Generalized Theories of Gravity and Supergravity in Three Dimensions

by

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Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a doctoral dissertation by

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To my beloved family

and

dearest friends...

ABSTRACT

Generalized Theories of Gravity and Supergravity in Three Dimensions
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Some generalizations inspired by the high and low energy modifications of general relativity have been studied in three dimensions. In particular modifications of massive gravity models have been considered. These generalizations have been studied using language of exterior algebra on Riemann-Cartan-Weyl space-times. All generalizations are defined in terms of a Lagrangian and variational field equations have been found using a first order variational formalism. Some exact solutions and properties of these models have been discussed.

ÖZETÇE

Üç Boyutlu Genelleşmiş Kütleçekimi ve Süperkütleçekimi Kuramları
Cem Yetişmişoğlu
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Genel görelilik kuramının yüksek ve düşük enerjilerdeki çeşitli modifikasyonları, üç boyutta incelenmiştir. Özellikle kütleli kütleçekimi kuramlarının genellemeleri çalışılmıştır. Çalışmaların formülasyonunda Riemann-Cartan-Weyl uzay-zamanları uzerinde dış cebir lisanı kullanılmıştır. Bütün genellemeler Lagrange fonksiyonelleri cinsinden ifade edilmiştir ve alan denklemleri birinci mertebeden varyasyonel prensip kullanılarak hesaplanmıştır. Genellemelerin bazı analitik çözümleri ve özellikleri tartışılmıştır.

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TABLE OF CONTENTS

ABBREVIATIONS

AdS	Anti de-Sitter
CS	Chern-Simons
CTMS	Cosmological Topologically Massive Supergravity
GR	General Relativity
MMG	Minimally Massive Gravity
NMG	New Massive Gravity
TMG	Topologically Massive Gravity
TMS	Topologically Massive Supergravity

Chapter 1

INTRODUCTION

General relativity, as proposed in 1916 by Einstein, has been verified by many solar tests such as advance of the precession of the perihelion of Mercury, deflection of light rays by massive objects, and lately the gravitational wave observations made by LIGO [Abbott et al., 2016]. However despite this success, as an effective field theory, in the extreme limits of both lower and higher energies, it still has some shortcomings. On the one hand, at the cosmic scales, it cannot explain the late time accelerated expansion of the universe with the given observed matter distribution [Astier and Pain, 2012]. In addition, observational rotation curves of disc galaxies are not in agreement with that of the theoretical predictions of general relativity [Casertano and Van Gorkom, 1991]. On the other hand, at high energies, it suffers from singularity theorems [Senovilla, 2012] as well as lacking a renormalizable, perturbative quantum field theory description [‘t Hooft and Veltman, 1993]. Consequently, different modifications of general relativity have been considered to tackle these problems.

For modifications at low energies, various alternative approaches have been studied. One of the phenomenological modifications for cosmological observations, leaving general relativity intact, is the extension by an addition of a dark sector which includes a cosmological constant, and non-baryonic cold dark matter [Padmanabhan, 2003]. The cosmological constant acts as a negative pressure dark energy, and cold dark matter does not interact via electromagnetic interactions. Together with a homogeneous, isotropic universe described by the constant (positive) curvature FLRW metric, it models the observed universe. This model is called lambda-cold dark matter, and although it provides a good fit for large set of cosmological data [Ade et al., 2014a, Ade et al., 2014b], it suffers from other problems such as the infamous cosmological constant problem [Weinberg, 1989] and the tension due to discrepancy between the observed and predicted values of the Hubble constant [Riess et al., 2021]. Another modification at long distances is obtained by allowing the graviton to become massive. Due to this reason such models are termed massive gravity models. Having mass, the gravitational fields meditate forces described by a Yukawa-like potential. This weakens the strength of gravitational interactions at larger distances. One of the earliest of such modification was constructed by Pauli and Fierz [Fierz and Pauli, 1939]. This model suffers from pathologies such as vDVZ discontinuity (not yielding general relativity in the limit the mass of the graviton vanishes) [Zakharov, 1970], and a ghost like instability termed Boulware-Deser ghost [Boulware and Deser, 1972].

For an improved behaviour at high energies, numerous modifications have been considered. Some of the important ones that motivated part of the studies in this thesis are quadratic curvature modifications [Utiyama and DeWitt, 1962], scale (Weyl) covariant generalizations [O'Raifeartaigh et al., 1996, Iorio et al., 1997], and supersymmetric models [Deser and Zumino, 1976, Freedman and Van Nieuwenhuizen, 1976]. Quadratic curvature gravity model is a higher derivative generalization which yields a renormalizable theory at the quantum level, however it suffers from ghost modes (Ostrogradsky instabilities). The scale covariant models are obtained by promoting the rigid scale invariance into a local one, yielding a gauge theory of scale transformations. Due to local scale covariance, these models are described by Lagrangians with dimensionless coupling constants. Therefore these models also provide viable alternatives for renormalizability purposes. Finally supersymmetric generalizations of gravity models are called supergravity models. It has been shown that they also have improved renormalizability properties due to having extra symmetries [Deser et al., 1977].

Despite a plethora of different generalizations to general relativity, they seem to have their own deficiencies. Due to this reason, three dimensions has the virtue of providing many interesting toy models to try new ideas without introducing too many difficulties [Carlip, 2003]. This is the main reason that we choose to work in three dimensions.

The organization of the thesis is as follows. In Chapter 2 we start by discussing main three dimensional gravity models (general relativity, topologically massive gravity, minimally massive gravity, and quadratic curvature gravity) models that the following works are based on. Then in Chapter 3 we discuss models given in [Dereli and Yetişmişoğlu, 2016] and [Ünlütürk and Yetişmişoğlu, 2022]. In [Dereli and Yetişmişoğlu, 2016] we study a model which combines topologically, minimally, and new massive gravity models. We derive its field equations both in the presence and absence of torsion, and show that there are exact constant negative curvature and constant torsion solutions to this model. In [Ünlütürk and Yetişmişoğlu, 2022] we study a model which combines topologically massive gravity, topologically massive electrodynamics with the most general non-minimal RF^2-type couplings between the gravitational and electromagnetic fields. We then show that this model admits constant negative curvature background solutions with self-dual electromagnetic fields. In Chapter 4 we discuss locally scale (Weyl) covariant generalizations of all the models noted in Chapter 2 which are studied in [Dereli and Yetişmişoğlu, 2019b] and [Dereli and Yetişmişoğlu, 2019a]. In [Dereli and Yetişmişoğlu, 2019b] we study scale covariant generalizations of topologically and minimally massive gravity models using Riemann-Cartan-Weyl space-times. We show that this way of gauging the scale group is consistent. Afterwards in [Dereli and Yetişmişoğlu, 2019a] we study the Weyl gauging of quadratic curvature gravity and note the complications coming from the quadratic curvature invariants when working on a Riemann-Weyl space-time. Moreover in the last section of Chapter 4, we present an unpublished work on an exact class of self-dual solutions to the scale covariant topologically massive gravity model. We show that these solutions form a non-trivial class. Then in

Chapter 5 we discuss supergravity models based on [Dereli and Yetişmişoğlu, 2021]. We study the supersymmetric generalizations of cosmological general relativity and topologically massive gravity. We derive the exact field equations and study properties of Cotton and Cottino tensors. In Chapter 6 we present our concluding remarks. In Appendix A we outline the definitions of mathematical structures that we use throughout this thesis. In Appendix B we give some useful identities. In Appendix C we derive the solution to the anti-symmetric connection equation that is used throughout the thesis. In Appendix D we explicitly derive the left-invariant 1-forms on AdS_3 making use of the homogeneous space structure. In Appendix E we give the Majorana realization of spinors that we use in Chapter 5.

Chapter 2

THREE DIMENSIONAL GRAVITY MODELS

Throughout this thesis, we use the language of exterior differential forms on Riemann-Cartan-Weyl space-times. We study models defined via an action principle and derive field equations using a first order variational formalism. That is, the basic fields associated to gravitation are co-frame and connection 1-forms and they are varied independently. The independent variations of connection and co-frames are important because we are interested in generalized theories of gravitation where one is not always interested to work in a Riemannian context. Particularly this importance is highlighted when we work on scale covariant, and supergravity models. Moreover, we can accompany first order order variational principle with constraint terms to impose conditions such as setting the torsion to zero. With the constraint terms, the first order variational field equations can be put in agreement with the more conventional second order variational field equations. We denote the variations of an independent variable by a dot over that variable, and we discard possible boundary terms. We now start describing our essential models that we will build upon in the following chapters.

2.1 General Relativity

Three dimensional general relativity (GR) is dynamically trivial. Outside matter sources, it describes constant curvature space-times [Deser et al., 1984, Deser and Jackiw, 1984]. It is equivalent to a Chern-Simons theory where the gauge group is identified with isometries of the model space-time [Witten, 1988, Wise et al., 2009]. In the presence of a negative cosmological constant, it demonstrates non-trivial global features of a four dimensional gravitational model. For asymptotically anti-de Sitter (AdS) boundary conditions, it is dual to a two dimensional conformal field theory at the boundary [Brown and Henneaux, 1986], and it admits a celebrated rotating charged black-hole solution [Banados et al., 1992] whose entropy is proportional to the perimeter of its horizon.

Independent variables that describe our model are the co-frame 1-forms $\{e^a\}$, metric connection 1-forms $\{\Omega^a{}_b\}$, and Lagrange multiplier valued 1-forms $\{\lambda_a\}$. The Lagrangian density 3-form is given by:

$$\mathcal{L}_{\rm GR} = \frac{1}{2K} R^a{}_b \wedge *e_a{}^b + \Lambda * 1 + \lambda_a \wedge T^a \,, \tag{2.1}$$

where K is the three dimensional gravitational constant, and Λ is the cosmological constant. We note that instead of starting with a metric connection 1-forms, we could have started with a general non-metric affine connection, and imposed metricity condition through another set of Lagrange multipliers. It can be shown that this method also yields the same results and for technical ease we will not do this here.

The total variational derivative of $\mathcal{L}_{GR}$ with respect to three independent variables is found to be:

$$\begin{aligned} \dot{\mathcal{L}_{\rm GR}} &= \dot{e}^a \wedge \left\{ \frac{1}{2K} R^b{}_c \epsilon_{ab}{}^c + \Lambda * e_a + D\lambda_a \right\} + \dot{\lambda}_a \wedge T^a \\ &\quad + \dot{\Omega^a{}_b} \wedge \left\{ \frac{1}{2K} \epsilon^a{}_{bc} T^c + \frac{1}{2}(e^b \wedge \lambda_a - e_a \wedge \lambda^b) \right\} . \end{aligned} \tag{2.2}$$

Here a dot over a field variable denotes the variation of that variable. As a result of the Lagrange constraint equation, torsion vanishes, that is $T^a = 0$. Consequently we work with the unique Levi-Civita connection 1-forms, *i.e.*, $\Omega^a{}_b = \omega^a{}_b$ and all curvature quantities and covariant derivatives are taken with respect to Levi-Civita connection. Looking at the connection variation equation we find that the Lagrange multiplier 1-forms vanish identically, $\lambda_a = 0$. Finally, the remaining part is the Einstein field equations:

$$\frac{1}{K} R^{ab} + \Lambda e^{ab} = 0 \,. \tag{2.3}$$

The only solution to Equation (2.3) are the spaces of constant curvatures. This implies that 3D GR is not a dynamical theory. Another way to see this is to express this model equivalently in terms of a Chern-Simons action where the gauge group is identified with the isometry group of 3D constant curvature space-times.

2.2 Topologically Massive Gravity

Topologically Massive Gravity (TMG) [Deser et al., 1982], is a dynamically non-trivial extension of GR obtained by augmenting the Lagrangian density (2.1) by a Lorentz Chern-Simons (CS) term. The Lagrangian of TMG reads:

$$\mathcal{L}_{\mathrm{TMG}} = \frac{1}{\mu}(\Omega^a{}_b \wedge d\Omega^b{}_a + \frac{2}{3}\Omega^a{}_b \wedge \Omega^b{}_c \wedge \Omega^c{}_a) + \frac{1}{2K} R^a{}_b \wedge * e_a{}^b + \Lambda * 1 + \lambda_a \wedge T^a \,. \quad (2.4)$$

Due to the CS term, it is a parity non-invariant extension which involves third derivative orders of the metric coefficients. TMG describes a single helicity, spin-2, massive propagating degree of freedom on a flat or negative curvature bulk. This degree of freedom has positive kinetic energy as long as the theory has the wrong sign for the Einstein-Hilbert term. With the asymptotically AdS_3 Brown-Henneaux boundary conditions, it yields two (chiral) copies of the Virasoro algebra, and one can guarantee that the bulk degree of freedom has positive energy by fine tuning the coupling constants to be at the chiral limit [Li et al., 2008]. At the chiral limit, it is shown that the boundary conformal field theory has log-modes with negative energies which make it non-unitary [Grumiller and Johansson, 2008]. This problem is called the bulk-boundary clash problem.

The total variation of the Lagrangian density (2.4) yields

$$\begin{aligned} \dot{\mathcal{L}_{\mathrm{TMG}}} &= \dot{e}^a \wedge \left\{ \frac{1}{2K} R^b{}_c \epsilon_{ab}{}^c + \Lambda * e_a + D\lambda_a \right\} + \dot{\lambda_a} \wedge T^a \\ &+ \dot{\Omega^a}{}_b \wedge \left\{ \frac{2}{\mu} R^b{}_a + \frac{1}{2K} \epsilon^a{}_{bc} T^c + \frac{1}{2}(e^b \wedge \lambda_a - e_a \wedge \lambda^b) \right\} . \end{aligned} \quad (2.5)$$

The Lagrange multiplier 1-forms makes sure that we are working with the Levi-Civita connection 1-forms. The equations coming from the variation of the connection can be put in the form of Equation (C.1) and it's solution involves the Schouten 1-forms $\{Y_a := Ric_a - (1/4)Re_a\}$ as

$$\lambda_a = -\frac{4}{\mu} Y_a \,. \quad (2.6)$$

Finally putting this result back in the co-frame variation equation, we obtain the Einstein field equations

$$\frac{1}{K}R^{ab} + \Lambda e^{ab} - \frac{4}{\mu}C_a = 0\,, \tag{2.7}$$

where $\{C_a := DY_a\}$ are the Cotton 2-forms.

2.3 Minimally Massive Gravity

Minimally Massive Gravity (MMG) model is introduced as an alternative to the TMG model which solves the bulk-boundary clash problem [Afshar et al., 2014, Bergshoeff et al., 2014]. Similar to TMG, it describes a single, massive, spin-2 degree of freedom at the bulk [Alishahiha et al., 2014]; and differently from TMG, the boundary conformal field theory around AdS_3 bulk is unitary. Its chiral limit and conserved charges are studied in [Tekin, 2014]. Due to lack of a Bianchi identity, consistency of its field equations in relation to certain solutions of TMG are investigated in [Altas and Tekin, 2015]. For the action formulation of MMG in Riemann-Cartan space-times we refer to [Baykal, 2014]:

$$\begin{aligned}\mathcal{L}_{\text{MMG}} = {} & \frac{1}{\mu}(\Omega^a{}_b \wedge d\Omega^b{}_a + \frac{2}{3}\Omega^a{}_b \wedge \Omega^b{}_c \wedge \Omega^c{}_a) + \frac{1}{2K}R^a{}_b \wedge *e_a{}^b + \Lambda * 1 \\ & + \lambda_a \wedge T^a + \frac{\nu}{2}\lambda_a \wedge \lambda_b \wedge *e^{ab}\,.\end{aligned} \tag{2.8}$$

The total variation of the Lagrangian density (2.8) yields:

$$\begin{aligned}\dot{\mathcal{L}_{\text{MMG}}} = {} & \dot{e}^a \wedge \left\{\frac{1}{2K}\epsilon_{abc}R^{bc} + \Lambda * e_a + D\lambda_a + \frac{\nu}{2}\epsilon_{abc}\lambda^b \wedge \lambda^c\right\} \\ & + \dot{\Omega}^{ab} \wedge \left\{\frac{2}{\mu}R_{ba} + \frac{1}{2K}\epsilon_{abc}T^c + \frac{1}{2}(e_b \wedge \lambda_a - e_a \wedge \lambda_b)\right\} \\ & + \dot{\lambda}_a \wedge \left\{T^a + \nu\lambda_b \wedge *e^{ab}\right\}.\end{aligned} \tag{2.9}$$

Notice that due to to the non-linear term in the 1-form field $\{\lambda_a\}$, there is a dynamical space-time torsion which can be solved as:

$$T^a = -\nu\lambda_b \wedge *e^{ab} \iff K_{ab} = \nu\epsilon_{abc}\lambda^c\,. \tag{2.10}$$

Putting this back into the connection variation equation, we can express it in the form of Equation (C.1), where

$$\Sigma_{ab} = -\left(\frac{1}{2} - \frac{\nu}{2K}\right)^{-1} \frac{2}{\mu} R_{ab} \,. \tag{2.11}$$

Its solution using Equation (C.4) involves the Schouten tensor $\{Y_a = Ric_a - 1/4Re_a\}$ as

$$\lambda_a = -\left(\frac{1}{2} - \frac{\nu}{2K}\right)^{-1} \frac{2}{\mu} Y_a \,. \tag{2.12}$$

Finally putting this back into the co-frame variation equations, we arrive at the Einstein field equations:

$$\frac{1}{2K}\epsilon_{abc}R^{bc} + \Lambda * e_a - \left(\frac{1}{2} - \frac{\nu}{2K}\right)^{-1} \frac{2}{\mu} C_a + \left(\frac{1}{2} + \frac{\nu}{2K}\right)^{-2} \frac{4}{\mu^2}\epsilon_{abc}Y^b \wedge Y^c = 0 \,. \tag{2.13}$$

We again stress that here all the covariant derivatives and curvature quantities in this model are calculated using the torsionful connection $\{\Omega^a{}_b = \omega^a{}_b + K^a{}_b\}$.

2.4 Quadratic Curvature Gravity

Quadratic curvature gravity is a fourth order modification of general relativity by the addition of quadratic curvature invariants:

$$\mathcal{L}_{\text{QCG}} = \frac{1}{2K} R^a{}_b \wedge * e_a{}^b + \Lambda * 1 + \lambda_a \wedge T^a + \kappa_1 R^a{}_b \wedge * R_a{}^b + \kappa_2 Ric^a \wedge * Ric_a + \kappa_3 R^2 * 1 \,. \tag{2.14}$$

where κ_1, κ_2 and κ_3 are coupling constants with dimensions of inverse length. These quadratic curvature modifications can be interpreted as high energy corrections to general relativity. Particularly, this family of Lagrangian densities also cover the Lagrangian density of the New Massive Gravity (NMG) model for the particular values of $\kappa_1 = 0$, $\kappa_2 = 1$ and $\kappa_3 = -3/8$ [Bergshoeff et al., 2009a, Bergshoeff et al., 2009b]. NMG model is a parity invariant model describing two massive, spin-2 degree of freedoms (with both helicities). It is a nonlinear unitary theory which extends the Pauli-Fierz model. The canonical analysis of this model is carried out it [Güllü et al., 2010].

The total variational derivative of $\mathcal{L}_{\text{QCG}}$ with respect to three independent variables is found to be:

$$\begin{aligned}\dot{\mathcal{L}}_{\text{QCG}} = \dot{e}^a \wedge \Big\{ \frac{1}{2K} R^b{}_c \epsilon_{ab}{}^c + \Lambda * e_a + D\lambda_a - \kappa_1 \hat{\tau}_a[R^b{}_c] + \kappa_2 \Big[\iota_a (Ric^b \wedge *Ric_b) \\ + 2\iota_a R_{bc} \wedge \iota^b * Ric^c \Big] + \kappa_3 (2RR^b{}_c \epsilon_{ab}{}^c - R^2 * e_a) \Big\} \\ + \dot{\Omega}^a{}_b \wedge \Big\{ \frac{1}{2K} D * e_a{}^b + e^b \wedge \lambda_a + 2D \Big[\kappa_1 * R_a{}^b - \kappa_2 \iota_a * Ric^b + \kappa_3 R * e_a{}^b \Big] \Big\} \\ + \dot{\lambda}_a \wedge T^a \,, \end{aligned} \tag{2.15}$$

where

$$\hat{\tau}_a[R^b{}_c] = \iota_a R^b{}_c \wedge *R_b{}^c - R^b{}_c(\iota_a * R_b{}^c) \tag{2.16}$$

denotes the stress-energy 2-form of the curvature 2-form coming from the quadratic term proportional to κ_1.

First, due to Lagrange constraint equation, torsion vanishes and we will be working with the unique Levi-Civita connection 1-forms $\{\Omega^a{}_b = \omega^a{}_b\}$. Then, we solve the Lagrange multiplier 1-forms from the connection variation equation. For this, we write the connection variation in the form of Equation (C.1), where

$$\Sigma_{ab} = 4D \left(\kappa_1 * R_{ab} - \kappa_2 \iota_{[a} * Ric_{b]} + \kappa_3 R * e_{ab} \right)$$

is anti-symmetric due to anti-symmetry of the Levi-Civita connection 1-forms. Then, using Equation (C.4) the unique solution for the Lagrange multiplier 1-forms reads:

$$\begin{aligned}\lambda_a = 4\iota^b D \left(\kappa_1 * R_{ba} - \kappa_2 \iota_{[b} * Ric_{a]} + \kappa_3 R * e_{ba} \right) \\ - \left[\iota^{bc} D \left(\kappa_1 * R_{cb} - \kappa_2 \iota_{[c} * Ric_{b]} + \kappa_3 R * e_{cb} \right) \right] e_a \,. \end{aligned} \tag{2.17}$$

Finally, Einstein field equations are determined to be:

$$\begin{aligned}-\frac{1}{K} G_a + \Lambda * e_a + D\lambda_a - \kappa_1 \hat{\tau}[R^b{}_c] + \kappa_3 (2RR^b{}_c \epsilon_{ab}{}^c - R^2 * e_a) \\ + \kappa_2 \left[\iota_a (Ric^b \wedge *Ric_b) + 2\iota_a R_{bc} \wedge \iota^b * Ric^c \right] = 0 \,, \end{aligned} \tag{2.18}$$

where the Lagrange multiplier 1-forms are given by Equation (2.17).

Chapter 3

TWO DIFFERENT EXTENSIONS OF MASSIVE GRAVITY MODELS

3.1 Non-minimally Coupled Electrodynamics and Gravity

In this section we are going to study a model where we extend the TMG model with electrodynamics together with RF^2-type non-minimal couplings between electrodynamics and gravity. Similar toy models in four dimensions has been considered in [Dereli and Sert, 2011c, Dereli and Sert, 2011a, Dereli and Sert, 2011b, Adak et al., 2017, Dereli and Şenikoğlu, 2020] where in general a single RF^2-type interaction is added to the Einstein-Maxwell theory. Although four dimensional Einstein-Maxwell theory, where the electromagnetic field is minimally coupled to gravity, is classically a well-established theory, non-minimal coupling terms between curvature and electrodynamics may be introduced, motivated by perturbative QED on curved backgrounds, in particular coming from vacuum polarization. On the tree level, photons propagate along null geodesics, whereas at the 1-loop level, due to vacuum polarization, a photon may exist as a virtual electron-positron pair. Consequently, photons attain a size at the order of magnitude of the Compton wavelength of the electron. Therefore, photon propagation can be influenced by the space-time curvature where a curved space-time acts as an optically active medium. Such effects can be conveniently described by non-minimal couplings of the electromagnetic field to gravity [Drummond and Hathrell, 1980]. Indeed, the effective action density that arises in QED at the 1-loop level is given by [Drummond and Hathrell, 1980]

$$\frac{\gamma_1}{2} R_{ab} \wedge F^{ab} * F + \frac{\gamma_2}{2} Ric_a \wedge \iota^a F \wedge *F + \frac{\gamma_3}{2} RF \wedge *F + \frac{\gamma_4}{2} d * F \wedge *(d * F) \ ,$$

where the coupling constants $\{\gamma_i\}$ depend on the fine structure constant and mass of the electron.

Motivated by these interactions, in [Dereli and Sert, 2011c, Dereli and Sert, 2011a, Dereli and Sert, 2011b, Adak et al., 2017, Dereli and Şenikoğlu, 2020] the authors consider various extensions of Einstein-Maxwell theory with RF^2-type interaction terms and look for solutions. Particularly these authors consider an $R_{ab} \wedge F^{ab} * F$ coupling term and show that a class of exact non-trivial pp-wave solutions exist [Dereli and Sert, 2011c]; add coupling terms of the form $Y(R)F \wedge *F$ and show that static, spherically symmetric solutions exist; look at exact solutions of anisotropic inflation for the same coupling [Adak et al., 2017]. Finally in [Dereli and Şenikoğlu, 2020], the authors consider $RF \wedge *F$ and $RF \wedge F$ couplings and show that there exists a charge screening solution (with vanishing total charge) for this model. That is, a spherically symmetric, asymptotically de Sitter solution with a constant electromagnetic field 2-form exists, provided that the coupling constant for the non-minimal coupling term is related to the radius of space-time in a certain way.

