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PHD THESIS

Strategic Customer Behavior in Service
Systems:
Externalities, Risk Sensitivity and
Heterogeneity

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September 20, 2022

KOÇ UNIVERSITY

Abstract

Industrial Engineering and Operations Management

Doctor of Philosophy

Strategic Customer Behavior in Service Systems: Externalities, Risk Sensitivity and Heterogeneity

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This thesis considers the analysis of models of service systems where customers are strategic risk-averse and decide whether to join the service system or balk, by taking into account various uncertainties and externalities generated by other potential customers. First, we propose a game-theoretic framework for a static service system with strategic risk-averse customers where we ignore the dynamics of the service system such as random arrivals or random service times and seek capturing the equilibrium joining probabilities in the proposed framework by utilizing other attributes of the service system such as reward, externalities and risk-sensitivity. Next, the pricing problem is presented and analyzed. We then, present an M/M/1 queueing model where similar to the previous static framework, customers are risk-averse and strategic, and address the equilibrium joining behaviour of customers in such a system. We also, come up with a pricing scheme for the service provider and social welfare maximizer. Interestingly, our results indicate the possibility of charging negative prices to achieve socially desired arrival rates with risk-averse strategic customers. Finally, we extend our results to the case where customers are heterogeneous in their risk-sensitivity degree in an M/M/1 model with strategic risk-averse customers.

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Chapter 1

Introduction

Service systems are impacted by several uncertainties including customer arrival times, service duration and quality. There is a huge literature based on queueing models analyzing the performance of service systems that treats customer-related uncertainties as exogenous. However, customers of service systems are strategic decision makers that react to system design and operations. The choice of capacity levels, prices or the current level of congestion affects the customers' decisions on whether to obtain service at a given time and their experience is determined by the overall quality offered at the time of service delivery. The literature on service systems that takes customers' choices explicitly into account has developed in the context of queueing models. The paper by Naor (1969) is the first to explore the customers' decisions of whether to join a service system modeled as a queue, or not based on the current queue length (which determines the expected wait for the customer). This literature has grown significantly exploring different queueing models. The book by Hassin and Haviv (2003) presents this modeling and analysis framework in a comprehensive manner. A recent book by Hassin (2016) provides a classification of the state-of-the-art research in this area. A comprehensive classification of the applied approaches in designing queueing systems and the related results was provided by Stidham Jr (2009).

When customers are modeled as strategic decision makers, they must not only react to the service-system properties but also to the other customers' decisions since resources

are usually shared or consumed simultaneously with other customers in a service system. This is a challenge in terms of modeling because a relevant model must capture the interactions among the customers in a game-theoretic framework in addition to capturing the usual uncertainties in arrivals, service processes, etc. While the ‘queueing with strategic customers’ literature has evolved significantly to address this dual challenge, there are many issues that are important from a service design point of view but cannot be addressed under a queueing model. There are other stochastic dynamic models incorporating strategic customer behavior, such as the transportation system in Manou, Canbolat, and Karaesmen (2017), which facilitate certain aspects of the problem but still leave many open issues. To fill this gap, first, we propose a class of static models of service systems and then extend our basic model into a queueing framework. Static models of service systems are conceptually simpler than queueing models, but enable the analysis of externalities (i.e., the utility or disutility of a customer from the presence of other customers in the system simultaneously), risk sensitivity and other service quality aspects. The static models we present and analyze ignore the dynamic aspects of the problem such as dynamic arrivals over time and individual service processes that take a random amount of time. This is a relative drawback but may be worth the additional benefits, considering that most queueing models with strategic customers also have to focus on the stationary regime of a Markovian system since otherwise the underlying joint queueing - game theoretic problem becomes intractable. After analyzing the static system, our second model is a joint queueing-game theoretic model which enables us to capture the relationship between externalities and risk aversion, in a dynamic framework. To enable addressing risk sensitive customers we would propose a risk-averse exponential utility function. Furthermore we extend our results into a service system with heterogeneous customers rather than homogeneous customers. Heterogeneous customers in our context pertains to customers with different degrees of risk aversion in their possessing utility functions. The remainder of the thesis is organized as follows. Chapter 2 presents

the literature review related to strategic queueing framework. Chapter 3 presents the results obtained for service systems with homogeneous customers and Chapter 4 treats the case of heterogeneous customers. Chapter 5 provides the results for an M/M/1 queueing system with risk averse strategic customers and finally, in chapter 6, the heterogeneous counterpart of chapter 5 is presented. Some main findings of this thesis is summarized as follows.

In chapter 3, we derive the symmetric equilibrium joining probabilities for the static service system with risk-averse strategic customers under different types of externalities. Under positive externalities, we encounter equilibrium joining probabilities that are not mutually exclusive, which restricts us in solving the revenue maximization problem with an exponential utility function. However, with no externalities, we were able to characterize the mutually exclusive equilibrium joining probabilities, which in turn enables us to address the unique revenue maximizing price set by the service provider with an exponential utility function. Furthermore, in case of negative externalities with two homogeneous customers we proved that the equilibrium joining probability is decreasing in the risk-aversion degree. In chapter 4, we extend our results under the assumption that the customers are heterogeneous, in the sense that one group of customers is more risk-averse than the other group. As a result, in the case of having heterogeneous customers, not only we explore the symmetric equilibrium joining probabilities but also we address the cases where asymmetric equilibrium joining probabilities can emerge, which in turn complicates the analysis of static service systems with heterogeneous customers. In chapter 5, we present an M/M/1 queueing system where customers are risk-averse and strategic. Then we derive the equilibrium arrival rate for our M/M/1 queueing system with risk-averse strategic customers and compare it to the socially desired arrival rate. Surprisingly, compared to the literature where the equilibrium arrival rate is always greater or equal than the socially desired arrival with risk-neutral strategic customers in an M/M/1 queue, the results in our case indicate something different. Under our assumption with

risk-averse strategic customers in an M/M/1 queue, the equilibrium arrival rate might be greater or even less than the socially desired arrival rate, which consequently implies the possibility of charging negative prices (providing subsidies) to fill the gap between the equilibrium arrival rate and socially desired arrival rate. In chapter 6, we continue with the the M/M/1 queueing system with risk-averse strategic customers, except that customers are heterogeneous in their risk-aversion degree. Under the heterogeneity assumption, we suppose that the risk-aversion degree of the pool of customers is reflected by an arbitrary continuous distribution. Afterwards, we explore an equilibrium threshold strategy under any given arbitrary price which enables us to address the equilibrium arrival rate in such a system and compare it to the socially desired arrival rate.

Chapter 2

Literature Review

This thesis is related to several streams of literature: *strategic queueing systems, externalities in service systems, risk sensitive customers, heterogeneous customers, pricing in service systems and finally social welfare maximization*. The literature related with strategic queueing systems is very rich and has evolved rapidly. However, the first research which took into account the existence of strategic customers was Naor (1969). In this paper, a toll is charged on newly arriving customers in an M/M/1 queueing system and it is shown that imposing such a strategy might lead to social optimality in many cases. To elaborate, In Naor (1969)'s Model, arriving customers are observing the number of other customers in a single-server queue (known as *Observable Queues*), and decide whether to join the the queue or balk, based on their expected obtained utility. In this M/M/1 model Customers take advantage of a risk neutral utility function, valuing service at R and incurring a waiting cost at rate C per unit time. Followed by this research, Yechiali (1971) relaxed the assumptions of Naor (1969)'s model by allowing inter-arrival times of customers to have a common distribution with finite mean. Chr (1972) studied an M/M/S queueing system where arriving customers to the service system are heterogeneous in their obtained benefit, based on the number of customers present in the system. Edelson and Hilderbrand (1975) studied a similar M/M/1 queueing system, except that customers are not able to see the number of other customers in the system (known as *Unobservable Queues*). It is worth mentioning that a queue is categorized as observable

if the queue length is known by customers, whether by direct observation or by indirect inference. Otherwise the queue is unobservable. The main part of the literature in rational queueing pertains to unobservable queueing models and observable queueing models are rare, due to the relative simplicity of observable models (Hassin, 2016). Furthermore, some researchers have relaxed the Markovian assumptions on service rates and have assumed that servers could have a general continuous distribution, known as M/G/1 queues (Kerner, 2011).

As it was discussed in the introduction, a majority of papers in the strategic queueing literature suppose that customers possess a linear utility function. However, there are rare papers which take into account the reality that customers might be risk-sensitive and base their strategic decisions on a nonlinear utility function. Chen and Frank (2004) study an M/M/1 strategic queue and address pricing (both in the short-run and long-run) and optimum service rate issues, under risk-neutrality and risk-aversion, and show that under risk-neutrality the equilibrium arrival rate is also socially desirable but with risk-averse customers the outcome is different. Guo and Zipkin (2007) study a strategic M/M/1 queue under three different levels of delay information, namely, no information, partial information and full information. Their results indicate that based on the cost-scale distribution (implying the customer preferences), customers or the provider could be hurt or better off by the revealed information. Their presented model is cable of addressing risk-averse behaviour of customers by assigning different cost functions to customers, where, similarly, this role is carried out in economics by assigning utility functions to customers. Wang and Zhang (2018) study an M/M/1 strategic queue by supposing that customers possess a quadratic utility function which enables addressing risk-sensitive customers. Afèche, Baron, and Kerner (2013) presented a lead-time based pricing scheme for a service provider with time-sensitive customers of different types. In this research, they relaxed the risk-neutrality assumption and also considered the case where customers are risk-averse with respect to their pay-offs. Furthermore, they supposed that the tariffs could

be arbitrary functions of the lead-time.

Kaman, Savaseneril, and Serin (2013), investigate a manufacturer operating under make-to-stock policy who quotes the lead-time to the customers, under two different settings. First, they model the case where the manufacturer has full information on the shop floor information, then they investigate the case where the manufacturer has limited information regarding the shop floor due to some information distortion . Savaseneril and Sayin (2017) study the dynamic lead-time quotation problem of the manufacturer with different class of lead-time-sensitive customers under responsive inventory. Savaşaneril, Griffin, and Keskinocak (2010) study a similar dynamic lead-time-quotation problem in an M/M/1 setting and under a base-stock inventory system and compare it to the make-to-order policy. Their results indicate that under a MTO system, the quoted lead-times are lower, and also the value of precise information on lead-time quotation with less lead-time sensitive customers is revealed; whereas with more lead-time-sensitive customers keeping a higher base-stock inventory is more advantageous. Kostami (2020) study a firm under two possible scenarios based on revealing the “pricing scheme/lead time information” to customers. In the first scenario, the firm is able to reveal the same price to all customers and conceal the lead time information, and in the second scenario, a dynamic pricing scheme is used and the lead time information is also revealed dynamically. Benioudakis, Burnetas, and Ioannou (2021a) model a production make-to-order system where strategic risk-averse customers decide whether to place their order or not, based on their earned payoff. In their research, the firm sets a price and quotes the lead-time and also compensates for those realized lead times which exceed the quoted lead-time. One of their main insights is that, providing an initial lead-time quotation along with reimbursement for exceeded delays, allows the service provider to charge higher prices. Benioudakis, Burnetas, and Ioannou (2021b) consider a similar problem, except that this time customers are able to see the state of the system before their decision. Furthermore, they address both the static and dynamic pricing problem with respect to the state of the system and compare

it to the social welfare maximization problem.

One of the topics which has drawn attention among researchers in strategic queueing systems is sharing delay information by service providers with their customers, about the upcoming waiting times. Hassin (1986) studied a revenue maximizing server who has the ability to prevent revealing information on the actual queue size to arriving customers, leaving them to decide whether to join the system or balk, based on the available distribution of the waiting time. Guo, Sun, and Wang (2011) investigate an M/G/1 strategic queue under partial information on the waiting time (e.g. moments and ranges) and identify whether the social welfare maximizer and service provider should reveal or conceal information regarding the waiting time. Furthermore, they enable analyzing different types of customer preferences by allowing the cost function to be linear or quadratic convex. A recent comprehensive review of this topic was done by Ibrahim (2018). The research in this thesis will also contribute to another stream of literature, namely pricing in service systems (Manou, Canbolat, and Karaesmen, 2017), (Naceur and Hayel, 2020) and (Wang, Wang, and Chen, 2020).

Our research contributes to the above-mentioned literature by relaxing the risk-neutrality assumption on the utility function possessed by customers in the strategic queueing framework, by proposing a concave exponential utility function. This enables us to address those cases where customers decide based on the possibility of worse outcomes (extreme waits in our setting) rather than the average waiting time, although there are some rare papers on the risk-averse behavior of customers in the literature. Furthermore, we first present a static-game theoretic setting which ignores the dynamics of the service system (e.g. random arrivals, random service durations) and instead, utilizes other attributes of the underlying service system (reward, externalities and risk-sensitivity) to capture equilibrium joining probabilities in our model. Afterwards, we present an M/M/1 strategic queueing system with risk-averse customers and characterize the equilibrium arrival rate both in the observable case and unobservable case. Furthermore, we present the

implications of dynamic pricing instead of static pricing in the observable case, and in the unobservable case we come up with a pricing scheme to obtain the socially desirable arrival rate with risk-averse strategic customers. Finally, we conclude with presenting an M/M/1 queueing system with risk-averse strategic customers who are heterogeneous in their risk-aversion degree, by letting the risk-aversion degree of customers to follow an arbitrary continuous distribution and then we characterize the equilibrium arrival rate.



Chapter 3

Static Models of service systems with Homogeneous Customers

In this chapter, we consider a static service system with strategic customers, who simultaneously decide whether to join the system or not. The static model focuses on the one-time joining decisions of the considered customers but ignores future events such as other customer arrivals and service completions. From the customer's perspective, her decision to join a service system or not is based on both factors that are observable and known before making the joining decision and those that are random at the time of the decision and are completely revealed only during the service process or after the completion of service. We denote by r the value of the known factors and by X the (random) value of the factors that are revealed during the process. For instance, for a restaurant, r can be the value associated with the location and the physical design of restaurant, while X can be the value associated with the quality of the food and the service experience. Typically, the overall experience of a customer in a service system depends not only on the practices adopted by the service provider, but also on the number of customers getting service at the same time. We incorporate the latter into our model by allowing the uncertain value X to be a function of the number of customers in the service system. Accordingly, we let X_n be the (random) value of the factors that are revealed during the process when n customers are simultaneously in the system. Furthermore, we enable capturing the ongoing

externality in the service system by the stochastic dominance among X_i s. For instance, $X_N >_{st} X_{N-1} >_{st} \dots >_{st} X_1$, implies that positive externality exists in the service system. In other words, customers are better off by the presence of other customers in the service system.

To model risk-sensitive preferences of customers, we assume that customer i decides whether to join the system or not according to a two-attribute utility function $u_i(r, x)$ that is increasing with respect to r and x . Given that exactly $n - 1$ other customers join, customer i experiences the expected utility $E[u_i(r, X_n)]$ from joining and $u_{i,0}$ from balking. Typical queuing models assume risk-neutral and homogeneous customers, and let $E[u_i(r, X_n)] = r - cE[X_n]$ be the expected utility of a customer from joining with r the reward from service, c the unit waiting cost, and $-X_n$ the waiting time of this customer when there are $n - 1$ other customers in the system. In cases where an arriving customer cannot observe the number of customers in the system prior to making a decision, her expected utility from joining becomes $E[u_i(r, X_Z)]$, where expectation is taken also over the random variable Z representing the number of customers in the system (including customer i).

In the framework described above, we explore the Nash equilibria of the game between a finite number of potential customers who simultaneously decide whether to join the service system or not. We focus on two particular issues that are very difficult to address by using queueing models. First, we relax the risk-neutrality assumption commonly made in queueing models, so we allow $u_i(r, X)$ to be nonlinear in r and/or X . This extension is useful in practice because there are many known cases where customers base their decisions on the possibility of extreme waits (or other disutilities) rather than on the average wait. The relaxation of the risk-neutrality assumption enables us to explore the customer decisions and the resulting equilibrium behavior under risk sensitivity. Second, we consider various types of externalities, or more specifically various types of relations between customer utility functions and the number of customers in the system.

In queueing systems, the service experience is typically negatively affected by the wait and the expected wait is usually a linear function of the expected queue length. This is a situation where each joining customer imposes a negative externality by contributing to the congestion and worsens the wait for new customers. However, in practice, there are other service systems where the service experience is non-decreasing in n . For example, an empty restaurant can be unattractive and discourage potential customers to go inside, or a crowded bus stop can indicate that the next bus is about to arrive and that the wait might be short for a newly arriving customer. These are examples with positive externalities, where a customer's decision to join a system positively influences the decisions of other potential customers. The case of positive externalities may lead to the so-called follow-the-crowd (FTC) behavior in the strategic customer literature, (Boudali and Economou, 2013), (Bountali and Economou, 2017) and (Hassin and Roet-Green, 2018).

We characterize the Nash equilibria of the resulting simultaneous-move game among finitely many potential customers under general utility functions. We then refine our results by assuming homogeneous customers making decisions according to risk-averse exponential utility functions with the same degree of risk sensitivity, and investigate the effect of changing the degree of risk sensitivity on the equilibrium joining behavior for cases with negative, positive, or no externalities. Next, we extend our analysis to the situation where customers are heterogeneous in the degree of risk sensitivity and compare the equilibrium joining probabilities of customers with different degrees of risk sensitivity. As pointed in the previous paragraph, in all these special cases, our work focuses on understanding the role of risk sensitivity and externalities. In section 3.1 we introduce a service system with two homogeneous customers and we characterize the equilibrium joining probabilities.

3.1 Service System with Two Homogeneous Customers

In this subsection, we consider the two customer case modeled as a game between two players. In this game, each player determines the probability of joining the service system. The game considered at this stage is static and assumes that both players join the system simultaneously or at least make decisions without being aware of each other's decisions. In addition, customers are modeled as strategic decision makers, so each customer makes his decision considering the existence of the other customer and that he is also a strategic individual. In the first model, it is assumed that the two customers are homogeneous, therefore, their preferences can be demonstrated with the same utility function.

The parameters and variables of this game can be defined as follows: $r > 0$ is the value of the service provided, X_1 is the experience random variable when only one customer joins the system, X_2 is the experience random variable when both customers join the system, $p_1 \in [0, 1]$ is the probability of the first customer joining, $p_2 \in [0, 1]$ the probability of the second customer joining the system, $u(r, X)$ represents the utility of the service with values r and experience variable X . X_1 and X_2 , which are defined as experience variables, are random variables and the customer learns their actual value only after experiencing the service. The utility of not joining the system is denoted by u_0 . This value is generally considered to be zero, but we do not make such an assumption in the general model, so our general results can cover situations where the customer can use an alternative and benefit positively, even when not using the system under consideration. Furthermore, since in this section we are interested in finding the Nash equilibria in an homogeneous game, players possess the same utility functions. Therefore, if we define the player set as N which represents the number of possible customers in our service system, with homogeneous customers we have:

$$u_i(r, X_n) = u_j(r, X_n) = u(r, X_n), \quad u_{i,0} = u_{j,0} = u_0 \quad \forall i, j, n \in N \quad i \neq j, \quad (3.1.1)$$

The expected utility of the first customer if the first customer joins the system with probability p_1 and the second customer with probability p_2 is

$$U_1(p_1, p_2) = p_1 [p_2 E[u_1(r, X_2)] + (1 - p_2) E[u_1(r, X_1)]] + (1 - p_1) u_{1,0} \quad (3.1.2)$$

and for the second customer the expected utility is

$$U_2(p_1, p_2) = p_2 [p_1 E[u_2(r, X_2)] + (1 - p_1) E[u_2(r, X_1)]] + (1 - p_2) u_{2,0} \quad (3.1.3)$$

Since the customers are assumed to be heterogeneous, we will consider (3.1.1) and rewrite equations (3.1.2) and (3.1.3) in the following format:

$$U_1(p_1, p_2) = p_1 [p_2 E[u(r, X_2)] + (1 - p_2) E[u(r, X_1)]] + (1 - p_1) u_0 \quad (3.1.4)$$

$$U_2(p_1, p_2) = p_2 [p_1 E[u(r, X_2)] + (1 - p_1) E[u(r, X_1)]] + (1 - p_2) u_0 \quad (3.1.5)$$

The fact that each customer is a strategic individual and optimizes his own expected utility function indicates that the first customer will choose the probability p_1 such that he maximizes the value of $U_1(p_1, p_2)$, and the second will choose the probability of p_2 , which maximizes the value of $U_2(p_1, p_2)$, which determines the best response mapping. Now, we want to determine the best response of a customer against a given strategy of the other customer. We denote by $BR_1(p_2)$ the best response of a customer against strategy p_2 of

the other customer. We have

$$BR_1(p_2) = \begin{cases} 0 & \text{if } p_2 E[u(r, X_2)] + (1 - p_2) E[u(r, X_1)] < u_0, \\ [0, 1] & \text{if } p_2 E[u(r, X_2)] + (1 - p_2) E[u(r, X_1)] = u_0, \\ 1 & \text{if } p_2 E[u(r, X_2)] + (1 - p_2) E[u(r, X_1)] > u_0. \end{cases} \quad (3.1.6)$$

Similarly, the best response function of the second customer is as follows:

$$BR_2(p_1) = \begin{cases} 0 & \text{if } p_1 E[u(r, X_2)] + (1 - p_1) E[u(r, X_1)] < u_0, \\ [0, 1] & \text{if } p_1 E[u(r, X_2)] + (1 - p_1) E[u(r, X_1)] = u_0, \\ 1 & \text{if } p_1 E[u(r, X_2)] + (1 - p_1) E[u(r, X_1)] > u_0. \end{cases} \quad (3.1.7)$$

We are interested in finding the Nash equilibrium of this homogeneous two-customer game. For this purpose, we have to find (p_1^*, p_2^*) that satisfies the constraints $p_1^* \in BR_1(p_2^*)$ and $p_2^* \in BR_2(p_1^*)$. A symmetric Nash equilibrium is a Nash equilibrium where each player uses the same strategy. In our model, it can be expressed as a single probability value as $p_1^* = p_2^* = p^*$. subsequently, we have the following theorem¹.

Theorem 3.1.1 *In the equilibrium of the symmetric game between two customers in a service system, we have the following cases.*

- (i) *If $E[u(r, X_1)] \leq u_0$, then strategy profile $(p_1^*, p_2^*) = (0, 0)$ is the equilibrium. Thus, in this case both customers balk.*
- (ii) *If $E[u(r, X_2)] \geq u_0$, then strategy profile $(p_1^*, p_2^*) = (1, 1)$ is the equilibrium. Thus, in this case both customers join.*
- (iii) *If $0 < p^* < 1$, satisfies $p^* E[u(r, X_2)] + (1 - p^*) E[u(r, X_1)] = u_0$, then strategy profile $(p_1^*, p_2^*) = (p^*, p^*)$ is the equilibrium. Thus in this case both customers join with probability p^* .*

¹The theorem was taken from the reports of a 117M583 numbered joint tübitak research with Fikri Karaesmen, Pelin Canbolat and Athanasia Manou

According to Theorem 3.1.1, the symmetric equilibrium in a 2-customer service system modeled by a static framework has three possible cases. First, if the expected utility when there is only one customer in the service system $E[u(r, X_1)]$, is less than or equal to the utility of balking, u_0 , then both customers balk. Second, if the expected utility when there are two customers in the service system $E[u(r, X_2)]$, is greater than or equal to the utility of balking, u_0 , then both customers join. Third, if there exists a p^* such that it satisfies

$$p^*E[u(r, X_2)] + (1 - p^*)E[u(r, X_1)] = u_0 \quad (3.1.8)$$

then both customers join with probability p^* . In other words, $E[u(r, X_1)]$ and $E[u(r, X_2)]$ are one-dimensional numbers which could take negative or positive or even zero values. Therefore, if there is a p^* such that sets “the convex combination of $E[u(r, X_1)]$ and $E[u(r, X_2)]$ ” equal to u_0 , that p^* is actually the equilibrium joining probability and both customers would join with p^* . As a result, the ongoing externality in the service system, which is determined by the relationship between $E[u(r, X_1)]$ and $E[u(r, X_2)]$ has a crucial role in deriving the possible equilibrium or equilibria. The role of externalities in defining the possible equilibrium joining probabilities will be presented in section 3.3. After deriving the equilibrium joining probabilities for the service system with 2 homogeneous customers, In the following section, we will extend our results to the case where there are N homogeneous customers in the service system.

3.2 Static service system with N homogeneous customers

After deriving the symmetric equilibrium joining probabilities for the case with 2 customers, in this section we will derive the equilibrium for the case with N customers, in a similar framework to section 3.1. By supposing there are N homogeneous customers in the service system, the expected utility of customer i if he follows strategy p_i and all other customers follow strategy p , is expressed as follows:

$$U_i(p_i, p) = p_i \left[\sum_{n=0}^{N-1} \binom{N-1}{n} p^n (1-p)^{N-n-1} E[u(r, X_{n+1})] \right] + (1-p_i)u_0$$

If the total number of customers other than Customer i is expressed with Y , the expected utility of customer i can also be written as:

$$U_i(p_i, p) = p_i E[u(r, X_{Y+1})] + (1-p_i)u_0$$

where Y is a binomial variable with parameters $N-1$ and p . The expected value is calculated based on both X and Y . The best response of customer i against strategy p of other customers is as follows:

$$BR_i(p) = \begin{cases} 0 & \text{if } E[u(r, X_{Y+1})] < u_0, \\ [0, 1] & \text{if } E[u(r, X_{Y+1})] = u_0, \\ 1 & \text{if } E[u(r, X_{Y+1})] > u_0. \end{cases}$$

As a result, we have the following theorem.²

Theorem 3.2.1 *the equilibrium or equilibria of the symmetric game between N homogeneous customers in a service system is as follows:*

(i) *If $E[u(r, X_1)] \leq u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (0, \dots, 0)$ is the equilibrium.*

Thus, in this case all customers balk.

(ii) *If $E[u(r, X_N)] \geq u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (1, \dots, 1)$ is the equilibrium.*

Thus, in this case all customers join.

(iii) *If binomial variable Y with parameters $N-1$ and $0 < p^* < 1$ provides $E[u(r, X_{Y+1})] = u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (p^*, \dots, p^*)$ is the equilibrium. Thus in this case all customers join with probability p^* .*

²The theorem was taken from the reports of a 117M583 numbered joint tübitak research with Fikri Karaesmen, Pelin Canbolat and Athanasia Manou

According to Theorem 3.2.1, the symmetric equilibrium in a N -customer service system modeled by a static framework has three possible cases. First, if the expected utility when there is only one customer in the service system $E[u(r, X_1)]$, is less than or equal to the utility of balking, u_0 , then all customers balk. Second, if the expected utility when there are N customers in the service system $E[u(r, X_N)]$, is greater than or equal to the utility of balking, u_0 , then all customers join. Third, if there exists a p^* such that it satisfies

$$E[u(r, X_{Y+1})] = \sum_{n=0}^{N-1} \binom{N-1}{n} p^{*n} (1-p^*)^{N-n-1} E[u(r, X_{n+1})] = u_0 \quad (3.2.1)$$

then all customers join with probability p^* .

After addressing the equilibrium joining probabilities for the case with N homogeneous customers with general utility functions, Now we are interested in first, extending our results to the cases with different kinds of externalities (e.g. no externality, negative externality and positive externality). We define the ongoing type of externality among customers by the stochastic dominance between X_i s. Afterwards, we will extend our results to the case where customers possess a specific utility function, namely an exponential utility function.

3.3 Externalities

The type of externality present in the service system is defined by the stochastic relationship between X_i s. In this regard, if $X_1 =_{st} X_2 =_{st} \dots =_{st} X_N$, for any non-decreasing function $u(r, X_n)$ in X_n , We have $E[u(r, X_1)] = E[u(r, X_2)] = \dots = E[u(r, X_N)]$. Therefore, for the cases where $X_1 =_{st} X_2 =_{st} \dots =_{st} X_N$, the implication is that there is no externality among customers and the experience variable X_i is independent from the number of customers in the service system. Similarly, if $X_1 >_{st} X_2 >_{st} \dots >_{st} X_N$, for any non-decreasing function $u(r, X_n)$ in X_n , We have $E[u(r, X_1)] > E[u(r, X_2)] > \dots > E[u(r, X_N)]$.

Hence, for the cases where $X_1 >_{st} X_2 >_{st} \dots >_{st} X_N$, the implication is that there is negative externality among customers. Finally, if we have $X_1 <_{st} X_2 <_{st} \dots <_{st} X_N$, the implication is that there is positive externality among customers. As a result we will derive the equilibrium joining probabilities under different types of externalities.

Remark. We define a strict stochastic dominance between X_i s by supposing X_i s have an infinite support e.g. X_i s follow an exponential distribution or etc.

3.3.1 No Externalities

Corollary 3.3.1 For the equilibrium of the symmetric game between 2 customers in a service system, when $X_1 =_{st} X_2 =_{st} X$, (no externality), we have the following equilibrium or equilibria:

(i) If $E[u(r, X)] < u_0$, then strategy profile $(p_1^*, p_2^*) = (0, 0)$ is the unique equilibrium.

Thus, in this case both customers balk.

(ii) If $E[u(r, X)] > u_0$, then strategy profile $(p_1^*, p_2^*) = (1, 1)$ is the unique equilibrium.

Thus, in this case both customers join.

(iii) If $E[u(r, X)] = u_0$, then strategy profile $(p_1^*, p_2^*) = (p^*, p^*)$ is the equilibrium, for every $p^* \in [0, 1]$. Thus in this case both customers join with probability p^* .

According to Corollary 3.3.1, the symmetric equilibrium in a 2-customer service system with no externality, has three possible cases. As we described, with no externality, X_n is independent of n (number of customers who have decided to be in the system simultaneously) and equals X for all n . First, if the expected utility $E[u(r, X)]$, is less than the utility of balking, u_0 , then both customers balk. Second, if the expected utility $E[u(r, X)]$, is greater than the utility of balking, u_0 , then both customers join. Third, since if $E[u(r, X)] = u_0$, every $p^* \in [0, 1]$ satisfies

$$p^* E[u(r, X_2)] + (1 - p^*) E[u(r, X_1)] = u_0 \quad (3.3.1)$$

with no externality, every $p^* \in [0, 1]$ is an equilibrium in case of $E[u(r, X)] = u_0$.

Corollary 3.3.2 *For the equilibrium of the symmetric game between N homogeneous customers in a service system when there is no externality, we have the following equilibrium or equilibria:*

(i) *If $E[u(r, X)] < u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (0, \dots, 0)$ is the unique equilibrium.*

Thus, in this case all customers balk.

(ii) *If $E[u(r, X)] > u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (1, \dots, 1)$ is the unique equilibrium.*

Thus, in this case all customers join.

(iii) *If satisfies $E[u(r, X)] = u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (p^*, \dots, p^*)$ is the equilibrium, for every $p^* \in [0, 1]$. Thus in this case all customers join with probability p^* .*

According to Corollary 3.3.2, the symmetric equilibrium in a N -customer service system with no externality, has three possible cases. As we described, with no externality, X_n is independent of n (number of customers who have decided to be in the system simultaneously) and equals X for all n . First, if the expected utility $E[u(r, X)]$, is less than the utility of balking, u_0 , then all customers balk. Second, if the expected utility $E[u(r, X)]$, is greater than the utility of balking, u_0 , then all customers join. Third, since if $E[u(r, X)] = u_0$, every $p^* \in [0, 1]$ satisfies

$$E[u(r, X_{Y+1})] = \sum_{n=0}^{N-1} \binom{N-1}{n} p^{*n} (1-p^*)^{N-n-1} E[u(r, X_{n+1})] = u_0 \quad (3.3.2)$$

with no externality, every $p^* \in [0, 1]$ is an equilibrium in case of $E[u(r, X)] = u_0$, and all customers join with $p^* \in [0, 1]$.

