STATISTICAL INFERENCE OF SOME COHERENT SYSTEMS IN STRESS-STRENGTH SETUP

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Submitted to Graduate School of Natural and Applied Sciences
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy in
Mathematics

Yeditepe University
2021

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To my mom...

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ACKNOWLEDGEMENTS

I would first like to thank my dear family, who has always supported me my entire life. I am very lucky and proud to have my beloved mom, Melek, my beloved dad, Şahbaz, my dear sister, Zehra and my lovely nephew, Mert, the person who motivates me the most. There are no adequate word to describe their unconditional love and belief on me. Without them, I could not have been with these valuable people who contributed to the creation of this thesis.

I would like to express my heartfelt gratitude to my supervisor, Fatih Kızılaslan for his encouragement, supportive guidance, insightful advice, time and effort throughout the thesis writing process.

I would also like to express my profound gratitude to my co-advisor, Ender Abadoğlu for his kindness, motivation and permanent support throughout my Ph.D.

I owe my heartfelt gratitude to Alexandros Papadopoulos for his inspiration of my research area. I am very lucky to learn from him.

Special thanks to K. İlhan İkeda not only for being a committee member but also encouraging me to start Ph.D. I would like to express heartfelt gratitude to him and Erol Serbest for their kindness and support. I would like to extend my sincere thanks to all committee members for their precious comments, time and patience that helped me improve this thesis.

I also wish to thank Demet Lüküslü, Tolga Uslu, Banu Koçer Reisman and Ahu Özmen Akalın, faculty members of the Sociology Department at Yeditepe University for their motivating support and belief. I had great pleasure to work with them during my assistantship.

Lastly, the support of my close friends was really precious during this process. I am so grateful to Serdar Nair, Baturay Yurtbay, Gülce Cüran, Seren Okan, Demet Çakır, Gülçin Akcan and Burcu Baş for pushing me up, no matter how and when.

ABSTRACT

STATISTICAL INFERENCE OF SOME COHERENT SYSTEMS IN STRESS-STRENGTH SETUP

This dissertation considers stress-strength reliability estimation of a consecutive k-out of-n system for identical and non-identical strength components when the stress and strength variables belong to the proportional hazard rate model. Estimation methods of the system reliability are applied under the classical and Bayesian perspectives. All point and interval estimations are derived in cases of the second parameters of underlying distributions are common and unknown, and different and known. Then, corresponding asymptotic confidence intervals and highest probability density credible intervals are also obtained for all cases. Comparison of the proposed estimates is presented through Monte Carlo simulations for each case. Finally, the implementation of proposed methods is analyzed by real data sets.

ÖZET

BAZI TUTARLI SİSTEMLERİN STRES-DAYANIKLILIK KURULUMU ALTINDA İSTATİKSEL ÇIKARIMI

Bu doktora tezi, özdeş ve özdeş olmayan dayanıklılık bileşenleri için ardıl n'den k'li bir sistemin güvenilirlik tahminini, stres ve dayanıklılık değişkenlerinin orantısal risk oran modeline sahip olmaları durumunda ele almaktadır. Sistem güvenilirliğinin tahmin yöntemleri, klasik ve Bayes bakış açılarına dayanarak uygulanmıştır. Bütün nokta ve aralık tahminleri, temel dağılımların ikinci parametrelerinin ortak ve bilinmeyen olması ve farklı ve bilinen olması durumunda elde edilmiştir. Ardından, asimptotik ve Bayes güven aralıkları da tüm durumlar için elde edilmiştir. Önerilen tahminlerin karşılaştırılması, her bir durum için Monte Carlo simülasyonları aracılığıyla sunulmaktadır. Son olarak, önerilen yöntemlerin uygulanması gerçek veri setleri ile analiz edilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

$(C, k, n : G)$	Consecutive k-out of-n:G system
$PHR(\alpha, \lambda)$	Proportional hazard rate family with positive interested parameters α and λ
$\Gamma(.)$	Gamma function
σ^2	Variance
ACI	Asymptotic confidence interval
A-D	Anderson-Darling
AL	Average length
ALs	Average lengths
cdf	Cumulative distribution function
CP	Coverage probability
CPs	Coverage probabilities
C-VM	Cramer-Von Mises
ER	Estimated risk
ERs	Estimated risks
HPD	Highest probability density
i.i.d.	Independent and identically distributed
K-S	Kolmogorov-Smirnov
MCMC	Markov chain Monte Carlo
ML	Maximum likelihood
MLE	Maximum likelihood estimation
MSE	Mean squared error
pdf	Probability density function
PHR	Proportional hazard rate
SE	Squared error
UMVU	Uniformly minimum variance unbiased
UMVUE	Uniformly minimum variance unbiased estimation

1. INTRODUCTION

In today's world, advances in science and technology make our lives easier and this increase our life standards. So, any failure in any modern system affects human life more than ever. At this point, reliability has become even more vital and it has been a common research trend in mathematical statistics and many applied fields. For a given system, the term reliability can be described as the probability that the system will operate successfully in a given time under the specified condition by Kumar et. al. in [1]. In this context, X and Y are assumed as random variables represents the "stress" and "strength" of the system, respectively. Formally, reliability refers to a measure described by the probability

$$R = P(X < Y) \tag{1.1}$$

which indicates the system success depends on the strength Y exceeding X. This probability, R is called **stress-strength reliability for one component**. Birnbaum [2] was first to introduce this idea and Birnbaum and McCarty [3] developed. Since Birnbaum's pioneering study, this problem has been discussed under several approaches to estimation and different assumptions on distributions. Kotz et al. [4] provides a comprehensive review of applications and theory in stress–strength models. Recent studies on this topic handled by Kundu and Gupta [5], Nadar et.al [6], Basirat et al. [7, 8], Çetinkaya and Genç [9] and Akgül and Şenoğlu [10]. Also, one can examine [11], [12] and [13] as current studies.

By extending the simple model with two or more components, multicomponent stress-strength model is developed. Reliability estimation in this model was introduced by Bhattacharyya and Johnson [14]. A **multicomponent system with n components** is a system includes n independent identical strengths $(Y_1, Y_2, \dots, Y_n)$ where each component experiences a random stress X. Suppose that this multicomponent system is alive if at least k of n strengths exceed the stress where $k < n$. The reliability in such a multicomponent stress-strength system, namely k-out of-n:G system is given by

$$R_{k,n} = P(\text{at least } k \text{ out of } n \text{ strength components exceed } X) \tag{1.2}$$

The estimation problems of multicomponent reliability were examined by many researchers under the different assumptions in the last decade. Some notable recent studies can be given as follows. For two parameter exponentiated Weibull distribution by Rao et al. [15], for the proportional reversed hazard rate model and a general class of inverse exponentiated distributions by Kızılaslan [16], [17], for Topp-Leone distribution by Akgül [18] and for Chen distribution by Kayal et al. [19]. Reliability estimation of a multicomponent system for generalized half-normal, modified Weibull, unit Gompertz distribution and Burr Type XII distributions based on progressive type-II censoring data were considered by Ahmadi and Ghafouri [20], Kotb and Raqab [21], Jha et al. [22] and Maurya and Tripathi [23], respectively. Similar problem was also considered by Akgül [24] for the exponentiated Pareto distribution based on complete sample. Reliability estimation of multicomponent system under a multilevel accelerated life testing was investigated by Wang et al [25] when the components follow Weibull distribution.

In reliability literature, "system" is described as a term refers to group of components that performs a particular function. There are plenty of real life applications and even logical problems that can be modelled as a "system" in this sense. Kuo and Zuo [26] provides a comprehensive review about system reliability models and methodologies to evaluate their reliabilities. Here are some fundamental concepts in this regard:

Definition 1.0.1. The state of the system or each component is a discrete random variable which takes only two possible values that indicates working state or failure state. If x_i defines the state of the component i for $1 \leq i \leq n$ and it is given by

$$x_i = \begin{cases} 1 & \text{if component } i \text{ works,} \\ 0 & \text{if component } i \text{ fails.} \end{cases} \tag{1.3}$$

The term component state vector $\mathbf{x} = (x_1, \dots, x_n)$ is used here to refer the states of all components. The state of the system is determined by the states of all components and it is a Bernoulli random variable. If ϕ denotes the state of the system, then

$$\phi = \begin{cases} 1 & \text{if the system works,} \\ 0 & \text{if the system fails.} \end{cases} \tag{1.4}$$

and $\phi = \phi(\mathbf{x}) = \phi(x_1, \dots, x_n)$ is defined as the **structure function** of the system.

A **series system** operates on condition that its every components work so that this series system fails whenever at least one component fails. The structure function of a series system is represented with the following equation:

$$\phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \min\{x_1, \dots, x_n\}. \tag{1.5}$$

A **parallel system** fails on condition that its each component fails so that this parallel system works whenever at least one component work. The structure function of a parallel system is represented with the following equation:

$$\phi(\mathbf{x}) = 1 - \prod_{i=1}^{n} (1 - x_i) = \max\{x_1, \dots, x_n\}. \tag{1.6}$$

By choosing k as 1 and n, series and parallel systems can be obtained from k-out of-n:G system, respectively.

Definition 1.0.2. A sequence of random variables $X_1, X_2, \dots, X_n$ can be described as **ex-**

changeable or **symmetric** if for each n

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = P(X_{\pi(1)} \leq x_1, \dots, X_{\pi(n)} \leq x_n), \tag{1.7}$$

for any permutation $\pi = (\pi(1), \dots, \pi(n))$ of $\{1, 2, \dots, n\}$.

Definition 1.0.3. A component in a system is called **irrelevant** if the state of this component does not influence the state of the system. If a component is **relevant**, then the system state is determined by at least one component i in the system. In other words, the system runs whenever component i runs and the system fails when the component i fails.

Definition 1.0.4. A system is defined as **coherent** on condition that its structure function $\phi(\mathbf{x})$ is non-decreasing in each x_i for $1 \leq i \leq n$ and each component is relevant. The reliability of a coherent system with n components which has structure function ϕ is

$$R_\phi = P\left(\phi\left(X_1, X_2, \dots, X_n\right) = 1\right). \tag{1.8}$$

where ith component of the state be denoted by X_i. In stress-strength setup, assume that a system with n strength components denoted by Y_i, $i = 1, 2, \dots, n$ and each strength subject to a stress X. The component Y_i fails if the applied stress exceeds its strength at any moment, otherwise fails. That is, the the reliability of the ith component is given by $P\left(Y_i > X\right)$. Define the indicators

$$\xi_i = \begin{cases} 1, & \text{if} \quad Y_i > X \\ 0, & \text{if} \quad Y_i \leq X \end{cases} \qquad i = 1, 2, \dots, n \tag{1.9}$$

where $Y_1, Y_2, \dots, Y_n$ are independent identical random strengths having cumulative distribution F_Y and independent of the random stress X having cumulative distribution F_X. It is clear that the random variables $\xi_1, \xi_2, \dots, \xi_n$ are exchangeable. A consecutive k-out-of-n:G system is an example of the coherent system and any coherent system is a linear combination of the series and parallel systems.

Definition 1.0.5. A linear (or circular) **consecutive k-out of-n: $G(F)$ system** is a system consisting of linearly (or circularly) connected n components such that it works (fails) if and only if at least its k consecutive components work(fail). This structure can be denoted by $(C, k, n : G)$ or $Con/k/n : G$. The classification of this system depends on the working principle of components, G refers good and F refers failure. Based on the definition above:

- If k is chosen as 1, then a consecutive 1-out of-n:$G(F)$ system becomes a parallel (series) system,
- If k is chosen as n, then a consecutive n-out of-n:$G(F)$ system becomes a series (parallel) system.

Estimating the reliability of the system has significant effects on society. There are plenty of real life applications and even logical problems that can be modelled as a $(C, k, n; G)$. Thus, exact expressions, approximations and bounds for reliability under this setup have been considered by many researchers with different methodology. However, computation of the system reliability is more complicated than k-out of-n:G system. Reliability analysis of this system was first considered by Kontoleon [27] and then Chiang and Niu [28]. A comprehensive review of earlier work in this context can be found in Chang et al. [29] and a research overview of reliability studies on consecutive k-out of-n systems was presented by Eryılmaz in [30]. In addition, some recent works on reliabilities of consecutive k-out of-n systems were discussed by the following authors: Zhu et al examined the system reliability with homogeneous Markov-dependent components in [31], Dui et al. studied importance measures in consecutive k-out of-n systems in [32] and Li et al. [33] considered the reliability modeling for consecutive k-out of-n:F-systems which can be applied to simulate the intelligent closed recurring water cooling automation system in industry.

As mentioned above, although the reliability properties of consecutive k-out-of-n:G system $(C, k, n; G)$ have been obtained in various studies, the estimation problem for this system has not been taken into consideration much until now except few studies. In our knowledge, the first study for the classical estimation of stress-strength reliability of consecutive k-out-of-n: G system was considered by Eryilmaz in [34]. Maximum likelihood estimate (MLE) and uniformly minimum variance unbiased estimate (UMVUE) were studied when the strength components were non-identical and distributed exponential. ML and UMVU estimates of stress-strength reliability of a consecutive k-out of-$n : G$ system with a change point in stress for the exponential distribution was studied by Akıcı [35].

Definition 1.0.6. Let X be a random variable from the proportional hazard rate (PHR) family of continuous distributions with cumulative distribution function (cdf)

$$F(x) = 1 - \left(\overline{F}_0(x;\lambda)\right)^{\alpha},\ x > 0, \tag{1.10}$$

and corresponding probability density function (pdf)

$$f(x) = \alpha f_0(x;\lambda)\left(\overline{F}_0(x;\lambda)\right)^{\alpha-1}, \tag{1.11}$$

where $\overline{F}_0(x;\lambda) = 1 - F_0(x;\lambda)$ defines the survival function of the baseline random variable, α and λ are the positive interested parameters. It is denoted by $X \sim PHR(\alpha,\lambda)$. It is known that some well-known distributions such as Burr Type XII, Gompertz, Kumaraswamy, Lomax, Pareto, Rayleigh, Weibull and so on belong to the PHR family [7]. Hence, usage of this kind of distribution families presents more general results to the researchers.

To the best of our knowledge, classical and Bayesian estimations for the stress-strength reliability of a consecutive k-out of-$n : G$ system has not been analyzed for PHR family. Since the PHR family includes exponential distribution, some result of this study is a generalization of Eryilmaz [34] from the statistical inference perspective. Also, it is a well-known fact that $(C, k, n; G)$ system include series and parallel systems as special cases and any coherent system is a linear combination of the series and parallel systems. That is why a part of this study is a generalization of some results on this topic.

In this thesis, our main consideration is reliability analysis of coherent systems in stress-strength setup which have underlying strength and stress variables following distributions from PHR family. Since $(C, k, n; G)$ system is an example of the coherent system, we consider point and interval reliability estimation of a $(C, k, n; G)$ system for PHR family. Based on this assumption,

- At first, we aim to consider reliability estimation of the system with independent and identically distributed (i.i.d.) strength components when both stress and strength components follow the proportional hazard rate model. This study is presented in Chapter 3.

In the reliability literature, most of the studies on multicomponent stress-strength reliability are based on the independent and identically distributed (i.i.d.) strength components. However, we can encounter that a system composed of different types of strength components in real-life applications. For example, meteorological measures such as temperature, sunshine duration, precipitation, humidity, wind speed are changed with regard to the season or month, time of day. In this case, consideration of this kind of meteorological data as iid is not been realistic. Hence, we can obtain more suitable models for real-life problems using non-identical components.

In recent years, the estimation problem of the multicomponent system reliability with non-identical components has been paid attention by many scholars under different assumptions. Some recent studies on this topic handled by the following authors: Rasethuntsa and Nadar considered the reliability estimation of this system based on upper record values from the Kumaraswamy generalized distribution family in [36], and Ali et al. studied the reliability estimation when both stress and strength variables follow Weibull and Burr-III distributions in [37]. Çetinkaya investigated the multicomponent reliability estimation under generalized progressive hybrid censoring scheme when the components follow Weibull distribution and strength components are non-identical in [38]. He also considered the same problem for the simple stress-strength reliability under jointly type-II censored Weibull components in [39]. Moreover, Kohansal et al. studied multicomponent stress-strength reliability with non-identical strength components based on bathtub-shaped distribution under adaptive Type-II hybrid progressive censoring samples in [40]. However, this non-identical reliability estimation problem for a consecutive k-out of-n system has not been considered in detail.

Even though the statistical inferences of multicomponent stress-strength reliability when the strength components have non-identical distributions have paid attention in recent years, the same problem for the consecutive k-out of-$n : G$ system has not been considered except few studies. Aforementioned studies based on exponential distribution when the change point in stress by Akıcı [35] and change point in strength by Eryilmaz [34].

- Our second aim is to consider stress-strength reliability estimation of a consecutive k-out of-n system with non-identical strength components when both stress and strength components follow the proportional hazard rate model. This study is presented in Chapter 4. In addition, this study provides new perspectives in the implementation of obtained methods to the NASA's POWER (Prediction Of Worldwide Energy Resource) data (for more detail see https://power.larc.nasa.gov/). For this aim, wind energy potentials of two locations are compared by using the considered reliability model results based on NASA's source data. From this point, this study will be the first study for using this kind of data in the reliability literature.

The content of present study is organized as follows: Chapter 2 introduces some fundamental concepts used in the study. Estimation of stress-strength reliability for the considered system is investigated in Chapter 3 based on identical strength components and in Chapter 4 based on non-identical strength components. Detailed information about these chapters are given in the introductions of both chapters. Then, concluding remarks are given in last chapter, Chapter 5.

2. PRELIMINARIES

This section introduces fundamental concepts that are used in this study.

2.1. ESTIMATION METHODS

One of the fundamental aim in statistics is estimating the parameters via point estimation or interval estimation. For a point estimation method, the main aim is to predict a value that is close to the real value of parameters. For instance, the maximum likelihood estimation method is one of the basic point estimation.

2.1.1 MLE

Let $X_1, \ldots, X_n$ be random variables with common probability density functions $f\left(x_1, \ldots, x_n \mid \theta\right)$ where $\theta \in \Theta$ is unknown. (Generally, θ represents a parameter vector, $\theta = (\theta_1, \ldots, \theta_k)$). The **likelihood function** is

$$L\left(\underline{x}; \theta\right) \equiv L\left(x_1, \ldots, x_n \mid \theta\right) = \prod_{i=1}^{n} f_{X_i}\left(x_i; \theta\right). \tag{2.1}$$

The **MLE** of parameters θ is the value $\hat{\theta}$ of θ such that

$$L\left(\underline{x}; \hat{\theta}\right) = \max_{\theta \in \Theta} L\left(\underline{x}; \theta\right). \tag{2.2}$$

The MLE of θ can be found by the following procedure:

1. A sample of size n is taken as input, say $X_1, X_2, \ldots, X_n$.

2. Find the likelihood function stated in the equation 2.2.

3. Find the derivative(s) with respect to the parameter(s) θ.

4. Find the values of θ maximizing the likelihood function.

5. The maximum likelihood estimator can be obtained as an output.

2.1.2 UMVUE

One major purpose in point estimation theory is to find a better estimate. For instance, unbiased estimators has a crucial role in estimation. An estimator $T\left(\underline{X}\right)$ is defined as an **unbiased estimator** for a function of the parameter $g(\theta)$ provided that $E_{\theta}\left(T(\underline{X})\right) = g(\theta)$ for every $\theta \in \Theta$.

