that contains $v$ and $\overline{T(v)}$ denotes the other of the parallel caterpillar. Fix an ordering of the vertices in each of $T_1$ and $T_2$ that is increasing in the same direction for both caterpillars in a standard drawing. With this ordering, let $first(T_i)$ and $last(T_i)$ denote the first and last vertices of $T_i$ , $i = 1, 2$. If $v, w \in V(T_i)$, then $v \prec w$ means $v$ precedes $w$ in the order on $T_i$. Furthermore, if $v \in V(T_i)$ and $v \neq last(T_i)$, $next(v)$ is the neighbor of $v$ in $T_i$ such that $v \prec next(v)$; $prev(v)$ is defined analogously (for $v \neq first(T_i)$). Note that if $v \in T_i$ is a support vertex, then $v$ precedes all of its pendant neighbors, and $prev(u)$ is $v$ if $u$ is a pendant neighbor of $v$.

**Theorem 2.43.** *Let $G$ be a graph. Then $Z_\ell(G) = 2$ if and only if $G$ is a graph on two parallel caterpillars.*

*Proof.* Suppose $Z_\ell(G) = 2$. Let $S$ be a minimum loop zero forcing set of a graph $G$. Each vertex in $S$ induces a caterpillar forcing chain. Since $G$ is not a caterpillar, $C(G) = 2$. Let $T_1$ and $T_2$ be the caterpillar forcing chains of $G$. Let $w_1 \in T_1$ and $w_2 \in T_2$ be pendant vertices. Since $w_1 \to w_1$ and $w_2 \to w_2$, $w_1w_2 \notin E(G)$. Therefore $G$ is a graph on two parallel caterpillars.

Suppose $G$ is a graph on two parallel caterpillars. Let $T_1$ and $T_2$ be the two parallel caterpillars of $G$. Draw $G$ on the plane so that $T_1$ and $T_2$ are horizontally parallel, and no two edges between $T_1$ and $T_2$ do not intersect. A vertex $u \in T_i$ can perform a force until it is adjacent to a white vertex $v$ in $T_j$ $(i \neq j)$, or $u$ is a pendant in $T_i$ but not in $G$. Since the edges between $T_1$ and $T_2$ do not intersect, forces can be performed along $T_i$ until $v$ is forced. Therefore, $Z_\ell(G) \leq 2$. Since $G$ is not a caterpillar, $Z_\ell(G) = 2$. □

**Theorem 2.44.** *Let $T$ be a tree. Then $pt_\ell(T) = |T| - 2$ if and only if $T$ is a caterpillar such that every support vertex has exactly one pendant neighbor except one support vertex with two pendant neighbors.*

*Proof.* Let $T$ be a caterpillar such that every support vertex has exactly one pendant neighbor except one support vertex with two pendant neighbors. By Theorem 2.8, $Z_\ell(T) = 1$. Suppose $S$ is an efficient loop zero forcing set of $T$. Each round, one force is performed except one round with two forces. Therefore, $pt_\ell(T) = |T| - 2$.

Assume $pt_\ell(T) = |T| - 2$. By Lemma 2.39, if $v \in V(T)$ is a support vertex, then $v$ have at most two neighbors. By Lemma 2.38 and Lemma 2.39, $T$ have only one support vertex with two pendant neighbors. Now, we need to show that $T$ is a caterpillar. Since $pt_\ell(T) = |T| - 2$, either $Z_\ell(T) = 2$ and one force is performed each round, or $Z_\ell(T) = 1$ and each round one force is performed except one with two forces. Suppose $Z_\ell(T) = 2$. By Lemma 2.38 and Lemma 2.39, there is a round two forces is performed, a contradiction. Hence, $Z_\ell(T) = 1$. By Theorem 2.8, $T$ is a caterpillar. □

The next observation is an immediate results of Theorem 2.44.

**Observation 2.45.** *Let $T$ be a tree such that $pt_\ell(T) = |T| - 2$, then*

$$PT_\ell(T) = \begin{cases} |T| - 2 & \textit{if } \Delta(T) = 4 \\ |T| - 1 & \textit{if } \Delta(T) \leq 3. \end{cases}$$

**Observation 2.46.** *Let $G$ be a graph such that $Z_\ell(G) = 2$ and $pt_\ell(G) = |G| - 2$. If $v \in V(G)$ is a support vertex, then $v$ has only one pendant neighbor.*

**Observation 2.47.** *Let $G$ be a graph. Then $pt_\ell(G) = |G| - 2$ if and only if $G$ is one of the following:*

1. *$G = T \cup K_1$ such that $T$ is a caterpillar such that every support vertex have exactly one pendant neighbor.*

2. *$G$ is a caterpillar such that $\Delta(G) \leq 4$ and there exists exactly one support vertex with two pendant neighbors.*

3. *$G$ is a graph on two parallel caterpillars satisfying the following conditions:*

   (a) *a support vertex of $G$ has exactly one pendant neighbor.*

   (b) *$G$ has path of neither support nor pendant vertices $Q = (z_1, z_2, \ldots, z_k)$ such that*

      i. *$z_1 \in T_1$ and $T(z_i) = T(z_{i+1})$ if and only if $z_i$ is a pendant of $z_{i+1}$ in $T(z_i)$ or vice versa.*

      ii. *For vertices $z_p$ and $z_q$ such that $T(z_p) = T(z_q)$, $z_p \prec z_q$ for $p < q$, and there is no vertex $z_r \in T(z_p)$ such that $z_p \prec z_r \prec z_q$ for $p < r < q$.*

   (c) *Every edge of $G$ is an edge of $T_1, T_2$ or $Q$, or an edge of the form $z_j w$ where $i < j < k$, $w \in \overline{T(z_j)}$ and $z_j \prec w \prec z_t$ for $z_j \prec z_t$.*

   (d) *$first(T_1)$ is not a pendant vertex or $first(T_2)$ is not a pendant vertex.*

   (e) *$last(T_1)$ is not a pendant vertex or $last(T_2)$ is not a pendant vertex.*

   (f) *if $z_2 \in \overline{T(z_1)}$, then $z_2 \neq first(T_2)$ or $z_2 \sim next(z_1)$.*

   (g) *if $z_2 \in T(z_1)$, then $z_3 \neq first(T_2)$ or $z_3 \sim next(z_1)$.*

   (h) *$z_{k-1}$ is not a pendant of $z_{k-2}$.*

   (i) *$z_{k-1} \neq last(T(z_{k-1}))$ or $z_{k-1} \sim prev(z_k)$ and $prev(z_k)$ is not a support vertex.*

Now, graphs with low loop propagation time will be characterized.

**Observation 2.48.** *Let $G$ be a graph. Then the following are equivalent.*

1. *$pt_\ell(G) = 0$.*

2. *$PT_\ell(G) = 0$.*

3. *$Z_\ell(G) = |G|$.*

4. *$G$ is $\overline{K_n}$.*

**Proposition 2.49.** *Let $G$ be a connected graph, $S$ be an efficient loop zero forcing set of $G$ such that $pt_\ell(G,S) = 1$. Then every vertex $v \in S$ either performs a force or every vertex in $V(G)\backslash S$ forces itself.*

*Proof.* Suppose $S$ is an efficient loop zero forcing of a graph $G$. If $v \in S$, then by Proposition 2.6, $|N(v) \cap V(G)\backslash S| \geq 1$. Since $pt_\ell(G) = 1$, $v$ performs a force if $|N(v) \cap V(G)\backslash S| = 1$. If $u \in V(G)\backslash S$ such that $u \to u$, then $N(u) \subseteq S$. For $v_1$ and $v_2$ in $S$ such that $v_1$ performs a force and $v_2$ does not perform any force, then $v_1$ is not a neighbor of $v_2$. If $v_1$ and $v_2$ are adjacent, let $S' = S\backslash\{v_1\}$. Then $S'$ is a loop zero forcing set of $G$ such that $|S'| < |S|$. Therefore every vertex $v \in S$ either performs a force or every vertex in $V(G)\backslash S$ forces itself. □

**Proposition 2.50.** *For a tree $T$ that is not a caterpillar, we have $pt_\ell(T) \geq 4$.*

*Proof.* Suppose $S$ is an efficient minimum loop zero forcing set of tree $T$. Let $v$ be an exterior general vertex of $T$ with a batch degree of two or more. Then there exists a batch of terminal vertices $B$ of $v$ in $T$ such that for any vertex in a path from $b \in B$ to $v$, $b \notin S$. Let $u \in B$ be a terminal vertex of $v$ in $T$. Then $S$ forces $v$ earliest at round $t = 2$, implying $S$ forces $u$ earliest at round $t = 4$. Thus $pt_\ell(T) \geq 4$. □

**Proposition 2.51.** *Let $T$ be a tree. Then $pt_\ell(T) = 1$ if and only if $T$ is a star.*

*Proof.* It is obvious that $pt_\ell(K_{1,n}) = 1$.

Assume $pt_\ell(T) = 1$. By Proposition 2.50, $T$ is a caterpillar. Since, every end-support of $T$ is an efficient minimum loop zero forcing of $T$, $T$ have only one end-support vertex. Thus, $T$ is a star. □

**Proposition 2.52.** *For a tree $T$, $pt_\ell(T) = 3$ if and only if $T$ is a double star.*

*Proof.* It is obvious that $pt_\ell(K(n,m)) = 3$.

Assume $pt_\ell(T) = 3$. By Propositions 2.50 and 2.51, $T$ is a caterpillar with two end-support vertices. Since every end-support vertex is efficient minimum loop zero forcing set of $T$, $T$ have adjacent end-support vertices. Thus, $T$ is a double star. □

The next Corollary is a consequence of Proposition 2.50, 2.51 and 2.52.

**Corollary 2.53.** *There is no tree $T$ such that $pt_\ell(T) = 2$.*

## 2.4. Loop Zero Forcing Number and Loop Propagation Time of Some Graph Families

In this section, the $Z_\ell(G)$, $pt_\ell(G)$ and $PT_\ell(G)$ for various families of graphs were determined.

**Proposition 2.54.** *Let $C_n$ be the cycle graph of $n \geq 3$ vertices. Then $Z_\ell(C_n) = 2$, $pt_\ell(C_n) = \lceil \frac{n-2}{2} \rceil$, and $PT_\ell(C_n) = \begin{cases} \lceil \frac{n-2}{2} \rceil & \text{if } n \text{ odd} \\ \lceil \frac{n-2}{2} \rceil + 1 & \text{if } n \text{ even.} \end{cases}$*

*Proof.* By Theorem 2.3 and Proposition 2.7, $Z_\ell(C_n) = 2$. Any pair of adjacent vertices or a pair of vertices of distance two is a minimum loop zero forcing set of $C_n$. Let $S_1$ be pair of adjacent vertices, $S_2$ be pair of vertices of distance two. Then $pt_\ell(C, S_1) = \lceil \frac{n-2}{2} \rceil$ and $pt_\ell(C_n, S_2) = \lceil \frac{n-2}{2} \rceil$ if $n$ is odd and $pt_\ell(C_n, S_2) = \lceil \frac{n-2}{2} \rceil + 1$ if $n$ is even. Therefore $pt_\ell(C_n) = \lceil \frac{n-2}{2} \rceil$, $PT_\ell(C_n) = \lceil \frac{n-2}{2} \rceil$ if $n$ is odd and $PT_\ell(C_n) = \lceil \frac{n-2}{2} \rceil + 1$ if $n$ is even. □

**Proposition 2.55.** *For $q \geq p \geq 1$, let $K_{p,q}$ be a complete bipartite graph. Then $Z_\ell(K_{p,q}) = p$ and $pt_\ell(K_{p,q}) = 1$.*

*Proof.* Assume $K_{p,q}$ for $q \geq p \geq 1$, is a complete bipartite graph. $K_{p,q}$ can be partitioned into two disjoint set of vertices $S = \{v_1, v_2, \ldots, v_p\}$ and $S' = \{u_1, u_2, \ldots, u_q\}$. Then every vertex in partition $S'$ have only neighbors in $S$. Thus by Proposition 2.7, $Z_\ell(K_{p,q}) = |S| = p$ and $pt_\ell(K_{p,q}) = 1$. □

**Proposition 2.56.** *For a wheel graph $W_n$, $Z_\ell(W_n) = 3$ and $pt_\ell(W_n) = \lceil \frac{n-2}{2} \rceil$.*

*Proof.* The proof of $Z_\ell(W_n) = 3$ directly follows from Theorems 2.3 and Proposition 2.7. For any efficient loop zero forcing set, each round at most two vertices can be forced. Thus $pt_\ell(G) = \lceil \frac{n-2}{2} \rceil$. □

Let $T_{p,q}$ denote the balanced spider with $p$ legs, each of length $q$.

