A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF ÇANKIRI KARATEKİN UNIVERSITY

# COMPLEX KERNEL OF ZZ INTEGRAL TRANSFORM AND ITS APPLICATIONS IN ORDINARY DIFFERENTIAL EQUATIONS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

BY

ALI HAMEED ALI ALI

ÇANKIRI

2022

COMPLEX KERNEL OF ZZ INTEGRAL TRANSFORM AND ITS APPLICATIONS IN ORDINARY DIFFERENTIAL EQUATIONS

By Ali Hameed ALI ALI

August 2022

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

**Advisor** **:** Asst. Prof. Dr. Şerifenur CEBESOY ERDAL
**Co-Advisor** **:** Asst. Prof. Dr. Emad ABBAS KUFFI AL-SAADI

**Examining Committee Members:**

**Chairman** : Assoc. Prof. Dr. Emre TAŞ
Mathematics
Kırşehir Ahi Evran University

**Member** : Asst. Prof. Dr. Şerifenur CEBESOY ERDAL
Mathematics
Çankırı Karatekin University

**Member** : Assoc. Prof. Dr. Faruk KARAASLAN
Mathematics
Çankırı Karatekin University

**Approved for the Graduate School of Natural and Applied Sciences**

**Prof. Dr. İbrahim ÇİFTÇİ**
**Director of Graduate School**

**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

**Ali Hameed ALI ALI**

## ABSTRACT

# COMPLEX KERNEL OF ZZ INTEGRAL TRANSFORM AND ITS APPLICATIONS IN ORDINARY DIFFERENTIAL EQUATIONS

Ali Hameed ALI ALI
Master of Science in Mathematics
Advisor: Asst. Prof. Dr. Şerifenur CEBESOY ERDAL
Co-Advisor: Asst. Prof. Dr. Emad ABBAS KUFFI AL-SAADI
August 2022

This master thesis follows the following construction: In the first section, an introduction to integral transforms with a literature review on numerous transform areas is given. A review of differential equations with their benefits in the scientific world is presented. In the second section, basic definitions and properties that will enrich the work, specifically on the ZZ transform that would be the seed for the entirety of the work are discussed. The third section discusses the basic properties of proposed integral transform (SEA) and its applications to the primary functions. Some of these properties are supported by practical examples for clarification. Finally, the fourth section includes the application of the SEA transformation to miscellaneous subjects from different fields of science.

**2022, 47 pages**

**Keywords**: SEA integral transform, ZZ transform, Complex kernel function, Newton's law of cooling, Beam deflection, Differential equations, Natural growth and decay, Mechanical engineering.

# ÖZET

## ZZ İNTEGRAL DÖNÜŞÜMÜNÜN KARMAŞIK ÇEKİRDEĞİ VE ADİ DİFERENSİYEL DENKLEMLERDE UYGULAMALARI

Ali Hameed ALI ALI
Matematik, Yüksek Lisans
Tez Danışmanı: Dr. Öğr. Üyesi Şerifenur CEBESOY ERDAL
Eş Danışman:  Dr. Öğr. Üyesi Emad ABBAS KUFFI AL-SAADI
Ağustos 2022

Bu yüksek lisans tezi aşağıdaki yapıyı takip eder: İlk bölümde, çok sayıda dönüşüm alanı üzerine bir literatür taraması ile integral dönüşümlere bir giriş verilmiştir. Diferensiyel denklemlerin bilim dünyasındaki faydalarıyla birlikte bir incelemesi sunulmaktadır. İkinci bölümde, özellikle işin bütününe tohum olacak ZZ dönüşümü üzerinde çalışmayı zenginleştirecek temel tanımlar ve özellikler ele alınmıştır. Üçüncü bölüm, önerilen integral dönüşümünün (SEA) temel özelliklerini ve temel fonksiyonlara uygulamasını tartışmaktadır. Bu özelliklerden bazıları, açıklama için pratik örneklerle desteklenmektedir. Son olarak dördüncü bölüm, SEA dönüşümünün farklı bilim alanlarından çeşitli konulara uygulamasını içermektedir.

**2022, 47 sayfa**

**Anahtar Kelimeler**: SEA integral dönüşümü, ZZ dönüşümü, Karmaşık çekirdek fonksiyonu, Newton'un soğuma yasası, Işın sapması, Diferensiyel denklemler, Doğal büyüme ve bozunma, Makine mühendisliği.

# PREFACE AND ACKNOWLEDGEMENTS

This thesis presents a novel integral transform with a kernel of complex parameters. The suggested integral transform, denoted by the acronym (Serifenur-Emad-Ali) SEA, introduces a complex parameter into the kernel of the ZZ transform. The addition of the complex parameter results in the creation of a new integral transform that performs the transformation differently and in a different domain than the ZZ transform. The essential features of the SEA transform will be examined, as well as its application to fundamental functions. It will also be explained how it may be used to solve certain actual differential equation problems.

I must first thank God Almighty, who enabled me to reach this high scientific stage, and paved the way for me to be among you today to discuss my master's thesis.

I also extend my highest thanks and gratitude to my thesis advisors, Asst. Prof. Dr. Şerifenur CEBESOY ERDAL and Asst. Prof. Dr. Emad ABBAS KUFFI AL-SAADI, for their patience, guidance and understanding.

**Ali Hameed ALI ALI**
**Çankırı-2022**

# CONTENTS

## LIST OF SYMBOLS

| | |
|---|---|
| I.T | Integral transform |
| SEA | Şerifenur-Emad-Ali integral transform |
| $\mathbb{C}$ | The set of complex numbers |
| $\mathbb{R}$ | The set of real numbers |

## LIST OF FIGURES

# 1. INTRODUCTION

The term "integral" refers to the process of combining or cohering into a whole functional unit, and the term "transform" refers to the converting of an object from its current form into another form (Oxford 2010).

In mathematics, the most inextricable link of integral transforms is with functional series. The summation that happens in the series between the multiplicated functions terms is replaced with variables of integration in the integral transforms (Mortimer 2013).

In an integral transformation, an equation of differential nature is metamorphosed into an equation of algebraic nature. The converted algebraic nature of the equation makes it more malleable to direct mathematical solutions. This transformation from the differential to algebraic forms resulted in the inevitable transformation from a domain where the solving of the intended equation is challenging into another domain where it is possible to process and solve the operation in a more manageable manner (Brychkov I. A and Brychkov Y. A. 1992, Zwillinger and Dobrushkin 1998, Hochstadt 2014).

An integral transform can be displayed mathematically as

$$F(s) = \int_{a}^{b} k(x,s)f(x)dx\ .$$

The transformation interval is determined by the integral limits ($a$ and $b$), the function that is intended for its domain to be transformed is $f(x)$, and the kernel function of the transformation is $k(x,s)$.

The distinguishing factors between integral transforms are the limits of the transformation ($a$ and $b$) and its kernel function $k(x,s)$. One or both of these factors are changed from one transformation to another. The prospect of creating a new integral transform only by manipulating these two factors encouraged the mathematicians to experiment in this area. These experiments suggested a lot of integral transformations that helped the scientific world solve a lot of problems in a lot of different fields of study (Kuffi *et al.* 2021).

As the heart of any integral transform, the kernel function plays a crucial role in defining the nature of the suggested integral transform (Saitoh and Sawano 2016). Most of the integral transforms that have been suggested in the early years and many in recent years are constraints on using a kernel that does not include any complex terms in them. Some of these transforms can be listed as follows: Rohit transform (Gupta *et al*. 2020), Sumudu transform (Aggarwal and Gupta 2019), Shehu transform (ALbukhuttar *et al*. 2020), Al-Zughair transform (Kuffi *et al*. 2020), Aboodh transform (Abbas *et al*. 2022a), SEE transform (Mansour *et al*. 2021b), Al-Tememe transform (Mohammed *et al.* 2016), ZZ transform (Zafar 2016), integral transformation in solving improved heat and Poisson PDEs (Abbas *et al*. 2022b), applying Al-Zughair transform into some engineering fields (Kuffi and Abbas 2022a) and many other transforms.

However, modern mathematicians have moved toward including a complex parameter in their suggested transforms. These transforms usually contain the term "complex" in their names to indicate this insertion. Some of the mentioned integral transforms can be listed as follows: Complex Al-Tememe transform (Mohammed and Makttoof 2017), Complex SEE transform (Mansour *et al*. 2021b), generalization of Complex Al-Tememe transform (Hussein *et al*. 2022), Complex EE transform (Kuffi and Abbas 2022b) and other transforms. It is noticeable that some of the complex transforms are the modifications of already existing transforms; that modification (the insertion of the complex parameter to the transform kernel) gave the modified transform a new domain that differes from the original transform which was based on it.

Differential equations are one of the equations in which integral transforms shine in handling them and solving them efficiently (Hochstadt 2014). Differential equations are known for representing problems in many scientific fields. That capability in representing these problems came from their containing variables, accompanied by the derivatives of these variables that represent the alteration that happened to the system represented by the differential equation with time (Clark 2017).

Differential equations are classified according to the number of variables with respect to which the equation is derived; an ordinary differential equation (O.D.E) is a differential equation that is derived with respect to one variable.

Ordinary differential equations have an essential role in representing many problems, including thermodynamics, electricity flowing, growth and decay of objects, civil engineering problems, cooling, and many other essential applications (Chicone 2006).

In this thesis, a new integral transform that has a complex parameter embedded in its kernel has been proposed. The new transform is named after the mathematicians that suggested it as the (Şerifenur-Emad-Ali) SEA integral transform. The SEA integral transform started with the well-known ZZ transform and modified its integral period and kernel to emerge a new transform with a domain distinct from the seed ZZ transform.