On the other hand, being unbiased is not enough to be a good estimator. So, there is a need for more efficient estimator, abbreviated as UMVUE. An unbiased estimator $T(\underline{X})$ of θ is called the **UMVUE** on condition that $\text{Var}(T(\underline{X})) \leq \text{Var}(U(\underline{X}))$ for any other unbiased estimator $U(\underline{X})$ of θ. If UMVUE exists, then the following ways can be applied to find it:

- The first way is considering the lower bound of variance. That is, if an unbiased estimator attains its Cramer-Rao lower bound, then it is the UMVUE.

- The second way is using sufficiency and completeness. That is, by finding an unbiased function of a complete sufficient statistic. (Rao-Blackwell and Lehmann-Scheffé Theorem)

2.1.3 Bayesian Estimation

As an alternative approach to the point estimation, Bayesian estimation is done by taking the parameters of the distribution as random variables having known prior probability density functions. Then the following machinery explains how to Bayesian estimation works:

1. A distribution of the parameter(s) with initialized parameter(s), which is called **prior distribution** and a loss function are taken as inputs.

2. Compute the **posterior** probability density function.

3. Compute the expected loss.

4. Find the values that minimizes the expected loss.

5. The Bayesian estimator can be obtained.

Often the posterior not be able to written in a closed form and thus numerical methods and an approximation are required. For example, Tierney-Kadane and Lindley's approximations and Markov Chain Monte Carlo (MCMC) methods can be useful in this case.

Definition 2.1.1. Let θ be an unknown parameter and $\hat{\theta}$ be an estimate of θ. Then the **squared error function** is defined by

$$L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2. \tag{2.3}$$

The Bayes estimator of θ with respect to SE loss function is the posterior mean of θ , given by

$$\hat{\theta} = E(\theta|\underline{x}). \tag{2.4}$$

2.1.3.1 Lindley's Approximation

For given data $x = (x_1, \dots, x_n)$, the posterior mean of a smooth, positive function $u(\theta)$ on the parameter space can be derived as follows:

$$E(u(\theta)|x) = \frac{\int u(\theta) e^{l(\theta)+\rho(\theta)} d\theta}{\int e^{l(\theta)+\rho(\theta)} d\theta} \tag{2.5}$$

where $l(\theta)$ is the logarithm of the likelihood function, $\rho(\theta)$ is the logarithm of the prior density of θ and $\theta = (\theta_1, \dots, \theta_m)$ is a parameter. Lindley developed the following theorem as an approximation for the calculations in (2.5).

Theorem 2.1.1. *For n sufficiently large and $l(\theta)$ defined in (2.5) concentrates around a unique maximum likelihood estimator $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_m)$ for θ, the ratio of integrals in (2.5) is given approximately as*

$$E(u(\theta)|x) = [u + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m} L_{ijk}\sigma_{ij}\sigma_{kl}u_l]_{\hat{\theta}} \tag{2.6}$$

where $u_i = \partial u(\theta)/\partial\theta_i, u_{ij} = \partial^2 u(\theta)/\partial\theta_i\partial\theta_j, L_{ijk} = \partial^3 l(\theta)/\partial\theta_i\partial\theta_j\partial\theta_k, \rho_j = \partial\rho(\theta)/\partial\theta_j$ and $\sigma_{ij} = (i,j)$th element in the inverse of matrix $\{-L_{ij}\}$ all evaluated at the MLE of the parameters.

2.1.3.2 MCMC Method

MCMC simulation technique aims to generate samples from a Markov chain that has posterior distribution as its stationary distribution and then use these samples to make inferences about the unknown quantities of interest. Here, a Markov chain involves a sequence of random variables where the distribution of each one depends on the previous one [41],[42].

MCMC techniques are useful in computing complicated integrals such as posterior and predictive distributions, finding expectations, model comparisons. Also, it can be applicable in high dimensions.

In implementation of MCMC, approximation of posterior distributions are constructed by the methods which differ on how the proposal distribution is defined. We used the popular ones: Gibbs sampling (introduced by Geman and Geman [43]) and Metropolis-Hastings algorithm (introduced by [44])

Gibbs sampling algorithm generates a dependent Markov chain to break the sampling problem from the high dimensional joint distribution into a series of sample from low-dimensional conditional distribution. Assume that $\theta = (\theta_1, \dots, \theta_p)$ with a probability density function $p(\theta) = p(\theta_1, \dots, \theta_p)$ (target distribution) and the full conditional distributions which are easier to sample from denoted by $\pi_1(\theta_1|\theta_2, \dots, \theta_p, \underline{x}), \dots, \pi_p(\theta_p|\theta_1, \dots, \theta_{p-1}, \underline{x})$. Then, given an initial value $\theta^{(0)} = (\theta_1^{(0)}, \dots, \theta_p^{(0)})$, Gibbs sampling algorithm can be applied as follows:

1. Set $i = 1$.
2. Generate $\theta_1^{(i)}$ from the conditional distribution $\pi_1(\theta_1|\theta_2, \dots, \theta_p, \underline{x})$.
3. Generate $\theta_2^{(i)}$ from the conditional distribution $\pi_2(\theta_2|\theta_1, \dots, \theta_p, \underline{x})$.
4. Generate $\theta_p^{(i)}$ from the conditional distribution $\pi_p(\theta_p|\theta_1, \dots, \theta_{p-1}, \underline{x})$.
5. Set $i = i + 1$ and repeat the steps 2-4, $i = 1, 2, \dots, N$.

Metropolis-Hastings algorithm is a general MCMC method which was proposed by Metropolis [44] and generalized by Hastings [45]. In this method, aim is to generate a candidate of the sequence based on a proposal distribution which enables to sample from easier. Then, acceptance or rejection rule is used to converge to the target distribution.

In this manner, let the proposal distribution $q(\theta_1^*|\theta_2, \dots, \theta_p, \underline{x})$ generate a candidate θ_1^* and given an itinial value $\theta^{(0)} = (\theta_1^{(0)}, \dots, \theta_p^{(0)})$, following steps are applied to implement the algorithm:

1. Propose: $\theta^* \sim q(\theta_1^*|\theta_2^{(i-1)}, \dots, \theta_p^{(i-1)}, \underline{x})$. Propose a value for θ_1 from the proposal distribution.

2. Compute the acceptance probability by the acceptance function $\alpha(\theta_1^{(i-1)}, \theta^*)$ based on full joint density $\pi()$ and the proposal distribution.

$$\alpha(\theta_1^{(i-1)}, \theta^*) = \min\left[1, \frac{\pi_1(\theta_1^*|\theta_2^{(i-1)}, \dots, \theta_p^{(i-1)}, \underline{x})q(\theta_1^{(i-1)}|\theta_2^{(i-1)}, \dots, \theta_p^{(i-1)}, \underline{x})}{\pi_1(\theta_1^{(i-1)}|\theta_2^{(i-1)}, \dots, \theta_p^{(i-1)}, \underline{x})q(\theta_1^*|\theta_2^{(i-1)}, \dots, \theta_p^{(i-1)}, \underline{x})}\right] \tag{2.7}$$

3. Generate $U \sim Uniform(0, 1)$.
   - If $U \leq \alpha(\theta_1^{(i-1)}, \theta^*)$, then **accept the proposal** and set $\theta_1^{(i)} = \theta^*$.
   - else, **reject the proposal** and set $\theta_1^{(i)} = \theta^{(i-1)}$.

4. Set $i = i + 1$ and repeat the steps 1-3, $i = 1, 2, \dots, N$.

When the proposal distribution is symmetric, $\alpha(\theta^{(i-1)}|\theta^*) = \alpha(\theta^*|\theta^{(i-1)})$, then the acceptance probability is as follows:

$$\alpha(\theta^{(i-1)}|\theta^*) = \min\left[1, \frac{\pi_1(\theta_1^*|\theta_2^{(i-1)}, \dots, \theta_p^{(i-1)}, \underline{x})}{\pi_1(\theta_1^{(i-1)}|\theta_2^{(i-1)}, \dots, \theta_p^{(i-1)}, \underline{x})}\right] \tag{2.8}$$

and $\theta_2, \dots, \theta_p$ are sampled using similar procedure.

Remark. Gibbs sampling with the acceptance probability is always 1 is a special case of

Metropolis-Hastings algorithm.

To minimize the effect of initial values on the posterior inference, **burn-in** method is applied by discarding an initial portion of a Markov chain sample. Theoretically, if we operate the Markov chain for an infinite amount of time, the effect of initial values decreases to zero. But, since we do not have infinite samples, after t iterations, the chain has reached its target distribution so that we can discard the early portion. Here, the value of t is **burn-in number** and the method is called **burn-in** which enables good samples for posterior inference.

2.2. SOFTWARE USED IN THE SIMULATION STUDY

Numerical computing and simulation is performed using MATLAB and statistical software R. [46]. Also, in real data analysis part, goodness-of-fit tests are applied using *stats* and *goftest* packages in R. [47]. The following data sources are used: California Data Exchange Center and Lyu's book [48] in Chapter 3 and NASA's POWER (Prediction Of Worldwide Energy Resource) in Chapter 4. Access information for these data are given in simulation study sections in chapters.

3. RELIABILITY ESTIMATION OF $(C,k,n;G)$ SYSTEM WITH IDENTICAL STRENGTH COMPONENTS

In this chapter, main consideration is the reliability estimation of a $(C,k,n;G)$ system in stress-strength setup when the strength and stress variables follow the PHR model and strength variables are i.i.d.. Based on this, let $Y_1,...,Y_n$ be independent and identical random strengths having cumulative distribution function (cdf) F_Y, and independent of the random stress X having cdf F_X. Then, the reliability of $(C,k,n;G)$ system in a stress-strength setup was obtained by Eryılmaz and Demir [49] as follows:

$$R_{n,k} = 1 - \sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\lambda_{l+i}, \tag{3.1}$$

where $t_l = \min\left(\left[l/k\right], n-l+1\right)$, $\lambda_m = P\left(Y_1 > X, \ldots, Y_m > X\right)$ and $[a]$ shows the integer part of a. They also derived the following easier formula for $2k \geq n$

$$R_{n,k} = \left(n-k+1\right)\lambda_k - (n-k)\lambda_{k+1}. \tag{3.2}$$

In this context, statistical inferences including point and interval estimation of $R_{n,k}$ are developed under both classical and Bayesian procedures when the second parameters of stress and strength variables are common and unknown, and different and known.

Chapter organization is given as follows:

- In Section 3.1, the MLE, asymptotic confidence interval and Bayes estimates of $R_{n,k}$ are derived when the second parameters of underlying distributions are common (λ) and unknown. Lindley's approximation and MCMC method are developed to obtain Bayes estimates of $R_{n,k}$ when the parameters α, β have statistically independent gamma prior distributions and λ has the log-concave density function. The HPD credible interval of $R_{n,k}$ is constructed by using the hybrid Metropolis-Hastings (M-H) and Gibbs sampling algorithm.

- In Section 3.2, the MLE, the asymptotic confidence interval, the UMVUE and the closed form of Bayes estimates $R_{n,k}$ are derived when the second parameters of underlying distributions λ_1 and λ_2 are different and known. The hypergeometric series are utilized to obtain the Bayes estimate of $R_{n,k}$ analytically. Moreover, Lindley's approximation and MCMC method are also applied for the comparison of the approximate methods of Bayes estimates with the exact Bayes estimate. The HPD credible interval of $R_{n,k}$ is constructed by using Gibbs sampling algorithm.

- In Section 3.3, simulation study is carried out for the comparison of the proposed estimators of $R_{n,k}$ via Monte Carlo simulations and findings are presented by tables and plots.

- In Section 3.4, two different real data sets are analyzed to illustrate the applicability of findings.

- Finally, the comments and conclusion of the chapter are given in Section 3.5.

3.1. ESTIMATION OF $R_{n,k}$ WHEN THE SECOND PARAMETERS ARE COMMON AND UNKNOWN

In this section, the ML and approximate Bayes estimates $R_{n,k}$ are considered when the second parameters of underlying distributions are common (λ) and unknown.

3.1.1 MLE of $R_{n,k}$

We assume that strength variables $Y_i \sim PHR(\alpha, \lambda)$, $i = 1, .., n$ and stress variable $X \sim PHR(\beta, \lambda)$. Then, λ_m is given as

$$\lambda_m = P\left(Y_1 > X, ..., Y_m > X\right) \tag{3.3}$$

$$= \int_0^\infty P\left(Y_1 > X, \dots, Y_m > X | X = x\right) dF_X(x) \tag{3.4}$$

$$= \int_0^\infty P\left(Y_1 > X | X = x\right) \dots P\left(Y_m > X | X = x\right) dF_X(x) \tag{3.5}$$

$$= \int_0^\infty \left(1 - F_Y(x)\right)^m f_X(x) dx \tag{3.6}$$

$$= \frac{\beta}{\alpha\, m + \beta} \tag{3.7}$$

and $R_{n,k}$ is given by using (3.1)

$$R_{n,k} = 1 - \sum_{l=0}^{n} \sum_{j=0}^{t_l} \sum_{i=0}^{n-l} (-1)^{i+j} \binom{n-l+1}{j} \binom{n-kj}{n-l} \binom{n-l}{i} \frac{\beta}{\alpha(l+i)+\beta}, \tag{3.8}$$

where $t_l = \min\left(\left[l/k\right], n-l+1\right)$, and for $2k \geq n$

$$R_{n,k} = \sum_{i=0}^{1} (-1)^i \frac{(n-k+1-i)\beta}{\alpha(k+i)+\beta} = (n-k+1)\frac{\beta}{\alpha k+\beta} - (n-k)\frac{\beta}{\alpha(k+1)+\beta}. \tag{3.9}$$

When m systems are subjected to a life-testing experiment for obtaining the ML estimators, we have the following observed data $Y_{i1}, Y_{i2}, \dots, Y_{in}$ and X_i, $i = 1, \dots, m$.

	strengths	stress
1^{st} experiment	$Y_{11}, Y_{12}, \dots, Y_{1n}$	X_1
2^{nd} experiment	$Y_{21}, Y_{22}, \dots, Y_{2n}$	X_2
$\vdots$	$\vdots$	$\vdots$
m^{th} experiment	$Y_{m1}, Y_{m2}, \dots, Y_{mn}$	X_m

Hence, the likelihood function of observed sample is

$$\begin{aligned} L\left(\alpha,\beta,\lambda;\underline{x},\underline{y}\right) =& \prod_{i=1}^{m}\left(\prod_{j=1}^{n}\left(f_Y(y_{ij})\right) f_X(x_i)\right. \\ =& \alpha^{nm}\beta^{m} \exp\left[\sum_{i=1}^{m}\sum_{j=1}^{n} \ln f_0(y_{ij};\lambda) + (\alpha-1)\sum_{i=1}^{m}\sum_{j=1}^{n}\ln \overline{F_0}(y_{ij};\lambda)\right] \\ & \exp\left[\sum_{i=1}^{m}\ln f_0(x_i;\lambda) + (\beta-1)\sum_{i=1}^{m}\ln\overline{F_0}(x_i;\lambda)\right] \end{aligned} \tag{3.10}$$

and the log-likelihood function is

$$l\left(\alpha,\beta,\lambda;\underline{x},\underline{y}\right) = nm\ln(\alpha) + m\ln(\beta) - w_\lambda - (\alpha-1)\,w_\lambda^* - v_\lambda - (\beta-1)\,v_\lambda^* \tag{3.11}$$

where

$$w_\lambda^* = -\sum_{i=1}^{m}\sum_{j=1}^{n}\ln\overline{F_0}(y_{ij};\lambda) \text{ and } w_\lambda = -\sum_{i=1}^{m}\sum_{j=1}^{n}\ln f_0(y_{ij};\lambda), \tag{3.12}$$

$$v_\lambda^* = -\sum_{i=1}^{m}\ln\overline{F_0}(x_i;\lambda) \text{ and } v_\lambda = -\sum_{i=1}^{m}\ln f_0(x_i;\lambda). \tag{3.13}$$

The ML estimates of α and β are given by

$$\hat{\alpha} = \frac{nm}{W_{\hat{\lambda}}^*} \text{ and } \hat{\beta} = \frac{m}{V_{\hat{\lambda}}^*} \tag{3.14}$$

The MLE of λ, named $\hat{\lambda}$, can be obtained from the solution of the following nonlinear equation

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{f_{0_\lambda}(y_{ij};\lambda)}{f_0(y_{ij};\lambda)}+\left(\frac{nm}{W_\lambda^*}-1\right)\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\overline{F_0}_\lambda(y_{ij};\lambda)}{\overline{F_0}(y_{ij};\lambda)}+\sum_{i=1}^{m}\frac{f_{0_\lambda}(x_i;\lambda)}{f_0(x_i;\lambda)} \\ +\left(\frac{m}{V_\lambda^*}-1\right)\sum_{i=1}^{m}\frac{\overline{F_0}_\lambda(x_i;\lambda)}{\overline{F_0}(x_i;\lambda)}=0. \tag{3.15}$$

where $\partial f_0(x;\lambda)/\partial\lambda \equiv f_{0_\lambda}(x;\lambda)$ and $\partial\overline{F_0}(x;\lambda)/\partial\lambda \equiv \overline{F_0}_\lambda(x;\lambda)$. In this nonlinear equation, numerical methods can be applied to compute $\hat{\lambda}$. Then, ML estimates of α and β can be obtained from (3.14). After the MLE of parameters are computed, the invariance property can be implemented in (3.8) and (3.9) to obtain the MLE of $R_{n,k}$, denoted $\hat{R}_{n,k}^{MLE}$.

$$\hat{R}_{n,k}^{MLE}=1-\sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\frac{\hat{\beta}}{\hat{\alpha}(l+i)+\hat{\beta}}, \tag{3.16}$$

where $t_l=\min\left(\left[l/k\right],n-l+1\right)$, and

$$\hat{R}_{n,k}^{MLE}=(n-k+1)\frac{\hat{\beta}}{\hat{\alpha}k+\hat{\beta}}-(n-k)\frac{\hat{\beta}}{\hat{\alpha}(k+1)+\hat{\beta}}, \tag{3.17}$$

for $2k\geq n$.