**Proposition 2.57.** *For $p, q \geq 2$, $Z_\ell(T_{p,q}) = p - 1$ and $pt_\ell(T_{p,q}) = 2q$.*

*Proof.* Let $T_{p,q}$ be a balanced spider. Then $eg(T) = 1$ with $bat(T) = p$. Let $v$ be an exterior general vertex. By Proposition 2.19, $Z_\ell(T_{p,q}) \geq p - 1$. Let $S$ be $p - 1$ terminal end-support vertices of $v$. Then $S$ is a loop zero forcing set of $T$ with $pt_\ell(T_{p,q}, S) = 2q$. Therefore, $Z_\ell(T_{p,q}) = p - 1$. Suppose $S'$ be a minimum loop zero forcing set of $T_{p,q}$ containing no pendants. Then $|S \cap S'| \geq 1$ and $pt_\ell(T_{p,q}, S') = 2q$. Thus $pt_\ell(T_{p,q}) = 2q$. □

**Definition 2.58.** The Banana tree graph $B_{n,k}$ is the graph obtained by connecting one pendant of each of $k$ copies of order $n$ star graph with a single root vertex that is distinct for all the stars.

**Proposition 2.59.** *For $B_{n,k}$ Banana tree, $Z_\ell(B_{n,k}) = k - 1$ and $pt_\ell(B_{n,k}) = 6$.*

*Proof.* Let $B_{n,k}$ be a Banana tree. Then $eg(B_{n,k}) = 1$ and $bat(B_{n,k}) = k$. By Proposition 2.19, $Z_\ell(B_{n,k}) \geq k-1$. Let $S$ be $k-1$ terminal end-support vertices of $B_{n,k}$. Then $S$ is a loop zero forcing set with $pt_\ell(B_{n,k}, S) = 6$. Thus $Z_\ell(B_{n,k}) = k-1$. Let $S'$ be a minimum loop zero forcing set $B_{n,k}$ different from $S$. Then $S'$ contains at least one pendant vertex. By Proposition 2.19 $pt_\ell(B_{n,k}, S') \geq 6$. Thus, $pt_\ell(B_{n,k}) = 6$. □

**Definition 2.60.** An $(n, k)$- firecracker tree is the graph obtained by the concatenation of $k$ order $n$ stars by linking one leaf from each.

**Proposition 2.61.** *For $n \geq 3$, $k \geq 2$, let $G$ be an $(n, k)$-firecracker tree. Then*

$$Z_\ell(G) = \left\lceil \frac{k}{2} \right\rceil \text{ and } pt_\ell(G) = 5.$$

*Proof.* Let $G$ be an $(n, k)$- firecracker tree. It is easy to show that the caterpillar cover number of $G$ is $C(G) = \lceil \frac{k}{2} \rceil$. Suppose $k = 2$. Then $G$ is a caterpillar. By Proposition 2.8, $Z_\ell(G) = 1$ and $pt_\ell(G) = 5$. Suppose $k = 3$. Then $eg(G) = 1$ and $bat(G) = 3$. By Proposition 2.19, $Z_\ell(G) \geq 2$. Any set of two support vertices of $G$ is a loop zero forcing set. Thus $Z_\ell(G) = 2$. By Lemma 2.38, $pt_\ell(G) = 5$. Now suppose $k \geq 2$. By Proposition 2.12, $Z_\ell(G) \geq \lceil \frac{k}{2} \rceil$. Label the support vertices of $G$ from left to right by $v_1, v_2, \ldots, v_k$. Let $S = \{v_i, v_{i+1} : i = 4j - 2 \text{ for } j = 1, 2, \ldots, \lfloor \frac{k}{4} \rfloor \text{ and } i = k \text{ if } k \text{ is odd}\}$. Then $S$ is a loop zero forcing set of $G$ and $pt_\ell(G, S) = 5$. Therefore $Z_\ell(G) = |S| = \lceil \frac{k}{2} \rceil$. By Lemma 2.38, we obtain $pt_\ell(G) = 5$. □

**Definition 2.62.** The sunlet graph of order $2n$ is obtained from a cycle $C_n$ by adjoining a pendant vertex to each vertex of $C_n$.

**Proposition 2.63.** *Let $G$ be a sunlet graph of order $2n$. Then $Z_\ell(G) = 2$ and*

$$pt_\ell(G) = PT_\ell(G) = \begin{cases} n & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* Suppose $G$ be a sunlet graph. Since $\delta(G) = 1$ and $G$ is not a caterpillar, $Z_\ell(G) \geq 2$. Label the vertices in a cycle of $G$ as $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in E(G)$ and $v_1 v_n \in E(G)$. Let $u_i$ be a pendant neighbor of $v_i$. Let $S = \{v_i, v_{i+1} : \text{for some } i \in \{1, 2, \ldots, n-1\}\}$. Then $S$ is a loop zero forcing set of $G$ with $pt_\ell(G, S) = n$ if $n$ is odd and $pt_\ell(G, S) = n - 1$ if $n$ is even. Thus $Z_\ell(G) = 2$. If $S'$ is a minimum loop zero forcing not isomorphic to $S$, then $S'$ is to isomorphic to the set $\{v_i, u_{i+1}\}$ or $\{u_i, u_{i+1}\}$. Similarly, $pt_\ell(G, S') = n$ if $n$ is odd and $pt_\ell(G, S') = n - 1$ if $n$ is even. Therefore, $pt_\ell(G) = PT_\ell(G) = n$ if $n$ is odd and $pt_\ell(G) = PT_\ell(G) = n - 1$ if $n$ is even. □

**Definition 2.64.** The pineapple graph $K_n^p$ is the union of the graphs $K_n$ and $K_{1,p}$ with the specification that $K_n \cap K_{1,p}$ is the vertex of $K_{1,p}$ that is degree $p$ (Thus $K_n^p$ can be obtained from $K_n$ by appending $p$ pendant edges to a vertex of $K_n$).

**Proposition 2.65.** *For $n \geq 3$ and $p \geq 1$, let $K_n^p$ be a pineapple graph. Then $Z_l(K_n^p) = n-1$ and $pt_l(K_n^p) = 1$.*

*Proof.* Suppose $K_n^p$ is a pineapple graph. Let $S$ a set of vertices in $K_n$ such that $|B| = n-1$. Then $S$ is a loop zero forcing set of $K_n^p$, so $Z_l(K_n^p) \leq n-1$. But since $Z_l(K_n) = n-1$, $Z_l(K_n^p) \geq n-1$. So $Z_l(K_n^p) = n-1$. To show that $pt_l(K_n^p) = 1$, Let $B$ the set containing the vertex of degree $p$ in $K_n^p$. Every vertex $w \in V(K_n^p) \backslash S$ has neighbors in $S$, thus $pt_l(K_n^p) = 1$. □

**Definition 2.66.** The barbell graph $B(n)$ is the graph obtained by connecting two copies of complete graphs $K_n$ by a bridge.

**Proposition 2.67.** *Let $B(n)$ be a barbell graph of order $2n$. Then $Z_\ell(B(n)) = 2n-3$ and $pt_\ell(B(n)) = PT_\ell(B(n)) = 3$.*

*Proof.* Let $B(n)$ be a barbell graph. Since $Z_\ell(K_n) = n-1$, at most $n-1$ vertices of each $K_n$ in $B(n)$ is containing in any loop zero forcing set of $B(n)$. But removing one more vertex in one of the $K_n$'s, is a loop zero forcing set of $B(n)$. Thus $Z_\ell(B(n)) = 2n-3$. If $S$ is a minimum loop zero forcing set of $B(n)$, then $pt_\ell(B(n), S) = PT_\ell(B(n), S) = 3$. □

**Definition 2.68.** A prism $Y_n$ is the graph obtained by taking two copies of $C_n$ (inner and outer cycle) with the same vertex labelings and joining each vertex of the inner cycle $C_n$ to the vertex of the outer vertex $C_n$ having the same label by an edge.

Let $Y_3$ be a prism graph. Let $Z$ be the vertices of the inner cycle (or outer cycle) of $Y_3$. Since every vertex of $Z$ is adjacent to only one vertex of the outer cycle (inner cycle, respectively), $Z$ is a loop zero forcing set and $pt_\ell(Y_3, Z) = 1$. Then by Proposition 2.7, $Z_\ell(Y_3) = 3$ and $pt_\ell(Y_3) = 1$. Now let $Z'$ be the set of vertices of the inner (or outer) cycle and one vertex of the outer (inner, respectively) cycle. Then $Z'$ is a loop zero forcing set and $pt_\ell(Y_3, Z') = 2$. Hence, every minimum loop zero forcing set of $Y_3$ is either $Z$ or $Z'$, $PT_\ell(Y_3) = 2$.

**Observation 2.69.** *For a prism graph $Y_3$, $Z_\ell(Y_3) = 3$, $pt_\ell(Y_3) = 1$, and $PT_\ell(Y_3) = 2$.*

**Proposition 2.70.** *Let $Y_n$ be the prism graph of $n \geq 4$ vertices. Then $Z_\ell(Y_n) = 4$, $pt_\ell(Y_n) = \lceil \frac{n-2}{2} \rceil$, $PT_\ell(Y_n) = \lceil \frac{n-2}{2} \rceil$ if $n$ is odd, and $PT_\ell(Y_n) = \lceil \frac{n-2}{2} \rceil + 1$ if $n$ is even.*

*Proof.* By Proposition 2.7, $Z_\ell(Y_n) \geq 3$. Let $S$ be a minimum loop zero forcing set of $Y_n$. Suppose $Z_\ell(Y_n) = 3$. Then $S$ consists of three vertices in the same cycle $C_i$ or $S$ consists of two vertices from cycle $C_i$, and one vertex from $C_j (i \neq j)$. Let $S$ consists of three vertices from the same cycle. If the induced subgraph $Y_n[S]$ is disconnected, then every vertex in $S$ has at least two neighbors in $V\backslash S$, and every vertex in $V\backslash S$ has at least one neighbor in $V\backslash S$. So, $S$ is not a loop zero forcing set. Hence, $Y_n[S]$ is connected ($Y_n[S] = P_3$). Now, $S$ performs one force at round one but can not perform any more forces. $S$ is not a loop zero forcing set. Let $S$ be two vertices from cycle $C_i$ and one vertex from cycle $C_j$. Then $S$ fails to force $Y_n$. Therefore $Z_\ell(Y_n) \geq 4$. Suppose $S = \{v_1, v_2, v_1', v_2'\}$. Then $S$ is a loop zero forcing set of $Y_n$ with $pt_\ell(Y_n, S) = \lceil \frac{n-2}{2} \rceil$. Therefore $Z_\ell(Y_n) = 4$. If $S'$ is a minimum loop zero forcing set of $Y_n$ not isomorphic to $S$, then $pt_\ell(Y_n, S') = \lceil \frac{n-2}{2} \rceil$ if $n$ is odd, and $pt_\ell(Y_n, S') = \lceil \frac{n-2}{2} \rceil$ if $n$ is even. Thus $pt_\ell(Y_n) = \lceil \frac{n-2}{2} \rceil$, $PT_\ell(Y_n) = \lceil \frac{n-2}{2} \rceil$ if $n$ is odd, and $PT_\ell(Y_n) = \lceil \frac{n-2}{2} \rceil + 1$ if $n$ is even. □

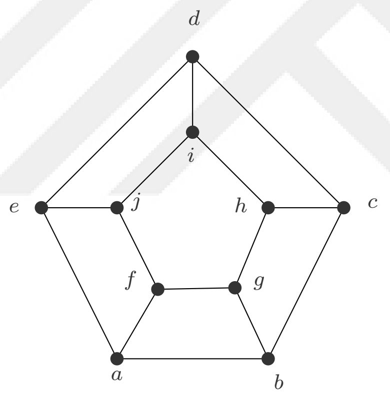



**Figure 2.11.** *Three isomorphic minimum loop zero forcing sets of $Y_5$: $S_1 = \{a, b, c, d\}$, $S_2 = \{b, e, f, g\}$ and $S_3 = \{b, e, f, h\}$*

**Definition 2.71.** An antiprism graph $A_n$ is made up of two $n$-gons on top and bottom, separated by a ribbon of $2n$ triangles, with the two $n$-gons being offset by one ribbon segment.

