

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

**DYNAMICAL SYSTEM ANALYSIS OF
COSMOLOGICAL INFLATION MODELS
WITH AXION-LIKE-PARTICLES (ALP)**

M.Sc. THESIS

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Department of Physics Engineering

Physics Engineering Programme

JANUARY 2022

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

**AXION BENZERİ PARÇACIKLARIN
KOZMOLOJİK ENFLASYON MODELLERİNİN
DİNAMİK SİSTEM ANALİZİ METODU İLE İNCELENMESİ**

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To my family,



FOREWORD

This thesis somehow survived a pandemic and its psychological toll of the quarantine days on me. Along with other bureaucratic problems this thesis has seen, currently I am writing its foreword section with full of stress but with also some relief that it is finally come to an end. There is nothing much left to be said by myself about this work, other than my appreciation to those who helped and supported me immensely during my studies.

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ABBREVIATIONS

GR	: General Relativity
SR	: Special Relativity
Λ-CDM	: Lambda - Cold Dark Matter
FLRW	: Friedmann-Lemaître-Robertson-Walker
FRW	: Friedmann-Robertson-Walker
CMB	: Cosmic Microwave Background
CMBR	: Cosmic Microwave Background Radiation
BBN	: Big Bang Nucleosynthesis
PQ	: Peccei-Quinn
EF	: Einstein Frame
JF	: Jordan Frame
CP	: Charge-Parity
UV	: Ultra-violet
EOS	: Equation of State
ALP	: Axion-Like-Particle
DSA	: Dynamical System Analysis
LST	: Linear Stability Theory
CMT	: Centre Manifold Theory
KCC	: Kosambi-Cartan-Chern
LFM	: Lyapunov Function Method
PSRA	: Potential Slow Roll Approximation
HSRA	: Hubble Slow Roll Approximation



SYMBOLS

c	: Speed of light in vacuum
G	: Gravitational Constant
m_{pl}	: Planck Mass
w	: Equation of state parameter
M_{pl}	: Reduced Planck Mass



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DYNAMICAL SYSTEM ANALYSIS OF COSMOLOGICAL INFLATION MODELS WITH AXION-LIKE-PARTICLES (ALP)

SUMMARY

Inflation theory, developed in 1980 by Alan Guth, solves the two biggest problems of the standard Big Bang cosmology called flatness and horizon problem. The flatness problem essentially is a fine-tuning of the initial value of the energy density problem. The name itself comes from the relation between energy density to critical energy density ratio and curvature parameter. From current observations, we know that the deviation of the ratio of the energy density of the content of the universe to the critical energy density from unity is of the order $O(10^{-3})$. Extrapolating this deviation back in time reveals that, in order to satisfy current observations, the value of the energy density has to be in agreement with the critical energy density of the order $O(10^{-62})$. Therefore this extreme sensitivity to initial conditions arises the flatness problem. The horizon problem is the problem regarding the inexplicability of isotropy and homogeneity in the observed cosmic microwave background radiation (CMBR/CMB). CMB is almost uniformly in agreement on temperature distribution with $T \approx 2.7$ K. One important fact of the CMB is that it contains regions that are separated by a distance larger than the particle horizon. Particle horizon is the definition of distance that light can reach from the start of the universe until now. Thus, regions or simply points in space-time that are separated more than the particle horizon are called causally disconnected regions. Causally disconnected points can never contact each other or ever be in contact previously. Therefore, CMB having causally disconnected patches that are almost in thermal equilibrium arises the question of how are the causally disconnected patches can reach a thermal equilibrium without the possibility of information exchange.

Inflation theory solves those two major problems by introducing an exponential accelerated expansion in the very early universe before the start of the Big Bang theory. This accelerated expansion eventually reveals that there is no need for extreme fine-tuning of initial conditions on the energy density. Furthermore, the theory explains the horizon problem as rapid early accelerated expansion separates regions that were actually in causal contact but now seems to be causally disconnected, by the process called shrinking Hubble radius. There is no shortage of cosmological inflation theory models in the literature, starting from the original inflation theory model called chaotic inflation with squared potential to string theory motivated axion monodromy inflation.

Axions are hypothetical pseudo-Nambu-Goldstone bosons that are emerged from solution to the CP problem, introduced by R.D. Peccei and H. Quinn in 1977. Axions in cosmology are regarded as the scalar field that enjoys the shift symmetry, i.e. $\phi \rightarrow \phi + \text{const}$ which solves the UV sensitivity of slow-roll inflation models.

Cosmological inflation models can be examined by employing a mathematical method called dynamical system analysis. In this thesis, we tried to work out dynamical system analysis of two main axion-like inflation theory models in the linear stability analysis framework. In linear stability analysis, one defines meaningful model variables so that the evolution of said dynamical variables can be written in terms of the defined variables, i.e. there is no explicit dependence on the independent variables of the dynamical variables. This differential equation system building is called an autonomous equation system. Solution of the autonomous equation system yields several or no critical points of the system that the behaviour of mentioned critical points in the phase space can be understood by examining the eigenvalues of the evaluated Jacobian matrix at critical points of the autonomous system. There is more advanced method to determine the behaviour of critical points that fails to be determined in linear stability analysis but the scope of this thesis does not include them and further discussion on the reason for not including them is clarified in the thesis.

We started with the linear stability analysis of a single scalar field having a natural inflation potential with several couplings to the gravity sector of the model. The analysis showed that having a cosine potential form is problematic in the definition of linear stability analysis therefore, we approximated to chaotic-like one. Results showed that in most of the configurations the critical points of the phase space behaves as an unstable point and in other cases linear stability theory fails to determine its behaviour. Moreover, we continued the analysis on the non-Abelian gauge field inflation model with extra scalar introduced to the model as an axion-like particle field with several different potential settings. We omitted the couplings to the gravity sector in this model for simplicity since most of the complexity comes from those said couplings and further difficulty comes from the fact that the model now has a multi-field form by definition. In a scenario where the extra scalar field is free, i.e. zero potential, with F^2 term has the coupling with the axionic field does not provide an inflationary period and by changing the potential to different forms, i.e. exponential, chaotic and general monomial we have found that in exponential case all critical points of the autonomous equation system becomes unstable and in chaotic-like and general monomial setting, none of the points' behaviour can be determined by linear stability analysis. The final attempt of linear stability analysis to axion-like field models was made to save the zero potential case by instead of coupling axion-like field to F^2 term we coupled it to a F^4 term which automatically solves the problem of not having an inflationary period since now the extra contribution coming from the F^4 has the equation of state parameter value of minus one. Although inflationary period is saved, linear stability method suffers from the non-minimal couplings since in order to observe the effect of newly introduced term one needs to use the same dynamical variables defined in the F^2 model, and while most of the equations can be written in required form, some explicit dependence to the coupling functions makes the model non-closed therefore none examinable with the same variables. Therefore, a direct comparison between those two models can not be made without defining a new variable set. As a result, we learned that the examination of axion-like cosmological model is not viable utilizing the dynamical system analysis with linear stability analysis constraint.

AXION BENZERİ PARÇACIKLARIN KOZMOLOJİK ENFLASYON MODELLERİNİN DİNAMİK SİSTEM ANALİZİ METODU İLE İNCELENMESİ

ÖZET

Enflasyon teorisi, standard Büyük Patlama kozmoloji modelinin açıklayamadığı iki problem olan düzlük ve ufuk problemlerine çözüm getirmek üzere 1980 yılında Alan Guth tarafından geliştirilmiştir. Düzlük problemi temelinde evrenin enerji yoğunluğunun başlangıç değerindeki ince ayar problemidir. Adının düzlük olmasının nedeni ise, enerji yoğunluğu ile kritik enerji yoğunluğunun oranının, evrenin eğrilik parametresi ile olan ilişkisinden gelmektedir. Güncel gözlemlerden bildiğimiz kadarı ile enerji yoğunluğunun kritik enerji yoğunluğuna oranı değerinin birden sapması $O(10^{-3})$ mertebesindedir. Bu sapmayı zamanı geri sararak ekstrapolasyon yöntemi ile hesaplırsak, gözlemlerin verdiği değer ile örtüşmesi için enerji yoğunluğu ile kritik enerji yoğunluğu arasında $O(10^{-62})$ mertebesinde bir uyum olmasının gerekliliği görülmektedir. Bu nedenle başlangıç koşullarında görülen aşırı boyutlardaki uyum mecburiyeti düzlük problemini ortaya çıkarmaktadır. Ufuk problemi ise gözlemlendiğimiz kozmik arkaalan ışınlamadaki açıklanamaz düzeydeki homojenlik ve izotropiden kaynaklanmaktadır. Kozmik arkaalan ışınlamı neredeyse mükemmel düzeyde eşit bir sıcaklık dağılımına sahip olup, ortalama sıcaklığı 2.7 Kelvin derecedir. Bu kozmik arkaalan ışınlamadaki bir önemli olgu ise, bazı bölgelerinin birbirinden parçacık ufkundan daha büyük bir uzaklık ile ayırık gözlemlenmesidir. Parçacık ufku, bir ışığın evrenin başlangıcından bugüne kadar ulaşabileceği mesafenin tanımıdır. Bu nedenle, uzay-zamanda parçacık ufkundan daha fazla ayrılan bölgelere veya basitçe noktalara nedensel olarak bağlantısız bölgeler denir. Nedensel olarak bağlantısı kesik olan noktalar asla birbirleriyle iletişime geçemez veya daha önce iletişim halinde olmuş olamazlar. Bu nedenle, neredeyse termal dengede olan nedensel olarak bağlantısız bölgelere sahip olan kozmik arkaalan ışınlamı, nedensel olarak bağlantısız bölgeler arasında bilgi alışverişi olasılığı olmadan termal dengeye nasıl ulaşabilir sorusunu ortaya çıkarmaktadır.

Enflasyon teorisi ise, Büyük Patlama teorisinin başlangıcından önceki çok erken evren dönemine, üstel olarak ivmelenerek genişleme dönemi getirerek bu iki ana sorunu çözer. Bu hızlandırılmış genişleme dönemi, enerji yoğunluğu üzerindeki başlangıç koşullarının aşırı ince ayarına gerek olmadığını ortaya koymaktadır. Ayrıca, enflasyon teorisinin ufuk problemi sorununa getirdiği çözüm ise, erken evrende ivmelenerek genişlemenin enflasyon döneminden önce nedensel ilişkide olan bölgeleri şu anda nedensel olarak kopuk gözlemlenecek kadar ayırması ile açıklıyor. 1980’de kurulan kozmolojik enflasyon modeli devamında 1981’de Andrei Linde tarafından ilk enflasyon modelinin problemlerinden biri olan tekrar ısınma problemi ele alınmıştır. Tekrar ısınma problemi kısaca, çok erken evrende enflasyon dönemi bitişinde ivmelenerek

genişleme nedeni ile soğuyan evrenin sıcaklığının büyük patlama teorisi ile devam edebilmesi ve skaler alanda toplanan enerjinin aktarılarak parçacık oluşumu gibi evrelere sakince bir geçiş yapması için gerekli mekanizma problemidir. Literatürde, kare potansiyeli olan kaotik enflasyon modeli olarak adlandırılan orijinal enflasyon teorisi modelinden başlayarak, sicim teorisi motivasyonu ile yazılan axion monodromi enflasyon modeline kadar kozmolojik enflasyon teorisi modelleri bulunmaktadır.

Axionlar, 1977’de R.D. Peccei ve H. Quinn tarafından tanıtılan, CP probleminin çözümünden ortaya çıkan varsayımsal sözde Nambu-Goldstone bozonlarıdır. Kozmoloji çerçevesinde axionlar ise, yavaş yuvarlanan skaler alan enflasyon modellerinin problemi olan ultraviyole hassaslığı problemini sahip oldukları öteleme benzeri bir sabit ile kaydırma simetrisi ile yok etme özelliğine sahip olan skaler alanlar olarak tanımlanabilir. Ayrıca bu özelliklerinin dışında Abelian olmayan ayar alanlarına birleşik eden skaler alanlar olarak da modellenebilir. Kozmolojik enflasyon modelleri, dinamik sistem analizi adı verilen matematiksel bir yöntem kullanılarak incelenebilir.

Bu tezde, lineer stabilite analizi çerçevesinde iki ana axion benzeri skaler alanın enflasyon teorisi modelinin dinamik sistem analizini yapmaya çalışıldı. Doğrusal kararlılık analizinde, modele özgü anlamlı değişkenler tanımlanır ve bu değişkenlerin evrim denklemleri çıkarılır. Çıkarılan denklemler, tanımlanan dinamik değişkenler cinsinden yazılır, yani evrim denklemlerinde tanımlanan dinamik değişkenlerin bağlı olduğu bağımsız değişkenler açıkça gözükmez. Oluşturulan bu diferansiyel denklem setine otonom denklem sistemi adı verilir. Otonom denklem sisteminin çözümünden bulunan kritik noktaların faz uzayındaki davranışlarının analizi, otonom sistemin Jacobian matrisinin kritik noktalarda hesaplandığı değerinin özdeğerlerine göre belirlenir. Otonom sistemin kritik noktalarının faz uzayı davranışlarının lineer stabilite analizi ile bulunamadığı noktalarda, bu noktaların davranışlarını incelemek için lineer stabilite analizinden daha ileri seviye dinamik sistem analizi metodları bulunmaktadır fakat bu metodlar ile sistem analizi bu tezin içeriğinin dışında olmakla beraber bu metodları neden kullanmayı tercih etmediğimiz ve fiziksel olarak anlamlı sonuç bulunması konusunda bu metodların verdiği sonuçlar ile ilgili kaygılar tezin içeriğinde bulunmaktadır. Lineer stabilite analizine doğal enflasyon modeli potansiyeline sahip tek skaler alanlı ve bu skaler alanın yerçekimi bölümüne minimal olmayan etkileşim fonksiyonu ile bağlı olan modeller ile başlandı. Bu analizin bize gösterdiği sonuç, doğal enflasyon potansiyelinin kosinüs formunda olması nedeni ile potansiyel için yazılması gereken dinamik değişkenin tanımlanamayacağı oldu. Bu nedenle, kosinüs formunda olan doğal enflasyon potansiyelini Taylor serisine açarak, kaotik formda bir potansiyele benzeterek analiz yapıldı. Sonuçlar, neredeyse bütün konfigürasyonlarda, otonom sistemin faz uzayındaki kritik noktalar stabil olmayan bir davranışa sahip, geri kalan kritik noktaların davranışları ise lineer stabilite analizi ile belirlenemedi. Devamında ele aldığımız model, Abelian olmayan ayar alanları ile axion-benzeri skaler alanların etkileşimini ve farklı potansiyel formlarını içeren enflasyon modelleri oldu. Bu modellerde ise, yerçekimi ile olan etkileşimi analizlere kolaylık olması açısından yok saydık ve modeli zorlaştıran en önemli yer, ayar alanlarının axion alanları ile etkileştiği fonksiyonlar oldu. Ayrıca, model kendi tanımı gereği çoklu skaler alan modelleri sınıfına girdiğinden lineer stabilite analizi zorlaştı. Ekstra skaler alanın serbest, yani potansiyelinin sıfır olduğu bir senaryoda skaler alanın durum denklemleri parametresi bir oldu ve axion ile ayar alanı etkileşimi teriminin durum denklemleri parametresi ise enflasyon dönemini tetikleyecek değere sahip aralıkta çıkmamıştır. Skaler alanın potansiyel formunu değiştirerek yapılan analizlerde ise

alınan üstel, kaotik ve genel kuvvet formundaki potansiyellerde bulunan sonuçlarda görüldüğü üzere üstel durumda hiçbir kritik noktanın davranışı stabil çıkmamıştır, kaotik ve genel kuvvet formunda yapılan analizde ise hiçbir noktanın davranışı lineer stabilite analizi ile tespit edilebilir cinsten olmadığı görülmüştür. Axion benzeri skaler alan modellerinde lineer stabilite analizini kurtarabilmek için son bir deneme olarak tekrar potansiyelin sıfır olduğu durum ele alındı. Fakat önceden de görüldüğü üzere serbest axion-benzeri skaler alan ile F^2 formundaki Abelian olmayan ayar alanları modelinde enflasyon dönemi oluşturabilecek içerik olmadığından, F^2 terimi yerine F^4 terimi konuldu. Bu terim, önceki modelden farklı olarak enflasyon dönemi oluşturamama sorununu otomatik olarak ortadan kaldırmaktadır. Bunun nedeni, F^4 teriminde önceden bulunan Yang-Mills katkısına ek olarak, eksi bir değerine sahip durum denklem parametresi ile enflasyon dönemi başlatabilecek katkı geldiği görüldü. Enflasyon dönemi başlatamama problemi çözülmesine rağmen, lineer stabilite analizi sırasında oluşturulmaya çalışılan otonom denklem sisteminde, F^4 terimli Abelian olmayan ayar alanına gelen minimal olmayan etkileşim fonksiyonu nedeniyle otonom sistemde açıkca bulunmaması gereken bağımsız değişkenlerin ortaya çıkmasına neden olmakta. Bunun nedeni ise, potansiyel olmayan F^2 modeli ile potansiyel olmayan F^4 modelinin direkt kıyası için aynı dinamik değişkenlerin kullanması gerekliliğinden kaynaklanmakta. Sonuç olarak ise bire bir kıyasın tanımlanan bu dinamik değişken seti ile yapılamadığı görülmektedir. İki model için de ortak dinamik değişken seti bulmak oldukça zor olduğu kanısına varılmıştır. Tek olarak modelin incelenmesi ise yeni bir dinamik değişken seti tanımlanarak yapılabilir fakat sonuçların kıyasının doğruluğu tartışmaya açık kalacağı için teze konulmamıştır.



1. INTRODUCTION

Cosmology is a branch of physics in which the interest is focused on the origin of the universe by examining its properties on scales that are significantly larger than those of astrophysics. Previously, cosmology was viewed as a part of philosophy [11] since the main quest of the subject is finding answers to questions that are along the lines of "How it all started?", "How it all came to be in this state?" and "How it can all end?". Therefore, cosmology in broad terms tries to answer one of the most fundamental questions of our existence, and although similar in context but with different motivations it is analogous to the work of Paul Gauguin titled "Where Do We Come From? What Are We? Where Are We Going?" in Figure 1.1 in which the painting roughly symbolic representation of circle of life.



Figure 1.1 : "Where Do We Come From? What Are We? Where Are We Going?" by Paul Gauguin (1897).

Currently cosmology is one of the most active research areas of physics in both theoretical and observational aspect. As a section of theoretical physics, it started with the static universe model of Einstein in 1917 [12] after the completion of his theory of general relativity, in which he modeled a universe as frozen in time and homogeneous (independent of the place of the observation) in matter content. In order to obtain a non-vanishing matter density and persistent solution to Einstein field equations, he introduced a term called cosmological constant, Λ . Even though we now know

that the cosmological constant term introduced by the Einstein is one of the popular candidates for dark energy or generally associated with dark energy, at that time after observational revelations made by Edwin Hubble in regards to expanding universe, Einstein dropped the cosmological constant term in his models and it is claimed that he thought of this term as his biggest blunder.

After this observation of expanding universe, exact solution of Einstein field equations for homogeneous, isotropic (independent of the direction of observation) and expanding universe led to Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The independently developed expanding universe model, by Friedmann, Lemaître, Robertson and Walker in 1920's and 1930's, is the basis of current standard model of cosmology called Λ -CDM that is based on the Big Bang theory in which the universe consists of a dark energy Λ , cold dark matter (CDM) and ordinary matter that starts with a singularity.

The fundamental source of information about history of our universe comes from the earliest light source that is called Cosmic Microwave Background Radiation (CMBR or CMB) discovered by Arno Penzias and Robert Wilson [13] that is formed approximately when the universe was 380.000 years old. Universe starts in a very dense, hot and energetic state and begins to expand and causing universe to cool down. Cooling down of the universe allows neutrinos to decouple around 1 second after singularity and neutrons and protons emerges. Furthermore, at about 2 minutes after, nucleosynthesis or Big Bang nucleosynthesis (BBN) occur, i.e. formation of heavier elements by nuclear fusion. After about 20 minutes, universe is expanded enough that nuclear fusion can not occur yet it is still hot for photons to move freely without interaction so that the universe is opaque. Around 18.000 years after singularity, so called recombination epoch starts. As the universe expands and cools down formation of elements continues, starting from helium and around when the universe is 380.000 years old formation of neutral hydrogen atoms end and the universe cools and expands enough that the photons can travel freely thus makes the universe transparent and forms the first source of light that we observe from universe called CMB.