The basic properties of the SEA integral transform, as well as their application to the basic function in mathematics, have been presented and demonstrated extensively. The SEA transformation has been used to solve some ordinary differential equations that represent applications in some scientific and engineering fields, in which it has proved its capability in handling the solving of these applications efficiently.

# 2. PRELIMINARIES

**Definition 2.1** (Goodwine 2010, Hiebert 1997). The condition that specifies a requirement that the solution must satisfy in a single instant of time is said to be the *initial condition.*

On the other hand, the boundary condition proclaims that the solution value is taken in some space region and time separation.

**Definition 2.2** (Brychkov I. A. And Bychkov Y. A. 1992) The relation between any two functions $\varphi$ and $\mu$ can be represented via integral transformation in the form:

$$\mu(p) = \int_a^b F(p,t)\varphi(t)dt,$$

where $a$ and $b$ are finite or infinite numbers.

Integral transform formula contains the following terms:

- The original function $\varphi(t)$ that has a domain called the original space.
- The transform function $\mu(p)$ that has a domain called the image space.
- The transformation kernel function $F(p,t)$.
- The real variable $t$.
- The complex parameter $p = \sigma + iw$, $\sigma$ and $w$ are real numbers.

It is possible to contemplate a shorter representation of the integral transformation that has kernel function $F(p,t)$ by implementing the symbol $T$ as

$$\mu(p) = T\{\varphi(t)\}.$$

**Property 2.1 Linearity of Integral Transformation**

(i) Linearity property of transforms can be summarized as follows (Pundir 2017).

- $T[a_1 f_1(t) \pm a_2 f_2(t) \pm \cdots \pm a_n f_n(t)] = a_1 T(f_1(t)) \pm a_2 T(f_2(t)) \pm \cdots \pm a_n T(f_n(t))$,

where $a_{1,}\ a_2, \ldots$ and $a_n$ are constants and functions $f_1(t), f_2(t)$ ..., and $f_n(t)$ are defined.

(ii) Linearity property of inverse transforms can be summarized as follows (Pundir 2017). If

$T^{-1}\Omega_1(s) = f_1(t),\ T^{-1}\Omega_2(s) = f_2(t), \ldots, T^{-1}\Omega_n(s) = f_n(t)$ and $a_1, a_2, \ldots, a_n$ are constants, then we have

$$T^{-1}[a_1\Omega_1(s) \pm a_2\Omega_2(s) \pm \cdots \pm a_n\Omega_n(s)] = a_1 f_1(t) \pm a_2 f_2(t) \pm \cdots \pm a_n f_n(t)\,.$$

**Definition 2.3 SEE Integral Transformation**

In 2021, Eman A. Mansour and her colleagues (Mansour *et al*. 2021b) introduced the (Sadiq-Emad-Eman) SEE integral transform.

The SEE integral transform is defined for a function $f(t)$ as

$$S\{f(t)\} = \frac{1}{v^n} \int_{t=0}^{\infty} e^{-vt} f(t)dt = F(v), \qquad t \geqslant 0\,, \qquad n \in Z,$$

where $v$ is a complex parameter.

**Definition 2.4 Complex SEE Integral Transform**

The complex (Sadiq-Emad-Eman) SEE transform is another integral transformation that had been introduced by Eman A. Mansour in 2021 (Mansour *et al.* 2021c).

The complex SEE integral transform is defined for a function $f(t)$ as

$$\mathrm{S^c}\{f(t)\} = F(iv) = \frac{1}{v^n} \int_{t=0}^{\infty} e^{-ivt} f(t)dt \,, \qquad t \geq 0 \;, \qquad \mathrm{i} = \sqrt{-1}\,, \qquad n \in Z \,,$$

where $v$ is a complex parameter.

### Definition 2.5 Al-Tememe Integral Transform

Al-Tememe integral transformation for the function $f(x)$; $x > 1$ is defined by the following integral (Kuffi *et al*. 2021)

$$T\{f(x)\} = \int_{x=1}^{\infty} x^{-p} f(x)dx = F(p),$$

such that this integral is convergent, $p$ is a positive constant.

### Definition 2.6 Complex Al-Tememe Integral Transform

Complex Al-Tememe integral transform for the function $g(x)$, where $x$ is greater than one and it is defined as (Mohammed and Makttoof 2017)

$$T^c\{g(x)\} = \int_{x=1}^{\infty} x^{-ip} g(x)dx = F(ip).$$

The Complex Al-Tememe transfomation is convergent in the interval $[1, \infty)$, in which $p$ is a positive constant, $x^{-ip}$ is the kernel of transform and $i = \sqrt{-1}$.

### Definition 2.7 General Complex Al-Tememe Integral Transform

The definition of the general complex Al-Tememe integral transform for the function $f(x)$ for $x > 1$ is (Hussein *et al*. 2022)

$$T_q^c[f(x)] = \int_{t=0}^{\infty} x^{-iq(p)} f(x)dx = F\big(iq(p)\big).$$

The general complex Al-Tememe transformation is convergent in the interval $[1,\infty)$, where $q(p)$ is a function of parameter $p$ and $x^{-iq(p)}$ is the kernel of this transform, $i = \sqrt{-1}$.

### Definition 2.8 EE Integral Transform

Consider the set A as

$$A = \left\{f(t): \exists M, L_{\mathrm{j}} > 0, |f(t)| < Me^{iL_j|t|}, \text{if t} \in [0,\infty), \mathrm{i} \in \mathbb{C} \text{ and } j = 1{,}2\right\},$$

where the constant $M$ must be a finite number and $L_j(j = 1{,}2)$ may be finite or infinite.

The EE integral transform denoted by the operator $E\{.\}$ is defined for a function $f(t)$ of exponential order by the equation (Kuffi and Abbas 2022a) as

$$E^c\{f(t)\} = E(iv) = \int_{t=0}^{\infty} f(t)e^{-iv^n t}dt, \qquad n \in \mathbb{Z}, \qquad t \geq 0,$$

where $v$ is a complex parameter and $i$ is a complex number. In the EE transformation, variable $v$ is used to factor the variable $t$ in the argument of the function $f$.

### Definition 2.9  ZZ Integral Transform

Let $f$ be a function defined for all $t \geq 0$. The ZZ transform $Z(v,s)$ for the function $f(t)$ is defined (Zafar 2016, Rao 2017) as

$$Z(v,s) = H\{f(t)\} = \mathrm{s}\int_{0}^{\infty} f(vt)e^{-\mathrm{st}}dt.$$

If the above integral exists, then the integral equation can be written as

$$Z(v,s) = H\{f(t)\} = \frac{\mathrm{s}}{v}\int_{0}^{\infty} f(t)e^{-\frac{st}{v}}dt\ ,$$

where $s$ and $v$ are complex parameters.

### Property 2.2 ZZ Integral Transform Existence

Let $f$ be a piecewise continuous function in every finite interval $0 \leq t \leq k$, and be of an exponential order $\rho$ for $t > k$.

The ZZ integral transform $Z(v,s)$ for the function $f(t)$ exists for all $s > \rho, v > \rho^2$ (Zafar 2016, Rao 2017).

### 2.1 ZZ Transform Properties

The basic properties of the ZZ transformation can be listed as follows (Zafar 2016, Rao 2017).

(i) Let $f(t) = e^{at}$, where $t \geq 0$ and $a$ is a constant, then

$$H\{e^{at}\} = \frac{s}{s - va}.$$

(ii) Let $f(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$, then

$$H\{1\} = 1.$$

(iii) Let $f(t) = t^n,\ n \in \mathbb{N}$, then

$$H\{t^n\} = n!\frac{v^n}{s^n}.$$

(iv) Linearity property: If $a, b$ are constants and $f(t)$ and $g(t)$ are functions, then

$$H\{af(t) \pm bg(t)\} = aH\{f(t)\} \pm bH\{g(t)\}.$$

(v) ZZ transform for derivatives: Let $H\{f(t)\} = Z(v, s)$, then

$$H\{f(t)\} = \frac{s}{v}Z(v, s) - \frac{s}{v}f(0).$$

# 3. (ŞERIFENUR-EMAD-ALI) SEA INTEGRAL TRANSFORM

This section will present the details of the complex kernel integral transform (Şerifenur-Emad-Ali) SEA integral transform as a new integral transform with a distinct domain from other transforms. The fundamental properties and theorems of the SEA integral transformation will be discussed and proved. Some of the properties will be explained via actual examples to demonstrate the further explained properties.

The SEA integral transformation takes another form in the process of transforming an equation and getting the solution of it, where in it follows the set of the exponential order functions $A$ that is defined as

$$A = \{f(t): \exists M, \lambda_{\mathrm{j}} > 0, |f(t)| < Me^{i\lambda_j|t|}, t \in [0,\infty), i \in \mathbb{C} \text{ and } \mathrm{j} = 1,2\}\,. \tag{3.1}$$

For set A which is defined in Equation (3.1), $M$ is a finite random constant, $\lambda_1, \lambda_2$ could be finite or infinite.

### Definition 3.1 SEA Transform Definition

Let $f(t)$ be a function defined for all $t \geq 0$. The SEA integral transform of $f(t)$ is defined as

$$H^c\{f(t)\} = \frac{s}{v}\int_{t=0}^{\infty} f(t)e^{-i\left(\frac{s}{v}\right)t}dt,$$

where $s$ and $v$ are complex parameters and $i$, $v$ are complex numbers.

The inverse of the SEA integral transform is defined as

$$H^{c^{-1}} = \frac{1}{2\pi \mathrm{i}}\int_{s=\delta-i\infty}^{\delta+i\infty} i\frac{v}{s}e^{i\frac{s}{v}t}F(s,v)ds = f(t)\,,$$

where $\delta$ is a positive real number.