Here, we extend three dimensional TMG with electrodynamics to a model that contains all of the non-minimal coupling terms coming from QED. It is noteworthy to see similar effects coming from the non-minimal coupling terms of this toy model. We derive the field equations using first order constrained variational principle. We then study an exact class of AdS solutions with a constant self-dual electromagnetic field. The notion of a self-dual electromagnetic 2-form in three dimensions is introduced for the study of exact solutions of Einstein-Maxwell-Chern-Simons [Dereli and Obukhov, 2000] and TMG with electrodynamics [Dereli and Sarıoğlu, 2000, Dereli and Sarıoğlu, 2001]. Here, what we mean by self-duality[1] is that the Faraday 2-form contains a single electric field component and a magnetic field whose magnitudes are equal. We show that similar to the result in [Dereli and Şenikoğlu, 2020], our model admits exact constant self-dual solutions on AdS space-time given that the coupling constants of the model are constrained by two algebraic relations.

[1]In three dimensions, electromagnetic 2-form cannot be self-dual in the sense that it cannot be in an eigenspace of the Hodge duality operator.

Action and Variational Field Equations: The action functional for our model,

$$S[e^a, \Omega^a{}_b, \lambda^a, A] = \int_M \mathcal{L}\,, \tag{3.1}$$

depends on co-frame 1-forms $\{e^a\}$, metric connection 1-forms $\{\Omega^a{}_b\}$, Lagrange multiplier 1-forms $\{\lambda^a\}$, and electromagnetic potential 1-form A with Faraday 2-form $F = dA$. The Lagrangian 3-form can be decomposed into four parts as

$$\mathcal{L} = \mathcal{L}_{\rm TMG} + \mathcal{L}_{\rm EM} + \mathcal{L}_{\rm coup} + \mathcal{L}_{\rm constraint}\,. \tag{3.2}$$

In this study we have set the 3D Newton's constant to unity, that is $K = 1$ in $\mathcal{L}_{\rm TMG}$. The electromagnetic action density contains the Maxwell term alongside Maxwell Chern-Simons term with the coupling constant m:

$$\mathcal{L}_{\rm EM} = -\frac{1}{2} F \wedge *F - \frac{m}{2} A \wedge F\,. \tag{3.3}$$

We consider the non-minimal couplings of type RF^2 between the electromagnetic and gravitational fields, *i.e.*,

$$\begin{aligned} \mathcal{L}_{\rm coup} &= \frac{\gamma_1}{2} F^{ab} R_{ab} \wedge *F + \frac{\gamma_2}{2} Ric_a \wedge \iota^a F \wedge *F + \frac{\gamma_3}{2} RF \wedge *F \\ &= \left(\frac{\gamma_1}{2} + \frac{\gamma_2}{4}\right) F^{ab} R_{ab} \wedge *F + \left(\frac{\gamma_2}{4} + \frac{\gamma_3}{2}\right) RF \wedge *F\,, \end{aligned} \tag{3.4}$$

where we use the curvature identity (B.11) to obtain the second equality. Finally, the constraint density

$$\mathcal{L}_{\rm constraint} = \lambda_a \wedge T^a \tag{3.5}$$

ensures that, upon the variation of Lagrange multiplier 1-forms that the torsion 2-forms vanish identically. Hence we will be working with the unique Levi-Civita connection 1-forms, that is $\Omega^a{}_b = \omega^a{}_b$. Therefore it is understood that all covariant derivatives and curvatures are calculated using the Levi-Civita connection.

The variation with respect to the electromagnetic potential 1-form yields modified Maxwell's equations

$$0 = -d * F - mF + \left(\frac{\gamma_1}{2} + \frac{\gamma_2}{4}\right) d\Big[R * F - 2\left(\iota^a * F\right) Ric_a + F_{ab} * R^{ab}\Big] + \left(\frac{\gamma_2}{4} + \frac{\gamma_3}{2}\right) d\left(2R * F\right) . \tag{3.6}$$

The variation with respect to the connection 1-forms $\{\omega^{ab}\}$ yields

$$\frac{1}{2}\left(e_a \wedge \lambda_b - e_b \wedge \lambda_a\right) = -\frac{2}{\mu}R_{ab} + \frac{1}{2}D * e_{ab} + \left(\frac{\gamma_1}{2} + \frac{\gamma_2}{4}\right) D\left(F_{ab} * F\right) + \left(\frac{\gamma_2}{4} + \frac{\gamma_3}{2}\right) D\iota_{ba}\left(F \wedge *F\right) . \tag{3.7}$$

This equation may be put in the form of Equation (C.1) and may be solved algebraically for the Lagrange multiplier 1-forms using Equation (C.4). The solution reads

$$\lambda_a = -\frac{4}{\mu}Y_a + \left(\gamma_1 + \frac{\gamma_2}{2}\right)\left[\iota^b D\left(F_{ba} * F\right) - \frac{1}{4}\iota^{cb} D\left(F_{bc} * F\right) e_a\right] + \left(\frac{\gamma_2}{2} + \gamma_3\right)\left[\iota^b D\iota_{ab}\left(F \wedge *F\right) - \frac{1}{4}\iota^{cb} D\iota_{cb}\left(F \wedge *F\right) e_a\right], \tag{3.8}$$

where $\{Y_a\}$ denote the Schouten 1-forms. Finally, the co-frame variation produces the Einstein field equations

$$\begin{aligned} 0 = &\frac{1}{2}\epsilon_{abc}R^{bc} - \frac{\Lambda}{2}\epsilon_{abc}e^{bc} - \frac{1}{2}\iota_a F \wedge *F + \frac{1}{2}\left(\iota_a * F\right) F + D\lambda_a \\ &+ \left(\frac{\gamma_1}{2} + \frac{\gamma_2}{4}\right)\left[\frac{1}{2}F^{bc}\left(\iota_a * R_{bc}\right) F - 2F_a{}^b Ric_b \wedge *F \right.\\ &\left. + 2F_a{}^b\left(\iota^c * F\right) R_{bc} + \frac{1}{2}F^{bc}\left(\iota_a * F\right) R_{bc}\right] \\ &+ \left(\frac{\gamma_2}{4} + \frac{\gamma_3}{2}\right)\Big[- \epsilon_{abc} * \left(F \wedge *F\right) R^{bc} - 2Ric_a F \wedge *F\Big], \end{aligned} \tag{3.9}$$

where the Lagrange multiplier 1-forms are given by (3.8). We note that when the non-minimal coupling constants are taken to be zero, the Einstein field equations (3.9) and modified Maxwell's equations (3.6) become that of TMG with topologically massive electrodynamics as expected. Now, we will be looking for exact solutions of these equations.

AdS_3 Solution with Constant Electromagnetic Field: The solutions that we will be investigating will be on constant negative curvature backgrgound, that is AdS_3 and we will be working with the left invariant 1-forms. A detailed analysis of this coordinate frame is given in Appendix D for AdS_3 of unit radius. On AdS_3 of radius ρ, the left-invariant 1-forms satisfy the Maurer-Cartan equation (notice the difference with Equation (D.15) due to radius ρ)

$$de^a = -\frac{1}{\rho}\epsilon^a{}_{bc}e^b \wedge e^c\,, \tag{3.10}$$

where $\rho = (-\Lambda)^{-1/2}$. Consequently the Levi-Civita connection 1-forms are given by

$$\omega^a{}_b = -\frac{1}{\rho}\epsilon^a{}_{bc}e^c\,. \tag{3.11}$$

Finally, the curvature 2-forms, Ricci 1-forms and curvature scalar are given by

$$R^a{}_b = -\frac{1}{\rho^2}e^a \wedge e_b\,, \quad Ric_a = -\frac{2}{\rho^2}e_a\,, \quad R = -\frac{6}{\rho^2}\,, \tag{3.12}$$

respectively. For the electromagnetic field, we take the "self-dual" ansatz

$$F = -Ee^{01} + Be^{12}\,, \tag{3.13}$$

with $E = kB$ where $k^2 = 1$. Furthermore we assume that the electric and magnetic fields are constant. These assumptions give, in particular, $F \wedge *F = 0$ which leads to great simplifications. After a tedious computation, Einstein field equations (3.9) and modified Maxwell's equation (3.6) become algebraic relations in terms of the AdS radius and coupling constants:

$$2\rho^2 + 14\gamma_1 + \gamma_2 - 12\gamma_3 = 0\,, \tag{3.14}$$

$$m\rho^3 + 2\rho^2 + 4\gamma_1 + 8\gamma_2 + 12\gamma_3 = 0\,. \tag{3.15}$$

These equations effectively carve out a three dimensional subspace of the parameter space $\{\rho, m, \gamma_1, \gamma_2, \gamma_3\}$. To be concrete, we can take the coupling constants $\gamma_1, \gamma_2, \gamma_3$ for the non-minimal couplings to be arbitrary and they completely determine the AdS radius and electromagnetic Chern-Simons coupling constant. Although the standard values of the parameters γ_i are calculated in four dimensions, to get a rough idea one may calculate the orders of magnitude of ρ and m using these values. This yields

$$\rho \sim 10^{-2} \lambda_{\mathrm{e}} \sim 10^{-14} \text{ m} ,$$
$$m \sim 10 \cdot m_{\mathrm{e}} \sim 10^{-29} \text{ kg} ,$$

where $\lambda_{\mathrm{e}} = \hbar / m_{\mathrm{e}} c$ is the reduced Compton wavelength of the electron.

Remarks: We showed that there exists exact constant negative curvature background with constant self-dual electromagnetic fields provided that the parameters of the model satisfy the constraints (3.14) - (3.15). Although the solution given in this work is a highly restrictive one, the existence of a solution is theoretically stimulating since this shows, in particular, that the model is not inconsistent. Moreover, this simple solution may serve as a starting point for looking for more general solutions. The solution we have found is similar to the work [Dereli and Şenikoğlu, 2020], where the existence of a spherically symmetric, asymptotically de Sitter solution in four dimensions with a constant electromagnetic field 2-form was shown, for a simple RF^2-type non-minimal interaction.

3.2 New Improved Massive Gravity

Here we consider a model which encompasses TMG, MMG, and NMG models as particular sub-cases. The action is given by the MMG action extended by the most general quadratic curvature terms, that is Riemann, Ricci and scalar curvature squared terms. For the quadratic curvature invariants, we make use of the curvature idendity (B.11) to replace Ricci squared term in terms of others. We make this choice for technical ease and note that this is different than how quadratic curvature terms are used in the formulation in NMG model. Moreover this choice also helps us present the variational field equations in a compact and geometrically transparent way. We show that this model admits exact classes of solutions that are Riemann-Cartan space-times of constant curvature and torsion. We then classify the possible solutions with respect to the free parameter of the model in the Lagrangian.

Action and Variational Field Equations: The field equations of our model will be determined by the constrained variations of an action integral,

$$I[e^a, \Omega^a{}_b, \lambda_a] = \int_M \mathcal{L}\,, \tag{3.16}$$

where M is a compact region on some chart on a (1+2)-dimensional Riemann-Cartan manifold. The independent variables on which the action depends are the co-frame 1-forms $\{e^a\}$, metric connection 1-forms $\{\Omega^a{}_b\}$, and Lagrange multiplier 1-forms $\{\lambda_a\}$. We consider a Lagrangian density 3-form

$$\mathcal{L} = \mathcal{L}_{\rm TMG} + \mathcal{L}_2 + \mathcal{L}_{\rm C}\,, \tag{3.17}$$

where

$$\mathcal{L}_2 = \alpha R^{ab} \wedge *R_{ab} + \beta Ric^a \wedge *Ric_a + \gamma R^2 * 1 \tag{3.18}$$

is a generic quadratic curvature term with three coupling constants α, β, and γ. It should be remarked that, there are alternative ways of specifying a generic quadratic curvature invariant in three dimensions. Due to the identity (B.11), either one of the terms $R^{ab} \wedge *R_{ab}$, $Ric^a \wedge *Ric_a$ or $R^2 * 1$ may be made redundant in favor of others. Therefore, still keeping the coupling constants α, β and γ, we may consider without loss of generality, any one of the following combinations:

$$\begin{aligned}
\mathcal{L}_2 &= (2\alpha + \beta) Ric^a \wedge *Ric_a + \left(\gamma - \frac{\alpha}{2}\right) R^2 * 1\,, \\
\mathcal{L}_2 &= \left(\alpha + \frac{\beta}{2}\right) R^{ab} \wedge *R_{ab} + \left(\gamma + \frac{\beta}{4}\right) R^2 * 1\,, \\
\mathcal{L}_2 &= (\alpha - 2\gamma) R^{ab} \wedge *R_{ab} + (\beta + 4\gamma) Ric^a \wedge *Ric_a\,.
\end{aligned} \tag{3.19}$$

For technical ease we prefer to work with the second alternative. Finally,

$$\mathcal{L}_{\rm C} = T^a \wedge \lambda_a + \frac{\nu}{2} \lambda_a \wedge \lambda_b \wedge *e^{ab} \tag{3.20}$$

is the constraint-like Lagrangian density involving terms proportional to the Lagrange multiplier 1-forms. In the case $\nu = 0$, it imposes the constraint that the connection is the torsion-free Levi-Civita connection. On the other hand if $\nu \neq 0$, the torsion 2-forms would be related with the Lagrange multiplier 1-forms in a non-trivial way. All the previously studied models such as TMG, NMG or MMG are covered as sub-cases of the action defined by Equation (3.17).

We evaluate the variational derivative of the total Lagrangian and find (upto a closed form)

$$\begin{aligned}
\dot{\mathcal{L}} = \dot{e}^a \wedge \Bigg\{ & \frac{1}{2K}\epsilon_{abc}R^{bc} + \Lambda * e_a + \left(\alpha + \frac{\beta}{2}\right)\left(\iota_a R^{bc} \wedge *R_{bc} - R^{bc} \wedge \iota_a * R_{bc}\right) \\
& + \left(\gamma + \frac{\beta}{4}\right)\left(2\epsilon_{abc}RR^{bc} - R^2 * e_a\right) + D\lambda_a + \frac{\nu}{2}\epsilon_{abc}\lambda^b \wedge \lambda^c \Bigg\} \\
+ \dot{\Omega}^{ab} \wedge \Bigg\{ & \frac{2}{\mu}R_{ba} + \frac{1}{2K}\epsilon_{abc}T^c + (2\alpha + \beta)D * R_{ab} \\
& + \left(2\gamma + \frac{\beta}{2}\right) D(R * e_{ab}) + \frac{1}{2}(e_b \wedge \lambda_a - e_a \wedge \lambda_b) \Bigg\} \\
+ \dot{\lambda}_a \wedge \Bigg\{ & T^a + \nu\lambda_b \wedge *e^{ab} \Bigg\}. \qquad (3.21)
\end{aligned}$$

Here a dot over a field variable denotes the variation of the corresponding field. We first impose the constraint

$$T^a = -\nu\lambda_b \wedge *e^{ab} \iff K_{ab} = \nu\epsilon_{abc}\lambda^c \, . \qquad (3.22)$$

Therefore unless $\nu = 0$, we are working with a connection with torsion. Then we go to connection variation equations which now read

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = Q^{-1}\Sigma_{ab} \, , \qquad (3.23)$$

where we set

$$Q = \frac{1}{2} - \frac{\nu}{2K} - \nu\left(2\gamma + \frac{\beta}{2}\right)R \, , \qquad (3.24)$$

and

$$\Sigma_{ab} = -\frac{2}{\mu}R_{ab} + (2\alpha + \beta)D * R_{ab} + \left(2\gamma + \frac{\beta}{2}\right) dR \wedge *e_{ab} \, . \qquad (3.25)$$

We solve (3.23) algebraically using equation (C.4) for the Lagrange multiplier 1-forms and determine

$$\lambda_a = Q^{-1}\Bigg(-\frac{2}{\mu}Y_a + (2\alpha+\beta)W_a + \left(2\gamma+\frac{\beta}{2}\right)(\iota_a * dR)\Bigg) , \tag{3.26}$$

where $\{Y_a\}$ denote the Schouten 1-forms and we further introduce the abbreviations

$$W_a = \iota^b(D*R_{ba}) - \frac{1}{4}(\iota^{bc}D*R_{cb})e_a \,. \tag{3.27}$$

Finally we substitute (3.26) into the co-frame variation equations and arrive at the Einstein field equations given as follows:

$$\begin{aligned}&\left(\frac{1}{2K}+\left(2\gamma+\frac{\beta}{2}\right)R\right)R^{bc}\wedge *e_{abc}+\left(\Lambda-\left(\gamma+\frac{\beta}{4}\right)R^2\right)*e_a\\ &\quad+\left(\alpha+\frac{\beta}{2}\right)\left(\iota_aR^{bc}\wedge *R_{bc}-R^{bc}\wedge\iota_a*R_{bc}\right)\\ &\qquad+D\lambda_a+\frac{\nu}{2}\lambda^b\wedge\lambda^c*e_{abc}=0\,.\end{aligned} \tag{3.28}$$

We note that these equations include among other terms, the Cotton 2-forms $\{C_a = DY_a\}$ calculated from the torsionful connection 1-forms $\{\Omega^a{}_b = \omega^a{}_b + K^a{}_b\}$ as well as the 2-forms

$$D_a \equiv DW_a = D\left(\iota^b(D*R_{ba}) - \frac{1}{4}\left(\iota^{bc}(D*R_{cb})\right)e_a\right) \tag{3.29}$$

that involve fourth derivatives of the metric components. We also note that Einstein field equations in the case of zero-torsion ($\nu = 0$) are given by

$$\begin{aligned}&\left(\frac{1}{2K}+\left(2\gamma+\frac{\beta}{2}\right)\hat{R}\right)\hat{R}^{bc}\wedge *e_{abc}+\left(\Lambda-\left(\gamma+\frac{\beta}{4}\right)\hat{R}^2\right)*e_a\\ &\quad+\left(\alpha+\frac{\beta}{2}\right)\left(\iota_a\hat{R}^{bc}\wedge *\hat{R}_{bc}-\hat{R}^{bc}\wedge\iota_a*\hat{R}_{bc}\right)\\ &\qquad-\frac{4}{\mu}\hat{C}_a+(4\alpha+2\beta)\hat{D}_a+(4\gamma+\beta)\hat{D}(\iota_a*d\hat{R})=0\,.\end{aligned} \tag{3.30}$$

For clarity, both in the above equation and up until the end of this chapter we denote the quantities calculated by the Levi-Civita connection with a hat. Field equations (3.30) descend consistently to the TMG field equations (2.7) if the quadratic curvature terms are absent, *i.e.*, if we set $\alpha = \beta = \gamma = 0$ above. Finally we re-write

Einstein field equations in two special cases of interest:

(i) NMG: $\frac{1}{\mu} \to 0$, $\Lambda = 0$, $\alpha = 0, \beta = 1, \gamma = -\frac{3}{8}, \nu = 0$.

$$\left(\frac{1}{2K} - \frac{1}{4}\hat{R}\right)\hat{R}^{bc} \wedge *e_{abc} + \frac{1}{8}\hat{R}^2 * e_a + \frac{1}{2}\left(\iota_a \hat{R}^{bc} \wedge *\hat{R}_{bc} - \hat{R}^{bc} \wedge \iota_a * \hat{R}_{bc}\right)$$
$$+ 2\hat{D}_a - \frac{1}{2}\hat{D}(\iota_a * d\hat{R}) = 0\,. \tag{3.31}$$

(ii) MMG: $\alpha = \beta = \gamma = 0$, $\nu \neq 0$.

$$-\frac{1}{K}G_a + \Lambda * e_a - \frac{4K}{\mu(K-\nu)}C_a + \frac{8K^2\nu}{\mu^2(K-\nu)^2}\epsilon_{abc}Y^b \wedge Y^c = 0\,,$$
$$K_{ab} + \frac{4K\nu}{\mu(K-\nu)}\epsilon_{abc}Y^c = 0\,. \tag{3.32}$$

Exact Solutions with Constant Curvature and Torsion: In order to proceed with the study of a quantized theory of gravity based on our model, its solutions should be found. Towards that end, here we consider three dimensional non-Riemannian space-times of constant curvature and constant torsion. This is a notion introduced many years ago, but has been overlooked up till now [Dereli and Verçin, 1991]. We also conveniently work with coordinate independent methods. That is to say, we evaluate curvatures and their derivatives without differentiating any functions. The relevant differential geometric techniques are briefly explained in an Appendix D for left-invariant 1-forms on AdS_3. The Riemannian part of the curvature 2-forms calculated from the Levi-Civita connection is:

$$\hat{R}^a{}_b = -\frac{1}{\rho^2}e^a \wedge e_b\,, \tag{3.33}$$

where ρ is the radius of AdS_3. Now we set the torsion 2-forms to be

$$T^a = \frac{2}{\sigma} * e^a\,, \quad \sigma^2 \neq \rho^2 \iff K^a{}_b = -\frac{1}{\sigma}\epsilon^a{}_{bc}e^c\,. \tag{3.34}$$

Then the full curvature 2-forms turn out to be

$$R^a{}_b = \left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right) e^a \wedge e_b\,. \tag{3.35}$$

Their contractions give

$$Ric_a = 2\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right) e_a\,, \quad R = 6\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right)\,. \tag{3.36}$$

Substituting these in (3.24), we find

$$Q = \frac{1}{2} - \frac{\nu}{2K} - 6\nu\left(2\gamma + \frac{\beta}{2}\right)\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right), \tag{3.37}$$

and in (3.26), we find

$$\lambda_a = -Q^{-1}\left(\frac{1}{\mu} + \frac{2\alpha + \beta}{\sigma}\right)\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right) e_a\,. \tag{3.38}$$

We must first check (3.22) for consistency:

$$\frac{1}{\nu\sigma} = Q^{-1}\left(\frac{1}{\mu} + \frac{2\alpha + \beta}{\sigma}\right)\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right). \tag{3.39}$$

Substituting for Q from Equation (3.37), we get an algebraic consistency equation as follows:

$$2\left(\frac{\sigma}{\mu} + 2\alpha + 4\beta + 12\gamma\right)\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right) = \frac{K - \nu}{K\nu}\,. \tag{3.40}$$

Next we consider the Einstein field Equations (3.28) with

$$\lambda_a = -\frac{1}{\nu\sigma} e_a\,, \tag{3.41}$$

and organise terms to arrive at

$$(4\alpha - \beta - 12\gamma)\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right)^2 + \frac{1}{K}\left(\frac{\rho^2 - \sigma^2}{\rho^2\sigma^2}\right) + \left(\Lambda - \frac{1}{\nu\sigma^2}\right) = 0\,. \tag{3.42}$$

Thus we have two algebraic Equations (3.40) and (3.42) to solve simultaneously for ρ and σ in terms of the coupling parameters $K, \Lambda, \alpha, \beta, \gamma, \mu$ and ν.

To proceed further, we concentrate first on a simpler case and discuss solutions of this algebraic system for MMG with $\alpha = \beta = \gamma = 0$. We rename our variables as

$$\frac{1}{\sigma} \equiv \xi\,, \quad \frac{1}{\rho} \equiv \eta\,, \tag{3.43}$$

and let

$$a \equiv \frac{\mu}{4}\left(\frac{1}{\nu} - \frac{1}{K}\right). \tag{3.44}$$

Then our coupled system of equations reduce to the equations of two conic sections in the $(\xi\,,\eta)$-plane given by

$$\frac{(\xi - a)^2}{a^2} - \frac{\eta^2}{a^2} = 1\,, \tag{3.45}$$

and

$$K\left(\frac{1}{\nu} - \frac{1}{K}\right)\frac{\xi^2}{K\Lambda} + \frac{\eta^2}{K\Lambda} = 1\,. \tag{3.46}$$

We take $K > 0$ without loss of generality at this point, since our model is not yet coupled to matter. We also point out that solutions come in pairs with values $\eta \leftrightarrow -\eta$, as a change in sign of η means going from one orientation of the co-basis to the other, or vice versa. In what follows, we restrict attention to the cases $0 < \eta$, but extension to cases $\eta < 0$ is easy. Then we classify possible pairs $(\xi\,,\eta)$ in accordance with the following ranges of our free parameters:

- For $\Lambda < 0$ and $-\infty < \mu < \infty$; no solution exists with $0 \leq \nu \leq K$.

- $\Lambda < 0\,, \mu > 0\,, \nu \leq 0$ or $K \leq \nu$.
  There is a solution for $0 \leq \xi \leq \sqrt{\left|\frac{(K-\nu)}{\nu} K\Lambda\right|}$ and $0 < \eta < \infty$.
  A second one may exist for $-\sqrt{|\frac{(K-\nu)}{\nu} K\Lambda|} \leq \xi \leq -\mu|\frac{(K-\nu)}{2K\nu}|$, depending on the magnitude of μ.

- $\Lambda < 0\,, \mu < 0\,, \nu \leq 0$ or $K \leq \nu$.
  There is a solution for $\xi \leq -\sqrt{|\frac{(K-\nu)}{\nu} K\Lambda|}$ and $0 < \eta < \infty$.
  A second one may exist for $\sqrt{|\frac{(K-\nu)}{\nu} K\Lambda|} \leq \xi \leq \mu\frac{(K-\nu)}{2K\nu}$, depending on the magnitude of μ.

- $\Lambda > 0\,, \mu > 0\,, 0 \leq \nu \leq K$.
  Solutions exist for $0 \leq \eta \leq \sqrt{K\Lambda}$. Then there is a solution for $-\sqrt{\frac{K-\nu}{\nu} K\Lambda} \leq \xi \leq 0$.
  A second one may exist for $\sqrt{\frac{K-\nu}{\nu} K\Lambda} \leq \xi \leq \mu\frac{(K-\nu)}{2K\nu}$, depending on the magnitude of μ.

- $\Lambda > 0\,, \mu < 0\,, \nu \leq 0$ or $K \leq \nu$.

  Two solutions exist for $\sqrt{K\Lambda} \leq \eta$ and with either $\xi \leq 0$ or $\mu\frac{(K-\nu)}{2K\nu} \leq \xi$.