**Proposition 2.72.** *Let $A_n$ be an antiprism graph for $n \geq 4$. Then $Z_\ell(A_n) = 4$, $pt_\ell(A_n) = n - 2$ and $PT_\ell(A_n) = n - 1$.*

*Proof.* Let $A_n$ be an antiprism graph. Since $\delta(A_n) = 4$, $Z_\ell(A_n) \geq 4$. Label the vertices of the inner cycle of $A_n$ as $v_1, v_2, \ldots, v_n$ such that $v_i v_{i+1} \in E(A_n)$ and $v_1 v_n \in E(A_n)$. Label a vertex of the outer cycle as $v_{ij}$ if $v_i v_{ij} \in E(A_n)$ and $v_j v_{ij} \in E(A_n)$. Let $S =$

$\{v_i, v_{i+1}, v_{i,i+1}, v_{i+1,i+2}\}$. Then $S$ is a loop zero forcing set with $pt_\ell(A_n, S) = n - 2$. Thus, $Z_\ell(A_n) = 4$. Let $S'$ is a minimum loop zero forcing set of $A_n$ not isomorphic to $S$. Then $|S \cap S'| = 3$ and $pt_\ell(A_n, S') = n - 1$. Every minimum loop zero forcing set of $A_n$ is isomorphic to $S$ or $S'$. Thus, $pt_\ell(A_n) = n - 2$ and $PT_\ell(A_n) = n - 1$. □

| Graphs $G$ | $Z_\ell(G)$ | $pt_\ell(G)$ | $PT_\ell(G)$ | Full Loop Interval |
|---|---|---|---|---|
| $C_n$ | $2$ | $\lceil \frac{n-2}{2} \rceil$ | $\begin{cases} \lceil \frac{n-2}{2} \rceil & n \text{ odd} \\ \lceil \frac{n-2}{2} \rceil + 1 & n \text{ even.} \end{cases}$ | Yes |
| $K_{n,m}$ | $\min\{n,m\}$ | $1$ | $2$ | Yes |
| $W_n$ | $3$ | $\lceil \frac{n-2}{2} \rceil$ | | Yes |
| $(n,k)$ firecracker | $\lceil \frac{k}{2} \rceil$ | $5$ | $6$ | Yes |
| $B(n,k)$ | $k-1$ | $6$ | | Yes |
| $Y_n$ | $4$ | $\lceil \frac{n-2}{2} \rceil$ | $\begin{cases} \lceil \frac{n-2}{2} \rceil & n \text{ odd} \\ \lceil \frac{n-2}{2} \rceil + 1 & n \text{ even.} \end{cases}$ | Yes |
| Sunlet graph of order $2n$ | $2$ | $\begin{cases} n & n \text{ odd} \\ n-1 & n \text{ even.} \end{cases}$ | $\begin{cases} n & n \text{ odd} \\ n-1 & n \text{ even.} \end{cases}$ | Yes |
| $n$ barbell graph | $2n-3$ | $3$ | $3$ | Yes |

**Figure 2.12.** *Summary table of loop propagation time of selected graphs*

## 3.1. Failed Loop Zero Forcing Number

The failed forcing number represents the number of vertices in the largest failed zero forcing set. Failed zero forcing numbers were first introduced in [18]. The standard failed zero forcing was investigated in [18, 19], while the failed skew zero forcing was examined in [2, 26]. Additionally, the failed positive semidefinite zero forcing was studied in [30].

In a simple, undirected graph $G = (V(G), E(G))$, we designate a set $F$ of vertices to be colored black, while the remaining vertices are colored white. We then apply the loop zero forcing color change rule ($CCR\text{-}Z_\ell$) to the graph. If the final coloring of $F$ under the $CCR\text{-}Z_\ell$ rule does not result in the entire vertex set $V(G)$, then the set $F$ is considered a $CCR\text{-}Z_\ell$ failed loop zero forcing set of $G$.

The $CCR\text{-}Z_\ell$ failed loop zero forcing number of $G$, denoted as $F_\ell(G)$, represents the maximum size of a $CCR\text{-}Z_\ell$ failed loop zero forcing set within $G$. Additionally, a failed loop zero forcing set $F$ is classified as stalled if no further color changes are possible from the set $F$.

**Observation 3.1.** *Every maximum failed loop zero forcing set is stalled.*

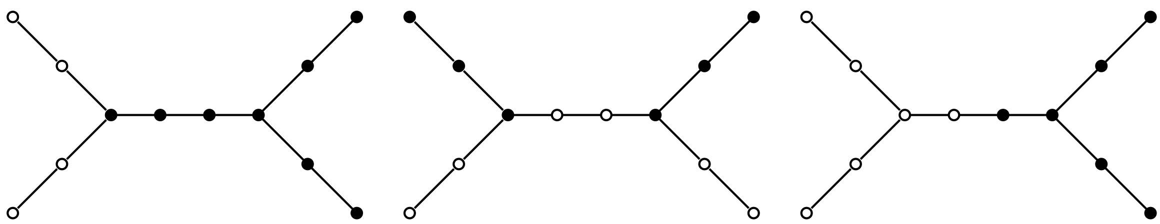


**Figure 3.13.** *Maximum stalled failed loop zero forcing set, maximal stalled failed loop zero forcing set, unstalled failed loop zero forcing set*

**Observation 3.2.** *If we has a loop zero forcing set $S \subseteq V(G)$, then any superset $S'$ of $S$ is also a loop zero forcing set. In other words, if $S$ is capable of forcing the entire graph to be black under the loop zero forcing color change rule, then any larger set $S'$ that includes $S$ will also achieve the same outcome and force the graph to be black.*

**Observation 3.3.** *If we has a failed loop zero forcing set $F \subseteq V(G)$, then any subset $F'$ of $F$ is also a failed loop zero forcing set. In other words, if $F$ fails to force the entire graph to be black under the loop zero forcing color change rule, then any smaller subset $F'$ that is taken from $F$ will also fail to achieve the color change and keep the graph from being completely black.*

**Observation 3.4.** *For any graph G,*

$$Z_\ell(G) - 1 \leq F_\ell(G) \leq |G| - 1.$$

**Proposition 3.5.** *For any graph G,*

$$F_\ell(G) \geq \delta(G) - 1.$$

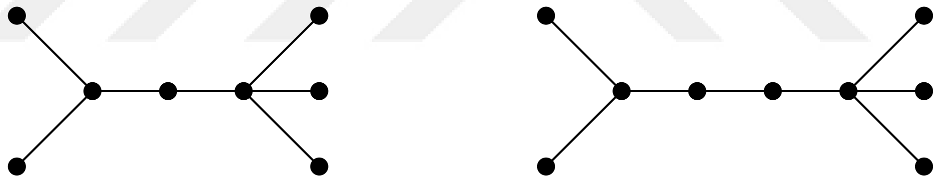
*Proof.* The proof follows from Proposition 2.7 and Observation 3.4. □

**Theorem 3.6.** *For any graph G,*

$$F_\ell(G) \leq F(G).$$

*Proof.* Assume $F$ is a maximum failed loop zero forcing set of a graph $G$. Then by definition, every vertex $f \in F$ with neighbors in $V(G)\backslash F$ has at least two neighbors in $V(G)\backslash F$. Hence, $F$ is failed standard zero forcing. Thus $F_\ell(G) \leq F(G)$. □

The graph $K_r(n,m)$ is the graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ with a path of order $r$ joining their centers. Figure 3.14 shows the $K_1(2,3)$ and $K_2(2,3)$ graphs.



$K_1(2,3)$ $K_2(2,3)$

**Figure 3.14.** *$K_1(2,3)$ and $K_2(2,3)$ graphs*

**Observation 3.7.** *For a graph G, $F_\ell(G) < Z_\ell(G)$ if and only if every vertex set of cardinality $Z_\ell(G)$ is a minimum loop zero forcing set.*

**Theorem 3.8.** *Let G be a graph. Then $F_\ell(G) < Z_\ell(G)$ if and only if G is one of the following graphs: $K_n$, $\bar{K_n}$, $K_{1,n}$, $K(n,m)$, $K_{3,3}$, $Y_3$, $C_4$, $C_5$ or $W_5$.*

*Proof.* It is obvious that $F_\ell(K_n) = n-2$, $Z_\ell(K_n) = n-1$, $F_\ell(\bar{K_n}) = n-1$ and $Z_\ell(\bar{K_n}) = n$. If $G$ is a star $K_{1,n}$ or a double star $K(n,m)$, by Theorem 2.8, $Z_\ell(K_{1,n}) = Z_\ell(K(n,m)) = 1$ and every one vertex set is a loop zero forcing set, thus $F_\ell(K_{1,n}) = F_\ell(K(n,m)) = 0$. If $G$ is $C_4$ or $C_5$, by Theorem 2.43, $Z_\ell(C_4) = Z_\ell(C_5) = 2$ and every two vertex set is a loop zero forcing set, thus $F_\ell(C_4) = F_\ell(C_5) = 1$. If $G$ is $K_{3,3}$ or $W_5$, by Theorem 2.43, $Z_\ell(K_{3,3}) = Z_\ell(W_5) = 3$ and every three vertex set is a loop zero forcing set, thus $F_\ell(K_{3,3}) = F_\ell(W_4) = 2$.

Assume $F_\ell(G) < Z_\ell(G)$. Let $S$ be a minimum loop zero forcing set. Suppose $|S| = 1$. By Theorem 2.8, $G$ is a caterpillar and every vertex of $G$ is a loop zero forcing set (i.e., a vertex of $G$ is either a support vertex or a pendant of a support vertex). Then is $G$ is a star graph $K_{1,n}$ or a double star $K(n,m)$.

Suppose $|S| = 2$. If $G$ is disconnected, then $G$ consists of two caterpillar components $G_1$ and $G_2$, otherwise $Z_\ell(G) \geq 3$. If either $|G_1| \geq 2$ or $|G_1| \geq 2$, then taking $V(G_1)$ or $V(G_2)$ fails to force $G$ and $F_\ell(G) \geq 2$. Thus $|G_1| = |G_2| = 1$, therefore $G = \bar{K}_2$. If $G$ is connected then $G$ is a two parallel caterpillars graph by Theorem 2.43 and any two vertices in $G$ is a loop zero forcing set. The graph $G$ does not have a pendant vertex, otherwise a vertex and its pendant vertex fail to force $G$ and $F_\ell(G) \geq 2$. If $G$ is a complete, then $G = K_3$ and if $G \neq K_3$ then for every two non-adjacent vertices $u$ and $v$ in $G$, there exist a vertex $w$ of degree two such that $N(u) \cup N(v) = \{w\}$. Then $G$ is either $C_4$ or $C_5$.

Suppose $|S| = 3$. If $G$ has a support vertex, then a support vertex and its pendant can not be both in any minimum loop zero forcing set. Therefore $\delta(G) \geq 2$. Suppose $|G| = 4$. Let $w$ be the vertex in $G$ but not in $S$. Then $w$ is adjacent to every vertex in $S$ and $G[S]$ is connected. If $G[S] = P_3$, then $Z_\ell(G) = 2$ contradicting the assumption that $Z_\ell(G) = 3$. Hence, $G[S] = K_3$ and $G = K_4$. By Theorem 3.8, $Z_\ell(G) = 3$ and $F_\ell(G) = 2$.
Suppose $|G| = 5$. Every vertex in $S$ has two neighbors in $V(G)\backslash S$. Let $u_1$ and $u_2$ be vertices in $V(G)\backslash S$. Let $u_1$ and $u_2$ are adjacent. Since $\{v, u_1, u_2\}$ for any $v \in S$ is a loop zero forcing set, $G[S] = K_3$. Therefore $\delta(G) = 4$ contracting Proposition 2.7. Now let $u_1$ and $u_2$ are not adjacent. Since $\{v, u_1, u_2\}$ for any $v \in S$ is a loop zero forcing set, $G[S] = P_3$ and two vertex set in $G$ fail to force $G$. Thus $G$ is $W_5$.
Suppose $|G| = 6$. Let $G[S] = \bar{K}_3$. a vertex $v \in S$ is adjacent to at least two vertices in $V(G)\backslash S$. But if $v \in S$ is adjacent to at least two vertices in $V(G)\backslash S$, then $N(v)$ is a loop zero forcing set or $N[v]$ is a failed loop zero forcing set, both producing contradictions. Hence, $v$ is adjacent to all vertices in $V(G)\backslash S$. Let $u_1$,$u_2$ and $u_3$ be vertices of $V(G)\backslash S$. If $e = u_1u_2$ is an edge of $G$, then $u_3$ combined with any two vertices in $S$, fails to force $G$. So, $G[V(G)\backslash S] = \bar{K}_3$. Therefore $G = K_{3,3}$ and any two vertex set fails to force $G$.

Suppose $|S| \geq 4$. Let $S'$ be the non-empty set of vertices such that for $w \in S'$, $N(w) \subseteq S$. Let $v \in S'$. Then $1 \leq \deg(v) \leq |S|$. Suppose $\deg(v) < |S|$. Construct a failed loop zero forcing set $F$ from $S$ such that $|F| = |S|$ as follows: let $F$ be the vertices $F = (S\backslash u) \cup \{v\}$ for a vertex $u \in S$ not adjacent to $v$. Then $F_\ell(G) \geq Z_\ell(G)$. Suppose $\deg(v) = |S|$. Then every vertex in $S$ has at least one neighbor in $V(G)\backslash(S \cup \{v\})$ or $G$ is a complete graph of order $|S| + 1$. Let every vertex in $S$ has a neighbor in $V(G)\backslash(S \cup \{v\})$. Construct a failed loop zero forcing set $F$ from $S$ such that $|F| = |S|$ as follows: let $F$ be the vertices $F = (S\backslash\{u_1, u_2\}) \cup \{v, v_1\}$ for distinct vertices $u_1, u_2 \in S$

and $v_1 \in V(G)\backslash(S \cup \{v\})$. Then $F_\ell(G) \geq Z_\ell(G)$, completing the proof. $\square$

## 3.2. Extreme Values

In this section, we investigate graphs with extreme failed loop zero forcing numbers. Specifically, we classify graphs with $F_\ell(G) \in \{|G|-1, |G|-2, |G|-3, 2, 1, 0\}$.