3.1.2 Asymptotic Distribution and Confidence Interval for $R_{n,k}$

The observed information matrix of $\theta=(\alpha,\beta,\lambda)$ denoted by $J(\theta)$ and its elements are derived as

$$J_{11}=nm/\alpha^2,\ J_{22}=m/\beta^2,\ J_{12}=J_{21}=0, \tag{3.18}$$

$$J_{13}=J_{31}=-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\overline{F_0}_\lambda(y_{ij};\lambda)}{\overline{F_0}(y_{ij};\lambda)},\ J_{23}=J_{32}=-\sum_{i=1}^{m}\frac{\overline{F_0}_\lambda(x_i;\lambda)}{\overline{F_0}(x_i;\lambda)}, \tag{3.19}$$

$$\begin{aligned}
J_{33} = &- (\alpha - 1) \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\overline{F_0}_{\lambda\lambda}(y_{ij};\lambda)\overline{F_0}(y_{ij};\lambda) - (\overline{F_0}_{\lambda}(y_{ij};\lambda))^2}{(\overline{F_0}(y_{ij};\lambda))^2} \\
&- \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{f_{0\lambda\lambda}(y_{ij};\lambda) f_0(y_{ij};\lambda) - (f_{0\lambda}(y_{ij};\lambda))^2}{(f_0(y_{ij};\lambda))^2} \\
&- (\beta - 1) \sum_{i=1}^{m} \frac{\overline{F_0}_{\lambda\lambda}(x_i;\lambda)\overline{F_0}(x_i;\lambda) - (\overline{F_0}_{\lambda}(x_i;\lambda))^2}{(\overline{F_0}(x_i;\lambda))^2} \\
&- \sum_{i=1}^{m} \frac{f_{0\lambda\lambda}(x_i;\lambda) f_0(x_i;\lambda) - (f_{0\lambda}(x_i;\lambda))^2}{(f_0(x_i;\lambda))^2}.
\end{aligned} \tag{3.20}$$

These elements include the baseline cdf F_0 and pdf f_0. Therefore, their expectations cannot be derived analytically and this implies the observed information matrix to be utilized as a consistent estimator of Fisher information matrix $I(\theta)$ under mild regularity conditions (see Lawless [50]). Hence, when the underlying distributions satisfy the mild regularity conditions, an asymptotic confidence interval of $R_{n,k}$ can be derived as below.

The $\hat{R}_{n,k}^{MLE}$ is asymptotically normal with mean $R_{n,k}$ and asymptotic variance

$$\sigma^2_{R_{n,k}} = \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial R_{n,k}}{\partial \theta_i} \frac{\partial R_{n,k}}{\partial \theta_j} I_{ij}^{-1}, \tag{3.21}$$

where I_{ij}^{-1} is the (i,j)th element of $I(\theta)$ (see Rao [51]). Then, for our case

$$\sigma^2_{R_{n,k}} = \left(\frac{\partial R_{n,k}}{\partial \alpha}\right)^2 I_{11}^{-1} + 2\frac{\partial R_{n,k}}{\partial \alpha} \frac{\partial R_{n,k}}{\partial \beta} I_{12}^{-1} + \left(\frac{\partial R_{n,k}}{\partial \beta}\right)^2 I_{22}^{-1} \tag{3.22}$$

where

$$\frac{\partial R_{n,k}}{\partial \alpha} = \sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\frac{(l+i)\beta}{(\alpha(l+i)+\beta)^2} \quad (3.23)$$

and

$$\frac{\partial R_{n,k}}{\partial \beta} = \sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j+1}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\frac{(l+i)\alpha}{(\alpha(l+i)+\beta)^2}. \quad (3.24)$$

When $I(\theta)$ cannot be obtained, $J(\theta)$ is substituted for $I(\theta)$. Therefore, $R_{n,k} \in (\hat{R}_{n,k}^{MLE} \pm z_{\gamma/2}\hat{\sigma}_{R_{n,k}})$ introduces an asymptotic 100(1- γ) % confidence interval of $R_{n,k}$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{\sigma}_{R_{n,k}}$ is the value at MLE of parameters.

3.1.3 Bayes Estimation of $R_{n,k}$

All parameters α, β and λ are assumed as random variable such that α and β have statistically independent gamma prior distributions with parameters (a_i, b_i), $i = 1, 2$, respectively and λ has the log-concave density function $\pi(\lambda)$. The pdf of a gamma random variable X with parameters (a_i, b_i) is

$$f(x) = \frac{{b_i}^{a_i}}{\Gamma(a_i)}x^{a_i-1}e^{-xb_i}, \quad (3.25)$$

where $x > 0$, $a_i, b_i > 0$, $i = 1, 2$. Then, the joint posterior density function of α, β and λ is

$$\pi(\alpha,\beta,\lambda|\underline{x},\underline{y}) = \frac{L\left(\alpha,\beta,\lambda;\underline{x},\underline{y}\right)\pi(\alpha)\pi(\beta)\pi(\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty L\left(\alpha,\beta,\lambda;\underline{x},\underline{y}\right)\pi(\alpha)\pi(\beta)\pi(\lambda)d\alpha d\beta d\lambda} \quad (3.26)$$

$$= [I(\underline{x}, \underline{y})]^{-1}\alpha^{nm+a_1-1}\beta^{m+a_2-1}e^{-\alpha(b_1+w^*_\lambda)}e^{-\beta(b_2+v^*_\lambda)}$$
$$e^{w^*_\lambda - w_\lambda}e^{v^*_\lambda - v_\lambda}\pi(\lambda), \tag{3.27}$$

where $I(\underline{x}, \underline{y})$ is the normalizing constant and given by

$$\frac{[I(\underline{x}, \underline{y})]^{-1}}{\Gamma(nm + a_1)\Gamma(m + a_2)} = \int_0^\infty \frac{e^{w^*_\lambda - w_\lambda}e^{v^*_\lambda - v_\lambda}\pi(\lambda)}{(b_1 + w^*_\lambda)^{nm+a_1}(b_2 + v^*_\lambda)^{m+a_2}} d\lambda. \tag{3.28}$$

Then, the Bayes estimator of $R_{n,k}$ under the SE loss function is given by

$$\hat{R}_{n,k,B} = \int_0^\infty \int_0^\infty \int_0^\infty R_{n,k}\pi\left(\alpha, \beta, \lambda|\underline{x}, \underline{y}\right) d\alpha d\beta d\lambda \tag{3.29}$$

Since the analytical computation of the integral in (3.29) is hard, alternative approaches can be applied to compute approximately. In this regard, Lindley's approximation and MCMC method are presented in next.

3.1.3.1 Lindley's Approximation

Lindley's approximation for three parameters, $\theta = (\theta_1, \theta_2, \theta_3)$ introduces

$$\hat{u}_B = E(u(\theta|\underline{x})) = u + (u_1a_1 + u_2a_2 + u_3a_3 + a_4 + a_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})] \tag{3.30}$$

evaluated at $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$, where

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, i = 1, 2, 3, \tag{3.31}$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},\ a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}), \tag{3.32}$$

$$A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{23}L_{231} + \sigma_{22}L_{221} + \sigma_{33}L_{331}, \tag{3.33}$$

$$B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{23}L_{232} + \sigma_{22}L_{222} + \sigma_{33}L_{332}, \tag{3.34}$$

$$C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{22}L_{223} + \sigma_{33}L_{333}. \tag{3.35}$$

In our problem, for $(\theta_1, \theta_2, \theta_3) \equiv (\alpha, \beta, \lambda)$ and $u \equiv u(\alpha, \beta, \lambda) = R_{n,k}$, we have $L_{11} = -nm/\alpha^2$, $L_{22} = -m/\beta^2$, $L_{13} = L_{31} = -J_{13}$, $L_{23} = L_{32} = -J_{23}$, $L_{33} = -J_{33}$, $\sigma_{ij}, i = 1, 2, 3$, are obtained by using $L_{ij}, i = 1, 2, 3$, $\rho_1 = ((a_1 - 1)/\alpha) - b_1$, $\rho_2 = ((a_2 - 1)/\beta) - b_2$, $\rho_3 = \pi'(\lambda)/\pi(\lambda)$ and $L_{111} = 2nm/\alpha^3$, $L_{222} = 2m/\beta^3$,

$$L_{133} = L_{331} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\overline{F_0}_{\lambda\lambda}(y_{ij};\lambda)\overline{F_0}(y_{ij};\lambda) - (\overline{F_0}_{\lambda}(y_{ij};\lambda))^2}{(\overline{F_0}(y_{ij};\lambda))^2} \tag{3.36}$$

$$L_{233} = L_{332} = \sum_{i=1}^{m} \frac{\overline{F_0}_{\lambda\lambda}(x_i;\lambda)(\overline{F_0}(x_i;\lambda)) - (\overline{F_0}_{\lambda}(x_i;\lambda))^2}{(\overline{F_0}(x_i;\lambda))^2} \tag{3.37}$$

$$\begin{aligned} L_{333} &= \sum_{i=1}^{m}\sum_{j=1}^{n} C_1(y_{ij};\lambda) + \sum_{i=1}^{m} C_1(x_i;\lambda) \\ &\quad + (\alpha - 1)\sum_{i=1}^{m}\sum_{j=1}^{n} C(y_{ij};\lambda) + (\beta - 1)\sum_{i=1}^{m} C(x_i;\lambda), \end{aligned} \tag{3.38}$$

where

$$C(x;\lambda) = \frac{\overline{F_0}_{\lambda\lambda\lambda}(x;\lambda)(\overline{F_0}(x;\lambda))^2 - 3\overline{F_0}(x;\lambda)\overline{F_0}_{\lambda}(x;\lambda)\overline{F_0}_{\lambda\lambda}(x;\lambda) + 2(\overline{F_0}_{\lambda}(x;\lambda))^3}{(\overline{F_0}(x;\lambda))^3}, \tag{3.39}$$

$$C_1(x;\lambda) = \frac{f_{0_{\lambda\lambda\lambda}}(x;\lambda)(f_0(x;\lambda))^2 - 3f_0(x;\lambda)f_{0_\lambda}(x;\lambda)f_{0_{\lambda\lambda}}(x;\lambda) + 2(f_{0_\lambda}(x;\lambda))^3}{(f_0(x;\lambda))^3}. \quad (3.40)$$

Moreover, $u_3 = \partial R_{n,k}/\partial\lambda = 0$, $u_{i3} = \partial^2 R_{n,k}/\partial\theta_i\partial\lambda = 0$, $i = 1, 2$, u_1 and u_2 are given in Equation (3.23) and (3.24), and

$$u_{11} = \frac{\partial^2 R_{n,k}}{\partial\alpha^2} = \sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j+1}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\frac{(l+i)^2 2\beta}{(\alpha(l+i)+\beta)^3}, \quad (3.41)$$

$$u_{22} = \frac{\partial^2 R_{n,k}}{\partial\beta^2} = \sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\frac{(l+i)^2 2\alpha}{(\alpha(l+i)+\beta)^3}, \quad (3.42)$$

$$u_{12} = u_{21} = \frac{\partial^2 R_{n,k}}{\partial\alpha\partial\beta} = \sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\frac{(l+i)(\alpha(l+i)-\beta)}{(\alpha(l+i)+\beta)^3}. \quad (3.43)$$

Hence,

$$a_4 = u_{12}\sigma_{12},\ a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}), \quad (3.44)$$

$$A = \sigma_{11}L_{111} + \sigma_{33}L_{331},\ B = \sigma_{22}L_{222} + \sigma_{33}L_{332}, \quad (3.45)$$

$$C = 2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{33}L_{333}. \quad (3.46)$$

Then, the Bayes estimate $R_{n,k}$ is computed approximately as follows:

$$\begin{aligned}\hat{R}_{n,k,B}^{Lin} = R_{n,k} + [u_1a_1 + u_2a_2 + a_4 + a_5] + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) \\ + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})].\end{aligned} \quad (3.47)$$

using (3.30) where all the parameters are evaluated at $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$.

3.1.3.2 MCMC Method

The joint posterior density function of α, β and λ is presented in (3.27) and the marginal posterior density functions of α, β and λ are shown respectively as

$$\alpha|\lambda, \underline{x}, \underline{y} \sim Gamma(nm + a_1, b_1 + w_\lambda^*),\ \beta|\lambda, \underline{x}, \underline{y} \sim Gamma(n + a_2, b_2 + v_\lambda^*) \quad (3.48)$$

and

$$\pi(\lambda|\alpha,\beta,\underline{x},\underline{y}) \propto \lambda^{m(n+1)+a_3-1} e^{-(\alpha-1)w^*_\lambda-(\beta-1)v^*_\lambda} \exp\left\{\lambda\left(-b_3+\sum_{i=1}^m \ln x_i + \sum_{i=1}^m \sum_{j=1}^n \ln y_{ij}\right)\right\} \quad (3.49)$$

Samples of α and β can be generated with the usage of gamma distribution. When the baseline pdf f_0 and F_0 are log-concave functions and $\alpha \geq 1, \beta \geq 1$, the posterior density of λ is log-concave function that is $\partial^2 ln\pi(\lambda|\alpha,\beta,\underline{x},\underline{y})/\partial\lambda^2 \leq 0$. As a result, we implement the M-H algorithm with the normal proposal distribution to generate a random sample from the posterior density of λ. The hybrid M-H and Gibbs sampling algorithm is suggested by Tierney[52].

Step 1: Start with initial guess $\lambda^{(0)}$.

Step 2: Set $i = 1$.

Step 3: Generate $\alpha^{(i)}$ from $Gamma(nm + a_1, b_1 + w^*_\lambda)$.

Step 4: Generate $\beta^{(i)}$ from $Gamma(n + a_2, b_2 + v^*_\lambda)$.

Step 5: Generate $\lambda^{(i)}$ from $\pi(\lambda|\alpha,\beta,\underline{x},\underline{y})$ using the Metropolis-Hastings with the proposal distribution $q(\lambda) \equiv N(\lambda^{(i-1)}, 1)$.

- Let $v = \lambda^{(i-1)}$
- Generate w from the proposal distribution q.
- Let $p(v,w) = \min\left\{1, \dfrac{\pi(w|\alpha^{(i)},\beta^{(i)},\underline{x},\underline{y})q(v)}{\pi(v|\alpha^{(i)},\beta^{(i)},\underline{x},\underline{y})q(w)}\right\}$
- Generate u from $U(0,1)$,
  - If $u \leq p(v,w)$, then accept the proposal and set $\lambda^{(i)} = w$;
  - otherwise, set $\lambda^{(i)} = v$.

Step 6: Compute the $R^{(i)}_{n,k}$ at $(\alpha^{(i)},\beta^{(i)},\lambda^{(i)})$.

Step 7: Set $i = i + 1$.

Step 8: Repeat Steps 2 through 7, N times and posterior sample $R^{(i)}_{n,k}$, $i = 1, \ldots, N$.

The Bayes estimate and the HPD credible interval for $R_{n,k}$ are acquired using this sample. The Bayes estimate of $R_{n,k}$ under a SE loss function is computed as follows:

$$\hat{R}_{n,k,B}^{MC} = \frac{1}{N-M} \sum_{i=M+1}^{N-M} R_{n,k}^{(i)}, \tag{3.50}$$

where M is the burn-in period. The HPD $100(1-\gamma)\%$ credible interval of $R_{n,k}$ is constructed by Chen and Shao's technique [53].

3.2. ESTIMATION OF $R_{n,k}$ WHEN THE SECOND PARAMETERS ARE DIFFERENT AND KNOWN

In this section, the ML, UMVU, exact and approximate Bayes estimates of $R_{n,k}$ are considered when the second parameters of underlying distributions are λ_1 and λ_2 are different and known.

3.2.1 MLE of $R_{n,k}$

We assume that strength variables $Y_i \sim PHR(\alpha, \lambda_1)$, $i = 1, .., n$ and stress variable $X \sim PHR(\beta, \lambda_2)$, λ_1 and λ_2 are known constants. Then, $R_{n,k}$ is the same as in Equations(3.8) and (3.9). In this case, the likelihood function of the observed sample is

$$L(\alpha, \beta; \lambda_1, \lambda_2, \mathbf{x}, \mathbf{y}) \propto \alpha^{nm} \beta^{m} \exp\left[-(\alpha-1)w^* - (\beta-1)v^*\right], \tag{3.51}$$

and the log-likelihood function is

$$l(\alpha, \beta; \mathbf{x}, \mathbf{y}) \propto nm \ln \alpha + m \ln \beta - (\alpha-1)\, w^* - (\beta-1)\, v^*, \tag{3.52}$$

where $w^* \equiv w^*_{\lambda_1} = -\sum_{i=1}^{m}\sum_{j=1}^{n} \ln \overline{F}_0(y_{ij};\lambda_1)$ and $v^* \equiv v^*_{\lambda_2} - \sum_{i=1}^{m} \ln \overline{F}_0(x_i;\lambda_2)$. The ML estimates of α and β are $\hat{\alpha} = nm/W^*$ and $\hat{\beta} = m/V^*$. Hence, the MLE of $\widehat{R}_{n,k}^{MLE}$ can be computed by application of the invariance property of MLE in (3.8) and (3.9).

After this, the derivation of the asymptotic confidence interval of $R_{n,k}$ is handled. The elements of the Fisher information matrix $I(\theta)$, $\theta = (\alpha, \beta)$, are $I_{11} = nm/\alpha^2$, $I_{22} = m/\beta^2$, $I_{12} = I_{21} = 0$. The $\widehat{R}_{n,k}^{MLE}$ is asymptotically normal with mean $R_{n,k}$ and asymptotic variance (see Rao [51])

$$\sigma^2_{R_{n,k}} = \left(\frac{\partial R_{n,k}}{\partial \alpha}\right)^2 \frac{\alpha^2}{nm} + \left(\frac{\partial R_{n,k}}{\partial \beta}\right)^2 \frac{\beta^2}{m}, \tag{3.53}$$

where $\partial R_{n,k}/\partial\alpha$ and $\partial R_{n,k}/\partial\beta$ are given in Equations (3.23) and (3.24). Thus, $R_{n,k} \in (\hat{R}_{n,k}^{MLE} \pm z_{\gamma/2}\hat{\sigma}_{R_{n,k}})$ introduces an an asymptotic $100(1-\gamma)\%$ confidence interval of $R_{n,k}$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{\sigma}_{R_{n,k}}$ is the value at MLE of parameters.