Observation of CMB showed that the universe at large scales is almost in a thermal equilibrium with $\delta T/T = 10^{-5}K$ [14], which ultimately surfaced problems of standard Big Bang cosmology in which the theory lacks the explanation of thermal equilibrium

between causally disconnected patches of CMB referred as horizon problem, and furthermore suffers from extreme fine tuning of initial conditions on the ratio of energy density of content and critical energy density of the universe generally referred as flatness problem since when Friedmann equation written in terms of density parameter, i.e. ratio of energy density to critical density, the deviation of density parameter from unity is related to curvature of the universe.

Resolution of the mentioned problems proposed by Alan Guth in 1980 with the theory of cosmological inflation [15]. Inflation theory introduces an rapid accelerated expansion in the very early universe starts roughly between 10^{-33} and 10^{-32} seconds after the singularity and lasts for about 10^{-36} seconds. This rapid expansion naturally evolves the universe towards a homogeneous and isotropic universe and gives plausible explanations to horizon and flatness problems of Big Bang cosmology. Inflationary period solves the horizon problem by stating that disconnected regions that are in thermal equilibrium in CMB once was actually inside their causal regions so that they exchanged information before and after inflation, universe expanded such that they now appear to us to be causally disconnected. Moreover, inflation period of accelerated expansion naturally removes the fine-tuning of initial conditions on the ratio of energy density to critical density by postulating that during inflation, evolution of Hubble parameter nearly none. Therefore, constant Hubble parameter during inflation results in with the exponential scale factor which rapidly converges the deviation of density parameter from unity to currently observed values and additionally extrapolating the deviation backwards in time yields expected values at nucleosynthesis time, radiation-matter equality time, radiation dominated era and Planck time without extreme fine-tuning.

Even though the inflation theory successfully solves the problems of standard Big Bang cosmology, the theory itself has problems such as after extreme cooling of the universe at the very early times, after inflation period universe must continue its evolution same as in the Big Bang cosmology, therefore there must be a transition from inflationary period to Big Bang evolution called reheating allowing to formation of particles and further development of the universe. Moreover, the actual driving force of the inflationary period is still an unanswered question and most popular candidate among others is a scalar field with nearly flat potential yet Higgs field, i.e. the only scalar

field in the standard model of particle physics proven to be not suitable for driving an inflationary period since first the Higgs potential is not flat and even introducing non-minimal couplings to gravity and transforming the theory to Einstein frame by conformal transformation and approximating the potential to become nearly flat, the coupling parameter needs to be extremely strong to obtain correct density fluctuation amplitude.

Additionally, models of inflation needs to last at least 50-60 e-folds (exponential expansion by a factor of e) to be considered as satisfactory inflation model and every scalar field driven inflation period with nearly flat potential model with viable e-folding count have its own problem known as ultra-violet (UV) sensitivity except a symmetry of the model avoids UV corrections. UV sensitivity is related with the model's susceptibility to corrections coming from Planck-scale physics and it shortens the duration of inflation period to levels that the model no longer be able to sustain enough e-foldings viable to obtain results matching to observations [16]. Therefore, one needs to avoid such correction while building an inflation model. UV corrections can be avoided if a symmetry of the theory in hand either eliminates the correction contributions or suppresses them and such symmetry can be introduced to the theory by employing scalar field that enjoys the shift symmetry, i.e. $\phi \rightarrow \phi + \text{const.}$ and the field with shift symmetry is called an "axion" in this context [16].

First introduction of shift symmetric scalar field inflation model is called natural inflation developed by Katherine Freese, Joshua Frieman and Angela Olinto in 1990 [17]. In natural inflation model the scalar field potential is of the form shown in equation (5.4). Furthermore, motivation for axion or axion-like particle's inflation models not constrained only to cosmology but also to string theory and supersymmetry with models like axion monodromy inflation [18]. Furthermore, axion-like inflation models can be formed by using 4-forms in the theory as done in [19]. Recently published paper by McDonough, Guth and Kaiser in [10] showed that axions can be thought as the phase of the complex scalar field where both radial field and phase field contributes to the inflation evolution in which the model can be categorized as the multi-field models of axion inflation. One popular axion inflation theory is that having non-Abelian gauge fields driving inflationary period as showed in [20], [21], [22]. One important application of this theory is that by employing an extra scalar field with

natural inflation potential along with non-Abelian gauge fields called chromo-natural inflation [23].

In this thesis, we employed dynamical system analysis in the constraint of linear stability analysis to analyze phase space attractor structure of several axion-like inflation models to understand and possibly compare models of axion inflation.

Dynamical system analysis is a widely used method in literature [24], [25], [26], [27], [28], as a mathematical tool of understanding evolutionary behaviour of complex systems by utilizing differential equations and as a comprehensive guide for the application of dynamical systems in cosmology one can refer to [1]. One starts the analysis with defining model specific meaningful dynamical variables and calculates their evolution in the phase space. Furthermore, by rewriting evolution equations in terms of dynamical variables without any explicit dependence of independent variable of defined variables, the resulting differential equation system is called autonomous equation system. Solution of the autonomous equation system yields the so called fixed points or critical points of the system. In the context of linear analysis, one finds the Jacobian matrix of the autonomous equation system and evaluates the matrix at critical points and eigenvalues of evaluated Jacobian matrix defines the behaviour of that critical points.

We tried to analyze axion-like inflation models such as single scalar field models with several different forms of non-minimally coupled scalar field with approximated natural inflation potential and further focused on non-abelian gauge field models with general couplings in the context of dynamical system analysis by constraining ourselves to linear stability analysis.

The structure of this thesis is as follows, in chapter 2 we give a rather quick introduction to cosmology followed by explanation of the cosmological inflation theory in chapter 3 and in chapter 4, we pay attention to models of inflation with scalar fields called scalar-tensor theory of inflation by providing examples of minimal and non-minimal coupled scalar field theories as well as conversion between Einstein and Jordan frames by applying a conformal transformation. In chapter 5, we give a summary regarding the origin of axion particle and adoption of axion particle in cosmology as an axion-like inflaton field presenting some of the axionic inflation potentials that is related to the

analysis done in this thesis. In chapter 6, we start by giving preliminary information about dynamical system analysis method and continue with application of the method to the axion-like inflation models with several different couplings and potentials accompanied by brief discussion on the result of each model. Thus, finishing up with general discussion on the results of our method of analysis to axion-like inflation models and possible future analysis that can be done in chapter 7.



2. FUNDAMENTALS OF COSMOLOGY

2.1 Notation

We will use the so called natural units throughout this thesis in which we will take the speed of light in vacuum as unity $c = 1$. Moreover, in the index notation, Latin indices $i, j, k, a, b, c, \dots$ will run from 1 to 3 and Greek indices $\mu, \nu, \rho, \dots$ will run from 0 to 3 indicating the 3D space and 4D space-time respectively. We will define 4-vectors in the contravariant form and will lower its index via the corresponding metric tensor as,

$$x^\mu \equiv (x^0, x^1, x^2, x^3) \quad (2.1)$$

$$x_\mu = g_{\mu\nu} x^\nu \quad (2.2)$$

Therefore, the basis vectors will be defined as covariant form, such as the partial derivative which is the basis vector of the tangent space,

$$\partial_\mu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \quad (2.3)$$

Metric signature will be mostly plus $(-, +, +, +)$ in agreement with mathematical language of cosmology in literature.

2.2 Expanding Universe

The most fundamental aspect of cosmology is that the universe is expanding. In 1929 paper, Edwin Hubble showed that the distance and velocity of galaxies have a linear relation [2], Hubble's law in equation (2.4). Which showed that the further the galaxy is, faster it is getting away from Earth, proving that the universe is expanding.

$$v = H_0 d \quad (2.4)$$

In equation (2.4), v is the velocity, d is the distance and H_0 is the Hubble's constant. Hubble, in his work [2], by plotting the graph from the observational data of around 20 galaxies, found the Hubble's constant to be approximately 500 km/s/Mpc which turned out to be inaccurate since he underrated the distances to particular galaxies [13].

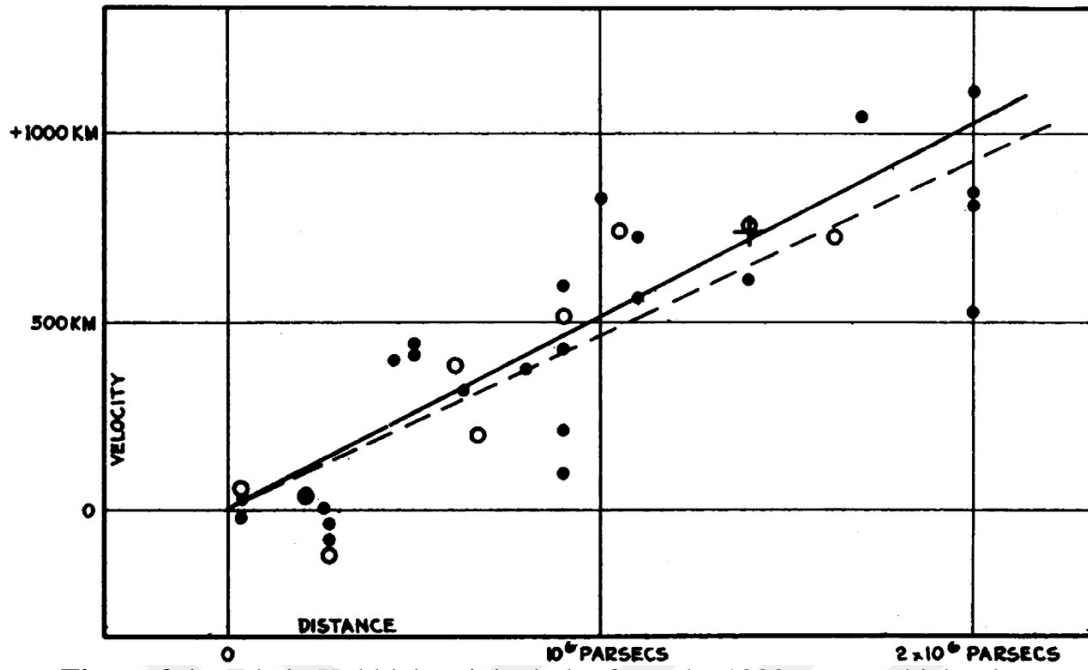


Figure 2.1 : Edwin Hubble's original plot from the 1929 paper, which relates distances of galaxies to its velocities [2].

In the more recent work from 2001, Hubble Space Telescope results showed that the Hubble's constant value is $H_0 = 72 \pm 8$ km/s/Mpc. Moreover, in 2018 paper published by the Planck collaboration gives the value of the Hubble's constant as $H_0 = 67.4 \pm 0.5$ km/s/Mpc [29]. To this day, the value of the Hubble's constant is an open question in cosmology which is called Hubble Tension. The dispute over the constant's value has some promising results over the recent years as the measured value of the constant reaches an asymptote as in 2019, Hubble Space Telescope measured the value $H_0 = 69.8 \pm 0.8$ km/s/Mpc [30].

2.3 Isotropic and Homogeneous Universe

Cosmological principle states that the universe is homogeneous and isotropic over large scales. In other terms, wherever we go in universe (homogeneity) and whichever direction we look at (isotropy), we see similar distribution of matter content. In the context of cosmology, large scales is of order 100 Mpc. To summarize, there are no preferred direction or location in universe when the scale is 100 Mpc or larger.

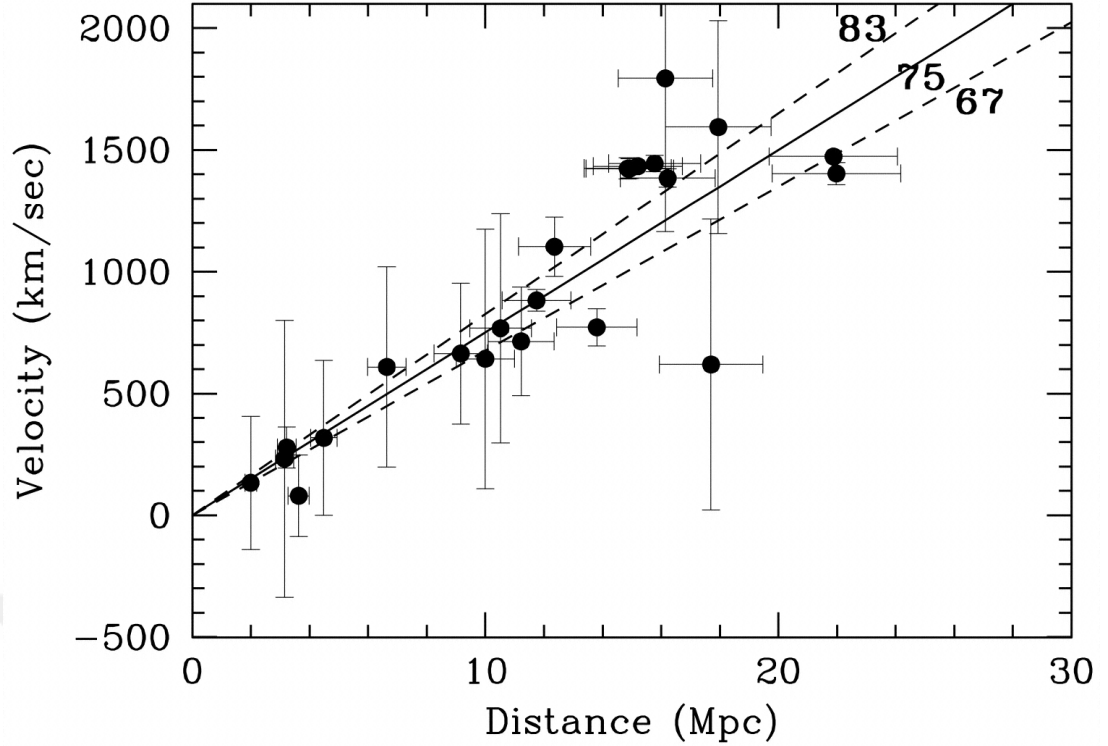


Figure 2.2 : Velocity vs. distance plot from the Hubble Space Telescope in 2001, $H_0 = 72 \pm 8 \text{ km/s/Mpc}$ [3].

2.4 Cosmic Microwave Background Radiation (CMBR or CMB)

Cosmic microwave background, CMB, is the oldest source of radiation we can observe, where photons released when the universe was just 380,000 years old. In early ages, universe was extremely hot, dense and was in thermal equilibrium most of the time. This extreme temperatures meant that there were enormous amount of interactions between particles so that there were no free photons and also no bound atoms. As the universe expanded, it got cooled enough to form atoms and mean free path of the photons got longer. Thus the epoch, which we now call recombination happened when the universe was around 380,000 years old and the cosmic microwave background created. Observations shows that mean temperature of the CMB is $T_0 = 2.72548 \pm 0.00057 \text{ K}$ [31] and fractional fluctuations of the CMB temperature is of order $\delta T/T = 10^{-5} \text{ K}$ [14]. Thus, this result of extreme closeness of temperatures all around the CMB created several obstacles to the standard model of cosmology, ΛCDM .

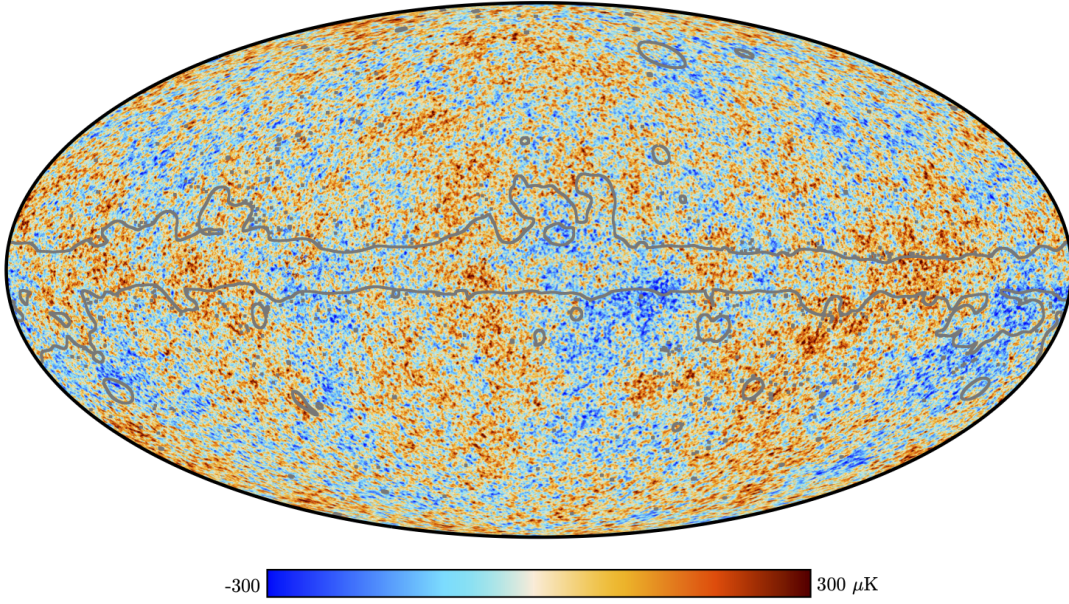


Figure 2.3 : Temperature fluctuations map of CMB from Planck 2018 data [4].

2.5 Geometry of Space-Time

Metric tensor or metric, defines the physical distance between two infinitesimally close spacetime points [14]. Metric provides a physical understanding of distance for example imagine two pair of points on a map (space) where the distance of each pair on the map is equal but the actual distance between those pair of points will differ when you consider the shape of the path (curvature), this is exactly why we have metric.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (2.5)$$

Metric is a (0,2)-tensor written as $g_{\mu\nu}$, whereas its inverse which is a (2,0)-tensor can be written as $g^{\mu\nu}$. Metric tensor is used to raise or lower the indices of a 4-vector or a tensor object as,

$$x_\mu = g_{\mu\nu} x^\nu \quad (2.6)$$

$$A^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} A_{\alpha\beta} \quad (2.7)$$

In Minkowski universe it has a specific symbol, $\eta_{\mu\nu}$, and defined as,

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \quad (2.8)$$

Minkowski metric given in equation (2.8) defines a homogeneous, isotropic and flat universe. For an expanding universe, solution of the Einstein field equations, later

be mentioned, for an universe homogeneous, isotropic and expanding, given by the Friedmann-Lemaître-Robertson-Walker metric in spherical polar coordinates,

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right) \quad (2.9)$$

where k is the curvature constant and has the values $\{-1, 0, 1\}$ corresponds respectively open, flat and closed universes and $a(t)$ is called the scale factor. From now on we will write the scale factor without explicitly writing the time dependence for simplicity in the notation but we know that it is dependent on time. In the Cartesian coordinates and flat universe $k = 0$, it has the following form,

$$ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2) \quad (2.10)$$

$$g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2) \quad (2.11)$$

2.5.1 Derivatives in curved space-time

Derivatives in curved space-time is similar to those in a flat one yet, one needs to take into account that in a curved space, movement of the object on a manifold in a particular chosen path, changes the direction of the resulting vector in accordance to the parallel transport. Similar to the parallel transport case, when one wants to take a derivative of an object in curved spacetime, a connection must be made between flat and curved spacetime. That connection, in the context of general relativity, is called Christoffel symbols, $\Gamma_{\mu\nu}^\rho$.

Instead of using ∂_μ symbol, one uses ∇_μ for derivatives in curved space which is called covariant derivative. Application of the covariant derivative changes according to the object which derivative is taken of. For instance, covariant derivative of a scalar function will yield partial derivative of the scalar alone as it was in a flat universe.

$$\nabla_\mu \phi = \partial_\mu \phi \quad (2.12)$$

On the other hand, covariant derivative of a 4-vector has the following form,

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\alpha}^\nu V^\alpha \quad (2.13)$$

Other variants can be written as,

$$\nabla_\mu T^{\alpha\beta} = \partial_\mu T^{\alpha\beta} + \Gamma_{\mu\lambda}^\alpha T^{\lambda\beta} + \Gamma_{\mu\lambda}^\beta T^{\alpha\lambda} \quad (2.14a)$$

$$\nabla_\mu T_{\alpha\beta} = \partial_\mu T_{\alpha\beta} - \Gamma_{\mu\alpha}^\lambda T_{\lambda\beta} - \Gamma_{\mu\beta}^\lambda T_{\alpha\lambda} \quad (2.14b)$$

For every contravariant index, a Christoffel symbol contracted with the object is added to the partial derivative and for every covariant index it is subtracted. The connection coefficients that are metric-compatible, i.e. covariant derivative of the metric with respect to that connection is zero everywhere ($\nabla_\mu g_{\alpha\beta} = 0$) and torsion free, i.e. torsion tensor ($T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$) is zero has the following form [32],

$$\Gamma^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\alpha} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \quad (2.15)$$

One important feature of the Christoffel symbol is that since it does not transform like a tensor, it is not a tensor even though it has three indices like a tensor. In its torsion free form, it is symmetric in the exchange of bottom two indices.