As it could be understood from corollaries 3.3.1 and 3.3.2, in case of no externalities the equilibrium joining probabilities are **comprehensive and mutually exclusive**. This enables us to derive the equilibrium joining probabilities under price τ , set by the service provider in section 3.4. In the following subsection, we will continue with deriving the equilibrium joining probabilities under negative externalities.

3.3.2 Negative Externalities

Let's now consider negative externalities where $X_1 >_{st} X_2$.

Corollary 3.3.3 *In a 2-customer service system with negative externality, $X_1 >_{st} X_2$, there exists a single Nash equilibrium:*

(i) *If $E[u(r, X_1)] \leq u_0$, then strategy profile $(p_1^*, p_2^*) = (0, 0)$ is the unique equilibrium.*

Thus, in this case both customers balk.

(ii) *If $E[u(r, X_2)] \geq u_0$, then strategy profile $(p_1^*, p_2^*) = (1, 1)$ is the unique equilibrium.*

Thus, in this case both customers join.

(iii) *If $E[u(r, X_2)] < u_0 < E[u(r, X_1)]$, then strategy profile $(p_1^*, p_2^*) = (p^*, p^*)$ is the unique equilibrium, where p^* is:*

$$p^* = \frac{E[u(r, X_1)] - u_0}{E[u(r, X_1)] - E[u(r, X_2)]} \in (0, 1)$$

Thus, in this case both customers join with probability p^ .*

According to Corollary 3.3.3, the symmetric equilibrium in a 2-customer service system modeled by a static framework has three possible cases. Note that $X_1 >_{st} X_2$, this implies $E[u(r, X_1)] > E[u(r, X_2)]$. First, if the expected utility when there is only one customer in the service system $E[u(r, X_1)]$, is less than or equal to the utility of balking, u_0 , then both customers balk. Second, if the expected utility when there are two customers in the service system $E[u(r, X_2)]$, is greater than or equal to the utility of balking, u_0 , then both customers join. Third, if there exists a p^* that it satisfies (3.1.8), then both customers join with probability p^* . Therefore, if there is a p^* such that sets “the convex combination of $E[u(r, X_1)]$ and $E[u(r, X_2)]$ ” equal to u_0 , that p^* is actually the equilibrium joining probability and both customers would join with p^* . But with negative externalities, since we have $E[u(r, X_1)] > E[u(r, X_2)]$, this p^* is unique.

Corollary 3.3.4 *In a N-customer service system with negative externality, $X_1 >_{st} X_2 >_{st} \dots >_{st} X_N$, there exists a single Nash equilibrium:*

- (i) *If $E[u(r, X_1)] \leq u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (0, \dots, 0)$ is the unique equilibrium. Thus, in this case all customers balk.*
- (ii) *If $E[u(r, X_N)] \geq u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (1, \dots, 1)$ is the unique equilibrium. Thus, in this case all customers join.*
- (iii) *If $E[u(r, X_N)] < u_0 < E[u(r, X_1)]$ then $(p_1^*, \dots, p_N^*) = (p^*, \dots, p^*)$ is the unique Nash equilibrium. here p^* is the single solution of $E[u(r, X_{Y+1})] = u_0$ where Y is binomial with parameters $N - 1$ and $0 < p^* < 1$. Thus in this case all customers join with probability p^* .*

According to Corollary 3.3.4, the symmetric equilibrium in a N-customer service system modeled by a static framework has three possible cases. Note that $X_1 >_{st} X_2 >_{st} \dots >_{st} X_N$, this implies $E[u(r, X_1)] > E[u(r, X_2)] > \dots > E[u(r, X_N)]$. First, if the expected utility when there is only one customer in the service system $E[u(r, X_1)]$, is less than or equal to the utility of balking, u_0 , then all customers balk. Second, if the expected utility when there are N customers in the service system $E[u(r, X_N)]$, is greater than or equal to the utility of balking, u_0 , then all customers join. Third, if there exists a p^* that it satisfies (3.2.1), then all customers join with probability p^* . Therefore, if there is a p^* such that it sets “the convex combination of $E[u(r, X_1)]$, $E[u(r, X_2)]$, ..., $E[u(r, X_N)]$ ” equal to u_0 , that p^* is actually the equilibrium joining probability and all customers would join with p^* . But with negative externalities, since we have $E[u(r, X_1)] > \dots > E[u(r, X_N)]$, this p^* is unique. Hence all customers will join with p^* .

Same as the case with no externalities, As it could be understood from corollaries 3.3.3 and 3.3.4, in case of negative externalities the equilibrium joining probabilities are **comprehensive and mutually exclusive**. Which in turn, enables us to derive the equilibrium

joining probabilities under price τ , set by the service provider in section 3.4. In the following section, we will extend our results by deriving the equilibrium joining probabilities under positive externalities.

3.3.3 Positive Externalities

Corollary 3.3.5 *In a 2-customer service system with positive externality, $X_2 >_{st} X_1$, the equilibrium or equilibria are as follows:*

- (i) *If $E[u(r, X_1)] < u_0$, then strategy profile $(p_1^*, p_2^*) = (0, 0)$ is the unique equilibrium. Thus, in this case both customers balk.*
- (ii) *If $E[u(r, X_2)] > u_0$, then strategy profile $(p_1^*, p_2^*) = (1, 1)$ is the unique equilibrium. Thus, in this case both customers join.*
- (iii) *If $E[u(r, X_1)] \leq u_0 \leq E[u(r, X_2)]$, then strategy profiles $(p_1^*, p_2^*) = (0, 0)$, $(p_1^*, p_2^*) = (1, 1)$ and $(p_1^*, p_2^*) = (p^*, p^*)$ are the equilibria, where p^* is:*

$$p^* = \frac{u_0 - E[u(r, X_1)]}{E[u(r, X_2)] - E[u(r, X_1)]} \in [0, 1]$$

specially when $E[u(r, X_1)] = u_0$ then $p^ = 0$ and when $E[u(r, X_2)] = u_0$ then $p^* = 1$.*

According to Corollary 3.3.5, the symmetric equilibrium in a 2-customer service system modeled by a static framework, under positive externality has three possible cases. Note that $X_1 <_{st} X_2$, implies $E[u(r, X_1)] < E[u(r, X_2)]$. First, if the expected utility when there is only one customer in the service system $E[u(r, X_1)]$, is less than the utility of balking, u_0 , then both customers balk. Second, if the expected utility when there are two customers in the service system $E[u(r, X_2)]$, is greater than the utility of balking, u_0 , then both customers join. Third, if there exists a p^* such that it satisfies (3.1.8), then both customers join with probability p^* . Therefore, if there is a p^* such that sets “the convex combination of $E[u(r, X_1)]$ and $E[u(r, X_2)]$ ” equal to u_0 , that p^* is actually the

equilibrium joining probability and both customers would join with p^* . But with positive externalities, since we have $E[u(r, X_1)] < E[u(r, X_2)]$, this p^* is unique.

Corollary 3.3.6 *In a N – customer service system with positive externalities, $X_N >_{st} X_{N-1} >_{st} \dots >_{st} X_1$ the symmetric Nash equilibrium or equilibria are as follows:*

- (i) *If $E[u(r, X_1)] < u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (0, \dots, 0)$ is the unique equilibrium. Thus, in this case all customers balk.*
- (ii) *If $E[u(r, X_N)] > u_0$, then strategy profile $(p_1^*, \dots, p_N^*) = (1, \dots, 1)$ is the unique equilibrium. Thus, in this case all customers join.*
- (iii) *If $E[u(r, X_1)] \leq u_0 \leq E[u(r, X_N)]$ then $(p_1^*, \dots, p_N^*) = (0, \dots, 0)$, $(p_1^*, \dots, p_N^*) = (1, \dots, 1)$ and $(p_1^*, \dots, p_N^*) = (p^*, \dots, p^*)$ are the Nash equilibria where Y is a binomial variable with parameters $N - 1$ and $0 < p^* < 1$ that provides $E[u(r, X_{Y+1})] = u_0$. specially if $E[u(r, X_1)] = u_0$ then $p^* = 0$ and if $E[u(r, X_N)] = u_0$ then $p^* = 1$.*

According to Corollary 3.3.6, the symmetric equilibrium in a N -customer service system modeled by a static framework has three possible cases. Note that $X_1 <_{st} X_2 <_{st} \dots <_{st} X_N$, implies $E[u(r, X_1)] < E[u(r, X_2)] < \dots < E[u(r, X_N)]$. First, if the expected utility when there is only one customer in the service system $E[u(r, X_1)]$, is less than the utility of balking, u_0 , then all customers balk. Second, if the expected utility when there are N customers in the service system $E[u(r, X_N)]$, is greater than the utility of balking, u_0 , then all customers join. Third, if there exists a p^* such that it satisfies (3.2.1), then all customers join with probability p^* . Therefore, if there is a p^* such that it sets “the convex combination of $E[u(r, X_1)]$, $E[u(r, X_2)]$, ..., $E[u(r, X_N)]$ ” equal to u_0 , that p^* is actually the equilibrium joining probability and all customers would join with p^* . But with positive externalities, since we have $E[u(r, X_1)] < \dots < E[u(r, X_N)]$, this p^* is unique. Hence all customers will join with p^* .

By referring to corollaries 3.3.5 and 3.3.6, it could be understood that in case of positive externalities, the equilibrium joining probabilities are comprehensive but **not mutually**

exclusive. Although this is an exiting result from the perspective of the equilibrium joining probabilities, but the existent overlap of equilibrium joining behaviour restricts us in the pricing section. Therefore, in the pricing section we will define the revenue maximization problem only under no externality and negative externality.

3.4 Exponential Utility Function

In this section, we will extend our results by supposing that the customers possess an exponential utility function, rather than a general utility function.

$$1 - e^{-\theta(r-\tau+cX)}$$

There are various measures of the degree of risk-aversion represented by a utility function. One of this common measures is the coefficient of absolute risk aversion defined as

$$\rho(x) = -\frac{u''(x)}{u'(x)}$$

The exponential utility function, has $\rho(x) = \theta$, which is constant in x . This attribute is peculiar to affine and exponential utility functions, called CARA (Constant Absolute Risk Aversion). Accordingly, the general utility function $u(r, X)$ mentioned in the previous sections is replaced by an exponential and risk-averse utility function. Where $\theta > 0$ represents the level of risk-aversion. This function is an increasing function with respect to r and X and its expected value is

$$E[u(r, X_n)] = E[1 - e^{-\theta(r+X_n)}] = 1 - e^{-\theta r} E[e^{-\theta X_n}]$$

Here, $E[e^{-\theta X_n}]$ is equal to the moment generating function of $-X_n$ with parameter θ (Ross, 2014). In some cases $-X_n$ represents the waiting time.

$$E[e^{-\theta X_n}] = M_{-X_n}(\theta),$$

After obtaining the equilibrium joining probabilities for cases with general utility functions, we will derive equilibrium joining probabilities for the cases with an exponential utility function and under certain types of ongoing externality (No externality and negative externality).

3.5 The pricing problem of the service provider

Consider a service system with N potential homogeneous customers, where the service provider sets the price equal to τ . In this case, the service provider is interested in maximizing her revenue, $R(\tau)$, which is obtained by multiplying the number of customers joining in the equilibrium times the service price. By considering a service system with N potential homogeneous customers, with exponential utility function, the expected value of the utility function, given $-X_n$ following an exponential variable with rate μ_n is

$$E[u(r, X_N)] = 1 - e^{-\theta(r-\tau)} \frac{\mu_n}{\mu_n - \theta}, \quad \mu_n > \theta$$

After defining the pricing problem of the service provider, we are able to address the equilibrium joining probabilities when the price is set equal to τ by the service provider. Furthermore, since we supposed that $-X_n$ follows an exponential distribution with rate μ_n , $\mu_1 = \mu_2$ would imply that $X_1 =_{st} X_2$, representing the case with no externality. Similarly, $\mu_1 > \mu_2$ implies $X_1 >_{st} X_2$ and $\mu_1 < \mu_2$ implies $X_1 <_{st} X_2$, which represents the cases with Negative externality and positive externality, respectively.

Corollary 3.5.1 *In a two-customer service system, where $\mu_1 = \mu_2 = \mu > \theta$, and there is no externality, with the same exponential utility function, we have a symmetric Nash equilibrium and these equilibrium or equilibria are as follows:*

- (i) *If $\tau > r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$, then $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium and in this case both customers balk. As a result, the revenue of the administrator is zero.*
- (ii) *If $\tau < r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$, then $(p_1^*, p_2^*) = (1, 1)$ is the unique Nash equilibrium and in this case both customers join. As a result, the revenue of the administrator is 2τ .*
- (iii) *If $\tau = r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$, for every $p^* \in [0, 1]$, $(p_1^*, p_2^*) = (p^*, p^*)$ is a Nash equilibrium. In this case both customers join with p^* . As a result, the revenue of the administrator is $2p^*\tau$.*

Corollary 3.5.2 *In a N -customer service system, where $\mu_n = \mu$, and there is no externality, with the same exponential utility function, we have a symmetric Nash equilibrium and these equilibrium or equilibria are as follows:*

- (i) *If $\tau > r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$, then $(p_1^*, \dots, p_N^*) = (0, \dots, 0)$ is the unique Nash equilibrium. As a result, the revenue of the administrator is zero.*
- (ii) *If $\tau < r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$, then $(p_1^*, \dots, p_N^*) = (1, \dots, 1)$ is the unique Nash equilibrium. As a result, the revenue of the administrator is $N\tau$.*
- (iii) *If $\tau = r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$, then for every $p^* \in [0, 1]$ $(p_1^*, \dots, p_N^*) = (p^*, \dots, p^*)$ is the Nash equilibrium. As a result, the revenue of the administrator is $Np^*\tau$.*

By rearranging the conditions of corollary 3.5.1 in terms of the service price, τ , we obtain the following revenue function:

$$R(\tau) = \begin{cases} 0 & \text{if } \tau > r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}, \\ 2\tau & \text{if } \tau < r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}, \\ 2p^*\tau \quad p^* \in [0, 1] & \text{if } \tau = r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}. \end{cases}$$

According to this revenue function, for the case with two potential homogeneous customers, for lower prices both customers join and the revenue is 2τ , for the boundary price, $\tau = r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$ both customers join with any positive probability, and for higher prices none of the customers join and the revenue is zero. Therefore the service provider should set the service price τ equal to $r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$ to ensure her revenue is maximized.

Now we will suppose there are N homogeneous customers with exponential utility function and there is no externality. By rearranging the conditions of corollary 3.5.2 in terms of τ , we obtain the following revenue function for the administrator:

$$R(\tau) = \begin{cases} 0 & \text{if } \tau > r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}, \\ N\tau & \text{if } \tau < r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}, \\ Np^*\tau \quad p^* \in [0, 1] & \text{if } \tau = r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}. \end{cases}$$

Therefore the service provider should set the service price τ equal to $r - \frac{\ln \frac{\mu}{\mu-\theta}}{\theta}$ to ensure her revenue is maximized.

Now we are interested in exploring the possible equilibrium joining probabilities in case of negative externalities and defining the pricing problem of the service provider, under an exponential utility function. Thus, we have the following corollary.

Corollary 3.5.3 *In a two customer service system, where $\mu_1 > \mu_2 > \theta$, with negative externality, and with the same exponential utility function, there is only one symmetric Nash equilibrium and this equilibrium is as follows:*

- (i) *If $e^{\theta(r-\tau)}(\mu_1 - \theta) \leq \mu_1$ then $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium and in this case both customers balk and the revenue of administrator is 0.*
- (ii) *If $e^{\theta(r-\tau)}(\mu_2 - \theta) \geq \mu_2$ then $(p_1^*, p_2^*) = (1, 1)$ is the unique Nash equilibrium and in this case both customers join and the revenue of administrator is 2τ .*

(iii) If $e^{\theta(r-\tau)}(\mu_2 - \theta) - \mu_2 < 0 < e^{\theta(r-\tau)}(\mu_1 - \theta) - \mu_1$ then $(p_1^*, p_2^*) = (p^*, p^*)$ is the unique Nash Equilibrium, where

$$p^* = \frac{e^{\theta(r-\tau)} - \frac{\mu_1}{\mu_1-\theta}}{\frac{\mu_2}{\mu_2-\theta} - \frac{\mu_1}{\mu_1-\theta}} \in (0, 1).$$

In this case both customers join with p^* and the revenue of the administrator is $2p^*\tau$.

By rearranging the the conditions of corollary 3.5.3 in terms of the service price, τ we obtain the following revenue function:

$$R(\tau) = \begin{cases} 0 & \text{if } \tau > r - \frac{\ln \frac{\mu_1}{\mu_1-\theta}}{\gamma}, \\ 2\tau & \text{if } \tau < r - \frac{\ln \frac{\mu_2}{\mu_2-\theta}}{\theta}, \\ 2 \left[\frac{e^{\theta(r-\tau)} - \frac{\mu_1}{\mu_1-\theta}}{\frac{\mu_2}{\mu_2-\theta} - \frac{\mu_1}{\mu_1-\theta}} \right] \tau & \text{if } r - \frac{\ln \frac{\mu_2}{\mu_2-\theta}}{\theta} < \tau < r - \frac{\ln \frac{\mu_1}{\mu_1-\theta}}{\theta}. \end{cases}$$

For determining the optimum service price, τ^* in the case of negative externalities, first we should make sure that there exists a unique maximizer for $R(\tau)$. Hence, we first present a numerical example for the 2-customer service system with negative externalities, to illustrate some possible cases which could occur for the shape of the revenue function. Furthermore, in corollary A.1.5, we prove that in case of two homogeneous customers with negative externalities, the equilibrium joining probability is decreasing in θ , the risk-aversion degree.

Example 3.5.1 We assume there are two homogeneous customers with the following information. and based on the possible intervals defined for r we will have 3 possible cases for the pricing problem with negative externalities. $\mu_1 = 0.8$, $\mu_2 = 0.5$, $\gamma = 0.2$.

case 1 if $r < \ln \frac{\mu_1}{\mu_1-\gamma}$ which means according to the given information above $r < 1.43$ then Both customers will balk regardless of the price offered by the administrator, τ .

case 2 if $1.43 < r < 3.55$ then τ^* is the unique solution of $e^{\gamma(r-\tau)}(1 - \gamma\tau) = \frac{\mu_1}{\mu_1-\gamma}$. For instance, if $r = 3$ then $\tau^* = 0.75$

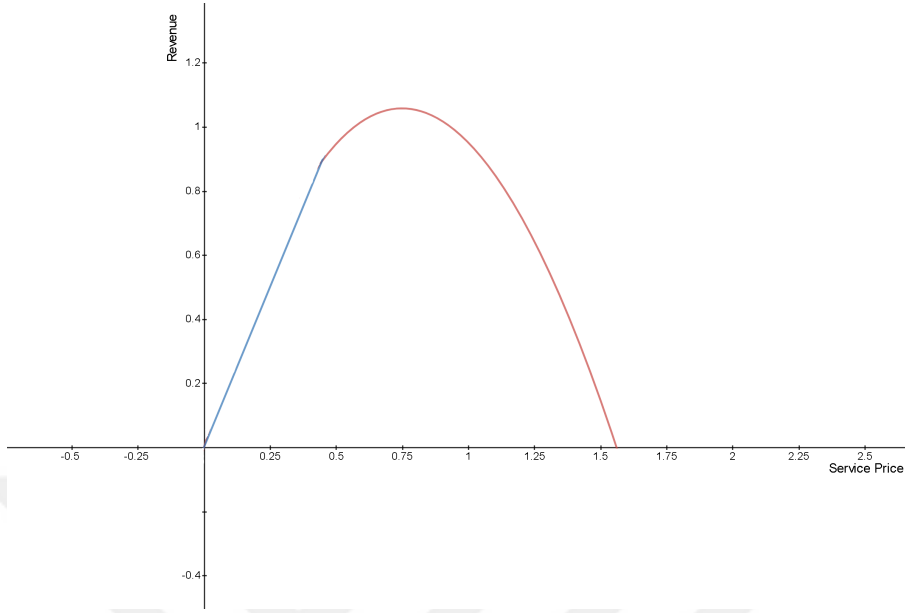


FIGURE 3.1: Revenue function with respect to service price for case 2 with $r = 3$

case 3 if $r > 3.55$ then $\tau^* = r - \frac{\ln \frac{\mu_2}{\mu_2 - \gamma}}{\gamma}$. For instance if $r = 5$ then $\tau^* = 2.44$ or for $r = 100$ we have $\tau^* = 97.44$

Figures 3.1 and 3.2, illustrate the revenue maximization problem with 2 homogeneous customers and under negative externalities. As it could be observed, based on the interval for which r lies in, the optimum price, τ^* is different. For instance, in case 2, τ^* is equal to 0.75, whereas in case 3, the revenue maximizing price is the cut-off point, $\tau^* = 2.44$

Corollary 3.5.4 In in a N -customer service system, where $\mu_1 > \mu_2 > \dots > \mu_N$, with negative externalities, and the same exponential utility function, there exists a unique Nash equilibrium which is as follows:

- (i) If $e^{\gamma r}(\mu_1 - \gamma) \leq \mu_1$ then $(p_1^*, \dots, p_N^*) = (0, \dots, 0)$ is the unique Nash equilibrium.
- (ii) If $e^{\gamma r}(\mu_N - \gamma) \geq \mu_N$ then $(p_1^*, \dots, p_N^*) = (1, \dots, 1)$ is the unique Nash equilibrium.
- (iii) If $e^{\gamma r}(\mu_N - \gamma) - \mu_N < 0 < e^{\gamma r}(\mu_1 - \gamma) - \mu_1$ then $(p_1^*, \dots, p_N^*) = (p^*, \dots, p^*)$ is the unique Nash equilibrium. where p^* is the unique solution of $E[u(r, X_{Y+1})] = u_0$ and

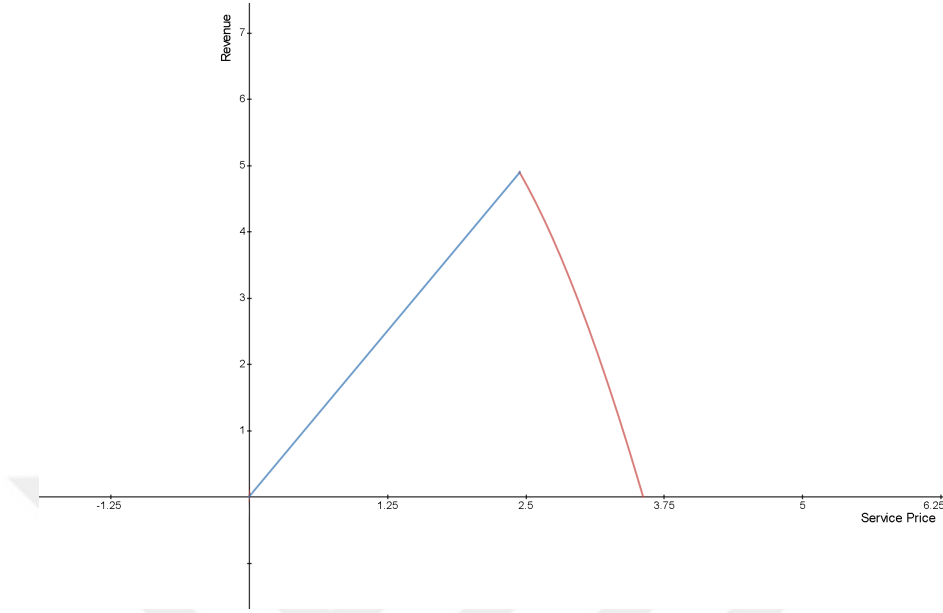


FIGURE 3.2: Revenue function with respect to service price for case 3 with $r = 5$

Y is Binomial with parameters $N - 1$ and $0 < p^* < 1$. in this case each customer joins with p^* .

By rearranging the the conditions of corollary 3.5.4 in terms of the service price, τ we obtain the following revenue function:

$$R(\tau) = \begin{cases} 0 & \text{if } \tau > r - \frac{\ln \frac{\mu_1}{\mu_1 - \theta}}{\gamma}, \\ N\tau & \text{if } \tau < r - \frac{\ln \frac{\mu_N}{\mu_N - \theta}}{\theta}, \\ N\bar{p}\tau, \tau \in (0, 1) & \text{if } r - \frac{\ln \frac{\mu_N}{\mu_N - \theta}}{\theta} < \tau < r - \frac{\ln \frac{\mu_1}{\mu_1 - \theta}}{\theta}. \end{cases} \quad (3.5.1)$$

where \bar{p} is the unique solution of $E[u(r, X_{Y+1})] = u_0$ and Y is Binomial with parameters $N - 1$ and $0 < p < 1$.

3.6 Conclusion

In this chapter, we presented a static game-theoretical model to investigate the symmetric equilibrium joining probabilities of customers in a service system. First, we were able to obtain the equilibrium joining probability under a general utility function and then we investigated the results under different types of externalities. From the theorems, we learned that under positive externalities, equilibrium joining probabilities that are not mutually exclusive emerge, which restricts us in solving the revenue maximization problem under this type of externality. However, under no externality, we were able to fully address the revenue maximization problem and derive the optimal price. Furthermore, in case of negative externalities we successfully addressed the equilibrium joining probabilities and defined the revenue maximization problem, but still there is some additional work to be done to fully characterize the optimum price. Furthermore, we proved that the equilibrium joining probability in case of negative externalities is decreasing in the risk-aversion degree. In the following chapter, we focus on addressing the heterogeneous static game among the customers of a service system which implies the interesting possibility of asymmetric equilibria.

Chapter 4

Static Models of Service Systems with Heterogeneous Customers

In this chapter, we will continue with the same class of static models for service systems with strategic customers as in chapter 3. In contrast to chapter 3, where customers were considered to be homogeneous and had the same utility functions, here we suppose that customers evaluate their gained pay-offs with different utility functions. Therefore, in this section we suppose that customers are heterogeneous in their utilities. We are interested in investigating the results connected with having heterogeneous customers since this case is more challenging. To clarify, by allowing customers to have different utility functions, we are actually allowing the Nash-equilibrium of our underlying game to have asymmetric results. Therefore, we expect to have more and various equilibrium joining probabilities in case of heterogeneous customers. We will first model our heterogeneous game with two customers and then extend it to N heterogeneous customers. Furthermore, we will address the effect of the existent externalities among customers (i.e. negative or positive) on the equilibrium joining probabilities in case of having heterogeneous customers.

4.1 Service System with Two Heterogeneous Customers

The parameters and variables of this heterogeneous game are the same as the homogeneous game, except that we allow customers to have different utility functions and we in fact relax assumption 3.1.1. Therefore, in the case of heterogeneous customers we would have the following assumption.

$$u_i(r, X_n) \neq u_j(r, X_n), \quad u_{i,0} \neq u_{j,0} \quad \forall i, j, n \in N \quad i \neq j, \quad (4.1.1)$$

As a result, the expected utility that the first customer gains when we have a service system with two heterogeneous customers is:

$$U_1(p_1, p_2) = p_1 [p_2 E[u_1(r, X_2)] + (1 - p_2) E[u_1(r, X_1)]] + (1 - p_1) u_{1,0} \quad (4.1.2)$$

and for the second customer the gained expected utility is

$$U_2(p_1, p_2) = p_2 [p_1 E[u_2(r, X_2)] + (1 - p_1) E[u_2(r, X_1)]] + (1 - p_2) u_{2,0} \quad (4.1.3)$$

Now, we want to determine the best response of a customer against a given strategy of the other customer. We proceed as in section 3.1, and denote by $BR_1(p_2)$, the best response of a customer against strategy p_2 of the other customer. We have

$$BR_1(p_2) = \begin{cases} 0 & \text{if } p_2 E[u_1(r, X_2)] + (1 - p_2) E[u_1(r, X_1)] < u_{1,0}, \\ [0, 1] & \text{if } p_2 E[u_1(r, X_2)] + (1 - p_2) E[u_1(r, X_1)] = u_{1,0}, \\ 1 & \text{if } p_2 E[u_1(r, X_2)] + (1 - p_2) E[u_1(r, X_1)] > u_{1,0}. \end{cases} \quad (4.1.4)$$

Similarly the best response function of the second customer is as follows:

$$BR_2(p_1) = \begin{cases} 0 & \text{if } p_1 E[u_2(r, X_2)] + (1 - p_1) E[u_2(r, X_1)] < u_{2,0}, \\ [0, 1] & \text{if } p_1 E[u_2(r, X_2)] + (1 - p_1) E[u_2(r, X_1)] = u_{2,0}, \\ 1 & \text{if } p_1 E[u_2(r, X_2)] + (1 - p_1) E[u_2(r, X_1)] > u_{2,0}. \end{cases} \quad (4.1.5)$$

Similar to section 3.1, to simplify the best response functions of both customers we would rewrite them after the following substitutions:

$$f_2(p_2) = p_2 E[u_1(r, X_2)] + (1 - p_2) E[u_1(r, X_1)] - u_{1,0} \quad (4.1.6)$$

and

$$f_2(p_1) = p_1 E[u_2(r, X_2)] + (1 - p_1) E[u_2(r, X_1)] - u_{2,0} \quad (4.1.7)$$

Therefore, after applying these replacements, the simplified best response functions are as follows:

$$BR_1(p_2) = \begin{cases} 0 & \text{if } f_2(p_2) < 0, \\ [0, 1] & \text{if } f_2(p_2) = 0, \\ 1 & \text{if } f_2(p_2) > 0. \end{cases} \quad (4.1.8)$$

Similarly, the best response function of the second customer is as follows:

$$BR_2(p_1) = \begin{cases} 0 & \text{if } f_2(p_1) < 0, \\ [0, 1] & \text{if } f_2(p_1) = 0, \\ 1 & \text{if } f_2(p_1) > 0. \end{cases} \quad (4.1.9)$$

Now, we are able to characterize the Nash equilibrium of the heterogeneous game among two potential customers in the service system, based on the best-response functions (4.1.8), (4.1.9).