**Theorem 3.9.** *For a graph $G$, $F_\ell(G) = |G|-1$ if and only if $\delta(G) = 0$.*

*Proof.* Assume $F_\ell(G) = |G|-1$. Let $F$ be a maximum failed loop zero forcing set such that $|F| = |G|-1$. Let $v \in F$. If $uv \in E(G)$ for some $u \in V(G)$, then $u$ forces $v$, violating our assumption that $F$ is failed. Hence, $\deg(v) = 0$.

Conversely, Assume $\delta(G) = 0$. Then there exist a vertex $v$ such that $\deg(v) = 0$. Let $F = V(G)\backslash\{v\}$. Then $F$ is failed. Hence by Observation 3.4, $F_\ell(G) = |G|-1$. $\square$

**Observation 3.10.** *For any connected graph $G$,*

$$F_\ell(G) \leq |G|-2.$$

**Theorem 3.11.** *For a connected graph $G$, $F_\ell(G) = |G|-2$ if and only if $G$ has a twin.*

*Proof.* Assume $F_\ell(G) = |G|-2$. Let $F$ be a maximum failed loop zero forcing set. Then $V(G)\backslash F = \{u, v\}$ for some vertices $u$ and $v$. Since neither $u$ nor $v$ can be forced, $u$ and $v$ are adjacent and every neighbor of $u$ in $F$ is also a neighbor of $v$, and vice versa. Therefore $u$ and $v$ form a twin.

On the converse, assume a connected graph $G$ has a twin vertices $u$ and $v$. Then the set $V(G)\backslash\{u, v\}$ is a loop zero forcing set of $G$ and $F_\ell(G) \geq |G|-2$. By Observation 3.10, $F_\ell(G) = |G|-2$. $\square$

**Example 3.12.** *Let $G$ be the graph on six vertices illustrated in Figure 3.15. The vertex set $\{a, b\}$ is a twin in $G$, so $F_\ell(G) = 4$.*

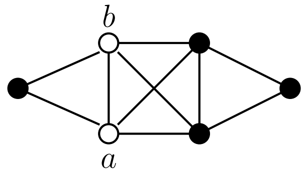



**Figure 3.15.** *Graph G in Example 3.12*

**Theorem 3.13.** *Let $G$ be a connected graph without twin. Then $F_\ell(G) = |G|-3$ if and only if $G$ has twin-free dependent triplet.*

*Proof.* Suppose $F_\ell(G) = |G| - 3$. Then $G$ does not have a twin. Let $F$ be a maximum failed loop zero forcing set. Then $V(G)\backslash F = \{u, v, w\}$ for some vertices $u$, $v$ and $w$. Since $F$ can not force $u$, $v$ or $w$, every vertex in $F$ either have no neighbors or has at least two neighbors in $V(G)\backslash F$, and every vertex in $V(G)\backslash F$ has a neighbor in $V(G)\backslash F$. Hence $G[V(G)\backslash F]$ is connected. Let $H$ be the induced subgraph $G[V(G)\backslash F]$. Then $H$ is a path or a cycle. If $H$ is a cycle then every vertex in $F$ is adjacent to two vertices in $H$ or there exists a vertex in $F$ adjacent to all vertices in $H$. If $H$ is a cycle and every vertex in $F$ is adjacent to two vertices in $H$, then $H$ is a twin-free dependent triplet. If $H$ is a cycle and there exists a vertex in $F$ adjacent to all vertices in $H$, then there exist two distinct vertices $f_1$ and $f_2$ in $F$ such that each of $f_1$ and $f_2$ is adjacent to two vertices in $H$ and every vertex in $H$ is adjacent to at least one of $f_1$ or $f_2$. Then $H$ is twin-free dependent triplet. If $H$ is a path, since a vertex in $F$ can not force a vertex in $H$, every neighbor of $H$ in $F$ has at least two neighbors in $H$. Then $H$ is a twin-free dependent triplet.

On the converse, let $G$ be a connected graph with twin- free dependent triplet vertices $u$, $v$ and $w$. By Theorem 3.11, $F_\ell(G) \leq |G| - 3$. Let $J = \{u, v, w\}$ and $F = V(G)\backslash J$. Since a vertex in $F$ with neighbors in $J$ has at least two neighbors in $J$, $F$ can not force $J$. Then $F$ is a failed loop zero set and $|F| = |G| - 3$. Thus $F_\ell(G) = |G| - 3$. □

**Example 3.14.** *Let G be the rising sun graph on seven vertices illustrated in Figure 3.16. The vertex set $\{a, b, c\}$ is a triplet in G, so $F_\ell(G) = 4$.*

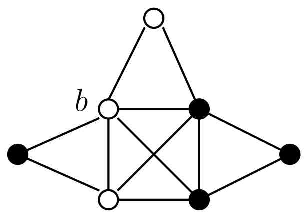



**Figure 3.16.** *The rising sun graph*

**Corollary 3.15.** *If $T$ is a tree, then $F_\ell(T) < |T| - 3$.*

*Proof.* Assume $F_\ell(T) \geq |T| - 3$. Let $F$ be a maximum failed loop zero forcing of $T$. Then two vertices $x_1, x_2 \in V(T)\backslash F$ and a vertex in $F$ adjacent to both $x_1$ and $x_2$ form a cycle in $T$. □

Let $T$ be a tree and $v \in V(T)$ be an exterior general vertex of batch degree two or more. Let $T_1, T_2, \ldots, T_k$ (for $k \geq 3$) be the components of $T - v$. A component $T_i$ of $T - v$ is called a branch if $T_i$ contains a terminal vertex of $v$ in $T$ and $T_i$ does not have any other general vertex. Let $v_{B_1}, v_{B_2}, \ldots, v_{B_k}$ such that $|v_{B_1}| \leq |v_{B_2}| \leq \cdots \leq |v_{B_k}|$ be the branches of $v$ in $T - v$. An exterior general vertex $v$ is called minimum exterior general vertex if $|v_{B_1}| + |v_{B_2}| \leq |u_{B_1}| + |u_{B_2}|$ for every other exterior general vertex $u$.

**Theorem 3.16.** *Let $T$ be a tree , $v \in V(T)$ an exterior general vertex of batch degree two or more. Then*

$$F_\ell(T) \geq |T| - (|v_{B_1}| + |v_{B_2}|).$$

*Proof.* Let $T$ be a tree. Suppose $v \in V(T)$ is an exterior general vertex such that $bat(v) \geq 2$. Then $v$ has at least two branches $v_{B_1}$ and $v_{B_2}$ in $T - v$. Let $F$ be the set of vertices $F = V(T) \backslash (V(v_{B_1}) \cup V(v_{B_2}))$. $F$ is a failed loop zero forcing set, since $v$ is adjacent a non-pendant neighbor in each of $v_{B_1}$ and $v_{B_2}$. Hence, $F_\ell(T) \geq |T| - (|v_{B_1}| + |v_{B_2}|)$. □

Theorem 3.16 immediately yields the following result.

**Theorem 3.17.** *For a tree $T$ and minimum exterior general vertex $v \in V(T)$,*

$$F_\ell(T) = |T| - (|v_{B_1}| + |v_{B_2}|).$$

**Proposition 3.18.** *For a tree $T$, $F_\ell(T) = |T| - 4$ if and only if $T$ is one of the following graphs:*

1. *$T = P_5$ or*
2. *$T$ is the graph obtained from $P_5$ by appending $m$ pendants to the center of $P_5$ or*
3. *$T$ has a minimum exterior general vertex $v$ such that $|v_{B_1}| = |v_{B_2}| = 2$.*

*Proof.* Let $T$ be a tree. Suppose $T = P_5$. Then it is easy to verify that $F_\ell(P_5) = |T| - 4 = 1$. Suppose $T$ is the graph obtained from $P_5$ by appending $m$ pendants to the center of $P_5$. Let $F$ be the center of $P_5$ and its $m$ pendants. Then $F$ is a failed loop zero forcing set and by Corollary 3.15, $F_\ell(T) \leq |F| = |T| - 4$. Thus $F_\ell(T) = |T| - 4$. Suppose $T$ is a tree with a minimum exterior general vertex $u$ of batch degree two or more. Let $F = V(T) \backslash (V(u_{B_1}) \cup V(u_{B_2}))$. Then $F$ is a failed loop zero forcing set and by Corollary 3.15, $F_\ell(T) \leq |F| = |T| - 4$.

For the converse, assume $F_\ell(T) = |T| - 4$. Let $F$ be a maximum failed loop zero forcing set. Suppose $T$ is a caterpillar. Let $s_1$ and $s_2$ be the end-support vertices of $T$. If $\deg(s_1) > 2$, then $F_\ell(T) < |T| - 4$ since $s_1$ has more than one pendant vertex. Hence $\deg(s_1) = 2$. Similarly. $\deg(s_2) = 2$. If $|N(s_1) \cap N(s_2)| = 0$. then there exist two distinct two non-pendant vertices $w_1, w_2 \in T$ such that $s_1$ is adjacent to $w_1$ and $s_2$ is adjacent to

$w_2$. $w_1 \notin F$, since if $w_1 \in F$ there exists a non-pendant neighbor $w_3$ different from $s_1$ (possible $w_3 = w_2$) such that $w_3 \notin F$. Similarly $w_2 \notin F$. Hence $|N(s_1) \cap N(s_2)| = 1$. Therefore $T$ is $P_5$ or $T$ is the graph obtained from $P_5$ by appending $m$ pendants to the center of $P_5$.

Now suppose $T$ is not a caterpillar. By Lemma 2.18, $T$ has a minimum exterior general vertex. Let $u \in V(T)$ be a minimum exterior general vertex, and let $u_{B_1}, u_{B_2} \ldots, u_{B_k}$ for $k \geq 2$, be the branches of $T - u$. Then by Theorem 3.17, $F_\ell(T) = |T| - (|u_{B_1}| + |u_{B_2}|) = |T| - 4$. Hence, $|u_{B_1}| + |u_{B_2}| = 2$. □

**Proposition 3.19.** *For a graph $G$, $F_\ell(G) = 0$ if and only if $G$ is a star or a double star.*

*Proof.* Assume $F_\ell(G) = 0$. Then $Z_\ell(G) = 1$ and every vertex is a loop zero forcing set. Then $G$ is a caterpillar such that every vertex $v \in V(G)$ is either an end-support vertex or a pendant of end-support vertex. Hence, $G$ is a star or a double star.

Conversely, assume $G$ be a star or a double star. Then $Z_\ell(G) = 1$ and every vertex is a loop zero forcing set. Hence, $F_\ell(G) = 0$. □

**Theorem 3.20.** *$F_\ell(G) = 1$ if and only if $G$ is one of the following graphs: two isolated vertices, $K_3$, $C_4$, $C_5$, $K_1(n,m)$, $K_2(n,m)$ or $K_3(n,m)$.*

*Proof.* If $G$ is a pair of isolated vertices, then we can pick at most one of them to be in the set, otherwise the graph is trivially forced. For $K_3$, $C_4$ and $C_5$, every two vertex set is a loop zero forcing set and none of $K_3$, $C_4$ or $C_5$ is a star or double star, thus $F_\ell(K_3) = F_\ell(C_4) = F_\ell(C_5) = 1$.