2.5.2 Riemann tensor, Ricci tensor and scalar

We have already said that result of the parallel transport action depends on the chosen path. Therefore, information about the local curvature of the manifold lies in the difference between two paths or in another way it can be thought of as the change in the vector in the direction of the parallel transport of both paths which can be found using the covariant derivatives [32].

$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \quad (2.16)$$

Above equation is the mathematical representation of the difference between the rate of change of the vector V^ρ in both paths if the vector had been parallel transported. The resulting tensor is called the Riemann tensor or curvature tensor as written as following,

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \quad (2.17)$$

Couple of most important properties are can be shown by lowering the upper index as,

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu} \quad (2.18)$$

Riemann tensor is antisymmetric in its first two and second two indices,

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \quad (2.19a)$$

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu} \quad (2.19b)$$

It is symmetric when first and second pair are exchanged,

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \quad (2.20)$$

Ricci tensor is a $(0, 2)$ -tensor constructed using the Riemann tensor by contracting the upper index with the second lower index,

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} \quad (2.21)$$

Ricci scalar is the contraction of the indices of Ricci tensor or the trace of the Ricci tensor,

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\mu} \quad (2.22)$$

2.5.3 Einstein tensor

Einstein tensor holds the information about geometry of the universe, constructed from the Ricci tensor and Ricci scalar,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (2.23)$$

By definition, it is divergence free as,

$$\nabla_{\mu} G^{\mu\nu} = 0 \quad (2.24)$$

This condition (2.24), as we will see later on, will imply the conservation of energy when we write the Einstein's field equations.

Einstein tensor can be obtained from the least action principle by taking the variation of the Einstein-Hilbert action with respect to the inverse metric,

$$\frac{\delta S}{\delta g^{\mu\nu}} = \int d^4x \frac{(\sqrt{-g}R)}{\delta g^{\mu\nu}} \quad (2.25a)$$

$$= \int d^4x \left[-\frac{1}{2} \sqrt{-g} g_{\mu\nu} R + \sqrt{-g} \frac{\delta R}{\delta g^{\mu\nu}} \right] \quad (2.25b)$$

$$= \int d^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \quad (2.25c)$$

where the term in paranthesis above is the definition of the Einstein tensor, the subtle but allowed cheating we did in this context is in the calculation of variation of the

Ricci scalar where we assumed the boundary term $(g_{\mu\nu}\square - \nabla_\mu \nabla_\nu) \delta g^{\mu\nu}$ vanishes at the limits of the integration and simply we take the variation as $\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu}$, but we will take this into account in more general models where a coupling of some scalar field to the gravity will prevents us to ignore boundary term. The symbol $\square$ corresponds to the D'Alembertian operator and defined as,

$$\square \equiv \nabla_\mu \nabla^\mu \quad (2.26)$$

2.6 Matter Content of Space-Time

2.6.1 Energy-Momentum tensor

As it was the case for the geometry of universe, the aim is to write matter content of universe in the form of a tensor so that we can construct a relation between each other. The tensor for that goal is called energy-momentum tensor, $T_{\mu\nu}$. In the cosmological context, it is convenient to adopt a particular form of energy-momentum tensor called perfect fluid, ultimately, observations are consistent with modelling cosmological matter content as perfect fluid [33].

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_{\mu\nu}P \quad (2.27)$$

where u^μ is the fluid 4-velocity, ρ is the energy density and P is the pressure. The density and pressure we wrote are in the rest frame of the fluid. Moreover, isotropic pressure is needed for it to agree with the metric that is FRW. Finally the normalized 4-velocity also in the rest frame of the fluid and can be written as,

$$u^\mu = (1, 0, 0, 0) \quad (2.28)$$

Therefore the energy-momentum tensor of a perfect fluid in matrix form can be written as,

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & a^2 P & 0 & 0 \\ 0 & 0 & a^2 P & 0 \\ 0 & 0 & 0 & a^2 P \end{pmatrix} \quad (2.29)$$

In general, one can construct the energy-momentum tensor by taking the variation of an action of the model in hand with respect to the inverse metric $g^{\mu\nu}$ and for the

next section when we are working with several models of inflation we will use this technique to obtain an expression for the energy-momentum tensor,

$$\frac{\delta S}{\delta g^{\mu\nu}} = \int d^4x \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \quad (2.30a)$$

$$= \int d^4x \left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \mathcal{L}_m + \sqrt{-g} \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \right] \quad (2.30b)$$

$$= \int d^4x \left[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\mathcal{L}_m + \sqrt{-g} \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \right] \quad (2.30c)$$

$$= -\frac{1}{2} \int d^4x \sqrt{-g} \left[g_{\mu\nu}\mathcal{L}_m - 2 \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \right] \quad (2.30d)$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = \int d^4x \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \quad (2.31a)$$

$$= \int d^4x \left[\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} \mathcal{L}_m + \sqrt{-g} \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \right] \quad (2.31b)$$

$$= \int d^4x \left[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\mathcal{L}_m + \sqrt{-g} \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \right] \quad (2.31c)$$

$$= -\frac{1}{2} \int d^4x \sqrt{-g} \left[g_{\mu\nu}\mathcal{L}_m - 2 \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \right] \quad (2.31d)$$

where the term in the paranthesis is the definition of the energy-momentum tensor and the subscript m stands for matter Lagrangian,

$$T_{\mu\nu} \equiv -2 \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \quad (2.32)$$

2.7 Einstein Field Equations

Einstein field equations can be obtained from the variation of the combination of previous two actions we saw while deriving Einstein tensor and energy-momentum tensor,

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} + \mathcal{L}_m \right) \quad (2.33)$$

Einstein field equations has an integral role in understanding universe as a whole in terms of both geometry and content. The connection between geometry and matter-energy content of universe is made by the field equations as,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \quad (2.34)$$

Left-hand side (LHS) of the equation (2.34) solely describes the geometry of the universe and right-hand side (RHS) characterizes the matter-energy content. Therefore, we can come to conclusions from the above equation such as "How matter content curves the fabric of space-time?" or "How gravity affects its matter content's evolution in space?" by just studying the field equations. Dynamics of universe thus related to the curvature properties by examining components of field equations.

Also a natural result of this equation tells us that, since LHS is divergence free by definition, $\nabla_\mu G^{\mu\nu} = 0$, therefore the RHS must also be divergence free. Thus we arrive at,

$$\nabla_\mu T^{\mu\nu} = 0 \quad (2.35a)$$

$$\partial_\mu T^{\mu\nu} + \Gamma_{\mu\alpha}^\mu T^{\alpha\nu} + \Gamma_{\mu\alpha}^\nu T^{\mu\alpha} = 0 \quad (2.35b)$$

Taking $\nu = 0$ yields,

$$\dot{\rho} + 3\frac{\dot{a}a}{\rho} + 3\frac{\dot{a}}{a}P = 0 \quad (2.36a)$$

$$\dot{\rho} + 3H(\rho + P) = 0 \quad (2.36b)$$

where we define the Hubble parameter, H , as $H \equiv \frac{\dot{a}}{a}$. It describes the rate of change in expansion and therefore we arrived is the famous continuity equation as we promised in the previous section.

2.7.1 Friedmann equations

Dynamical description of isotropic and homogeneous universes can be made using the Friedmann equations which are derived by the Russian physicist Alexander Friedmann in 1922. By taking the 00 component of the field equations we have the first Friedmann equation as,

$$H^2 = \frac{8\pi G}{3}\rho \quad (2.37)$$

Also taking the ii component of the field equations gives,

$$-(2a\ddot{a} + \dot{a}^2) = 8\pi a^2 P \quad (2.38)$$

Using the first Friedmann equation in the ii component of the field equations, we arrive at the second Friedmann equation called the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad (2.39)$$

The metric for the calculation above is the flat-FRW metric, $k = 0$, in Cartesian coordinates in equation (2.10). Furthermore, for the most part of the calculations, we will use the same metric unless otherwise mentioned.

In the case of a non-flat geometry, using the metric in equation (2.9), the Friedmann equation has the form,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (2.40)$$

In the above form we will relate the curvature of the universe to the energy-matter content in the cosmological inflation section to explain the flatness problem and to resolve it.

2.8 Cosmological Horizons

2.8.1 Comoving distance

We mentioned that the metric of expanding homogeneous and isotropic universe is given by the FRW metric as in the equation (2.9). Variables of the FRW metric in spherical coordinates are t is known as the cosmic time or cosmological proper time and r, θ, ϕ namely comoving coordinates. In the case of homogeneous and isotropic expansion of the universe, comoving coordinates stays constant during the expansion [13]. Thus, allows us to define a grid system called comoving grid and distance measure on this grid becomes a fundamental distance definition where all other definitions may be inferred from this fundamental distance [14], called comoving distance. The calculation of the comoving distance between two points, p_1 and p_2 , on the comoving grid is simply given as,

$$\chi = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad (2.41)$$

From the FRW metric we can write the comoving distance for light as,

$$\eta = \int \frac{dt'}{a(t')} \quad (2.42)$$

The comoving distance expression for light particles gives us an important measure which describes the distance that light could have travelled since $t = 0$ if there are no interaction [14]. Another important result we can deduce from this expression is

that, two points or regions located at a distance larger than the comoving distance for light, η , can not communicate at the time that is observed, since light can not reach to that distance even if it is emitted at time $t = 0$, which in this case we know call those points or regions *causally disconnected*. The definition of comoving distance equation (2.42) can also be thought of as a time measurement variable called *conformal time*, η , since it is monotonically increasing and it is convenient to use conformal time while presenting space-time diagrams considering the trajectory of light becomes straight lines aligned at a 45 degree angles, and also it is widely used in cosmological perturbation calculations as a reason of simple expressions in calculations.

2.8.2 Particle horizon

To emphasize the concept of horizons in cosmology, we can use the equation (2.42), to rewrite FRW metric Eq (2.10) as following,

$$ds^2 = a^2(\eta) [-d\eta^2 + dx^2 + dy^2 + dz^2] \quad (2.43)$$

$$= a^a(\eta) [-d\eta^2 + d\chi^2] \quad (2.44)$$

where we truncated the 3 dimensional spatial part of the metric to 1 dimension for practicality reasons. Photons travel along the null geodesic [7], meaning that the $ds^2 = 0$ for photons. Therefore, one can easily see the motivation of why we use the conformal time in our metric, since photons travel along the null geodesic photon world-lines sits at a 45° angle when one draws the $\eta - \chi$ plane [7] otherwise if one uses cosmic time t the lines are curved as both cases can be seen in middle and bottom plots of figure 2.5.

$$d\chi = \pm d\eta \quad (2.45)$$

Equation (2.45) tells us the maximum distance that a photon can travel between two times. Therefore, the maximum distance that a photon can travel since the "beginning of the universe", i.e. Big Bang if the universe starts from a singularity, in which the cosmic time is taken as $t_{BB} = 0$, where BB stands for Big Bang, to an observer at time η can be written as,

$$\chi_{ph} = \eta - \eta_{BB} = \int_0^t \frac{dt}{a(t)} \quad (2.46)$$

The calculation in equation (2.46) is the definition of the particle horizon. It sets a limit to the maximum distance that an information travelling at the speed of light can reach to an observer standing at time t . All causal interactions happens inside the particle horizon and in order to information exchange happen between two entities their past light cones must intersect [7].

2.8.3 Event horizon

Event horizon is sort of the opposite or the other way around of the particle horizon. Similarly to the limit to the affect of events happening in the past, i.e. particle horizon, there is a limit to the events that an observer can have an influence on or even able to observe in the future that we call event horizon. The mathematical definition of the event horizon can be shown as in equation (2.47).

$$\chi_{eh} = \eta_f - \eta = \int_t^{t_f} \frac{dt}{a(t)} \quad (2.47)$$

Interpretation of the event horizon is that the maximum distance that an observer sitting at time t_f can receive signals emitted from or to be influenced by the observer sitting at time t [7].

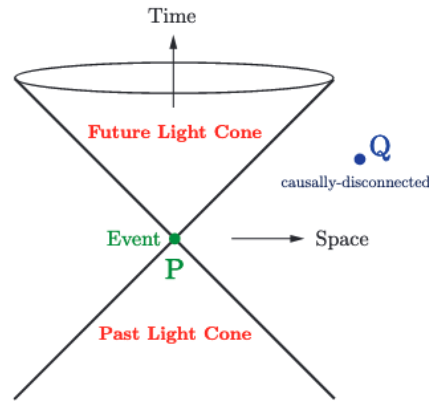


Figure 2.4 : Light cones of an observer or an event P , world lines of photons are the ones with 45° angled lines and space-like seperated regions becomes causally disconnected [5].

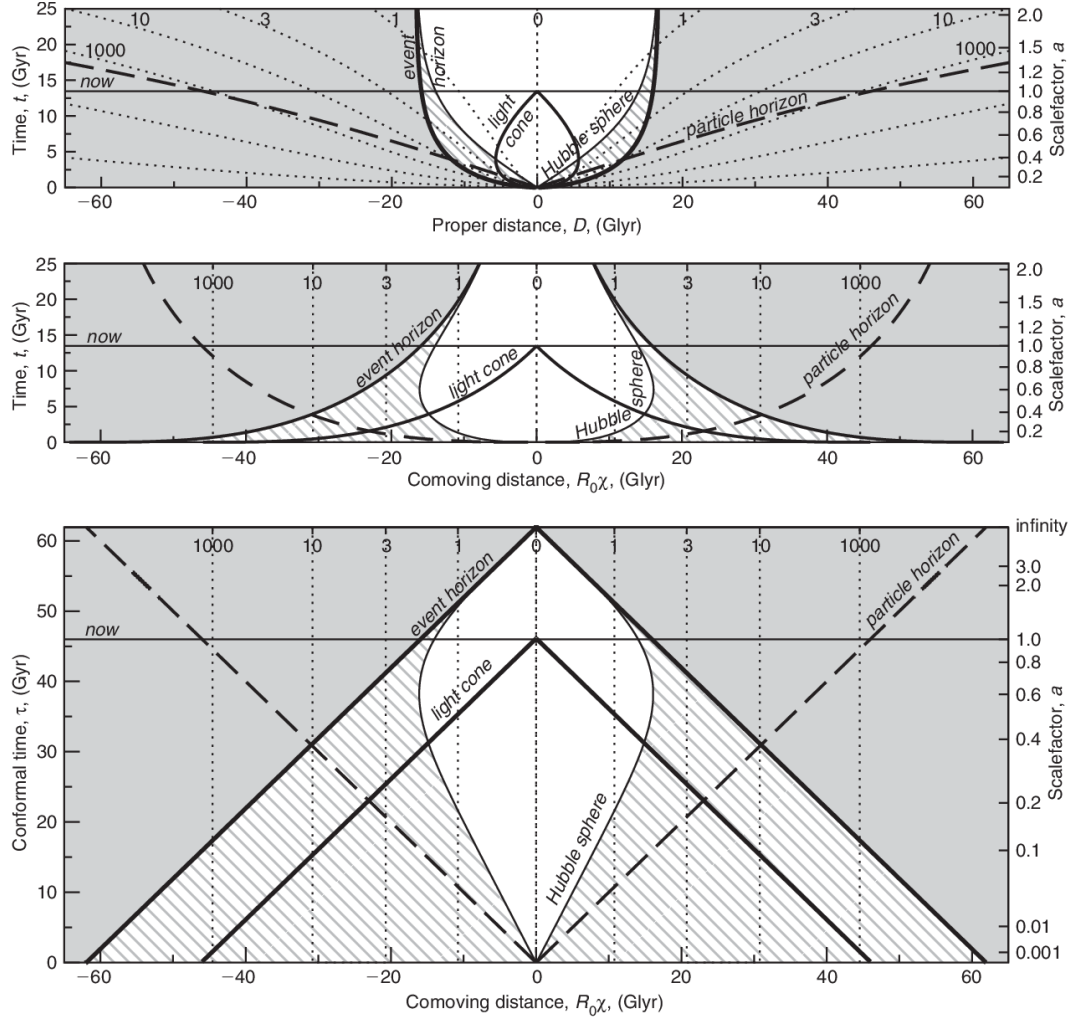


Figure 2.5 : Spacetime diagrams showing wordlines of comoving objects (dotted lines), Hubble spheres and light cones in different distance and time measures i.e. from top to bottom cosmic time t vs proper distance, cosmic time vs comoving distance and conformal time vs comoving distance. The universe model taken to be as $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$. Objects that are lying outside the Hubble sphere receding faster than speed of light and only objects that have overlapping past light cones can influence each other [6].

3. COSMOLOGICAL INFLATION THEORY

Cosmological inflation theory introduces a period at the very beginning of the universe of exponential accelerated expansion of the universe where aims to overcome problems of standard big bang model. The standard hot big bang model of the universe suffers from lacking of explaining the observational results of Cosmic Microwave Background Radiation. Standard model of cosmology can not explain why the universe is this much homogeneous and isotropic in large scale which the problem is called the "horizon problem" and why the geometry of the universe depends extremely fine tuned initial conditions which is called the "flatness problem". In 1981, Alan Guth published his paper on a possible solution to those unresolved problems of the standard cosmology, mainly the horizon and flatness problems, called Cosmological Inflation [15]. The theory proposes an accelerated expansion in a supercooled false vacuum state in the early universe which solves both problems in a very simple and elegant way but it also suffers a problem called the graceful exit [34]. The graceful exit problem is where the universe can not end its inflationary phase and does not allow reheating of the universe which prevents the formation of large-scale structure also one of the two cases of the problem results in a very inhomogeneous universe which contradicts the observations, thus it is now called "old inflation". Inflation theory further developed by Andrei Linde in 1981, where he proposed a modification of the original model what is now the inflation can be either start from an unstable state at the top of an effective potential or as it was in the original model in a false vacuum [34]. In the new inflation model, inflation driving field, inflaton, will be doing a so called slow-roll inflation where the field slowly rolls down to the minimum its effective potential [34, 35].

3.1 Problems of Standard Big Bang Cosmology

3.1.1 Horizon problem

Horizon problem, in essence a causality problem and first mentioned by Wolfgang Rindler in 1956 [36]. The problem at its simplest definition can be stated as the

problem of unexplained homogeneity and isotropy of the universe. Temperature fluctuation of the CMB is of order $\delta T \propto \mathcal{O}(10^{-5})$ i.e. it is nearly in a state of thermal equilibrium, therefore we understand that it is almost homogeneous and isotropic at the cosmological scale. In order to CMB to be in thermal equilibrium for some interval of time every part of it must be in contact or in other words every part of it must somehow exchange information with one another. On the other hand, if any two patches of CMB that have angular separation angle that is greater than 1 degrees or the distance of those patches are greater than the particle horizon distance χ_{ph} , then we know that those two patches can never be in a state of exchanging information. CMB has around 10^4 patches that in causal disconnection [5]. This state is known as causal disconnection and the patches in this context can not ever be in contact to send information or reach the equilibrium, let alone an agreement of temperature of order 10^{-5} . This is known as the horizon problem, the state of almost thermal equilibrium in a setting of causal disconnections. To see how the particle horizon is related to the horizon problem one can rewrite the particle horizon as,

$$\chi_{ph} = \int_0^t \frac{dt}{a} = \int_0^a \frac{da}{a\dot{a}} = \int_0^a (aH)^{-1} d \ln a \quad (3.1)$$

where $(aH)^{-1}$ is the Hubble radius. Hubble radius is the radius of the Hubble sphere is the region of the observable universe, objects situated beyond this region are in causal disconnection with the observer because of the Hubble expansion. The difference between Hubble radius $(aH)^{-1}$ and particle horizon χ_{ph} is that the observers at a distance greater than the particle horizon can never be in contact yet observers at a distance greater than the Hubble radius can never be in contact now but can be in contact at a later time or they might have been in contact beforehand [5]. Then the question is "How does the Hubble radius evolves with time in the standard Big Bang Cosmology?". From the Friedmann equation we have,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \propto \rho \quad (3.2a)$$

$$\frac{\dot{a}}{a} \propto \rho^{1/2} = a^{-\frac{3(1+w)}{2}} \quad (3.2b)$$

$$(aH)^{-1} = \frac{1}{\dot{a}} \propto a^{\frac{1}{2}(1+3w)} \quad (3.2c)$$

For all known matter sources $(1+3w) > 0$ thus Hubble radius is monotonically increasing. At the time of CMB, Hubble radius is greater than the particle horizon

$(aH)^{-1} > \chi_{ph}$ which creates the problem [7]. Therefore, some patches that looks in contact with each other i.e. that are inside the Hubble radius are actually at a distance that is larger than the causal contact distance i.e. outside the particle horizon.