- $\Lambda > 0\,, \mu > 0\,, \nu \leq 0$ or $K \leq \nu$.

  Two solutions exist for $\sqrt{K\Lambda} \leq \eta$ and with either $\xi \leq -\mu\frac{(K-\nu)}{2K\nu}$ or $0 \leq \xi$.

- $\Lambda > 0\,, \mu < 0\,, 0 \leq \nu \leq K$.

  Solutions exist for $\sqrt{K\Lambda} \leq \eta$. One solution has $0 \leq \xi \leq \sqrt{\frac{\nu}{K-\nu}K\Lambda}$.

  A second one may exist, for $-\sqrt{\frac{\nu}{K-\nu}K\Lambda} \leq \xi \leq -\sqrt{|\mu|\frac{(K-\nu)}{\nu}}$, depending on the magnitude of μ.

Remarks: It should be emphasised that our discussion based on the choice (3.17) of the action encompasses all currently studied models such as NMG or MMG as particular sub-cases. In recent literature, the generic quadratic curvature term in the action that is commonly used is given by the first alternative in Equation (3.19). Here we use the second alternative for technical ease and were able to present the variational field equations in a compact and geometrically transparent way. Moreover, the coupled algebraic equations (3.40) and (3.42) describe a cubic and a quartic curve, respectively, in the (ρ, σ)-plane. We chose to discuss the existence of solutions of the intersection points for these curves in the special case of MMG, that is $(\alpha = \beta = \gamma = 0)$, and similar analysis can be carried out with more technicalities for other cases as well.

Chapter 4

WEYL COVARIANT GRAVITY MODELS

In this chapter we study scale (Weyl) covariant generalizations of TMG, MMG and quadratic curvature gravity models. By a Weyl covariant generalization, we mean a gravity model extended by a gauge field, called the Weyl connection 1-form, compensating for the scale transformations. Due to Weyl connection 1-form, these models, with the addition of a dilaton scalar, field are ungauged generalizations of scalar-tensor models of gravity. That is, after fixing the Weyl gauge, one obtains a scalar-tensor model. Moreover, the number of symmetries these models contain are more than a pure gravity model and less than a conformal gravity model. Therefore these models provide a viable middle step where one may study gravity models without too many additional symmetries. Furthermore scale covariance mandates having dimensionless coupling constants, which helps construct low and high energy complete models of gravitation [O'Raifeartaigh et al., 1996,Dereli and Tucker, 2002].

The group of scale transformations is a non-compact, 1-parameter, abelian Lie group $\mathbb{R}_+$ called the Weyl[1] group. To obtain scale covariant theories, we promote global scale transformations into local ones. Consequently in a scale covariant field theory, there is a principal bundle structure over space-time where the structure group is the Weyl group. On this bundle, a Weyl connection 1-form taking values in the Lie algebra $\mathbb{R}$ introduces a scale covariant exterior derivative that is compatible with the action of the Weyl group. The fields over the space-time are allowed to carry a representation of this group, and their Weyl charge assignments are made

[1]Weyl group is generally used for the symmetry groups of Weyl chambers of root spaces of Lie algebras. Throughout this thesis, by a Weyl group we always mean the identity component of $GL(1,\mathbb{R})$ associated to scale transformations.

according to their physical dimensions.

We employ the Weyl gauging in the geometrical context of Riemann-Cartan-Weyl (RCW) space-times which provide a natural framework to discuss locally scale covariant theories of gravitation. A RCW space-time is a metric affine geometry (M, g, ∇) where the non-metricity tensor has only non-vanishing trace part that is identified with the Weyl connection 1-form Q via

$$\nabla g = 2Q \otimes g \qquad \Longleftrightarrow \qquad Q_{ab} = -Q\eta_{ab}\,. \tag{4.1}$$

This identification relates the origin of local changes of scale to the geometry of space-time. In a Weyl covariant model, we work with the non-metric connection 1-forms

$$\Lambda^a{}_b = \Omega^a{}_b - Q\eta^a{}_b = \omega^a{}_b + K^a{}_b - Q^a e_b + Q_b e^a - Q\eta^a{}_b\,. \tag{4.2}$$

To make the distinction clearer, throughout this chapter we put the symbol (Λ) over the quantities calculated from the connection 1-forms $\{\Lambda^a{}_b\}$.

As discussed above, a field Φ over the space-time is allowed to carry a representation of this group, that is under a local change of scale

$$\Phi \mapsto e^{-q\sigma}\Phi\,, \tag{4.3}$$

where the dimensionless parameter q is the Weyl charge of the field Φ and σ is a dimensionless real scalar field on space-time, sometimes called the scale factor. Conventionally, because it has dimensions of length squared, metric is assigned a Weyl charge of -2, that is

$$g \mapsto e^{2\sigma} g\,. \tag{4.4}$$

Then assignment of Weyl charges of other fields are made according to this choice. For instance, using Equations (4.1) and (4.4), we can determine the transformation rules of co-frame and connection 1-forms as

$$e^a \mapsto e^{\sigma} e^a\,, \quad \Lambda^a{}_b \mapsto \Lambda^a{}_b - \eta^a{}_b d\sigma\,, \tag{4.5}$$

respectively. Then, we can calculate the transformation rules for torsion 2-forms and curvature 2-forms as

$$T^a \mapsto e^{\sigma} T^a \,, \quad \overset{(\Lambda)}{R^a}_b \mapsto \overset{(\Lambda)}{R^a}_b \,. \tag{4.6}$$

For the affine connection, we adopt the Weyl transformation rule

$$\nabla \mapsto \nabla \,. \tag{4.7}$$

This choice is consistent with our framework, because connection is not a tensorial quantity, and therefore has no assigned dimension; it stays inert under local scale transformations [Dereli and Tucker, 2002]. Also, there need not be any correlations between metric scaling and transformation of the linear connection in a RCW space. To be able to make this choice, one has to have at least one of the torsion or non-metricity tensors to be nonzero. Otherwise, in a Riemannian space-time, the transformation property of the connection is determined by the metric tensor only.

Under a local change of scale the Weyl connection 1-form transforms as

$$Q \mapsto Q + d\sigma \,. \tag{4.8}$$

Space-time exterior covariant derivative $\overset{(\Lambda)}{D}$ does not transform covariantly under local changes of scale. Therefore by coupling it with the Weyl connection 1-form, we can define a Weyl exterior covariant derivative $\mathcal{D}$. In particular on a p-form Φ^p_q with Weyl charge q, its action is given by

$$\mathcal{D}\Phi^p_q = \overset{(\Lambda)}{D}\Phi^p_q + qQ \wedge \Phi^p_q \,, \tag{4.9}$$

so that under a local scale transformation, $\mathcal{D}\Phi^p_q \mapsto e^{-q\sigma}\mathcal{D}\Phi^p_q$. We note that, using the Weyl exterior covariant derivative with a slight abuse of notation, the Weyl geometry condition (4.1) can be equivalently expressed as $\mathcal{D}g = 0$.

The metric tensors related by a local change of scale form an equivalence class $[g]$ under the relation (4.4). This equivalence class is called a conformal structure on M.

For a conformal class, one can associate a class of Hodge stars [*]. Then, under the action of Hodge map, one has

$$*\Phi^p_q = \Phi^{3-p}_{q-(3-2p)}\,, \tag{4.10}$$

$$\mathcal{D} * \Phi^p_q = \overset{(\Lambda)}{D} * \Phi^p_q + (q-(3-2p))Q \wedge *\Phi^p_q\,. \tag{4.11}$$

In addition, under contractions with the interior product operation, Weyl charge of the fields increase by 1, *i.e.*,

$$\iota_a \Phi^p_q = \Phi^{p-1}_{q+1}\,. \tag{4.12}$$

To obtain a scale covariant generalization, we introduce two ingredients to the original models. First element is the dilaton field α. It has the dimensions of inverse length and is used to write scale invariant terms for Lagrangian. The second ingredient is the Weyl connection 1-form Q, which helps compensate for the local changes of scale. Both the dilaton field and Weyl connection 1-form are introduced as independent variables, the latter being introduced through the symmetric part of the connection 1-forms. Therefore, in comparison with Equation (4.2), we have

$$\dot{Q} = -\frac{1}{3}\eta^b{}_a \dot{\Lambda}^a{}_b\,. \tag{4.13}$$

As an explicit example let us see how the Einstein-Hilbert term

$$\mathcal{L}_{\text{EH}} = \frac{1}{2K}\overset{(\Omega)}{R^a}{}_b \wedge *e_a{}^b \tag{4.14}$$

would get modified. It depends on the co-frame $\{e^a\}$ and metric compatible connection 1-forms $\{\Omega^a{}_b\}$ and in three dimensions has scale weight 1. Therefore we can discard the dimensionful Newton's constant K and couple the dilaton field to obtain a scale invariant Einstein-Hilbert term as:

$$\mathcal{L}_{\text{WEH}} = \alpha \overset{(\Lambda)}{R^a}{}_b \wedge *e_a{}^b. \tag{4.15}$$

In (4.15) we also promoted the metric connection 1-forms $\{\Omega^a{}_b\}$ to the non-metric connection 1-forms $\{\Lambda^a{}_b = \Omega^a{}_b - Q\eta^a{}_b\}$. The scale covariant Einstein-Hilbert term depends on the co-frame fields $\{e^a\}$, dilaton field α, anti-symmetric part of connection 1-forms $\{\Omega^a{}_b\}$, and Weyl connection 1-form Q.

The consistency of any scale covariant generalization is checked by following the following diagram:

$$\begin{array}{ccc} \mathcal{L}_{\mathrm{T}} & \xrightarrow{\text{introduce } \alpha \ \& \ Q} & \mathcal{L}_{\mathrm{WT}} \\ \Big\downarrow \scriptstyle\text{variation} & & \Big\downarrow \scriptstyle\text{variation} \\ \dot{\mathcal{L}}_{\mathrm{T}} & \xleftarrow{\alpha=1 \ \& \ Q=0} & \dot{\mathcal{L}}_{\mathrm{WT}} \end{array}$$

We introduce scale invariant terms to the Lagrangian $\mathcal{L}_{\mathrm{T}}$ of the original model using the dilaton field α and Weyl connection 1-form Q, vary the scale invariant Lagrangian $\mathcal{L}_{\mathrm{WT}}$, and obtain the scale covariant variational field equations $\dot{\mathcal{L}}_{\mathrm{WT}}$. If these field equations agree with the field equations of original theory $\dot{\mathcal{L}}_{\mathrm{T}}$ for a fixed scale $\alpha = 1$, and vanishing non-metricity $Q = 0$, the above diagram commutes and we say that the generalization is consistent.

A consistent generalization means that the scale covariant theory contains the original theory in its vacuum configuration for the Weyl sector. The vacuum configuration means the Weyl connection 1-form has a vanishing field strength, *i.e.*, it is flat. In this case, any solution of the original theory defines an equivalence class of solutions for the scale covariant theory, where two solutions are related by a pure gauge transformation.

We note that scale covariant TMG and quadratic curvature theories have been studied using a similar technique in [Dengiz et al., 2012] and [Tanhayi et al., 2012], respectively. Scale covariant MMG is a new model. Due to the dynamical torsion present in the theory, it is more complicated but it becomes clear by the use of exterior algebra. Moreover in [Dengiz et al., 2012], contrary to our approach, the authors interpret the Weyl vector boson as a potential for the gauge group $U(1)$ of electrodynamics. Since Weyl connection 1-form is purely real we do not make this identification. Also, when interpreted as a linear connection, it has a geometrical meaning over space-time. For a $U(1)$ connection this geometrical meaning is rather obscure, because it is a purely imaginary 1-form. In the remainder of this chapter we present a formulation of scale covariant generalizations of the models considered in Chapter 2.

4.1 General Relativity

We consider the following Weyl invariant Lagrangian density:

$$\mathcal{L}_{\text{WGR}} = \alpha \overset{(\Lambda)}{R}{}^a{}_b \wedge *e_a{}^b + \alpha^3 \Lambda * 1 + \alpha \lambda_a \wedge T^a - \frac{\gamma}{2\alpha} \mathcal{D}\alpha \wedge *\mathcal{D}\alpha - \frac{\gamma'}{2\alpha} dQ \wedge *dQ \,. \tag{4.16}$$

In addition to modifying the terms in the original action (2.1), we also added the kinetic terms of the fields α and Q to promote them to dynamical fields. In (4.16), γ and γ' are dimensionless coupling constants due to scale invariance. The total variational of the above action density yield:

$$\begin{aligned}
\dot{\mathcal{L}_{\text{WGR}}} = \dot{e}^a \wedge \Big\{ &\alpha \overset{(\Omega)}{R}{}^b{}_c \epsilon_{ab}{}^c + \alpha^3 \Lambda * e_a + \overset{(\Omega)}{D}(\alpha \lambda_a) + \alpha Q \wedge \lambda_a + \frac{\gamma}{2\alpha} \tau_a[\mathcal{D}\alpha] \\
&+ \frac{\gamma'}{2\alpha} \hat{\tau}_a[dQ] \Big\} + \dot{\Omega}^a{}_b \wedge \Big\{ \overset{(\Omega)}{D}(\alpha * e_a{}^b) + \alpha e^b \wedge \lambda_a \Big\} \\
&+ \dot{Q} \wedge \Big\{ \alpha \lambda_a \wedge e^a - \gamma * \mathcal{D}\alpha - \gamma' d\Big(\frac{1}{\alpha} * dQ\Big) \Big\} + \dot{\lambda_a} \wedge \Big(\alpha T^a\Big) \\
&+ \dot{\alpha} \Big\{ \overset{(\Omega)}{R}{}^a{}_b \wedge *e_a{}^b + 3\alpha^2 \Lambda * 1 + \lambda_a \wedge T^a + \frac{\gamma}{2\alpha^2} \mathcal{D}\alpha \wedge *\mathcal{D}\alpha \\
&+ \gamma \mathcal{D}\Big(\frac{1}{\alpha} * \mathcal{D}\alpha\Big) + \frac{\gamma'}{2\alpha^2} dQ \wedge *dQ \Big\} ,
\end{aligned} \tag{4.17}$$

where the shorthand expressions

$$\tau_a[\mathcal{D}\alpha] = (\iota_a \mathcal{D}\alpha) * \mathcal{D}\alpha + \mathcal{D}\alpha \wedge \iota_a * \mathcal{D}\alpha \,, \tag{4.18}$$

$$\hat{\tau}_a[dQ] = \iota_a dQ \wedge *dQ - dQ(\iota_a * dQ) \,, \tag{4.19}$$

are the scale covariant stress-energy forms of the dilaton field α and the Weyl vector boson field Q, respectively.

We first note from Lagrange constraint equation, since $\alpha \neq 0$, that the torsion 2-forms identically vanish, *i.e.*, $T^a = 0$. The field equations are to be solved under this constraint. The scalar field equation can be replaced by a simpler expression. In order to demonstrate this, we take the trace of the co-frame equation by taking a left exterior multiplication with e^a, and compare this expression with the dilaton variation equation to obtain:

$$d(\alpha e_a \wedge \lambda^a + \gamma * \mathcal{D}\alpha) = 0\,. \tag{4.20}$$

Next, we solve the anti-symmetric connection equation. To do this, we lower an index using

$$\overset{(\Omega)}{D}(\alpha * e_a{}^b) = \overset{(\Lambda)}{D}(\alpha * e_a{}^b) = \mathcal{D}\alpha \wedge *e_a{}^b\,, \tag{4.21}$$

and write this equation in the form of Equation (C.1) as

$$\frac{\alpha}{2}(e_a \wedge \lambda_b - e_b \wedge \lambda_a) = \Sigma_{ab}\,, \tag{4.22}$$

where

$$\Sigma_{ab} := \mathcal{D}\alpha \wedge *e_{ab}\,. \tag{4.23}$$

The unique solution for the Lagrange multiplier 1-forms can be calculated using Equation (C.4), which is given by:

$$\lambda_a = \frac{2}{\alpha}\iota_a * \mathcal{D}\alpha\,. \tag{4.24}$$

Finally, after substituting the solution for Lagrange multiplier 1-forms, the variational field equations for the scale covariant general relativity theory read:

$$\alpha \overset{(\Omega)}{R^b}{}_c \epsilon_{ab}{}^c + \alpha^3 \Lambda * e_a + 2\overset{(\Omega)}{D}(\iota_a * \mathcal{D}\alpha) + 2Q \wedge (\iota_a * \mathcal{D}\alpha) + \frac{\gamma}{2\alpha}\tau_a[\mathcal{D}\alpha] + \frac{\gamma'}{2\alpha}\hat{\tau}_a[dQ] = 0\,, \tag{4.25}$$

$$(\gamma + 4) * \mathcal{D}\alpha + \gamma' d\left(\frac{1}{\alpha} * dQ\right) = 0\,, \tag{4.26}$$

$$(\gamma + 4) d * \mathcal{D}\alpha = 0 \,. \tag{4.27}$$

We note that, the dilaton field equation (4.27) is given by the Weyl vector field equation (4.26) by taking an exterior derivative. In addition, the dilaton field does not provide an extra degree of freedom [Jackiw and Pi, 2015] because it can be completely gauged away through gauge fixing the Weyl symmetry, *i.e.*, setting $\alpha = 1$. After gauge fixing, Equation (4.26) contains an effective mass term for the Weyl connection 1-form, which means that the Weyl vector boson is a Proca field.

Before checking the consistency of this generalization, we can gauge fix the Weyl sector by setting

$$\mathcal{D}\alpha = 0 \quad \Leftrightarrow \quad Q = -\frac{d\alpha}{\alpha} \,. \tag{4.28}$$

Consequently $dQ = 0$, and we are looking for a vacuum class (*i.e.*, with vanishing field strength) of solutions for the Weyl sector. For this choice, Lagrange multiplier 1-forms vanish identically, *i.e.*, $\lambda_a = 0$, and equations (4.26) and (4.27) are trivially satisfied. We are only left with the Einstein field equation:

$$\alpha \overset{(\Omega)}{R^b}{}_c \epsilon_{ab}{}^c + \alpha^3 \Lambda * e_a = 0 \,. \tag{4.29}$$

This is the field equation of a scalar-tensor model obtained by just introducing dilaton field and its kinetic term to the action of general relativity.

To check the consistency of our generalized model, we can either set $\alpha = 1$ in Equation (4.29), or set $\alpha = 1$ and $Q = 0$ in Equations (4.25), (4.26) and (4.27). In both cases, setting $\alpha = 1$ amounts to choosing a global units frame in which $2K = 1$. Moreover the field equations reduce to the original Einstein's Equations (2.3). Therefore this is a consistent Weyl gauged model.

4.2 Topologically Massive Gravity

For the the scale covariant TMG model, we analyze the following Weyl invariant Lagrangian density:

$$\mathcal{L}_{\mathrm{WTMG}} = \frac{1}{\mu}(\Lambda^a{}_b \wedge d\Lambda^b{}_a + \frac{2}{3}\Lambda^a{}_b \wedge \Lambda^b{}_c \wedge \Lambda^c{}_a) + \frac{1}{\mu'}Q \wedge dQ + \alpha \overset{(\Lambda)}{R^a}{}_b \wedge *e_a{}^b$$
$$+ \alpha^3 \Lambda * 1 + \alpha\lambda_a \wedge T^a - \frac{\gamma}{2\alpha}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha - \frac{\gamma'}{2\alpha} dQ \wedge *dQ. \tag{4.30}$$

Here we added the abelian Chern-Simons term for the Weyl connection 1-form Q in addition to the kinetic terms of the α and Q fields. When we separate the connection 1-forms $\{\Lambda^a{}_b\}$ into its skew-symmetric and symmetric parts, the abelian Chern-Simons term naturally arises with the same coupling constant μ as the gravitational one. To keep the final expressions coming from the gravitational and the abelian Chern-Simons terms independent, the abelian Chern-Simons term with the coupling constant μ' is also added.

The total variational derivative of the Lagrangian is found to be:

$$\dot{\mathcal{L}}_{\mathrm{WTMG}} = \dot{e}^a \wedge \Big\{ \alpha \overset{(\Omega)}{R^b}{}_c \epsilon_{ab}{}^c + \alpha^3\Lambda * e_a + \overset{(\Omega)}{D}(\alpha\lambda_a) + \alpha Q \wedge \lambda_a + \frac{\gamma}{2\alpha}\tau_a[\mathcal{D}\alpha]$$
$$+ \frac{\gamma'}{2\alpha}\hat{\tau}_a[dQ] \Big\} + \dot{\Omega}^a{}_b \wedge \Big\{ \frac{2}{\mu}\overset{(\Omega)}{R^b}{}_a + \overset{(\Omega)}{D}(\alpha * e_a{}^b) + \alpha e^b \wedge \lambda_a \Big\}$$
$$+ \dot{Q} \wedge \Big\{ \Big(\frac{6}{\mu} + \frac{2}{\mu'}\Big) dQ + \alpha\lambda_a \wedge e^a - \gamma * \mathcal{D}\alpha - \gamma' d\Big(\frac{1}{\alpha} * dQ\Big) \Big\}$$
$$+ \dot{\alpha} \Big\{ \overset{(\Omega)}{R^a}{}_b \wedge *e_a{}^b + 3\alpha^2\Lambda * 1 + \lambda_a \wedge T^a + \frac{\gamma}{2\alpha^2}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha$$
$$+ \gamma\mathcal{D}\Big(\frac{1}{\alpha} * \mathcal{D}\alpha\Big) + \frac{\gamma'}{2\alpha^2} dQ \wedge *dQ \Big\} + \dot{\lambda}_a \wedge \Big(\alpha T^a\Big). \tag{4.31}$$

First, from the Lagrange constraint equation, torsion 2-forms identically vanish. Then, by comparing the trace of the co-frame equation to the dilaton field equation, we obtain:

$$d(\alpha e_a \wedge \lambda^a + \gamma * \mathcal{D}\alpha) = 0\,. \tag{4.32}$$

Next, we solve the anti-symmetric part of connection equation for Lagrange multiplier 1-forms. To lower an index in this equation, we use (4.21) and write the anti-symmetric part of the connection equation in the form of Equation (C.1). From Equation (C.4), the unique solution for the Lagrange multiplier 1-forms is found to be:

$$\lambda_a = -\frac{4}{\mu\alpha}\overset{(\Omega)}{Y}_a + \frac{2}{\alpha}\iota_a * \mathcal{D}\alpha\,. \tag{4.33}$$

The Schouten 1-forms of the anti-symmetric part of connection 1-forms $\{\Omega^a{}_b = \omega^a{}_b + q^a{}_b\}$ are given by:

$$\overset{(\Omega)}{Y}_a = \iota^b \overset{(\Omega)}{R}_{ba} - \frac{1}{4}(\iota^{cb}\overset{(\Omega)}{R}_{bc})e_a\Omega \tag{4.34}$$

Finally, the variational field equations of the scale covariant TMG theory read:

$$\begin{aligned} -2\alpha\overset{(\Omega)}{G}_a + \alpha^3\Lambda * e_a - \frac{4}{\mu}\overset{(\Omega)}{D}\overset{(\Omega)}{Y}_a - \frac{4}{\mu}Q\wedge\overset{(\Omega)}{Y}_a + 2\overset{(\Omega)}{D}(\iota_a * \mathcal{D}\alpha) \\ + 2Q\wedge(\iota_a * \mathcal{D}\alpha) + \frac{\gamma}{2\alpha}\tau_a[\mathcal{D}\alpha] + \frac{\gamma'}{2\alpha}\hat{\tau}_a[dQ] = 0\,, \end{aligned} \tag{4.35}$$

$$(\gamma+4) * \mathcal{D}\alpha = \left(\frac{6}{\mu} + \frac{2}{\mu'}\right)dQ - \gamma' d\left(\frac{1}{\alpha} * dQ\right), \tag{4.36}$$

$$(\gamma+4)d * \mathcal{D}\alpha = 0\,. \tag{4.37}$$

To check the scalar-tensor generalization we set $\mathcal{D}\alpha = 0$. For this choice, equations (4.36) and (4.37) are trivially satisfied. We are only left with:

$$-2\alpha\overset{(\Omega)}{G}_a + \alpha^3\Lambda * e_a - \frac{4}{\mu}\overset{(\Omega)}{C}_a + \frac{4}{\mu\alpha}d\alpha\wedge\overset{(\Omega)}{Y}_a = 0\,. \tag{4.38}$$

Fixing the residual gauge freedom by setting $\alpha = 1$ annihilates the Weyl connection 1-form, *i.e.*, $Q = 0$, and we are left with the unique Levi-Civita connection 1-forms $\{\omega^a{}_b\}$. Consequently the last term in equation (4.38) vanishes, the third term becomes the Cotton 2-form of Levi-Civita connection, and the field equations consistently reduce to the field Equations (2.7) of original TMG model.

4.3 Minimal Massive Gravity

The Weyl invariant Lagrangian density that we consider for the scale covariant theory is given by:

$$\begin{aligned}\mathcal{L}_{\text{WMMG}} = &\frac{1}{\mu}(\Lambda^a{}_b \wedge d\Lambda^b{}_a + \frac{2}{3}\Lambda^a{}_b \wedge \Lambda^b{}_c \wedge \Lambda^c{}_a) + \frac{1}{\mu'}Q \wedge dQ + \alpha \overset{(\Lambda)}{R}{}^a{}_b \wedge *e_a{}^b \\ &+ \alpha^3 \Lambda * 1 - \frac{\gamma}{2\alpha}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha - \frac{\gamma'}{2\alpha} dQ \wedge *dQ + \alpha\lambda_a \wedge T^a \\ &+ \frac{\nu\alpha}{2}\lambda_a \wedge \lambda_b \wedge *e^{ab}\,. \end{aligned} \tag{4.39}$$

Here the coupling constants μ, μ', γ, γ' and ν are dimensionless due to Weyl invariance.