4.1.1 Negative Externalities

We are interested in finding the Nash equilibrium of this heterogeneous two-customer game. For this purpose, we have to find (p_1^*, p_2^*) that satisfies the constraints $p_1^* \in BR_1(p_2^*)$ and $p_2^* \in BR_2(p_1^*)$. For this reason, we need to first determine the type of externalities existing among the two customers. Therefore we will suppose that the first-order stochastic dominance will be applied to determine the relationship between X_1 and X_2 . Hence, we will model negative externalities by assuming $X_1 \geq_{st} X_2$, meaning X_1 stochastically dominates X_2 . As we know, if $X_1 \geq_{st} X_2$, then for any non-decreasing function $u(r, X_n)$ in X_n , we have $E[u(r, X_1)] \geq E[u(r, X_2)]$. We are seeking to find the Nash equilibrium of the heterogeneous game between two customers, so let p_2' be the root of $f_2(p_2) = 0$ and p_1' be the root of $f_1(p_1) = 0$. In addition, We can observe that since $E[u(r, X_1)] \geq E[u(r, X_2)]$ (as an implication of negative externalities), $f_2(p_2)$ is decreasing in p_2 . Therefore, we have a linear decreasing function $f_2(p)$ in p_2 , with p_2' defined as its unique root. To clarify, let's suppose the unique root p_2' is less than zero (the root maybe equal to zero, between zero and one, equal to one or greater than one). However, p_2 as the second customer's joining strategy can be assigned only values in $[0,1]$ and should not be confused with p_2' as the root of $f_2(p)$. Now, visualize the linear decreasing function in p_2 which has a root p_2' less than zero. Before p_2' , $f_2(p)$ is positive, at p_2' it is zero and after p_2' it is negative. As a result, $f_2(p)$ is always negative in $[0,1]$ (regardless of the probability p_2 that would be adopted by the second customer in $[0,1]$). Now that we have characterized $f_2(p)$'s behaviour when p_2' is negative, we would refer you back to 4.1.8 to elicit the first customer's best response. Since we argued that $f_2(p)$ is always negative, so the first customer's best response will be zero (balking) for all possible values for p_2 and we illustrate this in figure 4.1. As a result, based on the interval which the roots p_2' and p_1' lie, we will have different best response functions for the two customers. So we will define mutually exclusive possible intervals for p_2' and p_1' . Therefore, we have 5 possible cases, and p_2' and p_1' are whether less than zero, equal to zero, between zero and one, equal to one, or greater than one. These five

possible cases for the roots of $f_2(p_2)$ and $f_2(p_1)$, namely p'_2 and p'_1 are mutually exclusive and comprehensive. So we will have 5 possible best response functions for customer 1 and also five possible best response functions for customer 2. Therefore, we might have 25 possible cases for the intersection between best response functions of these two customers. In case of negative externalities, $f_2(p_2)$ is decreasing in p_2 and $f_2(p_1)$ is decreasing in p_1 . As we mentioned, $f_2(p'_2) = 0$, $f_2(p'_1) = 0$ and we will solve for p'_2 and p'_1 . So, we have:

$$p'_2 = \frac{u_{1,0} - E[u_1(r, X_1)]}{E[u_1(r, X_2)] - E[u_1(r, X_1)]} \quad (4.1.10)$$

and

$$p'_1 = \frac{u_{2,0} - E[u_2(r, X_1)]}{E[u_2(r, X_2)] - E[u_2(r, X_1)]} \quad (4.1.11)$$

According to equation (4.1.10), in case of having negative externalities, $p'_2 < 0$ implies $E[u_1(r, X_1)] < u_{1,0}$. Therefore, when we have $E[u_1(r, X_1)] < u_{1,0}$ the first customer best responds as in figure 4.1. By following the same arguments, $p'_2 = 0$ implies $E[u_1(r, X_1)] = u_{1,0}$, $0 < p'_2 < 1$ implies $E[u_1(r, X_2)] < u_{1,0} < E[u_1(r, X_1)]$, $p'_2 = 1$ implies $E[u_1(r, X_2)] = u_{1,0}$ and finally, $p'_2 > 1$ implies $E[u_1(r, X_2)] > u_{1,0}$. To have a better perception of how the first customer best responds, based on the interval in which p'_2 lies in, please refer to figures 4.1, 4.2, 4.3, 4.4 and 4.5. The same arguments are applied for the second customer and his best response function are derived based on the intervals defined for p'_1 and figures 4.6, 4.7, 4.8, 4.9 and 4.10 are obtained. To provide a better perception of the best response figures of both customers under different values of p'_1 and p'_2 , and consequently build the required background for deriving the equilibrium joining probabilities, each figure and its implications will be described separately.

Figure 4.1, represents the best response of the first customer against strategy p_2 of the second customer, given $p'_2 < 0$. As it could be seen, regardless of p_2 (strategy adopted by the second customer), the first customer always adopts strategy zero (balks). Furthermore, according to 4.1.10, $p'_2 < 0$ implies $E[u_1(r, X_1)] < u_{1,0}$. On the other hand, $E[u_1(r, X_1)] <$

$u_{1,0}$ implies $p'_2 < 0$ and the first customer will best respond like figure 4.1.

Figure 4.2, represents the best response of the first customer against strategy p_2 of the second customer, given $p'_2 = 0$. As it could be seen, when $p_2 = 0$, the first customers best-response could be any value in $[0,1]$, and when $0 < p_2 < 1$, the first customers best response is zero (balking). Furthermore, according to 4.1.10, $p'_2 = 0$ implies $E[u_1(r, X_1)] = u_{1,0}$. On the other hand, $E[u_1(r, X_1)] = u_{1,0}$ implies $p'_2 = 0$ and the first customer will best respond like figure 4.2.

Figure 4.3, represents the best response of the first customer against strategy p_2 of the second customer, given $0 < p'_2 < 1$. As it could be seen, when $p_2 = p'_2$, the first customers best-response could be any value in $[0,1]$, and when $p_2 > p'_2$, the first customers best response is zero (balking), and when $p_2 < p'_2$ the first customers best response is one (joining). Furthermore, according to 4.1.10, $0 < p'_2 < 1$ implies $E[u_1(r, X_2)] < u_{1,0} < E[u_1(r, X_1)]$. On the other hand, $E[u_1(r, X_2)] < u_{1,0} < E[u_1(r, X_1)]$ implies $0 < p'_2 < 1$ and the first customer will best respond like figure 4.3.

Figure 4.4, represents the best response of the first customer against strategy p_2 of the second customer, given $p'_2 = 1$. As it could be seen, when $p_2 = 1$, the first customers best-response could be any value in $[0,1]$, and when $0 < p_2 < 1$, the first customers best response is one (joining). Furthermore, according to 4.1.10, $p'_2 = 1$ implies $E[u_1(r, X_2)] = u_{1,0}$. On the other hand, $E[u_1(r, X_2)] = u_{1,0}$ implies $p'_2 = 1$ and the first customer will best respond like figure 4.4.

Figure 4.5, represents the best response of the first customer against strategy p_2 of the second customer, given $p'_2 > 1$. As it could be seen, regardless of p_2 (strategy adopted by the second customer), the first customer always adopts strategy one (joins). Furthermore, according to 4.1.10, $p'_2 > 1$ implies $E[u_1(r, X_2)] > u_{1,0}$. On the other hand, $E[u_1(r, X_2)] > u_{1,0}$ implies $p'_2 > 1$ and the first customer will best respond like figure 4.5.

Figure 4.6, represents the best response of the second customer against strategy p_1 of the first customer, given $p'_1 < 0$. As it could be seen, regardless of p_1 (strategy

adopted by the first customer), the second customer always adopts strategy zero (balks). Furthermore, according to 4.1.11, $p'_1 < 0$ implies $E[u_2(r, X_1)] < u_{2,0}$. On the other hand, $E[u_2(r, X_1)] < u_{2,0}$ implies $p'_1 < 0$ and the second customer will best respond like figure 4.6.

Figure 4.7, represents the best response of the second customer against strategy p_1 of the first customer, given $p'_1 = 0$. As it could be seen, when $p_1 = 0$, the second customer's best-response could be any value in $[0,1]$, and when $0 < p_1 < 1$, the second customer's best response is zero (balking). Furthermore, according to 4.1.11, $p'_1 = 0$ implies $E[u_2(r, X_1)] = u_{2,0}$. On the other hand, $E[u_2(r, X_1)] = u_{2,0}$ implies $p'_1 = 0$ and the second customer will best respond like figure 4.7.

Figure 4.8, represents the best response of the second customer against strategy p_1 of the first customer, given $0 < p'_1 < 1$. As it could be seen, when $p_1 = p'_1$, the second customer's best-response could be any value in $[0,1]$, and when $p_1 > p'_1$, the second customer's best response is zero (balking), and when $p_1 < p'_1$ the second customer's best response is one (joining). Furthermore, according to 4.1.11, $0 < p'_1 < 1$ implies $E[u_2(r, X_2)] < u_{2,0} < E[u_2(r, X_1)]$. On the other hand, $E[u_2(r, X_2)] < u_{2,0} < E[u_2(r, X_1)]$ implies $0 < p'_1 < 1$ and the second customer will best respond like figure 4.8.

Figure 4.9, represents the best response of the second customer against strategy p_1 of the first customer, given $p'_1 = 1$. As it could be seen, when $p_1 = 1$, the second customer's best-response could be any value in $[0,1]$, and when $0 < p_1 < 1$, the second customer's best response is one (joining). Furthermore, according to 4.1.11, $p'_1 = 1$ implies $E[u_2(r, X_2)] = u_{2,0}$. On the other hand, $E[u_2(r, X_2)] = u_{2,0}$ implies $p'_1 = 1$ and the second customer will best respond like figure 4.9.

Figure 4.10, represents the best response of the second customer against strategy p_1 of the first customer, given $p'_1 > 1$. As it could be seen, regardless of p_1 (strategy adopted by the first customer), the second customer always adopts strategy one (joins). Furthermore, according to 4.1.11, $p'_1 > 1$ implies $E[u_2(r, X_2)] > u_{2,0}$. On the other hand,

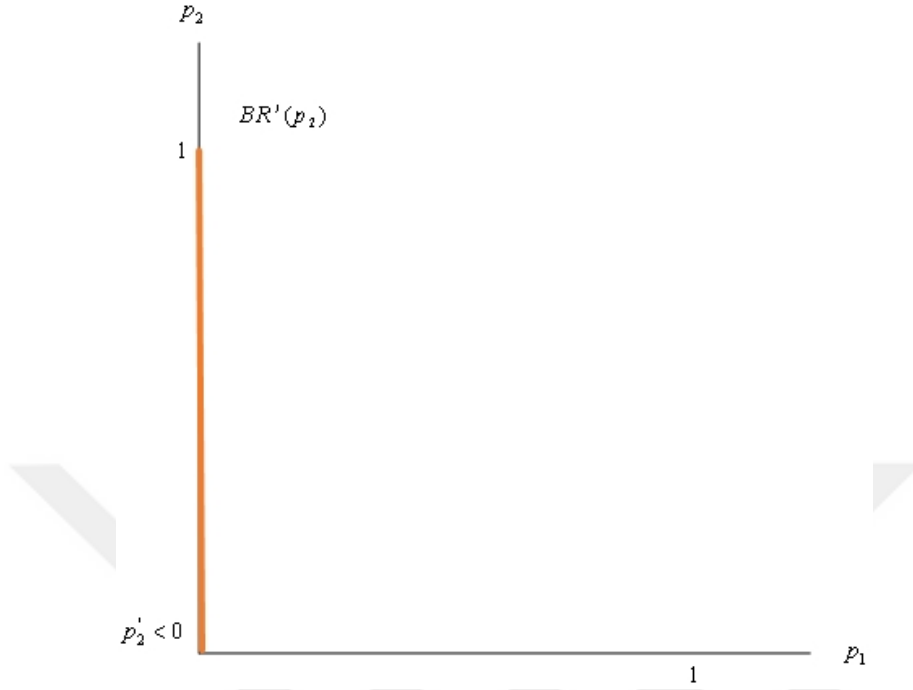


FIGURE 4.1: Best response function of customer 1 given $p_2' < 0$, under negative externality

$E[u_2(r, X_2)] > u_{2,0}$ implies $p_1' > 1$ and the second customer will best respond like figure 4.10.

After explaining the different possible cases for the best response functions of the two customers, we will provide an example of the 25 possible combinations of these best response functions in figure 4.11.

Figure 4.11, pertains to equilibrium joining strategies in the two-customer heterogeneous case, given $0 < p_1' < 1$ and $0 < p_2' < 1$, under negative externality. According to figure 4.11, there are 3 equilibrium joining strategies, namely $(0, 1)$, (p_1', p_2') and $(1, 0)$. At $(0, 1)$ the first customer balks and the second customer joins. At (p_1', p_2') , the first customer joins with probability p_1' and the second customer joins with probability p_2' . At $(1, 0)$, the first customer joins and the second customer decides balking. There are also 24 possible cases for the combination of best response functions and here we will not add all those possible cases, although they are easy to track. In this regard, in the next paragraph, we will address the equilibrium joining probabilities under exponential utility function with

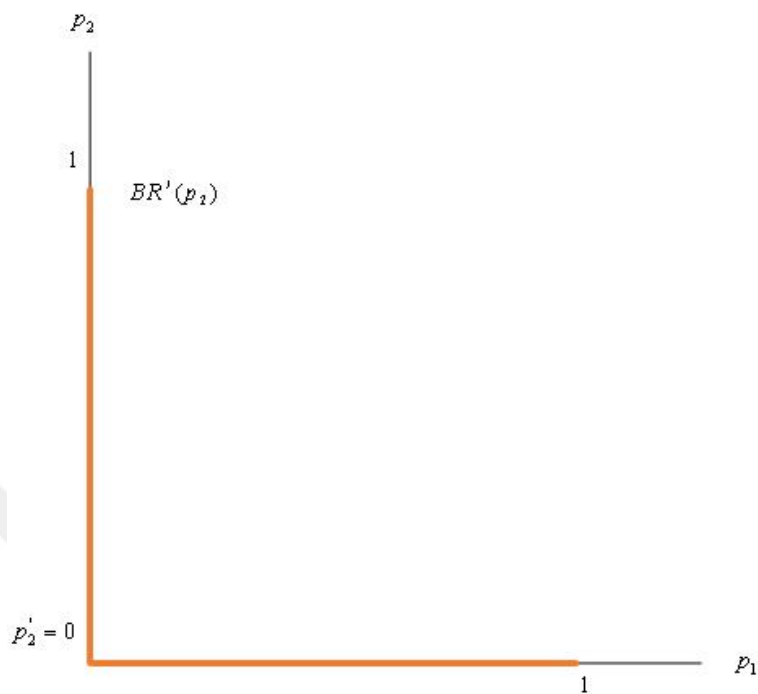


FIGURE 4.2: Best response function of customer 1 given $p'_2 = 0$, under negative externality

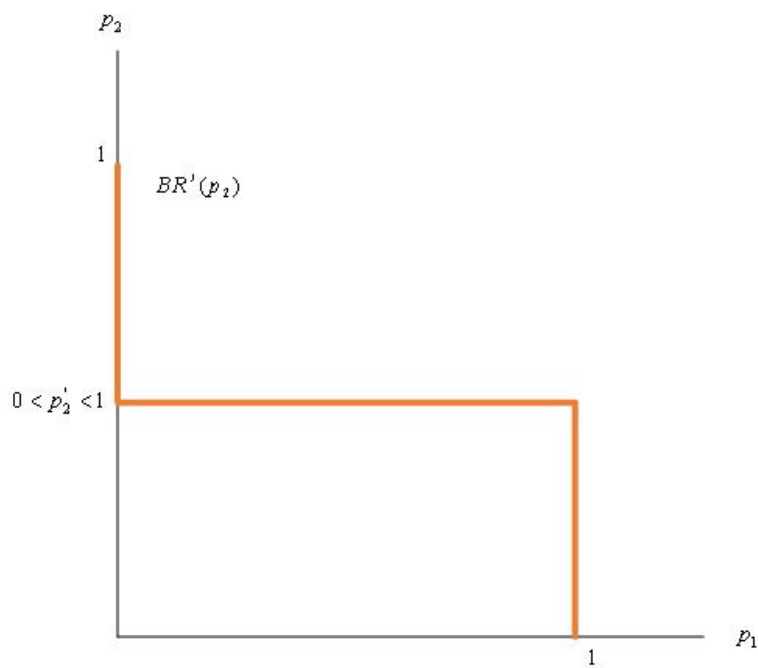


FIGURE 4.3: Best response function of customer 1 given $0 < p'_2 < 1$, under negative externality

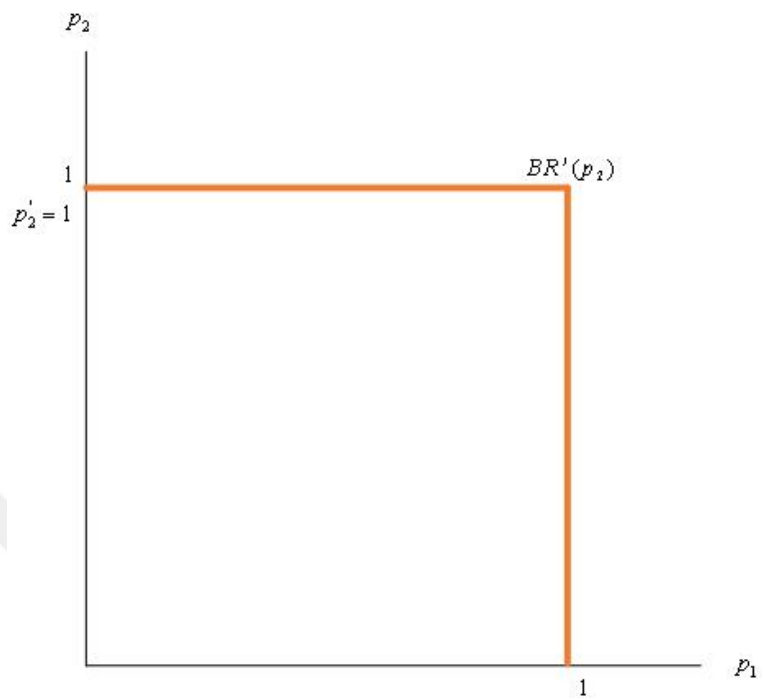


FIGURE 4.4: Best response function of customer 1 given $p'_2 = 1$, under negative externality

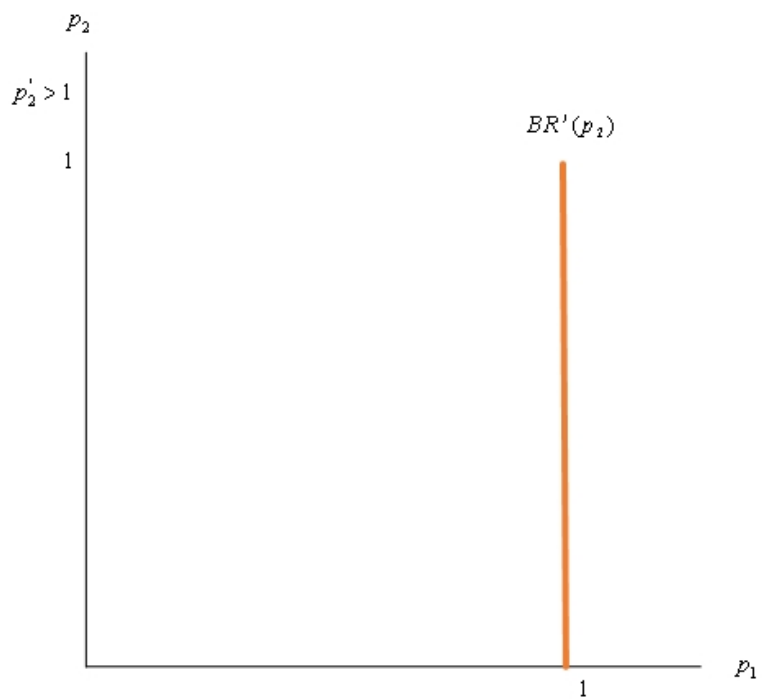


FIGURE 4.5: Best response function of customer 1 given $p'_2 > 1$, under negative externality

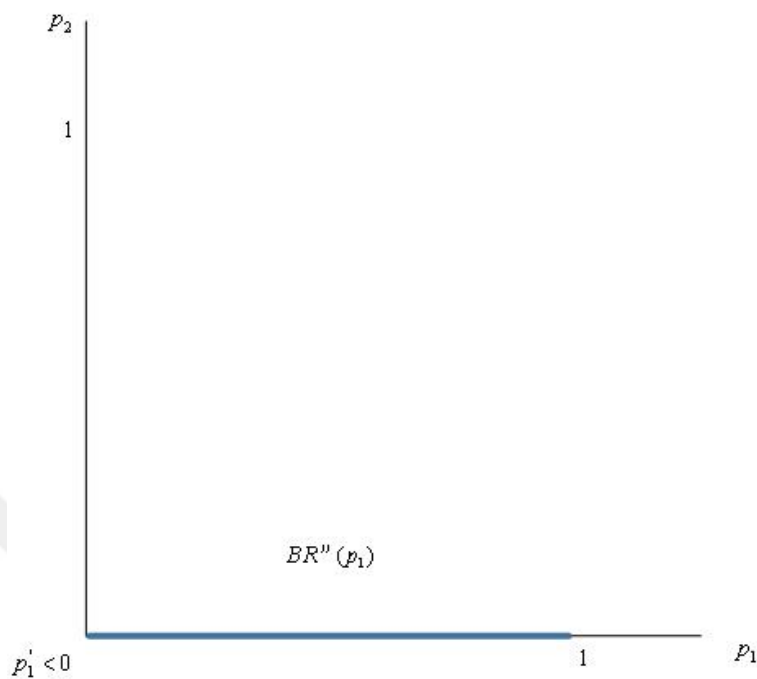


FIGURE 4.6: Best response function of customer 2 given $p'_1 < 0$, under negative externality

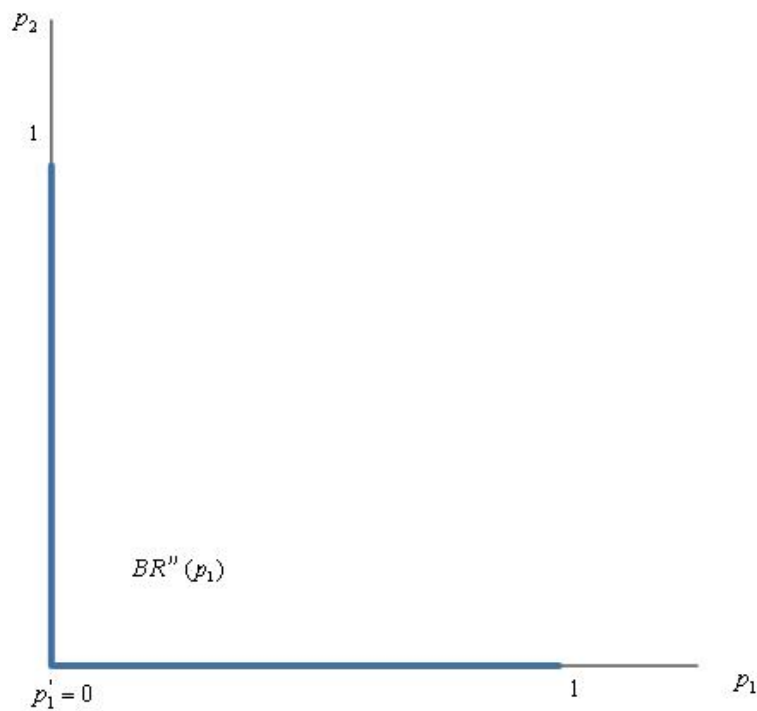


FIGURE 4.7: Best response function of customer 2 given $p'_1 = 0$, under negative externality

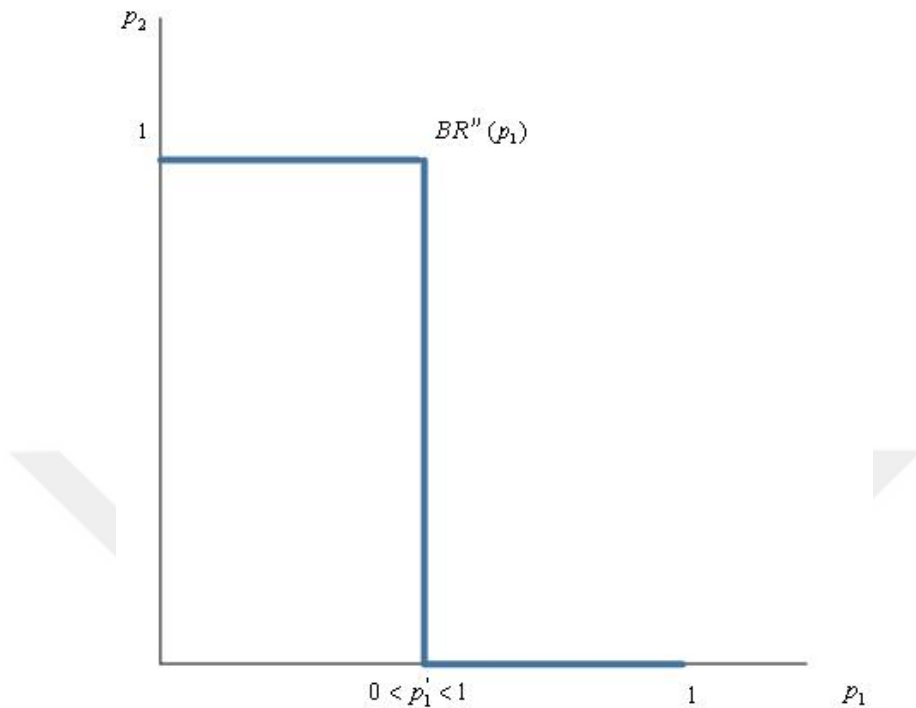


FIGURE 4.8: Best response function of customer 2 given $0 < p'_1 < 1$, under negative externality

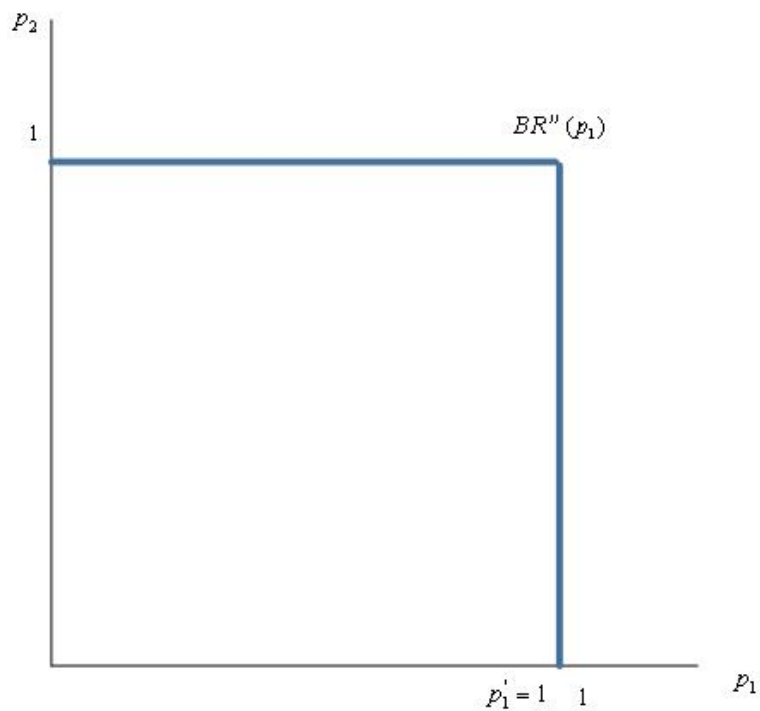


FIGURE 4.9: Best response function of customer 2 given $p'_1 = 1$, under negative externality

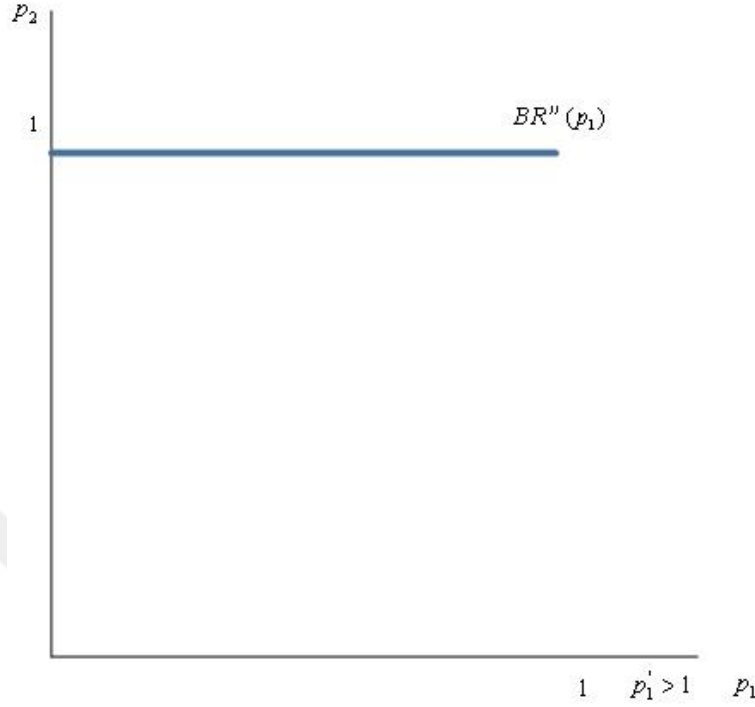


FIGURE 4.10: Best response function of customer 2 given $p'_1 > 1$, under negative externality

two heterogeneous customers. For this purpose, we will also use the properties we proved for the exponential utility functions in Lemma A.1.1 addressed in Appendix A.