### 3.1 Existence of the SEA Integral Transform

Assume an exponential piecewise continuous function $f(t)$ in every finite interval $0 \leq t \leq k$ of order $\rho$ for $t \geq k$. The SEA integral transform $H^c\{f(t)\}$ for the function $f(t)$ exists for all $s > \rho, v > \rho^2$.

### Property 3.1 SEA Integral Transform Linearity Property

If $a, b$ are constants and $f(t)$ and $g(t)$ are functions, then

$$H^c\{af(t) \pm bg(t)\} = aH^c\{f(t)\} \pm bH^c\{g(t)\}\,.$$

### Proof

From the definition of SEA transformation, we write

$$H^c\{f(t)\} = \frac{s}{v}\int_0^{\infty} e^{-i\frac{st}{v}}f(t)\,dt\,.$$

Then, we have

$$H^c\{af(t) + bg(t)\} = \frac{s}{v}\int_0^{\infty} e^{-i\frac{st}{v}}\big(af(t) + bg(t)\big)\,dt$$

$$= a\frac{s}{v}\int_{t=0}^{\infty} e^{-i\frac{st}{v}} f(t)dt + b\frac{s}{v}\int_{t=0}^{\infty} e^{-i\frac{st}{v}} g(t)dt$$
$$= aH^{c}\{f(t)\} + bH^{c}\{g(t)\},$$

which completes the proof.

### 3.2 The SEA Integral Transform for Fundamental Functions

(i) Let $f(t) = e^{\mathrm{at}}$, where $t \geqslant 0$ and $a$ is a constant, then

$$H^{c}\{e^{at}\} = -\left[\frac{asv}{a^2v^2 + s^2} + i\frac{s^2}{a^2v^2 + s^2}\right].$$

**Proof** From the definition of SEA transformation, we get

$$H^{c}\{e^{at}\} = \frac{s}{v}\int_{0}^{\infty} e^{-i\left(\frac{s}{v}\right)t} e^{at}\, dt\,.$$

Then we arrive at

$$H^{c}\{e^{at}\} = \frac{s}{v}\int_{t=0}^{\infty} e^{-\left(i\left(\frac{s}{v}\right)-a\right)t}\, dt = \frac{-s}{v\left(i\left(\frac{s}{v}\right)-a\right)} \quad \left[e^{-\left(i\left(\frac{s}{v}\right)-\mathrm{a}\right)t}\Big|_{0}^{\infty}\right]$$
$$= \frac{-s}{is - av}[0-1] = \frac{s}{is - av}\cdot\frac{-is - av}{-is - av}$$
$$= -\left[\frac{asv}{s^2 + a^2v^2} + i\frac{s^2}{s^2 + a^2v^2}\right].$$

(ii) Let $f(t) = k$, where $k$ is a constant, then

$$H^c\{k\} = -ik,\ i \in \mathbb{C}\,.$$

**Proof** From the definition of SEA transformation, we have

$$H^c\{k\} = \frac{s}{v}\int_0^\infty e^{-i\frac{s}{v}t} k\,dt = \frac{-ks}{vi\left(\frac{s}{v}\right)}\left[e^{-i\frac{s}{v}t}\Big|_0^\infty\right] \ = \frac{ks}{is} = \frac{k}{i} = -ik.$$

(iii) Let $f(t) = t$, then

$$H^c\{t\} = \frac{-v}{s}.$$

**Proof** From the definition of SEA transformation, it is easy to get

$$H^c\{t\} = \frac{s}{v}\int_0^\infty e^{-i\frac{s}{v}t} t\,dt = \frac{s}{v}\left[\frac{iv}{s}t + \frac{v^2}{s^2}\right] e^{-i\frac{s}{v}t}\Big|_0^\infty = \frac{s}{v}\left[0 - \left(\frac{v^2}{s^2}\right)\right].$$

(iv) Let $f(t) = t^2$, then

$$H^c\{t^2\} = 2i\left(\frac{v}{s}\right)^2.$$

**Proof** From the definition of SEA transformation, we write

$$H^c\{t^2\} = \frac{s}{v}\int_0^\infty e^{-i\frac{s}{v}t} t^2\,dt\,.$$

Performing integration by parts to the above equality, it is clear to see

$$H^c\{t^2\} = \frac{s}{v} \quad \left[ \left( \frac{it^2 v}{s} + \frac{2tv^2}{s^2} - 2\frac{iv^3}{s^3} \right) e^{-i\frac{s}{v}t} \Big|_0^{\infty} \right]$$

$$= \frac{s}{v}\left[0 - \left(-2i\frac{v^3}{s^3}\right)\right] = 2\frac{iv^2}{s^2} = 2i\left(\frac{v}{s}\right)^2.$$

(v) Let $f(t) = t^3$, then

$$H^c\{t^3\} = (3!)\left(\frac{v}{s}\right)^3.$$

**Proof** Using SEA transformation definition and performing partial integration, we obtain

$$H^c\{t^3\} = \frac{s}{v}\left[ \left( \frac{it^3 v}{s} + \frac{3t^2 v^2}{s^2} - \frac{i6tv^3}{s^3} - \frac{6v^4}{s^4} \right) e^{-i\frac{s}{v}t} \Big|_0^{\infty} \right]$$

$$= \frac{s}{v}\left[0 - \left(0 + 0 - 0 - \frac{6v^4}{s^4}\right)\right] = \frac{s}{v}\left[6\frac{v^4}{s^4}\right] = 6\frac{v^3}{s^3} = 3!\left(\frac{v}{s}\right)^3.$$

(vi) In general: $H^c\{t^n\} = (-1)^n (i)^{n-1} (n!) \left(\frac{v}{s}\right)^n$, where $n$ is a positive integer number.

(vii) Let $f(t) = \sin(at)$, where $a$ is a constant, then

$$H^c\{\sin(at)\} = \frac{-asv}{s^2 - a^2 v^2}.$$

**Proof** From SEA transformation definition, it is clear to write

$$H^c\{\sin(at)\} = \frac{s}{v}\int_0^{\infty} e^{-i\frac{s}{v}t} \sin(at)\, dt.$$

Then, we obtain

$$\begin{aligned} H^c\{\sin(at)\} &= \frac{s}{v}\int_{t=0}^{\infty} e^{-i\frac{s}{v}t}\cdot\left(\frac{e^{iat}-e^{-iat}}{2i}\right)dt \\ &= \left(\frac{s}{2iv}\right)\int_{t=0}^{\infty} e^{-i\left(\frac{s}{v}-a\right)t}dt-\left(\frac{s}{2iv}\right)\int_{t=0}^{\infty} e^{-i\left(\frac{s}{\mathrm{v}}+a\right)t}dt \\ &= \frac{-s}{2iv\cdot i\left(\frac{s}{v}-a\right)}e^{-i\left(\frac{s}{v}-a\right)t}\Big|_0^{\infty}+\frac{s}{2iv\ i\left(\frac{s}{v}+a\right)}e^{-i\left(\frac{s}{v}+a\right)t}\Big|_0^{\infty} \\ &= \frac{s}{2v\left(\frac{s}{v}-a\right)}[0-1]-\frac{s}{2v\left(\frac{s}{v}+a\right)}[0-1] \\ &= \frac{-s}{2v}\left[\frac{1}{\left(\frac{s}{v}-a\right)}-\frac{1}{\left(\frac{s}{v}+a\right)}\right]=\frac{-s}{2v}\left[\frac{\frac{s}{v}+a-\frac{s}{v}+a}{\left(\frac{s}{v}-a\right)\left(\frac{s}{v}+a\right)}\right]. \end{aligned}$$

Finally, we arrive at

$$H^c\{\sin(at)\}=\frac{-as}{v\left(\left(\frac{s}{v}\right)^2-a^2\right)}=\frac{-asv}{s^2-a^2v^2}.$$

(vii) Let $f(t)=\cos(at)$, where $a$ is a constant, then

$$H^c\{\cos(at)\}=\frac{-is^2}{s^2-a^2v^2}.$$

**Proof** Using the definition of SEA transformation, we write

$$H^c\{\cos(at)\}=\frac{s}{v}\int\limits_{t=0}^{\infty} e^{-i\frac{s}{v}t}\cos(at)dt$$

$$= \frac{s}{v}\int_{t=0}^{\infty} e^{-i\frac{s}{v}t}\cdot\left(\frac{e^{iat}+e^{-iat}}{2}\right)dt$$

$$= \left(\frac{s}{2v}\right)\int_{t=0}^{\infty} \mathrm{e}^{-i\left(\frac{s}{v}-a\right)t}dt + \left(\frac{s}{2v}\right)\int_{t=0}^{\infty} e^{-i\left(\frac{s}{v}+a\right)t}dt$$

$$= \frac{s}{2vi\left(\frac{s}{v}-a\right)} + \frac{s}{2vi\left(\frac{s}{v}+a\right)}$$

$$= \frac{s}{2iv}\left[\frac{\left(\frac{s}{v}+a\right)+\left(\frac{s}{v}-a\right)}{\left(\frac{s}{v}-a\right)\left(\frac{s}{v}+a\right)}\right]$$

$$= \frac{s}{2iv}\left[\frac{\frac{s}{v}+a+\frac{s}{v}-a}{\frac{s^2}{v^2}-a^2}\right] = \frac{s}{iv}\left[\frac{\frac{s}{v}}{\frac{s^2}{v^2}-a^2}\right].$$

At last, we find

$$H^c\{\cos(at)\} = -i\frac{s}{v^2}\left[\frac{v^2 s}{s^2-a^2v^2}\right] = \frac{-is^2}{s^2-a^2v^2}.$$