3.2.2 UMVUE of $R_{n,k}$

From Equations (3.8) and (3.9), finding the UMVUE of $\psi(\alpha,\beta) = \beta/\left(\alpha(l+i)+\beta\right)$ is enough when the linear property of UMVUE is taken into consideration. We obtain that (W^*, V^*) is a complete sufficient statistics for (α, β) and have Gamma distributions with parameters (nm, α) and (m, β), respectively. Let

$$\phi(V_1, W_1) = \begin{cases} 1, & W_1 > (l+i)V_1 \\ 0, & \text{otherwise} \end{cases}, \tag{3.54}$$

where $W_1 = -\ln \overline{F}_0(Y_{11})$ and $V_1 = -\ln \overline{F}_0(X_1)$. It is readily seen that W_1 and V_1 have exponential distribution with means $1/\alpha$ and $1/\beta$, respectively. Then, $\phi(V_1, W_1)$ is an unbiased estimator for $\psi(\alpha, \beta)$. Hence, the UMVUE of $\psi(\alpha, \beta)$, denoted $\hat{\psi}_U(\alpha, \beta)$, can be acquired with the application of Lehmann-Scheffé Theorem. We have

$$\hat{\psi}_U(\alpha, \beta) = E\left(\phi(V_1, W_1)|W^* = w^*, V^* = v^*\right) \tag{3.55}$$

$$= P\left(W_1 > (l+i)V_1|W^* = w^*, V^* = v^*\right) \tag{3.56}$$

$$= \iint_C f_{W_1|W^*=w^*}(w_1|w^*) f_{V_1|V^*=v^*}(v_1|v^*)\, dv_1\, dw_1, \tag{3.57}$$

where $C = \left\{(v_1, w_1) : 0 < v_1 < v^*, 0 < w_1 < w^*, v_1(l+i) < w_1\right\}$. From Lemma 1 in Basirat et al. [7], the conditional distribution of $W_1|W^*$ and $V_1|V^*$ are given by

$$f_{W_1|W^*=w^*}(w_1|w^*) = \frac{(nm-1)}{w^*}\left(1 - \frac{w_1}{w^*}\right)^{nm-2} \tag{3.58}$$

and

$$f_{V_1|V^*=v^*}(v_1|v^*) = \frac{(m-1)}{v^*}\left(1 - \frac{v_1}{v^*}\right)^{m-2}. \tag{3.59}$$

The double integral in (3.57) is examined in two cases $\left((l+i)v^*\right)/w^* \leq 1$ and $\left((l+i)v^*\right)/w^* > 1$. If $\left((l+i)v^*\right)/w^* \leq 1$, then

$$\hat{\psi}_U(\alpha, \beta) = \int_0^{v^*} \int_{v_1(l+i)}^{w^*} \frac{(m-1)(nm-1)}{v^* w^*}\left(1 - \frac{v_1}{v^*}\right)^{m-2}\left(1 - \frac{w_1}{w^*}\right)^{nm-2} dw_1 dv_1 \tag{3.60}$$

$$= (m-1)\int_0^1 (1-t)^{m-2}(1-ct)^{nm-1} dt \tag{3.61}$$

$$= \sum_{z=0}^{nm-1} (-1)^z \frac{\binom{nm-1}{z}}{\binom{m+z-1}{z}}\left(\frac{(l+i)v^*}{w^*}\right)^z, \tag{3.62}$$

where $c = ((l+i)v^*)/w^* \leq 1$, $t = v_1/v^*$. If $((l+i)v^*)/w^* > 1$, then

$$\hat{\psi}_U(\alpha,\beta) = \int_0^{w^*}\int_0^{w_1/(l+i)} \frac{(m-1)(nm-1)}{v^*w^*} 1\left(-\frac{v_1}{v^*}\right)^{m-2}\left(1-\frac{w_1}{w^*}\right)^{nm-2} dv_1 dw_1 \quad (3.63)$$

$$= 1-(nm-1)\int_0^1 (1-t)^{nm-2}(1-\frac{t}{c})^{m-1}dt \quad (3.64)$$

$$= 1-\sum_{z=0}^{m-1}(-1)^z \frac{\binom{m-1}{z}}{\binom{nm+z-1}{z}}\left(\frac{w^*}{(l+i)v^*}\right)^z, \quad (3.65)$$

where $c = (l+i)v^*/w^* > 1$, $t = w_1/w^*$. Thus, the UMVUE of $R_{n,k}$, say $\hat{R}^U_{n,k}$, is given by using (3.62) and (3.65)

$$\hat{R}^U_{n,k} = 1-\sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\hat{\psi}_U(\alpha,\beta). \quad (3.66)$$

For $2k \geq n$, $\hat{R}^U_{n,k}$ is obtained like in the general case. In this case, we have

$$\hat{R}^U_{n,k} = \sum_{i=0}^{1}(-1)^i(n-k+1-i)\hat{\psi}^U_i(\alpha,\beta), \quad (3.67)$$

where

$$\hat{\psi}^U_i(\alpha,\beta) = \begin{cases} \sum_{z=0}^{nm-1}(-1)^z \frac{\binom{nm-1}{z}}{\binom{m+z-1}{z}}\left(\frac{(k+i)\,v^*}{w^*}\right)^z & , \frac{(k+i)\,v^*}{w^*} \leq 1 \\ 1-\sum_{z=0}^{m-1}(-1)^z \frac{\binom{m-1}{z}}{\binom{nm+z-1}{z}}\left(\frac{w^*}{(k+i)\,v^*}\right)^z & , \frac{(k+i)\,v^*}{w^*} > 1 \end{cases}. \quad (3.68)$$

3.2.3 Bayes Estimation of $R_{n,k}$

The parameters α and β are assumed as random variables that have statistically independent gamma prior distributions with parameters (a_i, b_i), $i = 1, 2$, respectively. Then, the joint posterior density function of α and β is given as

$$\pi(\alpha, \beta \,|\mathbf{x}, \mathbf{y}) = \tfrac{(b_1+w^*)^{nm+a_1}(b_2+v^*)^{m+a_2}}{\Gamma(nm+a_1)\Gamma(m+a_2)}\alpha^{nm+a_1-1}\beta^{m+a_2-1}e^{-\alpha(b_1+w^*)-\beta(b_2+v^*)}, \tag{3.69}$$

and the marginal posterior densities of α and β have Gamma distributions with parameters $(nm + a_1, b_1 + w^*)$ and $(m + a_2, b_2 + v^*)$. Then, Bayes estimate of $R_{n,k}$ under the SE loss function, say $\widehat{R}_{n,k,B}$, is

$$\hat{R}_{n,k,B} = 1 - \textstyle\sum_{l=0}^{n}\sum_{j=0}^{t_l}\sum_{i=0}^{n-l}(-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}\int_0^\infty\int_0^\infty \dfrac{\beta\pi(\alpha, \beta\,|\mathbf{x}, \mathbf{y})}{\alpha(l+i)+\beta}d\alpha d\beta \tag{3.70}$$

If a one-to-one transformation is implemented for $u_1 = \beta/(\alpha\,(l+i)+\beta)$ and $u_2 = (\alpha\,(l+i)+\beta)$. Then, $0 < u_1 < 1$, $0 < u_2 < \infty$, $\alpha = u_2(1-u_1)/(l+i)$, $\beta = u_1u_2$ and the Jacobian of (u_1, u_2) is $J(u_1, u_2) = -u_2/(l+i)$. Hence, we can arrange the double integral in (3.70) as

$$\begin{aligned} & \frac{(b_1+w^*)^{nm+a_1}(b_2+v^*)^{m+a_2}}{\Gamma(nm+a_1)\Gamma(m+a_2)(l+i)^{nm+a_1}}\left\{\int_0^1\int_0^\infty u_1^{m+a_2}(1-u_1)^{nm+a_1-1}u_2^{p-1}\right. \\ & \left.\times \exp\left(-u_2\left\{\frac{(1-u_1)(b_1+w^*)}{(l+i)}+u_1(b_2+v*)\right\}\right)du_1du_2\right\} \qquad (3.71) \\ = \; & \frac{(1-z)^{m+a_2}}{B(nm+a_1, m+a_2)}\int_0^1 u_1^{m+a_2}(1-u_1)^{nm+a_1-1}(1-u_1z)^{-p}du_1, \qquad (3.72)\end{aligned}$$

where $z = 1 - ((b_2 + v^*)(l+i)/(b_1 + w^*))$ and $p = nm + a_1 + m + a_2$. It is known that the integral representation of the hypergeometric series is

$$_2F_1(\alpha, \beta; \gamma, s) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-ts)^{-\alpha} dt, \tag{3.73}$$

when $|s| < 1$, $Re(\gamma) > 0$ and $Re(\beta) > 0$, (see Gradshteyn and Ryzhik [54]). Therefore, we have

$$\hat{R}_{n,k,B} = 1 - \begin{cases} \sum_{l=0}^{n} \sum_{j=0}^{t_l} \sum_{i=0}^{n-l} c(l,j,i)\, c_1\, {}_2F_1\left(p, m + a_2 + 1; p+1, z\right), & |z| < 1 \\ \sum_{l=0}^{n} \sum_{j=0}^{t_l} \sum_{i=0}^{n-l} c(l,j,i)\, c_2\, {}_2F_1\left(p, nm + a_1; p+1, \dfrac{z}{z-1}\right), & z < -1 \end{cases}, \tag{3.74}$$

where $c(l,j,i) = (-1)^{i+j}\binom{n-l+1}{j}\binom{n-kj}{n-l}\binom{n-l}{i}$, $c_1 = \left((1-z)^{m+a_2}(m+a_2)\right)/p$ and $c_2 = (m+a_2)/\left(p(1-z)^{nm+a_1}\right)$. Next, we examine two approaches which are Lindley's approximation and MCMC methods to observe that how they approximate to the exact result that we obtained.

3.2.3.1 Lindley's Approximation

Lindley's approximation for two parameters $\theta = (\theta_1, \theta_2)$, introduces

$$\hat{u}_{Lin} = u(\theta) + \frac{1}{2}\left[B + Q_{30}B_{12} + Q_{21}C_{12} + Q_{12}C_{21} + Q_{03}B_{21}\right], \tag{3.75}$$

where $B = \sum_{i=1}^{2}\sum_{j=1}^{2} u_{ij}\tau_{ij}$, $Q_{ij} = \partial Q^{i+j}/\partial^i\theta_1\partial^j\theta_2$ for $i,j = 0,1,2,3$, $i+j = 3$, $u_i = \partial u/\partial\theta_i$, $u_{ij} = \partial^2 U/\partial\theta_i\partial\theta_j$ for $i,j = 1,2$ and $B_{ij} = (u_i\tau_{ii} + u_j\tau_{ij})\tau_{ii}$, $C_{ij} = 3u_i\tau_{ii}\tau_{ij} + u_j(\tau_{ii}\tau_{ij} + 2\tau_{ij}^2)$ for $i \neq j$. τ_{ij} is the (i,j)th element in the inverse of matrix $Q^* = (-Q^*_{ij})$, $i,j = 1,2$ such that $Q^*_{ij} = \partial Q^2/\partial\theta_i\partial\theta_j$. $\hat{u}_{Lin}$ which is the approximate Bayes estimate is evaluated at $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ which refers to the mode of the posterior density.

In our state, $(\theta_1, \theta_2) = (\alpha, \beta)$ and

$$Q = \ln \pi(\alpha, \beta \,|\mathbf{x}, \mathbf{y}) \propto (nm + a_1 - 1) \ln \alpha + (m + a_2 - 1) \ln \beta - \alpha(b_1 + w^*) - \beta(b_2 + v^*). \quad (3.76)$$

We can derive the posterior mode of (α, β) from Q as follows: $\tilde{\alpha} = (nm + a_1 - 1) \,/\, (b_1 + w^*)$ and $\tilde{\beta} = (m + a_2 - 1) \,/\, (b_2 + v^*)$.

Other elements are obtained as $\tau_{11} = \alpha^2/(nm+a_1-1)$, $\tau_{22} = \beta^2/(m+a_2-1)$, $\tau_{12} = \tau_{21} = 0$, $Q_{12} = Q_{21} = 0$, $Q_{03} = 2(m + a_2 - 1)/\beta^3$, $Q_{30} = 2(nm + a_1 - 1)/\alpha^3$, $B_{12} = u_1\tau_{11}^2$, $B_{21} = u_2\tau_{22}^2$, $B = u_{11}\tau_{11} + u_{22}\tau_{22}$. Therefore, the approximate Bayes estimate of $R_{n,k}$ under the SE loss function is demonstrated as

$$\widehat{R}_{n,k,B}^{Lin} = R_{n,k} + \frac{1}{2}\left[\frac{\alpha^2 u_{11} + 2\alpha u_1}{nm + a_1 - 1} + \frac{\beta^2 u_{22} + 2\beta u_2}{m + a_2 - 1}\right]_{(\alpha,\beta)=(\widetilde{\alpha},\widetilde{\beta})}, \quad (3.77)$$

where u_1, u_2, u_{11} and u_{22} are presented in (3.23), (3.24), (3.41) and (3.42).

3.2.3.2 MCMC Method

The marginal posterior densities of α and β are known as those possess gamma distributions with parameters $(nm+a_1, b_1+w^*)$ and $(m+a_2, b_2+v^*)$. Then, we implement the following Gibbs sampling algorithm to generate samples:

Step 1: Set $i = 1$.

Step 2: Generate $\alpha^{(i)}$ from $Gamma(nm + a_1, b_1 + w^*)$.

Step 3: Generate $\beta^{(i)}$ from $Gamma(m + a_2, b_2 + v^*)$.

Step 4: Compute the $R_{n,k}^{(i)}$ at $(\alpha^{(i)}, \beta^{(i)})$

Step 5: Set $i = i + 1$.

Step 6: Repeat Steps 2 through 5, N times and obtain the posterior sample $R^{(i)}_{n,k}$, $i = 1, ..., N$.

Then, Bayes estimate of $R_{n,k}$ under the SE loss function is

$$\widehat{R}^{MC}_{n,k,B} = \frac{1}{N-M} \sum_{i=M+1}^{N-M} R^{(i)}_{n,k}, \tag{3.78}$$

where M is the burn-in period. The HPD $100(1-\gamma)\%$ credible interval of $R_{n,k}$ is constructed by Chen and Shao's technique [53].

3.3. SIMULATION STUDY

This section presents some numerical results for Kumaraswamy and one parameter Weibull distributions from the PHR family when the second parameters of underlying distributions are common and unknown, and different and known.

All the estimates are presented along with mean square error (MSE) or estimated risks (ERs) values and biases. To compare the performances of the point estimates, MSE for ML and ER for ML Bayes estimates are used. The ER of θ, when θ is estimated by $\hat{\theta}$, is computed as

$$ER(\theta) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta_i)^2, \tag{3.79}$$

under the SE loss function. Average lengths(ALs) and coverage probabilities (CPs) are considered to compare the performances of asymptotic confidence and credible intervals. We use MATLAB and statistical software R to perform all the computations [46]. All the results are based on 2500 replications.

To implement MCMC procedures, two chains are run with fairly different initial values and generate 10000 iterations for each chain. In order to reduce the effect of the starting distribution, the first 5000 results of each sequence are discarded, which is called as burn-in. So as to remove the dependence between the results in the Markov chain, only every d^{th} draw of the chain is saved, which is called as thinning. In our cases, Bayesian MCMC estimates are evaluated by the means of the every 5^{th} sampled values after the discarding procedure. Moreover, the convergency of the MCMC chains has been monitored by using the scale reduction factor estimate in Gelman et al. [42]. The estimate is given by $\sqrt{Var(\psi)/W}$, where ψ is the estimand of interest, $Var(\psi) = (n-1)W/n + B/n$ with the iteration number n for each chain, the between-sequence variance B and the within-sequence variance W. In all cases, the scale factor values of the MCMC estimators are found to be below 1.1. It is an acceptable value for their convergence.

3.3.1 When the Second Parameters are Common and Unknown

In this subsection, the strength and stress variables are assumed generating from Kumaraswamy distributions with parameters (α, λ) and (β, λ) when the second parameters of underlying distributions are common (λ) and unknown. Then, the baseline survival functions are obtained as $\overline{F_0}(y;\lambda) = (1-y^{\lambda})$ and $\overline{F_0}(x;\lambda) = (1-x^{\lambda})$, respectively and

$$w_{\lambda}^{*} = -\sum_{i=1}^{m}\sum_{j=1}^{n} \ln(1-y_{ij}^{\lambda}) \text{ and } w_{\lambda} = -\left(nm\ln\lambda + (\lambda-1)\sum_{i=1}^{m}\sum_{j=1}^{n}\ln y_{ij}\right), \quad (3.80)$$

$$v_{\lambda}^{*} = -\sum_{i=1}^{m} \ln(1-x_{i}^{\lambda}) \text{ and } v_{\lambda} = -\left(m\ln\lambda + (\lambda-1)\sum_{i=1}^{m}\ln x_{i}\right). \quad (3.81)$$

The MLE of λ, $\hat{\lambda}$, is obtained from the solution of the nonlinear equation given below:

$$\begin{aligned} &\frac{m(n+1)}{\lambda} + \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\ln y_{ij}}{1-y_{ij}^{\lambda}} + \sum_{i=1}^{m}\frac{\ln x_i}{1-x_i^{\lambda}} - \frac{nm}{W_{\lambda}^{*}}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{y_{ij}^{\lambda} ln y_{ij}}{1-y_{ij}^{\lambda}} \\ &-\frac{m}{V_{\lambda}^{*}}\sum_{i=1}^{m}\frac{x_i^{\lambda} ln x_i}{1-x_i^{\lambda}} = 0 \end{aligned} \quad (3.82)$$

The elements of observed information matrix are given as

$$J_{13} = J_{31} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{y_{ij}^{\lambda} \ln y_{ij}}{1 - y_{ij}^{\lambda}},\ J_{23} = J_{32} = \sum_{i=1}^{m} \frac{x_i^{\lambda} \ln x_i}{1 - x_i^{\lambda}}, \tag{3.83}$$

$$J_{33} = \frac{m(n+1)}{\lambda^2} + (\alpha - 1)\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{y_{ij}^{\lambda}(\ln y_{ij})^2}{(1 - y_{ij}^{\lambda})^2} + (\beta - 1)\sum_{i=1}^{m} \frac{x_i^{\lambda}(\ln x_i)^2}{(1 - x_i^{\lambda})^2}. \tag{3.84}$$

Since Kumaraswamy distribution satisfies the mild regularity conditions, the observed information matrix can be used to obtain an an asymptotic confidence interval of $R_{n,k}$. Some elements for Bayes estimation of $R_{n,k}$ using Lindley's approximation are $L_{111} = 2nm/\alpha^3$, $L_{222} = 2m/\beta^3$

$$L_{133} = -\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{y_{ij}^{\lambda}(\ln y_{ij})^2}{(1 - y_{ij}^{\lambda})^2},\ L_{233} = -\sum_{i=1}^{m} \frac{x_i^{\lambda}(\ln x_i)^2}{(1 - x_i^{\lambda})^2}, \tag{3.85}$$

$$L_{333} = \frac{2m(n+1)}{\lambda^3} - (\alpha - 1)\sum_{i=1}^{m}\sum_{j=1}^{n} C(y_{ij}; \lambda) - (\beta - 1)\sum_{i=1}^{m} C(x_i; \lambda), \tag{3.86}$$

$$C_1(x; \lambda) = \frac{2}{\lambda^3},\ C(x; \lambda) = \frac{-x^{\lambda}(\ln x)^3(1 + x^{\lambda})}{(1 - x^{\lambda})^3}. \tag{3.87}$$

In Tables 3.1-3.3, we have presented the ML and Bayes estimates of $R_{n,k}$, and corresponding interval estimates for different sample sizes $m = 10, 15, 20, 25$. Bayes estimates are obtained by using both Lindley's approximation and MCMC method based on informative and non-informative priors. The true values of the parameters are taken as $(\alpha, \beta, \lambda) = (1.5, 3, 2)$, and $(2, 1.25, 3)$. In the Bayesian case, the following informative priors: Prior 1: $(a_1, b_1) = (3, 2), (a_2, b_2) = (6, 2), (a_3, b_3) = (4, 2)$, Prior 2: $(a_1, b_1) = (2, 1), (a_2, b_2) = (1.25, 1), (a_3, b_3) = (3, 1)$, and non-informative prior $(a_i, b_i) = (0, 0)$, $i = 1, 2, 3$ are used.