Figure 4.12, illustrates the equilibrium joining probabilities for two heterogeneous customers, where the second customer is more risk-averse and the risk sensitivity degree of the first customer and the second customer are shown with θ_1 and θ_2 , respectively ($\theta_2 > \theta_1$). Furthermore, θ_1^* and θ_2^* represent the roots of $E[u(r, X_1)] = 0$ and $E[u(r, X_2)] = 0$, respectively, with negative externality and under exponential utility function. We define possible intervals for the risk-aversion degree of the two customers and then based on the properties of our exponential utility function described in Lemma A.1.1, we explore the equilibrium joining probabilities in all those small partitions. For instance, in OAB , the triangle, where both customers possess a low risk-aversion degree, and the second customer is more risk-averse, both customers join in the equilibrium. In partition $ABCD$, where the second customer possesses a higher risk-aversion degree with respect to partition OAB , The first customer joins and the second customer balks. In partition BDE ,

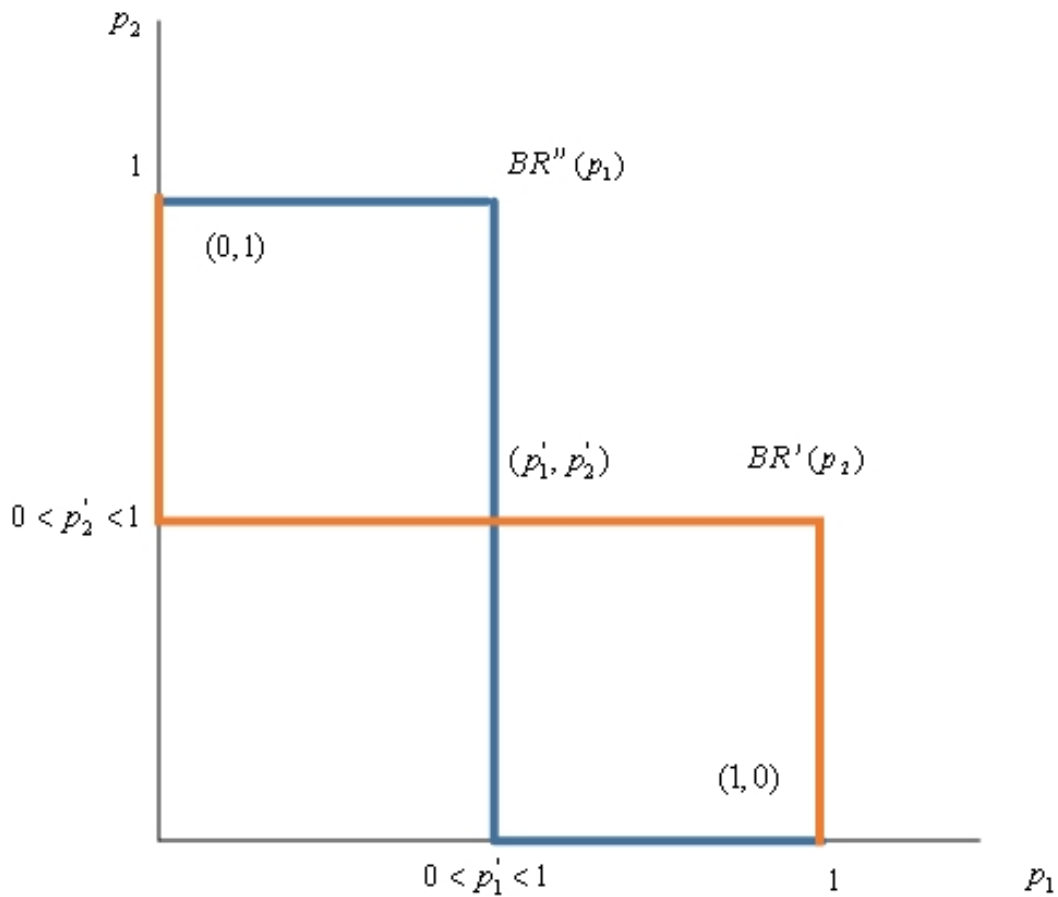


FIGURE 4.11: Equilibrium joining strategies in the two-customer heterogeneous case, given $0 < p_1' < 1$ and $0 < p_2' < 1$, under negative externality

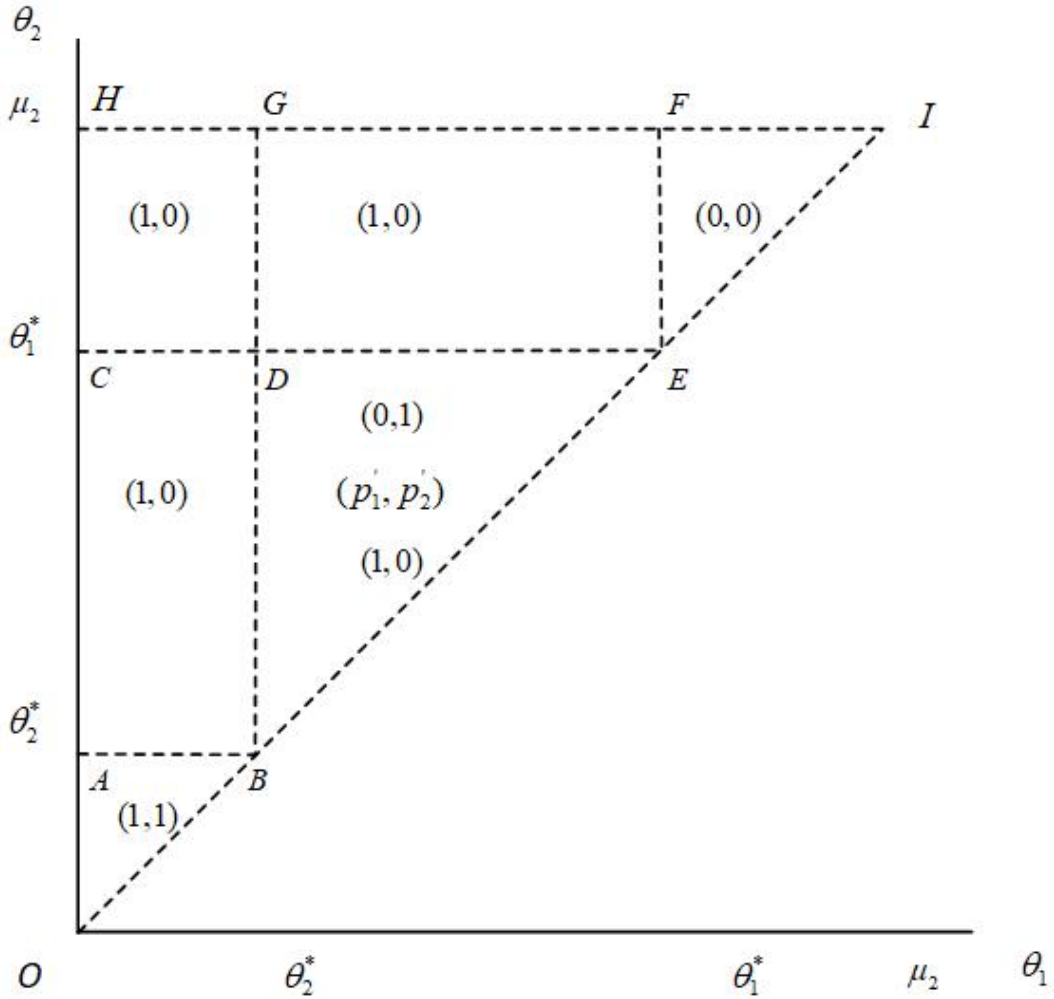


FIGURE 4.12: Equilibrium joining strategies in the two-customer heterogeneous case with exponential utility function, under negative externality

the middle triangle, where again the second customer is more risk-averse, there are three possible equilibrium joining strategies, $(0, 1)$, $(1, 0)$ and (p'_1, p'_2) . In triangle FEI , where both customers are highly risk-averse, but still the second customer is more risk-averse, both customers balk.

4.1.2 Positive Externalities

In this subsection, similar to section 4.1.1, the equilibrium joining strategies of two heterogeneous customers in case of positive externalities will be obtained, and the relevant figures will be presented. The derivation of the best response functions is very similar to

the case with negative externalities. Equilibrium joining strategies in the two-customer heterogeneous case with exponential utility function, under positive externality, is presented in figure 4.13. According to this figure, similar to figure 4.12, the second customer is supposed to be more risk-averse, except that the present externality among the customers is positive. We can observe that at triangle OAB , where the second customer is more risk-averse, both customers join. At partition $ABCD$, although the second customer gets more risk-averse with respect to partition OAB , but both customers still decide to join in the the equilibrium at this partition. At BDE , there are multiple equilibria, $(0, 0)$, $(1, 1)$ and (p'_1, p'_2) . At partition FEI , where both customers are highly risk-averse, both customers decide balking.

Furthermore, By comparing figures 4.12 and 4.13 we can observe the effect of externalities in the equilibrium joining probabilities. For instance, consider the same partition $ABCD$. With negative externalities, the equilibrium joining probability is $(1, 0)$, where as with positive externalities it is $(1, 1)$ and the reason also the second customer decides to join originates from the fact that with positive externality customers better-off by the presence of other customers.

After addressing the equilibrium joining probabilities with two heterogeneous customers, in section 4.2, we extend our results to the case where there are two groups of customers and one group is more risk-averse than the other group.

4.2 Service System with N Heterogeneous Customers

In this section, the results obtained in section 4.1 will be extended to N heterogeneous customers. In this section we will propose a service with N customers which are divided into two groups and each group has different level of risk-aversion. For instance, n_1 customers are in the first group and possess a risk-aversion degree of θ_1 and n_2 customers are in the second group and possess a risk-aversion degree of θ_2 , $\theta_2 > \theta_1$, $n_1 + n_2 = N$.

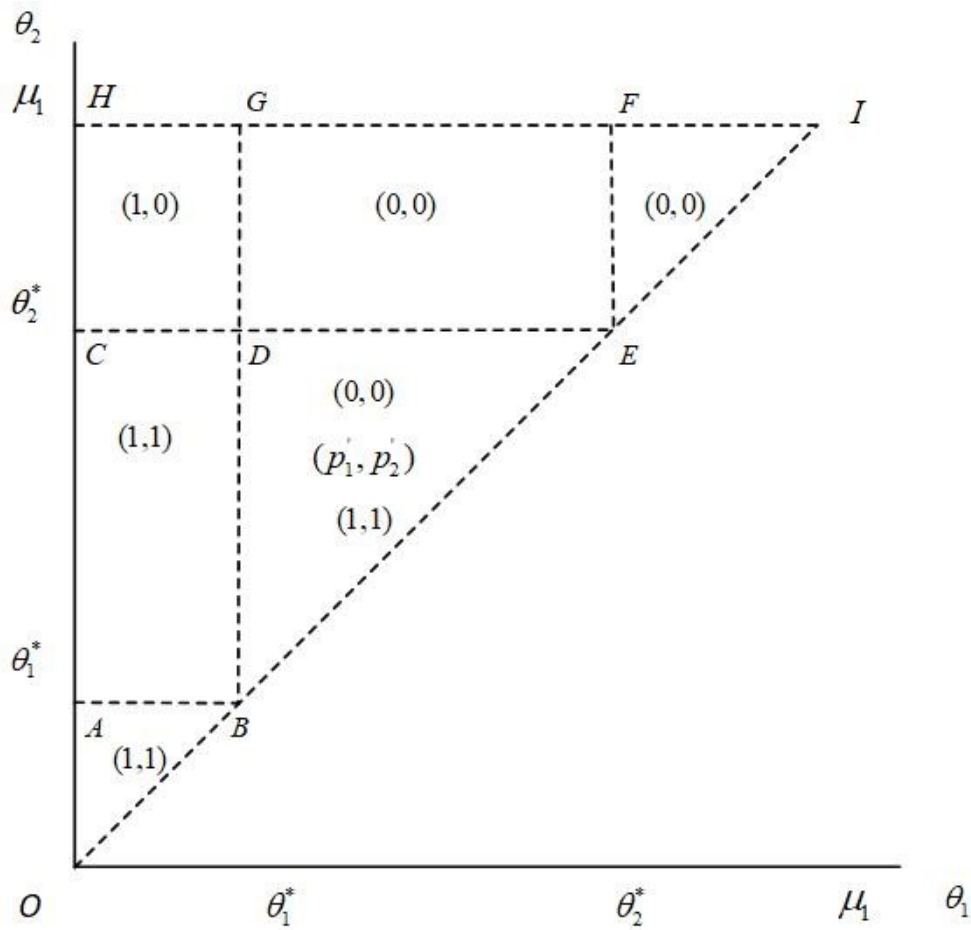


FIGURE 4.13: Equilibrium joining strategies in the two-customer heterogeneous case with exponential utility function, under positive externality

Similar to section 3.2, the expected utility gained by a customer of the first group, if she follows strategy p' , when all other customers in group one follow strategy p and all customers in group 2 follow strategy q , is

$$e_{N,1}(p, q) = p' \left[\sum_{n=0}^{n_2} \binom{n_2}{n} q^n (1-q)^{n_2-n} \sum_{m=0}^{n_1-1} \binom{n_1-1}{m} p^m (1-p)^{n_1-1-m} E[u_1(r, X_{n+m+1})] \right] + (1-p')u_1(0)$$

Where, n is the number of customers from the second class who decide joining and m is the number of customers of the first group who decide joining. By defining

$$f_{N,1}(p, q) = \sum_{n=0}^{n_2} \binom{n_2}{n} q^n (1-q)^{n_2-n} \sum_{m=0}^{n_1-1} \binom{n_1-1}{m} p^m (1-p)^{n_1-1-m} E[u_1(r, X_{n+m+1})] - u_1(0)$$

We have

$$e_{N,1}(p, q) = p' f_{N,1}(p, q) + u_1(0)$$

As a result, we are able to define the best response function of a customer in the first group against strategy p of the customers the first group and strategy q of the customers the second group.

$$BR_1(p, q) = \begin{cases} 0 & \text{if } f_{N,1}(p, q) < 0, \\ p' \in [0, 1] & \text{if } f_{N,1}(p, q) = 0, \\ 1 & \text{if } f_{N,1}(p, q) > 0. \end{cases}$$

Similarly, we are able to define the expected utility gained by a customer of the second group, if she follows strategy q' , when all other customers in group two follow strategy q and all customers in group 1 follow strategy p , the expected utility is equal to

$$e_{N,2}(p, q) = q' \left[\sum_{t=0}^{n_2-1} \binom{n_2-1}{t} q^t (1-q)^{n_2-t-1} \sum_{s=0}^{n_1} \binom{n_1}{s} p^s (1-p)^{n_1-s} E[u_2(r, X_{s+t+1})] \right] + (1-q')u_2(0)$$

Where, t is the number of customers from the second class who decide joining and s is the number of customers of the first group who decide joining. By defining

$$f_{N,2}(p, q) = \sum_{t=0}^{n_2-1} \binom{n_2-1}{t} q^t (1-q)^{n_2-t-1} \sum_{s=0}^{n_1} \binom{n_1}{s} p^s (1-p)^{n_1-s} E[u_2(r, X_{s+t+1})] - u_2(0)$$

We have

$$e_{N,2}(p, q) = q' f_{N,2}(p, q) + u_2(0)$$

As a result, we are able to define the best response function of a customer in the second group against strategy p of the customers the first group and strategy q of the customers the second group.

$$BR_2(p, q) = \begin{cases} 0 & \text{if } f_{N,2}(p, q) < 0, \\ q' \in [0, 1] & \text{if } f_{N,2}(p, q) = 0, \\ 1 & \text{if } f_{N,2}(p, q) > 0. \end{cases}$$

Therefore, we have the following theorem for the case with N heterogeneous customers.

Theorem 4.2.1 *The equilibrium or equilibria of the asymmetric game between N heterogeneous customers in a service system, where n_1 customers are in the first group and possess a risk-aversion degree of θ_1 and n_2 customers are in the second group and possess a risk-aversion degree of θ_2 , $\theta_2 > \theta_1$ is as follows:*

- (i) *if $E[u_1(r, X_N)] > u_1(0)$ and $E[u_2(r, X_N)] > u_2(0)$, then $(1, 1)$ is an equilibrium, and all customers in the first group and also all customers in the second group join.*
- (ii) *if $E[u_1(r, X_{n_1})] > u_1(0)$ and $E[u_2(r, X_{n_1+1})] < u_2(0)$, then $(1, 0)$ is an equilibrium and all customers of the first group will join and all of customers of the second group will balk.*
- (iii) *if $\sum_{n=0}^{n_2} \binom{n_2}{n} q^n (1-q)^{n_2-n} E[u_1(r, X_{n_1+n})] > u_1(0)$ and*

$$\binom{n_2-1}{n_2-1} q^{n_2-1} E[u_2(r, X_N)] + \binom{n_2-1}{n_2-2} q^{n_2-2} (1-q) E[u_2(r, X_{N-1})] + \dots + (1-q)^{n_2-1} E[u_2(r, X_{n_1+1})] =$$

$u_2(0)$, then $(1, q)$ is an equilibrium and all customers of the first group join and qn_2 of the second group joins.

(iv) if $E[u_1(r, X_{n_2+1})] < u_1(0)$ and $E[u_2(r, X_{n_2})] > u_2(0)$, then $(0, 1)$ is an equilibrium and all customers of group one balk and all customers of the second group join.

(v) if $E[u_1(r, X_1)] < u_1(0)$ and $E[u_2(r, X_1)] < u_1(0)$, then $(0, 0)$ is an equilibrium and all customers balk.

(vi) if $\sum_{n=0}^{n_2} \binom{n_2}{n} q^n (1-q)^{n_2-n} E[u_1(r, X_{n_1+n})] < u_1(0)$ and $\binom{n_2-1}{n_2-1} q^{n_2-1} E[u_2(r, X_{n_2})] + \binom{n_2-1}{n_2-2} q^{n_2-2} (1-q) E[u_2(r, X_{n_2-1})] + \dots + (1-q)^{n_2-1} E[u_2(r, X_1)] = u_2(0)$, then $(0, q)$ is an equilibrium and all customers of the first group balk and qn_2 customers of the second group join.

(vii) if $\binom{n_1-1}{n_1-1} p^{n_1-1} E[u_1(r, X_N)] + \binom{n_1-1}{n_1-2} p^{n_1-2} (1-p) E[u_1(r, X_{N-1})] + \dots + (1-p)^{n_1-1} E[u_1(r, X_{n_2+1})] = u_1(0)$ and $\sum_{s=0}^{n_1} \binom{n_1}{s} p^s (1-p)^{n_1-s} E[u_2(r, X_{n_2+s})] > u_2(0)$, then $(p, 1)$ is an equilibrium, and pn_1 of the first group join and all customers of the second group join.

(viii) if $\binom{n_1-1}{n_1-1} p^{n_1-1} E[u_1(r, X_{n_1})] + \binom{n_1-1}{n_1-2} p^{n_1-2} (1-p) E[u_1(r, X_{n_1-1})] + \dots + (1-p)^{n_1-1} E[u_1(r, X_1)] = u_1(0)$ and $\sum_{s=0}^{n_1} \binom{n_1}{s} p^s (1-p)^{n_1-s} E[u_2(r, X_{1+s})] < u_2(0)$, then $(p, 0)$ is an equilibrium and the pn_1 of the first group of customers join and all customers of the second group balk.

(ix) if $\sum_{n=0}^{n_2} \binom{n_2}{n} q^n (1-q)^{n_2-n} \sum_{m=0}^{n_1-1} \binom{n_1-1}{m} p^m (1-p)^{n_1-1-m} E[u_1(r, X_{n+m+1})] = u_1(0)$ and $\sum_{t=0}^{n_2-1} \binom{n_2-1}{t} q^t (1-q)^{n_2-t-1} \sum_{s=0}^{n_1} \binom{n_1}{s} p^s (1-p)^{n_1-s} E[u_2(r, X_{s+t+1})] = u_2(0)$, then (p, q) is an equilibrium and pn_1 of the customers of the first group join and qn_2 of the customers of the second group join.

According to Theorem 4.2.1, the asymmetric equilibrium joining probabilities in a N-customer heterogeneous game where n_1 of customers have a risk-sensitivity degree of θ_1 and n_2 of customers have risk-sensitivity degree θ_2 , and the the second group is more

risk-averse, has nine possible cases. In case (i), the expected earned utility by both group of customers is high and as a result all customer decide to join. In case (ii), the expected earned utility by the first group is high but the expected earned utility by the second group is low and all customers of the first group decide to join while all customers of the second group decide to balk. In case (iii), first group of customers all decide to join and the second group decide to join with probability q . In case (iv), in contrast with the second case, the second group earns higher expected utility while the first group earns low expected utility and as a result, all customers of the first group balk and all customers of the second group decide to join. In case (v), since the expected earned utility of both groups is low, all customers decide to balk. In case (vi), the first group of customers decide to balk, whereas, the second group of customers decide to join with q . In case (vii), the first group decide to join with probability p , while the second group decide to all join. In case (viii), the second group of customers decide to all balk, while the first group decide to join with probability p . Finally, in case (ix), the first group decide to join with probability p , whereas the second group of customers decide to join with probability q .

4.3 Conclusion

In this chapter, we investigated the equilibrium joining probabilities in a service system with heterogeneous customers, in a static-game theoretic framework. First, we addressed the case with two heterogeneous customers and derived the equilibrium joining probabilities under negative externalities and positive externalities. Then, we investigated the asymmetric equilibrium joining probabilities under an exponential distribution. Our results indicates that by supposing customers are heterogeneous the analysis gets much more challenging. Then we derived the equilibrium joining probabilities where we have N potential customers in the service system and presented theorem 4.2.1 which clearly indicates the cumbersome analysis of heterogeneous customers due to fact that diverse equilibrium joining probabilities emerge.

Chapter 5

A Queueing System with Homogeneous Risk-Averse Strategic Customers

In this chapter, we turn our attention to a service system modeled as a queue. In contrast to chapters 3 and 4, we investigate a dynamic system, with random customer arrivals over time, random service durations, waits and departures. Consider a queueing system where customers are able to decide whether to balk or join after their evaluation of the queue characteristics. As it is experienced by almost all, in many cases, the waiting time in a queue is random and has a crucial role in a customer's decision upon arrival. But a relevant question to be addressed is, how the decisions of strategic customers would change if they are risk-averse rather than risk-neutral? In this regard, a large part of the literature on strategic queueing systems assume that customers possess linear utility functions. However, the preferences of time-sensitive customers could be better reflected by supposing they have nonlinear preferences and possess a concave utility function rather than a linear one. To clarify, when customers are risk-neutral, they have linear preferences and upon arrival, decide based on their "expected monetary benefit". In this chapter, we are interested in filling this gap, by supposing that customers' utility function is concave in their pay-off. As a result, we suppose customers possess an exponential utility function, to be able to address the nonlinear preferences of customers. The remainder of this chapter is outlined as follows. In Section 5.1, our proposed model for addressing risk-averse strategic

customers is provided, for both the observable case and the unobservable case. In section 5.2, the Nash-equilibrium of the unobservable case is characterized. The social welfare maximization and revenue maximization problems are defined in sections 5.3 and 5.4, respectively. Finally, in section 5.5, the associated numerical examples are gathered.

5.1 The Model

We suppose risk-averse strategic customers are arriving according to a Poisson process with rate Λ and there is a single server, and the service durations are i.i.d exponential random variables with intensity μ . To reflect the risk-averse behavior of customers we would use an exponential utility function. The service price is τ and identical for all customers. The reward is r for a joining customer and identical for all customers. The waiting cost per unit of time in the system is c . θ , represents the risk-aversion degree of a customer and identical for all customers. The service price set by the service provider is τ and identical for all customers. The pay-off of a customer is indicated by

$$r - \tau - cX$$

The risk-averse customers evaluate their pay-offs with a an exponential utility function, as in section 3.4:

$$1 - e^{-\theta(r-\tau-cX)}. \tag{5.1.1}$$

In subsection 5.1.1, we will characterize our service system under an observable framework, whereas in subsection 5.1.2, we are interested in capturing and investigating an unobservable service system, with risk-averse strategic customers.

5.1.1 Observable Case

In the observable case, an arriving customer is able to see the number of customers in the queue, n_q . Therefore, a customer will evaluate his pay-off with the following utility function:

$$u(r - \tau - c(X|n_q)) = 1 - e^{-\theta(r-\tau-c(X|n_q))} \quad (5.1.2)$$

where, similar to chapter 3, θ represents the risk-aversion degree and the waiting time is denoted by X . The expected utility of an arbitrary joining customer with risk-sensitivity degree, θ is then

$$\mathbb{E}[u(r - \tau - c(X|n_q))] = 1 - e^{-\theta(r-\tau)} \mathbb{E} [e^{c\theta(X|n_q)}] \quad (5.1.3)$$

In (5.1.3), $X|n_q$ represents the waiting time of an arriving customer who observes n_q customers in the queue. Clearly, $X|n_q$ is the sum of $n_q + 1$ independent Exponential random Variables with parameter μ . As a result, $X|n_q$ follows an Erlang distribution with parameters $(n_q + 1, \mu)$. Therefore, by using the moment generating function for an Erlang distribution, the expected benefit is calculated as

$$\mathbb{E}[u(r - \tau - c(X|n_q))] = 1 - e^{-\theta(r-\tau)} \left(\frac{\mu}{\mu - c\theta} \right)^{n_q+1}, \quad \mu > c\theta \quad (5.1.4)$$

Upon arrival, a customer should decide whether to join or balk. Note that in (5.1.4), $\mathbb{E}[u(r - \tau - c(X|n_q))]$ is decreasing in n_q . The arriving customer, in case of pursuing self-interest, makes a decision based on his expected utility. An arbitrary customer who arrives, observes the queue size, N_q , at that instant and decides to join if it is under a value, n (the so-called threshold). In fact, the number of observed customers in the

system, N_q is a random variable which could take only integer values in the $[0, n]$ interval. Now, suppose we are interested in characterizing the customers who are pursuing self-interest, In the sense that, an arbitrary customer would join if $\mathbb{E}[u(r - \tau - c(X|n_q))] \geq 0$ and balk if $\mathbb{E}[u(r - \tau - c(X|n_q))] < 0$. Our purpose is to find an expression for the threshold queue length (n_e) which implies self-optimization. The threshold queue length which also implies self-interest is characterized in theorem 5.1.1.

We call n_e the threshold strategy adopted and if an arriving customer observes n_e or more customers he would balk, and for $n_q \leq n_e - 1$ he would join.

Theorem 5.1.1 *In the Observable case with risk-averse strategic customers:*

- (i) *the threshold strategy for self-optimizing customers is denoted by n_e , where n_e is equal to $\lfloor \frac{\theta(r-\tau)}{\ln \frac{\mu}{\mu-c\theta}} \rfloor$.*
- (ii) *n_e is non-decreasing in r and μ , and non-increasing in θ .*

Proof of Theorem 5.1.1.

(i) By referring to equation 5.1.4, it is obvious that a supposed n_e would imply self-optimization if for an arbitrary customer who observes n_q customers at his arrival instant, the following two equations are met simultaneously. Equation 5.1.5 below pertains to the case that is in favor of the arbitrary customer. In this case, the arbitrary customer observes $n_e - 1$ customers in the system and decides to join.

$$1 - e^{-\theta r} \left(\frac{\mu}{\mu - c\theta} \right)^{n_e} \geq 0 \quad , \quad \mu > c\theta \quad (5.1.5)$$

On the other hand, equation 5.1.6 pertains to the case that is not in favor of the arbitrary customer. To clarify, at the instant of his arrival, he observes already n_e customers in the system and he decides to balk.

$$1 - e^{-\theta r} \left(\frac{\mu}{\mu - c\theta} \right)^{n_e + 1} < 0, \quad \mu > c\theta \quad (5.1.6)$$

Equations 5.1.5 and 5.1.6 together imply

$$n_e \leq \frac{\theta r}{\ln \frac{\mu}{\mu - c\theta}} < n_e + 1$$

Now we are able to characterize the "self-optimization implying strategy", n_e , as

$$n_e = \left\lfloor \frac{\theta r}{\ln \frac{\mu}{\mu - c\theta}} \right\rfloor \quad (5.1.7)$$

(ii) Since the floor function $\lfloor X \rfloor$ is non-decreasing in X , it is enough to show that $\frac{\theta r}{\ln \frac{\mu}{\mu - c\theta}}$ is non-increasing in θ and non-decreasing in r and μ . the derivative of $\frac{\theta r}{\ln \frac{\mu}{\mu - c\theta}}$ with respect to θ is $r \left(\ln \frac{\mu}{\mu - c\theta} - \frac{c\theta}{\mu - c\theta} \right) < 0$ for $\mu > c\theta$. The fact that n_e is non-decreasing in r is self-evident. The derivative of $\frac{\theta r}{\ln \frac{\mu}{\mu - c\theta}}$ with respect to μ is, $\frac{\theta^2 r c}{\mu} \times \frac{1}{\mu - c\theta} \times \frac{1}{\left(\ln \frac{\mu}{\mu - c\theta} \right)^2} \geq 0$. ■

Corollary 5.1.1

n_e has the following features, where n_s is Naor's self-interest implying threshold.

$$(i) \lim_{\theta \rightarrow 0} n_e = n_s = \left\lfloor \frac{r\mu}{c} \right\rfloor$$

$$(ii) n_e \leq n_s$$

Proof of Corollary 5.1.1.

By using Hopital's rule we have:

$$\lim_{\theta \rightarrow 0} n_e = \lim_{\theta \rightarrow 0} \left\lfloor \frac{r}{\frac{c\mu}{(\mu - c\theta)^2} \times \frac{\mu - c\theta}{\mu}} \right\rfloor = \left\lfloor \frac{r\mu}{c} \right\rfloor$$

■

Corollary 5.1.1 implies that as the level of the risk-aversion degree, θ , approaches to zero, the threshold, n_e , converges to the threshold in the risk-neutral case.

In the observable case with risk-averse strategic customers, under the equilibrium threshold strategy, the number of customers in the system would not exceed n_e and we are interested in the steady state probabilities of this $M/M/1/n_e$ queue. The net arrival rate equals

$$\lambda = \Lambda(1 - P_{n_e})$$

where P_{n_e} is the steady state probability that there are n_e customers in the $M/M/1/n_e$ system and $\rho = \frac{\Lambda}{\mu}$.

$$P_{n_e} = \frac{\rho^{n_e}(1 - \rho)}{1 - \rho^{n_e+1}} \quad \rho \neq 1$$

and for $\rho = 1$ by using Hopital's rule we have

$$\lim_{\rho \rightarrow 1} P_{n_e} = \lim_{\rho \rightarrow 1} \frac{n_e \rho^{n_e-1}(1 - \rho) - \rho^{n_e}}{-(n_e + 1)\rho^n} = \frac{1}{n_e + 1}$$

Furthermore, by pursuing an arbitrary strategy, say n_e , the expected number of customers observed in the service system, $\mathbb{E}[N_q]$ is denoted by

$$\mathbb{E}[N_q] = \frac{\rho}{1 - \rho} - \frac{(n_e + 1)\rho^{n_e+1}}{1 - \rho^{n_e+1}} \quad (5.1.8)$$

In subsection 5.1.2, we will characterize the service system modeled as a queue with risk-averse strategic customers, under the unobservable setting.

5.1.2 Unobservable case

In the unobservable case, the arbitrary arriving customer is unable to see the queue length; and the waiting time, X is characterized in a different way. We again assume all customers

possess the same exponential utility function expressed in equation (5.1.1), where, X is the waiting time of an arbitrary customer. An arriving customer decides to "join, balk or join with a probability" based on a comparison of his expected utility. If his expected utility obtained by deciding to join is greater than his earned utility by balking, $u(0)$, he joins. Otherwise, the customer decides to balk, and in case of a tie he decides to join with a positive probability $0 < q < 1$, which is characterized based on the service parameters. Since we suppose all customers possess the same utility function, the Nash-Equilibrium of this game is symmetric. If we suppose that in the equilibrium all customers are joining with p , then the effective arrival rate is $\lambda = p\Lambda$. By using the standard approaches in an $M/M/1$ queue, given $\lambda < \mu$, it turns out that the waiting time, X , follows an exponential distribution with rate $\mu - \lambda$, where $\lambda \in [0, \Lambda]$ is the effective rate of the customers joining. As in section 5.1.1, λ is the effective rate of customers joining the queue based on the individual decision of customers. In other words, λ is the effective arrival rate which occurs when customers adopt an equilibrium strategy. We will calculate the expected utility of an arbitrary customer (with risk sensitivity degree θ) who decides to join as

$$\mathbb{E}[u(r - \tau - cX)_{RA}] = 1 - e^{-\theta(r-\tau)}\mathbb{E}[e^{c\theta X}] \quad (5.1.9)$$

As we mentioned above, X represents the waiting time of an arbitrary customer and follows an exponential distribution with rate $\mu - \lambda$. Therefore, by using the moment generating function for X we can rewrite the expected utility of an arbitrary customer who decides to join as

$$\mathbb{E}[u(r - \tau - cX)_{RA}] = \begin{cases} 1 - e^{-\theta(r-\tau)\frac{\mu-\lambda}{\mu-\lambda-c\theta}} & \text{if } \mu - \lambda - c\theta > 0 \\ -\infty & \text{if } \mu - \lambda - c\theta \leq 0. \end{cases} \quad (5.1.10)$$

suppose a tagged customer joins with probability p and all other customers are joining with probability q . We are interested in maximizing the expected utility, $U(p, q)$ or minimizing the expected disutility, $D(p, q)$ of the tagged customer. As a result, we have:

$$D(p, q) = pE_q[e^{-\theta(r-\tau-cX)}] + (1-p)$$

where $E_q[e^{-\theta(r-\tau-cW)}]$ is the expected disutility of a customer who decides to join with probability p , given all other customers are joining with probability q . Furthermore, the disutility of balking is 1. Therefore, for $\theta c < \mu - \lambda q$ we have:

$$D(p, q) = pe^{-\theta(r-\tau)} \frac{\mu - \lambda q}{\mu - \lambda q - c\theta} + (1-p). \quad (5.1.11)$$

Furthermore, for $\theta c \geq \mu - \lambda q$, the cases with a high risk aversion degree, θ we have:

$$D(p, q) = \begin{cases} \infty & \text{if } p > 0 \\ 1 & \text{if } p = 0 \end{cases}. \quad (5.1.12)$$

As a result, the tagged customer is trying to minimize his expected disutility and is interested in solving the following problem:

$$q^* \in \arg \min_{0 \leq p \leq 1} \left\{ p \left[e^{-\theta r} \left(\frac{\mu - \Lambda q^*}{\mu - \Lambda q^* - c\theta} \right) - 1 \right] \right\} \quad (5.1.13)$$

As a benchmark, we refer to the risk-neutral case by using a subscript *RN* (Risk Neutral) for the corresponding elements discussed. Furthermore, we use the subscript *RA* (Risk Averse) for the risk averse customers. According to non-strategic queueing theory assumptions, we need the arrival rate, Λ to remain below the service rate, μ . However, with strategic customers, there is no need for such an assumption and the arrival rate, Λ

could be more than the service rate, μ , provided that the effective arrival rate, λ remains below the service rate. It turns out that the cases $\Lambda < \mu$ and $\Lambda \geq \mu$ require different analysis. Therefore, to make sure we are not missing anything we consider two cases, the case where Λ is less than μ and the case where Λ exceeds μ .