### 3.3 The SEA Transform for Derivatives

**Theorem 3.1** Let $H^c\{f(t)\} = Z(v,s)$, then

$$H^c\{f'(t)\} = \frac{-sf(0)}{v} + i\frac{s}{v}Z(v,s)\,. \tag{3.2}$$

**Proof** Using the definition of SEA transformation, it is obvious to write

$$H^c\{f'(t)\} = \frac{s}{v}\int_{t=0}^{\infty} e^{-i\frac{s}{v}t}\, f'(t)dt.$$

Performing partial integration to the above equation as substituting

$$u = e^{-i\frac{s}{v}t}, \qquad dv = f'(t)dt\,,$$

$$du = -i\frac{s}{v}e^{-i\frac{s}{v}t}dt\,, \qquad v = f(t),$$

we arrive at

$$H^c\{f'(t)\} = \frac{s}{v}\int_0^\infty e^{-i\left(\frac{s}{v}\right)t} f'(t)\,dt = \frac{s}{v}\left[e^{-i\frac{s}{v}t}f(t)\Big|_{t=0}^{\infty} + i\frac{s}{v}\int_0^\infty f(t)e^{-i\left(\frac{s}{v}\right)t}dt\right],$$

that is,

$$H^c\{f'(t)\} = \frac{s}{v}[0 - f(0)] + i\frac{s}{v}H^c\{f(t)\} = \frac{-sf(0)}{v} + i\frac{s}{v}Z(v,s)\,.$$

To demonstrate this property practically, an example will be given where the property is going to be applied.

**Example 3.1** Solve the following first order differential equation:

$$y' + y = 0\,, \qquad y(0) = 1$$

**Solution**: Since

$$H^c\{y'\} + H^c\{y\} = 0,$$

then, using Equation (3.2) we write

$$-\frac{s}{v}y(0)+i\frac{s}{v}Z(v,\mathrm{s})+H^{c}\{y\}=0,$$

i.e.,

$$\frac{-s}{v}y(0)+i\frac{s}{v}H^{c}\{y\}+H^{c}\{y\}=0,$$

which verifies

$$\frac{-s}{v}+\left(i\frac{s}{v}+1\right)H^{c}\{y\}=0\,.$$

Thus, we get

$$H^{c}\{y\}=\frac{\frac{s}{v}}{\left(i\frac{s}{v}+1\right)}=\frac{\frac{s}{v}}{i\frac{s}{v}+\frac{v}{v}}=\frac{s}{is+v}=\frac{s}{is+v}\,.\frac{v-is}{v-is}=\frac{sv}{v^{2}+s^{2}}-i\frac{s^{2}}{v^{2}+s^{2}},$$

which is finally equal to

$$H^{c}\{y\}=-\left[\frac{-sv}{v^{2}+s^{2}}+i\frac{s^{2}}{v^{2}+s^{2}}\right].$$

Taking the inverse of the SEA transform gives

$$H^{c^{-1}}\left\{H^{c}\{y\}\right\}=e^{-t}\ \Rightarrow y=e^{-t},$$

which is the solution of the given differential equation.

**Theorem 3.2**

$$H^c\{f''(t)\} = \frac{-s}{v} f'(0) - i\frac{s^2}{v^2} f(0) - \frac{s^2}{v^2} H^c\{f(t)\}. \tag{3.3}$$

**Proof** From SEA transformation definition, we have

$$H^c\{f''(t)\} = \frac{s}{v}\int_0^\infty e^{-i\left(\frac{s}{v}\right)t} f''(t)\ dt\,.$$

Applying integration by parts by making the substitutions

$$\begin{aligned} u &= e^{-i\left(\frac{s}{v}\right)t}, \qquad dv = f''(t)dt\,, \\ du &= -i\left(\frac{s}{v}\right) e^{-i\left(\frac{s}{v}\right)t} dt\,, \qquad v = f'(t)\,, \end{aligned}$$

the above integral turns into

$$\begin{aligned} H^c\{f''(t)\} &= \frac{s}{v}\left[ e^{-i\left(\frac{s}{v}\right)t} f'(t) \Big|_0^\infty + i\left(\frac{s}{v}\right)\int_0^\infty f'(t) e^{-i\left(\frac{s}{v}\right)t}\ dt \right] \\ &= \frac{s}{v}[-f'(0)] + i\frac{s}{v} H^c\{f'(t)\} \\ &= \frac{-sf'(0)}{v} + i\frac{s}{v}\left[\frac{-s}{v} f(0) + i\frac{s}{v} H^c\{f(t)\}\right] \\ &= \frac{-sf'(0)}{v} - i\frac{s^2}{v^2} f(0) - \frac{s^2}{v^2} H^c\{f(t)\}\,. \end{aligned}$$

To demonstrate this property practically, an example will be given where the property is going to be applied.

**Example 3.2** Solve the following second order initial value problem.

$$y'' + y = 0 \qquad y(0) = 0\,,\ \ y'(0) = 1$$

**Solution**: Since

$$H^c\{y''\} + H^c\{y\} = 0\,,$$

Equation (3.3) implies

$$\frac{-s}{v}y'(0) - i\frac{s^2}{v^2}y(0) - \frac{s^2}{v^2}H^c\{y\} + H^c\{y\} = 0,$$

which verifies

$$-\frac{s}{v} - \frac{s^2}{v^2}H^c\{y\} + H^c\{y\} = 0.$$

We obtain from the last equation that

$$\left(1 - \frac{s^2}{v^2}\right)H^c\{y\} = \frac{s}{v}\,,$$

thus, it is clear that

$$H^c\{y\} = \frac{\frac{s}{v}}{1 - \frac{s^2}{v^2}} = \frac{\frac{s}{v}}{\frac{1}{v^2}(v^2 - s^2)} = \frac{vs}{v^2 - s^2} = \frac{-vs}{s^2 - v^2}\,.$$

Taking the inverse of the SEA transform gives

$$y(t) = \sin(t),$$

which gives the solution of the given second order differential equation.

**Theorem 3.3**

$$H^c\{f'''(t)\} = -\frac{s}{v}f''(0) - i\frac{s^2}{v^2}f'(0) + \frac{s^3}{v^3}f(0) - i\frac{s^3}{v^3}H^c\{f(t)\}\,.$$

**Proof** SEA transformation definition yields

$$H^c\{f'''(t)\} = \frac{s}{v}\int_0^\infty e^{-i\left(\frac{s}{v}\right)t} f'''(t)\ dt\,.$$

By using partial integration with the following substitutions

$$u = e^{-i\left(\frac{s}{v}\right)t}\,, \qquad dv = f'''(t)dt,$$
$$du = -i\left(\frac{s}{v}\right)e^{-i\left(\frac{s}{v}\right)t}dt\,, \qquad v = f''(t),$$

we obtain

$$H^c\{f'''(t)\} = \frac{s}{v}e^{-i\left(\frac{s}{v}\right)t}f''(t)\Big|_{t=0}^{\infty} + i\frac{s^2}{v^2}\int_0^\infty f''(t)e^{-i\left(\frac{s}{v}\right)t}dt$$
$$= \left(0 - \frac{s}{v}f''(0)\right) + i\frac{s}{v}\left[\frac{-s}{v}f'(0) - i\frac{s^2}{v^2}f(0) - \frac{s^2}{v^2}H^c\{f(t)\}\right].$$

Then, it is easy to get

$$H^c\{f'''(t)\} = -\frac{s}{v}f''(0) - i\frac{s^2}{v^2}f'(0) + \frac{s^3}{v^3}f(0) - i\frac{s^3}{v^3}H^c\{f(t)\}.$$

**Theorem 3.4**

$$H^c\left\{f^{(4)}(t)\right\} = -\frac{s}{v}f'''(0) - i\frac{s^2}{v^2}f''(0) + \frac{s^3}{v^3}f'(0) + i\frac{s^4}{v^4}f(0) + \frac{s^4}{v^4}H^c\{f(t)\}.$$

**Proof** Using the definition of SEA transformation, it is written that

$$H^c\left\{f^{(4)}(t)\right\} = \frac{s}{v}\int_0^\infty e^{-i\left(\frac{s}{v}\right)t} f^{(4)}(t)\, dt,$$

By applying partial integration with the substitutions

$$u = e^{-i\left(\frac{s}{v}\right)t}, \qquad dv = f^{(4)}(t)dt\,,$$
$$du = -i\left(\frac{S}{v}\right)e^{-i\left(\frac{s}{v}\right)t}dt\,, \qquad v = f'''(t),$$

we have

$$\begin{aligned} H^c\{f^{(4)}(t)\} &= \frac{s}{v}\left[e^{-i\left(\frac{s}{v}\right)t}f''(t)\Big|_{t=0}^{\infty} + i\frac{s}{v}\int_0^\infty f'''(t)e^{-i\left(\frac{s}{v}\right)t}\,dt\right] \\ &= \frac{s}{v}[0 - f''(0)] + i\frac{s^2}{v^2}\int_0^\infty f'''(t)e^{-i\left(\frac{s}{v}\right)t}dt \\ &= -\frac{s}{v}f'''(0) + i\frac{s}{v}\left[-\frac{s}{v}f''(0) - i\frac{s^2}{v^2}f'(0) + \frac{s^3}{v^3}f(0) - i\frac{s^3}{v^3}H^c\{f(t)\}\right]. \end{aligned}$$

Then, we finally get

$$H^c\left\{f^{(4)}(t)\right\} = -\frac{s}{v}f'''(0) - i\frac{s^2}{v^2}f''(0) + \frac{s^3}{v^3}f'(0) + i\frac{s^4}{v^4}f(0) + \frac{s^4}{v^4}H^c\{f(t)\}\,.$$

**In general**: $H^c\left\{f^{(n)}(t)\right\} = \sum_{m=1}^{n} \; -f^{(n-m)}(0)\frac{s^m}{v^m}(i)^{m-1} + \frac{i^n s^n}{v^n}H^c\{f(t)\}$ .