From Tables 3.1-3.2, it is observed that MSE, ERs and biases of the estimates generally decrease as the sample size increases, as expected. We observe that Bayes estimates of $R_{n,k}$ based on informative prior have the best performance in terms of error ER. We further observe that Bayes estimate using Lindley's approximation provide relatively better results than the MCMC method in terms of ERs. Bayes estimates based on non-informative prior and the ML estimate have similar ERs and MSE, and these values are close to each other when the sample size increases.

From Table 3.3, the ALs of all intervals decrease as the sample size increases, as expected. The CPs of all intervals are quite satisfactory. The HPD credible intervals based on informative priors provide the smallest AL. Furthermore, the HPD credible intervals based on non-informative prior and asymptotic confidence interval have similar performances. Hence, the HPD credible interval can be preferable when the prior information is available or not.

We have encountered convergence problems in the MCMC case for large sample sizes. Due to the term λ^{nm} in the marginal posterior density of λ, an indeterminate ratio in the Metropolis-Hastings algorithm is observed. For this reason, Bayes estimates of $R_{n,k}$ are computed by using Lindley's approximation for large sample sizes in Tables 3.4-3.6.

In Tables 3.4 and 3.5, we have presented the ML and Bayes estimates of $R_{n,k}$ for both informative and non-informative priors for $m = 30, 40, 50, 60$. The same parameters and priors as Tables 3.1-3.2 have been used. Moreover, the AL and CP of the asymptotic confidence intervals are presented in Table 3.6. From Tables 3.4-3.6, we observe that performance of the estimates behave similar as in Tables 3.1-3.3. Bayes estimate of $R_{n,k}$ based on informative prior has better results than that of non-informative prior and ML cases.

Table 3.1. Estimates of $R_{n,k}$ for Kumaraswamy distribution when $(\alpha, \beta, \lambda) = (1.5, 3, 2)$

			MLE			$Bayes$ (Prior 1)						$Bayes$ (Non-inf. prior)					
(k,n)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	Bias	MSE	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{MC}$	Bias	ER	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{MC}$	Bias	ER
(3,10)	0.72222	10	0.72965	0.00743	0.00873	0.69715	-0.02508	0.00340	0.71000	-0.01221	0.00390	0.71638	-0.00584	0.01020	0.71088	-0.01134	0.00840
		15	0.72775	0.00553	0.00603	0.70905	-0.01317	0.00284	0.71392	-0.00830	0.00323	0.71795	-0.00427	0.00666	0.71490	-0.00733	0.00585
		20	0.72578	0.00356	0.00474	0.71333	-0.00889	0.00268	0.71557	-0.00665	0.00287	0.71803	-0.00419	0.00509	0.71610	-0.00613	0.00464
		25	0.72673	0.00451	0.00379	0.71675	-0.00547	0.00238	0.71801	-0.00421	0.00247	0.72039	-0.00183	0.00400	0.71890	-0.00332	0.00371
(4,10)	0.57965	10	0.59607	0.01641	0.01076	0.53612	-0.04353	0.00400	0.57485	-0.00480	0.00415	0.56703	-0.01262	0.01067	0.58157	0.00191	0.00976
		15	0.59254	0.01288	0.00742	0.55792	-0.02174	0.00291	0.57813	-0.00152	0.00366	0.57244	-0.00722	0.00720	0.58261	0.00296	0.00691
		20	0.58852	0.00887	0.00521	0.56504	-0.01461	0.00255	0.57805	-0.00160	0.00297	0.57315	-0.00650	0.00509	0.58110	0.00144	0.00494
		25	0.58766	0.00801	0.00434	0.56873	-0.01093	0.00237	0.55973	-0.01992	0.00366	0.57399	-0.00567	0.00410	0.58044	-0.00078	0.00401
(7,10)	0.28889	10	0.30812	0.01923	0.00612	0.23426	-0.05463	0.00472	0.29241	0.00352	0.00203	0.26814	-0.02075	0.00387	0.30304	0.01415	0.00558
		15	0.30125	0.01236	0.00384	0.26115	-0.02774	0.00142	0.29176	0.00287	0.00169	0.27589	-0.01300	0.00282	0.29798	0.00909	0.00360
		20	0.29764	0.00875	0.00274	0.27161	-0.01728	0.00106	0.29153	0.00264	0.00146	0.27912	-0.00977	0.00219	0.29523	0.00634	0.00261
		25	0.29594	0.00705	0.00216	0.27643	-0.01246	0.00100	0.29129	0.00240	0.00130	0.28132	-0.00757	0.00180	0.29387	0.00510	0.00207
(9,10)	0.19697	10	0.21441	0.01744	0.00397	0.14538	-0.05159	0.00534	0.20134	0.00437	0.00119	0.17768	-0.01929	0.00195	0.21190	0.01493	0.00369
		15	0.20576	0.00879	0.00226	0.17233	-0.02464	0.00092	0.19927	0.00230	0.00099	0.18372	-0.01325	0.00151	0.20428	0.00731	0.00216
		20	0.20321	0.00624	0.00163	0.18117	-0.01580	0.00064	0.19895	0.00198	0.00087	0.18727	-0.00970	0.00122	0.20209	0.00512	0.00158
		25	0.20356	0.00659	0.00132	0.18660	-0.01037	0.00055	0.20016	0.00319	0.00078	0.19085	-0.00612	0.00101	0.20270	0.00573	0.00128

Table 3.2. Estimates of $R_{n,k}$ for Kumaraswamy distribution when $(\alpha, \beta, \lambda) = (2, 1.25, 3)$

			MLE			$Bayes$ (Prior 2)						$Bayes$ (Non-inf. prior)					
(k,n)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	Bias	MSE	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{MC}$	Bias	ER	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{MC}$	Bias	ER
(2,8)	0.49544	10	0.51302	0.01757	0.01218	0.49864	0.00320	0.00856	0.49973	0.00429	0.00862	0.50468	0.00924	0.01157	0.50291	0.00747	0.01096
		15	0.50410	0.00866	0.00748	0.49615	0.00071	0.00602	0.49626	0.00082	0.00599	0.49841	0.00297	0.00720	0.49758	0.00214	0.00699
		20	0.50315	0.00771	0.00600	0.49752	0.00207	0.00510	0.49748	0.00204	0.00505	0.49883	0.00339	0.00581	0.49821	0.00277	0.00568
		25	0.50043	0.00499	0.00449	0.49621	0.00077	0.00396	0.49616	0.00072	0.00393	0.49696	0.00152	0.00438	0.49656	0.00112	0.00431
(3,8)	0.33119	10	0.34633	0.01513	0.00814	0.33073	-0.00046	0.00498	0.33870	0.00751	0.00574	0.33621	0.00501	0.00708	0.34189	0.01070	0.00747
		15	0.34227	0.01108	0.00534	0.33326	0.00207	0.00393	0.33780	0.00661	0.00423	0.33568	0.00449	0.00484	0.33946	0.00826	0.00503
		20	0.33863	0.00743	0.00360	0.33258	0.00139	0.00292	0.33574	0.00454	0.00305	0.33380	0.00260	0.00335	0.33659	0.00540	0.00345
		25	0.33846	0.00728	0.00323	0.33375	0.00256	0.00272	0.33624	0.00504	0.00283	0.33462	0.00342	0.00304	0.33690	0.00570	0.00312
(5,8)	0.16143	10	0.17187	0.01044	0.00282	0.16031	-0.00112	0.00141	0.16869	0.00727	0.00197	0.16414	0.00272	0.00218	0.17126	0.00983	0.00269
		15	0.16926	0.00783	0.00170	0.16278	0.00136	0.00113	0.16767	0.00626	0.00136	0.16444	0.00302	0.00144	0.16893	0.00750	0.00165
		20	0.16754	0.00612	0.00126	0.16317	0.00175	0.00096	0.16663	0.00520	0.00108	0.16406	0.00264	0.00112	0.16732	0.00589	0.00123
		25	0.16509	0.00366	0.00095	0.16190	0.00048	0.00077	0.16452	0.00309	0.00084	0.16243	0.00100	0.00087	0.16494	0.00352	0.00093
(6,8)	0.11908	10	0.12776	0.00867	0.00165	0.11822	-0.00087	0.00078	0.12547	0.00638	0.00116	0.12137	0.00229	0.00122	0.12767	0.00858	0.00159
		15	0.12355	0.00447	0.00073	0.12007	0.00099	0.00054	0.12303	0.00395	0.00063	0.12076	0.00167	0.00063	0.12360	0.00451	0.00072
		20	0.12403	0.00494	0.00076	0.12048	0.00140	0.00056	0.12346	0.00437	0.00065	0.12121	0.00212	0.00066	0.12406	0.00498	0.00075
		25	0.12374	0.00466	0.00060	0.12103	0.00195	0.00047	0.12335	0.00427	0.00053	0.12152	0.00244	0.00053	0.12375	0.00467	0.00059

Table 3.3. Average confidence/credible lengths (ACL) and coverage probabilities (CP) of $R_{n,k}$ for Kumaraswamy distribution when $(\alpha, \beta, \lambda) = (1.5, 3, 2)$ and $(2, 1.25, 3)$

			ACI		HPD (Prior 1)		HPD (Non-inf.)				ACI		HPD (Prior 2)		HPD (Non-inf.)	
m	(k,n)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP	(k,n)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP
10	(3,10)	0.72222	0.36601	0.915	0.29622	0.978	0.35450	0.978	(2,8)	0.49544	0.40291	0.917	0.36831	0.946	0.38376	0.946
15			0.30400	0.926	0.25799	0.976	0.29634	0.976			0.33264	0.934	0.31063	0.947	0.32058	0.947
20			0.26582	0.930	0.23220	0.970	0.25962	0.970			0.28937	0.927	0.27382	0.939	0.28058	0.939
25			0.23864	0.933	0.21268	0.966	0.23363	0.966			0.25973	0.942	0.24772	0.948	0.25243	0.948
10	(4,10)	0.57965	0.38745	0.911	0.30831	0.976	0.37190	0.976	(3,8)	0.33119	0.32896	0.928	0.29889	0.947	0.31363	0.947
15			0.32049	0.923	0.26881	0.971	0.31061	0.971			0.26921	0.934	0.25082	0.944	0.25970	0.944
20			0.27984	0.939	0.24205	0.970	0.27243	0.970			0.23297	0.950	0.22024	0.953	0.22595	0.953
25			0.25112	0.934	0.23928	0.947	0.24506	0.946			0.20859	0.936	0.19874	0.937	0.20321	0.937
10	(7,10)	0.28889	0.28319	0.939	0.21599	0.979	0.27048	0.979	(5,8)	0.16143	0.18963	0.938	0.17157	0.949	0.18160	0.949
15			0.22957	0.943	0.18780	0.976	0.22163	0.976			0.15388	0.947	0.14329	0.950	0.14887	0.950
20			0.19805	0.948	0.16885	0.968	0.19217	0.968			0.13249	0.949	0.12530	0.952	0.12872	0.952
25			0.17685	0.953	0.15469	0.968	0.17211	0.950			0.11728	0.946	0.11190	0.950	0.11443	0.950
10	(9,10)	0.19697	0.21913	0.938	0.16384	0.977	0.20911	0.977	(6,8)	0.11908	0.14651	0.944	0.13224	0.958	0.14058	0.958
15			0.17430	0.938	0.14170	0.976	0.16833	0.976			0.10134	0.952	0.09576	0.954	0.09863	0.954
20			0.14984	0.944	0.12704	0.968	0.14555	0.968			0.10165	0.941	0.09611	0.944	0.09900	0.944
25			0.13439	0.949	0.11705	0.961	0.13093	0.941			0.09086	0.949	0.08668	0.952	0.08864	0.952

Table 3.4. Estimates of $R_{n,k}$ for Kumaraswamy distribution when $(\alpha, \beta, \lambda) = (1.5, 3, 2)$

			MLE			$Bayes$ (Prior 1)			$Bayes$ (Non-inf. prior)		
(k, n)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	Bias	MSE	$\widehat{R}^{Lin}_{n,k,B}$	Bias	ER	$\widehat{R}^{Lin}_{n,k,B}$	Bias	ER
(3,10)	0.72222	30	0.72700	0.00478	0.00316	0.71898	-0.00325	0.00213	0.72163	-0.00059	0.00329
		40	0.72430	0.00208	0.00237	0.71886	-0.00336	0.00179	0.72013	-0.00209	0.00245
		50	0.72519	0.00297	0.00188	0.72090	-0.00133	0.00150	0.72182	-0.00040	0.00193
		60	0.72414	0.00192	0.00153	0.72071	-0.00152	0.00126	0.72130	-0.00092	0.00156
(4,10)	0.57965	30	0.58517	0.00552	0.00370	0.57137	-0.00828	0.00232	0.57481	-0.00484	0.00363
		40	0.58197	0.00232	0.00263	0.57256	-0.00709	0.00189	0.57414	-0.00551	0.00262
		50	0.58412	0.00447	0.00228	0.57647	-0.00318	0.00172	0.57782	-0.00183	0.00223
		60	0.58241	0.00275	0.00177	0.57631	-0.00335	0.00142	0.57715	-0.00251	0.00175

Table 3.4 Continued

			MLE			$Bayes$ (Prior 1)			$Bayes$ (Non-inf. prior)		
(k,n)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	Bias	MSE	$\widehat{R}^{Lin}_{n,k,B}$	Bias	ER	$\widehat{R}^{Lin}_{n,k,B}$	Bias	ER
(7,10)	0.28889	30	0.29361	0.00472	0.00193	0.27836	-0.01053	0.00104	0.28161	-0.00728	0.00168
		40	0.29467	0.00578	0.00142	0.28358	-0.00531	0.00086	0.28565	-0.00324	0.00125
		50	0.29172	0.00283	0.00104	0.28364	-0.00525	0.00073	0.28465	-0.00424	0.00096
		60	0.29281	0.00393	0.00088	0.28593	-0.00296	0.00064	0.28689	-0.00200	0.00080
(9,10)	0.19697	30	0.20139	0.00442	0.00109	0.18831	-0.00866	0.00054	0.19107	-0.00590	0.00089
		40	0.20114	0.00417	0.00077	0.19194	-0.00503	0.00045	0.19347	-0.00350	0.00066
		50	0.20014	0.00317	0.00061	0.19308	-0.00389	0.00041	0.19409	-0.00288	0.00054
		60	0.19880	0.00183	0.00051	0.19323	-0.00374	0.00037	0.19383	-0.00314	0.00047

Table 3.5. Estimates of $R_{n,k}$ for Kumaraswamy distribution when $(\alpha, \beta, \lambda) = (2, 1.25, 3)$

			MLE			$Bayes$ (Prior 2)			$Bayes$ (Non-inf. prior)		
(k, n)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	Bias	MSE	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER
(2,8)	0.49544	30	0.49959	0.00415	0.00387	0.49615	0.00071	0.00349	0.49669	0.00125	0.00379
		40	0.49900	0.00356	0.00274	0.49649	0.00105	0.00253	0.49681	0.00136	0.00269
		50	0.49794	0.00250	0.00230	0.49600	0.00055	0.00216	0.49619	0.00075	0.00226
		60	0.49899	0.00355	0.00198	0.49735	0.00191	0.00188	0.49752	0.00208	0.00196
(3,8)	0.33119	30	0.33547	0.00427	0.00239	0.33174	0.00055	0.00208	0.33232	0.00112	0.00228
		40	0.33486	0.00367	0.00191	0.33221	0.00101	0.00173	0.33251	0.00132	0.00184
		50	0.33579	0.00459	0.00142	0.33368	0.00248	0.00131	0.33390	0.00270	0.00137
		60	0.33388	0.00269	0.00114	0.33219	0.00100	0.00107	0.33233	0.00113	0.00111

Table 3.5 Continued

			MLE			$Bayes$ (Prior 2)			$Bayes$ (Non-inf. prior)		
(k,n)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	Bias	MSE	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER
(5,8)	0.16143	30	0.16522	0.00379	0.00078	0.16264	0.00121	0.00066	0.16301	0.00159	0.00072
		40	0.16363	0.00220	0.00059	0.16182	0.00040	0.00052	0.16202	0.00059	0.00056
		50	0.16388	0.00246	0.00046	0.16246	0.00103	0.00041	0.16260	0.00117	0.00043
		60	0.16287	0.00144	0.00040	0.16174	0.00031	0.00037	0.16181	0.00039	0.00038
(6,8)	0.11908	30	0.12236	0.00328	0.00047	0.12027	0.00119	0.00039	0.12058	0.00149	0.00043
		40	0.12130	0.00221	0.00035	0.11983	0.00075	0.00031	0.12000	0.00091	0.00033
		50	0.12096	0.00187	0.00028	0.11982	0.00074	0.00025	0.11993	0.00085	0.00026
		60	0.12022	0.00114	0.00023	0.11932	0.00023	0.00021	0.11938	0.00030	0.00022

Table 3.6. Average confidence/credible lengths (ACL) and coverage probabilities (CP) of $R_{n,k}$ for Kumaraswamy distribution when $(\alpha, \beta, \lambda) = (1.5, 3, 2)$ and $(2, 1.25, 3)$

			ACI				ACI	
m	(k, n)	$R_{n,k}$	ACL	CP	(k, n)	$R_{n,k}$	ACL	CP
30	(3,10)	0.72222	0.21853	0.929	(2,8)	0.49544	0.23746	0.936
40			0.19069	0.934			0.20635	0.948
50			0.17079	0.946			0.18471	0.940
60			0.15640	0.946			0.16889	0.939
30	(4,10)	0.57965	0.22974	0.928	(3,8)	0.33119	0.18989	0.950
40			0.19974	0.944			0.16453	0.943
50			0.17886	0.936			0.14767	0.950
60			0.16358	0.946			0.13444	0.957
30	(7,10)	0.28889	0.16072	0.935	(5,8)	0.16143	0.10724	0.945
40			0.13978	0.943			0.09225	0.940
50			0.12436	0.948			0.08269	0.953
60			0.11387	0.946			0.07512	0.941
30	(9,10)	0.19697	0.12174	0.938	(6,8)	0.11908	0.08223	0.954
40			0.10548	0.950			0.07074	0.946
50			0.09403	0.949			0.06316	0.946
60			0.08542	0.950			0.05738	0.944

Furthermore, we have presented plots for comparing the performance of the Bayes estimates using Lindley's approximation based on informative and non-informative priors and MLE in terms of MSE and ERs for the sample sizes $m = 35$ and 70. In Figure 3.1, we chose the parameters as $\alpha_i = 11.2 - (2i/10)$, $\beta_i = 0.8 + (2i/10)$ and $\lambda_i = 0.9 + (i/10)$, $i = 1, ..., 50$ for $R_{9,3}$. Then, it takes the values 0.064 to 0.970. In Figure 3.2, we chose the parameters as $\alpha_i = 11.2 - (2i/10)$, $\beta_i = 0.8 + (2i/10)$ and $\lambda_i = 0.9 + (i/10)$, $i = 4, ..., 53$ for $R_{9,4}$. Then, it takes the values 0.070 to 0.973. We use the following algorithm to draw the plots.