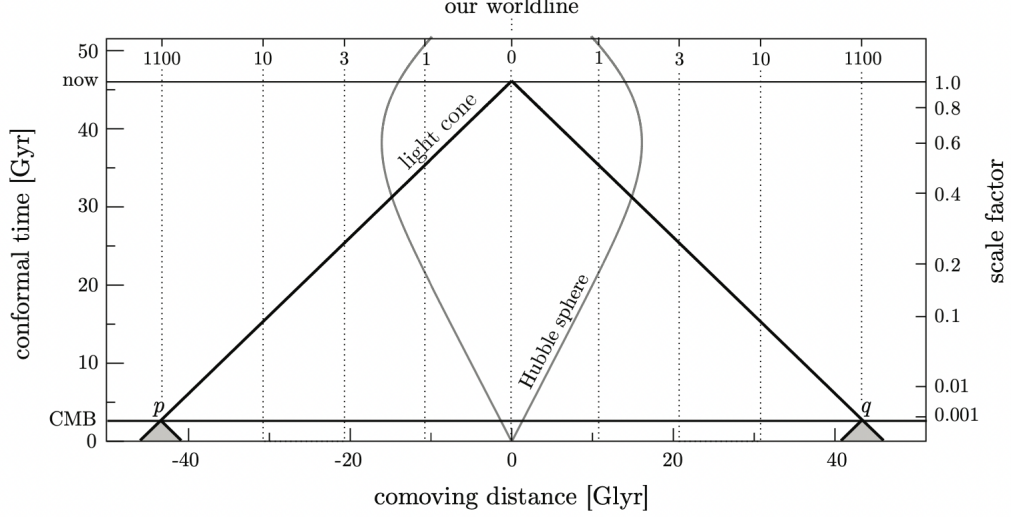


Figure 3.1 : Horizon problem illustration as two points on the CMB p and q has their past light cones do not intersect before the singularity in the standard Big Bang scenario, therefore there is no interaction/causal contact between them and they lie outside of the Hubble sphere [7].

3.1.2 Flatness problem

Flatness problem arises if we look into the Friedmann equation and infer conclusions from its geometrical interpretation. In its core, it is an initial value problem or fine-tuning problem of the current state of the universe observations being extremely dependent on the initial values. In general non-flat universe we write the following Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (3.3)$$

By rearranging the equation and defining the critical density and density parameter at current time as $\rho_c = 3H^2/(8\pi G)$, $\Omega = \rho/\rho_c$ we arrive at,

$$1 = \Omega - \frac{k}{a^2 H^2} \quad (3.4a)$$

$$(1 - \Omega) = -\frac{k}{a^2 H^2} \quad (3.4b)$$

Equation (3.4b) shows that the geometry of the universe whether it is flat, open or closed depends on the density parameter or energy density-critical energy density ratio.

Curvature constant k can take values $\{-1, 0, +1\}$ for open, flat and closed universes respectively. Thus, density parameter's deviation from unity decides the fate of the geometry of the universe. If we look at the current time, the equation takes the form as we define the current scale factor as $a(t = t_0) \equiv 1$,

$$(1 - \Omega_0) = -\frac{k}{H_0^2} \quad (3.5)$$

Current observations shows that density parameter's deviation from unity is,

$$|1 - \Omega_0| \leq 0.005 \quad (3.6)$$

If we go backwards in time and calculate the deviation we find that at the time of matter-radiation equality its value is $|1 - \Omega_{rm}| \leq 2.34 \times 10^{-6}$ and at the time of big bang nucleosynthesis $|1 - \Omega_{bbn}| \leq 7.2 \times 10^{-16}$ and finally at Planck time the deviation reduces to $|1 - \Omega_{pl}| \leq 2.22 \times 10^{-62}$. Probability of randomly choosing an initial value this exceedingly precise raises the problem and one can neither assume nor agree that this choice is a coincidence [37]. Therefore, this extremely small initial value dependence of the density parameter is called flatness problem.

3.2 Inflationary Solution

In the original paper [15], Guth proposes a solution for the above mentioned problems in terms of large entropy production. Alternatively, one can explain solutions of those problems in other ways in which we will cover. In section 3.2.1, we will explain one of the ways which one can solve the horizon problem called shrinking hubble sphere. In that method one postulates a time period where the Hubble radius gets smaller as opposed to getting larger and continues until a certain point after which the standard Big Bang theory evolution begins. This shrinkage makes it possible that in a time interval the Hubble radius is actually smaller than the particle horizon thus enables a possiblity of interaction between contents of the universe now known to us causally disconnected therefore explains the homogeneity. In section 3.2.2, we will examine the case where in order to solve the flatness problem, one can introduce a matter source that can drive the period of inflation where that source keeps the Hubble parameter constant in the inflation period. By keeping the Hubble parameter constant at a certain time period at the beginning of the universe one can see that the need of

the extremely precise initial conditions disappears and inflation period itself always evolves the universe to the state that we observe now.

3.2.1 Shrinking Hubble sphere

In Figure 3.1, one can see that the two points located at opposite sides of the CMB have non overlapping past light cones therefore no interactions between them and the cause of this was we observe CMB now and at this time the Hubble radius $(aH)^{-1}$ i.e. the area of current causal interactions, is larger than the particle horizon χ_{ph} i.e. the area of all possible causal interactions. If one thinks of a time interval where the Hubble radius decreases, the problem of having causally connected regions being bigger than the possible causal interaction region disappears.

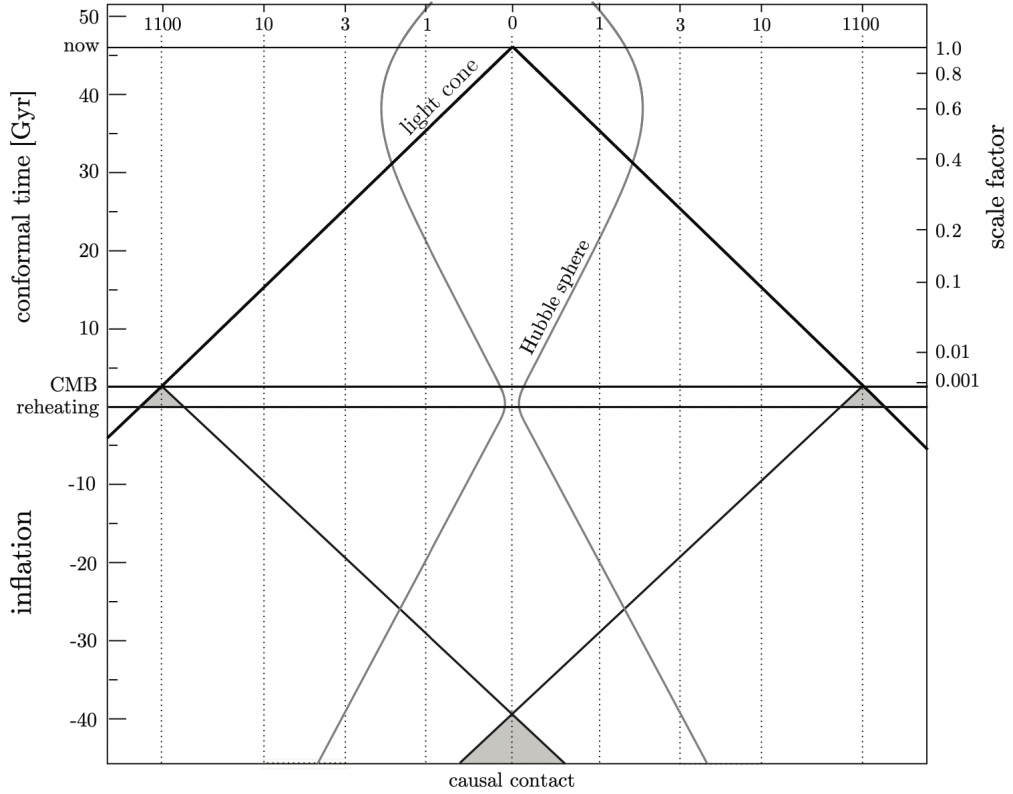


Figure 3.2 : Shrinking hubble sphere solution of the horizon problem. The points p and q before had non overlapping past light cones now have overlapping past light cones, thus the causal interaction happened at an earlier time allowing them to thermally stabilize [7].

By allowing the Hubble sphere decrease for at the early universe results in an accelerated expansion automatically. One can see that as,

$$(aH)^{-1} = (\dot{a})^{-1} \rightarrow \text{decreasing} \quad (3.7a)$$

$$\dot{a} \rightarrow \text{increasing} \quad (3.7b)$$

$$\ddot{a} > 0 \rightarrow \text{accelerated expansion} \quad (3.7c)$$

Thus, one can see that at early universe an accelerated expansion period solves the horizon problem by shirinking the Hubble radius [14].

3.2.2 Flatness problem

Flatness problem can be resolved by assuming that during the inflationary period the Hubble parameter nearly constant just as in the horizon problem solution. As a consequence of not allowing the Hubble parameter evolve with time in the inflation period we can solve the scale factor $a(t)$ as,

$$\frac{\dot{a}}{a} = H \sim \text{const.} \quad (3.8a)$$

$$\frac{da}{a} = H dt \quad (3.8b)$$

$$a(t) \propto e^{Ht} \quad (3.8c)$$

Turning back to the Friedmann equation written in terms of energy density parameters, we can plug in the scale factor in equation (3.8c) into equation (3.4b). One can immediatly see the evolution of deviation of density parameter from one as in Eq (3.9b). Noting that since we are in the inflation period Hubble parameter is assumed to be nearly constant.

$$1 - \Omega(t) = -\frac{k}{e^{2Ht} H^2} \quad (3.9a)$$

$$|1 - \Omega(t)| \propto e^{-2Ht} \quad (3.9b)$$

Therefore, one can see from the equation (3.9b), the energy density parameter's deviation from unity exponentially drops down with time. As a result, an exponential expansion of the universe in the beginning solves the flatness problem since, now in order to match the observational constraint to the deviation of energy density parameter, equation (3.6) one does not need to fine tune initial conditions.

4. SCALAR-TENSOR THEORIES OF INFLATION

Scalar-tensor theory by definition incorporates a non vanishing coupling between gravity sector and a scalar field. The idea of a scalar-tensor theory originally developed by Jordan [38]. He wrote the following Lagrangian for a scalar field living in the four dimensional space-time as [39],

$$\mathcal{L}_J = \sqrt{-g} \left[\varphi_J^\gamma \left(R - w_J \frac{1}{\varphi_J^2} g^{\mu\nu} \partial_\mu \varphi_J \partial_\nu \varphi_J \right) + L_{matter}(\varphi_J, \Phi) \right] \quad (4.1)$$

where w_J and γ is constant. The choice of γ is irrelevant in the case of matter Lagrangian is independent of φ_J [39]. Thus, by taking its value as one, $\gamma = 1$, we reach the fundamental Lagrangian for the scalar-tensor theory called Brans-Dicke Lagrangian [40].

$$\varphi \equiv \varphi_J \quad (4.2a)$$

$$L_{BD} = \sqrt{-g} \left[\varphi R - \frac{w}{\varphi} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + L_{matter} \right] \quad (4.2b)$$

Now instead of diving in detail of scalar-tensor theory and Brans-Dicke Lagrangian, in order to clearly make our point in distinguishing between Einstein and Jordan frames we can directly look for more simplistic scalar field actions. Therefore we can write the simplest action for a scalar field ϕ with a potential $V(\phi)$ as,

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \quad (4.3)$$

In cosmological scenarios we have the gravity part included as the Einstein-Hilbert action as,

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \quad (4.4)$$

From here we can derive the equations of motion for the scalar field and energy momentum tensor by taking the variation w.r.t. scalar field ϕ and inverse metric $g^{\mu\nu}$ respectively.

In scalar-tensor theories of inflation we have two main structures of action called minimally coupled and non-minimally coupled models. Models where the scalar field

explicitly couples with gravity are called non-minimally coupled and models where the scalar field is decoupled from the gravity are called minimally coupled models.

When there is an explicit scalar field - gravity couplings are in general expected to produce closer to actual physical result since we expect the scalar field interact with the gravity at some level yet explicit coupling results in complex equations of motion and makes the analytical and numerical calculations more challenging than minimal couplings. Therefore, most of the time even though the model has an explicit coupling to gravity, we do a so called conformal transformation to change the action from non-minimal to a minimal one mathematically. Moreover, an action with explicit coupling in general is called to be in Jordan Frame (JF) and after application of conformal transformation the resulting minimal coupled model is called to be in Einstein Frame (EF) where the analytical and numerical analysis significantly easier thus it is mostly preferred. In literature, there are extensive discussion on equivalency of the represented physics by both frames with curvature perturbation calculations [41], [42], [43], [44], [45]. Nevertheless, the contradicting arguments are also present that defends the inequivalency of both frames [46], [47], [48], [49], [50], [51].

4.1 Minimally Coupled Scalar Field Models

A very simple minimally coupled inflation model with a scalar field $\phi(t)$ in FRW space-time can be written as the following action,

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \quad (4.5)$$

By taking variation w.r.t. scalar field we have the equation of motion as,

$$\square \phi - V'(\phi) = 0 \quad (4.6a)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \quad (4.6b)$$

Where prime notation ($'$) denotes the derivation with respect to the scalar field ϕ . As one can see we have the Klein-Gordon equation as a result. Moreover, taking the variation w.r.t. inverse metric $g^{\mu\nu}$ we have,

$$G_{\mu\nu} = 8\pi G \left[\partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left(\frac{\dot{\phi}^2}{2} - V(\phi) \right) \right] \quad (4.7)$$

where if one inspects the components, 00 and ii , find the Friedmann equations as follows,

$$3H^2 = 8\pi G \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right) = 8\pi G \rho_\phi \quad (4.8a)$$

$$-2\dot{H} - 3H^2 = 8\pi G \left(\frac{\dot{\phi}^2}{2} - V(\phi) \right) = 8\pi G P_\phi \quad (4.8b)$$

where energy density ρ and pressure P of the scalar field can be found from the energy momentum tensor as,

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left(\frac{\dot{\phi}^2}{2} - V(\phi) \right) \quad (4.9)$$

as its temporal component being the energy density and its spatial component being the pressure,

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \quad (4.10a)$$

$$P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \quad (4.10b)$$

4.2 Non-minimally Coupled Scalar Field Models

Non-minimal coupling counterpart of the previous action 4.5 can be written as,

$$S = \int d^4x \sqrt{-g} \left(F(\phi) R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \quad (4.11)$$

where $F(\phi)$ is an arbitrary coupling function. By taking the variation again w.r.t. scalar field, we have the following equation of motion,

$$\square \phi - V'(\phi) + F'(\phi) R = 0 \quad (4.12a)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = F'(\phi) R \quad (4.12b)$$

We can see that, the non-minimal coupling manifested itself in the equation of motion and from now on scalar field dependence of the coupling function $F(\phi)$ will not be written in hopes to not clutter mathematical expressions. Moreover, if one further derives the Friedmann equations finds the following expressions,

$$\begin{aligned} F G_{\mu\nu} = & \left(F'' + \frac{1}{2} \right) \partial_\mu \phi \partial_\nu \phi + F' \left(\partial_\mu \partial_\nu \phi - \Gamma_{\mu\nu}^\alpha \partial_\alpha \phi \right) \\ & + g_{\mu\nu} \left(\frac{\dot{\phi}^2}{4} - \frac{V}{2} + F'' \dot{\phi}^2 + F' (\ddot{\phi} + 3H\dot{\phi}) \right) \end{aligned} \quad (4.13a)$$

$$6H^2F = \frac{\dot{\phi}^2}{2} + V - 6H\dot{\phi}F' \quad (4.14a)$$

$$-4\dot{H}F - 6H^2F = \frac{\dot{\phi}^2}{2} - V + 4HF'\dot{\phi} + 2F'\ddot{\phi} + 2F''\dot{\phi}^2 \quad (4.14b)$$

As we can see comparing with the equations of minimally coupled model, non-minimally coupled model have explicit coupling function present in its equations therefore making any calculation slightly challenging in the Jordan frame. General agreement in the literature is that both Einstein and Jordan frames represent the same physics and therefore one can write theory in Jordan frame with explicit coupling and make a conformal transformation to change the frame to Einstein and perform calculations there to overcome complexity of the Jordan frame.

4.3 Conformal Transformation

Conformal transformation is a way to map Jordan frame to Einstein frame by rescaling the metric $g_{\mu\nu}$. In general let us assume the action of our model in Jordan frame is given as,

$$S_J = \int d^4x \sqrt{-g} \left(F(\phi) R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V_J(\phi) \right) \quad (4.15)$$

where subscript J stands for Jordan frame. Our goal is to transform the action in equation (4.15) into the form of Einstein frame as in equation (4.16).

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - V_E(\phi) \right) \quad (4.16)$$

By making a conformal transformation of the form equation (4.17a) one can map Jordan frame to Einstein frame [52].

$$\tilde{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu} \quad (4.17a)$$

$$\Omega^2(\phi) = F(\phi) \quad (4.17b)$$

The conformal factor in Eq (4.17b) is chosen such that when the Ricci scalar is calculated in the Einstein frame with new metric $\tilde{g}_{\mu\nu}$, the factor in front of the Ricci scalar becomes unity. Expressions with tilde symbol remarks the symbols in Einstein frame counterpart of the ones in Jordan frame. To see that in mathematical form, one

calculates the Ricci scalar and metric determinant in Einstein frame with conformal transformed metric and finds,

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g} \quad (4.18a)$$

$$\tilde{R} = \frac{R}{\Omega^2} - 6 \frac{\square \Omega}{\Omega^3} \quad (4.18b)$$

Thus, inserting equation (4.18) into equation (4.15) we have the gravity part as,

$$S_{J \rightarrow E} = \int d^4x \sqrt{-\tilde{g}} \Omega^{-4} \left(F(\phi) \left(\tilde{R} \Omega^2 + 6 \frac{\square \Omega}{\Omega} \right) + \dots \right) \quad (4.19a)$$

$$= \int d^4x \sqrt{-\tilde{g}} \left(\frac{F}{\Omega^2} \tilde{R} + \dots \right) \quad (4.19b)$$

Therefore, we arrive at the conclusion equation (4.17b). Here we will not derive the full transformation of the kinetic part but for interested reader rest of the calculation can be followed in the third chapter in Ref. [39]. The scalar field relation is also given in terms of the canonically normalized field φ as,

$$\left(\frac{d\varphi}{d\phi} \right)^2 = \frac{1}{F(\phi)} + 3 \left(\frac{F'(\phi)}{F(\phi)} \right)^2 \quad (4.20)$$

Relation between two potentials can be found using the same method in the calculation of the term with gravity part and the result is given as in equation (4.22). The derivation is straightforward and can be showed by focusing on the potential part as,

$$S_{J \rightarrow E} = \int d^4x \sqrt{-\tilde{g}} \left(\dots + \frac{V_J}{\Omega^4} \right) \quad (4.21a)$$

where the Ω^4 comes directly from the metric determinant and to convert it to the form in equation (4.16) we define the Einstein frame potential V_E as following,

$$V_E(\varphi) = \frac{V_J(\phi)}{\Omega^4} \quad (4.22)$$

Thus we complete the conformal transformation between Jordan frame, equation (4.15), to Einstein frame, equation (4.16).