- If n=1, then

$$H^c\{f'(t)\} = -f(0)\frac{s}{v} + i\frac{s}{v}H^c\{f(t)\}\,.$$

- If n=2, then

$$\begin{aligned} H^c\{f''(t)\} &= -f'(0)\frac{s}{v} - f(0)\frac{s^2}{v^2}i + i^2\frac{s^2}{v^2}H^c\{f(t)\} \\ &= -\frac{s}{v}f'(0) - i\frac{s^2}{v^2}f(0) - i\frac{s^2}{v^2}H^c\{f(t)\}\,. \end{aligned}$$

- If n=3, then

$$\begin{aligned} H^c\{f'''(t)\} &= -f''(0)\frac{s}{v} - if'(0)\frac{s^2}{v^2} + f(0)\frac{s^3}{v^3} + i^3\frac{s^3}{\mathrm{v}^3}H^c\{f(t)\} \\ &= -f''(0)\tfrac{s}{v} - if'(0)\tfrac{s^2}{v^2} + f(0)\tfrac{s^3}{v^3} - i\tfrac{s^3}{v^3}H^c\{f(t)\}\,. \end{aligned}$$

# 4. SEA INTEGRAL TRANSFORMATION APPLICATIONS IN ORDINARY DIFFERENTIAL EQUATIONS

To prove the applicability of integral transforms in the real world, it is deemed essential to implement the transform in practical applications that have an impact on as many scientific fields as possible and to evaluate the performance of the integral transform in solving these applications (Pundir 2017, Kreyszig 2014, Saičev *et al*. 2018).

Ordinary differential equations (O.D.E's) represent an efficient tool to represent the problems of many applications in many fields, so the capability of any method to handle and solve them efficiently and adequately weighs the heftiness of that method in the scientific society (Hochstadt 2014, Zwillinger *et al*. 1998).

This section uses the SEA integral transformation to solve some ordinary differential equations with the most significant applications in numerous scientific areas, such as Newton's law of cooling, civil and mechanical engineering, and growth and decay problems.

**Application 4.1 Using the SEA Integral Transform to Solve Newton's Law of Cooling**

The heat changing of objects related to their surrounding environment represents such an exciting subject in many fields that the concept has been expressed mathematically by Newton through Newton's law of cooling. The law dictates that the heat lost by an object is put in a different temperature environment than that object is relative to the temperature difference between them (Maruyama and Moriya 2021).

Newton's law of cooling is expressed mathematically as

$$\frac{d\theta}{dt} \propto [\theta(t) - \theta_0(t)] \Rightarrow \frac{d\theta}{dt} = -\mu[\theta(t) - \theta_0(t)]\,. \tag{4.1}$$

The variables that appeared in the formula are

$\theta(t)$: body temperature at time $t$.
$\theta_0(t)$: the environment temperature.
$\mu$: proportionality constant ($\mu > 0$).

Newton's law of cooling problem has been solved by the ZZ transform (Rao 2017a). The ZZ transform represents the base transform from which the proposal for the SEA transform emerged. To prove the capability of the SEA transform in handling such a problem, it will be solved using the SEA transform, and the results should resemble those that the ZZ transform has deduced.

**Problem 4.1** For the ambient air temperature of 40 degrees Celsius and body temperature that has been decreased from 80 to 60 degrees Celsius in 20 minutes, calculate the body temperature after the passing of 40 minutes and calculate the time at which the body temperature reaches 55 degrees Celsius.

**Solution:** Following datas are given in the problem:

The environment temperature of the air is $\theta_0 = 40°\text{C}$.
The temperature at $t = 0$ is $\theta(0) = 80°\text{C}$.
After 20 minutes passed, the temperature would be $\theta(20) = 60°\text{C}$.
Newton's law of cooling in Equation (4.1) obtains

$$\frac{d\theta}{dt} \propto [\theta(t) - \theta_0(t)]$$
$$\Rightarrow \frac{d\theta}{dt} = -\mu[\theta(t) - \theta_0(t)],$$

where μ is a constant. Then we write

$$\frac{d\theta}{dt} = -\mu\theta(t) + \mu\theta_0(t),$$

that is,

$$\theta'(t) + \mu\theta(t) = 40\mu. \tag{4.2}$$

Applying the SEA transform to Equation (4.2) gives

$$H^c\{\theta'(t)\} + \mu H^c\{\theta(t)\} = 40\mu H^c\{1\}\,.$$

Using the derivatives property of SEA transform, we get

$$-\frac{s}{v}\theta(0) + i\frac{s}{v}H^c\{\theta(t)\} + \mu H^c\{\theta(t)\} = -40\mu i, \qquad i \in \mathbb{C} \tag{4.3}$$

Replacing $\theta(0) = 80°\mathrm{C}$ into Equation (4.3), it is easy to obtain

$$\begin{aligned} -\frac{s}{v}(80) + i\frac{s}{v}H^c\{\theta(t)\} + \mu H^c\{\theta(t)\}\} &= -40i\mu \\ \Rightarrow \left(i\frac{s}{v} + \mu\right)H^c\{\theta(t)\} &= -40i\mu + 80\frac{s}{v} \\ \Rightarrow \frac{1}{v}(is + v\mu)H^c\{\theta(t)\} &= \frac{-40i\mu v + 80s}{v} \\ \Rightarrow H^c\{\theta(t)\} &= \frac{-40i\mu v + 80s}{v} \cdot \frac{v}{is + v\mu}. \end{aligned}$$

Hence, it implies

$$H^c\{\theta(t)\} = 40\left(\frac{-i\mu v + 2s}{is + v\mu}\right). \tag{4.4}$$

So, we can write

$$\begin{aligned}
H^c\{\theta(t)\} &= 40\left[\frac{-i\mu v + 2s}{v\mu + is} \cdot \frac{v\mu - is}{v\mu - is}\right] \\
&= 40\left[\frac{-iv^2\mu^2 - \mu vs + 2sv\mu - 2is^2}{v^2\mu^2 + s^2}\right] \\
&= 40\left[\frac{-iv^2\mu^2 + \mu vs - 2is^2}{v^2\mu^2 + s^2}\right] \\
&= 40\left[\frac{-i(v^2\mu^2 + s^2) - is^2 + \mu vs}{v^2\mu^2 + s^2}\right] \\
&= 40\left[-i\frac{(v^2\mu^2 + s^2)}{(v^2\mu^2 + s^2)} + \frac{\mu vs - is^2}{(v^2\mu^2 + s^2)}\right],
\end{aligned}$$

which yields

$$H^c\{\theta(t)\} = 40\left[-i - \left(\frac{-\mu vs}{v^2\mu^2 + s^2} + i\frac{s^2}{v^2\mu^2 + s^2}\right)\right]. \tag{4.5}$$

Applying the inverse of the SEA transformation to Equation (4.5), we arrive at

$$H^{c^{-1}}\{H^c\{\theta(t)\}\} = H^{c^{-1}}\left\{40\left[-i - \left(\frac{-\mu sv}{v^2\mu^2 + s^2} + i\frac{s^2}{v^2\mu^2 + s^2}\right)\right]\right\}.$$

The resultant equation satisfies the linearity property. Then we have

$$\theta(t) = 40\left[H^{c^{-1}}\{-i\} - H^{c^{-1}}\left\{\frac{-\mu sv}{v^2\mu^2 + s^2} + i\frac{s^2}{v^2\mu^2 + s^2}\right\}\right],$$

which gives

$$\theta(t) = 40[1 + e^{-\mu t}]\,. \tag{4.6}$$

Due to the fact that it has been stated in the problem that at $t = 20$ min., $\theta(20) = 60$°C. Thus

$$60 = 40 + 40e^{-20\mu} \Rightarrow 20 = 40e^{-20\mu},$$

so that

$$\frac{1}{2} = e^{-20\mu} \Rightarrow (e^{-\mu})^{20} = \frac{1}{2} \Rightarrow e^{-\mu} = \left(\frac{1}{2}\right)^{\frac{1}{20}}. \tag{4.7}$$

(i) Equation (4.7) made it possible to find the body temperature after the passing of 40 minutes as

$$\begin{aligned}\theta(t) &= 40 + 40e^{-40\mu}\\ &= 40 + 40(e^{-\mu})^{40} = 40 + 40\left(\frac{1}{2}\right)^{\frac{40}{20}}\\ &= 40 + 40\left(\tfrac{1}{2}\right)^{20},\end{aligned}$$

Then,

$$\theta(\mathrm{t}) = 50°\mathrm{C}\,.$$

The concluded $\theta(\mathrm{t})$ represents the body temperature after 40 minutes.

(ii) The time required for the body to reach $50^\circ\mathrm{C}$ temperature can be calculated as

$$55 = 40 + 40^{-\mu t} \Rightarrow 15 = 40(e^{-\mu})^t \text{ ,}$$

which yields

$$\frac{15}{40} = (e^{-\mu})^t. \tag{4.8}$$

Substituting Equation (4.7) into Equation (4.8), we get

$$\frac{15}{40} = \left(\frac{1}{2}\right)^{\frac{t}{20}}.$$

By applying the natural logarithms to the resultant equation, it is concluded that

$$\ln\left(\frac{15}{40}\right) = \frac{t}{20}\ln\left(\frac{1}{2}\right) \Rightarrow t = \frac{20\ln\left(\frac{15}{40}\right)}{\ln\left(\frac{1}{2}\right)} \text{ .}$$

Then, $t = 28.3$ minutes is the time required for the body to reach $50^\circ\mathrm{C}$.