5.2 Equilibrium Analysis

In this section, we are interested in defining the Nash equilibrium of the unobservable case. By referring back to equations 5.1.11 and 5.1.12 we define the best response of a tagged customer against strategy q of all other customers as

$$BR(q) = \begin{cases} 0 & \text{if } e^{\theta(r-\tau)} < \frac{\mu-\Lambda q}{\mu-\Lambda q-c\theta} \text{ or } \theta c \geq \mu - \Lambda q \\ (0, 1) & \text{if } e^{\theta(r-\tau)} = \frac{\mu-\Lambda q}{\mu-\Lambda q-c\theta} \\ 1 & \text{if } e^{\theta(r-\tau)} \geq \frac{\mu-\Lambda q}{\mu-\Lambda q-c\theta} \end{cases} \quad (5.2.1)$$

A joining probability q^* is a Nash Equilibrium if $q^* \in BR(q^*)$. Now we are able to characterize the Nash-Equilibrium.¹

Theorem 5.2.1

(i) In an unobservable $M/M/1$ queueing system with risk-averse strategic customers with exponential utility functions the unique Nash-Equilibrium is characterized as

$$q^* = \begin{cases} 0 & \text{if } e^{\theta(r-\tau)} \leq \frac{\mu}{\mu-c\theta} \text{ or } \theta c \geq \mu \\ \bar{q} & \text{if } \frac{\mu}{\mu-c\theta} < e^{\theta(r-\tau)} < \frac{\mu-\Lambda}{\mu-\Lambda-c\theta} \\ 1 & \text{if } \frac{\mu-\Lambda}{\mu-\Lambda-c\theta} \leq e^{\theta(r-\tau)} \end{cases} \quad (5.2.2)$$

¹The theorem was taken from the reports of a 117M583 numbered joint tübitak research with Fikri Karaesmen, Pelin Canbolat and Athanasia Manou

where \bar{q} equals

$$\frac{1}{\Lambda} \left(\mu - \frac{\theta c}{1 - e^{-\theta(r-\tau)}} \right)$$

(ii) \bar{q} is decreasing in θ

Note: For $\theta c < \mu$ & $\tau \geq r$, $q^* = 0$

Proof of Theorem 5.2.1.

(i) Since, a joining probability q is a Nash Equilibrium if $q^* \in BR(q^*)$. By referring to equation 5.2.1 and plugging in zero and one instead of q , in the first and third case, respectively, the Nash-Equilibrium for the first and third case is obtained. Furthermore, in the second case we solve for

$$e^{\theta(r-\tau)} = \frac{\mu - \Lambda q}{\mu - \Lambda q - c\theta}$$

in terms of q .

(ii) for $\theta r > 0$, $e^{\theta r} - 1 - \theta r < 0$ always holds, as a result:

$$\frac{d\bar{q}(\theta)}{d\theta} = \frac{-ce^{-\theta r}}{\Lambda(1 - e^{-\theta r})^2} \times (e^{\theta r} - 1 - \theta r) < 0$$

■

In theorem 5.2.1, the first case, pertains to the situation where even if the service system is empty and an arbitrary customer only has to wait for his own service duration, he decides to balk. In other words, in such cases the risk-aversion degree, θ , is so high that even an empty system is not tempting for customers and in the equilibrium all decide to balk.

The third case, corresponds to situations where even if everybody in the system has decided joining, an arbitrary customer would be better-off by joining the service system.

Finally, the second case, is associated with the situation where neither pure joining nor pure balking is an equilibrium. To clarify, for moderate levels of the risk-aversion degree θ , the expected gained utility for the customer equals the utility gained by balking. As a result, the customers end up joining with a probability $0 < q < 1$. In other words, in the equilibrium, only a fraction of customers decide joining. In addition, q is decreasing in the risk-aversion degree, θ .

We will rearrange theorem 5.2.1 and rewrite the corresponding constraints based on Λ , to be able to compare it with the risk-neutral equilibrium arrival rate λ_{RN}^e .

Corollary 5.2.1 *In the unobservable case with risk-averse strategic customers the equilibrium arrival rate is as follows,*

$$\lambda_{RA}^e = \begin{cases} 0 & \text{if } \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} < 0 \\ \mu - \frac{\theta c}{1-e^{-\theta(r-\tau)}} & \text{if } 0 \leq \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} \leq \Lambda \\ \Lambda & \text{if } \Lambda \leq \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}}. \end{cases} \quad (5.2.3)$$

whereas in the risk-neutral benchmark model, (Hassin and Haviv, 2003) the equilibrium arrival rate, λ_{RN}^e is equal to

$$\lambda_{RN}^e = \begin{cases} 0 & \text{if } \mu - \frac{c}{r-\tau} < 0 \\ \mu - \frac{c}{r-\tau} & \text{if } 0 \leq \mu - \frac{c}{r-\tau} \leq \Lambda \\ \Lambda & \text{if } \Lambda \leq \mu - \frac{c}{r-\tau}. \end{cases} \quad (5.2.4)$$

Proposition 5.2.1

(i) λ_{RA}^e is non-increasing in θ

$$(ii) \lim_{\theta \rightarrow 0} \lambda_{RA}^e = \lambda_{RN}^e$$

Proof of Proposition 5.2.1.

(i) We differentiate with respect to θ

$$\frac{d\lambda_{RA}^e}{d\theta} = \frac{-c(1 - e^{-\theta(r-\tau)}) - c\theta(r-\tau)e^{-\theta(r-\tau)}}{[1 - e^{-\theta(r-\tau)}]^2} \leq 0$$

(ii) By using Hopital's rule we have:

$$\lim_{\theta \rightarrow 0} = \mu - \frac{c\theta}{1 - e^{-\theta(r-\tau)}} = \mu - \lim_{\theta \rightarrow 0} \left(\frac{c}{(r-\tau)e^{-\theta(r-\tau)}} \right) = \mu - \frac{c}{r-\tau}$$

■

5.3 Social Welfare Maximization

In this section, we will define the social welfare optimization problem for the social welfare planner, both in the observable case and unobservable case. We suppose, that the social welfare maximizer is interested in maximizing the following function and she is risk-neutral.

5.3.1 Observable Case

In the observable case, our social welfare maximization problem would be exactly Naor (1969)'s social welfare optimization problem.

$$S_O = \lambda r - c\mathbb{E}[n_q] \tag{5.3.1}$$

Accordingly, we would be able to take advantage of Naor's results. We will suppose that, in our case, n_t is the strategy that implies the maximum social welfare. Similarly, Naor uses

the following notations, He denotes the “self-interest implying strategy” with n_s , and the “social welfare optimizing strategy” as n_0 , and proved that $n_0 \leq n_s$. On the other hand, we proved that $n_e \leq n_s$, by providing Corollary 5.1.1. By taking into account the above arguments simultaneously, we could infer that unlike the risk-neutral case that we have $n_0 \leq n_s$, with risk-averse customers, cases such as $n_t \geq n_e$ could emerge. Surprisingly, this implies that, with risk-averse strategic customers, “socially optimal implying strategy” could be larger than the “self-interest implying strategy”. To wit, take θ , the risk-aversion degree, so high that n_e approaches zero. Please note that, n_t equals n_0 , and n_t , n_0 and n_s are independent of the risk-aversion degree, θ .

TABLE 5.1: Social Welfare, $\mathbb{E}[N_q]$ and λ for different values of the prescribed strategy, n with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

Observable Case	n							
	1	2	3	4	5	6	7	8
Social Welfare	0.2625	0.3245	0.3419	0.3452	0.3437	0.3409	0.3382	0.3359
$\mathbb{E}[N_q]$	0.3750	0.6735	0.9044	1.0784	1.2064	1.2984	1.3633	1.4084
λ	0.0750	0.0980	0.1081	0.1133	0.1161	0.1177	0.1186	0.1192

In table 5.1, values for the social welfare, expected number of customers in the queue and effective arrival rate are gathered with respect to n , under the observable case with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$. As it could be observed, for $n = 4$, the social welfare is maximized. The expected number of customers in the queue and effective arrival rate are both increasing in n .

5.3.2 Unobservable Case

After defining the equilibrium arrival rate, λ^e , our purpose is to define socially optimal arrival rate, λ^* . In other words, we are supposing that the social welfare maximizing entity thinks that the customers are risk-neutral. Note that in (5.3.2), the customers are

perceived as risk-neutral by the service provider. Therefore, we define the social welfare optimization problem as:

$$\lambda^* = \arg \max_{0 \leq \lambda \leq \Lambda} \left\{ \lambda \left(r - \frac{c}{\mu - \lambda} \right) \right\} \quad (5.3.2)$$

which implies

$$\lambda^* = \begin{cases} \mu - \sqrt{\frac{c\mu}{r}} & \text{if } \mu - \sqrt{\frac{c\mu}{r}} \leq \Lambda \\ \Lambda & \text{if } \mu - \sqrt{\frac{c\mu}{r}} \geq \Lambda. \end{cases} \quad (5.3.3)$$

(5.3.3), indicates the socially desired arrival rate and in the next section, we will define a pricing mechanism to reach this arrival rate.

5.3.3 Pricing scheme to obtain socially desired arrival rate

Now, in this subsection we will derive a pricing scheme for the social welfare maximizer to obtain socially desired arrival rate. By comparing $\lambda_{RN}^e(0)$ and λ^* We notice that

$$\lambda_{RN}^e(0) \geq \lambda^*$$

Therefore, this gap could be corrected by imposing a positive admission fee. However, When customers are risk-averse $\lambda_{RA}^e(0)$ might be greater or less than λ^* , the socially optimal arrival rate. Therefore, with risk-averse customers, the regulating price might be positive or negative (indicating a subsidy) or even might not exist. The regulating price for the risk-neutral case is equal to

$$\tau = r - \sqrt{\frac{cr}{\mu}} \quad (5.3.4)$$

and it is clear that this price is always non-negative. On the other hand, the regulating price for the risk-averse case is equal to:

$$\tau' = r + \frac{\ln(1 - \theta \sqrt{\frac{cr}{\mu}})}{\theta} \quad (5.3.5)$$

The sign of the regulating price depends on θ , the level of risk-aversion. For customers with low level of θ , a positive price exists and as θ increases filling the gap between λ_{RA}^e and λ^* is only possible by imposing a negative price (a subsidy). Finally, when θ is high enough, even imposing a negative price is not enough to fill the gap and no regulating price could be imposed.

Theorem 5.3.1 *In the Unobservable case with risk-averse strategic customers, based on the level of risk-aversion and the potential arrival rate, three mutually exclusive cases emerge for the sign of the regulating price*

- (i) *If $\Lambda < \text{Min}(\mu - \sqrt{\frac{c\mu}{r}}, \mu - \frac{c\theta}{1-e^{-\theta r}})$, then the regulating price equals zero.*
- (ii) *If $\Lambda > \text{Min}(\mu - \sqrt{\frac{c\mu}{r}}, \mu - \frac{c\theta}{1-e^{-\theta r}})$ and $\theta < \sqrt{\frac{\mu}{rc}}$, for lower levels of θ a positive price and for higher levels of θ a negative price (indicating a subsidy) should be imposed.*
- (ii) *If $\Lambda > \text{Min}(\mu - \sqrt{\frac{c\mu}{r}}, \mu - \frac{c\theta}{1-e^{-\theta r}})$ and $\theta > \sqrt{\frac{\mu}{rc}}$, the risk aversion degree is high and the regulating price does not exist.*

Proof of Theorem 5.3.1.

By comparing and setting equal the equations 5.3.3 and 5.2.3, the regulating price, τ' is obtained. Furthermore, by following equation 5.3.5 and noting that $1 - \theta \sqrt{\frac{cr}{\mu}}$ should be positive, the theorem is extracted. ■

Proposition 5.3.1 *As θ , approaches zero, the "risk-averse regulating price" converges to the "risk-neutral regulating price".*

$$\lim_{\theta \rightarrow 0} \tau' = \tau$$

Proof of Proposition 5.3.1.

By using Hopital's rule we have:

$$\lim_{\theta \rightarrow 0} r + \frac{\ln(1 - \theta \sqrt{\frac{cr}{\mu}})}{\theta} = \lim_{\theta \rightarrow 0} r - \sqrt{\frac{cr}{\mu}} \times \frac{1}{1 - \theta \sqrt{\frac{cr}{\mu}}} = r - \sqrt{\frac{cr}{\mu}}$$

■

5.4 Revenue Maximization

In this section, we are interested in addressing the revenue maximization problem of the service provider both in the observable case and unobservable case. In this regard, we will investigate the optimum τ^* which maximizes the revenue of the administrator which is obtained by multiplying the effective arrival rate by the price set, τ .

5.4.1 Observable Case

Static Pricing

We characterize the revenue maximization problem, for an arbitrary strategy, n , in the observable case with risk-averse customers as the following, where the price set by the revenue maximizer is denoted by τ . Furthermore, τ could be represented as a function of the pursued strategy, n . By setting the price equal to (5.4.1), the surplus of the customer " who decides to join and after he joins the number of customers in the system becomes n ", is zero.

$$\tau = r - \frac{n}{\theta} \ln \frac{\mu}{\mu - c\theta}. \quad (5.4.1)$$

As a result, for lower/higher values of the pursued strategy, n , we are implying a higher/lower price. Furthermore, since inside the Logarithm function should be positive, we must have $\mu > c\theta$.

$$R_n = \lambda\tau = \Lambda \left(\frac{1 - \rho^n}{1 - \rho^{n+1}} \right) \tau \quad (5.4.2)$$

$$R_n = \Lambda \left(\frac{1 - \rho^n}{1 - \rho^{n+1}} \right) \left(r - \frac{n}{\theta} \ln \frac{\mu}{\mu - c\theta} \right) \quad (5.4.3)$$

We are interested in finding an expression for the strategy, say n_m that leads to the maximum revenue. Naor, 1969 solved the revenue maximization problem for the risk-neutral case and denoted his pursued strategy by n_r . He proved that, $n_r \leq n_0 \leq n_s$.

Theorem 5.4.1 *The revenue function, R_n is concave in n , the prescribed strategy.*

Proof of Theorem 5.4.1.

To prove the concavity of R_n we need to show $R_n - R_{n-1} \geq R_{n+1} - R_n$. First, we define another function of n as $M_n = \frac{1 - \rho^n}{1 - \rho^{n+1}}$ and show that M_n is an increasing function of n . Furthermore, note that ρ^n is an increasing convex function for $\rho > 1$ and decreasing

convex for $\rho < 1$. Therefore we have

$$\begin{aligned}\rho^{n+2} - \rho^{n+1} &> \rho^{n+1} - \rho^n \\ 1 - 2\rho^{n+1} + \rho^{2n+2} &> 1 - \rho^{n+2} - \rho^n + \rho^{2n+2} \\ (1 - \rho^{n+1})(1 - \rho^{n+1}) &> (1 - \rho^n)(1 - \rho^{n+2}) \\ \frac{1 - \rho^{n+1}}{1 - \rho^{n+2}} &> \frac{1 - \rho^n}{1 - \rho^{n+1}}\end{aligned}$$

Therefore, $M_{n+1} > M_n$.

$$\begin{aligned}R_n - R_{n-1} &= \Lambda \left[\left(\frac{1 - \rho^n}{1 - \rho^{n+1}} - \frac{1 - \rho^{n-1}}{1 - \rho^n} \right) r + \left(-\frac{1 - \rho^n}{1 - \rho^{n+1}} + \frac{1 - \rho^{n-1}}{1 - \rho^n} \right) \frac{n}{\theta} \ln \frac{\mu}{\mu - c\theta} - \frac{1 - \rho^{n-1}}{1 - \rho^n} \frac{\mu}{\theta(\mu - c\theta)} \right] \\ R_{n+1} - R_n &= \Lambda \left[\left(\frac{1 - \rho^{n+1}}{1 - \rho^{n+2}} - \frac{1 - \rho^n}{1 - \rho^{n+1}} \right) r + \left(-\frac{1 - \rho^{n+1}}{1 - \rho^{n+2}} + \frac{1 - \rho^n}{1 - \rho^{n+1}} \right) \frac{n}{\theta} \ln \frac{\mu}{\mu - c\theta} - \frac{1 - \rho^{n+1}}{1 - \rho^{n+2}} \frac{\mu}{\theta(\mu - c\theta)} \right]\end{aligned}$$

Now, considering simultaneously the facts that M_n is an increasing function, and r being greater $\frac{n}{\theta} \ln \frac{\mu}{\mu - c\theta}$ (to impose positive prices by the administrator), $R_n - R_{n-1} > R_{n+1} - R_n$ holds. ■

Dynamic Pricing

Now, we will extend our results by setting a dynamic price, based on the number of customers observed in the system, by an arbitrary arriving customer. Suppose, n is the prescribed arbitrary strategy. In contrast to the static-pricing policy, here we will present a pricing policy which charges higher prices for shorter queue lengths. We look for policies that choose a maximum queue length and a corresponding price policy $(\tau_0, \tau_1, \tau_2, \dots$ and $\tau_{n_{max}})$. Therefore, the price will be a function of customers in the system as following

$$\tau_i = \max \left(r - \frac{i+1}{\theta} \ln \frac{\mu}{\mu - c\theta}, 0 \right) \quad (5.4.4)$$

By imposing the price (5.4.4), zero surplus is left for all customers in the queue, where as in the static pricing only the last customer who joins has zero surplus. And consequently, the average revenue of the administrator in the long-run for an arbitrary strategy n will be

$$R_n^d = \Lambda \sum_{i=0}^{n-1} P_i^n \tau_i \quad (5.4.5)$$

Where, P_i^n is the steady-state probability that there are i customers in the system when the prescribed strategy is n .

$$P_i^n = \frac{\rho^i (1 - \rho)}{1 - \rho^{n+1}} \quad \rho \neq 1$$

5.4.2 Unobservable Case

After addressing the equilibrium arrival rate for risk-averse customers, λ , we are able to characterize the revenue function of the administrator with risk-averse customers, who has set the price τ as

$$\pi_A(\tau) = \begin{cases} 0 & \text{if } \tau \geq r - \frac{1}{\theta} \ln \frac{\mu}{\mu - c\theta} \quad \text{or} \quad \theta c \geq \mu \\ \left[\mu - \frac{\theta c}{1 - e^{-\theta(r-\tau)}} \right] \tau & \text{if } r - \frac{1}{\theta} \ln \frac{\mu - \Lambda}{\mu - \Lambda - c\theta} < \tau < r - \frac{1}{\theta} \ln \frac{\mu}{\mu - c\theta} \\ \Lambda \tau & \text{if } \tau \leq r - \frac{1}{\theta} \ln \frac{\mu - \Lambda}{\mu - \Lambda - c\theta} \end{cases} \quad (5.4.6)$$

Proposition 5.4.1

(i) $\bar{q}(\tau)$ is decreasing and concave in τ .

(ii) $\bar{q}(\tau)\tau$ is concave in τ .

(iii) $\pi_A(\tau)$ has a unique maximizer, τ^*

Proof of Proposition 5.4.1.

(i) By taking the first and second derivative of $\bar{q}(\tau)$ with respect to τ , for $\tau \in [0, r]$, we have

$$\frac{d\bar{q}(\tau)}{d\tau} = \frac{-\theta^2 c}{\Lambda} \frac{1}{[1 - e^{-\theta(r-\tau)}]^2} e^{-\theta(r-\tau)} < 0$$

$$\frac{d^2\bar{q}(\tau)}{d\tau^2} = \frac{-\theta^2 c}{\Lambda} \left[\frac{2\theta e^{-2\theta(r-\tau)}}{[1 - e^{-\theta(r-\tau)}]^3} + \frac{\theta e^{-\theta(r-\tau)}}{[1 - e^{-\theta(r-\tau)}]^2} \right] < 0$$

(ii) By taking the second derivative of $\bar{q}(\tau)\tau$ with respect to τ we have

$$\frac{d^2[\bar{q}(\tau)\tau]}{d\tau^2} = \frac{d^2\bar{q}(\tau)}{d\tau^2}\tau + 2\frac{d\bar{q}(\tau)}{d\tau} < 0$$

(iii) According to the Revenue Function, 5.4.6, we observe that the function consists of three parts with respect to τ ; The first part (lower prices), is linearly increasing in τ , and the middle part (medium prices) of the function is concave in τ according to proposition 5.4.1 and finally, the third part (highest prices), is constant and equals zero. Therefore the revenue function has a unique maximizer, τ^* .

■

After characterizing the revenue maximization problem and presenting its properties, we are interested in providing the approach (presented in figure 5.1) for obtaining τ^* , for $\Lambda < \mu$. Although the first part of the revenue function is linearly increasing in τ and the middle part of the revenue function is concave in τ , and they intersect at $r - \frac{1}{\theta} \ln \frac{\mu - \Lambda}{\mu - \Lambda - c\theta}$, but based on the location of the extremum of the concave function $\Lambda\bar{q}\tau$, τ^* is characterized in a different way. If the maximum point of $\Lambda\bar{q}\tau$ is before the intersection point $r - \frac{1}{\theta} \ln \frac{\mu - \Lambda}{\mu - \Lambda - c\theta}$, then τ^* equals $\pi_A(\tau)$'s first cut-off point, $r - \frac{1}{\theta} \ln \frac{\mu - \Lambda}{\mu - \Lambda - c\theta}$. On the other hand, if the maximum point of $\Lambda\bar{q}\tau$ is after the intersection point $r - \frac{1}{\theta} \ln \frac{\mu - \Lambda}{\mu - \Lambda - c\theta}$, then τ^* is derived from the first order condition (FOC) of $\pi_A(\tau)$.

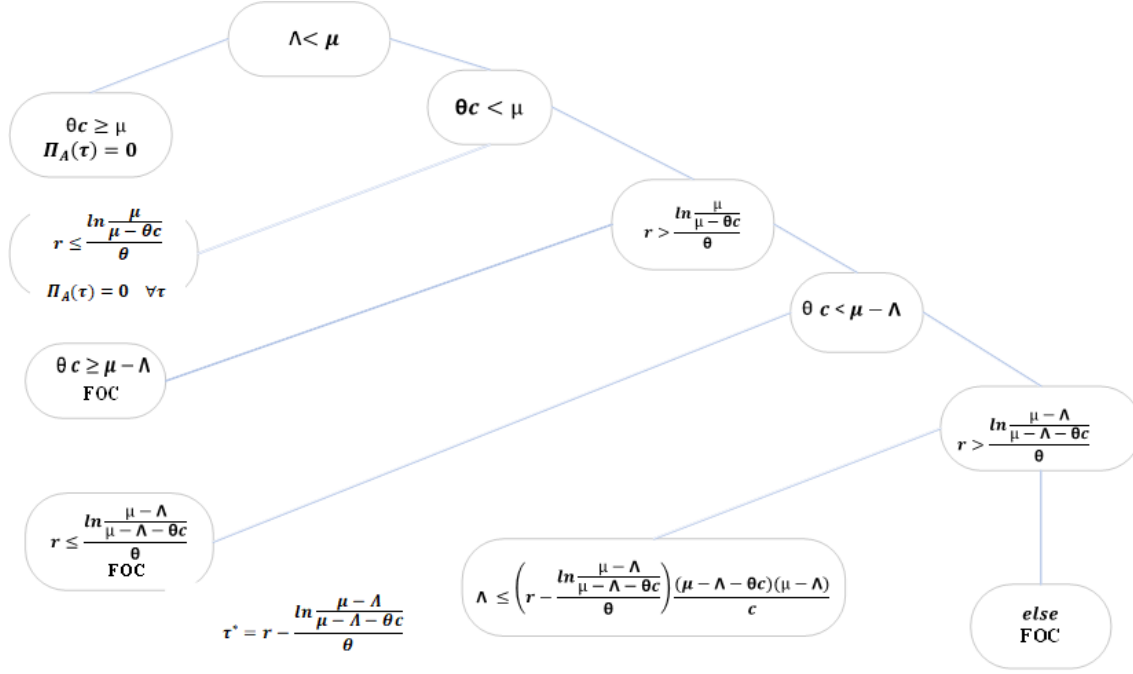


FIGURE 5.1: Solution Approach to characterize the revenue maximizing price, τ^* given $\Lambda < \mu$

As we discussed in the previous sections, since it turns out that based on whether $\Lambda < \mu$ or $\Lambda \geq \mu$, we need a quite different analysis of the problem. In the following corollary we provide an alternative representation of the revenue maximization problem.

Corollary 5.4.1 *In the Unobservable case with risk-averse strategic customers, the revenue maximization problem alternatively could be presented as*

$$\pi_A(p) = \begin{cases} 0 & \text{if } \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} < 0 \\ \left[\mu - \frac{\theta c}{1-e^{-\theta(r-\tau)}} \right] \tau & \text{if } 0 \leq \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} \leq \Lambda \\ \Lambda \tau & \text{if } \Lambda \leq \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}}. \end{cases} \quad (5.4.7)$$

By referring back to Corollary 5.4.1, it could be understood that by assuming $\Lambda \geq \mu$, since pure joining would be impossible, the $\Lambda \tau$ part of the function would be automatically

eliminated and for $\Lambda \geq \mu$, the revenue function $\pi_A(\tau)$ and the associated τ^* is presented in corollary 5.4.2.

Corollary 5.4.2 *In the Unobservable case with risk-averse strategic customers, for $\Lambda \geq \mu$, the revenue maximization problem could be presented as follows:*

$$\pi_A(p) = \begin{cases} 0 & \text{if } \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} < 0 \\ \left[\mu - \frac{\theta c}{1-e^{-\theta(r-\tau)}} \right] \tau & \text{if } 0 \leq \mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} \leq \Lambda \end{cases} \quad (5.4.8)$$

As a result, τ^* is derived by the first order condition of $\pi_A(\tau)$.

5.5 Numerical Examples

In this section, we will provide some numerical examples related to both observable case and unobservable case. In the observable case, the numerical examples will better illustrate the implications of risk-aversion on the effective arrival rate and the threshold for joining. Furthermore, the “relative/absolute change” (based on static and dynamic pricing) in the maximum extractable revenue with respect to θ , λ and r will be presented. In the unobservable case, numerical results related to the implications of risk-averse behavior on the effective arrival rate and the possible regulating prices is gathered.

5.5.1 Observable Case

In table 5.2, the threshold strategies, effective arrival rates and steady-state probabilities (n_e , λ and P_{n_e}) for risk-neutral and risk-averse customers are gathered, for different values of the risk-aversion degree, θ . In this particular example we assume that $r = 4$, $\mu = 0.2$, $\Lambda = 0.12$ and $c = 0.1$. Accordingly, for the risk-neutral case the threshold strategy, n_e is independent of the risk-aversion degree, θ . On the other hand, for the risk-averse

TABLE 5.2: n_e , λ and P_{n_e} for risk-neutral and risk-averse customers

θ	n_e		λ		P_{n_e}	
	r-neutral	r-averse	r-neutral	r-averse	r-neutral	r-averse
0.2	8	7	0.11919	0.11863	0.00679	0.01139
0.4	8	7	0.11919	0.11863	0.00679	0.01139
0.6	8	6	0.11919	0.11770	0.00679	0.01920
0.8	8	6	0.11919	0.11770	0.00679	0.01920
1	8	5	0.11919	0.11608	0.00679	0.03263
1.2	8	5	0.11919	0.11608	0.00679	0.03263
1.4	8	4	0.11919	0.11325	0.00679	0.05621
1.6	8	3	0.11919	0.10809	0.00679	0.09926
1.8	8	3	0.11919	0.10809	0.00679	0.09926

customers the situation is different, and the threshold strategy is non-increasing in θ . To clarify, in a service system with specified service characteristics, risk-averse customers tend to balk easier than risk-neutral customers, when self-interest is pursued by all customers. In other words, risk-averse customers have a lower threshold for balking, and as long as they get more risk-averse this integer threshold, n_e tends to decrease gradually. The reason behind this attitude is that customers who are risk-averse evaluate their expected gain with a concave exponential utility function rather than a linear risk-neutral utility function. For the effective arrival rate, λ , we have a similar story and the effective arrival rate for risk-neutral customers is independent of the risk-aversion degree, θ . However, for the risk-averse customers, the effective arrival rate is non-increasing in θ and as long as customers get more risk-averse, a lower portion of the initial arrival rate, Λ is accepted. Furthermore, in the third column, the values associated with P_{n_e} are gathered, for risk-neutral and risk-averse customers. The mentioned system has $n_e + 1$ states and the steady state probability of being in state n_e is P_{n_e} which is independent of the risk-aversion degree, θ for risk-neutral customers. However, for the risk-averse customers, P_{n_e} is non-decreasing in θ . The reason behind this is quite intuitive and as long as θ increases, n_e decreases and

we would have fewer states and as a result a higher probability is expected to be assigned to n_e , among fewer states.