Step 1: For given (α, β, λ), $R_{9,3}$ $(R_{9,4})$ is computed.

Step 2: For given m, samples from Kumaraswamy distribution are generated for the strength and the stress variables.

Step 3: Estimates of $R_{9,3}$ ($R_{9,4}$) are evaluated.

Step 4: Steps 2-3 are repeated 2500 times, the MSE or ER for estimates of $R_{n,k}$ are calculated as $MSE(\widehat{R}_{n,k}) = \sum_{i=1}^{N}(\widehat{R}_{n,k}^{(i)} - R_{n,k})^2/N$.

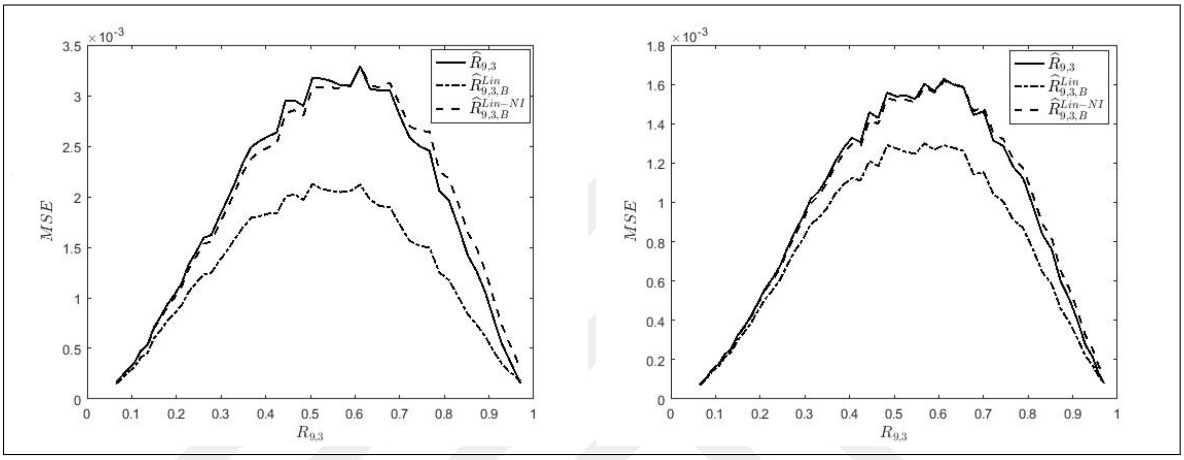

Figure 3.1. MSE (or ERs) of the estimates when $m = 35$ and 70

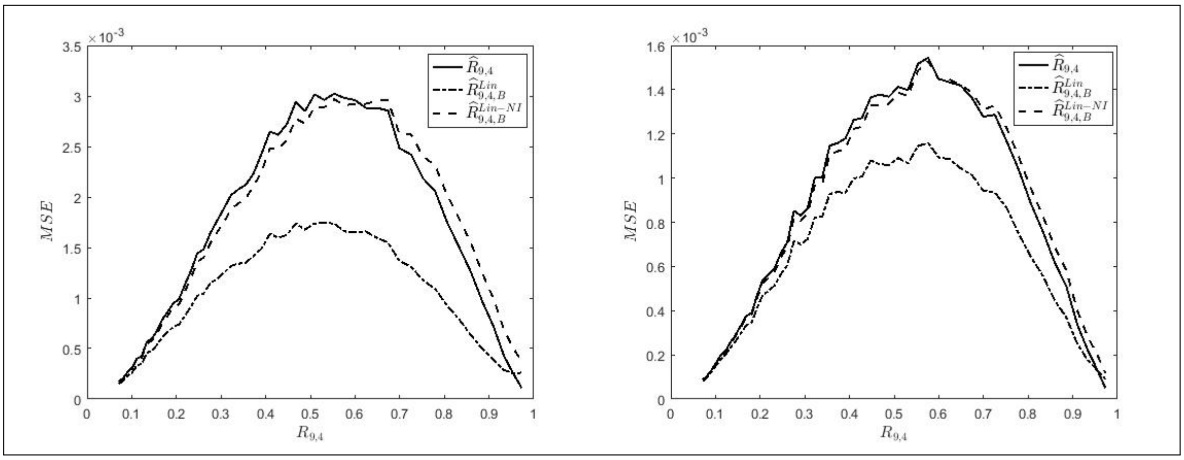

Figure 3.2. MSE (or ERs) of the estimates when $m = 35$ and 70

From Figures 3.1 and 3.2, we observe that all estimates have more error when $R_{n,k}$ tends to 0.5 but have small errors when $R_{n,k}$ tends to extreme values. Bayes estimate based on informative prior using Lindley's approximation have the smallest error. The similar results are also observed in Tables 3.4 and 3.5.

3.3.2 When the Second Parameters are Different and Known

In this subsection, the strength and stress variables are assumed generating from Weibull distributions with parameters (α, λ_1) and (β, λ_2), respectively when the second parameters of underlying distributions λ_1 and λ_2 are different and known. Then, the baseline survival functions are $\overline{F_0}(y; \lambda_1) = e^{-y^{\lambda_1}}$, $\overline{F_0}(x; \lambda_2) = e^{-x^{\lambda_2}}$, and then $w^* = \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^{\lambda_1}$ and $v^* = \sum_{i=1}^{m} x_i^{\lambda_2}$. Since Weibull distribution satisfies the mild regularity conditions, an asymptotic confidence interval of $R_{n,k}$ are obtained by using Fisher information matrix. All the estimates of $R_{n,k}$ are computed by using the obtained results in Section 3.

In Tables 3.7-3.9, we have presented the ML, UMVU and Bayes estimates of $R_{n,k}$, and corresponding interval estimates for different sample sizes $m = 10, 20, 30, 40$. Exact Bayes estimates based on informative prior are listed along with the approximate estimates which are obtained by using Lindley's approximation and MCMC method. The true values of the parameters are taken as $(\alpha, \lambda_1, \beta, \lambda_2) = (2, 2, 5, 3)$ and $(3, 6, 4.5, 4)$. In the Bayesian case, the following informative priors: Prior 3: $(a_1, b_1) = (2, 1)$, $(a_2, b_2) = (5, 1)$ and Prior 4: $(a_1, b_1) = (6, 2)$, $(a_2, b_2) = (9, 2)$ are used.

From Tables 3.7 and 3.8, it is observed that MSEs, ERs and biases of all estimates decrease as the sample size increases, as expected. Bayes estimates of $R_{n,k}$ have smaller ER than that of ML and UMVU estimates. The performances of all three Bayes estimates are similar. The exact Bayes estimates are very close to the Bayes estimates obtained by using Lindley's approximation and MCMC method in terms of the estimates and ERs. Therefore, we can say that the approximate Bayes estimates are good alternatives when the exact Bayes estimate cannot be obtained. Moreover, since the computation of MLE of $R_{n,k}$ is easy than UMVUE, it can be proposed to use MLE when the prior information is not available.

From Table 3.9, the ALs of all intervals decrease as the sample size increases, as expected. The CPs of all intervals are quite satisfactory. The ALs of the HPD credible intervals are shorter than asymptotic confidence interval. The HPD credible interval can be preferable to the asymptotic confidence interval with respect to AL and CP when the prior information is available.

Table 3.7. Estimates of $R_{n,k}$ for Weibull distribution when $(\alpha, \beta) = (2, 5)$ and $(\lambda_1, \lambda_2) = (2, 3)$

			MLE			$UMVUE$			$Bayes$ (Prior 3)								
(k,n)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	Bias	MSE	$\widehat{R}_{n,k}^{U}$	Bias	MSE	$\widehat{R}_{n,k,B}$	Bias	ER	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{MC}$	Bias	ER
(2,6)	0.84497	10	0.84545	0.00049	0.00495	0.84714	-0.00218	0.00545	0.83188	0.01308	0.00286	0.84207	0.00290	0.00302	0.83188	0.01309	0.00286
		20	0.84446	-0.00050	0.00253	0.84528	-0.00031	0.00266	0.83654	0.00843	0.00184	0.84283	0.00213	0.00192	0.83670	0.00827	0.00186
		30	0.84422	-0.00075	0.00169	0.84475	0.00022	0.00175	0.08382	0.00680	0.00130	0.84307	0.00190	0.00139	0.83871	0.00625	0.00136
		40	0.84461	-0.00036	0.00126	0.84501	-0.00004	0.00129	0.84000	0.00500	0.00100	0.84372	0.00125	0.00108	0.84034	0.00462	0.00106
(3,6)	0.66434	10	0.66764	0.00331	0.00890	0.66056	0.00378	0.00980	0.65385	0.01049	0.00460	0.64828	0.01606	0.00498	0.65391	0.01043	0.00462
		20	0.66724	0.00290	0.00453	0.66364	0.00069	0.00474	0.65951	0.00482	0.00307	0.65625	0.00810	0.00321	0.65952	0.00481	0.00307
		30	0.66736	0.00302	0.00304	0.66495	-0.00061	0.00313	0.66200	0.00234	0.00230	0.65968	0.00466	0.00237	0.66198	0.00236	0.00230
		40	0.66533	0.00099	0.00228	0.66349	0.00085	0.00233	0.66139	0.00294	0.00185	0.65957	0.00476	0.00190	0.66143	0.00290	0.00185
(4,6)	0.48718	10	0.49596	0.00878	0.00792	0.48431	0.00287	0.00837	0.48338	0.00380	0.00380	0.46811	0.01907	0.00393	0.48332	0.00386	0.00381
		20	0.49305	0.00587	0.00405	0.48717	0.00001	0.00414	0.48684	0.00034	0.00265	0.47771	0.00947	0.00266	0.48691	0.00270	0.00265
		30	0.49030	0.00312	0.00277	0.48635	0.00083	0.00282	0.48642	0.00076	0.00207	0.47992	0.00726	0.00208	0.48639	0.00079	0.00207
		40	0.48998	0.00280	0.00213	0.48701	0.00017	0.00216	0.48702	0.00016	0.00170	0.48197	0.00521	0.00170	0.48698	0.00020	0.00170
(5,6)	0.37255	10	0.38618	0.01363	0.00665	0.37286	-0.00031	0.00668	0.37484	-0.00229	0.00305	0.35522	0.01733	0.00291	0.37479	-0.00224	0.00305
		20	0.38132	0.00877	0.00343	0.37462	-0.00207	0.00341	0.37598	-0.00343	0.00219	0.36424	0.00831	0.00207	0.37595	-0.00340	0.00220
		30	0.37775	0.00520	0.00235	0.37327	-0.00072	0.00234	0.37465	-0.00211	0.00174	0.36634	0.00621	0.00167	0.37468	-0.00213	0.00174
		40	0.37555	0.00300	0.00163	0.37217	0.00038	0.00163	0.37345	-0.00090	0.00130	0.36702	0.00553	0.00126	0.37343	-0.00088	0.00130
(6,6)	0.29412	10	0.30793	0.01382	0.00564	0.29430	-0.00018	0.00549	0.29764	-0.00352	0.00247	0.27665	0.01747	0.00223	0.29768	-0.00018	0.00549
		20	0.29969	0.00558	0.00273	0.29288	0.00124	0.00270	0.29592	-0.00180	0.00177	0.28349	0.01062	0.00165	0.29588	-0.00177	0.00177
		30	0.29895	0.00483	0.00172	0.29440	-0.00028	0.00169	0.29664	-0.00252	0.00126	0.28786	0.00636	0.00118	0.29668	-0.00256	0.00126
		40	0.29771	0.00359	0.00122	0.29429	-0.00017	0.00120	0.29620	-0.00208	0.00096	0.28931	0.00481	0.00091	0.29622	-0.00210	0.00096

Table 3.8. Estimates of $R_{n,k}$ for Weibull distribution when $(\alpha, \beta) = (3, 4.5)$ and $(\lambda_1, \lambda_2) = (6, 4)$

			MLE			$UMVUE$			$Bayes$ (Prior 4)								
(k,n)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	Bias	MSE	$\widehat{R}_{n,k}^{U}$	Bias	MSE	$\widehat{R}_{n,k,B}$	Bias	ER	$\widehat{R}_{n,k,B}^{Lin}$	Bias	ER	$\widehat{R}_{n,k,B}^{MC}$	Bias	ER
(1,5)	0.91475	10	0.91124	-0.00351	0.00287	0.91681	0.00206	0.00303	0.90301	-0.01174	0.00129	0.91432	-0.00043	0.00117	0.90298	-0.01177	0.00129
		20	0.91222	-0.00254	0.00145	0.91507	0.00032	0.00149	0.90652	-0.00823	0.00091	0.91393	-0.00082	0.00085	0.90652	-0.00823	0.00091
		30	0.91385	-0.00090	0.00099	0.91578	0.00103	0.00101	0.90949	-0.00526	0.00079	0.91484	0.00008	0.00068	0.90933	-0.00543	0.00070
		40	0.91424	-0.00051	0.00074	0.91570	0.00094	0.00076	0.91050	-0.00430	0.00060	0.91500	0.00025	0.00055	0.91060	-0.00415	0.00056
(2,5)	0.67233	10	0.68240	0.01007	0.00949	0.67612	0.00379	0.01045	0.66703	-0.00530	0.00337	0.66837	-0.00396	0.00350	0.66697	-0.00536	0.00338
		20	0.67603	0.00370	0.00478	0.67266	0.00330	0.00503	0.66824	-0.00409	0.00258	0.66909	-0.00324	0.00265	0.66822	-0.00411	0.00258
		30	0.67649	0.00417	0.00333	0.67424	0.00191	0.00344	0.67082	-0.00151	0.00213	0.67146	-0.00086	0.00217	0.67082	-0.00151	0.00213
		40	0.67261	0.00028	0.00253	0.67086	-0.00147	0.00260	0.66884	-0.00349	0.00180	0.66933	-0.00300	0.00182	0.66890	-0.00343	0.00179
(3,5)	0.45454	10	0.47088	0.01634	0.00906	0.45813	0.00359	0.00937	0.45572	0.00118	0.00289	0.44946	-0.00509	0.00284	0.45567	0.00112	0.00289
		20	0.46130	0.00675	0.00430	0.45475	0.00207	0.00438	0.45490	0.00035	0.00221	0.45079	-0.00375	0.00219	0.45488	0.00033	0.00221
		30	0.46069	0.00614	0.00296	0.45632	0.00177	0.00298	0.45635	0.00181	0.00184	0.45329	-0.00125	0.00182	0.45635	0.00180	0.00184
		40	0.45650	0.00195	0.00218	0.45318	-0.00136	0.00221	0.45390	-0.00064	0.00152	0.45148	-0.00307	0.00152	0.45396	-0.00059	0.00152
(4,5)	0.31469	10	0.33139	0.01671	0.00657	0.31765	0.00296	0.00642	0.31843	0.00375	0.00197	0.30984	-0.00484	0.00182	0.31838	0.00370	0.00197
		20	0.32173	0.00704	0.00296	0.31481	0.00013	0.00293	0.31685	0.00217	0.00149	0.31126	-0.00342	0.00142	0.31684	0.00215	0.00149
		30	0.32073	0.00605	0.00201	0.31612	0.00144	0.00199	0.31762	0.00293	0.00124	0.31344	-0.00124	0.00119	0.31761	0.00292	0.00124
		40	0.31704	0.00235	0.00146	0.31358	-0.00111	0.00146	0.31533	0.00065	0.00101	0.31205	-0.00264	0.00099	0.31537	0.00069	0.00101
(5,5)	0.23077	10	0.24625	0.01548	0.00466	0.23320	0.00243	0.00434	0.23532	0.00455	0.00133	0.22644	-0.00433	0.00117	0.23528	0.00451	0.00134
		20	0.23732	0.00656	0.00202	0.23085	0.00008	0.00195	0.23354	0.00277	0.00100	0.22778	-0.00298	0.00092	0.23353	0.00276	0.00100
		30	0.23475	0.00398	0.00129	0.23045	-0.00032	0.00126	0.23275	0.00198	0.00079	0.22849	-0.00228	0.00074	0.23276	0.00199	0.00078
		40	0.23309	0.00232	0.00097	0.22987	-0.00090	0.00096	0.23192	0.00115	0.00067	0.22855	-0.00222	0.00064	0.23195	0.00118	0.00067

Table 3.9. Average confidence/credible lengths (ACL) and coverage probabilities (CP) of $R_{n,k}$ for Weibull distribution when $(\alpha, \beta) = (2, 5)$, $(\lambda_1, \lambda_2) = (2, 3)$ and $(\alpha, \beta) = (3, 4.5)$, $(\lambda_1, \lambda_2) = (6, 4)$

			ACI		HPD (Prior 3)				ACI		HPD (Prior 4)	
m	(k,n)	$R_{n,k}$	ACL	CP	ACL	CP	(k,n)	$R_{n,k}$	ACL	CP	ACL	CP
10	(2,6)	0.84497	0.26770	0.889	0.23248	0.971	(1,5)	0.91475	0.19025	0.869	0.16255	0.984
20			0.19760	0.922	0.17902	0.958			0.14974	0.913	0.12892	0.976
30			0.16296	0.931	0.15149	0.960			0.12249	0.917	0.10942	0.964
40			0.14158	0.940	0.13321	0.963			0.10640	0.924	0.09709	0.957
10	(3,6)	0.66434	0.36037	0.914	0.30274	0.969	(2,5)	0.67233	0.37486	0.906	0.28968	0.982
20			0.26037	0.930	0.23348	0.959			0.27319	0.936	0.23141	0.976
30			0.21407	0.938	0.19766	0.960			0.22454	0.936	0.19838	0.964
40			0.18642	0.947	0.17456	0.957			0.19593	0.944	0.17709	0.964
10	(4,6)	0.48718	0.34489	0.928	0.28608	0.973	(3,5)	0.45454	0.35682	0.926	0.27012	0.982
20			0.24768	0.936	0.22116	0.962			0.25584	0.942	0.21494	0.976
30			0.20317	0.943	0.18688	0.955			0.20994	0.943	0.18450	0.965
40			0.17640	0.942	0.16465	0.954			0.18194	0.946	0.16400	0.962
10	(5,6)	0.37255	0.31288	0.938	0.25683	0.977	(4,5)	0.31469	0.29737	0.937	0.22143	0.983
20			0.22312	0.942	0.19819	0.965			0.21020	0.945	0.17558	0.975
30			0.18230	0.941	0.16704	0.950			0.17208	0.947	0.15074	0.965
40			0.15797	0.945	0.14731	0.954			0.14843	0.948	0.13358	0.961
10	(6,6)	0.29412	0.27804	0.936	0.22648	0.978	(5,5)	0.23077	0.24604	0.942	0.18084	0.983
20			0.19614	0.934	0.17386	0.956			0.17190	0.947	0.14298	0.976
30			0.16070	0.944	0.14710	0.959			0.13984	0.949	0.12232	0.973
40			0.13916	0.955	0.12970	0.963			0.12071	0.949	0.10851	0.962

Moreover, we have presented plots for comparing the performance of the Bayes estimate using Lindley's approximation, ML and UMVU estimates in terms of ER and MSEs for the sample sizes $m = 25$ and 50. In Figure 3.3, we chose the parameters as $\alpha_i = 0.9 + (i/10)$, $\beta_i = 7.6 - (i/10)$, $i = 1, ..., 75$ and $(\lambda_1, \lambda_2) = (3, 5)$ for $R_{8,2}$. Then, it takes the values 0.014 to 0.990. In Figure 3.4, we chose the parameters as $\alpha_i = 0 + (i/10)$, $\beta_i = 25.3 - (3i/10)$, $i = 1, ..., 80$ and $(\lambda_1, \lambda_2) = (3, 5)$ for $R_{8,5}$. Then, it takes the values 0.047 to 0.992. We use the following algorithm to draw the plots.