The acquired results from applying the SEA integral transform to the problem of Newton's law of colling resemble the results that are concluded from applying the ZZ transform to the same problem, which proves the capability of the SEA transform in finding the exact solution to this problem.

### Application 4.2 Using the SEA Integral Transform to Solve Natural Growth and Decay

Natural growth and decay laws govern living things and many non-living objects. For that reason, the subject of natural growth and decay represents an important aspect of scientific society. Mathematicians succeed in representing the law of growth and decay using first order degree differential equations as follows (Mansour *et al*. 2021c, Rao 2017b).

Assuming $x(t)$ is the substance quantity that needed to be converted chemically at the time "$t$" based on the chemical conversion law $x(t)$ is changed at a rate proportional to the substance quantity at that time.

$$\frac{\mathrm{d}x}{dt} \propto x\,.$$

$k$ is the rate constant and $k > 0$. $k$ constant is the factor that governs the laws of growth and decay, when its value is positive, then the law is for growth, and when its value is negative, then the law is for decay, as follows.

The natural growth rule is

$$\frac{dx}{dt} = kx\,.$$

The natural decay rule is

$$\frac{dx}{dt} = -kx\,.$$

The problems of natural growth and decay have been processed and solved by integral transforms and specifically, the ZZ transform, where the ZZ transform possesses the ability to find the exact solution to these problems (Rao 2017b). The following problems represent the processing and solving of the growth and decay problems using the SEA integral transform to find the exact solution to these problems.

**Problem 4.2.1** The pace at which bacteria proliferate is relative to the count present at any given time. If the initial value doubles in two hours, how long will it take to triple?

**Solution:** Let $N$ be the bacteria number at any time $t$ and at the initial time $t = 0\,,\ N = x$. From natural growth law, we write

$$\frac{dN(t)}{dt} \propto N(t)$$
$$\Rightarrow \frac{dN(t)}{dt} = kN(t),$$

where $k$ is a constant. Hence, we get

$$N'(t) - kN(t) = 0. \tag{4.9}$$

By taking the SEA transformation into Equation (4.9), we have

$$H^c\{N'(t)\} - kH^c\{N(t)\} = 0$$
$$\Rightarrow \frac{-s}{v}N(0) + \frac{is}{v}H^c\{N(t)\} - kH^c\{N(t)\} = 0$$
$$\Rightarrow \left(i\tfrac{s}{v} - k\right)H^c\{N(t)\} = \tfrac{s}{v}N(0)\,,\ (N(0) = x)$$
$$\Rightarrow \left(i\frac{s}{v} - k\right)H^c\{N(t)\} = \frac{s}{v}x$$

$$\Rightarrow \left(\frac{is - vk}{v}\right) H^c\{N(t)\} = \frac{s}{v}x\,,$$

$$\Rightarrow H^c\{N(t)\} = \frac{sx}{v} \cdot \frac{v}{is - vk}$$

$$= \frac{sx}{is - vk} \cdot \frac{-is - vk}{-is - vk}$$

$$= \frac{-xis^2 - xvsk}{s^2 + v^2k^2} = \frac{-xvsk}{s^2 + v^2k^2} - \frac{xis^2}{s^2 + v^2k^2}$$

$$= -x\left[\frac{kvs}{s^2 + v^2k^2} + i\frac{s^2}{s^2 + v^2k^2}\right].$$

Taking the inverse of the SEA transformation gives the exact solution to Equation (4.9) which is

$$N(t) = xe^{kt}\,.$$

**Problem 4.2.2** Bacteria colonized the culture medium at a proportional rate to $N$. $N$ began at a value equal to 100 and rose to 332 after an hour. What value did $N$ have after 90 minutes?

**Solution:** Following datas are given in the problem:

At $t = 0\,,\ N = 100$.
At $t = 60\,,\ N = 332$ .

It is required that at $t = 90$, what would $N$ be?

Applying the SEA transform to Equation (4.9)  which represents the the law of natural growth, we write

$$H^c\{N'(t)\} - H^c\{kN(t)\} = H^c\{0\}$$

$$\Rightarrow \frac{-s}{v}N(0)+\frac{is}{v}H^c\{N(t)\}-kH^c\{N(t)\}=0$$

$$\Rightarrow \left(i\frac{s}{v}-k\right)H^c\{N(t)\}=\frac{s}{v}100\,, \qquad (N(0)=100)$$

$$\Rightarrow \left(\frac{is-vk}{v}\right)H^c\{N(t)\}=\frac{s}{v}100.$$

From the last equality, it is concluded that

$$\begin{aligned} H^c\{N(t)\} = 100\frac{s}{v}\,.\frac{v}{is-vk} &= \frac{100s}{is-vk}\,.\frac{-is-vk}{-is-vk} \\ &= \frac{-100is^2-100vsk}{s^2+v^2k^2} \\ &= -100\left[\frac{kvs}{s^2+v^2k^2}+i\frac{s^2}{s^2+v^2k^2}\right]. \end{aligned}$$

Taking the inverse of the SEA transform gives the exact solution as

$$N(t)=100e^{kt}. \tag{4.10}$$

Substituting $t=60\,min.$ and $N(t)=332$ into Equation (4.10), we see

$$332=100e^{60k} \;\Rightarrow\; \frac{332}{100}=e^{60k} \;\Rightarrow\; (e^k)^{60}=\frac{332}{100},$$

hence, we get

$$e^k=\left(\frac{332}{100}\right)^{\frac{1}{60}}. \tag{4.11}$$

By using Equation (4.11), it is possible to find $N(t)$ at t=90 minutes as

$$N(t) = 100e^{90k} \Rightarrow \mathrm{N(t)} = 100(e^k)^{90} = 100(\frac{332}{100})^{\frac{90}{60}} = 604.93,$$

i, e. ,

$$\mathrm{N(t)} \cong 605 \,.$$

The resulted $N(t)$ = 605 indicates the number of bacteria after 90 minutes.

**Problem 4.2.3** At each point in time, the rate at which a particular substance decomposes in a particular solution is relative to the quantity of that material present in the solution. There are initially 27 grams, but only 8 grams remain after three hours. How much substance will remain in one more hour?

**Solution:** Assuming $M$ grams are the substance amount left at any time $t$ in the solution. The SEA transformation will be applied to the natural decay equation: $N'(t) = -kN(t)$ as

$$H^c\{N'(t)\} + kH^c\{N(t)\} = 0 \,.$$

Then, we get

$$\begin{aligned}
\frac{-s}{v}N(0) + i\frac{s}{v}H^c\{N(t)\} + kH^c\{N(t)\} &= 0 \\
\Rightarrow \frac{-s}{v}27 + i\frac{s}{v}H^c\{N(t)\} + kH^c\{N(t)\} &= 0 \\
\Rightarrow \left(i\frac{s}{v} + k\right)H^c\{\mathrm{N(t)}\} &= \frac{s}{v}(27) \\
\Rightarrow \left(\frac{is + vk}{v}\right)H^c\{N(t)\} &= 27\frac{s}{v} \\
\Rightarrow H^c\{N(t)\} = 27\frac{s}{v}\cdot\frac{v}{is + vk} &= \frac{27\mathrm{s}}{is + vk}\cdot\frac{-is + vk}{-is + vk}
\end{aligned}$$

$$= \frac{-\mathrm{i}27s^2 + 27svk}{s^2 + v^2k^2}$$

$$= -27\left[\frac{-svk}{s^2 + k^2v^2} + i\,\frac{s^2}{s^2 + k^2v^2}\right].$$

By taking the inverse of the SEA transform, we find

$$N(t) = 27e^{-kt}. \tag{4.12}$$

Applying the conditions $t = 3$ hours and $N(t) = 8$ grams to Equation (4.12), it is found

$$8 = 27e^{-3k} \;\Rightarrow \frac{8}{27} = (e^{-k})^3 \;\Rightarrow \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = e^{-k},$$

that is,

$$\frac{2}{3} = e^{-k}\,. \tag{4.13}$$

To find $N(t)$ for the time $t = 4$ hours, we obtain from Equation (4.13) that

$$N(t) = 27e^{-4k} \;= 27\left(\frac{2}{3}\right)^4 = 27 \cdot \frac{16}{81},$$

thus, we get

$$N(t) = \frac{16}{3} = 5.333 \text{ grams.}$$

### Application 4.3 Using the SEA Integral Transform to Solve the Deflecting a Beam Problem

One of the most commonly discussed problems in civil engineering is the ordinary differential equation (O.D.E) of deflecting a hinged beam at its ends under a uniform load, as shown in Figure 4.1. This problem has been solved previously using integral transforms (Kuffi *et al*. 2019, Xu *et al*. 2021). And now, this problem is going to be solved using the SEA integral transform.