Weyl covariant field equations are found by taking the variational derivative of the Lagrangian density 3-form which reads:

$$\begin{aligned}\dot{\mathcal{L}}_{\text{WMMG}} = &\dot{e}^a \wedge \left\{ \alpha \overset{(\Omega)}{R}{}^b{}_c \epsilon_{ab}{}^c + \alpha^3 \Lambda * e_a + \overset{(\Omega)}{D}(\alpha\lambda_a) + \alpha Q \wedge \lambda_a + \frac{\gamma}{2\alpha}\tau_a[\mathcal{D}\alpha] \right. \\ &\left. + \frac{\gamma'}{2\alpha}\hat{\tau}_a[dQ] + \frac{\nu\alpha}{2}\epsilon_a{}^{bc}\lambda_b \wedge \lambda_c \right\} + \dot{\Omega}^a{}_b \wedge \left\{ \frac{2}{\mu}\overset{(\Omega)}{R}{}^b{}_a + \overset{(\Omega)}{D}(\alpha * e_a{}^b) + \alpha e^b \wedge \lambda_a \right\} \\ &+ \dot{\lambda}_a \wedge \left\{ \alpha(T^a + \nu\lambda_b \wedge *e^{ab}) \right\} + \dot{\alpha}\left\{ \overset{(\Omega)}{R}{}^a{}_b \wedge *e_a{}^b + 3\alpha^2\Lambda * 1 + \frac{\gamma}{2\alpha^2}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha \right. \\ &\left. + \gamma \mathcal{D}\left(\frac{1}{\alpha} * \mathcal{D}\alpha\right) + \frac{\gamma'}{\alpha^2} dQ \wedge *dQ + \lambda_a \wedge T^a + \frac{\nu}{2}\lambda_a \wedge \lambda_b \wedge *e^{ab} \right\} \\ &+ \dot{Q} \wedge \left\{ \left(\frac{6}{\mu} + \frac{2}{\mu'}\right) dQ + \alpha\lambda_a \wedge e^a - \gamma * \mathcal{D}\alpha - \gamma' d\left(\frac{1}{\alpha} * dQ\right) \right\}. \end{aligned} \tag{4.40}$$

We start by solving the Lagrange multiplier equation for torsion 2-forms.

$$T^a = -\nu\lambda_b \wedge *e^{ab}. \tag{4.41}$$

Therefore, the auxiliary field 1-forms are proportional to dualized contortion 1-forms. The dilaton field equation can be simplified greatly by comparing with the trace of co-frame variation equation, which reads:

$$\begin{aligned}&\alpha \overset{(\Omega)}{R}{}^a{}_b \wedge *e_a{}^b + 3\alpha^2 \Lambda * 1 - \frac{\gamma}{2\alpha}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha + \frac{\gamma'}{2\alpha} dQ \wedge *dQ \\ &\quad + \alpha\lambda_a \wedge T^a + \frac{\nu\alpha}{2}\lambda_a \wedge \lambda_b \wedge *e^{ab} - d(\alpha e_a \wedge \lambda^a) = 0\,. \end{aligned} \tag{4.42}$$

Therefore the simplified dilaton field equation reads:

$$d(\alpha e_a \wedge \lambda^a + \gamma * \mathcal{D}\alpha) = 0\,. \tag{4.43}$$

Next, using Equation (4.21) together with the expression for torsion 2-forms (4.41), we rewrite the anti-symmetric part of the connection equation in the form of Equation (C.1) as:

$$M(e_a \wedge \lambda_b - e_b \wedge \lambda_a) = \Sigma_{ab}\,, \tag{4.44}$$

where the shorthand expressions read:

$$M = \frac{\alpha}{2} - \alpha\nu, \quad \Sigma_{ab} = -\frac{2}{\mu}\overset{(\Omega)}{R}_{ab} + \mathcal{D}\alpha \wedge *e_{ab}\,. \tag{4.45}$$

From equation (4.44), auxiliary field 1-forms $\{\lambda_a\}$ can be solved uniquely as:

$$\lambda_a = -\frac{4}{\mu\alpha(1-2\nu)}\overset{(\Omega)}{Y}_a + \frac{2}{\alpha(1-2\nu)}\iota_a * \mathcal{D}\alpha\,. \tag{4.46}$$

Finally, the variational field equations of the scale covariant MMG theory read:

$$\begin{aligned} &-2\alpha\overset{(\Omega)}{G}_a + \alpha^3\Lambda * e_a + \frac{\gamma}{2\alpha}\tau_a[\mathcal{D}\alpha] + \frac{\gamma'}{2\alpha}\hat{\tau}_a[dQ] - \frac{4}{\mu(1-2\nu)}\left(\overset{(\Omega)}{D}\overset{(\Omega)}{Y}_a + Q \wedge \overset{(\Omega)}{Y}_a\right) \\ &\quad + \frac{2}{1-2\nu}\left(\overset{(\Omega)}{D}(\iota_a * \mathcal{D}\alpha) + Q \wedge (\iota_a * \mathcal{D}\alpha)\right) + \frac{2\nu}{\alpha(1-2\nu)}\epsilon_a{}^{bc}\Bigg[\frac{4}{\mu^2}\overset{(\Omega)}{Y}_b \wedge \overset{(\Omega)}{Y}_c \\ &\quad\quad - \frac{2}{\mu}\left(\overset{(\Omega)}{Y}_b \wedge \iota_c * \mathcal{D}\alpha - \overset{(\Omega)}{Y}_c \wedge \iota_b * \mathcal{D}\alpha\right) + \iota_b * \mathcal{D}\alpha \wedge \iota_c * \mathcal{D}\alpha\Bigg] = 0\,, \end{aligned} \tag{4.47}$$

$$\left(\gamma + \frac{4}{1-2\nu}\right) * \mathcal{D}\alpha = \left(\frac{6}{\mu} + \frac{2}{\mu'} - \frac{4}{\mu(1-2\nu)}\right)dQ - \gamma' d\left(\frac{1}{\alpha} * dQ\right), \tag{4.48}$$

$$\left(\gamma + \frac{4}{1-2\nu}\right)d * \mathcal{D}\alpha = 0\,. \tag{4.49}$$

Now, we show that original field equations of MMG theory lie in the vacuum configuration of Weyl sector. After we make the choice $\mathcal{D}\alpha = 0$, the field equations (4.48) and (4.49) vanish identically; and the Einstein field equations (4.47) reduce to:

$$-2\alpha\overset{(\Omega)}{G}_a + \alpha^3\Lambda * e_a - \frac{4}{\mu(1-2\nu)}\left(\overset{(\Omega)}{D}\overset{(\Omega)}{Y}_a - \frac{d\alpha}{\alpha}\wedge\overset{(\Omega)}{Y}_a\right) + \frac{8\nu}{\alpha\mu^2(1-2\nu)}\epsilon_a{}^{bc}\overset{(\Omega)}{Y}_b \wedge \overset{(\Omega)}{Y}_c = 0\,. \tag{4.50}$$

Furthermore by setting $\alpha = 1$, Weyl connection 1-form vanishes and connection 1-forms become the metric compatible connection 1-forms $\{\Omega^a{}_b = \omega^a{}_b + K^a{}_b\}$. Then, the Einstein field equations reduce to the original MMG field equations (2.13):

$$-2\overset{(\Omega)}{G}_a + \Lambda * e_a - \frac{4}{\mu(1-2\nu)}\overset{(\Omega)}{C}_a + \frac{8\nu}{\mu^2(1-2\nu)}\epsilon_a{}^{bc}\overset{(\Omega)}{Y}_b \wedge \overset{(\Omega)}{Y}_c = 0\,. \tag{4.51}$$

This calculation demonstrates that Weyl covariant MMG theory contains the original model in the vacuum configuration of its Weyl sector. Hence, scale covariant MMG theory defined by the Lagrangian (4.40) is a consistent generalization.

4.4 Quadratic Curvature Gravity

We start with the following Weyl invariant Lagrangian density 3-form

$$\begin{aligned}\mathcal{L}_{\text{WQCG}} = {} & \alpha\overset{(\Lambda)}{R}{}^a{}_b \wedge *e_a{}^b + \alpha^3\Lambda * 1 + \alpha\lambda_a \wedge T^a - \frac{\gamma}{2\alpha}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha - \frac{\gamma'}{2\alpha}dQ \wedge *dQ \\ & + \frac{1}{\alpha}\left[\kappa_1\overset{(\Lambda)}{R}{}^a{}_b \wedge *\overset{(\Lambda)}{R}{}_a{}^b + \kappa_2\overset{(\Lambda)}{Ric}{}^a \wedge *\overset{(\Lambda)}{Ric}_a + \kappa_3\overset{(\Lambda)}{R}{}^2 * 1\right].\end{aligned} \tag{4.52}$$

Notice that in earlier studies, we used the quadratic curvature identity (B.11) to replace the Ricci-squared term in terms of the other two. Instead of this, we keep all quadratic curvature invariants here. To justify this, let us study the decomposition of quadratic curvature invariants in a Weyl geometry.

First, we can uniquely decompose the curvature 2-forms into their symmetric and anti-symmetric parts according to

$$\overset{(\Lambda)}{R}{}^a{}_b = \overset{(\Omega)}{R}{}^a{}_b - \eta^a{}_b dQ\,. \tag{4.53}$$

Then we also have by contractions

$$\overset{(\Lambda)}{Ric}_a = \overset{(\Omega)}{Ric}_a - \iota_a dQ\,, \quad \overset{(\Lambda)}{R} = \overset{(\Omega)}{R}\,. \tag{4.54}$$

Moreover the co-frame variations of the Einstein-Hilbert term yield

$$-\frac{1}{2}\overset{(\Omega)}{R}{}^{bc}\epsilon_{abc} = *\overset{(\Omega)}{Ric}_a - \frac{1}{2}\overset{(\Omega)}{R} * e_a\,. \tag{4.55}$$

By virtue of working in three dimensions, the preceding equation can be inverted as

$$\overset{(\Omega)}{R}{}^{bc} = -\epsilon^{abc}\left(*\overset{(\Omega)}{Ric}_a - \frac{1}{2}\overset{(\Omega)}{R} * e_a\right)\,. \tag{4.56}$$

Squaring both sides and simplifying, we arrive at the following identity (*c.f.* Equation (B.11)) for the quadratic curvature invariants:

$$\overset{(\Omega)}{R^a}_b \wedge *\overset{(\Omega)}{R_a}^b = 2\overset{(\Omega)}{Ric_a} \wedge *\overset{(\Omega)}{Ric^a} - \frac{1}{2}\overset{(\Omega)}{R^2} * 1. \tag{4.57}$$

On the other hand from Equation (4.53), we have

$$\overset{(\Lambda)}{R^a}_b \wedge *\overset{(\Lambda)}{R_a}^b = \overset{(\Omega)}{R^a}_b \wedge *\overset{(\Omega)}{R_a}^b + 3dQ \wedge *dQ\,, \tag{4.58}$$

and

$$\overset{(\Lambda)}{Ric_a} \wedge *\overset{(\Lambda)}{Ric^a} = \overset{(\Omega)}{Ric_a} \wedge *\overset{(\Omega)}{Ric^a} + 2dQ \wedge *dQ - 2\overset{(\Omega)}{Ric^a} \wedge *\iota_a dQ\,. \tag{4.59}$$

In order to simplify the third term on the right hand side, we consider the second Bianchi identity written in the form

$$\overset{(\Omega)}{R^a}_b \wedge e^b - dQ \wedge e^a = \overset{(\Lambda)}{D}T^a\,, \tag{4.60}$$

and contract on both sides to get

$$\overset{(\Omega)}{Ric_a} \wedge e^a = dQ + \iota_a(\overset{(\Lambda)}{D}T^a)\,. \tag{4.61}$$

Therefore

$$-2\overset{(\Omega)}{Ric^a} \wedge *\iota_a dQ = 2\overset{(\Omega)}{Ric_a} \wedge e^a \wedge *dQ = 2dQ \wedge *dQ + 2\iota_a(\overset{(\Lambda)}{D}T^a) \wedge *dQ\,. \tag{4.62}$$

Putting all these back into our basic quadratic curvature identity B.11, we may write it as

$$\overset{(\Lambda)}{R^a}_b \wedge *\overset{(\Lambda)}{R_a}^b - 2\overset{(\Lambda)}{Ric_a} \wedge *\overset{(\Lambda)}{Ric^a} + \frac{1}{2}\overset{(\Lambda)}{R^2} * 1 = -5dQ \wedge *dQ - 4\iota_a(\overset{(\Lambda)}{D}T^a) \wedge *dQ\,. \tag{4.63}$$

Consequently the quadratic curvature invariants do not satisfy the identity (B.11), which holds in a Riemann-Cartan space-time. Therefore we will take generic quadratic curvature invariants in our action density (4.52).

While finding variational field equations, we vary the Lagrangian density with respect to the total connection 1-forms $\{\Lambda^a{}_b\}$. Then, we separate the connection variation equations according to

$$\dot{\Lambda^a}_b = \dot{\Omega^a}_b - \eta^a{}_b\dot{Q}\,. \tag{4.64}$$

Variation of the Lagrangian density (4.52) is found to be:

$$\begin{aligned}\dot{\mathcal{L}}_{\text{WQCG}} &= \dot{e}^a \wedge \Bigg\{ \alpha \overset{(\Lambda)}{R}{}^b{}_c \epsilon_{ab}{}^c + \alpha^3 \Lambda * e_a + \overset{(\Lambda)}{D}(\alpha\lambda_a) + \frac{\gamma}{2\alpha}\tau_a[\mathcal{D}\alpha] + \frac{\gamma}{2\alpha}\hat{\tau}_a[dQ] \\ &- \frac{\kappa_1}{\alpha}\hat{\tau}_a[\overset{(\Lambda)}{R}{}^b{}_c] + \frac{\kappa_2}{\alpha}\Big[\iota_a\Big(\overset{(\Lambda)}{Ric}_b \wedge *\overset{(\Lambda)}{Ric}{}^b\Big) + 2\iota_a \overset{(\Lambda)}{R}_{bc} \wedge \iota^b * \overset{(\Lambda)}{Ric}{}^c\Big] \\ &+ \frac{\kappa_3}{\alpha}\Big[2\overset{(\Lambda)}{R}\,\overset{(\Lambda)}{R}{}^b{}_c\epsilon_{ab}{}^c - \overset{(\Lambda)}{R}{}^2 * e_a\Big]\Bigg\} + \dot{\lambda}_a \wedge \big(\alpha T^a\big) \\ &+ \dot{\Lambda}^a{}_b \wedge \Bigg\{ \overset{(\Lambda)}{D}\Big[\alpha * e_a{}^b + \frac{2\kappa_1}{\alpha} * \overset{(\Lambda)}{R}_a{}^b - \frac{2\kappa_2}{\alpha}\iota_a * \overset{(\Lambda)}{Ric}{}^b + \frac{2\kappa_3}{\alpha}\overset{(\Lambda)}{R} * e_a{}^b\Big] \\ &+ \alpha e^b \wedge \lambda_a + \frac{1}{3}\eta^b{}_a \gamma * \mathcal{D}\alpha + \frac{1}{3}\eta^b{}_a\gamma' d\Big(\frac{1}{\alpha} * dQ\Big)\Bigg\} \\ &+ \dot{\alpha}\Bigg\{\overset{(\Lambda)}{R}{}^a{}_b \wedge *e_a{}^b + 3\alpha^2\Lambda * 1 + \frac{\gamma}{2\alpha^2}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha + \gamma\mathcal{D}\Big(\frac{1}{\alpha} * \mathcal{D}\alpha\Big) + \lambda_a \wedge T^a \\ &+ \frac{\gamma'}{2\alpha^2} dQ \wedge *dQ - \frac{1}{\alpha^2}\Big[\kappa_1 \overset{(\Lambda)}{R}{}^a{}_b \wedge *\overset{(\Lambda)}{R}_a{}^b + \kappa_2 \overset{(\Lambda)}{Ric}{}^a \wedge *\overset{(\Lambda)}{Ric}_a + \kappa_3 \overset{(\Lambda)}{R}{}^2 * 1\Big]\Bigg\}. \end{aligned} \tag{4.65}$$

We start simplifying this relation by first noting that the torsion 2-forms vanish. Then, we first go to dilaton field equation. To do this, we compare the trace of the co-frame equations,

$$\begin{aligned} e^a \wedge \frac{\delta\mathcal{L}_{\text{WQCG}}}{\delta e^a} &= \alpha \overset{(\Lambda)}{R}{}^a{}_b \wedge *e_a{}^b + 3\alpha^3\Lambda * 1 - \frac{\gamma}{2\alpha}\mathcal{D}\alpha \wedge *\mathcal{D}\alpha + \frac{\gamma'}{2\alpha} dQ \wedge *dQ \\ &+ \alpha\lambda_a \wedge T^a - d(\alpha e^a \wedge \lambda_a) - \frac{\kappa_1}{\alpha}\overset{(\Lambda)}{R}{}^a{}_b \wedge *\overset{(\Lambda)}{R}_a{}^b + \frac{3\kappa_2}{\alpha}\overset{(\Lambda)}{Ric}{}^a \wedge *\overset{(\Lambda)}{Ric}_a \\ &+ \frac{4\kappa_2}{\alpha}\overset{(\Lambda)}{R}{}^a{}_b \wedge \iota^a * \overset{(\Lambda)}{Ric}{}^b - \frac{\kappa_3}{\alpha}\overset{(\Lambda)}{R}{}^2 * 1 = 0\,, \end{aligned} \tag{4.66}$$

with the dilaton field equation in (4.65) and obtain:

$$d(\alpha e_a \wedge \lambda^a + \gamma * \mathcal{D}\alpha) = 0\,. \tag{4.67}$$

Next, we separate the symmetric and anti-symmetric parts of the connection variation equations by lowering an index. In order to lower an index inside a covariant derivative, we make use of Equation (4.21) together with the identities:

$$\overset{(\Lambda)}{D}\Big(\frac{1}{\alpha} * \overset{(\Lambda)}{R}_a{}^b\Big) = -\frac{2Q}{\alpha} \wedge *\overset{(\Lambda)}{R}_a{}^b - \frac{d\alpha}{\alpha^2} \wedge *\overset{(\Lambda)}{R}_a{}^b + \frac{1}{\alpha}\eta^{cb}\overset{(\Lambda)}{D} * \overset{(\Lambda)}{R}_{ac}\,, \tag{4.68}$$

$$\overset{(\Lambda)}{D}\Big(\frac{1}{\alpha}\iota_a * \overset{(\Lambda)}{Ric}{}^b\Big) = -\frac{2Q}{\alpha} \wedge \iota_a * \overset{(\Lambda)}{Ric}{}^b - \frac{d\alpha}{\alpha^2} \wedge \iota_a * \overset{(\Lambda)}{Ric}{}^b + \frac{1}{\alpha}\eta^{cb}\overset{(\Lambda)}{D}(\iota_a * \overset{(\Lambda)}{Ric}_c)\,, \tag{4.69}$$

$$\overset{(\Lambda)}{D}\left(\frac{1}{\alpha}\overset{(\Lambda)}{R} * e_a{}^b\right) = \frac{Q}{\alpha} \wedge \overset{(\Lambda)}{R} * e_a{}^b + \frac{1}{\alpha} d\overset{(\Lambda)}{R} \wedge *e_a{}^b - \frac{d\alpha}{\alpha^2} \wedge \overset{(\Lambda)}{R} * e_a{}^b . \tag{4.70}$$

On the right-hand side of these identities, there are terms proportional to the torsion 2-forms in general. These should be omitted as they identically vanish.

After lowering an index and using (4.64), the symmetric and anti-symmetric parts of the connection variation equations read:

$$\alpha e^a \wedge \lambda_a + \gamma * \mathcal{D}\alpha = (6\kappa_1 + 6\kappa_2 - \gamma')d\left(\frac{1}{\alpha} * dQ\right), \tag{4.71}$$

and

$$\frac{\alpha}{2}(e_a \wedge \lambda_b - e_b \wedge \lambda_a) = \Sigma_{ab} , \tag{4.72}$$

respectively, where

$$\Sigma_{ab} := \left[\mathcal{D}\alpha + \frac{2\kappa_3}{\alpha}\left(d\overset{(\Omega)}{R} + Q\overset{(\Omega)}{R} - \frac{d\alpha}{\alpha}\overset{(\Omega)}{R}\right)\right] \wedge *e_{ab} + 2\kappa_1 \overset{(\Omega)}{D}\left(\frac{1}{\alpha} * \overset{(\Omega)}{R}_{ab}\right)$$
$$+ 2\kappa_2 \overset{(\Omega)}{D}\left(\frac{1}{\alpha}\iota_{[a} * \iota_{b]} dQ - \frac{1}{\alpha}\iota_{[a} * \overset{(\Omega)}{Ric}_{b]}\right).$$

When writing the anti-symmetric part of the connection variation equations in (4.65), we used the fact that index raising and lowering operations commute with the covariant derivative operation with respect to the anti-symmetric connection 1-forms $\{\Omega^a{}_b\}$. We algebraically solve (4.72) for the Lagrange multiplier 1-forms as in Equation (C.4):

$$\lambda_a = \frac{2}{\alpha}\iota^b \Sigma_{ba} - \frac{1}{2\alpha}(\iota^{bc}\Sigma_{cb})e_a . \tag{4.73}$$

The substitution of (4.73) in Einstein field equations of the Weyl covariant theory (4.65) gives:

$$- 2\alpha \overset{(\Omega)}{G}_a + \alpha^3 \Lambda * e_a + \overset{(\Omega)}{D}(\alpha\lambda_a) + \alpha Q \wedge \lambda_a + \frac{\gamma}{2\alpha}\tau_a[\mathcal{D}\alpha] + \left(\frac{\gamma'}{2\alpha} - \frac{3\kappa_1}{\alpha}\right)\hat{\tau}_a[dQ]$$
$$- \frac{\kappa_1}{\alpha}\hat{\tau}_a[\overset{(\Omega)}{R}{}^b{}_c] + \frac{\kappa_2}{\alpha}\left[\iota_a\left(\overset{(\Omega)}{Ric}_b \wedge *\overset{(\Omega)}{Ric}{}^b\right) + 2\iota_a \overset{(\Omega)}{R}_{bc} \wedge \iota^b * \overset{(\Omega)}{Ric}{}^c + 4\iota_a(dQ \wedge *dQ)\right.$$
$$\left. - 2\iota_a \overset{(\Omega)}{R}_{bc} \wedge \iota^b * \iota^c dQ - 6\iota_a dQ \wedge *dQ\right] + \frac{\kappa_3}{\alpha}\left[2\overset{(\Omega)}{R}\overset{(\Omega)}{R}{}^b{}_c \epsilon_{ab}{}^c - \overset{(\Omega)}{R}{}^2 * e_a\right] = 0 . \tag{4.74}$$

Therefore the Weyl covariant quadratic curvature theory defined via the action (4.52) yields three sets of field equations: (4.67), (4.71), and (4.74). In order to show the consistency of this generalization, we restrict the Weyl sector of the theory to its vacuum sector and show that the original quadratic curvature theory field equations are contained in this configuration. To this end, we make the following choices:

$$\mathcal{D}\alpha = 0\,, \quad \alpha = 1 \quad \Longrightarrow \quad Q = 0\,. \tag{4.75}$$

The first choice sets the Weyl connection to be a flat connection, then the second choice fixes a global units scale. Consequently, the Weyl connection 1-form gets cancelled out and we are left with a pseudo-Riemannian geometry. Then the field equations reduce to:

$$\begin{aligned} -2\overset{(\omega)}{G}_a + \Lambda * e_a + \overset{(\omega)}{D}\lambda_a - \kappa_1 \hat{\tau}[\overset{(\omega)}{R}{}^b{}_c] + \kappa_3(2\overset{(\omega)}{R}\overset{(\omega)}{R}{}^b{}_c \epsilon_{ab}{}^c - \overset{(\omega)}{R}{}^2 * e_a) \\ + \kappa_2[\iota_a(\overset{(\omega)}{Ric}{}^b \wedge *\overset{(\omega)}{Ric}_b) + 2\iota_a \overset{(\omega)}{R}_{bc} \wedge \iota^b * \overset{(\omega)}{Ric}{}^c] = 0\,, \end{aligned} \tag{4.76}$$

and

$$e^a \wedge \lambda_a = 0\,, \qquad d(e^a \wedge \lambda_a) = 0\,, \tag{4.77}$$

where

$$\begin{aligned} \lambda_a &= 2\iota^b \Sigma_{ba} - \frac{1}{2}(\iota^{bc}\Sigma_{cb})e_a\,, \quad \text{and} \\ \Sigma_{ab} &= 2\overset{(\omega)}{D}(\kappa_1 * \overset{(\omega)}{R}_{ab} - \kappa_2 \iota_{[a} * \overset{(\omega)}{Ric}_{b]} + \kappa_3 \overset{(\omega)}{R} * e_{ab})\,. \end{aligned}$$

Although Equations (4.76) agree with the field equations (2.18) of quadratic curvature gravity, Equations (4.77) are extra. One must make sure that Equation (4.77) vanishes identically, so that the Weyl covariant generalization is consistent. Notice that vanishing of Equation (4.77) guarantees that $d(e^a \wedge \lambda_a) = 0$.