Assume $F_\ell(G) = 1$. If $G$ is disconnected, then $G$ has two connected components each of order one, otherwise, if $G$ has three or more connected components, taking two components fails to force $G$, and if a component has two or more vertices, taking this component fails to force $G$. Therefore, $G$ is a two isolated vertices. If $G$ is connected and complete, then $G = K_3$. Now assume $G$ is connected and $G \neq K_n$. Then either $Z_\ell(G) = 1$ or $Z_\ell(G) = 2$. If $Z_\ell(G) = 1$, then $G$ is a caterpillar. But since $F_\ell(G) = 1$, $G \neq K_{1,n}$ and $G \neq K(n,m)$, $G$ has at least one internal vertex. Let $u$ be an internal vertex of $G$. If $u$ is an internal support vertex, then the set $\{u \cup P(u)\}$ fails to force $G$, and $F_\ell(G) \geq 2$. So $\deg(u) = 2$. If $G$ has more than three internal vertices, choosing two internal vertices $v$ and $w$ such that $d(v,w) = 3$ fails to force $G$. Thus $G$ has one, two or three internal vertices. Therefore $G$ is $K_1(n,m)$, $K_2(n,m)$ or $K_3(n,m)$. If $Z_\ell(G) = 2$, since $F_\ell(G) \leq Z_\ell(G)$, $G = C_4$ or $G = C_5$. □

**Lemma 3.21.** *For a tree $T$ such that $Z_\ell(T) \geq 2$. Then*

$$F_\ell(T) > Z_\ell(T).$$

*Proof.* Assume $T$ is a tree such that $Z_\ell(T) \geq 2$. Then by Proposition 2.12, there are $T_1, T_2, \ldots, T_{Z_\ell(T)}$ vertex disjoint caterpillars that cover all vertices of $T$. Let $v$ be a pendant vertex of $T_1$, $u$ be a vertex in $T_1$ such that $u$ is adjacent to a vertex in $T_i$ (for $i \neq 1$) and the distance between $u$ and $v$ is minimum. Suppose $T_1$ and $T_2$ are adjacent at vertex $u$. Let $S = \{v\} \cup (V(T_3) \cup V(T_4) \cup \cdots \cup V(T_{Z_\ell(T)}))$. $S$ is a failed loop zero forcing set, since $u$ has two neighbors in $T_1$ and $T_2$, not in $S$. Note $T_i$ has at least two vertices. So $|S| > Z_\ell(T)$. Therefore, $F_\ell(T) > Z_\ell(T)$. □

**Observation 3.22.** *For a caterpillar $T$, $F_\ell(T) \geq 2$ if and only if $T$ has a central support vertex or $T$ has two central vertices of distance more than two.*

**Proposition 3.23.** *For a tree $T$, $F_\ell(T) = Z_\ell(T)$ if and only if $F_\ell(T) = 1$.*

*Proof.* Assume $F_\ell(T) = 1$ Then by Theorem 3.20, $T$ is one of the following caterpillars $K_1(n, m)$, $K_2(n, m)$ or $K_3(n, m)$. By Theorem 2.8 $Z_\ell(T) = 1$. For the converse, assume $F_\ell(T) = Z_\ell(T)$. If $Z_\ell(T) \geq 2$, then by Lemma 3.21, $F_\ell(T) > Z_\ell(T)$. If $F_\ell(T) = 0$, then by Theorem 3.20, $T$ is either $K_{1,n}$ or $K(n, m)$, but both $K_{1,n}$ and $K(n, m)$ has a loop zero forcing set of one. Therefore, $F_\ell(T) = 1$. □

**Definition 3.24.** Let $P_r$ be the path induced by the internal vertices of $K_r(n, m)$. Define $K_r^*(n, m)$ to be the class of graphs obtained from $K_r(n, m)$ by appending a pendant vertex to at least a vertex in $P_r$ with eccentric at most two in $P_r$.

**Theorem 3.25.** *For a tree $T$, $F_\ell(T) = 2$ if and only if $T$ is a tree in one of the following classes:$K_4(n, m)$, $K_5(n, m)$, $K_6(n, m)$, $K_1^*(n, m)$, $K_2^*(n, m)$, $K_3^*(n, m)$, $K_4^*(n, m)$ or $K_5^*(n, m)$.*

*Proof.* Assume $T$ is a tree in one of the following classes:$K_4(n, m)$, $K_5(n, m)$, $K_6(n, m)$, $K_1^*(n, m)$, $K_2^*(n, m)$, $K_3^*(n, m)$, $K_4^*(n, m)$ or $K_5^*(n, m)$. Then it is obvious that $F_\ell(T) = 2$.

Assume $F_\ell(T) = 2$. Then by Lemma 3.21 and Observation 3.22, $T$ is a caterpillar with either at least one internal support vertex or two internal vertices of distance more than two. If $T$ has an internal support vertex of degree four or more, then $F_\ell(T) \geq 3$. So, an internal support vertex of $T$ has only one pendant neighbor. Let $S$ be the set of internal vertices of $T$. Then induced subgraph $T[S]$ of $T$ is a path, otherwise $T$ has more than end-support vertices and $T$ is not a caterpillar. Assume $|S| \leq 3$. Then $T$ has at least one internal support vertex. A internal support vertex and its pendant fail to force $T$. Therefore $T$ is $K_1^*(n, m)$, $K_2^*(n, m)$ or $K_3^*(n, m)$. Assume $|S| = 4$. If $T$ has no internal support vertex, then the end-vertices of $T[S]$ fail to force $T$. Thus $T = K_4(n, m)$. If $T$ has internal support vertices, then $T$ can not have an internal support vertex at the ends of $T[S]$, otherwise the two end-vertices of $T[S]$ and a pendant of an end-vertex of $T[S]$ fail to force $T$ and $F_\ell(T) \geq 3$. Therefore $T = K_4^*(n,m)$. Assume $|S| = 5$. If $T$ has no internal support vertex, then $T = K_5(n,m)$. If $T$ has internal support vertices, then $T$ can not have an internal support vertex neither at the ends nor end-supports of $T[S]$, otherwise an end-vertex of $T[S]$, a support vertex of the other end-vertex of $T[S]$ and a pendant of one of these vertices fails to force. Therefore $T = K_5^*(n,m)$. Assume $|S| = 6$. Then a vertex of $T[S]$ is not an internal support vertex of $T$, otherwise an internal support vertex and its pendant, and other internal vertex of distance from the internal support vertex, fails to force $T$. Therefore $T = K_6(n,m)$. Assume $|S| \geq 7$. Then the two end-vertices of $T[S]$ and a vertex of distance three from one of the end-vertices of $T[S]$ fails to force $T$, contradicting the assumption that $F_\ell(T) = 2$. □

**Theorem 3.26.** *For a graph $G$, $F_\ell(G) = 2$ if and only if $G$ is a graph in one of the following classes: $K_4(n,m)$, $K_5(n,m)$, $K_6(n,m)$, $K_1^*(n,m)$, $K_2^*(n,m)$, $K_3^*(n,m)$, $K_4^*(n,m)$, $K_5^*(n,m)$ or a graph in Figures 3.17, 3.18, 3.19, 3.20, and 3.21.*

*Proof.* For the converse, assume $F_\ell(G) = 2$. If $G$ is disconnected, then by Theorem 3.9 $G = \bar{K}_3$. Suppose $G$ is disconnected and $G \neq \bar{K}_3$. If $G$ has more than two components, then $G$ has an edge between two vertices in some component. But taking two components with least one edge fails to force $G$ and $F_\ell(G) > 2$. So $G$ has two components $H_1$ and $H_2$. If $|H_i| \geq 3$ for $i = 1, 2$, then $|H_i|$ fails to force $G$ and $F_\ell(G) > 2$. Thus $|H_i| \leq 2$. Let $|H_1| \geq |H_2|$. Then either $|H_1| = |H_2| = 2$ or $|H_1| = 2$ and $|H_2| = 1$. Therefore $G = G_2$ or $G = G_3$ in Figure 3.17.

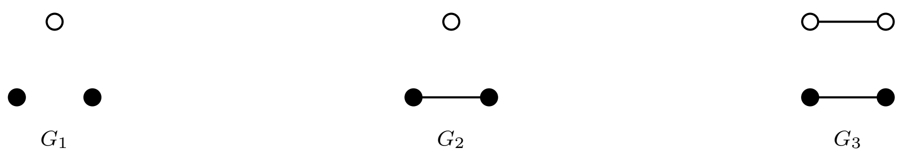
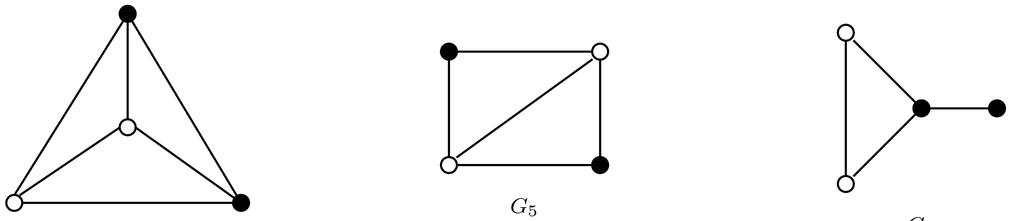



**Figure 3.17.** *The 3 disconnected graphs with $F_\ell(G) = 2$*

If $G$ is a connected and a tree, then by Theorem 3.25, $G$ is a tree in one of the classes : $K_4(n,m)$, $K_5(n,m)$, $K_6(n,m)$, $K_1^*(n,m)$, $K_2^*(n,m)$, $K_3^*(n,m)$, $K_4^*(n,m)$ or $K_5^*(n,m)$. Now assume $G$ is a connected and has a cycle. Then $|G| \geq 4$.

Assume $|G| = 4$. By Observation 3.10, $G$ has a twin. Let $S' = \{u, v\}$ be a twin of $G$. Then every vertex in $V(G)\backslash S'$ is completely adjacent or completely not adjacent to $S'$. If $G[V(G)\backslash S']$ is connected and every vertex in $V(G)\backslash S'$ is completely adjacent to $S'$, then $G = G_4$ Figure 3.18. If $G[V(G)\backslash S']$ is disconnected and every vertex in $V(G)\backslash S'$ is completely adjacent to $S'$, then $G = G_5$ Figure 3.18. If a vertex $w$ in $V(G)\backslash S'$ is completely non-adjacent to $S'$ , then there exist another vertex $w'$ in $V(G)\backslash S'$ adjacent to $w$ such that $w'$ is completely adjacent to $S'$. Thus $G = G_6$ in Figure 3.18.



$G_4$ $G_5$ $G_6$

**Figure 3.18.** *The 3 connected graphs of order 4 with* $F_\ell(G) = 2$

Assume $|G| = 5$. let $S = \{u, v\}$ be a maximum failed loop zero forcing set of $G$. The induced subgraph $G[V(G)\backslash S]$ is an either a path or a cycle. Suppose $u$ and $v$ are adjacent and every vertex in $S$ has a neighbor in $V(G)\backslash S$, then $u$ and $v$ has at least one common neighbor in $V(G)\backslash S$. If every vertex is adjacent to all vertices in $V(G)\backslash S$, then a vertex in $S$ and a vertex in $V(G)\backslash S$ of degree four in $G$ form a twin. So, $u$ and $v$ has at most two common neighbors in $V(G)\backslash S$.

Suppose $G[V(G)\backslash S]$ is a path. If $u$ and $v$ has two common neighbors in $V(G)\backslash S$, then $N[u] = N[v]$ or $N[u] \neq N[v]$. If $N[u] = N[v]$ then $u$ and $v$ is a twin and $F_\ell(G) = 3$. If $N[u] \neq N[v]$, let $w$ be the vertex such that $w \notin N[u] \cap N[v]$. If $w$ is an end-vertex of the path $G[V(G)\backslash S]$, then the other end-vertex of $G[V(G)\backslash S]$ and a vertex in $S$ not adjacent to $w$ form a twin and $F_\ell(G) = 3$. Hence, the common neighbors of $u$ and $v$ are the two ends of the path $G[V(G)\backslash S]$. Therefore $G$ is isomorphic to graph $G_{10}$ in Figure 3.19. If $u$ and $v$ has one common neighbor, let $x$ be the common neighbor. If $x$ is an end-vertex of $G[V(G)\backslash S]$, then $x$ and a vertex in $S$ not adjacent to the other end-vertex of $G[V(G)\backslash S]$ form a twin and $F_\ell(G) = 3$. Hence, $x$ is not an end-vertex of $G[V(G)\backslash S]$. Therefore, $G$ is isomorphic to graph $G_{13}$ in Figure 3.19.

Suppose $G[V(G)\backslash S]$ is a cycle. If $u$ and $v$ has two common neighbors in $V(G)\backslash S$, let $w_1, w_2 \in V(G)\backslash S$ be the common neighbors of $u$ and $v$. Then $w_1$ and $w_2$ form a twin and $F_\ell(G) = 3$. So $u$ and $v$ has one common neighbor. Then $G$ is isomorphic to graph $G_{10}$ in Figure 3.19. Now suppose $u$ and $v$ are adjacent and only one vertex in $S$ has neighbors in $V(G)\backslash S$. Let $u$ be the vertex in $S$ adjacent to vertices in $V(G)\backslash S$. If $G[V(G)\backslash S]$ is a cycle, then $u$ is adjacent to either two or three vertices of $V(G)\backslash S$, but any two vertices in $V(G)\backslash S$. adjacent to $u$ form a twin and $F_\ell(G) = 3$. So, $G[V(G)\backslash S]$ is a path. If $u$ is adjacent to all vertices in $V(G)\backslash S$, then $G$ does not have a twin and $G$ is isomorphic to graph $G_9$ in Figure 3.19. If $u$ is adjacent to the end-vertices of $G[V(G)\backslash S]$, then $G$ does not have a twin and $G$ is isomorphic to graph $G_7$ in Figure 3.19. If $u$ is adjacent to two adjacent vertices of $G[V(G)\backslash S]$, then $G$ does not have a twin and $G$ is isomorphic to graph $G_{12}$ in Figure 3.19.