Step 1: For given $(\alpha, \lambda_1, \beta, \lambda_2)$, $R_{8,2}$ (or $R_{8,5}$) is computed.

Step 2: For given m, samples from Weibull distribution are generated for the strength and the stress variables.

Step 3: Estimates of $R_{8,2}$ (or $R_{8,5}$) are evaluated.

Step 4: Steps 2-3 are repeated 2500 times, the MSE or ER for estimates of $R_{n,k}$ are calculated as $MSE(\widehat{R}_{n,k}) = \sum_{i=1}^{N}(\widehat{R}_{n,k}^{(i)} - R_{n,k})^2/N$.

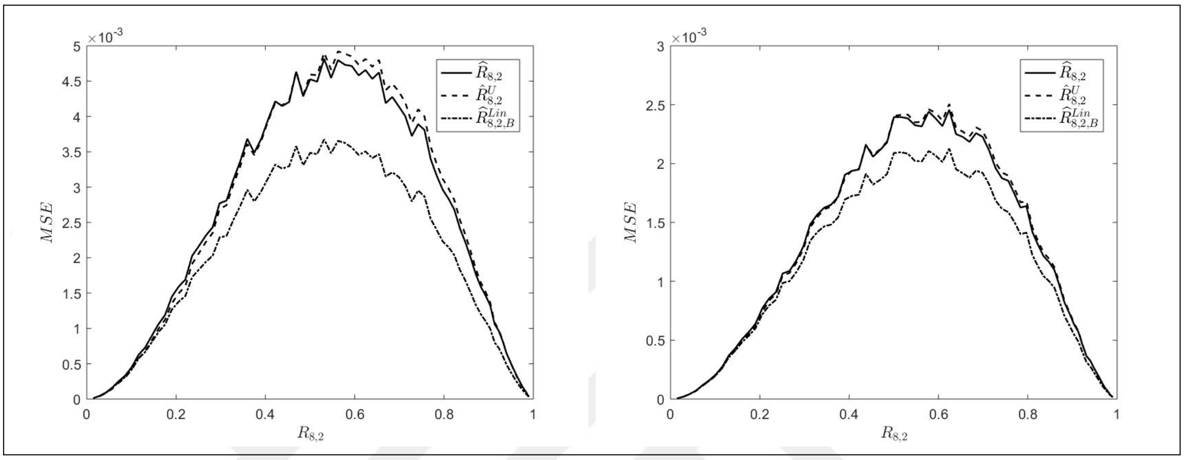

Figure 3.3. MSEs (or ER) of the estimates when $m = 25$ and 50

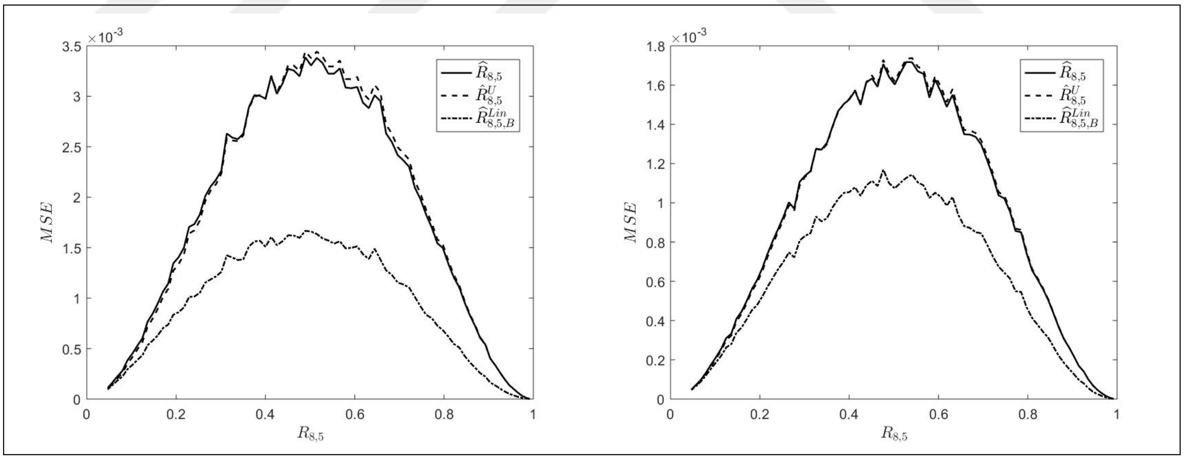

Figure 3.4. MSEs (or ER) of the estimates when $m = 25$ and 50

From Figures 3.3 and 3.4, ML, UMVU and Bayes estimates have more errors when $R_{n,k}$ tends to 0.5 with respect to $R_{n,k}$ tends to 0 or 1. The MSE of UMVUE are greater than that of MLE when $R_{n,k}$ is around 0.5, and the MSE of UMVU and ML estimates are close to each other when $R_{n,k}$ is around extreme values. The Bayes estimate based on Lindley's approximation have the smallest error. The similar outcomes are also observed in Tables 3.7-3.8.

3.4. REAL DATA ANALYSIS

In this section, two different real-life data sets are analyzed to illustrate the proposed methods for the cases of common parameter is unknown and known. First, the monthly water capacity of two different reservoirs in California, the US, has been studied. Then, the time-between-failures data of certain software model has been considered.

3.4.1 Real Data Set I

This data set represents the monthly water capacities of Trinity Lake and Shasta Dam in California, the US, from January 2000 until January 2020. All data sets are available in the California Data Exchange Center website (see: https://cdec.water.ca.gov/misc/monthly_res.html). These reservoirs are located close to each other and used for different aims such as irrigation, power generation, recreation and multi-purposes etc..

It is assumed that these two reservoirs supply all the water needs of a certain region. We consider the first six months of storage data for each year from 2000 to 2020. The first six months average values of Trinity Lake storage represent the stress data ($\mathbf{X}$) and the storage values of Shasta Dam for the first six months represent the strength data ($\mathbf{Y}$). In this case, the size of strength variables and sample size are 6 and 20, namely $n = 6$, $m = 20$ for this data sets. In this structure, we can construct the following scenario. We have the consecutive $k-$out-of$-6 : G$ system $k = 1, ..., 6$ based on these stress and strength data. If the stress-strength reliability of the consecutive $k-$out-of$-6 : G$ system exceeds the predefined threshold value, then Shasta Dam will be enough capacity for the region in the next six months and Trinity Lake storage will be used only for power generation. For example, if the system reliability is greater than 0.70, this scenario will happen.

Based on this scenario, our data are constructed as: the storage values of Shasta Dam from January 2000 to June 2000 are saved as $Y_{11}, ..., Y_{16}$, and X_1 is the average of Trinity Lake storage values in these six months. Then, the storage values from January 2001 to June 2001 are saved as $Y_{21}, ..., Y_{26}$ and X_2. When we carry on this data process up to January 2020, we have 20 and 120 units of data for $\mathbf{X}$ and $\mathbf{Y}$, respectively. These data have been used dividing by the total storage capacities 2447650 acre-foot (ac-ft) and 4552000 ac-ft for **X** and **Y**, respectively. Then, the real data sets for stress and strength variables are presented in Table 3.10.

Table 3.10. Real data set I

X	Y					
0.9021219	0.8121865	0.8474128	0.8242241	0.9124075	0.9032335	0.8403293
0.7607941	0.6668045	0.7680072	0.8690244	0.8831439	0.830958	0.7212386
0.7631484	0.7725362	0.8434851	0.9086448	0.9439185	0.9055044	0.8081819
0.8714552	0.7769710	0.7874084	0.9015176	0.9966221	0.9745138	0.8980881
0.8594681	0.7922891	0.8498682	0.8579051	0.8918699	0.8488137	0.7383168
0.7711141	0.6221509	0.6959701	0.8407384	0.9242605	0.9815435	0.9264464
0.8962444	0.7878018	0.8423163	0.8466388	0.8913236	0.9813491	0.9334583
0.7899102	0.7412731	0.8286891	0.8811498	0.8570160	0.7911191	0.6899453
0.6394267	0.4786975	0.5801935	0.6569767	0.6489701	0.6166151	0.5277210
0.4783290	0.3111039	0.4306806	0.6328238	0.6586659	0.6851804	0.6144802
0.5649707	0.5704925	0.7425630	0.8498787	0.9645813	0.9808313	0.9400652
0.8733742	0.7667395	0.8313078	0.8858025	0.9372285	0.9856872	0.9669442
0.8766279	0.6826305	0.6962291	0.8465321	0.9754161	0.9444884	0.8525299
0.8146628	0.7631123	0.7932001	0.8284563	0.8322390	0.7386931	0.6454804
0.4900054	0.3637522	0.3895681	0.4830246	0.5291762	0.4782979	0.4066929
0.4323364	0.4395496	0.5739708	0.5907476	0.5849031	0.5281489	0.4829042
0.4754467	0.5153833	0.6076714	0.8845940	0.9299453	0.9154385	0.8612250
0.8385771	0.7790308	0.8302603	0.8856210	0.9364721	0.9563014	0.9251973
0.7485033	0.7357485	0.7499117	0.8522939	0.9215215	0.8689047	0.8000437
0.8311339	0.6396175	0.8672348	0.8848357	0.9277579	0.9834857	0.9568100

Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Cramer-von Mises (C-VM) tests are applied for the goodness-of-fit by using the *stats* and *goftest* (see Faraway et al. [47]) packages in R. We check whether stress data set **X** and strength data set **Y** for $n = 6, m = 20$ come from Kumaraswamy distribution or not by goodness-of-fit test. The test statistics and corresponding $p-$values are computed based on the MLEs of the unknown parameters and compared by Weibull, Burr Type XII and exponential distributions. Since the computational method of *goftest* package, the test statistics and $p-$values of A-D and C-VM tests are random (see Faraway et al. [47]). For this reason, these tests are applied 10000 times and average values of test statistics (in the first row) and corresponding p-values (in the second row) are presented in Table 3.11. Moreover, the MLE of (α, β, λ), K-S test statistics and corresponding p-values are also given. From Table 3.11, it is observed that Kumaraswamy distribution provides a good fit than other lifetime distributions for both X and Y data sets.

Table 3.11. Goodnes-of-fit test for the real data set I

Distribution	**Data Y** (Strength)				**Data X** (Stress)			
	MLE	$K-S$	$A-D$	$C-VM$	MLE	$K-S$	$A-D$	$C-VM$
Kumaraswamy	$\hat{\alpha}$ =1.39005	0.06650	2.45401	0.42686	$\hat{\beta}$ =2.25456	0.21138	1.59498	0.31874
	$\hat{\lambda}$ =4.49264	0.66341	0.52993	0.55236	$\hat{\lambda}$ =4.49264	0.29041	0.55943	0.47876
Weibull	$\hat{\alpha}$ =2.18922	0.18339	3.05082	0.65318	$\hat{\beta}$ =2.720707	0.28430	1.71805	0.36506
	$\hat{\lambda}$ =3.90511	0.00062	0.32065	0.21880	$\hat{\lambda}$ =3.90511	0.06352	0.49917	0.38790
Burr Type XII	$\hat{\alpha}$ =4.00226	0.14296	3.08368	0.55742	$\hat{\beta}$ =5.51597	0.20175	2.06885	0.34134
	$\hat{\lambda}$ =6.92249	0.01482	0.34706	0.35709	$\hat{\lambda}$ =6.92249	0.34260	0.40958	0.44314
Exponential	$\hat{\alpha}$ =0.40921	0.40921	4.42252	0.96300	$\hat{\beta}$ =0.00708	0.44518	2.10408	0.45668
	-	0	0.06450	0.02659	-	0.00039	0.30497	0.18953

The ML and Bayes estimates of $R_{6,k}$, $k = 2, 3, 4, 5$ along with 95% asymptotic confidence and HPD credible intervals are presented in Table 3.12. Moreover, Bayes estimates are evaluated based on the non-informative prior $a_i = b_i = 0$, $i = 1, 2, 3$, informative priors Prior 5: $a_i = b_i = 1$, $i = 1, 2, 3$ and Prior 6: $(a_1, b_1) = (1, 1)$, $(a_2, b_2) = (2, 1)$, $(a_3, b_3) = (4, 1)$. It is observed that Bayes estimates based on non-informative prior and the MLE of $R_{n,k}$ are close to each other as in the previous tables. Bayes estimates of $R_{n,k}$ based on Prior 6 are greater than that of Prior 5. Furthermore, we can comment that if $k = 2$ is enough in the aforementioned scenario, we can use these reservoirs for given targets.

Table 3.12. Estimates of $R_{n,k}$ for the real data set I

		Non-informative prior		Informative prior (Prior 5)		Informative prior (Prior 6)	
(k,n)	$\widehat{R}_{n,k}^{MLE}/ACI$	$\widehat{R}_{n,k,B}^{Lind}$	$\widehat{R}_{n,k,B}^{MC}/HPD$	$\widehat{R}_{n,k,B}^{Lind}$	$\widehat{R}_{n,k,B}^{MC}/HPD$	$\widehat{R}_{n,k,B}^{Lind}$	$\widehat{R}_{n,k,B}^{MC}/HPD$
(2,6)	0.73679	0.73152	0.72973	0.70996	0.71064	0.72785	0.72607
	(0.60223,0.87135)		(0.58552,0.85088)		(0.57970,0.83592)		(0.59943,0.85482)
(3,6)	0.53818	0.52534	0.53243	0.50273	0.5117036	0.52149	0.52846
	(0.39714,0.67922)		(0.39325,0.66683)		(0.37772,0.64407)		(0.39168,0.65489)
(4,6)	0.37563	0.36165	0.37292	0.34282	0.35988	0.35844	0.37081
	(0.25817,0.49309)		(0.25584,0.48535)		(0.24602,0.47103)		(0.25979,0.47966)
(5,6)	0.27707	0.26380	0.27509	0.24831	0.26197	0.26116	0.27185
	(0.18041,0.37372)		(0.18777,0.37373)		(0.18153,0.35820)		(0.18389,0.36032)

3.4.2 Real Data Set II

In this example, we consider failure time data of certain software model which is discussed in Lyu [48]. We use the data sets CSR2 which is only failures due to software faults and CSR3 which are only failures due to Pascal programming. These data sets are also available in http://www.cse.cuhk.edu.hk/~lyu/book/reliability/data.html.

A computer engineer can want to compare the system failures times of certain software model with respect to the failure reasons. For example, she/he has two different failure time data as CSR2 and CSR3. 120 and 100 observations are taken from CSR2 and CSR3 data, respectively. The average of every ten observations of CSR3 data are used as the stress data ($\mathbf{X}$) and CSR2 data are used as the strength data ($\mathbf{Y}$). In this case, the size of the components in the system is 12 and $m = 10$. It is presented in Table 3.13. Hence, we have the consecutive $k-$out-of$-12 : G$ system $k = 1, ..., 12$ based on these stress and strength data. If the stress-strength reliability of this system smaller than the predefined threshold value, she/he can take precautions for future software modellings.

These data sets have been checked whether come from Weibull distribution or not as the same methods in the previous data. The MLE of $(\alpha, \beta, \lambda_1, \lambda_2)$, test statistics (in the first row) and corresponding p-values (in the second row) are presented in Table 3.14 and compared by Burr Type XII and exponential distributions. From Table 3.14, it is observed that Weibull distribution provides a good fit than other lifetime distributions for both X and Y data sets.

The ML, UMVU and Bayes estimates of $R_{12,k}$, $k = 2, 3, 4$ along with 95% asymptotic confidence and HPD credible intervals are presented in Table 3.15. Moreover, Bayes estimates are evaluated based on the non-informative prior, informative priors Prior 5: $a_i = b_i = 1$, $i = 1, 2$ and Prior 7: $(a_1, b_1) = (2, 2)$, $(a_2, b_2) = (2, 2)$. It is observed that Bayes estimates based on non-informative prior and the MLE of $R_{n,k}$ are close to each other as in the previous tables. Bayes estimates of $R_{n,k}$ based on Prior 7 are greater than that of Prior 5. Since the reliability of the system is very low, we can comment that the failure times of CSR3 is bigger than that of CSR2 with respect to the above scenario.