4.4 Slow-roll Approximation in Inflation Theory

Models of scalar field inflation models mostly based on the assumption that the field ϕ evolves slowly in a potential $V(\phi)$ [53] [54]. The approximation has its foundations on the premise of slowly changing terms can be neglected in the equations of motion,

therefore simplifies calculations. The method is widely used in the literature and two approaches developed over the years. First one puts constraints on the form of the potential and requires the field to evolve to its asymptotic form, it is called the Potential Slow Roll Approximation (PSRA) [53] [54]. Second one puts constraints on the Hubble parameter, thus called Hubble Slow Roll Approximation (HSRA) [54] [55]. In simplest scalar tensor theory action we have the action as in equation (4.5). Equations of motion of the action can be calculated as in equation (4.6b) and furthermore the Friedmann equations as in equation (4.8a). The assumption of the scalar field's slow evolution reduces the equation (4.6b) to the following form,

$$\dot{\phi} \approx -\frac{V'(\phi)}{3H} \quad (4.23)$$

and the form of the Friedmann equation takes the form,

$$H^2 \approx \frac{8\pi G}{3} V(\phi) \quad (4.24)$$

In the case of PSRA, the following parameters defined by [56], where the parameters smallness justifies the neglect of the acceleration term in the equations of motion, i.e. $(\ddot{\phi})$ and kinetic term in the Friedmann equation, i.e. $(\dot{\phi}/2)$, [54]. The PSRA parameters can be written as,

$$\varepsilon_V = \frac{1}{16\pi G} \left(\frac{V'(\phi)}{V(\phi)} \right)^2 = \frac{M_{pl}^2}{2} \left(\frac{V'(\phi)}{V(\phi)} \right)^2 = \frac{m_{pl}^2}{16\pi} \left(\frac{V'(\phi)}{V(\phi)} \right)^2 \quad (4.25a)$$

$$\eta_V = \frac{1}{8\pi G} \frac{V''(\phi)}{V(\phi)} = M_{pl}^2 \frac{V''(\phi)}{V(\phi)} = \frac{m_{pl}^2}{8\pi} \frac{V''(\phi)}{V(\phi)} \quad (4.25b)$$

Whereas, in the case of HSRA, the slow roll parameters are defined as a function of Hubble parameter and its derivative as in equation (4.26) [54],

$$\varepsilon_H = \frac{m_{pl}^2}{4\pi} \left(\frac{H'(\phi)}{H(\phi)} \right)^2 \quad (4.26a)$$

$$\eta_H = \frac{m_{pl}^2}{4\pi} \frac{H''(\phi)}{H(\phi)} \quad (4.26b)$$

In order for a slow roll approximation, the smallness of parameters defined in equation (4.25a) and equation (4.25b) necessary but not sufficient, since the initial value of the $\dot{\phi}$ remains as a free parameter and furthermore the equation (4.23) needs to hold [54]. Furthermore, calculation of number of e-foldings can be written as the

following integral,

$$N = \int_{t_i}^{t_f} H dt \quad (4.27)$$

By using the slow roll parameters and numerically solving the number of e-folds integral to find initial value of the field in equation (4.27) one can calculate the cosmological parameters spectral index of the primordial scalar perturbations $n_s \equiv 1 + 2\eta - 6\epsilon$ and the tensor-to-scalar ratio $r \equiv 16\epsilon$ to compare model in hand to observations. One example of this calculation can be seen in the Ref. [8]. There, a non-minimally coupled scalar field model with chaotic potential $V(\phi) \propto \phi^2$ is analyzed by first writing the model in Jordan frame with coupling function $F(\phi) = \frac{1}{2}(1 + \xi\phi^2)$ and by conformal transformation the cosmological parameters calculated in Einstein frame. For an inflation period of 50 e-foldings and 60 e-foldings, the spectral index n_s and tensor-to-scalar ratio r plotted as a function of the coupling parameter ξ to examine the affect of coupling parameter to the cosmological parameters as in figure 4.1

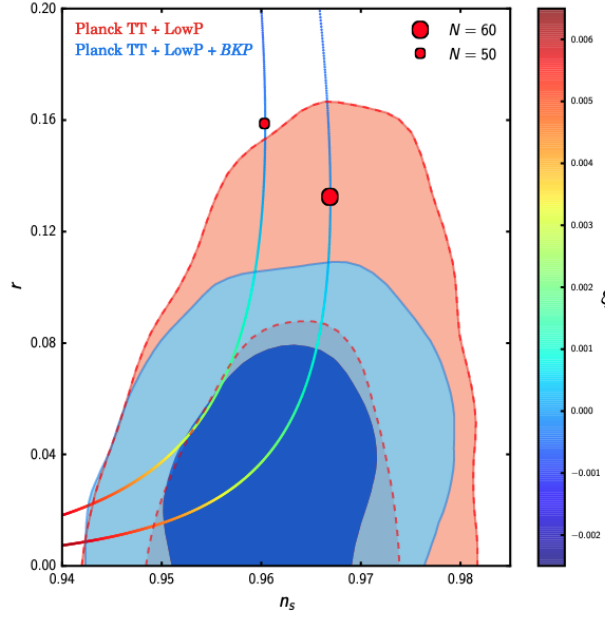


Figure 4.1 : n_s - r plane of a model with non-minimal coupling ξ in a chaotic inflation potential $V(\phi) \propto \phi^2$ where red circles are predictions of chaotic inflation model without non-minimal coupling, i.e. $\xi = 0$ [8].



5. AXION INFLATION

5.1 Axions

Axions are hypothetical pseudoscalar particles introduced in the context of particle physics as a solution to the strong CP problem by R.D. Peccei and Helen Quinn in 1977 [57]. The PQ theory proposes a new symmetry and a scalar field that breaks the symmetry spontaneously at low energies producing the axion that solves the strong CP problem. Quantum Chromo Dynamics (QCD) theory's strong-CP problem arises because of a total derivative term in its Lagrangian as can be seen in Eq (5.1) [58].

$$\mathcal{L}_{\text{QCD}} = \frac{\theta}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu} \quad (5.1)$$

The θ term is the actual CP violating term that results in with electric dipole moment for the neutron [58],

$$d_n \approx 3.6 \times 10^{-16} \theta \text{ e cm} \quad (5.2)$$

where e is the charge of the electron. The dipole moment has a constraint as in Eq (5.3) [59].

$$|d_n| < 2.9 \times 10^{-26} \text{ e cm} \quad (5.3)$$

Thus, resulting in a very small value of θ as $\theta \lesssim 10^{-10}$, which creates a fine-tuning problem. By promoting the θ to a field with a shift symmetry ϕ and minimizing the potential for the promoted field, strong CP problem can be resolved [58].

5.1.1 Spontaneous symmetry breaking

In general, symmetries leaves the physical laws unchanged, i.e. invariant, under certain transformations such as time translation, space translation, rotations, time reversal, charge conjugation and parity. Spontaneous symmetry breaking occurs when a transformation leaves the system invariant at the Lagrangian or equations of motion

level but symmetry disappears at the lowest energy vacuum solution thus breaking the symmetry. When a continuous symmetry breaks spontaneously in a system, it produces a massless boson called Nambu-Goldstone boson. Moreover if a symmetry is broken both explicitly and spontaneously then the boson is massive and called pseudo Nambu-Goldstone boson. The spontaneous symmetry breaking does not change the symmetry of the Lagrangian, only vacuum solution's symmetry is broken.

Spontaneous symmetry breaking can be shown with an example of Mexican hat potential as it is shown in the Figure (5.1).

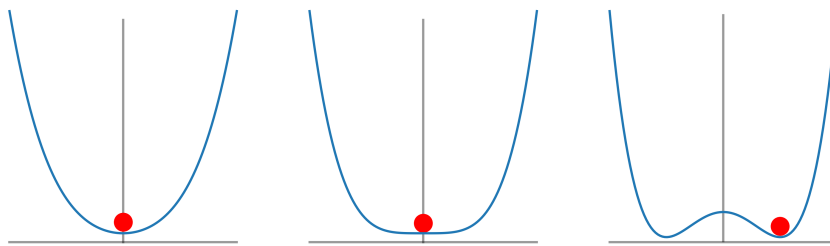


Figure 5.1 : Mexican hat potential where in the left-most figure one can see the stable equilibrium, at the middle the system is in between stable and unstable equilibrium. At the right-most figure the system still preserve its symmetry but symmetry is broken the vacuum solutions spontaneously where the particle has chosen between two vacuum states.

5.1.2 Strong CP Problem

The letter C stands for the charge conjugation. Charge conjugation is a transformation, when applied to a particle results in the particle's anti-particle and vice versa. In the case of C-symmetry, the physical laws will be the same when particles in the theory replaces with their anti-particles. The letter P stands for the parity. Parity, transforms a particles image in a way a mirror transforms objects image, i.e. reflection. In three dimensions, this transformation affects the particles spatial coordinate values to be their opposite signed values. Thus, when P-symmetry is present, laws of physics would remain the same when particle's spatial coordinate values reflected. Together the CP-symmetry stands for the simultaneous transformation of particle to anti-particle (or the other way around) and mirror image of the particle, makes no change in the physics. QCD, quantum chromodynamics, is the theory of strong interactions between quarks and gluons. Therefore, the term strong CP problem of QCD is the question of "Why does the theory of interactions between quarks and

gluons preserve the CP-symmetry?". The question can be answered with "Why not?" of course but, the troubling fact is that mathematically the violation of charge-parity can possible occur in strong interactions but never observed experimentally. The term causing this CP problem is given in equation (5.1) which theoretically violates the CP symmetry. The term results in with the electric dipole moment and comparison with the experimental constraint, equation (5.3), shows a fine tuning problem which is solved by the Peccei-Quinn theory by promoting the CP violating term to a scalar field with shift symmetry thus eliminating the contribution as long as the shift symmetry violation occurs via quantum effects [58].

5.2 Inflation from Axion or Axion-Like-Particle (ALP) Fields

Inflation models with an acceptable duration of inflationary period or sufficient number of e-foldings are very susceptible to affects of Planck-scale physics, i.e. where models with gravity plus scalar fields breaks down [16]. Constructing a model with a scalar field with potential where the field slowly rolls on its potential, the slow roll dynamics of the model is generally affected by higher dimensional radiative corrections if there are not any contradictions from a symmetry of the theory in hand. Therefore, in order for the theory to not be affected by those corrections, the model above the Planck scale, either does not allow to produce those terms or significantly suppresses their contribution [16]. As we said before, unless a symmetry prevents those corrections one needs to be careful while constructing an inflation model. The symmetry that avoids such corrections is the shift symmetry, i.e. $\phi \rightarrow \phi + \text{const.}$. In which, one can think of the axion field as the one which enjoys the shift symmetry, moreover without any attachment to the symmetry proposition of the axionic field, one can consider the scalar field that couples to gauge fields also as an axion as in the case of the models with non-Abelian gauge fields or gauge-flation [60]. In this regard, axions are a good inflaton candidate, since they can be source of the inflation period while avoiding the UV-sensitivity problem as its shift symmetric construction.

5.2.1 Natural inflation

Natural inflation is the first model of axion inflation, developed in 1990 by Freese, Frieman and Olinto [17]. Motivation of the model was to eliminate the shortcomings of

the slow roll inflation, i.e. fine tuning problems caused by the requirement of a nearly flat potential to achieve enough inflation [61]. The model introduces the axion field as a field that enjoys a periodic shift symmetry and the result of explicitly breaking of that shift symmetry at the early universe therefore making the inflaton a pseudo Nambu-Goldstone boson having a cosine form of scalar field potential $V(\phi)$ that drives an inflationary era naturally. The general form of the potential is given as,

$$V(\phi) = \Lambda^4 [1 \pm N \cos(\phi/f)] \quad (5.4)$$

where Λ is non-perturbatively generated scale, f is the axion decay constant, N can be taken as unity [16]. Sign of the potential does not change the outcome, thus one can employ either sign for the natural inflation potential. We will employ minus sign version of the natural inflation throughout this thesis whenever we incorporate cosine potential in our models. In Figure 5.2, one can see where the natural inflation model (purple lines and shaded area) lands on the $n_s - r$ plane from the Planck 2018 results.

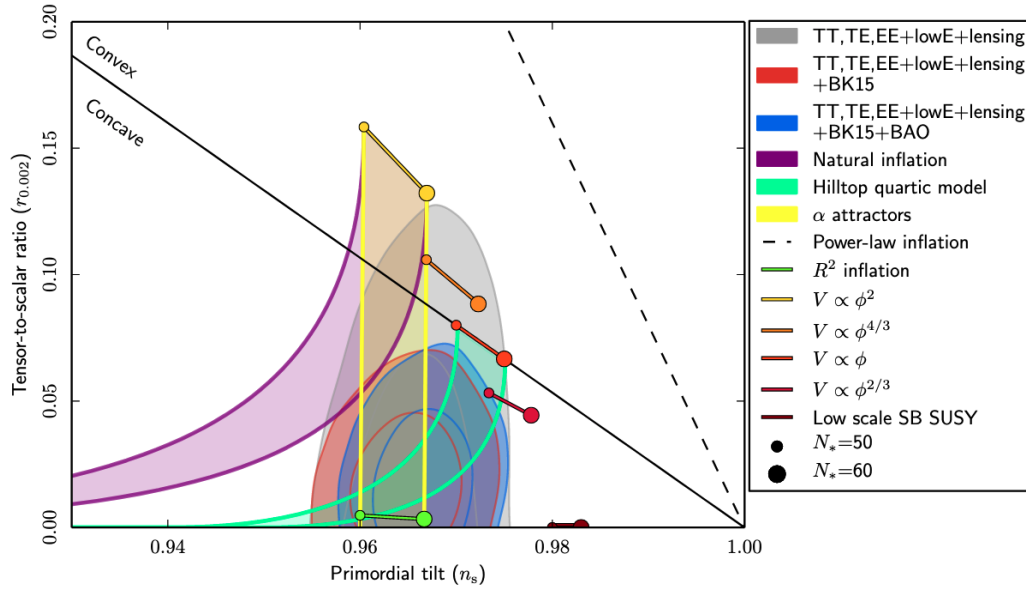


Figure 5.2 : Constraints on the primordial tilt n_s and tensor-to-scalar ratio r for several inflationary models from Planck 2018 results [9].

5.2.2 Non-Abelian gauge field inflation

Vector fields have been used widely in particle physics models yet the usage of said fields can be in contradiction with the homogeneous and isotropic universe because of its vector nature. To overcome this problem several solution methods is utilized, such as in a model called "Vector Inflation" where many vector fields is being introduced

so that the anisotropy is cancelled on average [62], or again in the same model one can define the vector fields as orthogonal to each other in hopes to preserve the rotational symmetry [62]. After all these methods considered, the vector inflation model still does not yield a successful inflationary period and at the quantum level produces instabilities in the background [22]. Construction of a vector inflation with successful inflationary period and at the same time elimination of instabilities is made by introducing non-Abelian gauge fields where also the anisotropy problem is solved by the introduction of three gauge fields that rotate among each other [22]. In order to have an inflationary period, the theory must have a component with the equations of state parameter with the following condition, $w < -1/3$. Therefore, the construction of the non-Abelian gauge field inflation must be made accordingly since only having a Yang-Mills like term in the action would not drive the inflation period. The simplest action with a component that is the gauge field that satisfies the equation of state requirement can be written as in equation (5.5).

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2} - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \frac{\kappa}{96} \left(F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \right)^2 \right] \quad (5.5)$$

where the field strength tensor $F_a^{\mu\nu}$ is defined as,

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g \epsilon_a^{bc} A_b^\mu A_c^\nu \quad (5.6)$$

and its dual $\tilde{F}_a^{\mu\nu}$ as,

$$\tilde{F}_a^{\mu\nu} = \frac{1}{2} \tilde{\epsilon}^{\mu\nu\rho\sigma} F_{\rho\sigma}^a = \frac{1}{2} \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F_{\rho\sigma}^a \quad (5.7)$$

At this point, one needs to make an ansatz to the definition of the gauge fields A_μ^a in order to introduce an effective inflaton field [22]. The model is widely worked in the temporal gauge, i.e. $A_0^a = 0$ so we will continue that tradition in our example calculations. Thus we have,

$$A_0^a = 0 \quad (5.8a)$$

$$A_i^a = \phi(t) \delta_i^a \quad (5.8b)$$

By inputting ansatz into the action in equation (5.5) and calculating the energy momentum tensor, one finds the equation of the state of the F^4 term is $w_\kappa = -1$.

Therefore, it is suitable to drive the inflationary period. Yang-Mills like term has the equation of state $w_{\text{YM}} = 1/3$, thus only contributing factor to the inflation is coming from the last term in equation (5.5). Modifications to this particular model is also studied in literature extensively. In the context of only gauge field driven inflation period [20], [22], cosmic no-hair conjecture in gauge inflation [60], leptogenesis [63], gauge inflation model with extra scalar field with natural inflation potential called "Chromo-Natural Inflation" [23], dynamical system analysis of non-Abelian gauge fields [25] and so on. In our work, we tried to compare models with non-Abelian gauge fields with an extra scalar field that is coupled to gauge fields with chaotic potential, i.e. $\propto \varphi^2$ by employing the dynamical system analysis method at linear order. The choice of chaotic potential is the approximation of the natural inflation potential made to overcome problems of dynamical system construction.

5.2.3 Multiple axionic fields

Axion models in cosmology can also be in the configuration of multiple scalar fields. There are different application methods of multi-field axion models in the literature. Some examples include models that are with two axion fields [64] and the model called Dante's Inferno [65], there are models with more than two axion fields called N-flation models [66]. Dynamics of multiple scalar fields can be examined in similar fashion to models that have single field by deriving equations of motion for each field and continue by deriving Friedmann equations. In this thesis for the multi-field axion inflation, we will summarize the calculations of recently published paper by E. McDonough, A. Guth and D. Kaiser [10] and examine its results. In the paper, axion field, θ , appears as the phase of a complex scalar field Φ : $\Phi = \varphi e^{i\theta}$. Theory describes the general definition of the axion in particle physics context as the pseudo Nambu-Goldstone boson appearing after a spontaneously broken symmetry [10]. The action for the theory can be written in Jordan frame as,

$$S_J = \int d^4x \sqrt{-\tilde{g}} \left[f(\Phi) \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) \right] \quad (5.9)$$

By decomposing the complex field into real fields using $\Phi = \varphi e^{i\theta}$ the action in equation (5.9) turns into,

$$S_J = \int d^4x \sqrt{-\tilde{g}} \left[f(\varphi) \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) - \frac{1}{2} \varphi^2 \tilde{g}^{\mu\nu} \partial_\mu \theta \partial_\nu \theta - V(\theta) \right] \quad (5.10)$$

The complex field potential $V(\Phi)$ when decomposed into real parts consists of contribution from spontaneous symmetry breaking Higgs potential $V(\varphi) = \frac{\lambda}{4} (\varphi^2 - v^2)^2$ and the natural inflation potential $V(\theta) = \Lambda^4 (1 - \cos \theta)$. The chosen non-minimal coupling in the paper [10] is the generic $f(\varphi) \propto 1 + \xi \varphi^2$ form. After making a conformal transformation and derivation of the field equations in Einstein frame, one can numerically solve the equations of motion along with the evolution of Hubble parameter. Figure 5.3 shows that in the case of multi fields interacting with each other and both contributing the inflation period, we have a multi-phased inflation and by combining with the results from figure 5.4 we can clearly see that in about 70 e-foldings of inflation period, for almost 60 e-folds dominant field is the radial field φ and drives the inflation and inflation begins its second phase with axion field θ for about 10 e-folds. Moreover, one can analyze the effect of non-minimal coupling parameter ξ 's role in the inflation phases and can see that radial inflation phase (driven by the radial field φ) is controlled by the ξ parameter Figure 4 in [10]. On the other hand, the dynamics of axion inflation phase is controlled by the fields initial value θ_i and axion decay constant, Figure 5 in [10].