Now suppose $u$ and $v$ are not adjacent. Suppose $G[V(G)\backslash S]$ is a path. If $u$ and $v$ has

two or three common neighbors, then $G$ can has a twin only if $u$ is adjacent to all vertices $V(G)\backslash S$ but $v$ is adjacent to only the two end-vertices of the path $G[V(G)\backslash S]$, other $G$ does not have a twin and vertices in $V(G)\backslash S$ form a triplet . Therefore $G$ is isomorphic to one of the following graphs $G_8, G_9, G_{10}$ or $G_{13}$ in Figure 3.19. If $u$ and $v$ has one common neighbor, then $G$ can has a twin if there exists a vertex $w \in V(G)\backslash S$ such that $w$ is an end-vertex of $G[V(G)\backslash S]$ and $w \notin N(u) \cap N(v)$. Therefore $G$ is isomorphic to $G_{11}$ in Figure 3.19.
Suppose $G[V(G)\backslash S]$ is a cycle. If $u$ and $v$ has two or three common neighbors, then two vertices $w_1$ and $w_2$ both adjacent to $u$ and $v$ form a twin. So $u$ and $v$ has one common neighbor and vertices $V(G)\backslash S$ form a triplet. Therefore $G$ is isomorphic to the graph $G_{13}$ in Figure 3.19.

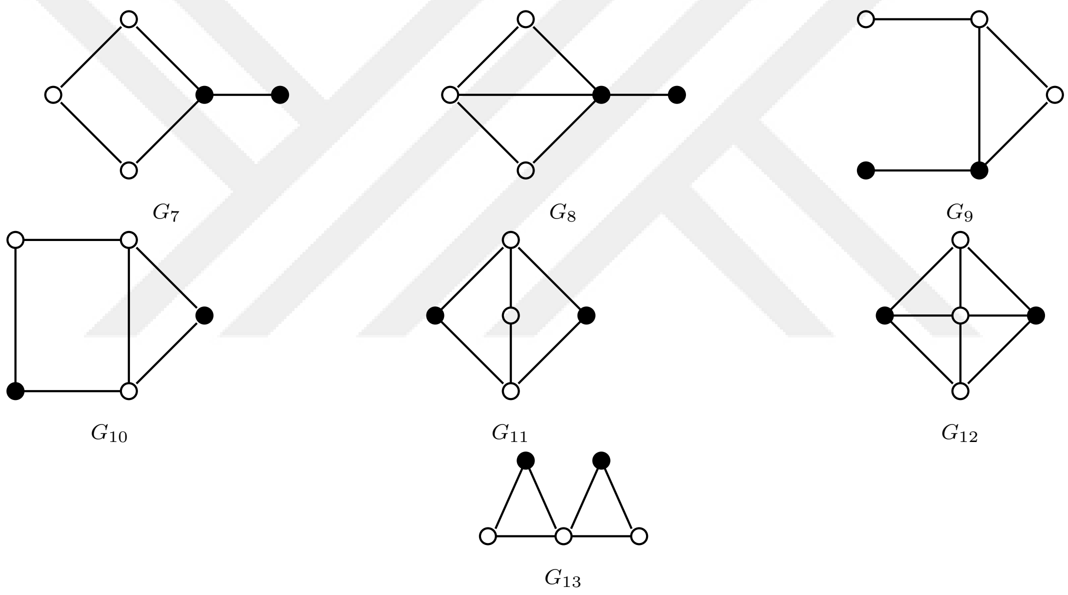



**Figure 3.19.** *The* 7 *graphs of order* 5 *with* $F_\ell(G) = 2$

□

## 3.3. Failed Loop Zero Forcing Number of Some Graph Families

**Proposition 3.27.** *For any path* $P_n$ *of order* $n \geq 5$*,*

$$F_\ell(P_n) = \left\lceil \frac{n-4}{3} \right\rceil.$$

*Proof.* Consider the path graph $P_n$ with $n \geq 5$ vertices. Let $F$ be a maximum failed loop zero forcing set of $P_n$. In this case, neither pendant vertices nor support vertices of $P_n$ can be in $F$, as any set containing a pendant or support vertex would be a loop zero forcing set

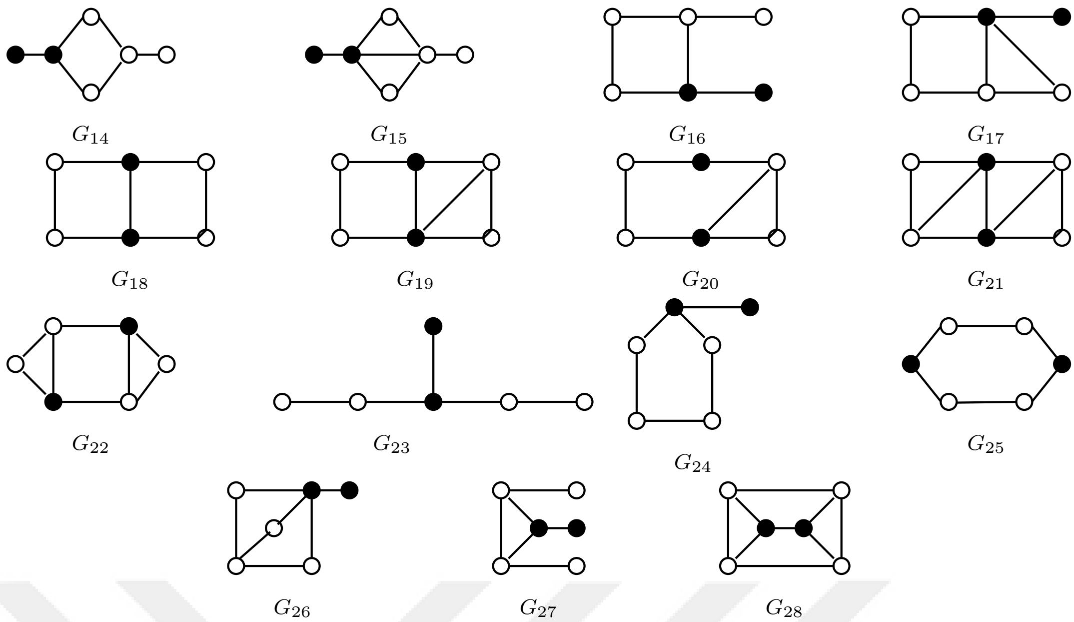



**Figure 3.20.** *The 18 graphs of order 6 with $F_\ell(G) = 2$*

in $P_n$. Additionally, $F$ can not contain pairs of adjacent vertices or pairs of vertices with a common neighbor, as any such pairs would also form a loop zero forcing set in $P_n$. Hence, $F$ can has at most $\lceil \frac{n-4}{3} \rceil$ vertices. Construct $F$ as follows: Start with a degree two vertex adjacent to a support vertex in $P_n$ and add it to $F$. Then, select one furthest neighbor from $F$ out of every three vertices until $F$ reaches the other support vertex, which is not included in $F$. Consequently, $|F| = \lceil \frac{n-4}{3} \rceil$, and it does not force the graph since every vertex in $F$ has exactly two neighbors that are not in $F$, and every vertex not in $F$ has at least one neighbor that is not in $F$. Therefore, $F_\ell(P_n) = \lceil \frac{n-4}{3} \rceil$. □

**Proposition 3.28.** *For any cycle $C_n$,*

$$F_\ell(C_n) = \left\lfloor \frac{n}{3} \right\rfloor .$$

*Proof.* Suppose $F$ is a maximum failed loop zero forcing set of $C_n$. In this case, $F$ can not contain a pair of adjacent vertices or a pair of vertices with one common neighbor, as each such pair would form a loop zero forcing set of $C_n$. Hence, the cardinality of $F$ is at most $|F| \leq \lfloor \frac{n}{3} \rfloor$. Construct $F$, begin with any vertex in $C_n$ and include every other third vertex in $F$. Since every vertex in $F$ has two neighbors that are not in $F$, and every vertex not in $F$ has at least one neighbor in $V(C_n) \backslash F$, $F$ fails to force $C_n$. Therefore, $F_\ell(C_n) = \lfloor \frac{n}{3} \rfloor$. □

**Proposition 3.29.** *Let $G$ be a sunlet graph of order $2n$. Then*

$$F_\ell(G) = 2 \left\lfloor \frac{n}{3} \right\rfloor .$$

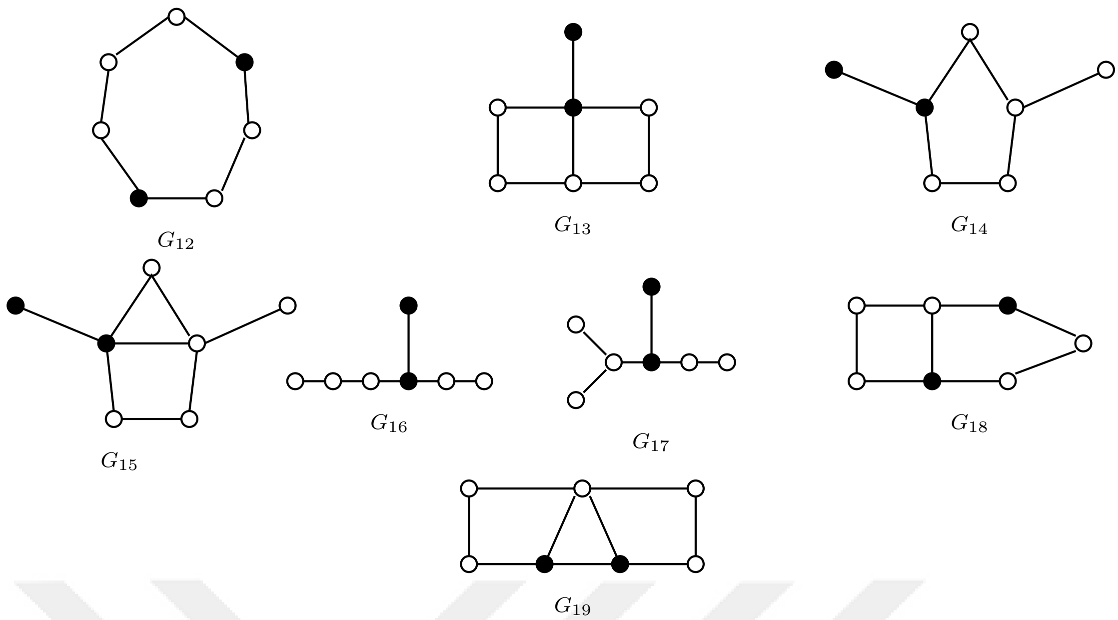



**Figure 3.21.** *The 8 graphs of order 7 with $F_\ell(G) = 2$*

The proof of Proposition 3.29 can be constructed using the same techniques used to prove Proposition 3.28.

A full $m-$ary tree is a rooted tree such that each vertex of degree greater than one has exactly $m$ children and all degree-one vertices are of equal distance (height) to the root.

**Proposition 3.30.** *Let $T$ be a full $m-$ary tree of height $h \geq 2$.*

$$F_\ell(T) = |T| - 2(m+1).$$

*Proof.* Let $T$ be a full $m-$ary tree of height $h \geq 2$. Let $u$ be a support vertex of $T$. Then $u$ has $m$ pendant neighbors. A non-pendant neighbor $v$ of $u$ is a minimum exterior general vertex of a batch degree $m$ such that $|v_{B_i}| = m+1$ for $i = 1, 2, \ldots, m$. By Theorem 3.17, $F_\ell(T) = |T| - 2(m+1)$. □

**Proposition 3.31.** *For $p, q \geq 2$, let $T_{p,q}$ be a balanced spider. Then*

$$F_\ell(T_{p,q}) = q(p-2) + 2\left\lceil \frac{q-4}{3} \right\rceil + 1.$$

*Proof.* Let $T_{p,q}$ be a balanced spider. Then $eg(T) = 1$. Let $v$ be an exterior general vertex of $T$. Hence, $bat(v) = p$ such that $|v_{T_i}| = q$ for $i = 1, 2, \ldots, p$. By Theorem 3.16 $F_\ell(T_{p,q}) \geq q(p-2)+1$. Since $v_{T_i}$ is a path of length $q$. By Theorem 3.16, $F_\ell(v_{T_i}) = \left\lceil \frac{q-4}{3} \right\rceil$. Thus $F_\ell(T_{p,q}) = q(p-2) + 2\left\lceil \frac{q-4}{3} \right\rceil + 1$. □

**Proposition 3.32.** *For $m \geq n \geq 2$,*

$$F_\ell(K_{n,m}) = \begin{cases} m-1 & \text{if } n = 2 \\ n+m-4 & \text{if } n \geq 3. \end{cases}$$

*Proof.* Let $K_{n,m}$ be a complete bipartite graph with partitions $V_1$ and $V_2$ such that $|V_1| = n$ and $|V_2| = m$. If $n = 2$, $K_{n,m}$ does not have a twin and the vertices in $V_1$ and a vertex in $V_2$ is a triplet. Hence, by Theorem 3.13, $F_\ell(K_{n,m}) = m-1$. If $n \geq 3$, $K_{n,m}$ does not have a twin or a triplet. Construct a failed loop zero forcing set $F$ by excluding two vertices in each partition. Then by Theorem 3.13, $F_\ell(K_{n,m}) \leq n+m-4$ but $|F| = n+m-4$, thus $F_\ell(K_{n,m}) = n + m - 4$. □

**Proposition 3.33.** *For a wheel graph $W_n$,*

$$F_\ell(W_n) = \left\lfloor \frac{2n}{3} \right\rfloor.$$

*Proof.* Construct such failed loop zero forcing set $F$ by starting with two adjacent vertices along the cycle. Skip one vertex in the clockwise direction and add the next two adjacent vertices to $F$. Repeat this process until three consecutive vertices are not in $F$. Then $|F| = \lfloor \frac{2n}{3} \rfloor$. Since every vertex in $F$ has two neighbors not in $F$, and every vertex not in $F$ is adjacent to the center $v_0$, $F$ is a failed loop zero forcing set and $F_\ell(W_n) \geq \lfloor \frac{2n}{3} \rfloor$. If $F_\ell(W_n) > \lfloor \frac{2n}{3} \rfloor$, then either $v_0 \in F$ or there is three consecutive vertices along the cycle that are in $F$. In both cases, $F$ is a loop zero forcing set. Thus $F_\ell(W_n) = \lfloor \frac{2n}{3} \rfloor$. □

**Definition 3.34.** The Helm graph $H_n$ is the graph obtained from the wheel $W_n$ by adjoining a pendant vertex to each vertex of degree three.