Before starting to calculate (4.77), we note that

$$e^a \wedge \lambda_a = 2e^a \wedge \iota^b \Sigma_{ba} = 2\iota^b(e^a \wedge \Sigma_{ab})\,. \tag{4.78}$$

Now, we redecorate the anti-symmetric 2-forms $\{\Sigma_{ab}\}$. First using identity (B.1), we see

$$\iota_a * \overset{(\omega)}{Ric}_b = *(\overset{(\omega)}{Ric}_b \wedge e_a)\,, \tag{4.79}$$

where we also made use of $**\chi = -\chi$ for a 2-form χ. Using (4.79), anti-symmetric 2-forms $\{\Sigma_{ab}\}$ now read

$$\Sigma_{ab} = 2\overset{(\Omega)}{D} * \left[\kappa_1 \overset{(\omega)}{R}_{ab} + \frac{\kappa_2}{2}(e_a \wedge \overset{(\omega)}{Ric}_b - e_b \wedge \overset{(\omega)}{Ric}_a) + \kappa_3 \overset{(\omega)}{R} e_{ab}\right],. \tag{4.80}$$

Then using the fact that geometry is torsion free, we can write

$$e^a \wedge \Sigma_{ab} = -2\overset{(\Omega)}{D}\left[e^a \wedge *\left(\kappa_1 \overset{(\omega)}{R}_{ab} + \frac{\kappa_2}{2}(e_a \wedge \overset{(\omega)}{Ric}_b - e_b \wedge \overset{(\omega)}{Ric}_a) + \kappa_3 \overset{(\omega)}{R} e_{ab}\right)\right]. \tag{4.81}$$

Identity (B.1) implies $e^a \wedge *\chi = -*\iota^a\chi$ and we can put Equation (4.81) in the form:

$$\begin{aligned} 2\iota^b(e^a \wedge \Sigma_{ab}) &= 4\overset{(\Omega)}{D^b} * \iota^a\left[\kappa_1 \overset{(\omega)}{R}_{ab} + \frac{\kappa_2}{2}(e_a \wedge \overset{(\omega)}{Ric}_b - e_b \wedge \overset{(\omega)}{Ric}_a) + \kappa_3 \overset{(\omega)}{R} e_{ab}\right] \\ &= 4\overset{(\Omega)}{D^a} * \left[\left(\kappa_1 + \frac{\kappa_2}{2}\right)\overset{(\omega)}{Ric}_a + \left(2\kappa_3 + \frac{\kappa_2}{2}\right)\overset{(\omega)}{R} e_a\right]. \end{aligned} \tag{4.82}$$

Finally equating (4.82) to zero, we deduce

$$e^a \wedge \lambda_a = 0 \quad \Leftrightarrow \quad (2\kappa_1 + 3\kappa_2 + 8\kappa_3)\iota_a * d\overset{(\omega)}{R} = 0\,. \tag{4.83}$$

Then, either the space-time has constant curvature or else in a generic space-time, only certain combinations of quadratic curvature invariants for which

$$2\kappa_1 + 3\kappa_2 + 8\kappa_3 = 0 \tag{4.84}$$

are allowed. It is remarkable that NMG model, with $\kappa_1 = 0\,, \kappa_2 = 1\,, \kappa_3 = -\frac{3}{8}$, meets this condition.

4.5 A Class of Nontrivial Exact Solutions to TMG

Now we consider scale covariant TMG model, and look for exact solutions. We start by restating the variational field equations of the model:

$$\begin{aligned} -2\alpha\overset{(\Omega)}{G}_a - \alpha^3\Lambda * e_a - \frac{4}{\mu_1}\overset{(\Omega)}{D}\overset{(\Omega)}{Y}_a - \frac{4}{\mu_1}Q \wedge \overset{(\Omega)}{Y}_a + 2\overset{(\Omega)}{D}(\iota_a * \mathcal{D}\alpha) \\ + 2Q \wedge (\iota_a * \mathcal{D}\alpha) + \frac{\gamma_1}{2\alpha}\tau_a[\mathcal{D}\alpha] + \frac{\gamma_2}{2\alpha}\hat{\tau}_a[dQ] = 0\,, \end{aligned} \tag{4.85}$$

$$(\gamma_1 + 4) * \mathcal{D}\alpha = \left(\frac{2}{\mu_1} + \frac{2}{\mu_2}\right) dQ - \gamma_2 d\left(\frac{1}{\alpha} * dQ\right), \tag{4.86}$$

$$(\gamma_1 + 4) d * \mathcal{D}\alpha = 0\,, \tag{4.87}$$

where the stress-energy 2-forms of dilaton α and Weyl connection 1-form Q read

$$\tau_a[\mathcal{D}\alpha] = (\iota_a \mathcal{D}\alpha) * \mathcal{D}\alpha + \mathcal{D}\alpha \wedge \iota_a * \mathcal{D}\alpha\,, \tag{4.88}$$

$$\hat{\tau_a}[dQ] = \iota_a dQ \wedge *dQ - dQ(\iota_a * dQ)\,, \tag{4.89}$$

respectively. Notice that we replace μ and μ' with μ_1 and μ_2; and γ and γ' with γ_1 and γ_2, respectively.

At this point we take a negative cosmological constant $\Lambda = -2/l^2$ and choose a constant local units frame by setting $\alpha = 1$. As the dilaton field equation (4.87) is already contained in the Weyl 1-form equation (4.86), the choice $\alpha = 1$ also amounts to a gauge fixing. Furthermore we set $\gamma_1 = -4$ which is the value of the coupling constant of the scalar field in 3D Brans-Dicke theory which makes that theory scale covariant. These conditions make significant simplifications to the field equations. For instance we have $\mathcal{D}\alpha = Q$ and many terms directly drop out. The simplified form of the field equations read:

$$\begin{aligned} -2\overset{(\Omega)}{G}_a + \frac{2}{l^2} * e_a - \frac{4}{\mu_1}\overset{(\Omega)}{D}\overset{(\Omega)}{Y}_a - \frac{4}{\mu_1} Q \wedge \overset{(\Omega)}{Y}_a + 2\overset{(\Omega)}{D}(\iota_a * Q) \\ + 2Q \wedge (\iota_a * Q) - 2\tau_a[Q] + \frac{\gamma_2}{2}\hat{\tau_a}[dQ] = 0\,, \end{aligned} \tag{4.90}$$

$$\left(\frac{2}{\mu_1} + \frac{2}{\mu_2}\right) dQ - \gamma_2 d * dQ = 0\,, \tag{4.91}$$

$$d * Q = 0\,. \tag{4.92}$$

Notice that after setting $\gamma_1 = -4$, we do not get rid of the dilaton field equation (4.87). Later we use this equation for the component Q_1 of the Weyl 1-form which cannot be obtained from the Weyl connection equation (4.91). Actually (4.92) is just the partial gauge fixing condition for the Lorenz gauge. We now move to our ansatz for the solution of metric functions and Weyl 1-form.

We choose a local coordinate system (t, ρ, ϕ) to express our ansatz for the co-frame 1-forms:

$$e^0 = f(\rho)dt\,, \quad e^1 = d\rho\,, \quad e^2 = h(\rho)(d\phi + a(\rho)dt)\,. \tag{4.93}$$

The resulting metric

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \tag{4.94}$$

is well-suited for the study of stationary and rotationally symmetric solutions. Furthermore the ansatz for the Weyl 1-form is

$$Q = Q_0(\rho)e^0 + Q_1(\rho)e^1 + Q_2(\rho)e^2 \tag{4.95}$$

whose field strength 2-form may be expressed as:

$$dQ = Ee^{01} + Be^{12}\,. \tag{4.96}$$

Here we denote the components of field strength with E and B but note that they take values in a purely real algebra, not a pure imaginary one. Although they are analogous to electric and magnetic fields, they are different. The components

$$E = -Q_0' - \delta Q_0 - \beta Q_2\,, \quad B = Q_2' + \sigma Q_2 \tag{4.97}$$

of the field strength for the Weyl connection are defined in terms of the components of Weyl 1-form and Levi-Civita connection functions

$$\delta = \frac{f'}{f}\,, \quad \beta = \frac{a'h}{f}\,, \quad \sigma = \frac{h'}{h}\,, \tag{4.98}$$

where a prime over a function denotes derivation with respect to the variable ρ. In terms of the Levi-Civita part of the connection components (4.98), we define the Riemann curvature functions which are useful when expressing the field equations:

$$\begin{aligned} W &= \delta' + \delta^2 - \frac{3\beta^2}{4}\,, \qquad X = \frac{\beta'}{2} + \beta\sigma\,, \\ Y &= \delta\sigma + \frac{\beta^2}{4}\,, \qquad Z = \sigma' + \sigma^2 + \frac{\beta^2}{4}\,. \end{aligned} \tag{4.99}$$

Under this choice, Einstein field equations (4.90) read:

$$-2Z + \frac{2}{l^2} + \frac{4}{\mu_1}\left[X' + \sigma X + \frac{\beta}{2}(Y - W)\right] = \frac{\gamma_2}{2}\left(E^2 + B^2\right), \tag{4.100}$$

$$-2X + \frac{4}{\mu_1}\left[\frac{1}{2}(Z - W - Y)' + \delta(Z - Y) + \frac{3}{2}\beta X\right] = -\gamma_2 EB, \tag{4.101}$$

$$2Y - \frac{2}{l^2} + \frac{4}{\mu_1}\left[(\sigma - \delta)X + \frac{\beta}{2}(W - Z)\right] = -\frac{\gamma_2}{2}\left(E^2 - B^2\right), \tag{4.102}$$

$$2W - \frac{2}{l^2} + \frac{4}{\mu_1}\left[X' + \delta X + \frac{\beta}{2}(Y + Z - 2W)\right] = \frac{\gamma_2}{2}\left(E^2 + B^2\right). \tag{4.103}$$

Weyl connection 1-form equations (4.91) read:

$$\left(\frac{2}{\mu_1} + \frac{2}{\mu_2}\right)B + \gamma_2(E' + \sigma E) = 0, \tag{4.104}$$

$$\left(\frac{2}{\mu_1} + \frac{2}{\mu_2}\right)E + \gamma_2(B' + \delta B - \beta E) = 0. \tag{4.105}$$

Finally, the dilaton field equation (4.92) reads:

$$Q_1' + (\delta + \sigma)Q_1 = 0. \tag{4.106}$$

The field equations (4.100 - 4.105), are very similar to the ones coming from TMG coupled with Maxell-Chern-Simons theory. This is mainly because we imposed $\gamma_1 = -4$, otherwise the field equations would be too complicated. However the additional dilaton field equation (4.106) is new and will be used to determine the Q_1 component of the Weyl 1-form. It can be integrated to yield:

$$Q_1 = \frac{\tilde{c_0}}{fh}, \tag{4.107}$$

where $\tilde{c_0}$ is an integration constant and from now on quantities with a tilde on top denote integration constants till the end of this section.

The remaining components Q_0 and Q_2 are related to each other by imposing the self-duality condition

$$Q_0 = -kQ_2, \qquad E = kB, \qquad k^2 = 1, \quad k \in \mathbb{R}. \tag{4.108}$$

In earlier studies of the self-dual solutions [Dereli and Sarıoğlu, 2000, Dereli and Sarıoğlu, 2001, Dereli and Obukhov, 2000], the self-duality condition is imposed directly on the components of the field strength dQ, however here we are interested in the solution of the components of the Weyl connection 1-form as it solely governs the non-metric contributions coming to the geometry. Consequently we have to impose the self-duality condition at the level of components of Weyl connection 1-form. Self-duality (4.108) is satisfied given that connection functions (4.98) are related by $\sigma + k\beta = \delta$. Putting this in (4.102) and noting that the right hand side vanishes, we find that the Levi-Civita connection components are described by the single function β as:

$$\delta = \frac{1}{l} + \frac{k\beta}{2}\,, \qquad \sigma = \frac{1}{l} - \frac{k\beta}{2}\,. \tag{4.109}$$

As a result the components (4.99) of the Riemannian part of curvature 2-forms simplify and read

$$W = u + \frac{1}{l^2}\,, \quad X = ku, \quad Y = \frac{1}{l^2}\,, \quad Z = -u + \frac{1}{l^2}\,, \tag{4.110}$$

where we defined the function

$$u = \frac{k\beta'}{2} + \frac{k\beta}{l} - \frac{\beta^2}{2}\,. \tag{4.111}$$

Imposing (4.110), we find that the field equations (4.100 - 4.105) are satisfied simultaneously given that the following coupled equations hold:

$$u' - k\beta u + \left(\frac{1}{l} + \frac{k\mu_1}{2}\right)u = 0\,, \tag{4.112}$$

$$y' - k\beta y + \left[\frac{2}{l} + \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\right] = 0\,. \tag{4.113}$$

where $y = (k\mu_1\gamma_2/4)E^2$. First, the equations (4.112) and (4.113) are solved for the variables β and y and afterwards using the solution for y, we are going to solve for the remaining Weyl 1-form components.

In (4.112), we make the change of variable $y = ku/z$ to obtain a first order differential equation for the function z:

$$z' + \left[-\frac{1}{l} + \frac{k\mu_1}{2} - \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\right]z = k\,. \tag{4.114}$$

The solution of equation (4.114) is

$$z = \frac{k}{-\frac{1}{l} + \frac{k\mu_1}{2} - \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)} \left[1 + \tilde{c_1} e^{\left(\frac{1}{l} - \frac{k\mu_1}{2} + \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\right)\rho}\right] \tag{4.115}$$

for some integration constant $\tilde{c_1}$. Putting this result back in (4.112), we obtain

$$u' - k\beta u + \left[\frac{1}{l} + \frac{k\mu_1}{2} + \frac{\frac{1}{l} - \frac{k\mu_1}{2} + \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)}{1 + \tilde{c_1} e^{\left(\frac{1}{l} - \frac{k\mu_1}{2} + \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\right)\rho}}\right] u = 0\,. \tag{4.116}$$

Setting $u = k\beta/v$, in (4.116) we find a linear equation for the variable v:

$$v' + \left[\frac{1}{l} - \frac{k\mu_1}{2} - \frac{\frac{1}{l} - \frac{k\mu_1}{2} + \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)}{1 + \tilde{c_1} e^{\left(\frac{1}{l} - \frac{k\mu_1}{2} + \frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\right)\rho}}\right] v = 2\,, \tag{4.117}$$

which we can solve as

$$v = \frac{2\varphi}{\varphi'}, \quad \varphi = \frac{\tilde{c_1} e^{\left(\frac{1}{l} - \frac{k\mu_1}{2}\right)\rho}}{\frac{1}{l} - \frac{k\mu_1}{2}} - \frac{e^{\left(-\frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\right)\rho}}{\frac{4k}{\gamma_2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)} + \tilde{c_2}\,, \tag{4.118}$$

where $\tilde{c_2}$ is some integration constant and φ is a function that we have defined for simplifying the forthcoming calculations. Putting this into the expression (4.111), we obtain a non-linear equation for the function β:

$$\beta' + \left(\frac{2}{l} - \frac{\varphi'}{\varphi}\right)\beta - k\beta^2 = 0\,. \tag{4.119}$$

Making the change of variable $w = 1/\beta$, we obtain a linear equation

$$w' + \left(\frac{\varphi'}{\varphi} - \frac{2}{l}\right) w = -k\,. \tag{4.120}$$

The solution of equation (4.120) may be expressed as:

$$w = \frac{k\Omega}{e^{-\frac{2\rho}{l}}\varphi}\,, \qquad \Omega = \tilde{c_3} - \int^{\rho} e^{-\frac{2\rho'}{l}} \varphi(\rho') d\rho'\,. \tag{4.121}$$

Using (4.121), we can express the sole connection function as:

$$\beta = \frac{k e^{-\frac{2\rho}{l}}\varphi}{\Omega}\,. \tag{4.122}$$

Using (4.122) in (4.109) and then using the definition of metric functions (4.98), we obtain first order equations for the metric functions which we integrate to yield:

$$f = \tilde{f}_0 e^{\frac{\rho}{l}} \Omega^{-1/2} , \tag{4.123}$$

$$h = \tilde{h}_0 e^{\frac{\rho}{l}} \Omega^{1/2} , \tag{4.124}$$

$$a = \frac{k\tilde{f}_0}{\tilde{h}_0} \Omega^{-1} + \tilde{a}_0 . \tag{4.125}$$

Going back to the definition $y = (k\mu_1\gamma_2/4)E^2$, we find the components of the field strength as

$$E = \left[\frac{\left(-\frac{2k}{\mu_1\gamma_2 l} + \frac{1}{\gamma_2} - \frac{8}{\mu_1\gamma_2^2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right) \varphi' e^{-\frac{2\rho}{l}}}{\Omega \left(1 + \tilde{c}_1 e^{\left(\frac{1}{l} - \frac{k\mu_1}{2} + \frac{4k}{\gamma_2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right) \rho} \right)} \right]^{1/2} , \qquad B = kE . \tag{4.126}$$

Using the definition (4.97) together with the self-duality condition (4.108), we can obtain a first order linear equation for the remaining component of the Weyl 1-form:

$$Q_2' + \left(\frac{\Omega'}{2\Omega} - \frac{1}{l} \right) Q_2 = \left[\frac{\left(-\frac{2k}{\mu_1\gamma_2 l} + \frac{1}{\gamma_2} - \frac{8}{\mu_1\gamma_2^2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right) \varphi' e^{-\frac{2\rho}{l}}}{\Omega \left(1 + \tilde{c}_1 e^{\left(\frac{1}{l} - \frac{k\mu_1}{2} + \frac{4k}{\gamma_2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right) \rho} \right)} \right]^{1/2} . \tag{4.127}$$

Equation (4.127) may be integrated to obtain a solution for the remaining components of the Weyl 1-form:

$$\begin{aligned} Q_2 = & \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right)^{-1} \left[-\frac{k\gamma_2}{2\mu_1 l} + \frac{\gamma_2}{4} - \frac{2}{\mu_1} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right]^{\frac{1}{2}} \\ & \times \frac{e^{-\left(\frac{1}{l} + \frac{2k}{\gamma_2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right) \rho}}{\Omega^{1/2}} \left[1 + \tilde{c}_4 e^{\frac{2k}{\gamma_2} \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \rho} \right] , \end{aligned} \tag{4.128}$$

$$Q_0 = -kQ_2 . \tag{4.129}$$

Finally using (4.123) and (4.124) in (4.107) we obtain the last component of Weyl connection as

$$Q_1 = \frac{\tilde{c}_0}{\tilde{f}_0 \tilde{h}_0} e^{-\frac{2\rho}{l}} . \tag{4.130}$$

Remarks: We start by showing that our solution is nontrivial in the sense that it cannot be obtained from a solution of TMG theory by a local gauge transformation. This can be observed by noting that such solutions satisfy the condition $\mathcal{D}\alpha = 0$. Because when this condition is satisfied, the Weyl 1-form is given by

$$Q = -d(\log \alpha) \qquad \Longrightarrow \qquad dQ = 0\,. \tag{4.131}$$

Such solutions are pure gauge and consequently can be obtained directly starting from a solution of the TMG model. In our case, we have $\alpha = 1$, $dQ \neq 0$ and therefore our solution is not a pure gauge in the Weyl sector.

Next, we make some remarks about the self-duality condition for the Weyl connection and field strength. For the solutions we have found, if one calculates the Chern-Simons and Chern-Pontryagin densities for the Weyl connection, one obtains

$$Q \wedge dQ = 0\,, \qquad dQ \wedge *dQ = 0\,. \tag{4.132}$$

This is because we take our ansatz to be only dependent on the radial coordinate on a rotationally symmetric and stationary background. If we were to take a Weyl 1-form which also depended on an angular variable, then this would not be the case in general. Furthermore the conditions (4.132) show that self-dual ansatz is particularly effective to study models that have abelian gauge symmetries. We suspect that non-abelian analogues of self-dual ansatz would be as effective and note that it would be interesting to see such examples.

As advertised, this solution describes a non-metric geometry because Weyl 1-form contributes to the connection 1-forms and curvature 2-forms, which may be readily seen from the decompositions:

$$\Lambda^a{}_b = \hat{\omega}^a{}_b + Q^a e_b - Q_b e^a - Q\eta^a{}_b\,, \tag{4.133}$$

$$\overset{(\Lambda)}{R}{}^a{}_b = \overset{(\omega)}{R}{}^a{}_b + \overset{(\omega)}{D}(Q^a e_b - Q_b e^a) - dQ\eta^a{}_b\,. \tag{4.134}$$

In spite of these contributions, it is curious to see that the Chern-Pontryagin density of the full linear connection is completely determined by the Riemannian piece. To verify this, we can decompose the Chern-Pontryagin density as

$$\overset{(\Lambda)}{R}{}^a{}_b \wedge *\overset{(\Lambda)}{R}{}_a{}^b = \overset{(\Omega)}{R}{}^a{}_b \wedge *\overset{(\Omega)}{R}{}_a{}^b + 3dQ \wedge *dQ\,, \tag{4.135}$$

and each term on the right hand side reads

$$\overset{(\Omega)}{R}{}^a{}_b \wedge *\overset{(\Omega)}{R}{}_a{}^b = \overset{(\omega)}{R}{}^a{}_b \wedge *\overset{(\omega)}{R}{}_a{}^b = \frac{3}{l^4}\,, \qquad dQ \wedge *dQ = 0\,. \tag{4.136}$$

This calculation also shows that the Chern-Pontryagin invariant is regular everywhere for our solution.

Chapter 5

SUPERGRAVITY MODELS

We now move on to the formulation of supergravity models. Supergravity models are supersymmetric generalizations of gravity models. Supersymmetry is a symmetry between the bosonic and fermionic degrees of freedom in a model which is implemented by the use of a $\mathbb{Z}_2$-graded symmetry algebras. The even symmetry generators are associated with bosonic symmetries whereas the odd ones with fermionic ones. For details and more motivation on supergravity models, we refer to [Gates Jr et al., 2001, Freedman and Van Proeyen, 2012] and now move on to three dimensional models.

As noted earlier, Einstein's GR in three dimensions has no propagating gravitational degree of freedom. In a similar way, its simple ($\mathcal{N} = 1$) locally supersymmetric generalization has no propagating degree of freedom either. It may be possible to introduce a cosmological constant Λ and a mass constant m for the gravitino as well and it is still possible to maintain local supersymmetry provided $\Lambda = -m^2$ [Howe and Tucker, 1978]. Then for $\Lambda \neq -m^2$, this would mean that there is one massive gravitino degree of freedom with a non-dynamical background metric and an algebraic torsion that is quadratic in the gravitino field [Dereli and Deser, 1978].

As discussed before, a propagating graviton degree of freedom to GR can be induced by introducing a gravitational Chern-Simons density 3-form. This term generates the Cotton tensor with components that involve third derivatives of the metric components. These higher derivatives are the reason why the theory is now dynamical. Despite having third order field equations in metric components, the theory turns out to be ghost-free and implies causal propagation and has been extensively studied in the literature. An obvious question could be raised at this point: Is there a simple ($\mathcal{N} = 1$) cosmological topologically massive supergravity theory? The answer is yes. This theory is discovered many years ago, without the cosmological constant in [Deser and Kay, 1989] and with the cosmological constant in [Deser, 1984]. Both in [Deser and Kay, 1989] and [Deser, 1984], a locally supersymmetric action density 3-form with both of its bosonic and fermionic sectors is written down and treated using second order variational formalism. That is, the connection coefficients were assumed to be fixed in terms of the dreibein and gravitino fields and not treated as independent variables. Some families of its supersymmetric solutions and several generalizations of it were studied since then [Aragone, 1987, Gibbons et al., 2008, Abecasis and Zandron, 2010, Becker et al., 2009, Routh, 2013, Percacci et al., 2014]. Yet the complete, explicit expressions for the variational field equations of the theory are still missing to the best of our knowledge.

Our aim in this chapter is a pedagogical one. Firstly, we explicitly demonstrate the local supersymmetry of cosmological topologically massive supergravity action. Next we derive the consistent set of its variational field equations. All the details of calculation are shown. The fact that long time has passed since the theory has been introduced, and yet there is an the apparent lack of explicit derivation of the complete set of variational field equations may be considered sufficient. For the variational principle in our formulations, we use the so-called 1.5-formalism. That is for the variational formalism, connection, co-frame and gravitino fields are taken as independent. However the local supersymmetry transformation of the connection field is not independent and will be solved from that of co-frame and gravitino fields. Now we move on to formulate supergravity models.

5.1 Cosmological Supergravity

The action for cosmological supergravity theory [Howe and Tucker, 1978, Dereli and Deser, 1978]

$$S[e^a, \omega^{ab}, \chi] = \int_M \mathcal{L}_{\mathrm{CSG}}\,, \tag{5.1}$$

where the Lagrangian density $\mathcal{L}_{\mathrm{CSG}} = \mathcal{L}_{\mathrm{SG}} + \mathcal{L}_{\mathrm{C}}$ can be decomposed in terms of the action densities for the simple supergravity and the cosmological sectors given by

$$\mathcal{L}_{\mathrm{SG}} = -\frac{1}{2} R^{ab} \wedge *e_{ab} - \frac{i}{2}\bar{\chi} \wedge D\chi\,,\,, \tag{5.2}$$

and

$$\mathcal{L}_{\mathrm{C}} = \Lambda * 1 - \frac{im}{4}\bar{\chi} \wedge \gamma \wedge \chi\,, \tag{5.3}$$

respectively. Here we set the gravitational constant $K = 1$, Λ is a cosmological constant and m is a mass parameter. The fermionic part of the supergravity and cosmological sectors are the kinetic and non-topological mass terms for the Rarita-Schwinger (gravitino) field. The gravitino field χ and its field strength $D\chi$ are Majorana spinor valued 1- and 2-form fields, respectively:

$$\chi = (\iota_a \chi) e^a = \chi_a e^a\,, \qquad D\chi = \frac{1}{2}(\iota_{ba} D\chi) e^{ab} = (D\chi)_{[ab]} e^{ab}\,. \tag{5.4}$$

Furthermore we introduced a gamma matrix valued 1-form $\gamma = \gamma_a e^a$ to write down the mass term for the gravitino field. The cosmological constant and the "mass" of the gravitino field shall be related below via $\Lambda = -m^2$ for local supersymmetry.