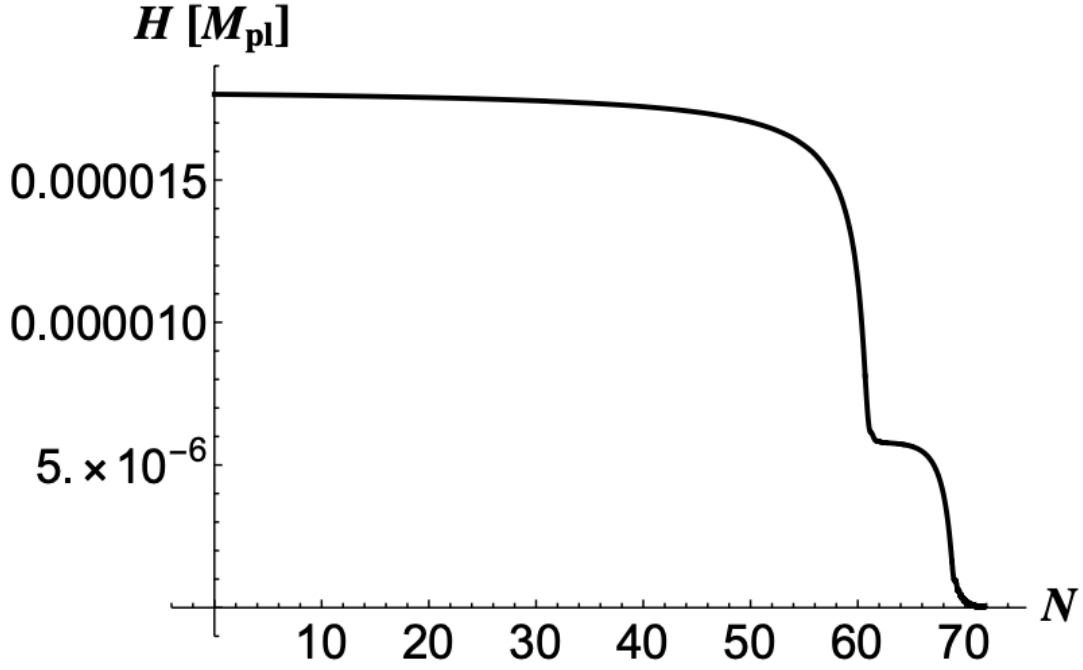


Figure 5.3 : Hubble parameter evolution of multi-field configuration, two phases of inflation in which radial inflation lasts for 60 e-folds and axion inflation lasts for 10 e-folds [10].

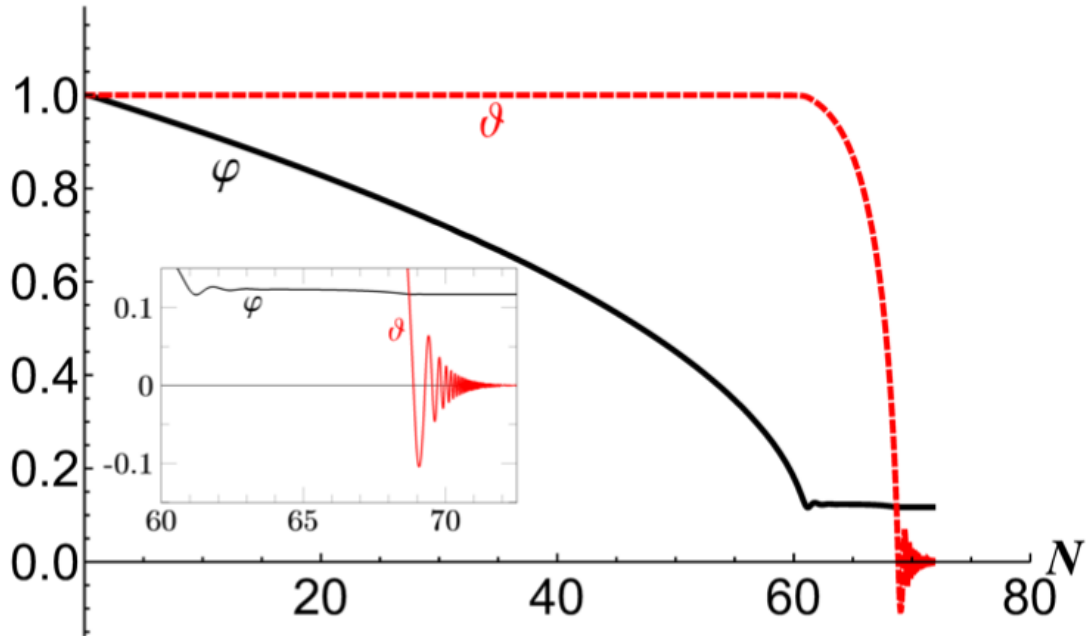


Figure 5.4 : Scalar field evolutions of multi-field configuration in which radial field drives the inflation period for about 60 e-folds and after 60 e-folds axion field begins to dominate and drives the inflation for 10 more e-folds [10].

6. DYNAMICAL SYSTEM ANALYSIS OF AXION-LIKE INFLATION MODELS

Dynamical system analysis (DSA) is a mathematical tool to investigate the evolution of defined phase space variables that has a wide use case in cosmology. In general, a dynamical system consist of two important parts, first is a space and second is a mathematical rule that describes the evolution of any points in that system [1]. Dynamical system is constructed first by defining parameters that represents important characteristics of the model in hand i.e. choosing the phase space variables. Secondly, by using those defined parameters one needs to construct a so called autonomous equation system where the evolution of defined parameters are written without explicit dependence of any independent variable i.e. construction of the mathematical rule that represents the evolution of that space points. One does that by substitution of explicit dependencies of independent variables by the defined phase space variables. After construction of variables and autonomous equation system, we look for critical points of the system and calculate Jacobian of the autonomous equation system and evaluate Jacobian at the critical points. Eigenvalues of the evaluated Jacobian defines the characteristic of that critical point in the phase space. We look at the Jacobian because underneath the surface, calculation performed here is the linear stability theory (LST) where the Taylor expansion of the system at each critical point is taken up to linear term thus results in with first derivatives of the system with respect to phase space variables. Let us define the dynamical system as $\dot{\bar{x}} = \bar{f}(\bar{x})$ where $\bar{x} = \{x_1, \dots, x_N\}$ being the phase space variables. Therefore, Taylor expansion with linear approximation results in with the following expressions,

$$f_i(x) = f_i(x_0) + \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(x_0) (x - x_j) + \dots \quad (6.1a)$$

Since we are expanding at the critical point, evaluation at the critical point is automatically vanishes, $f_i(x_0) = 0$, leaving us with the first derivative. Therefore, the Jacobian evaluation at the critical point is made. Eigenvalues of the Jacobian evaluated at the critical points determines the characteristic of that point, the classification of

those said points at the top level divides into to categories, (i) Hyperbolic and (ii) Non-hyperbolic. Hyperbolic critical point is defined as one where the eigenvalues of the Jacobian at that critical point have no zero real part and non-hyperbolic critical point is therefore defined as the contrary where at least one of the eigenvalues of Jacobian evaluated at critical point is null [67]. Linear stability theory only applies to hyperbolic critical points, thus if any eigenvalue of Jacobian at any critical point contains zero real part stability of that said critical point can not be determined by the linear stability theory. To analyze non-hyperbolic points one must refer to other methods such as centre manifold theory (CMT) [1], Lyapunov function method (LFM) [67] or Kosambi-Cartan-Chern (KCC) theory [24]. Beyond linear stability is outside of the scope of this theses and all systems will be investigated in the context of linear stability theory. In LST, all possible cases for a 2-dimensional system listed in the Table 6.1.

Table 6.1 : Stability properties of critical points for a 2-dimensional system with eigenvalues λ_1 and λ_2 [1].

Point	Eigenvalues	Behaviour
A	$\lambda_1 < 0, \lambda_2 < 0$	Stable
B	$\lambda_1 > 0, \lambda_2 > 0$	Unstable
C	$\lambda_1 < 0, \lambda_2 > 0$	Saddle
D	$\lambda_1 = 0, \lambda_2 > 0$	Unstable, non-hyperbolic
E	$\lambda_1 = 0, \lambda_2 < 0$	Unstable, non-hyperbolic
F	$\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$	Depends on α and β
G	$\lambda_1 = i\beta, \lambda_2 = -i\beta$	Oscillatory solutions

Point A: Characteristics of this critical point is asymptotically stable, meaning that phase space trajectories near this point will approach to it, behaving like an attractor.

Point B: This point is unstable and trajectories near this point will repelled from the point, behaving like a past time attractor, source point.

Point C: Saddle point, meaning that some trajectories near this point will be repelled and other will be attracted.

Point D: The point is non-hyperbolic but positive eigenvalues ensure that there is at least one unstable direction.

Point E: The point is non-hyperbolic and since there is no positive eigenvalue, the point remains undetermined until further analysis made to determine its stability.

Point F: Assuming $\beta \neq 0$, stability of the point is determined by the α being negative

of positive. In the case of $\alpha > 0$ then the point is unstable spiral, otherwise if $\alpha < 0$ then the point is stable spiral.

Point G: Solution around this critical point is oscillatory and the point is called centre without any relation with centre manifolds.

Generalization of the stability conditions of critical points can be inferred from the Table 6.1, meaning that for a dynamical system of dimension N , any critical point where all eigenvalues are negative (positive) considered as stable (unstable) but in the case of M eigenvalues ($M < N$) are negative and $N - M$ eigenvalues are positive, the stability condition of that point whether it is saddle or not can not be determined without plotting the phase space but to say the least it is definitely unstable as a consequence of the positive eigenvalues. Here, we will start the dynamical system analysis of first non-minimally coupled single field model to understand what if we couple axion-like scalar field to gravity with several coupling forms and potentials, and move on to the axion-like scalar field non-minimally coupling to non-Abelian gauge fields.

6.1 Non-minimally Coupled Single Field Model

For a system with scalar field non-minimally coupled to gravity with an arbitrary function $F(\varphi)$ can be written as,

$$S_J = \int d^4x \sqrt{-g} \left\{ F(\varphi)R - \frac{1}{2}(\partial\varphi)^2 - V(\varphi) \right\} \quad (6.2)$$

The "J" subscript denotes that we are working in the Jordan frame. Moreover, by taking the variation w.r.t. inverse metric $g^{\mu\nu}$ we have the following Friedmann equations,

$$6H^2F(\varphi) = \frac{1}{2}\dot{\varphi}^2 - 6H\dot{\varphi}F' + V(\varphi) \quad (6.3a)$$

$$2(2\dot{H} + 3H^2)F = -\frac{1}{2}\dot{\varphi}^2 - 4H\dot{\varphi}F' - 2\ddot{\varphi}F' - 2\dot{\varphi}^2F'' + V(\varphi) \quad (6.3b)$$

$$4\dot{H}F = -\dot{\varphi}^2 + 2H\dot{\varphi}F' - 2\ddot{\varphi}F' - 2\dot{\varphi}^2F'' \quad (6.3c)$$

Continuing the standard derivation of the model, we move on to the equation of motion for the scalar field by taking the variation of action w.r.t. scalar field φ an arrive at,

$$\ddot{\varphi} + 3H\dot{\varphi} - 6F'(\dot{H} + 2H^2) + V' = 0 \quad (6.4)$$

To analyze the model in hand using DSA, we furthermore define the following dimensionless variables,

$$x \equiv \frac{\phi}{H\sqrt{F}} \quad y \equiv \frac{\sqrt{V}}{H\sqrt{F}} \quad (6.5a)$$

$$\lambda_F \equiv -\frac{F'}{\sqrt{F}} \quad \lambda_V \equiv -\frac{V'}{V}\sqrt{F} \quad (6.5b)$$

$$\Gamma_F \equiv \frac{F''F}{(F')^2} \quad \Gamma_V \equiv \frac{V''V}{(V')^2} \quad (6.5c)$$

Thus, autonomous equation system can be derived by taking the derivative of dimensionless variables w.r.t. $' \equiv \frac{1}{H} \frac{d}{dt}$ as,

$$x' = \frac{\dot{\phi}}{H^2\sqrt{F}} - x\frac{\dot{H}}{H^2} + \frac{1}{2}\lambda_F x^2 \quad (6.6a)$$

$$y' = \frac{1}{2}xy(\lambda_F - \lambda_V) - y\frac{\dot{H}}{H^2} \quad (6.6b)$$

$$\lambda_\phi' = \lambda_V^2 x (1 - \Gamma_\phi) - \frac{1}{2}\lambda_F \lambda_V x \quad (6.6c)$$

$$\lambda_F' = x\lambda_F^2 \left(\frac{1}{2} - \Gamma_F \right) \quad (6.6d)$$

As one can see the terms related to scalar field equation of motion and Friedmann equation left as it is, thus we need to write their expressions in terms of dynamical variables to close our system.

$$\frac{\dot{H}}{H^2} = \frac{1}{1+3\lambda_F^2} \left\{ -\frac{1}{2}x^2 \left[\frac{1}{2} + \Gamma_F \lambda_F^2 \right] - 2\lambda_F [x + 3\lambda_F] + \frac{1}{2}\lambda_F \lambda_\phi y^2 \right\} \quad (6.7a)$$

$$\frac{\dot{\phi}}{H^2\sqrt{F}} = -3x - 6\lambda_F \frac{\dot{H}}{H^2} - 12\lambda_F + \lambda_\phi y^2 \quad (6.7b)$$

By using the equation (6.3a), we now have a constraint for the phase space of this system as,

$$\frac{x^2}{12} + \frac{y^2}{6} + x\lambda_F = 1 \quad (6.8)$$

We now have everything that we need to analyze this model using DSA with this configuration of phase space variables. In our study we choose the scalar field potential to be $V(\phi) = \phi^2$ with different coupling function and studied models are show below case by case. Axion inflation potentials like natural, i.e. cosine potential, or monodromy does not allow one to analyze it utilizing the dynamical system analysis method without approximating the potential by expanding it its Taylor series and taking

the second order contribution. Approximation to those potentials readily yields the chaotic-like form if one employs the negative sign in the natural inflation potential equation (5.4), i.e. $\approx \phi^2$. Therefore, the choice of potential as ϕ^2 for dynamical system analysis of axion-like scalar field is justified and an example of approximating the cosine natural inflation potential to chaotic one can be seen in Ref. [68].

6.1.1 Case I: Brans-Dicke like coupling to gravity

For our system to look like Brans-Dicke theory, we take the general coupling function $F(\phi) = \frac{\phi}{2}$. 4-dimensional phase space critical points are shown in Table 6.2.

Table 6.2 : Stability properties of Brans-Dicke like coupled model.

x	y	λ_V	λ_F	Eigenvalues	Behaviour
0	$\sqrt{6}$	2	λ_F	$\{0, 0, 0, -3\}$	LST fails
0	$-\sqrt{6}$	$2\lambda_F$	λ_F	$\{0, 0, 0, -3\}$	LST fails
$2\sqrt{3}$	0	0	0	$\{0, 0, 6, 3\}$	Unstable
$-2\sqrt{3}$	0	0	0	$\{0, 0, 6, 3\}$	Unstable

Even though we have a 4-dimensional system, we can reduce the dimension by 1, using the constraint equation to eliminate λ_F from the equations. Result of the reduced 3-dimensional phase space critical points are shown in Table 6.3.

Table 6.3 : Stability properties of the Brans-Dicke like coupled model with reduced dimension.

x	y	λ_V	Eigenvalues	Behaviour
0	0	0	$\{5/2, -1/2, -1/2\}$	Unstable/Saddle
$2\sqrt{3}$	0	0	$\{0, 0, 3\}$	Unstable
$-2\sqrt{3}$	0	0	$\{0, 0, 3\}$	Unstable
0	$\sqrt{30}$	0	$\{-5, 2, 2\}$	Unstable/Saddle
0	$-\sqrt{30}$	0	$\{-5, 2, 2\}$	Unstable/Saddle

6.1.1.1 Conclusion of Case I:

Phase space dimensions does not allow us to visually observe the behaviour of critical points even though we can reduce its dimensions from 4 to 3. In Table 6.2, one can see the model, without reducing the dimensions, has 2 unstable critical points which the conclusion is inferred from the existence of only non-zero eigenvalues being positive and rest is zero, even though the linear stability theory fails for that points, we can say that they are unstable. The behaviour of remaining two critical points in the 4

dimensional result are can not be determined by the analysis of linear stability theory. By reducing the systems dimension from 4 to 3 by using the constraint equation, equation (6.8), we have results as shown in figure 6.3. Five critical points are found, three of them can be categorized as saddle points and remaining two of them can be categorized as unstable in the context of linear stability theory. To determine the actual behaviour of the critical points where their behavioural analysis fails in the linear stability theory must be made using the CMT or other before mentioned analysis methods, yet the physical interpretation of the further analysis is questionable and represents itself as more of an applied mathematics rather than theoretical physics.

6.1.2 Case II: Induced coupling to gravity model

We now take the coupling function of the form $F(\varphi) = \frac{\xi\varphi^2}{2}$ where ξ can be thought as the coupling strenght constant. This special case of choices of both scalar field potential $V(\varphi) = \varphi^2$ and coupling function $F(\varphi) = \frac{\xi\varphi^2}{2}$ makes both λ_V and λ_F constant and equal to each other with the value $\lambda_V = \lambda_F = -\sqrt{2\xi}$. Therefore, 4-dimensional system immediatly reduces to 2-dimensions. Table 6.4 shows the critical points for the 2-dimensional system without specifying the parameter ξ .

Table 6.4 : Stability properties of induced coupled like model with general ξ coupling.

x	y	Eigenvalues
$6\sqrt{2\xi} + 2\sqrt{3+18\xi}$	0	$\left\{ 2\sqrt{6\xi+36\xi^2} + 12\xi + 6, 2\sqrt{6\xi+36\xi^2} + 12\xi + 3 \right\}$
$6\sqrt{2\xi} - 2\sqrt{3+18\xi}$	0	$\left\{ -2\sqrt{6\xi+36\xi^2} + 12\xi + 6, -2\sqrt{6\xi+36\xi^2} + 12\xi + 3 \right\}$
$\frac{2\sqrt{2\xi}}{4\xi+1}$	$\frac{\sqrt{192\xi^2+68\xi+6}}{4\xi+1}$	$\left\{ -\frac{16\xi+3}{4\xi+1}, -\frac{4\xi}{4\xi+1} \right\}$
$\frac{2\sqrt{2\xi}}{4\xi+1}$	$-\frac{\sqrt{192\xi^2+68\xi+6}}{4\xi+1}$	$\left\{ -\frac{16\xi+3}{4\xi+1}, -\frac{4\xi}{4\xi+1} \right\}$

For $\xi = \{0.001, 1/6, 1\}$ values which corresponds to very weak coupling, conformal coupling and strong coupling, we have their critical points and behaviour shown in the table 6.5, 6.6, 6.7.

Since the phase space of the system is 2-dimensional we can plot their phase diagram for weak, conformal and strong coupling cases.

Table 6.5 : Stability properties of induced like coupled model with weak coupling $\xi = 0.001$.

Point	x	y	Eigenvalues	Behaviour
S1	3.74	0.00	$\{6.17, 3.17\}$	Repeller
S2	-3.21	0.00	$\{5.85, 2.86\}$	Repeller
A1	0.09	2.45	$\{-3.00, 0.00\}$	LST fails
A2	0.09	-2.45	$\{-3.00, 0.00\}$	LST fails

Table 6.6 : Stability properties of the induced model with conformal coupling $\xi = 1/6$.

Point	x	y	Eigenvalues	Behaviour
S1	8.36	0.00	$\{10.83, 7.83\}$	Repeller
S2	-1.43	0.00	$\{5.17, 2.17\}$	Repeller
A1	0.69	2.86	$\{-3.40, -0.40\}$	Attractor
A2	0.69	-2.86	$\{-3.40, -0.40\}$	Attractor

Table 6.7 : Stability properties of induced model with strong coupling $\xi = 1$.