**Proposition 3.35.** *Let $H_n$ be a Helm graph. Then $F_\ell(H_4) = 5$ and for $n \geq 4$ $F_\ell(H_n) = 2 \lfloor \frac{2n}{3} \rfloor$.*

The proof of Proposition 3.35 can be constructed using the same techniques used to prove Proposition 3.33.

**Theorem 3.36** ([18])**.** *Let G be the Petersen graph. Then*

$$F(G) = 6.$$

**Proposition 3.37.** *Let $G$ be the Petersen graph. Then*

$$F_\ell(G) = 6.$$

*Proof.* Figure 3.22 shows a maximal failed loop zero forcing set of six vertices for the Petersen graph. Then by Theorem 3.36 and Theorem 3.6, $F_\ell(G) = 6$. □

**Proposition 3.38.** *For $n \geq 2$, $F_\ell(P_n \Box P_n) = (n-1)^2$.*

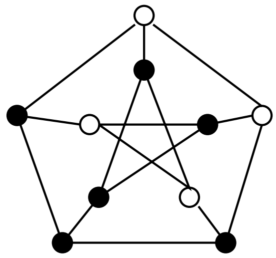


**Figure 3.22.** *A maximum failed loop zero forcing set of Petersen graph*

*Proof.* Construct such failed loop zero forcing set $S$ by putting in the set every vertex in the graph except those vertices $v_{i,j}$ such that $i = j$ or $j = i+1$ (the diagonal vertices and vertices above the diagonals) See Figure 3.23. Take any vertex $v_{i,i}$. Then $v_{i,i}$ is adjacent to four vertices $v_{i,i-1}$, $v_{i,i+1}$, $v_{i-1,i}$ and $v_{i+1,i}$ expect when $i = 1$ or $i = n$. Take any vertex $v_{i,j}$ such that $j = i+1$ for $i = 1, 2, \ldots n-1$. Then $v_{i,j}$ is adjacent to four vertices $v_{i,i}$, $v_{i,j+1}$, $v_{i-1,j}$ and $v_{j,j}$ expect when $i = 1$ or $i = n-1$. $v_{i+1,i}$ is adjacent to $v_{i,i}$ and $v_{i+1,i+1}$. Therefore neither $v_{i+1,i}$ nor $v_{i,i-1}$ will force $v_{i,i}$. $v_{i,i+1}$ is adjacent to $v_{i,i}$ and $v_{i+1,i+1}$. Therefore $N(v_{i,i})$ will force not $v_{i,i}$. Similarly, $v_{i,j+1}$ is adjacent to $v_{i,j}$ and $v_{j,j+1}$. Therefore neither $v_{i-1,j}$ nor $v_{i,j+1}$ will force $v_{i,j}$. $v_{i,i}$ is adjacent to $v_{i-1,i}$ and $v_{i,j}$. Therefore $N(v_{i,j})$ will not force $v_{i,j}$. Thus $S$ is a maximum failed loop zero forcing set. □

The same construction works for $P_n \square P_m$ if $m = n+1$. However, the construction does not work for rectangular grids in general.

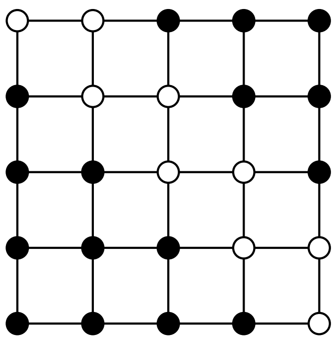


**Figure 3.23.** *A maximum failed loop zero forcing set of* $P_5 \square P_5$

**Proposition 3.39.** *Let $B(n)$ be a barbell graph. Then*

$$F_\ell(B_n) = 2(n-1).$$

*Proof.* Let $u$ and $v$ be two adjacent non-cut vertices of $B(n)$. Then $u$ and $v$ form a twin in $B(n)$. By Theorem 3.11, $F_\ell(B_n) = 2(n-1)$. □

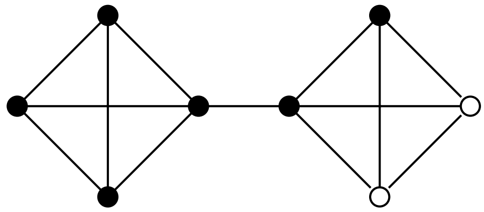


**Figure 3.24.** *The maximum failed loop zero forcing of* $B(4)$

For $2 \leq k_1 \leq k_2 \leq \cdots \leq k_n$, let $B_n = (k_1, k_2, \ldots, k_n)$ be a bouquet of $n \geq 2$ circles $C_1, C_2, \ldots, C_n$ with cut-vertex $v$, where $k_i$ is the number of vertices of $C_i - \{v\}, (1 \leq i \leq n)$. Note that $|V(B_n)| = 1 + \sum_{i=1}^{n} k_i$, where $k_i \geq 2$.

Figure 3.25 shows a bouquet of three circles $B_4 = (2, 3, 5, 6)$.

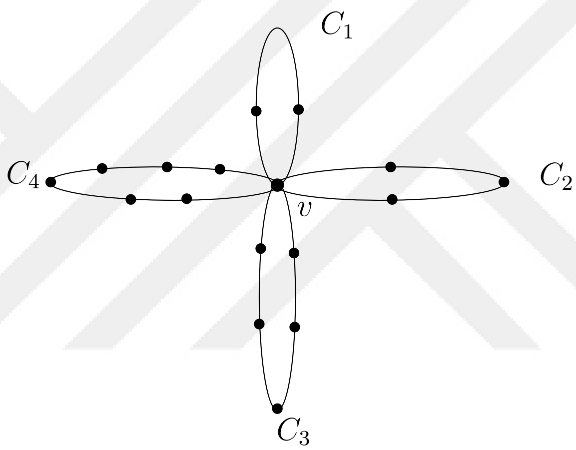



**Figure 3.25.** *A bouquet of three circles* $B_4 = (2, 3, 5, 6)$

**Proposition 3.40.** *Let* $B_n = (k_1, k_2, \ldots, k_n)$ *be a bouquet of n circles with cut-vertex* $v$*. Then*

$$F_\ell(B_n) = \begin{cases} |V(B_n)| - k_1 & , \textit{if } k_1 \leq 4 \\ |V(B_n)| - \left\lfloor \frac{2k_1+4}{3} \right\rfloor & , \textit{if } k_1 > 4. \end{cases}$$

*Proof.* Let $F = V(B_n) - V(C_1 - \{v\})$. Since $F$ can not perform a force, $F$ is a failed loop zero forcing set of $B_n$. So, $F_\ell(B_n) \geq |V(B_n)| - k_i$. The induced subgraph $C_1 - \{v\}$ of $B_n$ is a path. By Proposition 3.27, we get $F_\ell(C_1 - \{v\}) = \left\lceil \frac{k_1-4}{3} \right\rceil$ and $\left\lfloor \frac{k_1-4}{3} \right\rfloor$ vertices are not in the failed loop zero forcing set of $C_1 - \{v\}$. Thus $F_\ell(B_n) = |V(B_n)| - \left\lfloor \frac{2k_i+4}{3} \right\rfloor$. $\square$

Friendship graph $F_k$ of order $2k+1$ is a class of bouquet of $k$ circles of three vertices. By Proposition 3.40 $F_\ell(F_k) = 2k - 1$.

**Proposition 3.41.** *For Banana tree $B_{n,k}$,*

$$F_\ell(B_{n,k}) = n(k-2)+1.$$

*Proof.* Let $B_{n,k}$ be a Banana tree. Suppose $k=2$. Then $B_{n,2} = K_3(n-2,n-2)$. By Theorem 3.20, $F_\ell(B_{n,2}) = 1$. Suppose $k \geq 3$. Then $B_{n,k}$ has only one exterior general vertex $u$ such that $|u_{B_1}| = |u_{B_2}| = \cdots = |u_{B_k}| = n$. By Theorem 3.17, $F_\ell(B_{n,k}) = nk+1-2n = n(k-2)+1$. □

Figure 3.26 shows a maximum failed loop zero forcing set of $B(5,5)$ banana tree.

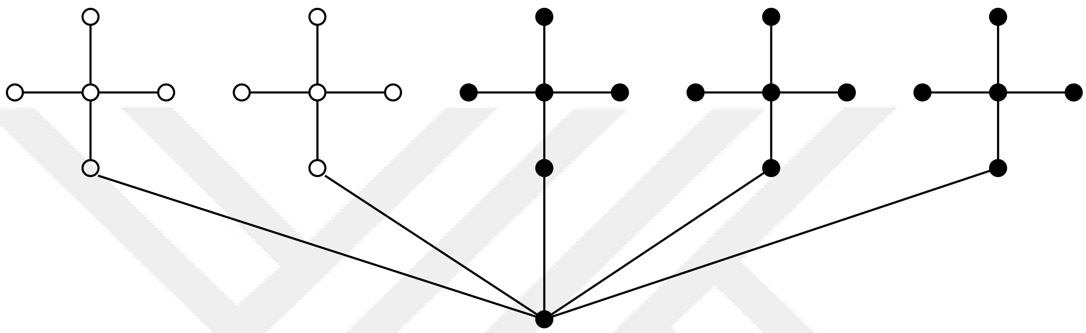


**Figure 3.26.** *An $B(5,5)$-Banana tree with failed loop zero forcing set of* 16

**Proposition 3.42.** *For $n \geq 3$, $k \geq 2$, let $G$ be an $(n,k)$-firecracker tree. Then*

$$F_\ell(G) = n(k-2)+1.$$

*Proof.* Let $G$ be an $(n,k)$-firecracker tree. Suppose $k=2$. Then $G = K_2(n-2,n-2)$. By Theorem 3.20, $F_\ell(G) = 1$. Suppose $k \geq 3$. Let $v$ be a support vertex adjacent to an end-support vertex. Then $v$ is a minimum exterior general vertex such that $|v_{B_1}| = n-1$ and $|v_{B_2}| = n$. By Theorem 3.17, $F_\ell(G) = nk+1-2n = n(k-2)+1$. □

Figure 3.27 shows a maximum failed loop zero forcing set of $(5,5)$ firecracker tree is given .

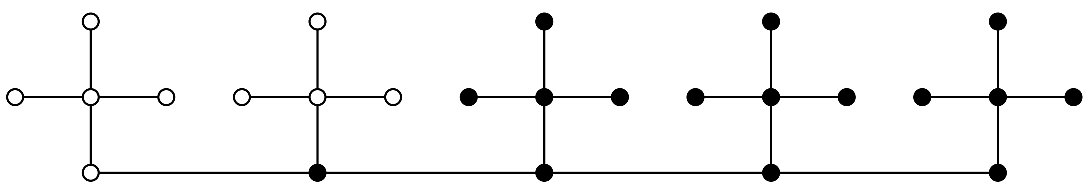


**Figure 3.27.** *An $(5,5)$-firecracker tree with failed loop zero forcing set of* 16

**Definition 3.43.** The Sierpiński gasket graph is defined as follows: $S_n$ has vertex set $\{1,2,3\}^n$ and there is an edge between two vertices $u = (u_1,\dots,u_n)$ and $v = (v_1,\dots,v_n)$ if there is an $h \in [n]$ such that

- $u_j = v_j$ for $j = 1,2,\dots,h-1$
- $u_h \neq v_h$ and
- $u_j = v_h$ and $v_j = u_h$ for $j = h+1,\dots,n$.