TABLE 5.3: Static Price with respect to the prescribed strategy, n and the risk-aversion degree, θ in an observable service system with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

Price	n								
	0	1	2	3	4	5	6	7	8
0.1	4	3.4871	2.9741	2.4612	1.9483	1.4353	0.9224	0.4095	0
0.2	4	3.4732	2.9464	2.4196	1.8928	1.366	0.8392	0.3124	0
0.3	4	3.4583	2.9165	2.3748	1.8331	1.2914	0.7496	0.2079	0
0.4	4	3.4421	2.8843	2.3264	1.7686	1.2107	0.6528	0.095	0
0.5	4	3.4246	2.8493	2.2739	1.6985	1.1232	0.5478	0	0
0.6	4	3.4055	2.8111	2.2166	1.6222	1.0277	0.4333	0	0
0.7	4	3.3846	2.7692	2.1538	1.5384	0.923	0.3076	0	0
0.8	4	3.3615	2.7229	2.0844	1.4459	0.8073	0.1688	0	0
0.9	4	3.3357	2.6715	2.0072	1.3429	0.6787	0.0144	0	0
1	4	3.3069	2.6137	1.9206	1.2274	0.5343	0	0	0
θ 1.1	4	3.2741	2.5482	1.8223	1.0963	0.3704	0	0	0
1.2	4	3.2364	2.4728	1.7093	0.9457	0.1821	0	0	0
1.3	4	3.1924	2.3849	1.5773	0.7698	0	0	0	0
1.4	4	3.14	2.28	1.4201	0.5601	0	0	0	0
1.5	4	3.0758	2.1516	1.2274	0.3032	0	0	0	0
1.6	4	2.9941	1.9882	0.9823	0	0	0	0	0
1.7	4	2.884	1.7681	0.6521	0	0	0	0	0
1.8	4	2.7208	1.4416	0.1624	0	0	0	0	0
1.9	4	2.4233	0.8466	0	0	0	0	0	0
1.98	4	1.6742	0	0	0	0	0	0	0

Table 5.3, pertains to the static prices with respect to the prescribed strategy, n and the risk-aversion degree, θ in an observable service system with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

TABLE 5.4: Administrator's Revenue (Static Pricing) with respect to the prescribed strategy, n and the risk-aversion degree, θ in an observable service system with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

Revenue	n								
	0	1	2	3	4	5	6	7	8
0.1	0	0.26153	0.291344	0.266027	0.220651	0.166621	0.108563	0.048577	0
0.2	0	0.26049	0.288626	0.261529	0.214367	0.15857	0.098769	0.037059	0
0.3	0	0.25937	0.285702	0.256689	0.207605	0.149906	0.088227	0.024663	0
0.4	0	0.258161	0.282542	0.251459	0.200298	0.140545	0.076837	0.011269	0
0.5	0	0.256848	0.279112	0.245783	0.192368	0.130384	0.064476	0	0
0.6	0	0.255416	0.275371	0.239591	0.183718	0.119301	0.050992	0	0
0.7	0	0.253845	0.271268	0.232799	0.174229	0.107144	0.0362	0	0
0.8	0	0.25211	0.266737	0.2253	0.163752	0.09372	0.019868	0	0
0.9	0	0.25018	0.261695	0.216956	0.152095	0.078785	0.001697	0	0
1	0	0.248014	0.256036	0.20759	0.13901	0.06202	0	0	0
θ 1.1	0	0.245556	0.249616	0.196964	0.124165	0.043	0	0	0
1.2	0	0.242732	0.242238	0.184752	0.107105	0.021142	0	0	0
1.3	0	0.239433	0.233622	0.170491	0.087181	0	0	0	0
1.4	0	0.235501	0.223351	0.153492	0.063431	0	0	0	0
1.5	0	0.230685	0.21077	0.132669	0.034341	0	0	0	0
1.6	0	0.224558	0.194763	0.106175	0	0	0	0	0
1.7	0	0.216304	0.173201	0.070489	0	0	0	0	0
1.8	0	0.204059	0.141215	0.017549	0	0	0	0	0
1.9	0	0.181747	0.082932	0	0	0	0	0	0
1.98	0	0.125562	0	0	0	0	0	0	0

Table 5.4, pertains to the administrator's revenue (static pricing) with respect to the prescribed strategy, n and the risk-aversion degree, θ in an observable service system with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

TABLE 5.5: Steady State Probability, P_i^n for different values of n (Prescribed Strategy) and the number of customers in the system, i with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

P_i^n	n										
	0	1	2	3	4	5	6	7	8	9	10
0	1	0.625	0.5102	0.4596	0.4337	0.4196	0.4115	0.4068	0.4041	0.4024	0.4015
1	0	0.375	0.3061	0.2757	0.2602	0.2517	0.2469	0.2441	0.2424	0.2415	0.2409
2	0	0	0.1837	0.1654	0.1561	0.151	0.1481	0.1465	0.1455	0.1449	0.1445
3	0	0	0	0.0993	0.0937	0.0906	0.0889	0.0879	0.0873	0.0869	0.0867
4	0	0	0	0	0.0562	0.0544	0.0533	0.0527	0.0524	0.0522	0.052
5	0	0	0	0	0	0.0326	0.032	0.0316	0.0314	0.0313	0.0312
6	0	0	0	0	0	0	0.0192	0.019	0.0189	0.0188	0.0187
7	0	0	0	0	0	0	0	0.0114	0.0113	0.0113	0.0112
8	0	0	0	0	0	0	0	0	0.0068	0.0068	0.0067
9	0	0	0	0	0	0	0	0	0	0.0041	0.004
10	0	0	0	0	0	0	0	0	0	0	0.0024

Table 5.5, pertains to the steady state probability, P_i^n for different values of n (prescribed strategy) and the number of customers in the system, i with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

Table 5.6, pertains to the administrator's revenue (dynamic pricing) with respect to the prescribed strategy, n and the risk-aversion degree, θ in an observable service system with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

5.5.2 Unobservable Case

Example 5.5.1 Consider a service system with a single server and risk-averse strategic customers are arriving with the following parameters, $r = 1.5$, $\mu = 0.2$, $c = 0.1$. Based on

TABLE 5.6: Administrator's Revenue (Dynamic Pricing) with respect to the prescribed strategy, n and the risk-aversion degree, θ in an observable service system with $\Lambda = 0.12$, $\mu = 0.2$, $c = 0.1$ and $R = 4$.

Revenue	n								
	0	1	2	3	4	5	6	7	8
0.1	0	0.26153	0.32275	0.33957	0.34239	0.34058	0.33759	0.33467	0.3324
0.2	0	0.26049	0.32088	0.33706	0.3394	0.33724	0.33398	0.33089	0.32865
0.3	0	0.25937	0.31887	0.33436	0.33618	0.33363	0.33011	0.32682	0.3246
0.4	0	0.25816	0.3167	0.33145	0.3327	0.32974	0.32592	0.32242	0.32024
0.5	0	0.25685	0.31434	0.32828	0.32892	0.32552	0.32137	0.31771	0.31556
0.6	0	0.25542	0.31177	0.32483	0.3248	0.32091	0.31641	0.31281	0.31069
0.7	0	0.25384	0.30895	0.32104	0.32029	0.31586	0.31098	0.30743	0.30535
0.8	0	0.25211	0.30583	0.31685	0.3153	0.31028	0.30497	0.3015	0.29945
0.9	0	0.25018	0.30236	0.3122	0.30975	0.30407	0.29829	0.29489	0.29289
1	0	0.24801	0.29847	0.30697	0.30352	0.2971	0.2914	0.28808	0.28612
θ 1.1	0	0.24556	0.29406	0.30105	0.29645	0.2892	0.28364	0.28041	0.27851
1.2	0	0.24273	0.28899	0.29424	0.28833	0.28011	0.27473	0.2716	0.26976
1.3	0	0.23943	0.28306	0.28628	0.27884	0.26974	0.26457	0.26155	0.25978
1.4	0	0.2355	0.276	0.2768	0.26754	0.25881	0.25384	0.25095	0.24924
1.5	0	0.23069	0.26735	0.26518	0.25368	0.24541	0.2407	0.23796	0.23634
1.6	0	0.22456	0.25635	0.2504	0.23633	0.22862	0.22423	0.22167	0.22017
1.7	0	0.2163	0.24152	0.2305	0.21754	0.21044	0.2064	0.20405	0.20267
1.8	0	0.20406	0.21953	0.20097	0.18967	0.18348	0.17996	0.17791	0.1767
1.9	0	0.18175	0.17946	0.16165	0.15256	0.14759	0.14475	0.1431	0.14213
1.98	0	0.12556	0.1025	0.09232	0.08714	0.08429	0.08267	0.08173	0.08118



FIGURE 5.2: Relative (a, b)/Absolute change (c, d) in maximum extractable revenue with respect to θ

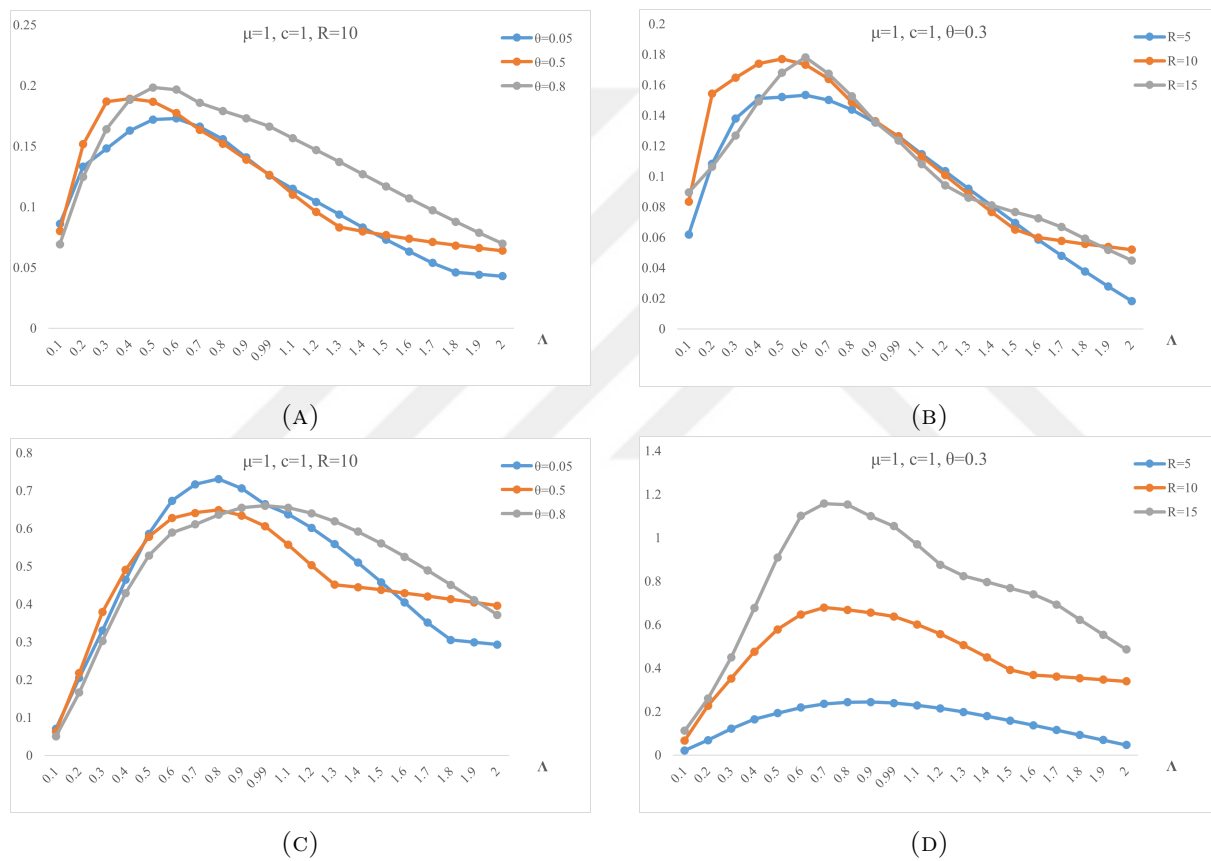


FIGURE 5.3: Relative (a, b)/Absolute change (c, d) in maximum extractable revenue with respect to Λ

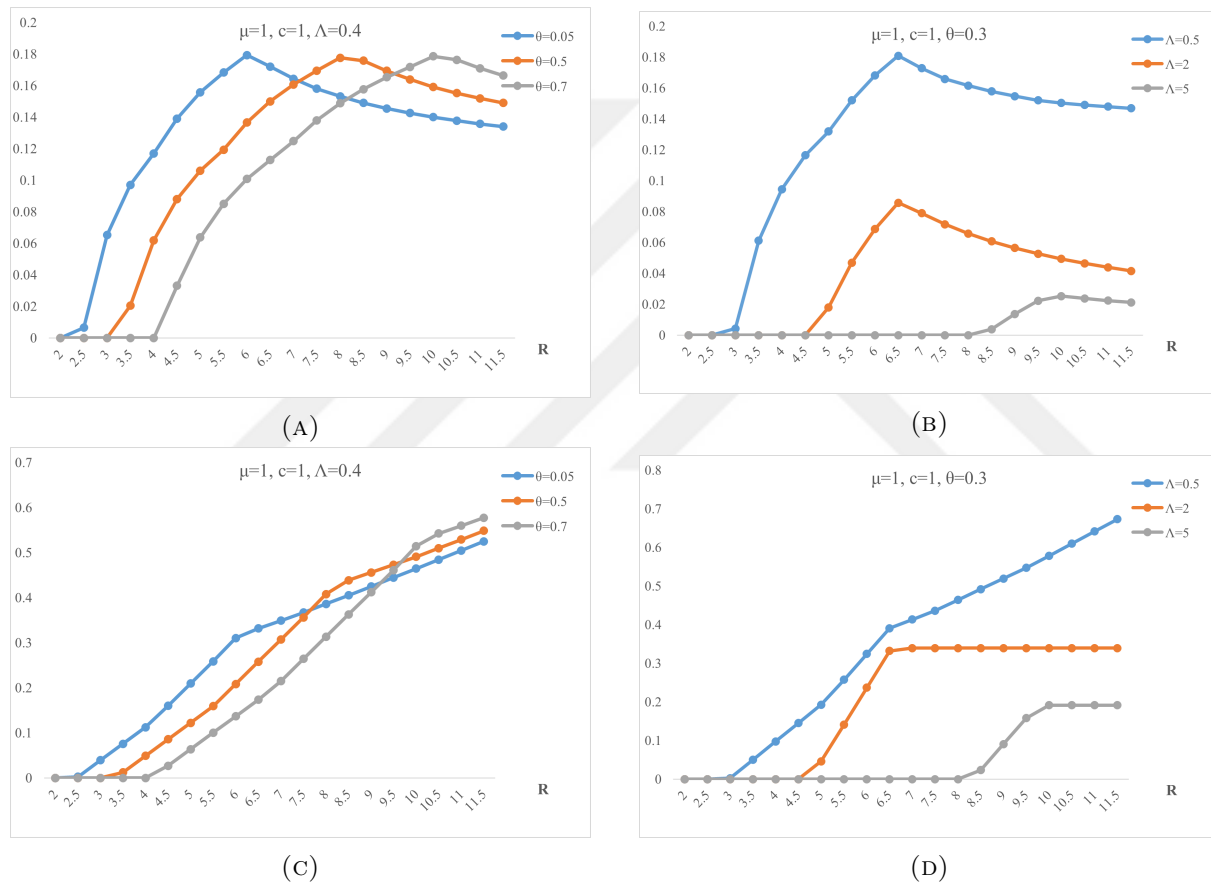


FIGURE 5.4: Relative (a, b)/Absolute change (c, d) in maximum extractable revenue with respect to R

different values for Λ and θ , the equilibrium arrival rates and socially optimal arrival rates are gathered in the following tables. For better perception, the arrival rates are divided to two cases compared to the service rate μ .

TABLE 5.7: Equilibrium Arrival rate for risk-averse customers given $\Lambda < \mu$

λ_{RA}^e	Λ								
	0.00001	0.01	0.02	0.04	0.06	0.08	0.1	0.14	0.16
0.001	0.00001	0.01000	0.02000	0.04000	0.06000	0.08000	0.10000	0.13328	0.13328
0.1	0.00001	0.01000	0.02000	0.04000	0.06000	0.08000	0.10000	0.12821	0.12821
0.2	0.00001	0.01000	0.02000	0.04000	0.06000	0.08000	0.10000	0.12283	0.12283
0.4	0.00001	0.01000	0.02000	0.04000	0.06000	0.08000	0.10000	0.11135	0.11135
0.6	0.00001	0.01000	0.02000	0.04000	0.06000	0.08000	0.09889	0.09889	0.09889
0.8	0.00001	0.01000	0.02000	0.04000	0.06000	0.08000	0.08552	0.08552	0.08552
θ 1	0.00001	0.01000	0.02000	0.04000	0.06000	0.07128	0.07128	0.07128	0.07128
1.2	0.00001	0.01000	0.02000	0.04000	0.05624	0.05624	0.05624	0.05624	0.05624
1.4	0.00001	0.01000	0.02000	0.04000	0.04046	0.04046	0.04046	0.04046	0.04046
1.6	0.00001	0.01000	0.02000	0.02404	0.02404	0.02404	0.02404	0.02404	0.02404
1.87	0.00001	0.00096	0.00096	0.00096	0.00096	0.00096	0.00096	0.00096	0.00096
1.89	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

In table 5.7 the equilibrium arrival rate for risk-averse customers, λ_{RA}^e is gathered according to different values of the risk-aversion degree, θ and potential arrival rate, Λ , for $\Lambda < \mu$. as it could be noticed, for low levels of risk-aversion degree, θ there exists a positive equilibrium arrival rate, λ_{RA}^e but for higher risk-aversion degrees the equilibrium arrival rate is zero. Furthermore, for a constant level of θ , λ_{RA}^e is non-decreasing in Λ . In addition, for a constant level of θ , for $\mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} \geq \Lambda$, λ_{RA}^e equals Λ and for $\mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}} \leq \Lambda$, the equilibrium arrival rate is constant and equal to $\mu - \frac{c\theta}{1-e^{-\theta(r-\tau)}}$.

In table 5.8 the equilibrium arrival rate for risk-averse customers, λ_{RA}^e is gathered according to different values of the risk-aversion degree, θ and potential arrival rate, Λ , given $\Lambda \geq \mu$. as it could be noticed, for low levels of risk-aversion degree, θ there exists a

TABLE 5.8: Equilibrium Arrival rate for risk-averse customers given $\Lambda \geq \mu$

λ_{RA}^e	Λ								
	0.2	0.22	0.24	0.26	0.28	0.3	0.32	0.36	1
0.001	0.13328	0.13328	0.13328	0.13328	0.13328	0.13328	0.13328	0.13328	0.13328
0.1	0.12821	0.12821	0.12821	0.12821	0.12821	0.12821	0.12821	0.12821	0.12821
0.2	0.12283	0.12283	0.12283	0.12283	0.12283	0.12283	0.12283	0.12283	0.12283
0.4	0.11135	0.11135	0.11135	0.11135	0.11135	0.11135	0.11135	0.11135	0.11135
0.6	0.09889	0.09889	0.09889	0.09889	0.09889	0.09889	0.09889	0.09889	0.09889
0.8	0.08552	0.08552	0.08552	0.08552	0.08552	0.08552	0.08552	0.08552	0.08552
θ 1	0.07128	0.07128	0.07128	0.07128	0.07128	0.07128	0.07128	0.07128	0.07128
1.2	0.05624	0.05624	0.05624	0.05624	0.05624	0.05624	0.05624	0.05624	0.05624
1.4	0.04046	0.04046	0.04046	0.04046	0.04046	0.04046	0.04046	0.04046	0.04046
1.6	0.02404	0.02404	0.02404	0.02404	0.02404	0.02404	0.02404	0.02404	0.02404
1.87	0.00096	0.00096	0.00096	0.00096	0.00096	0.00096	0.00096	0.00096	0.00096
1.89	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

positive equilibrium arrival rate, λ_{RA}^e but for higher risk-aversion degrees the equilibrium arrival rate is zero. In addition, for a constant level of θ , λ_{RA}^e is constant and equals

$$\mu - \frac{\theta c}{1 - e^{-\theta(r-\tau)}}.$$

TABLE 5.9: Socially desired arrival rate for risk-averse customers given $\Lambda < \mu$

Λ								
0.00001	0.01	0.02	0.04	0.06	0.08	0.1	0.14	0.16
0.00001	0.01000	0.02000	0.04000	0.06000	0.08000	0.08453	0.08453	0.08453

In table 5.9, the Socially desired arrival rate for risk-averse customers, λ^* , given $\Lambda < \mu$ is gathered. As it could be noticed, λ^* is non-decreasing in Λ . Furthermore, for $\mu - \sqrt{\frac{c\mu}{r}} \geq \lambda$, the socially desired arrival rate, λ^* , equals Λ and for $\mu - \sqrt{\frac{c\mu}{r}} \leq \Lambda$, λ^* equals $\mu - \sqrt{\frac{c\mu}{r}}$.

In table 5.10, the Socially desired arrival rate for risk-averse customers, λ^* , given $\Lambda \geq \mu$ is gathered. As it could be noticed, λ^* is constant in Λ and equals $\mu - \sqrt{\frac{c\mu}{r}}$.

In table 5.11, the regulating price imposed for the sake of filling the gap between the equilibrium arrival rate and socially optimal arrival rate, is gathered, given $\Lambda < \mu$. As it could be noticed, for cases which the equilibrium arrival rate, λ_{RA}^e , already equals the socially optimal arrival rate, λ^* , the regulating price is zero. Furthermore, for cases which λ_{RA}^e exceeds λ^* , a positive price is imposed, and when λ_{RA}^e has remained below λ^* , a negative price (considered as a subsidy) is imposed to incentivize customers to join. More surprisingly, when the risk-aversion degree is high than a specified threshold presented in theorem 5.3.1, even imposing a negative price is not encouraging enough for customers to join more. Therefore, for high levels of risk-aversion degree, θ , a regulating price does not exist and maybe other mechanisms should be pursued by the social welfare planner.

TABLE 5.12: Regulating price for risk-averse customers given $\Lambda \geq \mu$

Price	Λ								
	0.2	0.22	0.24	0.26	0.28	0.3	0.32	0.36	1
0.001	0.6336	0.6336	0.6336	0.6336	0.6336	0.6336	0.6336	0.6336	0.6336
0.1	0.5942	0.5942	0.5942	0.5942	0.5942	0.5942	0.5942	0.5942	0.5942
0.2	0.5490	0.5490	0.5490	0.5490	0.5490	0.5490	0.5490	0.5490	0.5490
0.4	0.4368	0.4368	0.4368	0.4368	0.4368	0.4368	0.4368	0.4368	0.4368
0.6	0.2781	0.2781	0.2781	0.2781	0.2781	0.2781	0.2781	0.2781	0.2781
0.8	0.0246	0.0246	0.0246	0.0246	0.0246	0.0246	0.0246	0.0246	0.0246
θ 1	-0.5101	-0.5101	-0.5101	-0.5101	-0.5101	-0.5101	-0.5101	-0.5101	-0.5101
1.2	IM	IM	IM	IM	IM	IM	IM	IM	IM
1.4	IM	IM	IM	IM	IM	IM	IM	IM	IM
1.6	IM	IM	IM	IM	IM	IM	IM	IM	IM
1.87	IM	IM	IM	IM	IM	IM	IM	IM	IM
1.89	IM	IM	IM	IM	IM	IM	IM	IM	IM

In table 5.12, the regulating price imposed for the sake of filling the gap between the equilibrium arrival rate and socially optimal arrival rate, is gathered, given $\Lambda \geq \mu$. As it could be noticed, in contrast to table 5.11, cases which the equilibrium arrival rate, λ_{RA}^e , already equals the socially optimal arrival rate, λ^* , does not exist. Furthermore,

for cases which λ_{RA}^e exceeds λ^* , a positive price is imposed, and when λ_{RA}^e has remained below λ^* , a negative price (considered as a subsidy) is imposed to incentivize customers to join. More surprisingly, when the risk-aversion degree is high than a specified threshold presented in theorem 5.3.1, even imposing a negative price is not encouraging enough for customers to join more. Therefore, for high levels of risk-aversion degree, θ , a regulating price does not exist and maybe other mechanisms should be pursued by the social welfare planner.

5.6 Conclusion

In this chapter, we presented an M/M/1 queueing system with homogeneous risk-averse strategic customers and investigated the proposed framework under both the observable setting and the unobservable counterpart. In the observable case we addressed the threshold for joining with risk-averse strategic customers. We proved that the threshold, n_e is non-increasing in θ . Furthermore, we obtained the effective arrival rate for the observable case and proved that it is decreasing in θ . Afterwards, we solved the revenue maximization problem under static and dynamic pricing and illustrated the implications of dynamic pricing. In the unobservable case, we characterized the equilibrium arrival rate for risk-averse strategic customers and proved that it is decreasing in the risk-aversion degree θ . Then we presented a pricing scheme for the social welfare maximizer to reach the socially desired arrival rate. We proved that with risk-averse strategic customers in an M/M/1 queue, unlike the case with risk-neutral customers, there is a possibility of charging negative prices (a subsidy) to meet the socially desirable arrival rate.

Chapter 6

A Queueing System with Heterogeneous Risk-Averse Strategic Customers

In this chapter, similar to chapter 5, we will provide a modeling framework to capture the equilibrium joining probabilities in an M/M/1 queue, with the novelty that customers are supposed to be heterogeneous rather than homogeneous. We suppose that risk-averse strategic customers are being served in an M/M/1 queueing system, where customers are supposed to be heterogeneous in their risk-sensitivity degree. In this regard, there are some papers which have relaxed the assumption of customer homogeneity by letting the service rewards to be random, Larsen, 1998, Mendelson, 1985 or by letting the waiting cost to be random, Gavirneni and Kulkarni, 2016. However, there are some papers which investigate the implications of customer heterogeneity by letting the service reward and waiting cost to be random simultaneously, Afeche and Pavlin, 2016, Nazerzadeh and Randhawa, 2018, Zhou and Huang, 2015 and Bueker, Kim, and Kim, 2020. The remainder of this chapter is outlined as follows. In section 6.1, the the model associated with the risk-averse heterogeneous customers in an M/M/1 queue is presented. In section 6.2, the equilibrium strategy of customers is characterized. Profit maximization and social optimization are discussed in section 6.3 and finally the numerical examples connected

with our model are gathered in section 6.4.

6.1 The Model

In this model, customers are heterogeneous in their risk sensitivity degree, $\theta > 0$, which follows a general distribution with cumulative distribution $F(\theta)$. The risk sensitivity degree of customers is private information, and an arbitrary customer knows only his own θ , and about other customers he only knows that their risk-sensitivity degree follows $F(\theta)$. Similar to chapter 5, we will characterize our model under two settings, Observable and unobservable.

6.1.1 Observable case

We suppose the arriving customer is able to observe the number of customers in the queue, n_q . Therefore, similar to 5.1.1 the expected utility of an arbitrary customer who decides joining with risk-sensitivity degree θ is calculated as

$$\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q] = 1 - e^{-\theta(r-\tau)} \left(\frac{\mu}{\mu - c\theta} \right)^{n_q+1}, \quad \mu > c\theta \quad (6.1.1)$$

Hence, a customer with risk-sensitivity degree θ , who observes n_q customers in the system upon arrival, decides joining if $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q] \geq 0$ and balks if $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q] < 0$.

In the following Proposition 6.1.1, we will prove some features of the $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]$ which enables us to define a threshold θ_O^* for the Observable case.

Proposition 6.1.1 *Consider an observable M/M/1 system where an arbitrary customer with risk-sensitivity θ , observes n_q customers in the system upon arrival, with r, μ and c as the service characteristics. Then we have the following.*

(i) $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]$ is concave with respect to θ .

(ii) a customer with risk-sensitivity degree $\frac{\mu}{c} - \frac{(n_q+1)}{r-\tau}$ gains the highest $\mathbb{E}[u(r - \tau - cX)]$.

Proof of Proposition 6.1.1.

(i) We will differentiate $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]$ with respect to θ

$$\frac{d\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]}{d\theta} = (r - \tau)e^{-\theta(r-\tau)} \left(\frac{\mu}{\mu - c\theta}\right)^{n_q+1} - c(n_q+1)\mu^{n_q+1}(\mu - c\theta)^{-(n_q+2)} e^{-\theta(r-\tau)}$$

Next we obtain the second derivative of $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]$ with respect to θ

$$\begin{aligned} \frac{d^2\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]}{d\theta^2} &= -e^{-\theta(r-\tau)} \left(\frac{\mu}{\mu - c\theta}\right)^{n_q+1} \left[(r - \tau)^2 - \frac{2cr(n_q + 1)}{\mu - c\theta} + \frac{c^2(n_q + 1)(n_q + 2)}{(\mu - c\theta)^2} \right] \\ &= -e^{-\theta(r-\tau)} \left(\frac{\mu}{\mu - c\theta}\right)^{n_q+1} \left[(r - \tau)^2 - \frac{2cr(n_q + 1)}{\mu - c\theta} + \frac{c^2(n_q + 1)^2}{(\mu - c\theta)^2} + \frac{c^2(n_q + 1)}{(\mu - c\theta)^2} \right] \\ &= -e^{-\theta(r-\tau)} \left(\frac{\mu}{\mu - c\theta}\right)^{n_q+1} \left[\left(r - \tau - \frac{c(n_q + 1)}{\mu - c\theta} \right)^2 + \frac{c^2(n_q + 1)}{(\mu - c\theta)^2} \right] < 0 \end{aligned}$$

(ii) By setting the first derivative equal to zero we obtain the stationary point

$$\theta' = \frac{\mu}{c} - \frac{(n_q + 1)}{r - \tau}$$

■

Theorem 6.1.1 *In an observable M/M/1 system with heterogeneous risk-averse customers and the above-mentioned characteristics, given $\frac{\mu}{c} > \frac{(n_q+1)}{(r-\tau)}$, for each $\tau \in \mathbb{R}$ and $n_q \in \mathbb{Z}^+$, there exists a unique non-zero $\theta \in (0, \infty)$ such that*

$$1 - e^{-\theta(r-\tau)} \left(\frac{\mu}{\mu - c\theta}\right)^{n_q+1} = 0$$

Let's denote this solution with θ_O^* , then we have the following:

Customers with risk sensitivity degree less than θ_O^* would join and otherwise they would balk.

Proof of Theorem 6.1.1. According to Proposition 6.1.1, $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]$ is concave with respect to θ , and $\theta = 0$ is always a root of $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]$ and for $\frac{\mu}{c} > \frac{(n_q+1)}{(r-\tau)}$, there is a unique non-zero root, say θ_O^* . Since $\mathbb{E}[u(r - \tau - cX) \mid \theta, n_q]$ is negative after θ_O^* , customers with risk-sensitivity less than θ_O^* will join and will balk with risk-sensitivity greater than θ_O^* . ■

Note that the number of customers in such a system would never exceed $\lfloor \frac{(r-\tau)\mu}{c} \rfloor$. The reason is that if a customer with a very low risk-aversion degree (Risk-neutral) arrives and upon arrival wants to decide whether to join or balk, he would do his trade-off based on a linear utility function, $r - \tau - cX$.

6.1.2 Unobservable case

In the unobservable case, customers are heterogeneous in the sense that their risk sensitivity degree, $\theta > 0$, follows a general distribution with cumulative distribution $F(\theta)$. The risk sensitivity degree of customers is private information, and an arbitrary customer knows only his own θ , and about other customers he only knows that their risk-sensitivity degree follows a general distribution, $F(\theta)$. Therefore, similar to (5.1.1), the net benefit of a joining customer with risk-sensitivity degree, θ is

$$u(r - \tau - cX \mid \theta) = 1 - e^{-\theta(r-\tau-cX)} \quad (6.1.2)$$

Where, X , similar to 5.1.2, is the potential waiting time of an arbitrary customer and according to the M/M/1 queueing standards it follows an exponential distribution with rate $\mu - \lambda_q$, where $\lambda_q \in [0, \Lambda]$ is the effective rate of the customers joining. Actually, λ_q

is the effective rate of customers joining the queue based on the individual decision of customers. Furthermore, we define λ_e as the equilibrium joining rate which occurs when customers adopt an equilibrium strategy. In this case an arriving customer's strategy, s , is defined by a mapping from $\theta : (0, \infty)$ to $[0, 1]$. To be more precise, $s(\theta)$ is the joining probability of an arbitrary customer with realized risk sensitivity degree θ . We will calculate the expected value of an arbitrary customer (with realized risk sensitivity degree θ) who decides to join as

$$\mathbb{E}[u(r - \tau - cX) \mid \theta] = 1 - e^{-\theta(r-\tau)} \mathbb{E} [e^{c\theta X}] \quad (6.1.3)$$

As we mentioned above X represents the potential waiting time of an arbitrary customer and follows an exponential distribution with rate $\mu - \lambda_q$. Therefore, similar to chapter 5, by using the moment generating function for X we can rewrite the expected value of an arbitrary customer with realized risk-sensitivity θ as

$$\mathbb{E}[u(r - \tau - cX) \mid \theta] = \begin{cases} 1 - e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} & \text{if } \mu - \lambda_q - c\theta > 0 \\ -\infty & \text{if } \mu - \lambda_q - c\theta \leq 0. \end{cases} \quad (6.1.4)$$

For $\mu - \lambda_q - c\theta > 0$, let's represent the $\mathbb{E}[u(r - \tau - cX) \mid \theta]$ as a function of θ and denote it with $M(\theta)$. Then, we have the following Proposition 6.1.2.

Proposition 6.1.2

- (i) $M(\theta)$ is concave with respect to θ .
- (ii) Given $r > \tau + \frac{c}{\mu - \lambda_q}$, $M(\theta)$ has a positive root denoted by θ^q .

Proof of Proposition 6.1.2.