Then the total variation of the action reads (upto a closed form)

$$\begin{aligned}\dot{\mathcal{L}} = \dot{e^a} \wedge \Big\{ -\frac{1}{2}\epsilon_{abc} R^{bc} + i\frac{m}{4}\bar{\chi} \wedge \gamma_a \chi + \Lambda * e_a \Big\} \\ + \dot{\omega}^{ab} \wedge \Big\{ -\frac{1}{2}\epsilon_{abc}\Big(T^c - \frac{i}{4}\bar{\chi} \wedge \gamma^c \chi \Big)\Big\} + \dot{\bar{\chi}} \wedge \Big\{ -iD\chi - \frac{im}{2}\gamma \wedge \chi \Big\}.\end{aligned} \tag{5.5}$$

We determine from this expression the coupled field equations of the cosmological supergravity theory:

$$G_a + \Lambda * e_a + i\frac{m}{4}\bar{\chi} \wedge \gamma_a \chi = 0\,, \tag{5.6}$$

$$D\chi + \frac{m}{2}\gamma \wedge \chi = 0\,, \tag{5.7}$$

$$T^a = \frac{i}{4}\bar{\chi} \wedge \gamma^a \chi\,. \tag{5.8}$$

The (infinitesimal) local supersymmetry transformations of the supergravity multiplet are given as usual by:

$$\dot{e^a} = i\bar{\alpha}\gamma^a\chi\,, \qquad \dot{\chi} = 2D\alpha + m\gamma\alpha\,, \tag{5.9}$$

where the local supersymmetry parameter $\alpha = \alpha(x)$ is an arbitrary odd-Grassmann valued Majorana spinor. In order to determine the supersymmetry transformation law for the connection field, we look at the variation of the first Cartan structure equation **correct this ref. later in the appendix** (4.119) which yields

$$\dot{\omega}_{ab} \wedge e^b = -iD(\bar{\alpha}\gamma_a\chi) + \dot{T}_a = \frac{im}{2}\bar{\alpha}\gamma\gamma_a \wedge \chi - i\bar{\alpha}\gamma_a D\chi =: Z_a\,. \tag{5.10}$$

The final simplification follows from the field equations (5.8). The solution to the system of equations (5.10) is obtained algebraically as

$$2\dot{\omega}_{ab} = \iota_a Z_b - \iota_b Z_a - e^c(\iota_{ab} Z_c)\,. \tag{5.11}$$

that explicitly yields

$$\begin{aligned}\dot{\omega}_{ab} = &\frac{i}{2}\Big(\bar{\alpha}\gamma_a\iota_b(D\chi) - \bar{\alpha}\gamma_b\iota_a(D\chi) + \bar{\alpha}\gamma\iota_{ab}(D\chi)\Big) \\ &- \frac{im}{2}\Big(\epsilon_{abc}(\bar{\alpha}\gamma^c\chi) + e_a(\bar{\alpha}\chi_b) - e_b(\bar{\alpha}\chi_a)\Big)\,.\end{aligned} \tag{5.12}$$

Although this does not contribute to the transformation of the action density on-shell, we give the transformation law (5.12) for the connection 1-forms for completeness. This result will be relevant when we discuss cosmological topologically massive supergravity theory in what follows.

We now prove the local supersymmetry of cosmological supergravity theory. Let us first consider the contributions that are independent of m in the variations of the action density under our local supersymmetry transformations:

$$\dot{\mathcal{L}}_{\rm SG}(m=0) = -\frac{i}{2}\epsilon_{abc}(\bar{\alpha}\gamma^a\chi)\wedge R^{bc} - 2iD\bar{\alpha}\wedge D\chi\,. \tag{5.13}$$

This particular combination may be shown to add up to a closed form by noting that:

$$-\frac{i}{2}\epsilon_{abc}(\bar{\alpha}\gamma^a\chi)\wedge R^{bc} = -i\bar{\alpha}R^{ab}\sigma_{ab}\wedge\chi\,, \tag{5.14}$$

and

$$-2iD\bar{\alpha}\wedge D\chi = i\bar{\alpha}R^{ab}\sigma_{ab}\wedge\chi + d(-2i\bar{\alpha}D\chi)\,. \tag{5.15}$$

The rest of the contributions ($m \neq 0$) are given by

$$i\Lambda\bar{\alpha} * \gamma\wedge\chi + \frac{im^2}{2}\bar{\alpha}\gamma\wedge\gamma\wedge\chi - \frac{m}{4}(\bar{\alpha}\gamma^a\chi)\wedge(\bar{\chi}\wedge\gamma_a\chi)$$
$$- imD\bar{\alpha}\wedge\gamma\wedge\chi + im\bar{\alpha}\wedge\gamma\wedge D\chi\,. \tag{5.16}$$

The first two terms cancel each other out when we set $\Lambda = -m^2$ and use the identity $\gamma\wedge\gamma = 2 * \gamma$. The last two terms on the other hand can be combined to give

$$-imD\bar{\alpha}\wedge\gamma\wedge\chi + im\bar{\alpha}\wedge\gamma\wedge D\chi = -d(im\bar{\alpha}\gamma\wedge\chi) - \frac{m}{4}(\bar{\alpha}\gamma^a\chi)\wedge(\bar{\chi}\wedge\gamma_a\chi)\,. \tag{5.17}$$

When all the above contributions are put together, we are left with a closed form plus a non-linear spinorial expression

$$-\frac{m}{2}(\bar{\alpha}\gamma^a\chi)\wedge(\bar{\chi}\wedge\gamma_a\chi)\,. \tag{5.18}$$

It is not difficult to verify that (5.18) vanishes identically by performing a Fierz rearrangement. However, some care is needed for signs during the Fierz rearrangements because we are dealing with spinor valued differential forms. One must first open up an expression in the co-frame basis, apply the Fierz rearrangement formula to the components and then bring back in the basis forms. The final outcome reads

$$-\frac{m}{2}(\bar{\alpha}\gamma^a\chi)\wedge(\bar{\chi}\wedge\gamma_a\chi) = -\frac{m}{2}(\bar{\alpha}\chi)\wedge(\bar{\chi}\wedge\chi) = 0\,. \tag{5.19}$$

With this result, the local supersymmetry of the cosmological supergravity action (5.1) is proven.

5.2 Cosmological Topologically Massive Supergravity

The action functional of this model is given by

$$S[e^a, \omega^{ab}, \chi, \lambda_a] = \int_M \mathcal{L}_{\text{Total}} \tag{5.20}$$

that will be varied independently with respect to the co-frames $\{e^a\}$, connection 1-forms $\{\omega^{ab}\}$ and the gravitino 1-form χ as before. We further introduce below Lagrange multipliers 1-forms $\{\lambda_a\}$ that are also varied as independent variables.

Now our Lagrangian density 3-form decomposes according to,

$$\mathcal{L}_{\text{Total}} = \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{SG}} + \mathcal{L}_{\text{C}} + \mathcal{L}_{\text{Constraint}}\,, \tag{5.21}$$

where we added on to the cosmological supergravity action density (5.2) of the previous section, the topological Chern-Simons density 3-form

$$\mathcal{L}_{\text{CS}} = \frac{1}{\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \wedge \omega^b{}_c \wedge \omega^c{}_a\right) - \frac{i}{\mu}\left(D\bar{\chi} \wedge {*D\chi} + {*D\bar{\chi}} \wedge \gamma \wedge {*D\chi}\right). \tag{5.22}$$

Here μ is a new coupling constant. It should be noted that the fermionic part of the Chern-Simons density (5.22) contains derivatives of order 2 of the gravitino field. This is consistent with the fact that third order derivatives of metric components appear in the usual bosonic part of Chern-Simons density. Alternatively, the fermionic part of the Chern-Simons density 3-form could have been expressed as

$$-\frac{i}{\mu}\left(2D\bar{\chi} \wedge {*D\chi} - D\bar{\chi} \wedge \gamma * (\gamma \wedge D\chi)\right),$$

which seems more suitable for a Hamiltonian description. However, as far as the variations of the action are concerned this form is considerably harder to work with and we prefer to use our form of the topological action density. We furthermore introduced a set of Lagrange multiplier 1-forms $\{\lambda_a\}$ that appear linearly in the constraint Lagrangian density

$$\mathcal{L}_{\text{Constraint}} = \left(T^a - \frac{i}{4}\bar{\chi} \wedge \gamma^a \chi\right) \wedge \lambda_a\,. \tag{5.23}$$

Then independent variations of the action relative to the Lagrange multipliers impose the constraint that the space-time torsion 2-forms are given algebraically by (5.8) as in the previous section. The remaining variational field equations are to be solved subject to this Lagrangian constraint. In Riemannian space-times, in a similar way, one may introduce a constraint term of the form $T^a \wedge \lambda_a$ in the action whose variations with respect to the multipliers set the space-time torsion to zero in a first order constrained variational formulation of gravitational theories. However, since we are working with a supergravity theory we don't want the torsion to vanish, but be equal to the quadratic expression given by (5.8). As we are going to show later on, the origin of the Cotton tensor which involves third derivatives of the metric components in the Einstein field equations is due to this constraint term. The torsion constraint furthermore ensures that the supersymmetry transformation law (5.12) of the connection 1-forms remains as it is in the previous section.

The variation of the total action with respect to the independent field variables turns out to be, modulo a closed form,

$$\begin{aligned}
\dot{\mathcal{L}}_{\text{Total}} = \dot{e^a} \wedge \Bigg\{ & -\frac{1}{2}\epsilon_{abc}R^{bc} - m^2 * e_a + i\frac{m}{4}\bar{\chi} \wedge \gamma_a \chi \\
& + \frac{i}{\mu}\Big(\iota_a D\bar{\chi} \wedge *D\chi - D\bar{\chi}\iota_a * D\chi + *D\bar{\chi} \wedge \gamma_a * D\chi \\
& - 2\iota_a * D\bar{\chi}\gamma \wedge *D\chi + 2\iota_a D\bar{\chi} \wedge *(\gamma \wedge *D\chi)\Big) + D\lambda_a \Bigg\} \\
+ \dot{\omega}^{ab} \wedge \Bigg\{ & -\frac{1}{2}\epsilon_{abc}\Big(T^c - \frac{i}{4}\bar{\chi} \wedge \gamma^c\chi\Big) - \frac{2}{\mu}R_{ab} - \frac{1}{2}\Big(\lambda_a \wedge e_b - \lambda_b \wedge e_a\Big) \\
& + \frac{i}{2\mu}\epsilon_{abc}\Big(\bar{\chi} \wedge \gamma^c * D\chi + \bar{\chi} \wedge \gamma^c * (\gamma \wedge *D\chi)\Big)\Bigg\} \\
+ \dot{\bar{\chi}} \wedge \Bigg\{ & -iD\chi - i\frac{m}{2}\gamma \wedge \chi - \frac{2i}{\mu}\Big(D * D\chi + D * (\gamma \wedge *D\chi)\Big) + \frac{i}{2}\lambda_a \wedge \gamma^a\chi \Bigg\} \\
+ \dot{\lambda}_a \wedge \Bigg\{ & T^a - \frac{i}{4}\bar{\chi} \wedge \gamma^a\chi \Bigg\}. \qquad (5.24)
\end{aligned}$$

We will first demonstrate the local supersymmetry of the action (5.20) under the usual transformations of the co-frame, connection and gravitino fields given by (5.9) and (5.12). The explicit supersymmetry transformations of the Lagrange multiplier

1-forms $\{\lambda^a\}$ are not necessary because transformation of the connection is obtained using the torsion field. Therefore the last term in (5.24) does not make any contribution to the variations on-shell.

Under local supersymmetry transformations (5.9) and (5.12), the variation of the action density decomposes as follows:

$$\dot{\mathcal{L}}_{\text{Total}} = \dot{\mathcal{L}}_{\text{CS}} + \dot{\mathcal{L}}_{\text{SG}} + \dot{\mathcal{L}}_{\text{C}} + \dot{\mathcal{L}}_{\text{Constraint}} \, , \tag{5.25}$$

where the contribution $\dot{\mathcal{L}}_{\text{SG}}+\dot{\mathcal{L}}_{\text{C}}$ from the cosmological supergravity sector is already shown to yield a closed form given by (5.15). We are left to deal with contributions coming from the topological sector and the constraint term. The contributions from the constraint term can be seen to produce just a closed form by a straightforward computation:

$$\begin{aligned} \dot{\mathcal{L}}_{\text{Constraint}} &= i(\bar{\alpha}\gamma^a\chi) \wedge D\lambda_a + iD\bar{\alpha} \wedge \lambda^a \wedge \gamma_a\chi - i\frac{m}{2}\bar{\alpha}\gamma\gamma^a \wedge \chi \wedge \lambda_a \\ &- \frac{i}{2}\left(\bar{\alpha}\gamma^a\iota^b(D\chi) - \bar{\alpha}\gamma^b\iota^a(D\chi) + \bar{\alpha}\gamma\iota^{ab}(D\chi)\right) \wedge \lambda_a \wedge e_b \\ &- \frac{im}{2}\left(\epsilon^{abc}(\bar{\alpha}\gamma_c\chi) + e^a(\bar{\alpha}\chi^b) - e^b(\bar{\alpha}\chi^a)\right) \wedge e_b \wedge \lambda_a \\ &= iD(\bar{\alpha}\lambda^a) \wedge \gamma_a\chi - i\bar{\alpha}\lambda^a \wedge \gamma_a D\chi = id(\lambda^a \wedge \bar{\alpha}\gamma_a\chi) \, . \end{aligned} \tag{5.26}$$

We note that the result (5.26) does not depend on the explicit form of the Lagrange multiplier 1-forms. We therefore must only check those contributions coming from the topological Chern-Simons density. In order to ease the discussion, we are going to deal separately with terms obtained when $m = 0$ and rest of the terms for $m \neq 0$.

Case: $m = 0$

The supersymmetry transformation of the Chern-Simons density gives

$$\dot{\mathcal{L}}_{\mathrm{CS}}(m=0) = -\frac{i}{\mu}\bigg(2\bar{\alpha}\gamma^a\iota^b(D\chi) + \bar{\alpha}\gamma\iota^{ab}(D\chi)\bigg)\wedge R_{ab} - \frac{4i}{\mu}D^2\bar{\alpha}\wedge\bigg(*D\chi$$
$$+*(\gamma\wedge *D\chi)\bigg) - \frac{1}{\mu}(\bar{\alpha}\gamma^a\chi)\wedge\bigg(\iota_a D\bar{\chi}\wedge *D\chi - D\bar{\chi}\wedge\iota_a * D\chi$$
$$+*D\bar{\chi}\wedge\gamma_a * D\chi - 2\iota_a * D\bar{\chi}\gamma\wedge *D\chi + 2\iota_a D\bar{\chi}\wedge *(\gamma\wedge *D\chi)\bigg)$$
$$-\frac{1}{2\mu}\bigg(2\bar{\alpha}\gamma^a\iota^b(D\chi) + \bar{\alpha}\gamma\iota^{ab}(D\chi)\bigg)\wedge\bigg(\epsilon_{abc}\bar{\chi}\gamma^c\wedge *(D\chi + \gamma\wedge *D\chi)\bigg). \tag{5.27}$$

Showing that this unpromising expression vanishes in fact is laborious but can be done by pursuing the following steps. First we start by manipulating the third and fourth terms in (5.27) to cancel the first two terms which are proportional to the curvature 2-forms. Next we perform a Fierz rearrangement once on the remaining terms to bring them into the form of $(\bar{\alpha}\chi)(D\bar{\chi}D\chi)$. Then we will be able to group terms into three distinct generic types that do not mix with each other. Then we finally show that each of these group of terms vanish on their own.

Let us start by manipulating the third term in (5.27):

$$\begin{aligned}
-\frac{4i}{\mu}D^2\bar{\alpha}\wedge *D\chi &= \frac{i}{\mu}\epsilon_{abc}(\bar{\alpha}\gamma^c * D\chi)\wedge R^{ab} \\
&= \frac{i}{2\mu}(\bar{\alpha}\gamma^c\iota_{lk}D\chi)\epsilon_{abc} * e^{kl}\wedge R^{ab} \\
&= -\frac{i}{\mu}(\bar{\alpha}\gamma\iota_{ba}D\chi)\wedge R^{ab} + \frac{2i}{\mu}(\bar{\alpha}\gamma^b\iota_{ba}D\chi)R^{ac}\wedge e_c \\
&= -\frac{i}{\mu}(\bar{\alpha}\gamma\iota_{ba}D\chi)\wedge R^{ab} - \frac{1}{\mu}(\bar{\alpha}\gamma^b\iota_{ba}D\chi)\wedge(D\bar{\chi}\wedge\gamma^a\chi)\,.
\end{aligned} \tag{5.28}$$

Above, in the first equality we used the Ricci's identity (E.19), in the second line we opened the gravitino field in terms of the co-frame basis as in (5.4), in the third line we used the co-frame identity (B.6) and in the final line we used the first Bianchi identity (4.127) together with the torsion expression (5.8). We note that the first term in the final equality in (5.28) cancels out the second term in (5.27).

Now we manipulate the fourth term in (5.27):

$$
\begin{aligned}
-\frac{4i}{\mu} D^2\bar{\alpha} \wedge *(\gamma \wedge *D\chi) &= \frac{i}{\mu}\epsilon_{abc}(\bar{\alpha}\gamma^c\gamma^d\iota_d D\chi) \wedge R^{ab} \\
&= \frac{i}{\mu}\epsilon_{abc}(\bar{\alpha}\iota^c D\chi) \wedge R^{ab} - \frac{2i}{\mu}(\bar{\alpha}\gamma_b\iota_a D\chi) \wedge R^{ab} \\
&= -\frac{i}{\mu}(\bar{\alpha}D\chi) \wedge \epsilon_{abc}\iota^a R^{bc} - \frac{2i}{\mu}(\bar{\alpha}\gamma_b\iota_a D\chi) \wedge R^{ab} \\
&= \frac{1}{\mu}(\bar{\alpha}\gamma_b\iota_a D\chi) \wedge *\iota_a(D\bar{\chi} \wedge \gamma^a\chi) - \frac{2i}{\mu}(\bar{\alpha}\gamma_b\iota_a D\chi) \wedge R^{ab} \,. \qquad (5.29)
\end{aligned}
$$

Above, in the first equality we used Ricci's identity (E.19), while in the second line we used the Clifford product rule (E.3) together with the co-frame identity **check here.**(B.2). In the third line we distributed the interior product operation in the first term and made use of the fact that a 4-form vanishes identically. In the final equality we made use of the contracted Bianchi identity (E.20) to show $\epsilon_{abc}\iota^a R^{bc} = 2 * (\iota_a DT^a)$ and used the torsion expression (5.8). We note that the last term in (5.29) cancels the first term in the total variation (5.27) and we have no terms remaining proportional to curvature 2-forms. (By looking at these cancellations, we fixed the relative sign between the bosonic and fermionic terms in Chern-Simons action density (5.22).)

The remaining terms can be brought into a generic form $(\bar{\alpha}\chi)(D\bar{\chi}D\chi)$ by applying the Fierz rearrangement formula (E.17). While doing such calculations, we make ample use of the co-frame identities (B.2)-(B.6). Finally bringing everything back together, the relevant piece of the supersymmetry transformation of the topological

Chern-Simons density will be put into the following form:

$$\begin{aligned}
\dot{\mathcal{L}}_{\text{CS}}(m=0) = \frac{1}{\mu}\bigg\{ & (\bar{\alpha}\gamma^a\chi) \wedge \left[-\frac{1}{4}(\iota_a D\bar{\chi} \wedge *D\chi) - \frac{1}{2}(D\bar{\chi}\iota_a * D\chi) \right] \\
& - \frac{1}{4}(\bar{\alpha}\gamma \wedge \chi) \wedge (\iota_a D\bar{\chi}\iota^a * D\chi) - \frac{1}{4} * e^{ab} \wedge (\bar{\alpha}\gamma_a\chi) \wedge (\iota^c D\bar{\chi}\iota_{cb} D\chi) \\
& + (\bar{\alpha}\chi) \wedge \bigg[D\bar{\chi} * (\gamma \wedge D\chi) - \frac{1}{4}\iota_a D\bar{\chi} \wedge \gamma\iota^a * D\chi \\
& + \frac{5}{4} * D\bar{\chi} \wedge *(\gamma \wedge *D\chi) \bigg] + \frac{1}{4} * e^{ab} \wedge (\bar{\alpha}\chi) \wedge (\iota^c D\bar{\chi}\gamma_a\iota_{cb} D\chi) \\
& + (\bar{\alpha}\gamma^a\chi) \wedge \bigg[-\frac{3}{2} * D\bar{\chi} \wedge \gamma_a * D\chi - \frac{3}{2}\iota_a D\bar{\chi} \wedge *(\gamma \wedge *D\chi) \\
& + \frac{9}{4}\iota_a * D\bar{\chi}\gamma \wedge *D\chi + \frac{1}{4}\iota^b D\bar{\chi} \wedge \gamma\iota_{ba} D\chi \bigg] \\
& + \frac{1}{4}(\bar{\alpha}\gamma \wedge \chi) \wedge \left[* D\bar{\chi} * (\gamma \wedge D\chi) + \iota^a D\bar{\chi}\gamma^b\iota_{ab} D\chi \right] \\
& + \frac{1}{4} * e^{ab} \wedge (\bar{\alpha}\gamma_a\chi) \wedge (\iota_c D\bar{\chi}\gamma_b\iota^c * D\chi) \\
& - \frac{1}{2}\epsilon^{abc}(\bar{\alpha}\gamma_a\chi) \wedge (\iota_b D\bar{\chi} \wedge \gamma_c * D\chi) \bigg\}.
\end{aligned} \tag{5.30}$$

To show that the right hand side vanishes, we group the terms into three generic types which read as follows:

$$1. \quad \big(\bar{\alpha}\gamma\chi\big)\big(D\bar{\chi}D\chi\big) \tag{5.31}$$

$$2. \quad \big(\bar{\alpha}\chi\big)\big(D\bar{\chi}\gamma D\chi\big) \tag{5.32}$$

$$3. \quad \big(\bar{\alpha}\gamma\chi\big)\big(D\bar{\chi}\gamma D\chi\big) \tag{5.33}$$

Again these generic types do not mix with each other under Fierz rearrangements and each group of terms vanish on their own. In particular, the terms of the types (5.31) and (5.32) can be shown to vanish by expanding each term in the co-frame basis and then using co-frame identities, taking $*1$ out of these expressions. However, when applying the same method for terms of the type (5.33), some simplifications occur and we obtain the following combination:

$$\begin{aligned}
\frac{1}{\mu}\bigg\{ & (\bar{\alpha}\gamma^a\chi) \wedge \bigg[3\iota_a * D\bar{\chi}\gamma \wedge *D\chi - \frac{3}{2} * D\bar{\chi} \wedge \gamma_a * D\chi \\
& - \frac{3}{2}\iota_a D\bar{\chi} \wedge *(\gamma \wedge *D\chi) \bigg] + \frac{3}{4} * e^{ab} \wedge (\bar{\alpha}\gamma_a\chi) \wedge (\iota_c D\bar{\chi}\gamma_b\iota^c * D\chi) \bigg\}.
\end{aligned} \tag{5.34}$$

To show that this combination vanishes we move an interior product operation in the first three terms and use the fact that a 4-form field identically vanishes. Then the resulting combination cancels out the fourth term. Thus the local supersymmetry of the Chern-Simons density (5.27) under the local supersymmetry transformations (5.9) and (5.12) with $m = 0$ is established.