Point	x	y	Eigenvalues	Behaviour
S1	17.65	0.00	$\{30.96, 27.96\}$	Repeller
S2	-0.68	0.00	$\{5.04, 2.04\}$	Repeller
A1	0.56	3.26	$\{-3.80, -0.80\}$	Attractor
A2	0.56	-3.26	$\{-3.80, -0.80\}$	Attractor

6.1.2.1 Conclusion of Case II:

In the presence of induced non-minimal coupling to the gravity, three different cases have been analyzed. Special case for the choice of scalar field potential and coupling reduced the dimension of phase space to 4 thus allowed us to visually inspect the behaviour of critical points even though linear stability theory fails to analyze some of the critical points behaviour. In general the model has 4 distinct critical points

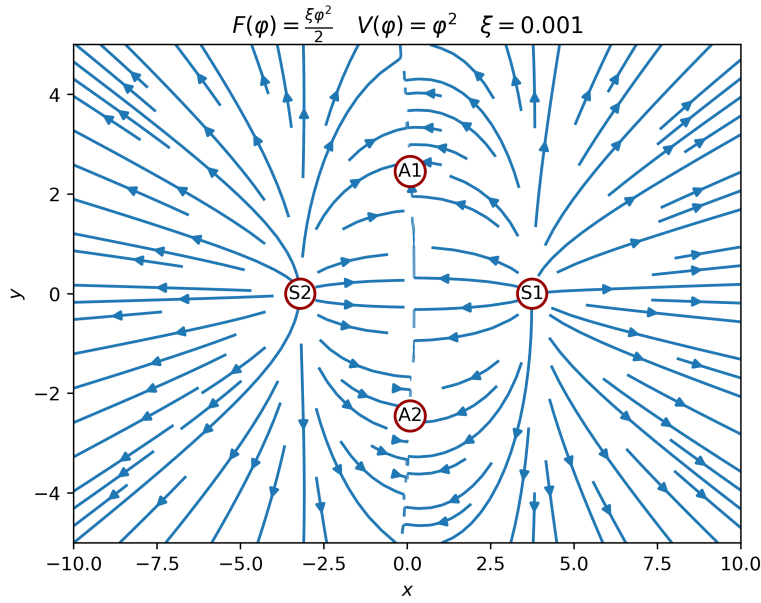


Figure 6.1 : Phase plot of the system in the weak coupling case of induced model.

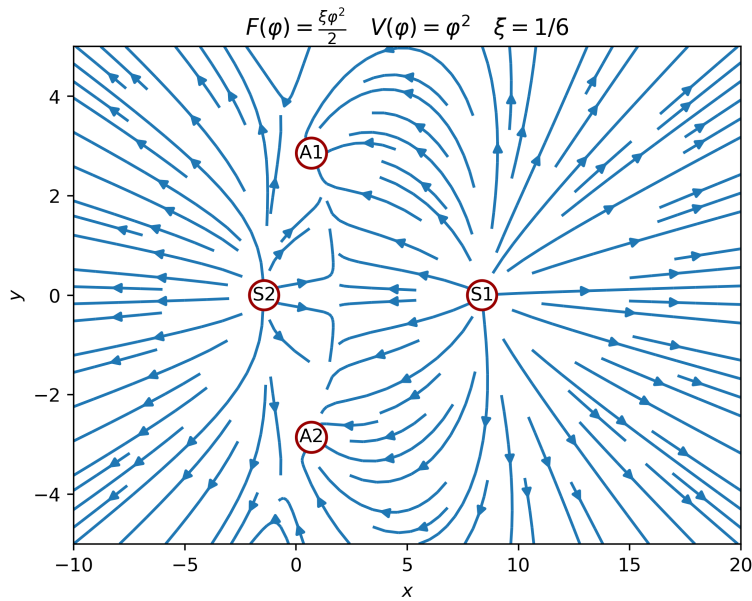


Figure 6.2 : Phase plot of system in the conformal coupling case of induced model.

where the behaviour dependent on the coupling constant ξ . In the weak coupled configuration, Table 6.5, showed us that 2 of the 4 critical points acts as repeller/source points, whereas other 2 critical point's phase space behaviour can not be determined by LST. Thus, determination of stability for those points should be made using the CMT. By examining the figure 6.1 we can clearly see the centre manifold crossing through the critical points where LST fails and shows that the centre manifold is unstable and have at least two directions. In the conformal coupling configuration, Table 6.6, and strong coupling configuration, Table 6.7, the model has two attractor and two repeller critical points and one can see in the figure 6.2 and figure 6.3.

6.1.3 Case III: Dilaton like coupling to gravity

To analyze a model in which a dilaton like scalar field function couples to gravity, we take the coupling function $F(\varphi)$ to be of the form $F(\varphi) = e^{-\alpha\varphi}$ where α is a constant. Initial phase space is 4-dimensional and critical points of the system can be found in Table 6.8. Moreover, using the constraint equation, we can again reduced the system from 4-dimension to 3-dimension by eliminating λ_F and corresponding system's critical points and stability properties is shown in Table 6.9.

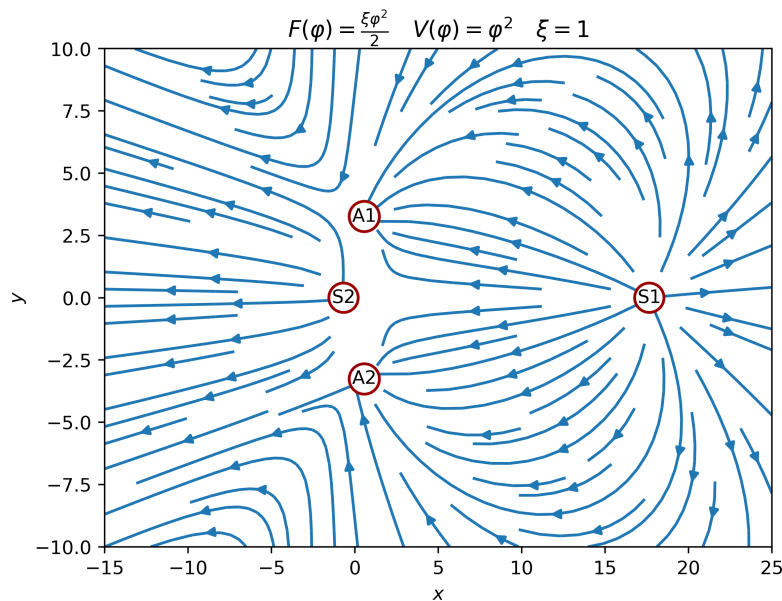


Figure 6.3 : Phase plot of the system in the strong coupling case of induced model.

Table 6.8 : Stability properties of dilaton model.

x	y	λ_V	λ_F	Eigenvalues
0	$\sqrt{6}$	$2\lambda_F$	λ_F	$\left\{ 0, 0, \frac{-9\lambda_F^2-3+\sqrt{225\lambda_F^4+102\lambda_F^2+9}}{6\lambda_F^2+2}, \frac{-9\lambda_F^2-3-\sqrt{225\lambda_F^4+102\lambda_F^2+9}}{6\lambda_F^2+2} \right\}$
0	$-\sqrt{6}$	$2\lambda_F$	λ_F	$\left\{ 0, 0, \frac{-9\lambda_F^2-3+\sqrt{225\lambda_F^4+102\lambda_F^2+9}}{6\lambda_F^2+2}, \frac{-9\lambda_F^2-3-\sqrt{225\lambda_F^4+102\lambda_F^2+9}}{6\lambda_F^2+2} \right\}$
$2\sqrt{3}$	0	0	0	$\{0, 0, 6, 3\}$
$-2\sqrt{3}$	0	0	0	$\{0, 0, 6, 3\}$

Table 6.9 : Stability properties of dilaton model reduced to 3-dimensions.

x	y	λ_V	Eigenvalues	Behaviour
0	0	0	$\{5/2, -1/2, 1/2\}$	Unstable/Saddle
$2\sqrt{3}$	0	0	$\{0, 0, 3\}$	Unstable
$-2\sqrt{3}$	0	0	$\{0, 0, 3\}$	Unstable
0	$\sqrt{30}$	0	$\{-5, 2, -2\}$	Unstable/Saddle
0	$-\sqrt{30}$	0	$\{-5, 2, -2\}$	Unstable/Saddle

6.1.3.1 Conclusion of Case III:

In the case of a coupling between scalar field and gravity with dilaton like coupling function, the phase space is 4 dimensional and eigenvalues dependent on the dynamical variable λ_F . The result of the 4 dimensional phase space can be seen in figure 6.9. Four dimensional system has a total of 4 critical points but behaviours are not specified because of some critical point's dependence of λ_F . Further analysis made by utilizing the constraint equation, equation (6.8), to eliminate λ_F dynamical variable and results of the 3 dimensional system can be seen in figure 6.9. In total, 5 critical points are found and three of them appear to be a saddle point, while remaining two unstable yet further analysis required to analyze in completion with CMT.

6.2 Non-Abelian Gauge Field Model with $F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}$

Axion-like scalar fields can also be taught as, without considering its particle physics definition of a field that enjoys shift symmetry, one can think of it as a scalar field coupled to SU(2) gauge fields. For a system consisting of general scalar field couplings, $F_1(\chi)$, to gravity and $F_2(\chi)$, to non-Abelian gauge field sector can be

written as,

$$S = \int d^4x \sqrt{-g} \left\{ F_1(\chi) R - \frac{1}{2} (\partial\chi)^2 - V(\chi) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{\lambda}{4} F_2(\chi) F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \right\} \quad (6.9)$$

gauge field strength tensor, $F_{\mu\nu}^a$, and its Hodge-dual $\tilde{F}_a^{\mu\nu}$ defined as,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon_{bc}^a A_\mu^b A_\nu^c \quad (6.10a)$$

$$\tilde{F}_a^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^a = \frac{1}{2} \frac{\tilde{\epsilon}^{\mu\nu\rho\sigma}}{\sqrt{-g}} F_{\rho\sigma}^a \quad (6.10b)$$

where $\epsilon^{\mu\nu\rho\sigma}$ is totally anti-symmetric tensor and $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is Levi-Civita symbol. At this stage of calculation, we need to make an ansatz to the gauge fields A_μ^a so that we preserve the homogeneity and isotropy to use FRW space-time by the choice of temporal gauge equation (6.11a), thus allowing us to restore the rotation symmetry with the choice equation (6.11b), even though we incorporated gauge fields to the model [20].

$$A_0^a = 0 \quad (6.11a)$$

$$A_i^a = a(t) \psi(t) \delta_i^a = \phi(t) \delta_i^a \quad (6.11b)$$

Using the ansatz, Lagrangian of the system concerning the gauge fields reduces and denoted as $\mathcal{L}_{red}$,

$$\mathcal{L}_{red} = a^3 \left\{ \frac{3}{2} \left[\frac{\dot{\phi}^2}{a^2} - g^2 \frac{\phi^4}{a^4} \right] - 3\lambda F_2(\chi) g \frac{\phi^2 \dot{\phi}}{a^3} \right\} \quad (6.12)$$

Furthermore, we can calculate terms $F_{\mu\nu}^a F_a^{\mu\nu}$ and $\tilde{F}_a^{\mu\nu} F_{\mu\nu}^a$ and their variations w.r.t. inverse metric $g^{\mu\nu}$ as,

$$F_{\mu\nu}^a F_a^{\mu\nu} = 6 \left(-\frac{\dot{\phi}^2}{a^2} + \frac{g^2 \phi^4}{a^4} \right) \quad (6.13a)$$

$$\tilde{F}_a^{\mu\nu} F_{\mu\nu}^a = 12g \frac{\phi^2 \dot{\phi}}{a^3} \quad (6.13b)$$

$$\delta(F_{\mu\nu}^a F_a^{\mu\nu}) = 2g^{\alpha\beta} F_{\mu\alpha}^a F_{\nu\beta}^a \delta g^{\mu\nu} \quad (6.14a)$$

$$\delta(\tilde{F}_a^{\mu\nu} F_{\mu\nu}^a) = \frac{1}{2} g_{\mu\nu} \left(\tilde{F}_a^{\alpha\beta} F_{\alpha\beta}^a \right) \delta g^{\mu\nu} \quad (6.14b)$$

Friedmann equations of the model calculated as,

$$6H^2F_1(\chi) = \frac{\dot{\chi}^2}{2} + V(\chi) + \frac{3}{2} \left(\frac{\dot{\phi}^2}{a^2} + g^2 \frac{\phi^4}{a^4} \right) - 6HF_1'\dot{\chi} \quad (6.15a)$$

$$4\dot{H}F_1 + 6H^2F_1 = - \left(\left[2F_1'' + \frac{1}{2} \right] \dot{\chi}^2 + 2F_1'[\dot{\chi} + 2H\dot{\chi}] - V(\chi) + \frac{1}{2} \frac{\dot{\phi}^2}{a^2} + \frac{1}{2} g^2 \frac{\phi^4}{a^4} \right) \quad (6.15b)$$

$$4\dot{H}F_1 = - \left((2F_1'' + 1) \dot{\chi}^2 + 2F_1'(\dot{\chi} - H\dot{\chi}) + 2\frac{\dot{\phi}^2}{a^2} + 2g^2 \frac{\phi^4}{a^4} \right) \quad (6.15c)$$

Equations of motion,

$$\ddot{\chi} + 3H\dot{\chi} + V'(\chi) - 6(\dot{H} + 2H^2)F_1'(\chi) + \frac{\lambda}{4}F_2'(\chi)F_{\mu\nu}^a\tilde{F}_a^{\mu\nu} = 0 \quad (6.16a)$$

$$\frac{\ddot{\phi}}{a} + \frac{H\dot{\phi}}{a} - \lambda F_2'\dot{\chi}g\frac{\phi^2}{a^2} + 2g^2\frac{\phi^3}{a^3} = 0 \quad (6.16b)$$

$$\ddot{\psi} + 3H\dot{\psi} + (\dot{H} + 2H^2)\psi + 2g^2\psi^3 - \lambda gF_2'\dot{\chi}\psi^2 = 0 \quad (6.16c)$$

where in equation (6.16c) calculated by inserting ansatz definition equation (6.11b) into equation (6.16b).

Now to construct the autonomous equation system of the model, we define the following dimensionless phase space variables as,

$$x_1 = \frac{1}{\sqrt{12}} \frac{\dot{\chi}}{H\sqrt{F_1}} \quad x_2 = \frac{V}{6H^2F_1} \quad (6.17a)$$

$$y_1 = \frac{1}{2} \frac{\dot{\psi}}{H\sqrt{F_1}} \quad y_2 = \frac{1}{2} g \frac{\psi}{H} \quad y_3 = \frac{1}{\sqrt{2}} \frac{\psi}{\sqrt{F_1}} \quad (6.17b)$$

$$\lambda_{F_1} = -\frac{F_1'}{\sqrt{F_1}} \quad \lambda_V = -\frac{V'}{V}\sqrt{F_1} \quad (6.17c)$$

$$\Gamma_{F_1} = \frac{F_1''F_1}{(F_1')^2} \quad \Gamma_V = \frac{V''V}{(V')^2} \quad (6.17d)$$

Rewriting Friedmann equation 6.15a using the dynamical variables above, we have the system's phase space constraint as,

$$1 = x_1^2 + x_2 + \frac{1}{2}y_3^2 + \sqrt{2}y_1y_3 + y_1^2 + 2y_2^2y_3^2 + 12\sqrt{3}x_1\lambda_{F_1} \quad (6.18)$$

Furthermore, we define a parameter ε in order to clean up the equations at some degree as,

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{1+3\lambda_{F_1}^2} \left(3x_1^2 (2\Gamma_{F_1} \lambda_{F_1}^2 + 1) + 4\lambda_{F_1} \sqrt{3}x_1 - 3\lambda_{F_1} \lambda_V x_2 + 6\lambda_{F_1}^2 \right. \\ \left. + 6\lambda \lambda_{F_1} F_2' \sqrt{F_1} y_2 (y_3^2 + \sqrt{2}y_1 y_3) + y_3^2 + 2y_1^2 + 2\sqrt{2}y_1 y_3 + 4y_2^2 y_3^2 \right) \quad (6.19)$$

Thus, we write the equations of motion 6.16a, 6.16b, 6.16c using the variables as,

$$\frac{\ddot{\chi}}{H^2 \sqrt{F_1}} = -6\sqrt{3}x_1 + 6\lambda_V x_2 + 6\lambda_{F_1} (\varepsilon - 2) - 12\lambda F_2' \sqrt{F_1} y_2 (y_3^2 + \sqrt{2}y_1 y_3) \quad (6.20a)$$

$$\frac{\ddot{\psi}}{H^2 \sqrt{F_1}} = -6y_1 + \sqrt{2}y_3 (\varepsilon - 2) - 8\sqrt{2}y_2^2 y_3 + 4\sqrt{6}\lambda F_2' x_1 y_2 y_3 \quad (6.20b)$$

Autonomous equation system can now be written as where prime (') derivative denotes $' \equiv \frac{1}{H} \frac{d}{dt}$,

$$x_1' = \frac{1}{\sqrt{12} H^2 \sqrt{F_1}} \ddot{\chi} + x_1 \varepsilon + \sqrt{3} \lambda_{F_1} x_1^2 \quad (6.21a)$$

$$x_2' = \sqrt{12} x_1 x_2 (\lambda_{F_1} - \lambda_V) + 2x_2 \varepsilon \quad (6.21b)$$

$$y_1' = \frac{1}{2 H^2 \sqrt{F_1}} \ddot{\psi} + y_1 \varepsilon + \sqrt{3} x_1 y_1 \lambda_{F_1} \quad (6.21c)$$

$$y_2' = y_2 \left(\sqrt{2} \frac{y_1}{y_3} + \varepsilon \right) \quad (6.21d)$$

$$y_3' = y_1 + \sqrt{3} x_1 y_3 \lambda_{F_1} \quad (6.21e)$$

$$\lambda_{F_1}' = 2\sqrt{3} x_1 \lambda_{F_1}^2 \left(\frac{1}{2} - \Gamma_{F_1} \right) \quad (6.21f)$$

$$\lambda_V' = 2\sqrt{3} x_1 \lambda_V^2 \left(1 - \Gamma_V - \frac{1}{2} \frac{\lambda_{F_1}}{\lambda_V} \right) \quad (6.21g)$$

Autonomous system is completed and we calculated critical points and corresponding eigenvalues of the Jacobian matrix to determine their stability for some configurations of coupling functions and potentials.

6.2.1 Case I: $F_1 = 1$ $F_2 = \chi$ $V = 0$

6.2.1.1 Conclusion of Case I:

For numeric computational reasons we take the λ as unity but this is not restricted to unity when the actual physics of the model established, λ can take any value and after that shown results must be re-written accordingly. Even though we have some attractor

Table 6.10 : Stability properties of Non-Abelian Gauge Field model with free scalar field.

x_1	y_1	y_2	Eigenvalues	Behaviour
0	0	0	$\{-1, -1, 2\}$	Unstable
0	$-\sqrt{2}$	1/2	$\{89444.72, -1, -2\}$	Unstable
0	$\sqrt{2}$	1/2	$\{-89440.72, -1, -2\}$	Stable attractor
0	$-\sqrt{2}$	-1/2	$\{89444.72, -1, -2\}$	Unstable
0	$\sqrt{2}$	-1/2	$\{-89440.72, -1, -2\}$	Stable attractor

like fixed points and some unstable fixed points where their eigenvalues have positive values, the model can not produce inflationary epoch since the model does not consist of a component that has the equation of state valued, $w \leq -1/3$.

6.2.2 Case II: $F_1 = 1$ $F_2 = \chi$ $V = \exp(-\chi)$

Table 6.11 : Stability properties of Non-Abelian Gauge Field with exponential scalar field potential.

x_1	x_2	y_1	y_2	Eigenvalues	Behaviour
0	0	0	0	$\{-1, -1, 2, 4\}$	Unstable
0	0	$-\sqrt{2}$	1/2	$\{89444.72, -1, -2, 4\}$	Unstable
0	0	$\sqrt{2}$	1/2	$\{-89440.72, -1, -2, 4\}$	Unstable
0	0	$-\sqrt{2}$	-1/2	$\{89444.72, -1, -2, 4\}$	Unstable
0	0	$\sqrt{2}$	-1/2	$\{-89440.72, -1, -2, 4\}$	Unstable

6.2.2.1 Conclusion of Case II:

In this model the introduction of scalar field potential fixes the problem of not having an inflationary epoch, but this results in a 4-dimensional phase space with the new x_2 variable in play thus graphical presentation is not possible. The reason that λ_V not showing up in the fixed points is that the choice of exponential potential makes λ_V constant.

6.2.3 Case III: $F_1 = 1$ $F_2 = \chi$ $V = \chi^2$

6.2.3.1 Conclusion of Case III:

In this setup, the monomial potential χ^2 results in with λ_V not being a constant, yet in the fixed points λ_V shows itself as a free parameters with the inclusion of 0 in the eigenvalues. This makes the fixed points indeterminable without centre manifold

Table 6.12 : Stability properties of Non-Abelian Gauge Field with chaotic scalar field potential.

x_1	x_2	y_1	y_2	λ_V	Eigenvalues	Behaviour
0	0	0	0	λ_V	$\{-1, -1, 2, 4, 0\}$	LST fails
0	0	$-\sqrt{2}$	1/2	λ_V	$\{89444.72, -1, -2, 4, 0\}$	LST fails
0	0	$\sqrt{2}$	1/2	λ_V	$\{-89440.72, -1, -2, 4, 0\}$	LST fails
0	0	$-\sqrt{2}$	-1/2	λ_V	$\{89444.72, -1, -2, 4, 0\}$	LST fails
0	0	$\sqrt{2}$	-1/2	λ_V	$\{-89440.72, -1, -2, 4, 0\}$	LST fails

theory. Further analysis is required to completely understand this setup's fixed point behaviour.