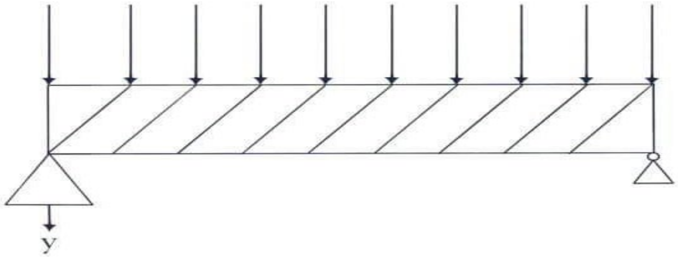

**Figure 4.1** A hinged beam subjected to a constant load.

The ordinary differential equation (O.D.E) with boundary conditions that express the hinged beam at its ends $x = 0$ and $x = L$, where a uniformed load applied on it $w_0$ (per length unit) is represented as

$$y^{(4)} = \frac{\omega_0}{EI} \quad or \quad \frac{d^4y}{dx^4} = \frac{\omega_0}{EI}, \qquad 0 < x < L\,, \tag{4.14}$$

$$y(0) = y_0 = 0, y''(0) = y_0'' = 0\,, \qquad y(L) = 0\,, \qquad y''(L) = y_L'' = 0\,, \tag{4.15}$$

where $E$ is Young's modulus, $I$:normal bending plain moment of inertia for an axis-crossed cross-section, $EI$ is beam's flexural rigidity.

Some more abundances required for the problem are:

$y'(x)$: beam's slop.

$M(x) = EI\acute{\acute{y}}(x)$: beam's bending moment.

$S(x) = \acute{M}(x) = EI\acute{\acute{\acute{y}}}(x)$: beam's shear point.

Applying the SEA transform to Equation (4.14), it implies

$$H^c\left\{\frac{d^4y}{dx^4}\right\} = \frac{\omega_0}{EI}H^c\{1\}$$

$$\Rightarrow -\frac{s}{v}y'''(0) - i\frac{s^2}{v^2}y''(0) + \frac{s^3}{v^3}y'(0) + \frac{is^4}{v^4}y(0) + \frac{s^4}{v^4}H^c\{y(x)\} = \frac{\omega_0}{EI}H^c\{1\}\,. \tag{4.16}$$

Applying to Equation (4.16), Equation (4.15) conditions coupled with the following unknown conditions obtains

$$y'(0) = C_1\,, y'''(0) = C_2\,.$$

Then, we write

$$-C_2\frac{s}{v} + C_1\frac{s^3}{v^3} + \frac{s^4}{\mathrm{v}}H^c\{y(x)\} = \frac{-w_0}{EI}i$$

$$\Rightarrow \frac{s^4}{v^4}H^c\{y(x)\} = \frac{-w_0 i}{EI} + C_2\frac{s}{v} - C_1\frac{s^3}{v^3}$$

$$\Rightarrow H^c\{y(x)\} = \frac{-w_0 i v^4}{EIs^4} + C_2\frac{sv^4}{vs^4} - C_1\frac{s^3v^4}{v^3s^4}$$

$$\Rightarrow H^c\{y(x)\} = \frac{-w_0 i v^4}{EIs^4} + C_2\frac{v^3}{s^3} - \frac{C_1 v}{s}\,.$$

Using the SEA integral transform inverse on the concluded equation, we arrive at

$$y(x) = \frac{w_0}{EI(24)}x^4 + C_2\frac{x^3}{6} + C_1x$$

$$\Rightarrow y(x) = \frac{w_0x^4}{24EI} + \frac{C_2}{6}x^3 + C_1x.$$

Applying Equation (4.15) conditions $y(L) = 0$ , $y''(L) = y_L'' = 0$ gives

$$C_1 = \frac{w_\circ L^3}{24EI} \text{ and } C_2 = \frac{w_\circ L}{2EL}.$$

The deflecting of a beam under uniform load equation to find the shear and bending instant at any point is

$$y(x) = \frac{w_0}{24EI}x(L-x)(L^2 - Lx - x^2)\,.$$

**Application 4.4 Using the SEA Integral Transform in Mechanical Engineering**

The SEA integral transform will be utilized to solve a mechanical engineering problem to demonstrate the capability of using this transform in such a field.

**Problem 4.4** Figure 4.2 illustrates that a particle P of mass 2 grams travels down the x-axis and is pulled towards the origin O by force quantitatively equivalent to 8x.

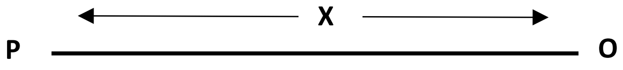

**Figure 4.2** A particle P pulled towards the origin O through the x-axis.

Assuming the particle's initial rest is at $x = 10$, find the particle position at any succeeding time if

a) There is no other forces intervene.
b) A damping force equivalent to eight times the instantaneous velocity is applied.

**Solution:**

a) By taking the right direction in Figure 4.2, which is the positive direction, when $x > 0$, then the net force is given by $-8x$.

From Newton's law: net force= (mass)(acceleration), we write

$$2\frac{d^2x}{dt^2} = -8x \;\Rightarrow\; \frac{d^2x}{dt^2} = -4x \;\Rightarrow\; \frac{d^2x}{dt^2} + 4x = 0\,. \tag{4.17}$$

The initial conditions that are going to be applied are

$$x(0) = 10\,, \qquad x'(0) = \frac{dx(0)}{dt} = 0\,. \tag{4.18}$$

By using the initial conditions in Equation (4.18) and applying the SEA transform to Equation (4.17), we have

$$\begin{aligned} -\frac{s}{v}x'(0) - i\frac{s^2}{v^2}x(0) - \frac{s^2}{v^2}H^c\{x(t)\} + 4H^c\{x(t)\} &= 0 \\ \Rightarrow -10i\frac{s^2}{v^2} - \frac{s^2}{v^2}H^c\{x(t)\} + 4H^c\{x(t)\} &= 0 \\ \Rightarrow \left(4 - \frac{s^2}{v^2}\right)H^c\{x(t)\} &= 10i\frac{s^2}{v^2} \\ \Rightarrow \frac{4v^2 - s^2}{v^2}H^c\{x(t)\} &= 10i\frac{s^2}{v^2} \end{aligned}$$

$$\Rightarrow H^{c}\{x(t)\} = \frac{10is^2}{4v^2 - s^2} = -\frac{10\mathrm{i}}{s^2 - 4v^2} = 10\left(\frac{-is^2}{s^2 - (2)^2v^2}\right).$$

Then, we get

$$x(t) = 10H^{c^{-1}}\left\{\frac{-is^2}{s^2 - (2)^2v^2}\right\} = 10\cos(2t)\,.$$

The concluded equation motion graph is demonstrated in Figure 4.3. The cosine signal that appeared in the graph has an amplitude (maximum distance from 0) of 10 and a period (cycle completing time) of $\pi$ and a frequency (cycles per second count) of $\frac{1}{\pi}$.

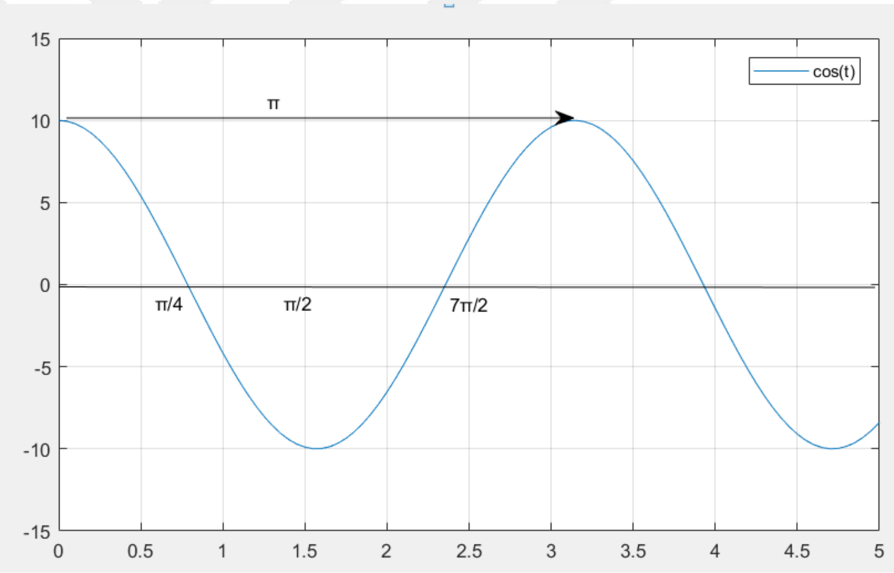

**Figure 4.3** A cosine signal.

b) Considering

- The damping force would be negative being left directed and given by $-8\frac{dx}{dt}$. When x and its derivative with respect to t is on the right of 0 and moving to the right ($x > 0$ and $\frac{dx(t)}{dt} > 0$).

- The damping force would be posetive being left directed and given by $-8\frac{dx}{dt}$. When $x$ and its derivative with respect to $t$ is on the left of 0 and moving to the left ($x < 0$ and $\frac{dx}{dt} < 0$).
- Also, damping force equals $-8\frac{dx}{dt}$ for $x > 0$, $\frac{dx}{dt} < 0$ and $x < 0$ , $\frac{dx}{dt} > 0$, and since net force = (Mass) (Acceleration),

or

$$2\frac{d^2x}{dt^2} = -8x - 8\frac{dx}{dt},$$

that is,

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0\,. \tag{4.19}$$

Applying the initial conditions

$$x(0) = 10\,, x'(0) = 0$$

and taking into account the SEA transform to Equation (4.19), we obtain

$$\frac{-s}{v}x'(0) - i\frac{s^2}{v^2}x(0) - \frac{s^2}{v^2}H^c\{x(t)\} + 4i\frac{s}{v}H^c\{x(t)\} - 4\frac{s}{v}x(0) + 4H^c\{x(t)\} = 0$$

$$\Rightarrow -i\frac{s^2}{v^2}(10) - \frac{s^2}{v^2}H^c\{x(t)\} + 4i\frac{s}{v}H^c\{x(t)\} - 40\frac{s}{v} + 4H^c\{x(t)\} = 0$$

$$\Rightarrow \left(\frac{-s^2}{v^2} + 4i\frac{s}{v} + 4\right) + H^c\{x(t)\} = 10\frac{is^2}{v^2} + 40\frac{s}{v}$$

$$\Rightarrow \left(\frac{-s^2}{v^2} + 4i\frac{s}{v} + 4\right) + H^c\{x(t)\} = \frac{10is^2 + 40sv}{v^2}$$

$$\Rightarrow (-s^2 + 4isv + 4v^2)H^c\{x(t)\} = 10is^2 + 40sv$$

$$\Rightarrow H^c\{x(t)\} = \frac{10is^2 + 40sv}{-s^2 + 4isv + 4v^2}.$$

Then

$$x(t) = 10H^{c^{-1}}\left\{\frac{is^2 + 4sv}{-s^2 + 4isv + 4v^2}\right\}$$

$$\Rightarrow x(t) = 10H^{c^{-1}}\left\{\frac{\mathrm{i}s^2 + 4vs}{(is + 2v)^2}\right\}.$$

Finally we find

$$x(t) = 10e^{-2t} + 20te^{-2t}.$$

Figure 4.4 shows the relationship between $x$ and $t$. It shows a non-oscillatory motion, where the particle approaches but never makes contact with O.