Case: $m \neq 0$

Now we move on to take care of $m \neq 0$ terms coming from topological sector. They explicitly read

$$\begin{aligned}
&\frac{im}{2}\Big[\epsilon_{abc}(\bar{\alpha}\gamma^c\chi) + 2e_a(\bar{\alpha}\chi_b)\Big] \wedge \Big\{\frac{2}{\mu}R^{ab} - \frac{i}{2\mu}\epsilon^{abd}\Big[\bar{\chi}\wedge\gamma_d * D\chi \\
&+\bar{\chi}\wedge\gamma_d * (\gamma\wedge *D\chi)\Big]\Big\} + \frac{2im}{\mu}D(\bar{\alpha}\gamma)\wedge *\Big[D\chi + \gamma\wedge *D\chi\Big] \\
&\quad = -\frac{m}{2\mu}(\bar{\alpha}\gamma^a\chi)\wedge\Big[(\bar{\chi}\wedge\gamma_a * D\chi) + (\bar{\chi}\wedge\gamma_a\gamma_b\iota^b D\chi)\Big] \\
&\quad + \frac{m}{2\mu}\epsilon_{abc}e^a\wedge(\bar{\alpha}\chi^b)\Big[(\bar{\chi}\wedge\gamma^c * D\chi) + (\bar{\chi}\wedge\gamma^c\gamma^d\iota_d D\chi)\Big] \\
&\quad - \frac{m}{\mu}(\bar{\alpha}\chi^a)(\bar{\chi}\wedge\gamma_a D\chi) - \frac{m}{2\mu}(\bar{\alpha}\gamma^a * D\chi)\wedge(\bar{\chi}\wedge\gamma_a\chi) \\
&\quad - \frac{m}{2\mu}(\bar{\alpha}\gamma^a\gamma^b\iota_b D\chi)\wedge(\bar{\chi}\wedge\gamma_a\chi) + \frac{im}{\mu}\epsilon_{abc}(\bar{\alpha}\gamma^c\chi)\wedge R^{ab} \\
&\quad + \frac{2im}{\mu}D\bar{\alpha}\wedge\gamma\wedge *D\chi + \frac{2im}{\mu}D\bar{\alpha}\wedge\gamma\gamma^a\wedge\iota_a D\chi\,.
\end{aligned} \tag{5.35}$$

To show that the right hand side of the above equality adds up to a closed form, we will start by manipulating the very last term. Using $\gamma\gamma^a = e^a + \epsilon^{acb}e_b\gamma_c$, we can write

$$\begin{aligned}
\frac{2im}{\mu}D\bar{\alpha}\wedge\gamma\gamma^a\wedge\iota_a D\chi &= \frac{4im}{\mu}D\bar{\alpha}\wedge D\chi + \frac{2im}{\mu}(D\bar{\alpha}\gamma_c\iota_{ba}D\chi)\wedge *e^{ac}\wedge e^b \\
&= d\Big(\frac{4im}{\mu}\bar{\alpha}D\chi\Big) - \frac{4im}{\mu}\bar{\alpha}D^2\chi - \frac{2im}{\mu}D\bar{\alpha}\gamma^b * e^a\iota_{ba}D\chi \\
&= d\Big(\frac{4im}{\mu}\bar{\alpha}D\chi\Big) - \frac{im}{\mu}\epsilon_{abc}(\bar{\alpha}\gamma^c\chi)\wedge R^{ab} - \frac{2im}{\mu}D\bar{\alpha}\wedge\gamma\wedge *D\chi\,.
\end{aligned} \tag{5.36}$$

Above in the second equality we distributed a covariant derivative in the first term and made use of the identity B.5). Then in the third equality we used the Ricci identity and (E.19) with the interior product identity (B.1). The result (5.36) shows that the last three terms in (5.35) combine to yield a closed form. The remaining terms of (5.35) can be brought into the form $(\bar{\alpha}D\chi)(\bar{\chi}\chi)$ by Fierzing once. The resulting expression can be shown to vanish identically after taking $*1$ out of each term. We have thus proven, with this final observation, the local supersymmetry of the cosmological topologically massive supergravity action (5.20).

Now we are ready to derive the complete set of field equations of cosmological topologically massive supergravity. The field equations are read off from the variations of the total action (5.24) and they consist of the Einstein field equations

$$G_a - m^2 * e_a + D\lambda_a = \frac{i}{\mu}\Big(- \iota_a D\bar{\chi} \wedge *D\chi + D\bar{\chi}(\iota_a * D\chi) - *D\bar{\chi} \wedge \gamma_a * D\chi + 2(\iota_a * D\bar{\chi})\gamma \wedge *D\chi - 2\iota_a D\bar{\chi} \wedge *(\gamma \wedge *D\chi)\Big) - i\frac{m}{4}\bar{\chi} \wedge \gamma_a \chi \,, \tag{5.37}$$

together with the gravitino field equation

$$D\chi + \frac{m}{2}\gamma \wedge \chi + \frac{2}{\mu}\Big(D * D\chi + D * (\gamma \wedge *D\chi)\Big) - \frac{1}{2}\lambda^a \wedge \gamma_a \chi = 0 \,, \tag{5.38}$$

subject to the constraint that the space-time torsion is given by

$$T^a = \frac{i}{4}\bar{\chi} \wedge \gamma^a \chi \,. \tag{5.39}$$

We note that the $D\lambda_a$ term in the Einstein field equations (5.37) is precisely the term that produces the Cotton tensor in topologically massive gravity theory. It is obtained by solving the Lagrange multiplier 1-forms λ_a from the connection field equations, which are of the form Equation (C.1) with

$$\Sigma_{ab} = -\frac{4}{\mu}R_{ab} + \frac{i}{\mu}\epsilon_{abc}\bar{\chi} \wedge \gamma^c * (D\chi + \gamma \wedge *D\chi) \,. \tag{5.40}$$

We then substitute the result back in the other field equations. Using Equation (C.4), the solution for the Lagrange multiplier 1-forms read:

$$\lambda_a = -\frac{4}{\mu}\Big(Y_a + \frac{i}{4}W_a\Big) \,, \tag{5.41}$$

where

$$Y_a = Ric_a - \frac{1}{4}Re_a \,, \tag{5.42}$$

are the Schouten curvature 1-forms and

$$W_a = \iota_a * (\bar{\chi} \wedge \gamma)\Big(* D\chi + *(\gamma \wedge *D\chi)\Big) + \bar{\chi}\Bigg(\iota_a * \Big(\gamma \wedge *D\chi + \gamma \wedge *(\gamma \wedge *D\chi)\Big)\Bigg)$$
$$- \frac{1}{2} * \Bigg(\bar{\chi} \wedge \gamma \wedge *(D\chi + \gamma \wedge *D\chi)\Bigg) e_a\,, \tag{5.43}$$

are the corresponding contributions coming from the fermionic sector, respectively. Having obtained the solution (5.41) for Lagrange multiplier 1-forms in hand, we can now write down the final form of the variational field equations of cosmological topologically massive gravity. We have the Einstein field equations

$$G_a - m^2 * e_a + i\frac{m}{4}\bar{\chi} \wedge \gamma_a \chi$$
$$- \frac{4}{\mu} DY_a - \frac{i}{\mu}\Bigg(DW_a - \iota_a D\bar{\chi} \wedge *D\chi + D\bar{\chi}(\iota_a * D\chi) - *D\bar{\chi} \wedge \gamma_a * D\chi$$
$$+ 2(\iota_a * D\bar{\chi})\gamma \wedge *D\chi - 2\iota_a D\bar{\chi} \wedge *(\gamma \wedge *D\chi)\Bigg) = 0, \tag{5.44}$$

coupled to the gravitino field equation

$$D\chi + \frac{m}{2}\gamma \wedge \chi + \frac{2}{\mu}\Bigg(D * D\chi + D * (\gamma \wedge *D\chi) + \Big(Y^a + \frac{i}{4}W^a\Big) \wedge \gamma_a \chi\Bigg) = 0, \tag{5.45}$$

where the torsion 2-forms of space-time are

$$T^a = \frac{i}{4}\bar{\chi} \wedge \gamma^a \chi. \tag{5.46}$$

The Schouten curvature 1-forms $\{Y_a\}$ and their fermionic counterparts $\{W_a\}$ are given by the expressions (5.42) and (5.43), respectively.

5.3 Remarks

In the present work we formulate the cosmological topologically massive supergravity theory using a torsion-constrained first order variational formalism in the language of exterior differential forms on three dimensional Riemann-Cartan space-times. In particular, we regard the connection 1-forms as independent field variables thus treating them at the same level as local Lorentz co-frames and the gravitino field.

However, the space-time torsion is constrained algebraically to its standard form by the method of Lagrange multipliers. This is an essential feature of our approach giving rise to contributions of the Lagrange multiplier fields in the final set of field equations. We first prove the invariance of the action under infinitesimal local supersymmetry transformations of the co-frame, connection and the gravitino fields. This we did in explicit detail. We also present and simplify the final set of variational field equations since the field equations in their complete form had been lacking in the previous literature.

In particular the field equations that come from the connection variations are solved algebraically for the Lagrange multiplier fields. We substitute them into the coupled Einstein and Rarita-Schwinger field equations which arise from the co-frame and gravitino field variations, respectively. The terms that appear on their right hand sides are identified as the Cotton 2-forms and their fermionic counterpart, the so-called, Cottino 2-form. We note that the variations of the complete Chern-Simons density imply some further non-linear terms besides those coming from the Lagrange multipliers. We wish to make a few remarks concerning these. We read off from the final version of the field equations the Cotton 2-forms

$$C_a = DY_a + \frac{i}{4}DW_a - \frac{i}{4}\Big(\iota_a D\bar{\chi} \wedge *D\chi - D\bar{\chi}(\iota_a * D\chi) + *D\bar{\chi} \wedge \gamma_a * D\chi - 2(\iota_a * D\bar{\chi})\gamma \wedge *D\chi + 2\iota_a D\bar{\chi} \wedge *(\gamma \wedge *D\chi)\Big), \tag{5.47}$$

and the Cottino 2-form

$$C = Y^a \wedge \gamma_a \chi + \frac{i}{4}W^a \wedge \gamma_a \chi + D * D\chi + D * (\gamma \wedge *D\chi). \tag{5.48}$$

Let us discuss the Cotton 2-forms first. The Cotton 2-forms that one obtains in the formulation of topologically massive gravity theory are given by only the first term in (5.47). The second term governs the higher order contributions in the gravitino field that is due to the fermionic part of the topological action and contains second, fourth and sixth powers of the gravitino field. This may be observed by separating the connection 1-forms according to (4.114) and expanding the covariant derivatives of the gravitino field as:

$$D\chi = \hat{D}\chi + \frac{i}{8}\left[(\bar{\chi}^a\gamma^b - \bar{\chi}^b\gamma^a)\chi + \bar{\chi}^a\gamma\chi^b\right] \wedge \sigma_{ab}\chi \tag{5.49}$$

where $\hat{D}$ denotes the covariant exterior derivative operation with respect to Levi-Civita connection and the second term is the contribution of contortion. An important feature of the Cotton 2-forms in the Riemannian case (that is, with no torsion present in the geometry) is that they are traceless. The trace of Cotton 2-forms can be taken by wedging them with the co-frame from the left as follows:

$$e^a \wedge \hat{D}\hat{Y}_a = -d(e^a \wedge \hat{Y}_a) = -d(e^{ab}\hat{Y}_{a,b}) = 0, \tag{5.50}$$

because the components of the Schouten 1-forms are symmetric. Again, a hat over a quantity means that it is obtained by using the Levi-Civita connection. Of course this does not hold when there is torsion present in the geometry, however, one is tempted to ask whether this property still holds for the full Cotton 2-forms given by (5.47), provided we take our geometry to be Riemannian. Unfortunately even this is not the case. The trace of the modified Cotton 2-forms read:

$$\begin{aligned} e^a \wedge \hat{C}_a = e^a \wedge \frac{i}{4}\Big(&\hat{D}\hat{W}_a + \iota_a\hat{D}\bar{\chi} \wedge *\hat{D}\chi - \hat{D}\bar{\chi}(\iota_a * \hat{D}\chi) + *\hat{D}\bar{\chi} \wedge \gamma_a * \hat{D}\chi \\ &- 2(\iota_a * \hat{D}\bar{\chi})\gamma \wedge *\hat{D}\chi + 2\iota_a\hat{D}\bar{\chi} \wedge *(\gamma \wedge *\hat{D}\chi)\Big) \\ = -\frac{i}{4}&\Big(d(e^a \wedge \hat{W}_a) + \hat{D}\bar{\chi} \wedge *\hat{D}\chi + *\hat{D}\bar{\chi} \wedge \gamma \wedge \hat{D}\chi\Big) \\ = \frac{i}{4}&\Big(2d(\bar{\chi}^a\gamma_a\hat{D}\chi) - 2 * \hat{D}\bar{\chi} \wedge \gamma \wedge \hat{D}\chi - \bar{\chi} \wedge \hat{D} * (\hat{D}\chi - \gamma \wedge *\hat{D}\chi)\Big). \end{aligned} \tag{5.51}$$

There are two reasons for the modified Cotton 2-forms to be not traceless. The first and main impediment is due to the fact that the components of the contributions $\{W_a = W_{a,b}e^b\}$ are not symmetric unlike the components of Schouten 1-forms. Explicitly they read:

$$\begin{aligned} W_{a,b} = &-\left[(\bar{\chi}^k\gamma^l\iota_{lk}D\chi) + \frac{1}{2}\epsilon_{klm}(\bar{\chi}^k\iota^{ml}D\chi)\right]\eta_{ab} + (\bar{\chi}^k\gamma_a\iota_{bk}D\chi) + (\bar{\chi}^k\gamma_b\iota_{ak}D\chi) \\ &+ \left[(\bar{\chi}^k\iota_b{}^lD\chi) + (\bar{\chi}_b\iota^{lk}D\chi)\right]\epsilon_{akl} + (\bar{\chi}_b\gamma^k\iota_{ka}D\chi) + (\bar{\chi}^k\gamma_k\iota_{ab}D\chi). \end{aligned} \tag{5.52}$$

In general the variation of a fermionic action with respect to the co-frame field yields an asymmetric tensor. The asymmetry of (5.52) is an example to this fact. The second reason is that we included the terms coming from co-frame variation of the fermionic part of Chern-Simons 3-form. The contribution of these terms to trace is the fermionic part of Chern-Simons 3-form itself as can be seen from the second equality in (5.51). This contribution is also asymmetric. One final remark that we will make about the modified Cotton 2-forms is that, due to fermionic contributions they also cease to be symmetric and divergence free when working in a Riemannian geometry. We do not write down the divergence and anti-symmetric part of Cotton 2-forms here because their expressions are not very instructive.

The Cottino 2-form, unlike its superpartner, is not discussed abundantly in the literature. Only in the references [**?**, **?**], the part that is linear in the gravitino field is discussed. It reads in a Riemannian geometry,

$$\hat{C} = \hat{D} * \hat{D}\chi + \hat{D} * (\gamma \wedge *\hat{D}\chi) + \hat{Y}^a \wedge \gamma_a \chi \tag{5.53}$$

and is linear only when we are working in a Riemannian geometry. Otherwise the full Cottino 2-form (5.48) contains terms that are to the first, third and fifth powers in the gravitino field. Similar to the Cotton 2-forms, the higher order contributions are encoded in the term that is proportional to $\{W_a\}$. Again in the Riemannian context, the linear part (5.53) of Cottino 2-forms are γ-traceless. This is the spinorial version of the Cotton tensor being traceless. The γ-trace operation is given by wedging the Cottino 2-form from the left with the γ-matrix valued 1-form $\gamma = \gamma_a e^a$:

$$\begin{aligned} \gamma \wedge \hat{C} &= \gamma \wedge \hat{D}[*\hat{D}\chi + *(\gamma \wedge *\hat{D}\chi)] + \gamma \wedge \hat{Y}^a \wedge \gamma_a \wedge \chi \\ &= -\hat{D}[\gamma \wedge \sigma^{ab}\gamma\iota_{ba}\hat{D}\chi] + \epsilon_{abc}e^a \wedge \hat{Y}^b \wedge \gamma^c\chi \\ &= 2\hat{D}^2\chi - *\hat{Ric}^a \wedge \gamma_a\chi + \frac{\hat{R}}{2} * \gamma \wedge \chi = 0. \end{aligned} \tag{5.54}$$

When showing γ-tracelessness, in the second equality we used the fact that the connection is torsion-free together with the identity $*\hat{D}\chi + *(\gamma \wedge *\hat{D}\chi) = \sigma^{ab}\gamma\iota_{ba}\hat{D}\chi$ and the fact that Schouten tensor is symmetric. In the final equality we made use of the curvature identity (B.9) and the Ricci identity (E.19). The γ-tracelessness does not hold at the presence of torsion, but we again calculate the γ-trace of the full Cottino 2-form (5.48) when there is no torsion. The final result is

$$\begin{aligned} \gamma \wedge \hat{C} &= \frac{i}{4}\gamma \wedge \hat{W}^a \wedge \gamma_a \chi \\ &= \frac{i}{4}\Bigg[\frac{1}{2}\epsilon_{abc}\Bigg(\gamma_d \iota^{ab}\hat{D}\chi(\bar{\chi}^c \chi_d) + \iota^{ad}\hat{D}\chi(\bar{\chi}_b \gamma_d \chi_c)\Bigg) \\ &\quad + \frac{1}{2}\gamma^a \iota_{ab}\hat{D}\chi(\bar{\chi}^b \gamma^c \chi_c) + \gamma^a \iota^{bc}\hat{D}\chi\Bigg(\bar{\chi}_b \gamma_a \chi_c - \bar{\chi}_a \gamma_b \chi_c\Bigg)\Bigg] * 1. \end{aligned} \tag{5.55}$$

When calculating the γ-trace we opened the expression in co-frame basis and Fierzed once to bring every term into the form $D\chi(\bar{\chi}\chi)$. The expression (5.55) shows that, like in the case of Cotton 2-forms, the contributions due to terms $\{W_a\}$ spoil the γ-tracelessness of the linear part of the Cottino 2-forms (5.48).

Finally, we wish to emphasize once again the importance of the torsion constraint (5.23) for our first order variational formulation of the cosmological topologically massive supergravity theory. For instance if torsion constraint hasn't been imposed by the method of Lagrange multipliers, then one would have obtained by first order variations a completely different set of field equations. In this new theory the space-time torsion would be dynamical rather than being determined algebraically by the gravitino fields. One can see by looking at the variation field equations that this would be the case:

$$\begin{aligned} R^{ab} &= \Lambda e^{ab} - \frac{i}{\mu}\epsilon^{abc}\Bigg(\iota_c D\bar{\chi} \wedge *D\chi - D\bar{\chi}(\iota_c * D\chi) + *D\bar{\chi} \wedge \gamma_c * D\chi \\ &\quad - 2(\iota_c * D\bar{\chi})\gamma \wedge *D\chi + \iota_c D\bar{\chi} \wedge *(\gamma \wedge *D\chi)\Bigg) - i\frac{m}{4}\epsilon^{abc}\bar{\chi} \wedge \gamma_c \chi \end{aligned} \tag{5.56}$$

$$T^a = \frac{i}{4}\bar{\chi} \wedge \gamma_a \chi + \frac{2}{\mu}\epsilon^{abc}R_{bc} + \frac{i}{\mu}\bar{\chi} \wedge \gamma\Bigg(* D\chi + *(\gamma \wedge *D\chi)\Bigg) \tag{5.57}$$

$$D\chi + \frac{m}{2}\gamma \wedge \chi + \frac{2}{\mu}D\Bigg(* D\chi + *(\gamma \wedge *D\chi)\Bigg) = 0. \tag{5.58}$$

This is a generalized version of our usual torsion expression (5.46) with contributions coming from the topological term. Similar to what we have done before, it is possible to solve for the contortion 1-forms from (5.57). Then using this result, one can further solve for the supersymmetry transformation of connection 1-forms. It is clear that this new transformation for connection 1-forms will have terms proportional to the Chern-Simons coupling constant μ at different orders. This would be a completely different theory both at the level of the action and with different supersymmetry transformations of fields. At this point it is not even apparent whether this action will be invariant or not under supersymmetry transformations because of the highly non-linear terms present in the expression above for the torsion.

As far as we are aware, the full set of field equations (5.44),(5.45) and (5.46) of the cosmological topologically massive supergravity theory has not been given explicitly before in the literature. In the references [Deser and Kay, 1989] and [Deser, 1984], the field equations of the theory are not discussed. In [Gibbons et al., 2008], the Einstein field equation is devoid of fermionic contributions and the gravitino field equation only covers the linear part (5.48) of Cottino 2-forms. Furthermore all the equations are written in terms of Levi-Civita connection so the higher order contributions in the gravitino field are omitted. In [Becker et al., 2009], higher order contributions to the Einstein field equations are not explicitly given and Cottino tensor is again given by the linear expression (5.48). The field equations are Taylor expanded to second order and the exact field equations are not discussed. Lastly in the remaining references regarding TMS and CTMS theories, the field equations are not discussed. In our formulation we achieve to express the full set of field equations consistently from a variational principle. By doing so, we are able to find the contributions coming to the Cotton and Cottino 2-forms.

Chapter 6

CONCLUDING REMARKS

In this thesis, various three dimensional gravity and supergravity models have been studied. The models that are studied have two important inspirations: different modifications to four dimensional general relativity at both high and low energies (discussed at the introduction), and three dimensional massive gravity models. As three dimensions provide us with important toy models, it is reasonable to study all these generalizations there.

In all of our work, we use the concise language of exterior differential forms on Riemann-Cartan-Weyl space-times. Firstly usage of differential forms makes the derivations and expressions more geometrically transparent. Secondly, Riemann-Cartan-Weyl space-times provide a natural geometrical ground to study all of our generalizations. We believe Weyl covariant generalizations provide the finest example to this fact. Moreover, using the Lagrange multiplier constrained variations, although not very conventional, are very convenient. This fact is best demonstrated in the formulation of cosmological topologically massive supergravity model.

The possible directions for all the models studied have been discussed at the end of each section. Given the diverse generalizations we studied in this thesis, there are several different directions that we may pursue in the future. Probably the most important one is the study of range of parameters for which the models are unitarily quantizable. This requires the linearization of models around background geometries together with boundary analysis. Moreover, generalization of these models to four dimensions is not too hard, but applying the possible known quantization schemes is. It would be interesting and physically relevant to see what can be done in this direction as well.

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Appendix A

MATHEMATICAL STRUCTURES

In all of our work we use the Einstein summation convention, that is repeated indices are summed over the whole range of the index set. Furthermore if some indices are squeezed between square brackets or parentheses then it means the normalized anti-symmetrization or normalized symmetrization of those indices, respectively. For differential geometry and tensors on manifolds, we refer to [Kobayashi and Nomizu, 1963] and for applications of parabolic geometries in physics, we refer to [Cap and Slovák, 2009].

We use language of exterior differential forms on a manifold M. The exterior bundle on M is denoted by $\Omega M = \bigoplus_{k=0}^{3} \Omega^k M$ and its sections are form (totally anti-symmetric tensor) fields on M. There are important operations on exterior bundle, which generalize vector calculus on manifolds. For the explicit results of these operations, let $\alpha = \frac{1}{k!}\alpha_{[i_1,..,i_k]}dx^{i_1} \wedge .. \wedge dx^{i_k}$ be a k-form field on some open chart of M.

1. *Exterior derivative operator*: $d : \Omega^k M \to \Omega^{k+1} M$ is the generalisation of differential of a function to arbitrary form fields. In local coordinates,

$$d\alpha = \frac{1}{(k+1)!}\partial_{[j}\alpha_{i_1,..,i_k]}dx^j \wedge dx^{i_1} \wedge .. \wedge dx^{i_k} \,. \tag{A.1}$$

An important identity of the exterior derivative is $d^2 = 0$.

2. *Interior product operation with respect to a vector field* $\{X_a\}$: $\iota_{X_a} : \Omega^k M \to \Omega^{k-1} M$ is the generalisation of a pairing between a vector and a covector. In local coordinates,

$$\iota_{X_a}\alpha = \frac{1}{(k-1)!}X_a^{i_1}\alpha_{[i_1,..,i_k]}dx^{i_2} \wedge .. \wedge dx^{i_k} \,. \tag{A.2}$$

3. Hodge duality operator: Given that M is equipped with a metric field, Hodge duality operation $* : \Omega^k(M) \to \Omega^{n-k}(M)$ maps a k-form field uniquely to a $(n-k)$-form field. In local coordinates,

$$*\alpha = \frac{1}{k!(n-k)!}\epsilon^{i_1 \dots i_k}{}_{i_{k+1}\dots i_n}\alpha_{[i_1,..,i_k]}dx^{i_{k+1}} \wedge .. \wedge dx^{i_n} \,, \tag{A.3}$$

where $\{\epsilon_{i_1 \dots i_n}\}$ is the totally anti-symmetric Levi-Civita symbol and its indices are raised using the metric on M. An important property of Hodge duality operation is that it squares to the identiy operator up to a sign, that is $** = \pm 1$. With the help of Hodge duality operation we can define a top-form valued bilinear form $\langle \alpha_1, \alpha_2 \rangle$ and an L^2-inner product[1] (α_1, α_2) of two k-forms α_1 and α_2 as

$$\alpha_1 \wedge *\alpha_2 = \langle \alpha_1, \alpha_2 \rangle * 1 \,, \tag{A.4}$$

$$(\alpha_1, \alpha_2) = \int_M \alpha_1 \wedge *\alpha_2 \,, \tag{A.5}$$

respectively.

4. Exterior co-derivative operator: $d^\dagger : \Omega^k M \to \Omega^{k-1} M$ is the adjoint operator of exterior derivative operator with respect to the L^2-inner product on a manifold without boundary. That is for a k-form field α_1 and $(k-1)$-form field α_2, we have

$$(\alpha_1, d\alpha_2) = (d^\dagger \alpha_1, \alpha_2) \,. \tag{A.6}$$

The exterior co-derivative operation may be expressed in terms of exterior derivative and Hodge duality operations up to a sign, that is $d^\dagger = \pm * d *$.