(i) We first find the derivative of $M(\theta)$ with respect to θ :

$$\frac{dM(\theta)}{d\theta} = (r - \tau)e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} - e^{-\theta(r-\tau)} \frac{c[\mu - \lambda_q]}{(\mu - \lambda_q - c\theta)^2}$$

By setting the derivative equal to zero we will obtain the stationary point:

$$\theta' = \frac{\mu - \lambda_q}{c} - \frac{1}{r - \tau}$$

Now, we obtain the second derivative of $M(\theta)$:

$$\begin{aligned} \frac{d^2M(\theta)}{d\theta^2} &= -(r - \tau)^2 e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} + \frac{2c(r - \tau)(\mu - \lambda_q)e^{-\theta(r-\tau)}}{(\mu - \lambda_q - c\theta)^2} - \frac{2c^2(\mu - \lambda_q)e^{-\theta(r-\tau)}}{(\mu - \lambda_q - c\theta)^3} \\ &= \left[-(r - \tau)^2 + \frac{2c(r - \tau)}{\mu - \lambda_q - c\theta} + \frac{-2c^2}{(\mu - \lambda_q - c\theta)^2} \right] e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} \\ &= - \left[(r - \tau)^2 + \frac{2c(r - \tau)}{\mu - \lambda_q - c\theta} + \frac{c^2}{(\mu - \lambda_q - c\theta)^2} + \frac{c^2}{(\mu - \lambda_q - c\theta)^2} \right] e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} \\ &= - \left[\left(r - \tau - \frac{c}{\mu - \lambda_q - c\theta} \right)^2 + \frac{c^2}{(\mu - \lambda_q - c\theta)^2} \right] e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} \end{aligned}$$

The second derivative of $M(\theta)$ is negative with respect to θ for $\mu - \lambda_q - c\theta > 0$.

(ii) According to $M(\theta)$, $M(0)=0$ and since we proved that $M(\theta)$ is concave, for $r > \tau + \frac{c}{\mu - \lambda_q}$ there exists another root for $M(\theta)$. We denote this positive root by θ^a . ■

In section 6.2, we will address the equilibrium joining probabilities in the unobservable case with heterogeneous risk-averse customers.

6.2 Equilibrium Strategies of Customers

According to proposition 6.1.2, we can have a better perception of $\mathbb{E}[u(r - \tau - cX) | \theta]$. For $\mu - \lambda_q - c\theta > 0$, $\mathbb{E}[u(r - \tau - cX) | \theta]$ is a concave function and for $\mu - \lambda_q - c\theta \leq 0$ its value is $-\infty$. These results imply that $\mathbb{E}[u(r - \tau - cX) | \theta]$ is a uni-modal function with θ' as the maximum. After characterizing $\mathbb{E}[u(r - \tau - cX) | \theta]$, we are able to define an

equilibrium strategy s as following. For a given τ , a strategy s is an equilibrium strategy if the following holds for any θ .

$$\begin{cases} s(\theta) = 1 & \text{if } 1 - e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} > 0 \\ s(\theta) = 0 & \text{if } 1 - e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} < 0 \text{ or } \mu - \lambda_q - c\theta \leq 0 \\ 0 < s(\theta) < 1 & \text{if } 1 - e^{-\theta(r-\tau)} \frac{\mu - \lambda_q}{\mu - \lambda_q - c\theta} = 0. \end{cases} \quad (6.2.1)$$

Corollary 6.2.1 *For each $\tau \in \mathbb{R}$, there exists a unique non-zero $\theta \in (0, \infty)$ such that*

$$1 - e^{-\theta(r-\tau)} \frac{\mu - \Lambda F(\theta)}{\mu - \Lambda F(\theta) - c\theta} = 0$$

Let's denote this solution with θ^ , then we have the following. A strategy s is an equilibrium for the service price τ , if we have the following*

$$\begin{cases} s(\theta) = 1 & \text{if } \theta < \theta^* \\ s(\theta) = 0 & \text{if } \theta > \theta^* \\ 0 < s(\theta) < 1 & \text{if } \theta = \theta^*. \end{cases} \quad (6.2.2)$$

Proof of Corollary 6.2.1. After some manipulation, $1 - e^{-\theta(r-\tau)} \frac{\mu - \Lambda F(\theta)}{\mu - \Lambda F(\theta) - c\theta} = 0$ could be written as

$$\theta = \frac{1}{r - \tau} \ln \frac{\mu - \Lambda F(\theta)}{\mu - \Lambda F(\theta) - c\theta}$$

For each τ , we define a function $L_\tau(x) : [0, \infty] \rightarrow [0, \infty]$ as

$$L_\tau(x) = \begin{cases} \frac{1}{r-\tau} \ln \frac{\mu - \Lambda F(x)}{\mu - \Lambda F(x) - cx} & \text{if } \mu - \Lambda F(x) - cx > 0 \\ 0 & \text{if } \mu - \Lambda F(x) - cx \leq 0. \end{cases} \quad (6.2.3)$$

We will show that $L_\tau(x)$ is an increasing function for $\mu - \Lambda F(x) - cx > 0$:

$$\frac{dL_\tau(x)}{dx} = \frac{1}{r - \tau} \frac{\mu - \Lambda F(x) - cx}{\mu - \Lambda F(x)} \frac{c[\mu - \Lambda F(x) + x\Lambda f(x)]}{(\mu - \Lambda F(x) - cx)^2} > 0$$

Therefore, $1 - e^{-\theta(r-\tau)} \frac{\mu - \Lambda F(\theta)}{\mu - \Lambda F(\theta) - c\theta}$ has a unique solution.

■

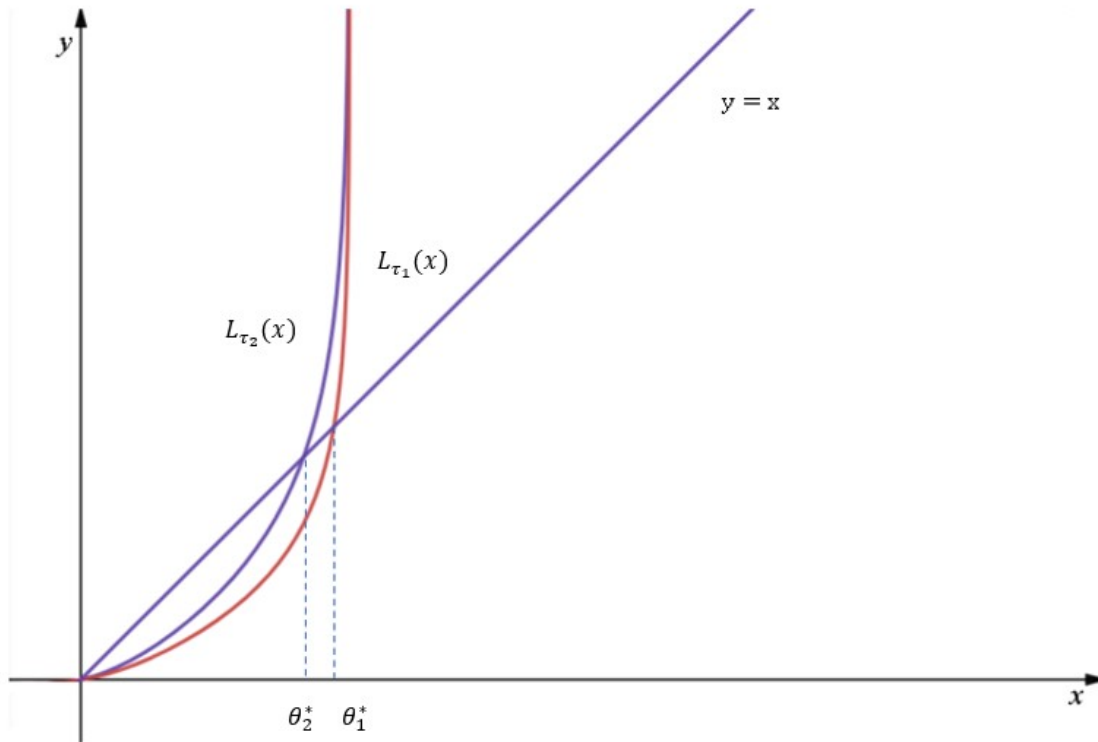


FIGURE 6.1: The graph of $L_\tau(x)$ for different values of τ

Theorem 6.2.1 θ^* is decreasing in τ .

Proof of Theorem 6.2.1. As we proved in corollary 6.2.1, If a θ^* exists, it must be unique. By referring to figure 6.1 it is observed that for $\tau_2 > \tau_1$, $L_{\tau_2}(x) > L_{\tau_1}(x)$. Therefore, if $\tau_2 > \tau_1$, $\theta_2^* < \theta_1^*$.

■

6.3 Profit Maximization and Social Optimization

As in 5.4, the objective of the service provider is to maximize the profit rate.

$$\Pi(\tau) = \Pi(\theta^*) = \lambda^e \tau = \Lambda F(\theta^*) \tau = \Lambda F(\theta^*) \left(r - \frac{1}{\theta^*} \ln \frac{\mu - \Lambda F(\theta^*)}{\mu - \Lambda F(\theta^*) - c\theta^*} \right) \quad (6.3.1)$$

On the other hand, the social welfare planner is interested in solving

$$\lambda^* = \arg \max_{0 \leq \lambda \leq \Lambda} \left\{ \lambda \left(r - \frac{c}{\mu - \lambda} \right) \right\}$$

After defining the equilibrium strategies of customers, revenue maximization problem and social welfare maximization problem, we provide some numerical examples in section 6.4.

6.4 Numerical Examples

In this subsection, we provide some numerical examples regarding a service system modeled as a queue with risk-averse heterogeneous customers. In this regard, we present numerical examples on the equilibrium threshold, effective arrival rate, optimal price, maximum revenue, socially desired arrival rate and maximum social welfare, for different value of r and Λ .

TABLE 6.1: Threshold θ^* for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution

θ^*	Λ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	0.565	0.5191	0.478	0.4412	0.4084	0.3792	0.3531	0.3298	0.309
r 4	0.6437	0.5954	0.5515	0.5117	0.4758	0.4435	0.4143	0.3881	0.3644
5	0.6937	0.644	0.5985	0.5571	0.519	0.485	0.454	0.426	0.401

Table 6.1, gathers the threshold θ^* for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution.

TABLE 6.2: Equilibrium Arrival Rate, $\lambda_e(\tau^*)$ for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution

$\lambda_e(\tau^*)$	Λ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	0.04316	0.08099	0.11399	0.14269	0.16764	0.18935	0.20825	0.22475	0.23924
r 4	0.04747	0.08973	0.12717	0.16021	0.18931	0.21493	0.23744	0.25732	0.27485
5	0.05003	0.09496	0.13511	0.17085	0.20244	0.23058	0.25544	0.27751	0.29731

Table 6.2, gathers equilibrium arrival rates, $\lambda_e(\tau^*)$ for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution.

TABLE 6.3: Optimal Price, τ^* , for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution

τ^*	Λ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	1.41984	1.39712	1.37774	1.36162	1.34811	1.33679	1.32756	1.3199	1.31338
r 4	2.25024	2.21702	2.18807	2.16325	2.14192	2.12352	2.10831	2.09498	2.084
5	3.1113	3.06939	3.03235	2.9996	2.97316	2.94806	2.92734	2.90919	2.89146

Table 6.3, pertains to the optimal price, τ^* , for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution.

TABLE 6.4: Maximum Revenue, $\Pi(\tau^*)$ for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution

$\Pi(\tau^*)$	Λ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	0.06129	0.11315	0.15705	0.19429	0.226	0.25313	0.27646	0.29664	0.31421
r 4	0.10681	0.19894	0.27827	0.34657	0.40548	0.45641	0.50059	0.53909	0.57278
5	0.15565	0.29148	0.4097	0.51249	0.60189	0.67977	0.74777	0.80732	0.85967

Table 6.4, gathers the maximum revenue, $\Pi(\tau^*)$ for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution.

TABLE 6.5: Socially desired arrival rate for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution

		Λ								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
r	λ^*									
	3	0.1	0.2	0.3	0.4	0.42265	0.42265	0.42265	0.42265	0.42265
	4	0.1	0.2	0.3	0.4	0.5	0.5	0.5	0.5	0.5
	5	0.1	0.2	0.3	0.4	0.5	0.55279	0.55279	0.55279	0.55279

Table 6.5, gathers the socially desired arrival rate for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution.

TABLE 6.6: Maximum Social Welfare for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution

		Λ								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
r	SW^*									
	3	0.1889	0.35	0.4714	0.5333	0.5359	0.5359	0.5359	0.5359	0.5359
	4	0.2889	0.55	0.7714	0.9333	1	1	1	1	1
	5	0.3889	0.75	1.0714	1.3333	1.5	1.5279	1.5279	1.5279	1.5279

Table 6.6, pertains to the maximum social welfare for a service system with $c = 1$, $\mu = 1$ and heterogeneous customers with Expo(1) distribution.

6.5 Conclusion

In this chapter, we characterized a service system as a queue with heterogeneous risk-averse strategic customers and derived the equilibrium joining strategy and effective arrival rate of customers joining in the equilibrium. We proved that the equilibrium arrival rate is decreasing in τ . Furthermore, we defined the revenue maximization problem and social welfare maximization problem and provided some numerical examples for the equilibrium

joining threshold, equilibrium arrival rate, optimal revenue maximizing price and also socially desired arrival rate.



Chapter 7

Conclusions

In this chapter, first we provide the summary of this thesis and point out the related main findings, then some future research directions will be introduced in the next section. Apart from the static models presented in chapters 3 and 4 and the M/M/1 strategic queueing framework in chapters 5 and 6, in this chapter, we will discuss some possible alternatives for modeling static service systems namely, "customers possessing linear utility functions" and "boundedly rational customers in service systems".

7.1 Summary and Conclusion

In this thesis, we first presented a class of static models to address the equilibrium joining behaviour of customers in a service system with strategic risk-averse customers. In such a modeling framework, although we ignored the dynamics of the service system such as random arrivals and random service times, but we utilized other attributes of the service system such as externalities, reward and risk-aversion to address the equilibrium joining behaviour. Then we extended the same static framework to the case with heterogeneous risk-averse strategic customers, where our results indicated the challenging analysis of asymmetric equilibria. At first glance, the framework might seem inefficient, since it ignores the ongoing queueing evolution. However, after deriving the equilibrium joining strategies under the static setting, the structure of the equilibria seems pretty similar to

the equilibrium joining strategies in the queueing framework. The intuition behind this similarity might stem from the fact that while analyzing strategic queueing frameworks, the underlying assumption is characterizing the queueing system in the long-run, which has an implication of a snapshot of the queueing system after some considerable amount of time has been spent. Similarly, in our static modeling framework, we implicitly ignore the queueing evolution, but indeed incorporate the service experience by utilizing a service experience variable X_n .

Furthermore, After addressing the equilibrium joining probabilities in a service system modeled as a M/M/1 queue with risk-averse strategic customers, and comparing it to the socially desired arrival rate, we noticed an interesting result. In contrast to the previous research papers which modeled a service system as an M/M/1 queue with risk-neutral customers, in our setting with risk-averse customers, the gap can between the socially desired arrival rate and equilibrium arrival rate can be filled with charging negative prices (indicating a subsidy).

7.2 Future Research Directions

Reviewing the literature related to service systems, it is understood that, a part of research in this area seeks to model customer behaviour. In particular, a significant portion of papers assume customers as fully rational entities who are maximizing their payoffs and accomplish their task flawlessly. However, this assumption seems not to be always true and could be relaxed by assuming customers are error-prone entities in the sense that they might miscalculate their expected utility of joining a service system. As a result, modeling customers as boundedly rational decision-makers has drawn great attention among researchers. Ren and Huang (2018) provide a classification of the applied approaches in modeling boundedly rational customers in the context of operations management. Canbolat (2020) investigates the implications of boundedly rational customers in a clearing

service system. The following research papers incorporate the concept of bounded rationality in their modeling frameworks. (Chen, Friedman, and Thisse, 1997), (Li, Guo, and Lian, 2017), (Li, Guo, and Lian, 2016), (Ren and Huang, 2018), (Huang, Allon, and Bassamboo, 2013), (Li et al., 2017), (Goeree, Holt, and Palfrey, 2020) and (Rubinstein, 1998). In the following subsection, we will provide some brief implications of bounded rationality in our proposed static service system with 2-homogeneous customers.

7.2.1 Boundedly Rational Customers in Service Systems

In theorem 3.1.1, we are supposing that each customer is able to calculate $E[u(r, X_n)]$ exactly. In a boundedly rational framework, we try to model the error-prone feature of customers by incorporating an error term, ϵ_n , in calculating $E[u(r, X_n)]$. By ϵ_n we mean the error which a customer has in calculating $E[u(r, X_n)]$ when n customers are in the service system. We suppose ϵ_n is a zero-mean random variable following a logistic distribution, as following:

$$F(x) = \frac{1}{1 + e^{-\frac{x}{\beta}}} \quad (7.2.1)$$

Here, β represents the level of rationality for a customer and $\beta > 0$. We suppose that ϵ_n is the same for all n and $\epsilon_n = \epsilon$. At this stage, we will define the joining probability of the first customer, p_1 based on the adopted strategy of the other customer, p_2 . We would refer back to equations 3.1.2 and 3.1.3 and calculate the expected utility of the first and second customer. In case of supposing customers are error-prone (boundedly rational), the expected utility of the first customer if the first customer joins the system with probability p_1 and the second customer with probability p_2 is

$$U_1(p_1, p_2) = p_1 [p_2(E[u_1(r, X_2)] + \epsilon) + (1 - p_2)(E[u_1(r, X_1)] + \epsilon)] + (1 - p_1)u_{1,0} \quad (7.2.2)$$

and for the second customer the expected utility is

$$U_2(p_1, p_2) = p_2 [p_1(E[u_2(r, X_2)] + \epsilon) + (1 - p_1)(E[u_2(r, X_1)] + \epsilon)] + (1 - p_2)u_{2,0} \quad (7.2.3)$$

After applying equations 4.1.6 and 4.1.7, the expected utilities of the first and second customer could be rewritten as:

$$U_1(p_1, p_2) = p_1 [f_2(p_2) + \epsilon] + u_{1,0} \quad (7.2.4)$$

$$U_2(p_1, p_2) = p_2 [f_2(p_1) + \epsilon] + u_{2,0} \quad (7.2.5)$$

Now, the logistic response for joining of the first customer is equal with:

$$P_1(Join) = \frac{1}{1 + e^{\frac{p_2 E[u(r, X_2)] + (1 - p_2) E[u(r, X_1)] - u_0}{-\beta}}} \quad (7.2.6)$$

And the logistic response for joining of the second customer is equal with:

$$P_2(Join) = \frac{1}{1 + e^{\frac{p_1 E[u(r, X_2)] + (1 - p_1) E[u(r, X_1)] - u_0}{-\beta}}} \quad (7.2.7)$$

Therefore, based on the externality type, the equilibrium joining strategies may change. We would first discuss a service system with no externalities. As we discussed in chapter 4, no externalities implies $E[u(r, X_1)] = E[u(r, X_2)]$. Therefore, in case of no externalities, equations 7.2.6 and 7.2.7 boil down to:

$$P_1(Join) = \frac{1}{1 + e^{\frac{E[u(r, X)] - u_0}{-\beta}}} \quad (7.2.8)$$

$$P_2(Join) = \frac{1}{1 + e^{\frac{E[u(r, X)] - u_0}{-\beta}}} \quad (7.2.9)$$

On the other hand, the logistic responses for joining of the first and second customer, 7.2.6 and 7.2.7 will be different for negative and positive externalities. With negative externalities, $f_2(p_2) = p_2 E[u(r, X_2)] + (1 - p_2) E[u(r, X_1)] - u_0$ is decreasing in p_2 , and with positive externalities $f_2(p_2)$ is increasing in p_2 . Therefore, in case of negative externalities, 7.2.6 and 7.2.7 are decreasing in p_2 and p_1 , respectively. However, with positive externalities, 7.2.6 and 7.2.7 are increasing in p_2 and p_1 , respectively.

To be more precise, we are interested in addressing the effects of bounded rationality in equilibrium joining strategies, the revenue of service provider and social welfare. For future research, we propose to investigate models in chapter 3 and chapter 4 in this framework.

Figure 7.1, illustrates Equilibrium joining probabilities for 2 boundedly rational customers with negative externalities, $E[u(r, X_2)] = -0.6$, $E[u(r, X_1)] = -0.2$, $\beta = 0.1$. As it could be observed, the symmetric equilibrium in this case is a positive symmetric value less than (0.5, 0.5). However, if customers were supposed to be fully rational, the equilibrium would be (0,0).

Figure 7.2, illustrates Equilibrium joining probabilities for 2 boundedly rational customers with positive externalities, $E[u(r, X_2)] = 0.2$, $E[u(r, X_1)] = 0.6$, $\beta = 0.1$. As it could be observed, the symmetric equilibrium in this case is a positive symmetric value close to (1,1). However, if customers were supposed to be fully rational, the equilibrium would be (1,1).

Figure 7.3 illustrates Equilibrium joining probabilities for 2 boundedly rational customers with positive externalities, $E[u(r, X_2)] = -0.5$, $E[u(r, X_1)] = 2$, $\beta = 0.1$. As it could be observed, the symmetric equilibrium in this case is a positive symmetric value close to (0.8, 0.8).

Figure 7.4 illustrates Equilibrium joining probabilities for 2 boundedly rational customers with positive externalities, $E[u(r, X_2)] = 0$, $E[u(r, X_1)] = 1$, $\beta = 0.1$. As it could be seen, the symmetric equilibrium in this figure is close to (0.8, 0.8). However, under fully

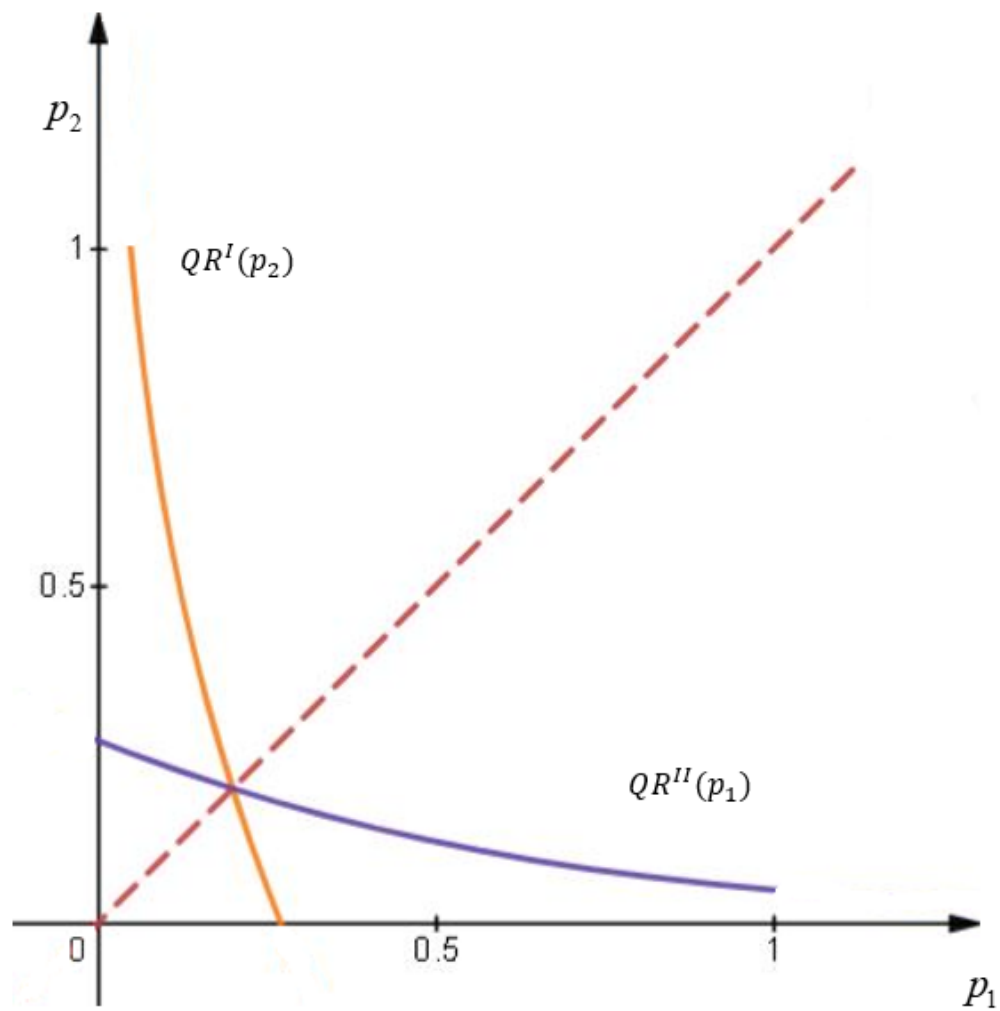


FIGURE 7.1: Equilibrium joining probabilities for 2 boundedly rational customers with negative externalities, $E[u(r, X_2)] = -0.6$, $E[u(r, X_1)] = -0.2$, $\beta = 0.1$

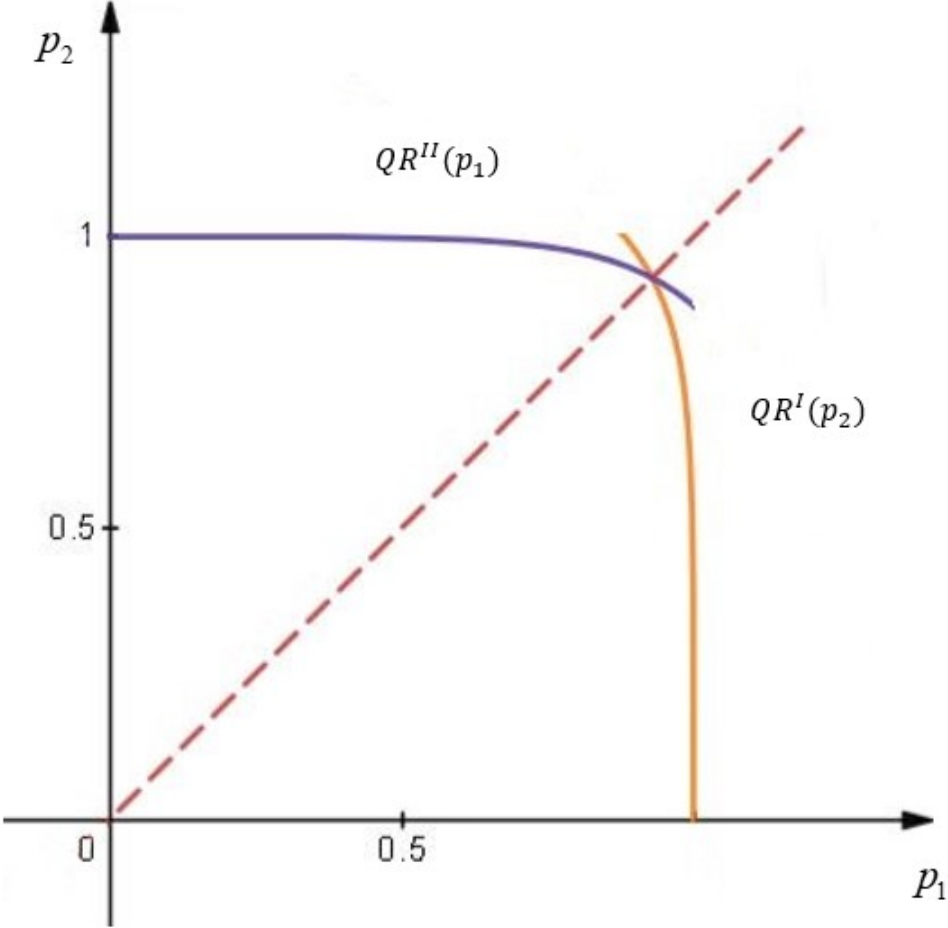


FIGURE 7.2: Equilibrium joining probabilities for 2 boundedly rational customers with negative externalities, $E[u(r, X_2)] = 0.2$, $E[u(r, X_1)] = 0.6$, $\beta = 0.1$

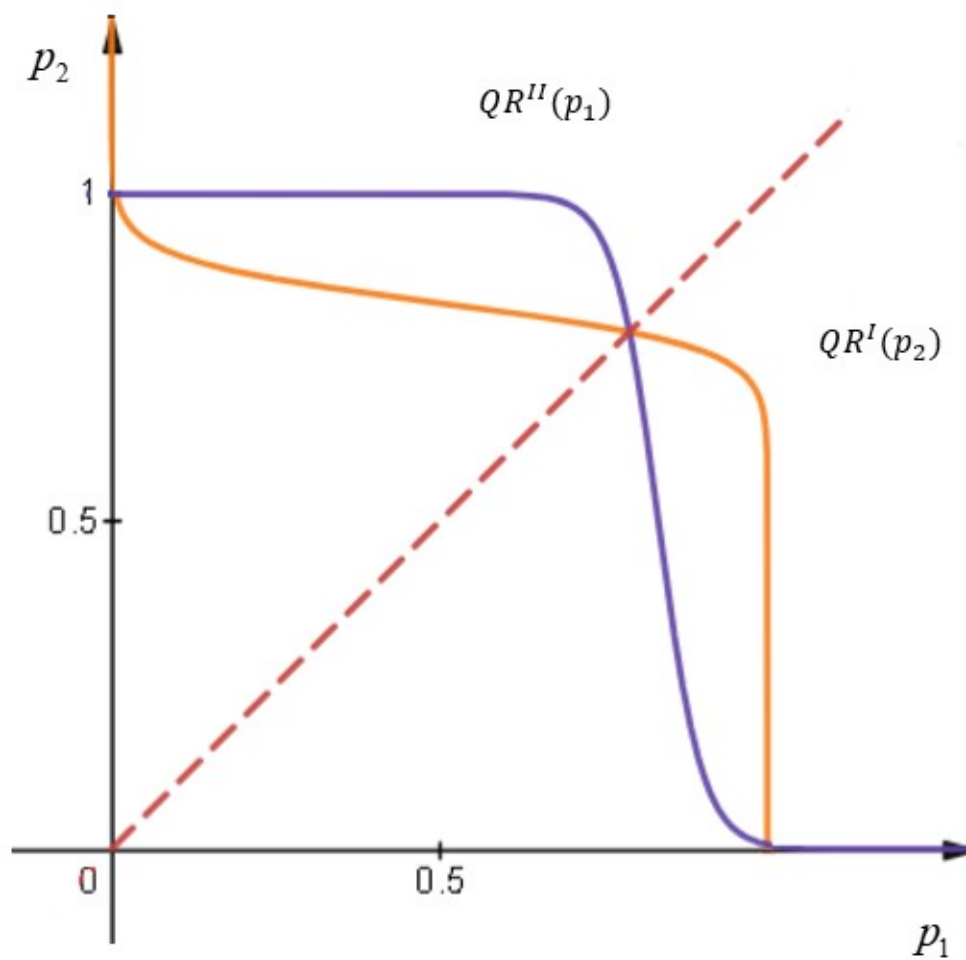


FIGURE 7.3: Equilibrium joining probabilities for 2 boundedly rational customers with negative externalities, $E[u(r, X_2)] = -0.5$, $E[u(r, X_1)] = 2$, $\beta = 0.1$