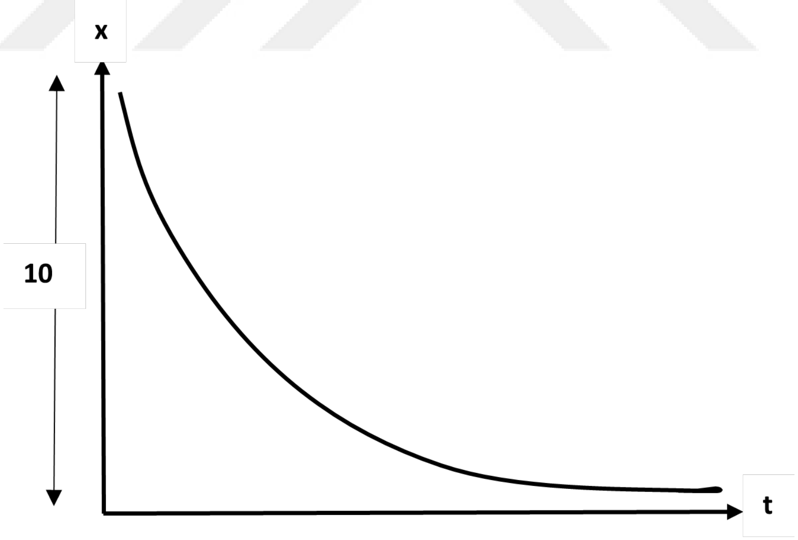

**Figure 4.4** The relationship between $x$ and $t$.

# 5. CONCLUSIONS AND RECOMMENDATION

The proposed SEA integral transform is a new integral transform that changes the ZZ transform kernel by incorporating the complex parameter. To get a better grasp of the required qualities, the SEA integral transform properties were illustrated by applying them to basic functions and derivatives, which were supplemented with practical examples. The efficiency and capability of the SEA transform to handle practical applications have been demonstrated by using it to solve some important ordinary differential equations applications, such as Newton's law of cooling, the problem of a hinged beam that deflects under a uniform load, the growth and decay problem, and a mechanical engineering problem.

# REFERENCES

Abbas, E. S., Kuffi, E. A. and Abdlrasol, L. B. 2022a. General Solution of Telegraph Equation Using Aboodh Transform. Mathematical Statistician and Engineering Applications, 71(2): 267-271.

Abbas, E .S., Kuffi, E. A. and Hanna, E. 2022b. Al-Zughair integral transformation in solving improved heat and Poisson PDEs. In AIP Conference Proceedings . 2386(1): 040041. AIP Publishing LLC.

Aggarwal, S. and Gupta, A. R. 2019. Sumudu transform for the solution of Abel's integral equation. Journal of Emerging Technologies and Innovative Research, 6(4): 423-431.

ALbukhuttar, A. N., Ridha, Z. D. and Kadhim, H. N. 2020. Applications of a shehu transform to the heat and transport equations. International Journal of Psychosocial Rehabilitation, 24(05): 4254-4263.

Brychkov, I. A. and Brychkov, Y. A. 1992. Multidimensional integral transformations. CRC Press.

Chicone, C. 2006. Ordinary Differential equations with Applications, 2nd edition, Springer.

Clark, D. N. 2017. Dictionary of analysis, calculus, and differential equations. CRC Press.

Goodwine, B. 2010. Engineering differential equations: theory and applications. Springer Science and Business Media.

Gupta, R. and Verma, D. 2020. Solving Schrodinger equation for a quantum mechanical particle by a new integral transform: Rohit Transform,. ASIO Journal of Chemistry, Physics, Mathematics and Applied Sciences (ASIO-JCPMAS), 4(1): 32-36.

Hiebert, J. 1997. Making sense: Teaching and learning mathematics with understanding. Heinemann, 361 Hanover Street, Portsmouth, NH 03801-3912.

Hochstadt, H. 2014. Differential equations. Courier Corporation.

Hussein, M. J., Atun, Y. and Kuffi E. A. 2022. Generalization of Complex Al-Tememe Transformation and its Applications, International Journal of Health and Science, 6(52): 2378-2349.

Kreyszig, E. 2014. Advanced Engineering Mathematics, 9th ed., WILEY.

Kuffi, E. A., Abbas, E. S. and Maktoof, S. F. 2019. Solving the Beam deflection problem using Al-Tememe transform, Proc. The 6th International Arab Conference on Mathematics and Computations (IACMC) (14(4): 519-527).

Kuffi, E. A. and Abbas, E. S. 2022a. Applying Al-Zughair transform into some engineering fieldes. In AIP Conference Proceedings. 2386(1): 040027. AIP Publishing LLC.

Kuffi, E. A. and Abbas, E. S. 2022b. A Complex Integral Transform "Complex EE Transform" and Its Applications. Mathematical Statistician and Engineering Applications, 71(2): 263-266.

Kuffi, E. A., Meftin, N. K., Abbas, E. S. and Mansour, E. A. 2021. A Review on the Integral Transforms. Eurasian Journal of Physics, Chemistry and Mathematics, 1: 20-28.

Kuffi, E. A., Mohammed, A. H., Majde, A. Q. and Abbas, E. S. 2020. Applying al-zughair transform on nuclear physics. International Journal of Engineering and Technology, 9(1): 9-11.

Mansour, E. A., Kuffi, E. A. and Mehdi, S. A. 2021b. The new integral transform "SEE transform" and its applications. Periodicals of Engineering and Natural Sciences, 9(2): 1016-1029.

Mansour, E. A., Kuffi, E. A. and Mehdi, S. A. 2021c. Solving Population Growth and Decay Problems Using Complex SEE Transform. In 2021 7th International Conference on Contemporary Information Technology and Mathematics (ICCITM) (pp. 243-246). IEEE.

Mansour, E. A., Mehdi, S. A., Kuffi, E. A. 2021a. The New Integral Transform and its Applications, Int. J. Nonlinear Anal. Appl. 12(2): 849-856.

Maruyama, S. and Moriya, S. 2021. Newton's Law of Cooling: Follow up and exploration. International Journal of Heat and Mass Transfer, 164: 120544.

Mohammed, A. H. and Makttoof, S. F. 2017. A Complex Al-Tememe Transform, International J. of Pure and Eng. Mathematics, 5(2): 17-30.

Mohammed, A. H., Hadi, A. S. and Rasoul, H. N. 2016. Differentiation of the Al-Tememe Transformation And Solving Some Linear Ordinary Differential equation With or Without Initial Conditions. journal of kerbala university, 14: 18-27.

Mortimer, R. G. 2013. Mathematics for Physical Chemistry (Fourth Edition), 4th edition, Elsevier.

Oxford Languages, 2010. Oxford Dictionary of English, 3rd edition, Oxford University Press.

Pundir, S. K. 2017. Integral Transform Methods in Science and Engineering, CB.

Rao, R. U. 2017a. Application of ZZ Transform Method for Newton's Law of Cooling. International Journal of Engineering, Science and Mathematics, 6(5): 166-169.

Rao, R. U. 2017b. ZZ Transform Method for Natural Growth and Decay Problems, International Journal of Progressive Sciences and Technologies (IJPSAT), 5(5): 147-150.

Saičev, A. I. and Woyczyński, W. A. 2018. Distributions in the Physical and Engineering Sciences: Distributional and Fractal Calculus, Integral Transforms and Wavelets/Alexander I. Saichev, Wojbor Woyczynski. Birkhäuser.

Saitoh, S. and Sawano, Y. 2016. Theory of reproducing kernels and applications. Singapore: Springer Singapore.

Xu, K., Liu, H. and Xiao, J. 2021. Static deflection modeling of combined flexible beams using elliptic integral solution. International Journal of Non-Linear Mechanics, 129, 103637.

Zafar, Z. U. 2016. ZZ Transform Method, International Journal of Advanced Engineering and Global Technology, 4(1): 1605-1611.

Zwillinger, D. and Dobrushkin, V. 1998. Handbook of differential equations. Chapman and Hall/CRC.

## CURRICULUM VITAE

### Personal Information

Name and Surname : Ali Hameed ALI ALI

### Education

| | | |
|---|---|---|
| MSc | Çankırı Karatekin University<br>Graduate School of Natural and Applied Sciences<br>Mathematics | 2020-Present |
| Undergraduate | University Mustansiriya<br>College of Education<br>Mathematics | 2015-2019 |

### Academic Activities

1. Ali, A. H and Kuffi, E. A. 2022. The SEA Integral Transform and its Application on Differential equations. International Journal of Health Sciences, 6(S3): 613-622.