Note that, from the definition of Sierpiński gasket graph, $|S_n| = n^3$.

**Proposition 3.44.** *For $n \geq 2$, let $S_n$ be a Sierpiński gasket graph. Then*

$$F_\ell(S_n) = |S_n| - 3.$$

*Proof.* For $n \geq 2$, let $S_n$ be a Sierpiński gasket graph, let $S = \{v_{1\dots12}, v_{1\dots13}, v_{1\dots123}\}$. Then $S$ forms a triplet in $S_n$. $F_\ell(S_n) = |S_n| - 3$ by Theorem 3.13. □

# 4. LOOP THROTTLING

## 4.1. Loop Throttling Number

Throttling zero forcing is a technique used to optimize the size of the set of initially black colored vertices used in the zero forcing process and the number of steps taken to color the entire graph black. There are several different throttling techniques based on the type of color change rule. Standard throttling uses the standard color change rule: If a white vertex $v$ is the only one white neighbor of a black vertex $u$, then change the color of $v$ to black. Positive semidefinite throttling uses the positive semidefinite color change rule: Let $B$ be the set of black vertices and let $W_1, W_2, \ldots, W_k$ be the set of vertices of the components $G - B$. if $v \in W_i$ is the only white neighbor of $u \in B$ in $G[W_i \cup B]$, then change the color of $v$ to black. Skew throttling uses the skew color change rule: If a white vertex $v$ is the only one white neighbor of vertex $u$, then change the color of $v$ to black.

For a graph $G$ and set $S \subseteq V(G)$, define

$$th_\ell(G, S) = |S| + pt_\ell(G, S)$$

and

$$th_\ell(G) = \min_{S \subseteq V(G)} th_\ell(G, S).$$

## 4.2. Trees

**Lemma 4.1.** *Let $T$ be a tree of order $n \geq 3$. Then there exists a loop zero forcing set $S \subseteq V(T)$ that contains no pendant vertices such that $th_\ell(T) = th_\ell(T, S)$.*

*Proof.* Assume $S'$ is a set of vertices such that $th_\ell(T) = |S'| + pt_\ell(T, S')$. Suppose $v \in S'$ is a pendant vertex. Let $u$ be a neighbor of $v$. If $u \in S'$, then $v$ is the only pendant neighbor of $u$ and every neighbor of $u$ except $v$ have at least one neighbor not in $S'$, otherwise $S'' = S' \backslash \{v\}$ is a loop zero forcing set such that $|S''| < |S'|$ and $pt_\ell(T, S'') < pt_\ell(T, S')$ contradicting the initial assumption. Then $S = S' \backslash \{v\}$ is a loop zero forcing set such that $pt_\ell(T, S) = pt_\ell(T, S') + 1$. Similarly, if $u \notin S'$, $v$ is the only pendant neighbor of $u$ and every neighbor of $u$ except $v$ have at least one neighbor not in $S'$, otherwise $S'' = (S' \backslash \{v\}) \cup \{u\}$ is a loop zero forcing set such that $|S''| = |S'|$ but $pt_\ell(T, S'') < pt_\ell(T, S')$ again contradicting the initial assumption. Then $S = (S' \backslash \{v\}) \cup \{u\}$ is a loop zero forcing set such that $pt_\ell(T,S) = pt_\ell(T,S')+1$. Repeat this pendant removal process as needed. □

**Corollary 4.2.** *Let $T$ be a tree, $S$ is a loop zero forcing set such that $th_\ell(T) = th_\ell(T,S)$. Then a pendant vertex containing in $S$ if and only if it is the only pendant neighbor of a support vertex.*

**Proposition 4.3.** *Let $G$ be a graph. Then $th_\ell(G) = Z_\ell(G) + pt_\ell(G)$ if $pt_\ell(G) \leq 2$.*

*Proof.* It is obvious when $pt_\ell(G) \leq 1$. Assume $pt_\ell(G) = 2$. For any loop zero forcing set $S$ with $Z_\ell(G) < |S| < |G|$, $pt_\ell(G,S) \geq 1$. Thus $pt_\ell(G,S) \geq Z_\ell(G) + 2$ completing the proof. □

## 4.3. Extreme Loop Throttling

**Proposition 4.4.** *For $q \geq p \geq 1$, let $K_{p,q}$ be a complete bipartite graph. Then $Z_\ell(K_{p,q}) = p$ and $pt_\ell(K_{p,q}) = 1$.*

The proof is direct consequences of the definitions.

**Observation 4.5.** *For any graph $G$, $th_\ell(G) = 1$ if and only if $G$ is the trivial graph (i.e., has only one vertex).*

**Theorem 4.6.** *For a graph $G$, $th_\ell(G) = 2$ if and only if $G$ is a star or two isolated vertices.*

*Proof.* Assume $th_\ell(G) = 2$. Then either $Z_\ell(G) = 1$ and $pt_\ell(G) = 1$ or $Z_\ell(G) = 2$ and $pt_\ell(G) = 0$. If $Z_\ell(G) = 1$ and $pt_\ell(G) = 1$, then by Theorem 2.8 and Proposition 4.4, $G$ is a star. If $Z_\ell(G) = 2$ and $pt_\ell(G) = 0$, then $|V(G)| = 2$. So $G$ is two isolated vertices. For the converse, let $G$ be a star. Then by Theorem 2.8 and Proposition 4.4, $Z_\ell(G) = 1$ and $pt_\ell(G) = 1$. Now let $G$ is isolated vertices, then $Z_\ell(G) = 2$ and $pt_\ell(G) = 0$. □

**Proposition 4.7.** *Let $G$ be a graph with $Z_\ell(G) \leq 2$ and $pt_\ell(G) = 1$. Then there exists an efficient loop zero forcing set $S$ such that every vertex $v \in V(G)\backslash S$, $N(v) \subseteq S$.*

*Proof.* This is immediate if $Z_\ell(G) = 1$, so assume $Z_\ell(G) = 2$. By Proposition 2.51, $G$ have at least one cycle. Let $S = \{u_1, u_2\}$ be a loop zero forcing set such that $pt_\ell(G,S) = 1$. If both $u_1$ and $u_2$ perform a force, then $G = C_4$. Let $S' = V(G)\backslash S$. Then $|S'| = 2$, $pt_\ell(G,S') = 1$ and every $u \in S$, $N(u) \subseteq S'$. If only one vertex of $S$ performs force, say $u_1$, then $G = K_3$ or $G$ is a tree. If both $u_1$ and $u_2$ does not perform a force, then every vertex $v \in V(G)\backslash S$ is a pendant of $u_1$ or $u_2$, or $v$ is adjacent to both $u_1$ and $u_2$ completing the proof. □

Define $\mathcal{G}_1$ and $\mathcal{G}_2$ be the class of graphs of order greater than three obtained from $P_2$ and $P_3$, respectively, by adding

1. $r \geq 0$ degree two vertices adjacent to an end vertex of $P_2$ or $P_3$
2. $p \geq 0$ leaves adjacent to an end vertex of $P_2$ or $P_3$

such that $p + r \geq 1$.

**Theorem 4.8.** *For a graph $G$, $th_\ell(G) = 3$ if and only if $G$ is one of the given graphs*

1. *$G$ is disconnected, and*
   (a) *$G$ is $\overline{K_3}$, or*
   (b) *$G$ have two components, each component is a star or a trivial graph, and at least one component is a star.*
2. *$G$ is connected, $G \neq P_3$, and*
   (a) *$G \in \mathcal{G}_1$, or*
   (b) *$G \in \mathcal{G}_2$.*

*Proof.* It is easy to show that if $th_\ell(G) = 3$ if $G$ is one of the given graphs. Assume $th_\ell(G) = 3$. Let $S$ be a loop zero forcing set such that $th_\ell(G, S) = th_\ell(G)$. Note that $|S| \leq 3$, $pt_\ell(G, S) \geq 0$ if $S$ is an efficient loop zero forcing set and $pt_\ell(G, S) \geq 1$ if $S$ is not efficient loop zero forcing set. Suppose $S$ is an efficient loop zero forcing set. Then $|S| \in \{1, 2, 3\}$, and

1. If $|S| = 3$, then $pt_\ell(G, S) = 0$. By Observation 2.48, $G = \overline{K_3}$.
2. If $|S| = 2$, then $G$ is a graph on two parallel caterpillars such that $pt(G, S) = 1$. Let $S = \{u, v\}$. If $|S| > |V(G)\backslash S|$ then $G = K_3 \in \mathcal{G}_2$. If $|S| = |V(G)\backslash S|$ then $G = C_4 \in \mathcal{G}_2$ and $|S| < |V(G)\backslash S|$ then $G \in \mathcal{G}_1$ or $G \in \mathcal{G}_2$.
3. If $|S| = 1$, then $G$ is a caterpillar. By Corollary 2.53, there is no such graph.

Now suppose $S$ is not efficient loop zero forcing set. Then $|S| \leq 2$ and $Z_\ell(G) = 1$. $G$ is a caterpillar such that every vertex in $V(G)\backslash S$ is adjacent to at least one vertex in $S$. Thus $G \in \mathcal{G}_1$ if $G[S]$ is connected and $G \in \mathcal{G}_2$ if $G[S]$ is disconnected. □

**Corollary 4.9.** *For a tree $T$, $th_\ell(T) = 3$ if and only if $T = K(n, m)$ or $T = K_1(n, m)$.*

**Theorem 4.10.** *For a graph $G$, $th_\ell(G) = 4$ if and only if $G$ is one of the given graphs*

1. *$G$ is disconnected, and*

    (a) *$G$ is $\overline{K_4}$, or*

    (b) *$G$ have three components, each component is a star or a trivial graph, and at least one component is a star, or*

    (c) *$G$ have two components, one component is $\mathcal{G}_1$ or $\mathcal{G}_2$ and the other component is a star or trivial graph.*

2. *$G$ is connected, and*

    (a) *$G$ have a set of three vertices $W = \{w_1, w_2, w_3\}$ such that $W$ dominates $G$, every vertex in $W$ have a neighbor in $V(G)\backslash W$ and two distinct vertices in $V(G)\backslash W$ are not adjacent , or*

    (b) *$G = Y_3$.*

*Proof.* It is easy to show that if $th_\ell(G) = 4$ if $G$ is one of the given graphs. Assume $th_\ell(G) = 4$. Suppose $S$ is a loop zero forcing set such that $th_\ell(G, S) = th_\ell(G)$. Suppose $G$ is disconnected. Let $G_1, G_2, \ldots, G_k$ be the connected components of $G$. Note that $2 \leq k \leq 4$. If $k = 4$, then by Observation 2.48, $G = \overline{K_4}$. If $k = 3$, then each component $G_i$ has only vertex in $S$, and $pt_\ell(G_i, S) \leq 1$ and there exists at least one component $G_i$ with $pt_\ell(G_i, S) = 1$. Thus $G_i$ is a star or trivial graph and $G_i$ is a star if $pt_\ell(G_i, S) = 1$. If $k = 2$, then each component $G_i$ has at least one vertex in $S$, and $pt_\ell(G_i, S) \leq 2$ and there exists at least one component $G_i$ with $pt_\ell(G_i, S) = 2$. Suppose $|G_i \cap S| = 2$, then $G_i \in \mathcal{G}_1$ or $G_i \in \mathcal{G}_2$, and $G_2$ is a star or trivial.

Now suppose $G$ is connected. Then $|S| \in \{1, 2, 3\}$. If $|S| = 1$, then $pt_\ell(G, S) = 3$. By Proposition 2.52 $G$ is a double star. But by Corollary 4.9, $th_\ell(G) = 3$ contradicting the assumption that $th_\ell(G) = 4$. If $|S| = 2$ and $S$ is an efficient loop zero forcing set of $G$, then by Corollary 4.9 and Corollary 2.53, $G$ is a graph on two parallel caterpillars that is not a tree. Let $S = \{v_1, v_2\}$. Since $pt(G, S) = 2$ there exists distinct vertices $u_1$ and $u_2$ such that $v_1 \to u_1$ and $v_2 \to u_2$.

1. If both $v_1$ and $v_2$ does not perform any force in around one, let $W = \{v_1, v_2, u_1\}$.

2. If $v_1$ does not perform any force at round one and $v_2$ performs a force at round one, let $W = \{v_1, u_1, u_2\}$.

3. If $G$ is not isomorphic to graphs in Case 1 and 2, let $W$ be three independent vertices of $G$.

The graphs in Cases $1, 2$ and $3$ satisfies $2(a)$. If $|S| = 2$ and $S$ is not efficient loop zero forcing set of $G$, then $G$ is a caterpillar. Let $v_1$ and $v_2$ be the end-support vertices of $G$. Now let $W = \{v_1, v_2, u_1\}$. Then $G$ satisfies $2(a)$. If $|S| = 3$, let $S = W$. Then $G$ satisfies $2(a)$ or $G = Y_3$. □

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