The geometrical framework that we express our gravitational models is metric affine geometry $\{M, g, \nabla\}$. Here (M, g) is a three dimensional locally Lorentzian manifold modeled on the vector space $(\mathbb{R}^2, \eta)$ where $\eta = \text{diag}(-, +, +)$, and ∇ denotes an affine connection on M, that is, a connection on the frame bundle FM of M. The metric tensor can be expressed as $g = \eta_{ab} e^a \otimes e^b$ where $\eta_{ab} = g(X_a, X_b)$, in terms of a g-orthonormal frame $\{X_a\}$ that are dual to the co-frame 1-forms $\{e^a\}$

[1] Here we misuse the term inner product as this map is not positive definite in general.

so that $e^a(X_b) = \iota_b e^a = \delta^a_b$. For simplicity, we use the following abbreviations for the exterior products $e^{ab\cdots} \equiv e^a \wedge e^b \wedge \ldots$, and the interior products $\iota_{ab\ldots} \equiv \iota_a \iota_b \ldots$, respectively. The space-time orientation is fixed by the choice of a volume form $*1 = e^{012}$. Lastly, the connection ∇ is given by a set of connection 1-forms $\{\Lambda^a{}_b\}$ so that $\nabla_{X_a} X_b = \Lambda^c{}_b(X_a)X_c$. We define the non-metricity, torsion, and curvature forms of a linear connection by the Cartan's structure equations below:

$$Q_{ab} = \frac{1}{2}(\Lambda_{ab} + \Lambda_{ba})\,, \tag{A.7}$$

$$T^a = de^a + \Lambda^a{}_b \wedge e^b\,, \tag{A.8}$$

$$\overset{(\Lambda)}{R}{}^a{}_b = d\Lambda^a{}_b + \Lambda^a{}_c \wedge \Lambda^c{}_b\,. \tag{A.9}$$

Here $\overset{(\Lambda)}{D}$ is the exterior covariant derivative operator that acts on any rank (p, q) tensor field $T_{i_1\ldots i_p}{}^{j_1\ldots j_q}$ as:

$$\begin{aligned}\overset{(\Lambda)}{D} T_{i_1\ldots i_p}{}^{j_1\ldots j_q} &= dT_{i_1\ldots i_p}{}^{j_1\ldots j_q} - \Lambda^i{}_{i_1} \wedge T_{ii_2\ldots i_p}{}^{j_1\ldots j_q} - \cdots \\ &\quad + \Lambda^{j_1}_j \wedge T_{i_1\ldots i_p}{}^{jj_2\ldots j_q} + \cdots\,,\end{aligned} \tag{A.10}$$

and $\{\overset{(\Lambda)}{R}{}^a{}_b\}$ denote the curvature of the above connection 1-forms $\{\Lambda^a{}_b\}$. Bianchi identities of the structure equations are:

$$\overset{(\Lambda)}{D} Q_{ab} = \frac{1}{2}(\overset{(\Lambda)}{R}_{ab} + \overset{(\Lambda)}{R}_{ba})\,, \tag{A.11}$$

$$\overset{(\Lambda)}{D} T^a = \overset{(\Lambda)}{R}{}^a{}_b \wedge e^b\,, \tag{A.12}$$

$$\overset{(\Lambda)}{D}\,\overset{(\Lambda)}{R}{}^a{}_b = 0\,. \tag{A.13}$$

The Ricci 1-forms and the curvature scalar are obtained by contracting the curvature 2-forms:

$$\overset{(\Lambda)}{Ric}_a = \iota_b \overset{(\Lambda)}{R}{}^b{}_a\,, \quad \overset{(\Lambda)}{R} = \iota^a \overset{(\Lambda)}{Ric}_a = \iota^{ab} \overset{(\Lambda)}{R}_{ba}\,. \tag{A.14}$$

Moreover, the Einstein $(n-1)$-forms of our non-Riemannian connection are defined through the variation of Einstein-Hilbert term as:

$$\overset{(\Lambda)}{G}_a = \overset{(\Lambda)}{G}_{ab} * e^b = -\frac{1}{2}\overset{(\Lambda)}{R}_{bc} \wedge *e^{abc}\,. \tag{A.15}$$

We note that, although curvature 2-forms can have both anti-symmetric and symmetric parts, similar to the Riemannian case, only the anti-symmetric part contributes to the Einstein tensor.

In a metric affine geometry, the most general linear connection is fixed uniquely by the metric tensor field g, the torsion tensor field T and the non-metricity tensor field $S = \overset{(\Lambda)}{D} g$. To observe this, we separate the anti-symmetric and symmetric parts of the connection 1-forms as follows:

$$\Lambda^a{}_b = \Omega^a{}_b + Q^a{}_b \,, \tag{A.16}$$

where the anti-symmetric part further decomposes in a unique way according to

$$\Omega^a{}_b = \omega^a{}_b + K^a{}_b + q^a{}_b \,. \tag{A.17}$$

Here, the Levi-Civita connection 1-forms $\{\omega^a{}_b\}$ are determined completely by the co-frames from the Cartan structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0 \,. \tag{A.18}$$

The contortion 1-forms $\{K^a{}_b\}$ are fixed by the torsion 2-forms

$$K^a{}_b \wedge e^b = T^a \,. \tag{A.19}$$

The anti-symmetric 1-forms $\{q^a{}_b\}$ are completely determined in terms of the symmetric non-metricity 1-forms $\{Q^a{}_b\}$ by the equations

$$q^a{}_b = -(\iota^a Q_{bc})e^c + (\iota_b Q^a{}_c)e^c \,. \tag{A.20}$$

We term a geometry where metric (Riemann), torsion (Cartan), and non-metricity (Weyl) tensors are all present a Riemann-Cartan-Weyl (RCW) space-time.

Appendix B

USEFUL IDENTITIES

Co-frame identities: We are going to give some identities regarding the exterior algebra that are helpful:

$$\iota_a \xi = (-1)^p * (e_a \wedge *\xi), \quad \xi \in \Omega^p M \tag{B.1}$$

$$\epsilon^{abc}\epsilon_{klm} = -\eta^a_k(\eta^b_l\eta^c_m - \eta^c_l\eta^b_m) + \eta^a_l(\eta^b_k\eta^c_m - \eta^b_m\eta^c_k) - \eta^a_m(\eta^b_k\eta^c_l - \eta^c_l\eta^b_k) \tag{B.2}$$

$$\epsilon^{abc} * 1 = -e^{abc} \tag{B.3}$$

$$\epsilon^{abc} * e_k = -\delta^a_k e^{bc} + \delta^b_k e^{ac} - \delta^c_k e^{ab} \tag{B.4}$$

$$e^a \wedge *e_{kl} = -\eta^a_k * e_l + \eta^a_l * e_k \tag{B.5}$$

$$\epsilon^{abc} * e_{kl} = -2\eta^{ab}_{[kl]}e^c + 2\eta^{ac}_{[kl]}e^b - 2\eta^{bc}_{[kl]}e^a \tag{B.6}$$

where $\epsilon_{abc} = *e_{abc}$ denotes the totally anti-symmetric Levi-Civita symbol with the choice $\epsilon_{012} = 1$ and square bracket around some indices means the normalized total anti-symmetrization of those indices.

Curvature Identities: We will prove that in a Riemann-Cartan space-time the following curvature identity holds:

$$G_a = -\frac{1}{2}R^{bc} \wedge *e_{abc} = *\left(Ric_a - \frac{1}{2}Re_c\right) . \tag{B.7}$$

To show this, we start by expanding the curvature 2-forms in the co-frame basis:

$$\begin{aligned}
-\frac{1}{2}R^{bc} \wedge *e_{abc} &= -\frac{1}{4}R^{bc}{}_{,mn}e^{mn} \wedge *e_{abc} \\
&= \frac{1}{4}\left[\left(-R^{bc}{}_{,bc} + R^{bc}{}_{,cb}\right) * e_a + \left(R^{bc}{}_{,ac} - R^{bc}{}_{,ca}\right) * e_b + \left(-R^{bc}{}_{,ab} + R^{bc}{}_{,ba}\right) * e_c\right] \\
&= \frac{1}{2}\left[R * e_a + \left(R^{bc}{}_{,ac} - R^{cb}{}_{,ac}\right) * e_b\right] \\
&= -\frac{1}{2}R * e_a + *Ric_a\, .
\end{aligned} \tag{B.8}$$

Above, when going to the second line we make use of the identity (B.6) and do the contractions when going to the third line. Finally in the last equality we make use of the definitions of Ricci 1-forms and scalar curvature. Also dualizing the (B.9), we obtain

$$*R^{ab} = -\epsilon^{abc}\left(Ric_a - \frac{1}{2}Re_a\right) . \tag{B.9}$$

Now we move on to proving the identity between quadratic curvature invariants:

$$R^{ab} \wedge *R_{ab} - 2Ric^a \wedge *Ric_a + \frac{1}{2}R * 1 = 0\,. \tag{B.10}$$

To show this we make use the identity (B.9).

$$\begin{aligned} R^{ab} \wedge^* R_{ab} &= -\epsilon^{cab} * \left(Ric_c - \frac{1}{2}Re_c\right) \wedge \epsilon^d{}_{ab}\left(Ric_d - \frac{1}{2}Re_d\right) \\ &= -2\eta^{ab}\left[Ric_a \wedge *Ric_b - \frac{R}{2}\left(*e_a \wedge Ric_b + *Ric_a \wedge e_b\right) + \frac{R^2}{4} * e_a \wedge e_b\right] \\ &= 2Ric^a \wedge *Ric_a - 2R^2 * 1 + \frac{1}{2}R^2 e^a \wedge *e_a \\ &= 2Ric^a \wedge *Ric_a - \frac{1}{2}R^2 * 1\,. \end{aligned} \tag{B.11}$$

Above when going to the second equality we use the identity (B.2). After that when going to the third equality we use $Ric_a \wedge *e^a = R * 1$ and then in the final equality we make use of $e^a \wedge *e_a = 3$.

Appendix C

SOLUTION TO ANTI-SYMMETRIC CONNECTION VARIATION FIELD EQUATION

In all of the models considered in this thesis, the Lagrangian densities always contain a constraint term involving torsion 2-forms. In this case, the variational field equations coming from an anti-symmetric connection 1-forms $\{\omega^{ab}\}$ are always in the form

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = \Sigma_{ab} \,, \tag{C.1}$$

for a 2-form $\Sigma_{ab} = \Sigma_{[ab]}$. To solve algebraically for the Lagrange multiplier 1-forms $\{\lambda_a\}$, first contract the Equation (C.1) with the interior product operation ι^a to get

$$\lambda_b + e_b(\iota^a \lambda_a) = \iota^a \Sigma_{ab} \,. \tag{C.2}$$

Contracting Equation (C.2) once more with the interior product operator ι^b, we obtain

$$4(\iota^a \lambda_a) = \iota^{ba} \Sigma_{ab} \,. \tag{C.3}$$

Putting Equation (C.3) in Equation (C.2), we obtain the solution as

$$\lambda_a = \iota^b \Sigma_{ba} - \frac{1}{4}(\iota^{bc} \Sigma_{cb}) e_a \,. \tag{C.4}$$

Appendix D

LEFT-INVARIANT CO-FRAMES ON AdS_3

AdS_3 is a homogeneous space as $\mathrm{AdS}_3 \cong SO(2,2)/SO(1,2)$ where $SO(2,2)$ is the isometry subgroup acting transitively on AdS_3 and $SO(1,2)$ is the stabilizer of any point. It can be concretely realized as an embedded hypersurface of $\mathbb{R}^{2,2}$ with metric $g = -dU^2 - dV^2 + dX^2 + dY^2$ expressed in the Cartesian coordinate chart $\{\xi^A = (U, V, X, Y)\}$. The embedding equation is for AdS_3 with unit radius is

$$-U^2 - V^2 + X^2 + Y^2 = -1\,. \tag{D.1}$$

The isometry algebra has $\mathfrak{so}(2,2)$ is generated by the Killing vector fields

$$J_{AB} = \xi_A \frac{\partial}{\partial \xi^B} - \xi_B \frac{\partial}{\partial \xi^A}\,, \tag{D.2}$$

and satisfy the commutation relations

$$[J_{AB}, J_{BC}] = \begin{cases} -J_{AC}, & \text{for } \; B \in \{1,2\} \text{ and } A \neq B \neq C\,, \\ J_{AC}, & \text{for } \; B \in \{3,4\} \text{ and } A \neq B \neq C\,. \end{cases} \tag{D.3}$$

It is a straightforward exercise to divide these generators into two mutually commuting sets as:

$$X_0 = -J_{UV} - J_{XY}\,, \quad X_1 = J_{XU} + J_{YV}\,, \quad X_2 = J_{YU} - J_{XV}\,, \tag{D.4}$$

$$Y_0 = -J_{UV} + J_{XY}\,, \quad Y_1 = J_{XU} - J_{YV}\,, \quad Y_2 = -J_{YU} - J_{XV}\,. \tag{D.5}$$

Both the left-invariant vector fields $\{X_a\}$ and the right-invariant vector fields $\{Y_a\}$ satisfy the same $\mathfrak{so}(1,2)$ commutation relations, that is

$$[X_0, X_1] = 2X_2\,, \quad [X_1, X_2] = -2X_0\,, \quad [X_0, X_2] = -2X_1\,, \tag{D.6}$$

$$[Y_0, Y_1] = 2Y_2\,, \quad [Y_1, Y_2] = -2Y_0\,, \quad [Y_0, Y_2] = -2Y_1\,. \tag{D.7}$$

To concretely realize these vector fields, let us choose a local coordinate chart $\{x^\mu = (t, \chi, \theta)\}$ on AdS_3 such that

$$U = \cos t\,,\ V = \sin t \cosh \chi\,,\ X = \sin t \sinh \chi \cos \theta\,,\ Y = \sin t \sinh \chi \sin \theta\,. \quad \text{(D.8)}$$

Then we compute the following explicit expressions for $\{X_a\}$:

$$\begin{aligned} X_0 &= \cosh \chi \partial_t - \cot t \sinh \chi \partial_\chi - \partial_\theta\,, \\ X_1 &= -\sinh \chi \cos \theta \partial_t + (\cot t \cosh \chi \cos \theta + \sin \theta)\partial_\chi + (\coth \chi \cos \theta - \cot t \chi \sin \theta)\partial_\theta\,, \\ X_2 &= -\sinh \chi \sin \theta \partial_t + (\cot t \cosh \chi \sin \theta - \cos \theta)\partial_\chi + (\coth \chi \sin \theta + \cot t \chi \cos \theta)\partial_\theta\,, \end{aligned}$$

and for $\{Y_a\}$:

$$\begin{aligned} Y_0 &= \cosh \chi \partial_t - \cot t \sinh \chi \partial_\chi + \partial_\theta\,, \\ Y_1 &= -\sinh \chi \cos \theta \partial_t + (\cot t \cosh \chi \cos \theta - \sin \theta)\partial_\chi + (-\coth \chi \cos \theta - \cot t \chi \sin \theta)\partial_\theta\,, \\ Y_2 &= \sinh \chi \sin \theta \partial_t + (-\cot t \cosh \chi \sin \theta - \cos \theta)\partial_\chi + (\coth \chi \sin \theta - \cot t \chi \cos \theta)\partial_\theta\,. \end{aligned}$$

Finally, exploiting the canonical pairings $e^b(X_a) = \delta^b_a$ and $\tilde{e}^b(Y_a) = \delta^b_a$, we determine in a unique way the following set of left-invariant co-frame 1-forms:

$$e^0 = \cosh \chi dt + \cos t \sin t \sinh \chi d\chi + \sin^2 t \sinh^2 \chi d\theta\,, \quad \text{(D.9)}$$

$$\begin{aligned} e^1 &= \sinh \chi \cos \theta dt + (\cos t \sin t \cosh \chi \cos \theta + \sin^2 t \sin \theta)d\chi \\ &\quad + \sin^2 t \sinh \chi(\cosh \chi \cos \theta - \cot t \sin \theta)d\theta\,, \end{aligned} \quad \text{(D.10)}$$

$$\begin{aligned} e^2 &= \sinh \chi \sin \theta dt + (\cos t \sin t \cosh \chi \sin \theta - \sin^2 t \cos \theta)d\chi \\ &\quad + \sin^2 t \sinh \chi(\cosh \chi \sin \theta + \cot t \cos \theta)d\theta\,, \end{aligned} \quad \text{(D.11)}$$

and the right-invariant co-frame 1-forms:

$$\tilde{e}^0 = \cosh \chi dt + \cos t \sin t \sinh \chi d\chi - \sin^2 t \sinh^2 \chi d\theta\,, \quad \text{(D.12)}$$

$$\begin{aligned} \tilde{e}^1 &= \sinh \chi \cos \theta dt + (\cos t \sin t \cosh \chi \cos \theta - \sin^2 t \sin \theta)d\chi \\ &\quad - \sin^2 t \sinh \chi(\cosh \chi \cos \theta + \cot t \sin \theta)d\theta\,, \end{aligned} \quad \text{(D.13)}$$

$$\begin{aligned} \tilde{e}^2 &= -\sinh \chi \sin \theta dt - (\cos t \sin t \cosh \chi \sin \theta + \sin^2 t \cos \theta)d\chi \\ &\quad + \sin^2 t \sinh \chi(\cosh \chi \sin \theta - \cot t \cos \theta)d\theta\,. \end{aligned} \quad \text{(D.14)}$$

It is now straightforward to verify that these basis 1-forms satisfy the Maurer-Cartan equations

$$de^a = -\epsilon^a{}_{bc} e^b \wedge e^c\,, \qquad d\tilde{e}^a = -\epsilon^a{}_{bc} \tilde{e}^b \wedge \tilde{e}^c\,. \tag{D.15}$$

Moreover in the local coordinate chart we picked, the metric tensor becomes

$$\begin{aligned} g_{\mathrm{AdS}_3} &= -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \\ &= -\tilde{e}^0 \otimes \tilde{e}^0 + \tilde{e}^1 \otimes \tilde{e}^1 + \tilde{e}^2 \otimes \tilde{e}^2 \\ &= -dt^2 + \sin^2 t (d\chi^2 + \sinh^2 \chi d\theta^2)\,. \end{aligned} \tag{D.16}$$

Appendix E

MAJORANA SPINORS IN 3D

The spinor fields on a Riemann-Cartan space-time M are related to the spinors on the model space $(\mathbb{R}^{1,2}, \eta)$ whose Clifford algebra is denoted by $Cl(1,2)$. We are going to use a real realization generated by the Pauli matrices given by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{E.1}$$

These generators satisfy the Clifford product rule

$$\gamma_a \gamma_b = \eta_{ab} I + \epsilon_{abc} \gamma^c , \tag{E.2}$$

so that

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} I , \quad [\gamma_a, \gamma_b] = 2\epsilon_{abc}\gamma^c . \tag{E.3}$$

The Clifford algebra $Cl(1,2)$ can be spanned by a basis $\{I, \gamma_a, 2\sigma_{ab}, \gamma_5\}$ where I is the 2×2 identity operator, $\{\sigma_{ab}\}$ and γ_5 are the Lorentz generators and the volume element, given explicitly by

$$\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\epsilon_{abc}\gamma^c , \quad \gamma_5 = \gamma_0\gamma_1\gamma_2 = I , \tag{E.4}$$

respectively. The following identities are satisfied by the generators of the Clifford algebra:

$$2\gamma_a \sigma_{bc} = \eta_{ab}\gamma_c - \eta_{ac}\gamma_b + \epsilon_{abc} I , \tag{E.5}$$

$$2\sigma_{ab}\gamma_c = -\eta_{ac}\gamma_b + \eta_{bc}\gamma_a + \epsilon_{abc} I , \tag{E.6}$$

$$[\sigma_{ab}, \sigma_{cd}] = -\eta_{ac}\sigma_{bd} + \eta_{ad}\sigma_{bc} + \eta_{bc}\sigma_{ad} - \eta_{bd}\sigma_{ac} . \tag{E.7}$$

Furthermore the following summation identities are also satisfied:

$$\gamma^a\gamma_a = 3\,, \quad \gamma^a\gamma_b\gamma_a = -\gamma_b\,, \quad \gamma^a\gamma_b\gamma_c\gamma_a = 3\eta_{bc} - 2\sigma_{bc}\,, \quad \gamma^a\sigma_{bc}\gamma_a = -\sigma_{bc}\,,$$
$$\gamma^a\sigma_{ab} = \gamma_b\,, \quad 2\sigma^{ab}\sigma_{ab} = -3\,, \quad 2\sigma^{ab}\gamma_c\sigma_{ab} = \gamma_c\,. \tag{E.8}$$

Since the rank-2 and rank-3 elements of the Clifford basis are linearly dependent on the rank-0 and rank-1 elements, we use the basis $\{\gamma_A\} = \{I, \gamma_a\}$.

The spin group of our model space, $Spin(1,2) \cong SL(2,\mathbb{R})$ is the double cover of the local Lorentz group $SO(1,2)$. $Spin(1,2)$ is generated by the elements $\{I, 2\sigma_{ab}\}$ of the even Clifford subalgebra $Cl_0(1,2)$ and the spinors carry its irreducible representations. In our case the representation space will be $\mathbb{R}^2$, however, components of the spinors should be odd-Grasmann valued. That is, given $\psi = (\psi_1, \psi_2)^T \in \mathbb{R}^2$, both components are nilpotent and they anti-commute:

$$\psi_1^2 = 0 = \psi_2^2\,, \quad \psi_1\psi_2 = -\psi_2\psi_1\,. \tag{E.9}$$

An adjoint spinor is defined as an element of the dual space of the spinor space. The map between the representation space $\mathbb{R}^2$ and its dual space $(\mathbb{R}^2)^*$ is given by an anti-symmetric operator

$$\begin{aligned}\mathcal{C} : \mathbb{R}^2 &\to (\mathbb{R}^2)^* \\ \psi &\mapsto \bar{\psi} = \psi^T\mathcal{C}\end{aligned} \tag{E.10}$$

called the charge conjugation operator. In the Majorana realization that we use,

$$\mathcal{C} = \gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{E.11}$$

which satisfies $\mathcal{C}^{-1} = \mathcal{C}^T = -\mathcal{C}$ and $\mathcal{C}^2 = -I$. The inverse map acts on the conjugate spinors from the left and defines charge conjugated spinors

$$\begin{aligned}\mathcal{C}^{-1} : (\mathbb{R}^2)^* &\to \mathbb{R}^2 \\ \bar{\psi} &\mapsto \psi_C = \mathcal{C}(\bar{\psi})^T\,.\end{aligned} \tag{E.12}$$

Note that $\mathcal{C}^{-1}\mathcal{C} = \text{Id}$ as expected. Because we are using real Clifford generators, our spinors are self-charge conjugate, $\psi_C = \psi$. That is to say, all our spinors will be odd-Grassmann valued Majorana (real) spinors.

Furthermore we note that under the action of the charge conjugation operator, the Clifford basis elements $\{\gamma_A\}$ are transposed:

$$\mathcal{C}\gamma_A\mathcal{C}^{-1} = -{\gamma_A}^T \,. \tag{E.13}$$

Another useful way to think about the charge conjugation matrix is to consider it as a metric on space of the spinors. Using this property, we may pair spinors to obtain objects which have tensorial behavior under local Lorentz transformations. In fact, in 3-dimensions there are only two spinor bi-linears that one may write:

$$\bar{\psi}\phi\,, \quad \bar{\psi}\gamma_a\phi\,. \tag{E.14}$$

The first bi-linear is a pseudoscalar and the second one is a Lorentz vector. The other bi-linears $\bar{\psi}\sigma_{ab}\phi$ and $\bar{\psi}\gamma_5\phi$ may be expressed in terms of the above ones. Any two arbitrary spinors ψ and ϕ satisfy the Majorana flip identities, given by

$$\bar{\psi}\phi = \bar{\phi}\psi\,, \quad \bar{\psi}\gamma_a\phi = -\bar{\phi}\gamma_a\psi\,. \tag{E.15}$$

The complex conjugation operation is an anti-linear anti-involution acting on the Clifford algebra. Consequently, the spinor bilinears in (E.15) are pure imaginary. In order to obtain real quantities instead, we have to introduce a factor of complex unit i into these expressions. Therefore

$$i(\bar{\psi}\phi) \in\,, \quad i(\bar{\psi}\gamma_a\phi) \in \mathbb{R}^{1,2}\,. \tag{E.16}$$

Considering the product of three or more spinors, the order of the products may be arranged according to the Fierz rearrangement formula. Suppose U and V are real valued 2×2 matrices and $\alpha, \beta, \phi, \psi$ are arbitrary Majorana 2-spinors. Then

$$(\bar{\alpha}U\beta)(\bar{\phi}V\psi) = -\frac{1}{2}\sum_{A=1,a}(\bar{\alpha}U\gamma^A V\psi)(\bar{\phi}\gamma_A\beta)\,. \tag{E.17}$$

When considering spinor fields on M, we consider sections of the spin bundle on M which takes values in our spinor space and is acted upon by the spin group $SL(2,\mathbb{R})$ fibrewise. We define the spin covariant exterior derivative operation that acts on a Majorana spinor valued p-form section, for instance the gravitino 1-form over the Riemann-Cartan space-time as:

$$D\chi = d\chi + \frac{1}{2}\omega^{ab}\sigma_{ab} \wedge \chi\,, \tag{E.18}$$

where $\{\omega^{ab}\}$ is a set of metric compatible connection 1-forms on M and χ is a section of Majorana spinor valued 1-forms. The Ricci's identity takes the following form on the spinor valued differential form fields:

$$D^2\chi = \frac{1}{2}R^{ab}\sigma_{ab} \wedge \chi \tag{E.19}$$

where $\{R^{ab}\}$ are the curvature 2-forms on M.

Finally, we also present the contracted version of the Bianchi identity (A.12) because it is relevant for our supergravity calculations:

$$2(\iota_{dc}R_{ab} - \iota_{ba}R_{cd}) = \iota_{bdc}(DT_a) + \iota_{abd}(DT_c) + \iota_{acd}(DT_b) + \iota_{acb}(DT_d)\,. \tag{E.20}$$

The contracted Bianchi identity (E.20) shows us that in the presence of torsion, the two-two symmetry of the components of the Riemann curvature tensor fails in general.