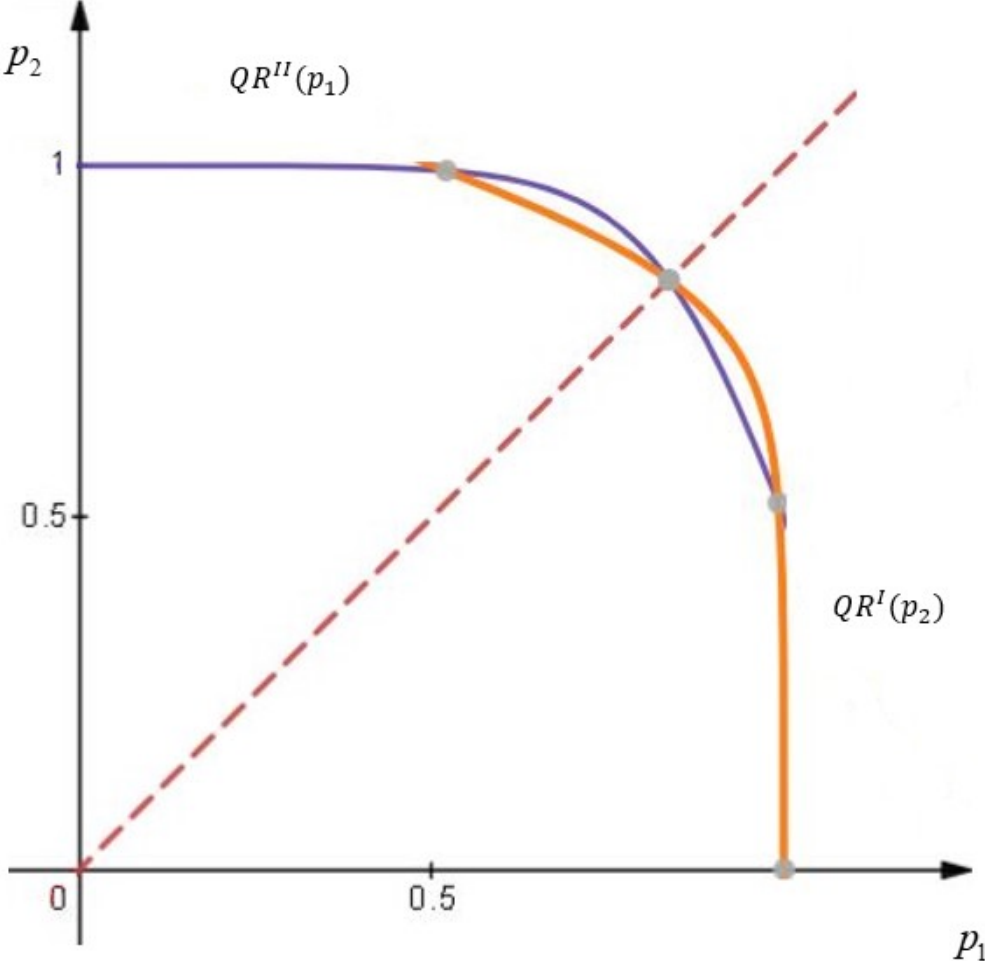


FIGURE 7.4: Equilibrium joining probabilities for 2 boundedly rational customers with negative externalities, $E[u(r, X_2)] = 0$, $E[u(r, X_1)] = 1$, $\beta = 0.1$

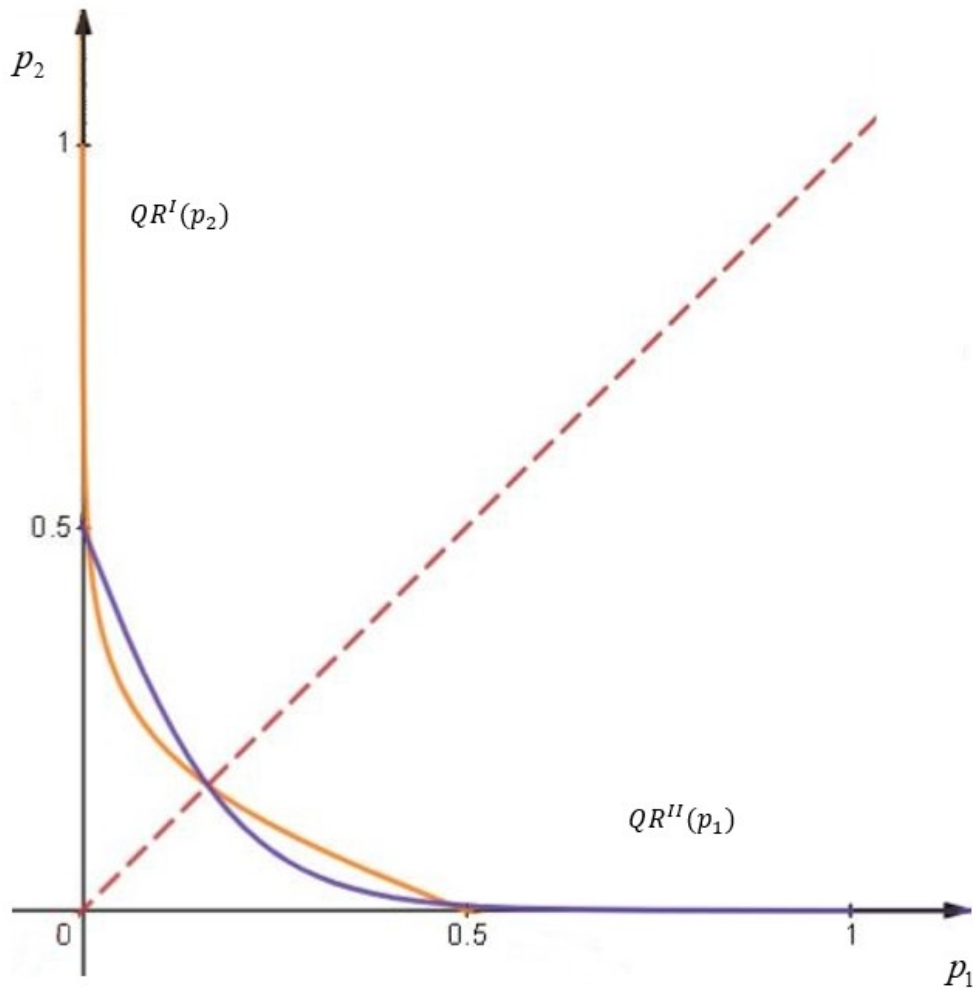


FIGURE 7.5: Equilibrium joining probabilities for 2 boundedly rational customers with negative externalities, $E[u(r, X_2)] = -1$, $E[u(r, X_1)] = 0$, $\beta = 0.1$

rational customer behavior the equilibrium would be $(1,1)$.

Figure 7.5, illustrates Equilibrium joining probabilities for 2 boundedly rational customers with negative externalities, $E[u(r, X_2)] = -1$, $E[u(r, X_1)] = 0$, $\beta = 0.1$. As it could be seen, the symmetric equilibrium in this figure is close to $(0,0)$. However, under full rationality, the equilibrium would be exactly $(0,0)$.

7.2.2 Customers Possessing Other Utility Functions

In section 3.4, we presented the exponential utility function and mentioned that it is categorized as a constant absolute risk averse (CARA) utility function. We took advantage of this utility function to model the preferences of our risk-averse customers both in the homogeneous case and heterogeneous case. However, in reality, it might be plausible that in service systems customers have an attitude to get less risk-averse as their waiting time in the service system decreases. By applying an exponential utility function, we assume that the risk-aversion degree of customers remains constant as their waiting time increases. To be capable of modeling customers with decreasing absolute risk-aversion features, we propose a Linex utility function which is in the class of DARA utility functions and might better address the customer behaviour in service systems. Linex utility functions are a convex combination of a linear utility function and an exponential utility function, as can be seen in equation 7.2.10 (Canbolat, 2019) and (Denuit, Eeckhoudt, and Schlesinger, 2013). We are interested in capturing the implications of applying a linex utility function. To clarify, what would be the effect of increasing α on the equilibrium joining strategies?

$$u(r, X_n) = (1 - \alpha)(r + X_n) - \alpha e^{-\theta(r+X_n)} \quad (7.2.10)$$

In future research we propose to carry out this extension to the models presented in chapter 3 and chapter 4.

Appendix A

A.1

Corollary A.1.1 *Symmetric Nash equilibria in two customer service systems with the same exponential utility function is as follows*

- (i) *If $e^{\theta r}(\mu_1 - \theta)^+ \leq \mu_1$ then $(p_1^*, p_2^*) = (0, 0)$ is a Nash equilibrium and in this case both customers balk.*
- (ii) *If $e^{\theta r}(\mu_2 - \theta)^+ \geq \mu_2$ then $(p_1^*, p_2^*) = (1, 1)$ is a Nash equilibrium and in this case both customers join.*
- (iii) *If $0 < p^* < 1$ provides $p \frac{\mu_2}{(\mu_2 - \theta)^+} + (1 - p) \frac{\mu_1}{(\mu_1 - \theta)^+} = e^{\theta r}$ then $(p_1^*, p_2^*) = (p^*, p^*)$ is a Nash equilibrium. In this case both customers joins with p^* .*

Proof. In theorem 3.1.1 we replace $E[u(r, X_n)]$ with $1 - e^{-\theta r} \frac{\mu_n}{(\mu_n - \theta)^+}$ and set u_0 equal to zero and then rearrange the inequalities. ■

In the remainder of this section, we assume that the degree of risk aversion θ is less than μ_1 and μ_2 and we rewrite Corollary A.1.1.

Corollary A.1.2 *Symmetric Nash equilibria in a two customer service systems with the same exponential utility function and $\mu_1 > \theta$ and $\mu_2 > \theta$, is as follows*

- (i) *If $e^{\theta r}(\mu_1 - \theta) \leq \mu_1$ then $(p_1^*, p_2^*) = (0, 0)$ is a Nash equilibrium and in this case both customers balk.*

- (ii) If $e^{\theta r}(\mu_2 - \theta) \geq \mu_2$ then $(p_1^*, p_2^*) = (1, 1)$ is a Nash equilibrium and in this case both customers join.
- (iii) If $0 < p^* < 1$ provides $p \frac{\mu_2}{(\mu_2 - \theta)} + (1 - p) \frac{\mu_1}{(\mu_1 - \theta)} = e^{\theta r}$ then $(p_1^*, p_2^*) = (p^*, p^*)$ is a Nash equilibrium. In this case both customers joins with p^* .

Corollary A.1.3 *In the case of $\mu_1 = \mu_2 = \mu > \theta$ where there is no externality, two customer service systems with the same exponential utility function have a symmetric Nash equilibrium and these equilibrium or equilibria are as follows:*

- (i) If $e^{\theta r}(\mu - \theta) < \mu$ then $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium and in this case both customers balk.
- (ii) If $e^{\theta r}(\mu - \theta) > \mu$ then $(p_1^*, p_2^*) = (1, 1)$ is the unique Nash equilibrium and in this case both customers join.
- (iii) If $e^{\theta r}(\mu - \theta) = \mu$, for every $p^* \in [0, 1]$, $(p_1^*, p_2^*) = (p^*, p^*)$ is a Nash equilibrium. In this case both customers joins with p^* .

Proof. In corollary A.1.2 by setting $\mu = \mu_1 = \mu_2$ the symmetric Nash equilibria are obtained. the 3 cases in this corollary are mutually exclusive and comprehensive and in the first and second case there exists a unique Nash equilibrium. ■

Corollary A.1.4 *In the case of $\mu_1 > \mu_2 > \theta$ with negative externality, there is only one symmetric Nash equilibrium in two customer service systems with the same exponential utility function and this equilibrium is as follows:*

- (i) If $e^{\theta r}(\mu_1 - \theta) \leq \mu_1$ then $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium and in this case both customers balk.
- (ii) If $e^{\theta r}(\mu_2 - \theta) \geq \mu_2$ then $(p_1^*, p_2^*) = (1, 1)$ is the unique Nash equilibrium and in this case both customers join.

(iii) If $e^{\theta r}(\mu_2 - \theta) - \mu_2 < 0 < e^{\theta r}(\mu_1 - \theta) - \mu_1$ then $(p_1^*, p_2^*) = (p^*, p^*)$ is the unique Nash Equilibrium, where

$$p^* = \frac{e^{\theta r} - \frac{\mu_1}{\mu_1 - \theta}}{\frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_1}{\mu_1 - \theta}} \in (0, 1).$$

In this case both customers join with p^* .

Proof. We have $-X_1 \sim \text{Exp}(\mu_1)$, $-X_2 \sim \text{Exp}(\mu_2)$ and $-X_2 >_{st} -X_1$ and $X_1 >_{st} X_2$. by using corollary 3.3.3 and substituting $E[u(r, X_1)]$ and $E[u(r, X_2)]$ and setting $u_0 = 0$ we have corollary A.1.4. ■

Lemma A.1.1 for $\mu_i > 0$ the function defined as $f_i(\theta) = e^{\theta r}(\mu_i - \theta) - \mu_i$ has the following properties:

- (i) If $r\mu_i \leq 1$ then for every $\theta > 0$, $f_i(\theta) < 0$.
- (ii) If $r\mu_i > 1$ then $f_i(\theta) = 0$ has a unique solution. if we specify the unique solution with θ_i , $\theta_i \in (\mu_i - \frac{1}{r}, \mu_i)$, for every $\theta < \theta_i$ we have $f_i(\theta) > 0$, and for every $\theta > \theta_i$ we have $f_i(\theta) < 0$.
- (iii) If $r\mu_i > r\mu_j > 1$ then $f_i(\theta) = 0$ and $f_j(\theta) = 0$ both have a unique solution and these unique solutions provides $\theta_i > \theta_j$.

Proof. The derivative of the defined function $f_i(\theta)$ with respect to θ , and the value of this derivative at $\theta = 0$ is as following.

$$\frac{df_i(\theta)}{d\theta} = e^{\theta r}[r(\mu_i - \theta) - 1] \quad \text{and} \quad \left. \frac{df_i(\theta)}{d\theta} \right|_{\theta=0} = r\mu_i - 1$$

this derivative is positive for $\theta < \mu_i - \frac{1}{r}$, and is negative for $\theta > \mu_i - \frac{1}{r}$. Therefore $f_i(\theta)$ is an increasing function until $\mu_i - \frac{1}{r}$ and it is decreasing after this point. Furthermore, if $r\mu_i \leq 1$ then the first derivative for any $\theta > 0$ is negative and as a result $f_i(\theta)$ is a decreasing function for positive values of θ , and since $f_i(0) = 0$, $f_i(\theta) < 0$ for all $\theta > 0$. If $r\mu_i \geq 1$ then $f_i(\theta)$ is increasing in the interval $(0, \mu_i - \frac{1}{r})$ and decreasing in the interval $(\mu_i - \frac{1}{r}, \infty)$. In

addition, $f_i(0) = 0$ and $f_i(\mu_i) = -\mu_i < 0$, so according to the intermediate value theorem $f_i(\theta)$ has a root in the interval $(\mu_i - \frac{1}{r}, \mu_i)$. To prove the third property of the lemma, for any $\theta > 0$ by assuming $\mu_i > \mu_j$ we have $f_i(\theta) = \mu_i(e^{\theta r} - 1) - \theta e^{\theta r} > \mu_j(e^{\theta r} - 1) - \theta e^{\theta r} = f_j(\theta)$ and according to this we have $f_j(\theta_i) < f_i(\theta_i) = 0$.

■

Theorem A.1.1 *In the case of $\mu_1 > \mu_2 > \theta$ with negative externality, the equilibrium joining possibilities of a two customer service system with the same exponential utility function are as follows:*

- (i) *If $r\mu_1 \leq 1$ then for every $\theta \in (0, \mu_2)$, $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium.*
- (ii) *If $r\mu_2 \leq 1 < r\mu_1$ then for every $\theta \in (0, \theta_1)$ $(p_1^*, p_2^*) = (p^*, p^*)$ and for every $\theta \in [\theta_1, \mu_2)$ $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium. where θ_i is the unique positive solution of $f_i(\theta) = 0$ and*

$$p^* = \frac{e^{\theta r} - \frac{\mu_1}{\mu_1 - \theta}}{\frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_1}{\mu_1 - \theta}} \in (0, 1).$$

- (iii) *If $r\mu_2 > 1$ then for every $\theta \in (0, \theta_2]$ $(p_1^*, p_2^*) = (1, 1)$ and for every $\theta \in (\theta_2, \theta_1)$ $(p_1^*, p_2^*) = (p^*, p^*)$ and for every $\theta \in [\theta_1, \mu_2)$ $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium. where θ_i is the unique positive solution of $f_i(\theta) = 0$ and*

$$p^* = \frac{e^{\theta r} - \frac{\mu_1}{\mu_1 - \theta}}{\frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_1}{\mu_1 - \theta}} \in (0, 1).$$

Proof. In the first case as it was defined in lemma A.1.1, for every $\theta > 0$ we have $f_i(\theta) < 0$, which means the condition in the first case of corollary A.1.4 is met. in the second case, $f_2(\theta) < 0$ is valid for every $\theta > 0$ and the condition in the second case of corollary A.1.4 is never met. On the other hand according to lemma A.1.1 $f_1(\theta) = 0$ has a unique solution and it is in the $(\mu_1 - \frac{1}{r}, \mu_1)$ interval. For every $\theta < \theta_1$ we have $f_1(\theta) > 0$ and as a result the condition in the third case of corollary A.1.4 is met. furthermore, if

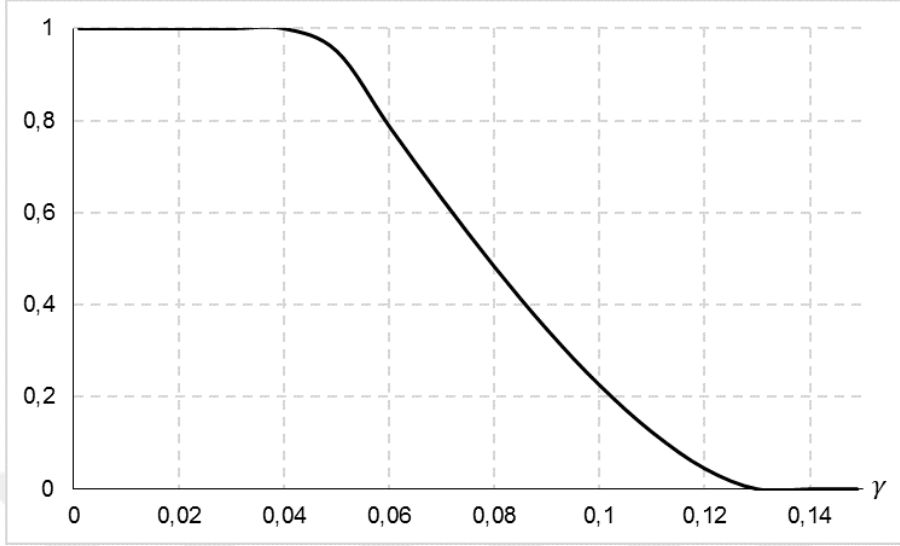


FIGURE A.1: joining probability is decreasing in θ for negative externalities

$\theta \geq \theta_1$ then the first case is valid. Finally in the third case, $f_1(\theta) = 0$ and $f_2(\theta) = 0$ each has a unique positive solution. According to the second and third properties of lemma A.1.1 we have $\mu_2 - \frac{1}{r} < \theta_2 < \theta_1 < \mu_1$ and $\theta_2 < \mu_2$. For every $\theta \leq \theta_2$ we have $f_1(\theta) > f_2(\theta) \geq 0$ and therefore the second case in corollary A.1.4 is valid. for every $\theta \in (\theta_2, \theta_1)$ we have $f_1(\theta) > 0 > f_2(\theta)$ and the third condition in corollary A.1.4 is met. In addition, for every $\theta \geq \theta_1$ we have $f_2(\theta) < f_1(\theta) \leq 0$ and as a result the first condition in corollary A.1.4 is met. The possibility of θ_1 being greater than μ_2 should be taken into consideration in the second and third cases.

■

Corollary A.1.5 *In the case of $\mu_1 > \mu_2 > \theta$ with negative externality, equilibrium joining probability in a two-customer service system with the same exponential utility functions is decreasing in θ , the risk aversion degree.*

Proof. Based on the notation in theorem A.1.1 the equilibrium joining probability is at most a three-piece function. The first piece of the interval assigns probability 1 to sufficiently small θ and the third piece of the interval assigns 0 probability for sufficiently large θ . and for the θ values in the second piece it assigns a probability according to

$$p(\theta) = \frac{e^{\theta r} - \frac{\mu_1}{\mu_1 - \theta}}{\frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_1}{\mu_1 - \theta}} = \frac{\mu_2 - \theta}{\theta(\mu_1 - \mu_2)} [e^{\theta r}(\mu_1 - \theta) - \mu_1].$$

and the derivative is

$$\frac{dp(\theta)}{d\theta} = \frac{\mu_1\mu_2 + e^{\theta r}[\theta^2 - \mu_1\mu_2 + \theta r(\mu_1 - \theta)(\mu_2 - \theta)]}{\theta^2(\mu_1 - \mu_2)}$$

To determine the sign of the derivative we only need to determine the sign of the numerator, because $\mu_1 > \mu_2$ as an assumption. The numerator could be handles as a function of μ_2 :

$$g(\mu_2) = \mu_1\mu_2 + e^{\theta r}[\theta^2 - \mu_1\mu_2 + \theta r(\mu_1 - \theta)(\mu_2 - \theta)]$$

since the other parameters of $g(\mu_2)$ are constant it could be treated as a linear function of μ_2 and it will have its greatest value whether in the lower bound or upper bound of μ_2 . Since, $\theta < \mu_2$ according to the assumption we have θ as our upper bound for μ_2 and will define an upper bound for it too. Furthermore, for the sake of $p(\theta)$ to be a an equilibrium joining probability we must have $f_2(\theta) < 0$ so

$$\begin{aligned} e^{\theta r}(\mu_2 - \theta) - \mu_2 &< 0 \\ \mu_2 &< \frac{\theta e^{\theta r}}{e^{\theta r} - 1} = \theta + \frac{\theta}{e^{\theta r} - 1} \end{aligned}$$

and this provides us the upper bound. Now we will examine the values of $g(\mu_2)$ in its lower and upper bound.

$$g(\theta) = \theta\mu_1 + \theta e^{\theta r}(\theta - \mu_1) = -\theta f_1(\theta) < 0,$$

as we need $p(\theta)$ to be an equilibrium joining probability $f_1(\theta) > 0$. Similarly, we have

$$g\left(\theta + \frac{\theta}{e^{\theta r} - 1}\right) = \frac{\theta e^{\theta r}}{e^{\theta r} - 1}(\mu_1 - \theta)(1 - e^{\theta r} + \theta r) < 0,$$

since $\theta < \mu_1$ and $e^x > 1 + x$. Therefore $g(\mu_2)$ is negative for any valid μ_2 and this proves that $p(\theta)$ is decreasing in θ . ■

Corollary A.1.6 *In the case of $\mu_2 > \mu_1 > \theta$ with positive externality, in a two-customer service system with the same exponential utility function there is symmetric Nash equilibrium, these equilibria are as follows:*

- (i) *If $e^{\theta r}(\mu_2 - \theta) < \mu_2$ then $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium and in this case both customers balk.*
- (ii) *If $e^{\theta r}(\mu_1 - \theta) > \mu_1$ then $(p_1^*, p_2^*) = (1, 1)$ is the unique Nash equilibrium and in this case both customers join.*
- (iii) *If $e^{\theta r}(\mu_1 - \theta) - \mu_1 \leq 0 \leq e^{\theta r}(\mu_2 - \theta) - \mu_2$ then $(p_1^*, p_2^*) = (0, 0)$ and $(p_1^*, p_2^*) = (1, 1)$ and $(p_1^*, p_2^*) = (p^*, p^*)$ are the Nash Equilibria, where*

$$p^* = \frac{e^{\theta r} - \frac{\mu_1}{\mu_1 - \theta}}{\frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_1}{\mu_1 - \theta}} \in [0, 1]$$

Specially, if $e^{\theta r}(\mu_1 - \theta) = \mu_1$ then $p^ = 0$ and if $e^{\theta r}(\mu_2 - \theta) = \mu_2$ then $p^* = 1$*

Proof. The condition required for positive externality as $\mu_2 > \mu_1$ is equivalent to $X_2 > X_1$ and therefore corollary 3.3.5 could be applied and it is sufficient to substitute $E[u(r, X_1)]$, $E[u(r, X_2)]$ and set $u_0 = 0$. ■

Theorem A.1.2 *In case of positive externality $\mu_2 > \mu_1 > \theta$ in a two-customer service system with the same exponential utility functions, the equilibrium joining probabilities are as follows:*

- (i) *If $r\mu_2 \leq 1$ then for every $\theta \in (0, \mu_1)$, $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium.*

(ii) If $r\mu_1 \leq 1 < r\mu_2$ then for every $\theta \in (0, \theta_2)$, $(p_1^*, p_2^*) = (0, 0)$, $(p_1^*, p_2^*) = (p^*, p^*)$, $(p_1^*, p_2^*) = (1, 1)$ are the Nash equilibria, for $\theta \in (\theta_2, \mu_1)$ $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium. where θ_2 is the unique positive solution of $f_2(\theta) = 0$ and

$$p^* = \frac{e^{\theta r} - \frac{\mu_1}{\mu_1 - \theta}}{\frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_1}{\mu_1 - \theta}}$$

(iii) If $r\mu_1 > 1$ then for every $\theta \in (0, \theta_1)$, $(p_1^*, p_2^*) = (1, 1)$ is the unique Nash equilibrium, for every $\theta \in [\theta_1, \theta_2]$, $(p_1^*, p_2^*) = (0, 0)$, $(p_1^*, p_2^*) = (p^*, p^*)$, $(p_1^*, p_2^*) = (1, 1)$ are the Nash equilibria, and for every $\theta \in (\theta_2, \mu_1)$, $(p_1^*, p_2^*) = (0, 0)$ is the unique Nash equilibrium. where θ_i is the unique positive solution of $f_i(\theta) = 0$ and

$$p^* = \frac{e^{\theta r} - \frac{\mu_1}{\mu_1 - \theta}}{\frac{\mu_2}{\mu_2 - \theta} - \frac{\mu_1}{\mu_1 - \theta}}.$$

Proof. In the first case as it was defined in lemma A.1.1, for every $\theta > 0$ we have $f_i(\theta) < 0$, which means the condition in the first case of corollary A.1.6 is met. in the second case, $f_1(\theta) < 0$ is valid for every $\theta > 0$ and the condition in the second case of corollary A.1.6 is never met. On the other hand according to lemma A.1.1 $f_2(\theta) = 0$ has a unique solution and it is in the $(\mu_2 - \frac{1}{r}, \mu_2)$ interval. For every $\theta \leq \theta_2$ we have $f_2(\theta) \geq 0$ and as a result the condition in the third case of corollary A.1.6 is met. furthermore, if $\theta > \theta_2$ then the first case is valid. Finally in the third case, $f_1(\theta) = 0$ and $f_2(\theta) = 0$ each has a unique positive solution. According to the second and third properties of lemma A.1.1 we have $\mu_1 - \frac{1}{r} < \theta_1 < \theta_2 < \mu_2$ and $\theta_1 < \mu_1$. For every $\theta < \theta_1$ we have $f_2(\theta) > f_1(\theta) > 0$ and therefore the second case in corollary A.1.6 is valid. for every $\theta \in [\theta_1, \theta_2]$ we have $f_2(\theta) \geq 0 \geq f_1(\theta)$ and the third condition in corollary A.1.6 is met. In addition, for every $\theta > \theta_2$ we have $f_1(\theta) < f_2(\theta) < 0$ and as a result the first condition in corollary A.1.6 is met. The possibility of θ_2 being greater than μ_1 should be taken into consideration in the second and third cases. ■

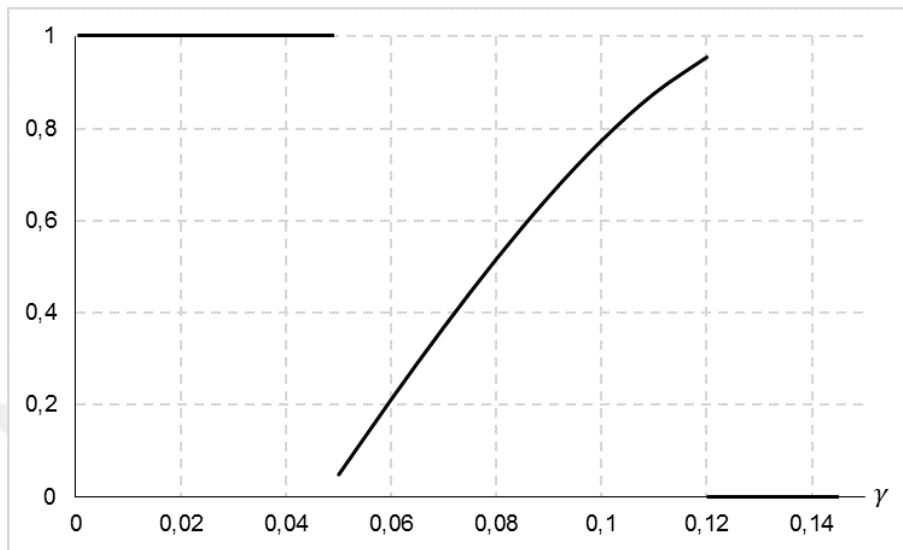


FIGURE A.2: joining probability is increasing in θ for positive externalities

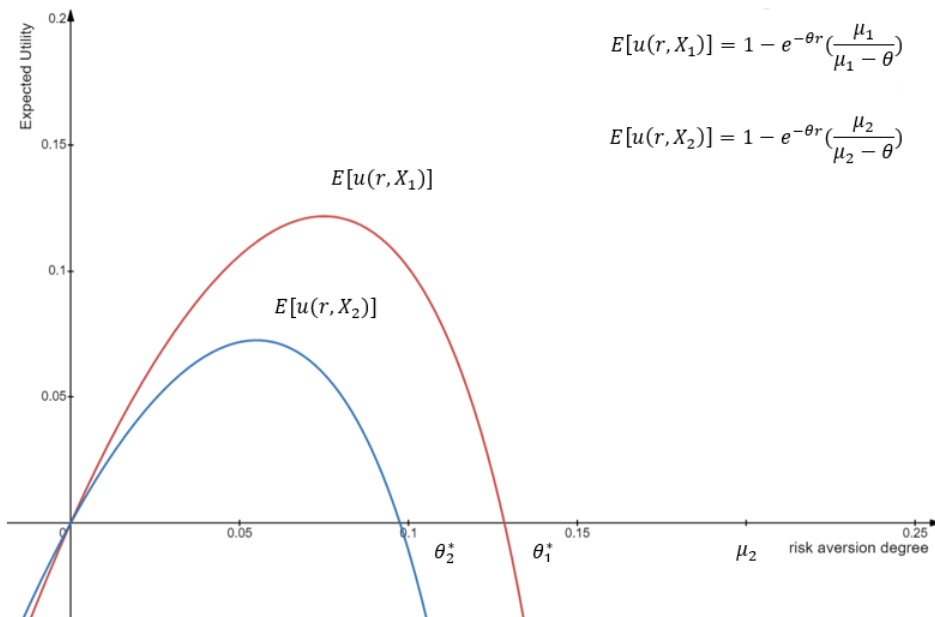


FIGURE A.3: $E[u(r, X_1)]$, $E[u(r, X_2)]$, θ_1^* and θ_2^* under negative externalities

According to figure A.1, as long as the customers get more risk-averse in case of negative externalities, they tend to join with lower probability.

According to figure A.2, as long as the customers get more risk-averse in case of positive externalities, they tend to join with higher probability.

Figure A.3, represents the properties of $f_i(\theta)$, presented in Lemma A.1.1. As it can be observed, in case of negative externalities, the expected utility functions are illustrated with respect to the risk-aversion degree and θ_1^* is greater than θ_2^* .



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