6.2.4 Case IV: $F_1 = 1$ $F_2 = \chi$ $V = \chi^n$

Table 6.13 : Stability properties of Non-Abelian Gauge Field with monomial scalar field potential.

x_1	x_2	y_1	y_2	λ_V	Eigenvalues	Behaviour
0	0	0	0	λ_V	$\{-1, -1, 2, 4, 0\}$	LST fails
0	0	$-\sqrt{2}$	1/2	λ_V	$\{89444.72, -1, -2, 4, 0\}$	LST fails
0	0	$\sqrt{2}$	1/2	λ_V	$\{-89440.72, -1, -2, 4, 0\}$	LST fails
0	0	$-\sqrt{2}$	-1/2	λ_V	$\{89444.72, -1, -2, 4, 0\}$	LST fails
0	0	$\sqrt{2}$	-1/2	λ_V	$\{-89440.72, -1, -2, 4, 0\}$	LST fails

6.2.4.1 Conclusion of Case IV:

Arbitrary choice of monomial potential with parameter n did no affect on the fixed points or eigenvalues. λ_V remained as free parameter and 0 eigenvalue still needs to be addressed. Further conclusions can be derived by applying alternative methods mentioned before. Beyond linear stability analysis is out of the scope of this thesis and physical interpretation of said advanced models is questionable.

6.2.5 Conclusion

In the non-minimal coupling to the gravity case, i.e. $F_1 \neq 1$, none of the configurations above can be used in the dynamical system analysis without some form of χ explicitly appearing in the expressions, thus results in a non-complete / non-closed dynamical system. In order to fix this problem one needs to redefine the dynamical variables and try to find a suitable variable system and close the autonomous equation system.

Furthermore, the analysis in the boundary of linear stability theory does not yield compelling results since for the significant amount of critical points linear stability theory fails. Even though analysis can be made for the problematic critical points by centre manifold theory or other mathematical methods, we feel that the conclusions made from the analysis of methods that are not linear stability theory might overshadow the physical interpretation. Thus, we stop at the linear stability theory and choose not to move on with calculations of CMT for the points that their behavioral analysis fails to be determined in linear stability theory.

6.3 Non-Abelian Gauge Field Model with $\left(F_{\mu\nu}^a \tilde{F}_a^{\mu\nu}\right)^2$

In the case of F^2 model with free scalar field, i.e. $V(\chi) = 0$ there were no component that can drive the inflation. This problem can be resolved by introducing F^4 term instead of F^2 . Equation of state of the extra term coming from the expansion of F^4 is can be seen as $w = -1$. Thus, now there is a source that can produce inflationary epoch even though the scalar field potential taken as zero. The action for such model can be written as,

$$S = \int d^4x \sqrt{-g} \left\{ F_1(\chi) R - \frac{1}{2} (\partial\chi)^2 - V(\chi) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \frac{\lambda}{96} F_2(\chi) \left(F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \right)^2 \right\} \quad (6.22)$$

By the standart procedure of calculations we find Friedmann equations,

$$6H^2 F_1 = \frac{\dot{\chi}^2}{2} + V(\chi) + \frac{3}{2} \left(\frac{\dot{\phi}^2}{a^2} + g^2 \frac{\phi^4}{a^4} \right) - \frac{3}{2} \lambda F_2 g^2 \frac{\phi^4 \dot{\phi}^2}{a^6} - 6H F_1' \dot{\chi} \quad (6.23a)$$

$$-4\dot{H} F_1 - 6H^2 F_1 = \frac{\dot{\chi}^2}{2} - V(\chi) + \frac{1}{2} \left(\frac{\dot{\phi}^2}{a^2} + g^2 \frac{\phi^4}{a^4} \right) + \frac{3}{2} \lambda F_2 g^2 \frac{\phi^4 \dot{\phi}^2}{a^6} + 2F_1' (\ddot{\chi} + 2H\dot{\chi}) + 2F_1'' \dot{\chi}^2 \quad (6.23b)$$

$$-4\dot{H} F_1 = \dot{\chi}^2 + 2 \left(\frac{\dot{\phi}^2}{a^2} + g^2 \frac{\phi^4}{a^4} \right) + 2F_1' (\ddot{\chi} - H\dot{\chi}) + 2F_1'' \dot{\chi}^2 \quad (6.23c)$$

Moreover, we again define the ε in the same fashion as before,

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{4} \frac{\dot{\chi}^2}{H^2 F_1} + \frac{1}{2} \left(\frac{\dot{\phi}^2}{a^2 H^2 F_1} + g^2 \frac{\phi^4}{a^4 H^2 F_1} \right) + \frac{1}{2} \frac{F_1'}{\sqrt{F_1}} \left(\frac{\ddot{\chi}}{H^2 \sqrt{F_1}} - \frac{\dot{\chi}}{H \sqrt{F_1}} \right) + \frac{1}{2} F_1'' \frac{\dot{\chi}^2}{H^2 F_1} \quad (6.24)$$

As we can see that by defining the density and the pressure of the contents of the universe, we have,

$$\rho_\chi = \frac{\dot{\chi}^2}{2} + V \quad P_\chi = \frac{\dot{\chi}^2}{2} - V \quad \rightarrow \quad w_\chi = -1 \quad (1 \text{ when } V = 0) \quad (6.25a)$$

$$\rho_{\text{YM}} = \frac{3}{2} \left(\frac{\dot{\phi}^2}{a^2} + g^2 \frac{\phi^4}{a^4} \right) \quad P_{\text{YM}} = \frac{1}{2} \left(\frac{\dot{\phi}^2}{a^2} + g^2 \frac{\phi^4}{a^4} \right) \quad \rightarrow \quad w_{\text{YM}} = \frac{1}{3} \quad (6.25b)$$

$$\rho_{\text{inf}} = -\frac{3}{2} \lambda F_2 g^2 \frac{\phi^4 \dot{\phi}^2}{a^6} \quad P_{\text{inf}} = \frac{3}{2} \lambda F_2 g^2 \frac{\phi^4 \dot{\phi}^2}{a^6} \quad \rightarrow \quad w_{\text{inf}} = -1 \quad (6.25c)$$

It is apparent that we can now freely take $V = 0$ since there is now the coupling component between non-Abelian gauge fields and axion field to derive the inflation period. Equations of motion of both axion field χ and gauge field ϕ and ψ can be written as,

$$\ddot{\chi} + 3H\dot{\chi} + V'(\chi) - F'_1(\chi)R = \frac{\lambda}{96} F'_2(\chi) \left(F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} \right)^2 \quad (6.26a)$$

$$\begin{aligned} \frac{\ddot{\phi}}{a} \left(1 - \lambda g^2 F_2(\chi) \frac{\phi^4}{a^4} \right) + \frac{H\dot{\phi}}{a} \left(1 + 3\lambda g^2 F_2(\chi) \frac{\phi^4}{a^4} \right) \\ + 2g^2 \frac{\phi^3}{a^3} \left(1 - \lambda F_2(\chi) \frac{\dot{\phi}^2}{a^2} - \frac{1}{2} \lambda F'_2(\chi) \dot{\chi} \frac{\phi \dot{\phi}}{a} \right) = 0 \end{aligned} \quad (6.26b)$$

$$\begin{aligned} \ddot{\psi} (1 - \lambda g^2 F_2 \psi^4) + 3H\dot{\psi} (1 + \lambda g^2 F_2 \psi^4) + (-\dot{H} + 2H^2) \lambda g^2 F_2 \psi^5 \\ + (\dot{H} + 2H^2) \psi + 2g^2 \psi^3 \left(1 - \lambda F_2 (H\psi + \dot{\psi})^2 - \frac{1}{2} \lambda F'_2 \dot{\chi} \psi (H\psi + \dot{\psi}) \right) = 0 \end{aligned} \quad (6.26c)$$

Now our aim is to concentrate on the same phase space variables and create the similar autonomous system using those to compare exactly the affect of F^4 term in the different potential settings including the absence of scalar potential case. Thus, we have the constraint equation as,

$$1 = x_1^2 + x_2 + \left(\frac{y_3^2}{2} + \sqrt{2} y_1 y_3 + y_1^2 \right) (1 - 4\lambda g^2 F_1^2 F_2 y_3^4) + 2y_2^2 y_3^2 + 2\sqrt{3} \lambda_{F_1} x_1 \quad (6.27)$$

ε and equations of motion can be rewritten with the dynamical variables,

$$\begin{aligned} \varepsilon = \frac{1}{1 + 3\lambda_{F_1}^2} \left(3x_1^2 (2\Gamma_{F_1} \lambda_{F_1}^2 + 1) + 4\lambda_{F_1} \sqrt{3} x_1 - 3\lambda_{F_1} \lambda_V x_2 + 6\lambda_{F_1}^2 \right. \\ \left. + y_3^2 + 2y_1^2 + 2\sqrt{2} y_1 y_3 + 4y_2^2 y_3^2 \right. \\ \left. + 6\lambda g^2 F_2' F_1^{5/2} \lambda_{F_1} y_3^4 (y_3^2 + 2\sqrt{2} y_1 y_3 + 2y_1^2) \right) \end{aligned} \quad (6.28)$$

$$\begin{aligned} \frac{\ddot{\chi}}{H^2 \sqrt{F_1}} = -6\sqrt{3} x_1 + 6\lambda_V x_2 + 6\lambda_{F_1} (\varepsilon - 2) \\ - 12\lambda g^2 F_1^{5/2} F_2' y_3^4 (y_3^2 + 2\sqrt{2} y_1 y_3 + 2y_1^2) \end{aligned} \quad (6.29)$$

$$\begin{aligned}
\frac{\ddot{\psi}}{H^2\sqrt{F_1}} = \frac{1}{1-4\lambda g^2 F_1^2 F_2 y_3^4} & \left(-6y_1 (1+4\lambda g^2 F_1^2 F_2 y_3^4) + \sqrt{2}(\varepsilon-2)y_3 \right. \\
& -4\sqrt{2}(\varepsilon+2)\lambda g^2 F_1^2 F_2 y_3^5 - 8\sqrt{2}y_2^2 y_3 \\
& + 4\sqrt{2}F_1^{3/2}y_3^3 \left(2\lambda\sqrt{F_1}F_2 (y_3^2 + 2\sqrt{2}y_1 y_3 + 2y_1^2) \right. \\
& \left. \left. + 2\sqrt{3}\lambda F_1 F_2' x_1 y_3 (y_3 + \sqrt{2}y_1) \right) \right) \quad (6.30)
\end{aligned}$$

We do not need to re-derive the autonomous equation system because we used the same variables as the previous one and that derivation includes closed version in which equation of motion terms left as a so-called "parameter" that we can just write into them. Yet, we can not directly compare those models since the system now has coupling functions explicitly. Therefore, once a non-minimal coupling introduced, axion field χ will start to appear in the expressions that yields non-complete/non-closed system. In order to make a dynamical system analysis of this particular model, one needs to redefine dynamical variables suitable to this configuration and rederive autonomous equation system. On the other hand, even though the model can be analysed with a new set of variables, it is important for us to see the behavioural change in phase space attractor structure between F^2 and F^4 models which could have been made directly with same phase space variables. Since this is not possible, we will not continue the analysis of this model.

7. CONCLUSIONS AND RECOMMENDATIONS

Axion and axion-like field's cosmological inflation models is an attractive field of study from several points of view. Most importantly it is the shift symmetric field definition that naturally dismisses the slow roll inflation model's UV sensitivity problem. In this thesis, we tried to analyze cosmological models with single axion-like inflaton field and two versions of chromo-natural inflation model with F^2 and F^4 non-Abelian gauge field with different configurations of coupling terms and potentials by utilizing the linear stability analysis of dynamical system analysis.

The motivation of linear stability analysis comes from the fact that the method provides clear fixed point behaviour when applied to simpler cosmological models such that attractor structure required by inflation theory is apparent. It is clear to us now that application of such method at least in the linear stability analysis constraint suffers mostly from complexity coming from non-minimal couplings introduced to the model either to gravity sector or gauge field sector. Complex systems such as non-minimally coupled single field models or non-Abelian gauge field incorporated models require at least 4 dynamical variables along with roll and track parameters in order to create a closed form of autonomous equation system thus automatically increasing the difficulty of analysis. As a result of phase space dimensions being higher than 3, prevents the visualization option of phase space. Although, determination of behaviour of critical points of autonomous system can still be achievable without visualization unless there is an eigenvalue set that fails linear stability analysis, i.e. zero eigenvalue(s) along with negative one(s), the method loses its appeal to some degree.

Nevertheless, for the model with non-minimally coupled single scalar field model with ϕ^2 potential, we have tried to apply linear stability analysis to three different coupling functions. Phase space of the model with arbitrary coupling function with the dependence to ϕ constructed to be 4 dimensional with the possibility of dimension reduction by eliminating one of the variables using the constraint equation coming from Friedmann equation.

In the first case, the coupling function chosen to be of the form of a Brans-Dicke like coupling with the function $F(\varphi) = \varphi/2$. Four dimensional autonomous system has a total of 4 critical points in which 2 of the critical point's behaviour is undeterminable in the context of linear stability theory since eigenvalue set has zero value and only non-zero terms are negative. The remaining 2 has an unstable behaviour and the results in table form can be found in Table 6.2. Reduced system has a total of 5 critical points where there are no points that fails the LST, yet almost all of the points are unstable with 3 of them can be considered as saddle points.

For the second case of non-minimally coupled single field model, we tried to analyze a coupling to the gravity of the form $F(\varphi) = \xi \varphi^2/2$ which we call it induced coupling. This choice of coupling function yields an interesting result for the autonomous system since initial 4 dimensional system automatically reduces to 2 dimensions without the usage of any constraint equation. In the coupling function we introduced a parameter called ξ to analyze the affect of the coupling strength to phase space structure. Thus, for three cases of coupling strength, i.e. weak ($\xi = 0.001$), conformal ($\xi = 1/6$) and strong ($\xi = 1$), we have 4 critical points for each coupling. In weak coupling case, 2 of the 4 critical point's behaviour is undeterminable in LST and the other 2 behaves as a repeller. By examining the phase space diagram in figure 6.1 one can see that around the origin there is a vertical line that is almost behaves like an attractor but instead of a point it is a line. The behaviour of this line is almost exactly corresponds to the behaviour of a centre manifold. Focusing around the attractor points on this line numerically showed us that field lines coming towards the "centre manifold" like line eventually converges to the said attractor points. Therefore, without further analysis with CMT, we can have an idea what to expect in terms of behaviour of a critical points that its behaviour fails to be determined in LST. In conformal case, phase space has 2 attractors and 2 repellers and finally in strong coupling case again the phase space has 2 attractors and 2 repellers. The phase space diagrams for each coupling strength can be found in figure 6.1, figure 6.2 and figure 6.3 respectively.

Final form of coupling we dealt with in the non-minimally coupled single scalar field model of inflation is the dilaton like coupling function, i.e. $F(\varphi) = e^{-\alpha\varphi}$. Linear stability analysis of this particular system showed us that 4 dimensional phase space has 4 distinct critical points with 2 of the point's behaviour is dependent on the value

of dynamical variables and other two has the characteristics of an unstable point with zero and positive eigenvalues that can be seen in Table 6.8. By eliminating one of the variables using constraint equation we are left with 3 dimensional system and LST yield that the reduced system has a total of 5 fixed points, 3 of them has unstable/saddle behaviour and remaining 2 are unstable.

As a result, we can see that in the non-minimally coupled single scalar field model other than the very specifically defined induced coupling case along with chatic-like approximated natural inflation potential, does not have any attractor points in our configuration of phase space from the application of linear stability analysis. Furthermore, introduction of any non-minimal couplings to the system increases difficulty of analysis along with making complete analysis of the system in the perspective of linear stability nearly impossible.

In the case of non-Abelian gauge field inflation models, we tried to analyze two settings of similar models one of which has a coupling to its F^2 term and the other has coupling to its F^4 term. In the F^2 model, we opted out of coupling to the gravity sector by choosing gravity coupling function to unity, $F_1 = 1$ and we set coupling to the gauge sector to $F_2 = \chi$ for simplicity. Therefore, we performed linear stability analysis to a model with zero potential, $V(\chi) = 0$, exponential potential, $V(\chi) = \exp(-\chi)$, chaotic potential, $V(\chi) = \chi^2$, and finally general monomial potential, $V(\chi) = \chi^n$.

In the first case, zero potential removes the possibility of inflation driven by the extra scalar field since only its kinetic term is left and therefore its equation of state parameter becomes 1. Thus, only possible inflaton candidate becomes the gauge field, but F^2 term alone can not drive inflation since its equation of state parameter does not satisfy the condition of inflation, i. e. $w \not\leq -1/3$. Since both components of the model in this particular configuration can not drive inflation period, this model is can be ruled out for now.

Next, we analyzed the same model with the scalar field potential taken to be as exponential. Phase space created by defined dynamical variables results in with 4 dimensional space that was initially 7 dimensional in general potential function, but the choice of coupling along with potential reduces its dimension naturally. This autonomous system created with relevant dynamical variables yields 5 distinct fixed

points with all having unstable behaviour. Linear stability analysis again does not give useful information about the configuration in hand.

In the third setting, we set the potential to be the one of chaotic inflation potential. Unlike the exponential potential, the phase space has one extra dimension coming from the chaotic potential does not make λ_V variable constant. Therefore, in this scenario the phase space is 5 dimensional and have 5 critical points in which the variable λ_V remains as a free parameter for all and behavioral analysis fails in the context of LST for all of the critical points. As same as the general result of the previous cases, in this model one needs further analysis to understand behaviour completely by using one of the alternative methods such as centre manifold theory (CMT), Kosambi-Cartan-Chern (KCC) or Lyapunov function method (LFM).

Finally, the last tried model is not that different from the one where we set the potential to be chaotic-like since now instead of squared field, we have general power law potential as $V(\chi) = \chi^n$ and every aspect of the model is same as before where behaviour of all 5 critical points fails to be determined in LST.

In our last attempt of redemption of linear stability analysis, we turn our calculation back to the model with zero potential. In the absence of any potential we know that there are no content of the model that can drive inflation. One way to resolve this problem is to have F^4 term instead of F^2 . To save the zero potential model, we introduced F^4 term in the last model and we see that now even if the potential of zero where the equation of state of scalar field becomes one, $w_\chi = 1$, and contribution of Yang-Mills term's equation of state is the one of radiation, i.e. $w_{YM} = 1/3$, we have the extra contribution coming from the F^4 term which has the equation of state of minus one, $w_{inf} = -1$, which is exactly what we want. To compare both zero potential model phase spaces, i.e. zero potential in F^2 and F^4 models, we need to use the same dynamical variables. Unfortunately, this constraint that we put on ourselves creates a fundamental problem that the only solution is to redefine dynamical variables that suits the new model with F^4 term since F^2 model variables does not close the autonomous equation system where the coupling to gauge sector explicit in autonomous equation system which prevents any introduction of non-minimal coupling. Therefore, we concluded our calculations for this model.

To sum up, we now undoubtedly know that dynamical system analysis in the linear stability analysis frame is not a viable method of evaluation of complex inflation models. The method is not suitable to analyze axion-like inflation potentials without any approximations to its potential in the case of natural inflation since cosine potential is not a good form where one can define a parameter with derivatives of the potential and come up with a closed form, i.e. independent variables becomes explicit for potential variable λ_V for cosine form. In the case of non-Abelian gauge field axionic inflation models, the method immensely suffers from non-minimal couplings and complexity of multi field nature of the model. For future work, one can continue the analysis of the axion-like inflation models by application of CMT or other methods to determine their phase space behaviour of critical points that are failed to be resolved in LST, yet along with it being crossing the boundaries of this thesis, we think that analysis beyond the linear stability, might make the work done in hopes for physical interpretation of inflation models with axion-like character more mathematical rather than physical.



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