Table 3.13. Real data set II

X	Y											
35.85	760	758	303	6	22	14	42	4	84	15	221	14
43.50	15	41	1	153	409	54	24	44	180	397	19	145
45.30	36	54	1337	163	8	1	17	16	87	19	29	5
56.40	360	10	11	100	252	460	179	3	24	253	163	54
94.90	137	328	3	9	12	18	9	75	15	366	428	212
173.40	115	264	269	279	1	999	30	495	472	344	550	131
236.30	47	92	863	991	35	9549	249	607	83	614	352	673
184.90	4179	111	75	407	288	897	1314	845	55	409	36	15
346.40	1960	60	19	20	79	24	1737	7984	10	20	338	250
196.20	1682	212	287	56	4973	3500	59	98	2439	1812	6203	385

Table 3.14. Goodnes-of-fit test for the real data set II

Distribution	**Data Y** (Strength)				**Data X** (Stress)			
	MLE	$K-S$	$A-D$	$C-VM$	MLE	$K-S$	$A-D$	$C-VM$
Weibull	$\hat{\alpha}$ =0.04765	0.06862	2.29560	0.43070	$\hat{\beta}$ =0.00056	0.20490	1.27607	0.23864
	$\hat{\lambda}_1$ =0.53638	0.62430	0.57296	0.54641	$\hat{\lambda}_2$ =1.47938	0.73590	0.62716	0.59761
Burr Type XII	$\hat{\alpha}$ =0.02636	0.30446	3.50511	0.75345	$\hat{\beta}$ =0.07158	0.53523	1.42581	0.30990
	$\hat{\lambda}_1$ =8.06617	0	0.18434	0.10679	$\hat{\lambda}_2$ =2.99039	0.00333	0.48432	0.33608
Exponential	$\hat{\alpha}$ =0.00172	0.32562	10.87642	1.52565	$\hat{\beta}$ =0.00708	0.22407	1.01053	0.20811
	-	0	0.00100	0.00296	-	0.62060	0.73383	0.65075

Table 3.15. Estimates of $R_{n,k}$ for the real data set II

			Non-informative prior		Informative prior (Prior 5)		Informative prior (Prior 7)	
(k,n)	$\widehat{R}_{n,k}^{MLE}/ACI$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}^{Lind}$	$\widehat{R}_{n,k,B}^{MC}/HPD$	$\widehat{R}_{n,k,B}^{Lind}$	$\widehat{R}_{n,k,B}^{MC}/HPD$	$\widehat{R}_{n,k,B}^{Lind}$	$\widehat{R}_{n,k,B}^{MC}/HPD$
(2,12)	0.01672	0.01508	0.01684	0.01689	0.01836	0.01817	0.01985	0.01990
	(0.00604,0.02740)			(0.00684,0.02726)		(0.00859,0.03033)		(0.00850,0.03106)
(3,12)	0.01010	0.00910	0.01017	0.01011	0.01109	0.01116	0.01200	0.01203
	(0.00362,0.01657)			(0.00395,0.01613)		(0.00522,0.01814)		(0.00520,0.01920)
(4,12)	0.00681	0.00614	0.00686	0.00698	0.00749	0.00747	0.00810	0.00804
	(0.00244,0.01119)			(0.00284,0.01154)		(0.00290,0.01185)		(0.00383,0.01262)

3.5. CONCLUSIONS

In this chapter, we studied the estimation of stress-strength reliability of a consecutive k-out of-n system when the distributions of both strength and stress variables belong to the PHR model. The estimation of system reliability has been considered under classical and Bayesian approaches when the second parameters of underlying distributions are common and unknown, and different and known. ML and UMVU estimation methods have been used in regard of classical approach. In Bayesian estimation procedure, exact Bayes estimate is obtained in the known second parameter case, and Bayesian estimates obtained by using Lindley's approximation and MCMC method are obtained in both the unknown and known second parameter cases. The ML and Bayes estimates of the reliability have been compared with respect to biases and ERs. The asymptotic confidence intervals and credible intervals are also constructed. Two real data analysis have been done for the data following Kumaraswamy and Weibull distributions.

From simulation study, it is observed that average ERs for the estimates of $R_{n,k}$ and average confidence intervals decrease as the sample size increases. It is also observed that all estimates have large ER values when $R_{n,k}$ is around 0.5 and have small ER values when $R_{n,k}$ is near to extreme values. The simulation study shows that Bayes estimates based on informative prior have smaller risks than the ML estimates while Bayes estimates based on non-informative prior and the ML estimates show similar performances. When the second parameters of underlying distributions are unknown, Bayesian estimation procedure for large sample sizes is constructed only using Lindley's approximation because of the convergence problem of MCMC method. Therefore, Lindley's approximation is recommended for large sample sizes in this case. However, the exact Bayes and other Bayes estimates have very similar performance when the second parameters of underlying distributions are known. Moreover, in all cases credible intervals are shorter than asymptotic confidence intervals.

4. RELIABILITY ESTIMATION OF $(C, k, n; G)$ SYSTEM WITH NON-IDENTICAL STRENGTH COMPONENTS

We consider stress-strength reliability estimation of a $(C, k, n; G)$ system with non-identical strength components when both stress and strength components follow the proportional hazard rate model.

Chapter organization is given as follows:

- In Section 4.1, frequentist and Bayesian estimates the stress-strength reliability for the considered system are derived when the common parameter λ is unknown. Lindley's approximation and MCMC method with Metropolis-Hastings algorithm are used in Bayesian part. The asymptotic confidence intervals and highest posterior density credible intervals are also constructed.

- In Section 4.2, all the estimates are studied when the second parameters λ_1, λ_2 and λ_3 of the underlying distributions are different and known. The ML and UMVU estimates are obtained in view of classical manner. In Bayesian approach, exact Bayes and also approximate Bayes estimates (Lindley's approximation and MCMC method via Gibbs algorithm) are obtained. In addition, the asymptotic confidence intervals and HPD credible intervals are developed.

- In Section 4.3, We provide comprehensive simulation experiments for investigating the performances of the considered estimates.

- In Section 4.4, Wind speed data from NASA's satellite data source project is used in the application of the considered model and methods. Wind energy capacities of two districts in the Aegean coast of Turkey are investigated, and results are presented.

- Finally, comments and concluding remarks of the study are presented in Section 4.5.

4.1. ESTIMATION OF $R_{n,k}$ WHEN THE SECOND PARAMETERS ARE COMMON AND UNKNOWN

In this section, reliability estimates of a consecutive k-out-of-n:G system with non-identical strengths are examined in case of the common parameter λ of underlying distributions is unknown. The reliability formula of this system is given in next before the presentation of our results.

Suppose that a consecutive k-out-of-n:G system with n strength components such that first n_1 ones follow a common distribution $F^{(1)}$ and the remaining $n_2 = n - n_1$ ones follow another common distribution $F^{(2)}$, and random stress X having cdf F_X independently. Then, stress-strength reliability of consecutive k-out-of-n:G system under these assumptions when n_1 is known and $2k \geq n = n_1 + n_2$ was obtained by Eryılmaz [34] as follows:

$$R_{n,k} = \sum_{j=k}^{n} p(j,k), \tag{4.1}$$

where

$$\text{p(j,k)=}\begin{cases} g(0,0,1,0) - \sum\limits_{m=0}^{k-1} g(m,0,2,0), & \text{, if } k \leq j \leq n_1 - 1 \\ g(0,0,0,1) - \sum\limits_{m=j-n_1}^{k-1} g(m+n_1-j, j-n_1, 1, 1) - \sum\limits_{m=0}^{j-n_1-1} g(0,m,0,2), & \text{, if } n_1 \leq j \leq n_1+n_2-1, \\ 1 - \sum\limits_{m=n-n_1}^{k-1} g(m+n_1-n, n-n_1, 1, 0) - \sum\limits_{m=0}^{n-n_1-1} g(0,m,0,1), & \text{, if } j = n_1+n_2 \end{cases} \tag{4.2}$$

and $g(a,b,c,d)$ can be computed by the following function:

$$g(a,b,c,d) = \int (1-(F^{(1)}(x))^a(1-(F^{(2)}(x))^b((F^{(1)}(x))^c((F^{(2)}(x))^d dF_X(x). \tag{4.3}$$

4.1.1 MLE of $R_{n,k}$

Let the system consists of n strength variables such that first n_1 ones $Y_{j_1}^{(1)} \sim PHR(\alpha_1, \lambda)$, $j_1 = 1, .., n_1$, the remaining $n_2 = n - n_1$ ones $Y_{j_2}^{(2)} \sim PHR(\alpha_2, \lambda)$, $j_2 = n_1 + 1, .., n$ and, stress variable $X \sim PHR(\beta, \lambda)$. Then, $g(a, b, c, d)$ is obtained by using (4.3) as follows:

$$g(a,b,c,d) = \sum_{i=0}^{c} \sum_{j=0}^{d} (-1)^{i+j} \binom{c}{i} \binom{d}{j} \frac{\beta}{\alpha_1(a+i) + \alpha_2(b+j) + \beta}. \tag{4.4}$$

The following observed data $\underline{Y_1} = \left(Y_{i1}^{(1)}, Y_{i2}^{(1)}, \dots, Y_{in_1}^{(1)}\right)$, $\underline{Y_2} = \left(Y_{i1}^{(2)}, Y_{i2}^{(2)}, \dots, Y_{in_2}^{(2)}\right)$, $i = 1, ..., m$ and $\underline{X} = (X_1, ..., X_m)$ are obtained when m systems are subjected to a life-testing experiment to obtain the estimates of $R_{n,k}$. Then, the likelihood function of the observed sample is expressed as

$$\begin{aligned} L\left(\alpha_1, \alpha_2, \beta, \lambda; \underline{x}, \underline{y_1}, \underline{y_2}\right) &= \prod_{i=1}^{m} \left(\prod_{j_1=1}^{n_1} \left(f_{Y^{(1)}}(y_{ij_1}^{(1)}\right) \prod_{j_2=1}^{n_2} \left(f_{Y^{(2)}}(y_{ij_2}^{(2)}\right) \right) f_X(x_i) \\ &= \alpha_1^{n_1 m} \alpha_2^{n_2 m} \beta^m \exp\left[\sum_{i=1}^{m} \sum_{j_1=1}^{n_1} \ln f_0(y_{ij_1}^{(1)}; \lambda) \right. \\ &+ (\alpha_1 - 1) \sum_{i=1}^{m} \sum_{j_1=1}^{n_1} \ln \overline{F_0}(y_{ij_1}^{(1)}; \lambda) + \sum_{i=1}^{m} \sum_{j_2=1}^{n_2} \ln f_0(y_{ij_2}^{(2)}; \lambda) \\ &+ (\alpha_2 - 1) \sum_{i=1}^{m} \sum_{j_2=1}^{n_2} \ln \overline{F_0}(y_{ij_2}^{(2)}; \lambda) + \sum_{i=1}^{m} \ln f_0(x_i; \lambda) \\ &\left. + (\beta - 1) \sum_{i=1}^{m} \ln \overline{F_0}(x_i; \lambda) \right], \end{aligned} \tag{4.5}$$

and the log-likelihood function is

$$l\left(\alpha_1,\alpha_2,\beta,\lambda;\underline{x},\underline{y_1},\underline{y_2}\right)=n_1m\ln\left(\alpha_1\right)+n_2m\ln\left(\alpha_2\right)+m\ln\left(\beta\right)$$
$$-t_\lambda^{(1)*}-\left(\alpha_1-1\right)t_\lambda^{(1)}-t_\lambda^{(2)*}-\left(\alpha_2-1\right)t_\lambda^{(2)} \quad (4.6)$$
$$-t_\lambda^{*}-(\beta-1)t_\lambda$$

where

$$t_\lambda^{(1)}=-\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\ln\overline{F_0}(y_{ij_1}^{(1)};\lambda),\ t_\lambda^{(1)*}=-\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\ln f_0(y_{ij_1}^{(1)};\lambda), \quad (4.7)$$

$$t_\lambda^{(2)}=-\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\ln\overline{F_0}(y_{ij_2}^{(2)};\lambda),\ t_\lambda^{(2)*}=-\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\ln f_0(y_{ij_2}^{(2)};\lambda), \quad (4.8)$$

$$t_\lambda=-\sum_{i=1}^{m}\ln\overline{F_0}(x_i;\lambda),\ t_\lambda^{*}=-\sum_{i=1}^{m}\ln f_0(x_i;\lambda). \quad (4.9)$$

The ML estimates of α_1, α_2 and β are given by

$$\hat{\alpha_1}=\frac{n_1m}{T_{\hat{\lambda}}^{(1)}},\ \hat{\alpha_2}=\frac{n_2m}{T_{\hat{\lambda}}^{(2)}}\text{ and }\hat{\beta}=\frac{m}{T_{\hat{\lambda}}}. \quad (4.10)$$

The MLE of λ, say $\hat{\lambda}$, can be obtained from the solution of the following nonlinear equation

$$\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\frac{f_{0\lambda}(y_{ij_1}^{(1)};\lambda)}{f_0(y_{ij_1}^{(1)};\lambda)}+\left(\frac{n_1m}{T_\lambda^{(1)}}-1\right)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\frac{\overline{F_0}_\lambda(y_{ij_1}^{(1)};\lambda)}{\overline{F_0}(y_{ij_1}^{(1)};\lambda)}$$
$$+\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\frac{f_{0\lambda}(y_{ij_2}^{(2)};\lambda)}{f_0(y_{ij_2}^{(2)};\lambda)}+\left(\frac{n_2m}{T_\lambda^{(2)}}-1\right)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\frac{\overline{F_0}_\lambda(y_{ij_2}^{(2)};\lambda)}{\overline{F_0}(y_{ij_2}^{(2)};\lambda)} \quad (4.11)$$
$$+\sum_{i=1}^{m}\frac{f_{0\lambda}(x_i;\lambda)}{f_0(x_i;\lambda)}+\left(\frac{m}{T_\lambda}-1\right)\sum_{i=1}^{m}\frac{\overline{F_0}_\lambda(x_i;\lambda)}{\overline{F_0}(x_i;\lambda)}=0,$$

where $\partial f_0(x;\lambda)/\partial\lambda \equiv f_{0_\lambda}(x;\lambda)$ and $\partial\overline{F_0}(x;\lambda)/\partial\lambda \equiv \overline{F_0}_\lambda(x;\lambda)$. Then, $\hat{\lambda}$ can be obtained using numerical methods such as fixed point method or Newton-Raphson method. By obtaining $\hat{\lambda}$, we get the ML estimates of α_1, α_2 and β from Equation (4.10). Thus, the invariance property can be implemented in Equations (4.1) and (4.4) to obtain the MLE of $R_{n,k}$, say $\hat{R}^{MLE}_{n,k}$.

4.1.2 Asymptotic Distribution and Confidence Interval for $R_{n,k}$

The observed information matrix of $\theta = (\alpha_1, \alpha_2, \beta, \lambda)$ is given by

$$J(\theta) = -\begin{pmatrix} \frac{\partial^2 l}{\partial\alpha_1^2} & \frac{\partial^2 l}{\partial\alpha_1\partial\alpha_2} & \frac{\partial^2 l}{\partial\alpha_1\partial\beta} & \frac{\partial^2 l}{\partial\alpha_1\partial\lambda} \\ \frac{\partial^2 l}{\partial\alpha_2\partial\alpha_1} & \frac{\partial^2 l}{\partial\alpha_2^2} & \frac{\partial^2 l}{\partial\alpha_2\partial\beta} & \frac{\partial^2 l}{\partial\alpha_2\partial\lambda} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha_1} & \frac{\partial^2 l}{\partial\beta\partial\alpha_2} & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\partial\lambda} \\ \frac{\partial^2 l}{\partial\lambda\partial\alpha_1} & \frac{\partial^2 l}{\partial\lambda\partial\alpha_2} & \frac{\partial^2 l}{\partial\lambda\partial\beta} & \frac{\partial^2 l}{\partial\lambda^2} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{pmatrix}. \tag{4.12}$$

Then, we obtain the elements of $J(\theta)$ as $J_{11} = n_1 m/\alpha_1^2$, $J_{22} = n_2 m/\alpha_2^2$, $J_{33} = m/\beta^2$, $J_{12} = J_{21} = J_{23} = J_{32} = J_{13} = J_{31} = 0$,

$$J_{14} = J_{41} = -\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{\overline{F_0}_\lambda(y^{(1)}_{ij_1};\lambda)}{\overline{F_0}(y^{(1)}_{ij_1};\lambda)}, \tag{4.13}$$

$$J_{24} = J_{42} = -\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{\overline{F_0}_\lambda(y^{(2)}_{ij_2};\lambda)}{\overline{F_0}(y^{(2)}_{ij_2};\lambda)}, \tag{4.14}$$

$$J_{34} = J_{43} = -\sum_{i=1}^{m} \frac{\overline{F_0}_\lambda(x_i;\lambda)}{\overline{F_0}(x_i;\lambda)}, \tag{4.15}$$

$$\begin{aligned}
J_{44} = &-(\alpha_1-1)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\frac{\overline{F}_{0\lambda\lambda}(y_{ij_1}^{(1)};\lambda)\overline{F}_0(y_{ij_1}^{(1)};\lambda)-(\overline{F}_{0\lambda}(y_{ij_1}^{(1)};\lambda))^2}{(\overline{F}_0(y_{ij_1}^{(1)};\lambda))^2}\\
&-\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\frac{f_{0\lambda\lambda}(y_{ij_1}^{(1)};\lambda)f_0(y_{ij_1}^{(1)};\lambda)-(f_{0\lambda}(y_{ij_1}^{(1)};\lambda))^2}{(f_0(y_{ij_1}^{(1)};\lambda))^2}\\
&-(\alpha_2-1)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\frac{\overline{F}_{0\lambda\lambda}(y_{ij_2}^{(2)};\lambda)\overline{F}_0(y_{ij_2}^{(2)};\lambda)-(\overline{F}_{0\lambda}(y_{ij_2}^{(2)};\lambda))^2}{(\overline{F}_0(y_{ij_2}^{(2)};\lambda))^2}\\
&-\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\frac{f_{0\lambda\lambda}(y_{ij_2}^{(2)};\lambda)f_0(y_{ij_2}^{(2)};\lambda)-(f_{0\lambda}(y_{ij_2}^{(2)};\lambda))^2}{(f_0(y_{ij_2}^{(2)};\lambda))^2}\\
&-(\beta-1)\sum_{i=1}^{m}\frac{\overline{F}_{0\lambda\lambda}(x_i;\lambda)\overline{F}_0(x_i;\lambda)-(\overline{F}_{0\lambda}(x_i;\lambda))^2}{(\overline{F}_0(x_i;\lambda))^2}\\
&-\sum_{i=1}^{m}\frac{f_{0\lambda\lambda}(x_i;\lambda)f_0(x_i;\lambda)-(f_{0\lambda}(x_i;\lambda))^2}{(f_0(x_i;\lambda))^2}.
\end{aligned} \tag{4.16}$$

The expectation of these elements cannot always be obtained analytically since they consist of the baseline pdfs and cdfs. Based on the considered baseline distribution, we compute the Fisher information matrix of θ, $I(\theta)=E(J(\theta))$ numerically or we can use the observed information matrix as a consistent estimator. In case of not being able to compute the Fisher information matrix analytically, $J(\theta)$ is used in the asymptotic normality of the MLE.

$\hat{R}_{n,k}^{MLE}$ is asymptotically normal with mean $R_{n,k}$ and asymptotic variance

$$\sigma_{R_{n,k}}^2=\sum_{j=1}^{4}\sum_{i=1}^{4}\frac{\partial R_{n,k}}{\partial\theta_i}\frac{\partial R_{n,k}}{\partial\theta_j}I_{ij}^{-1}, \tag{4.17}$$

where I_{ij}^{-1} is the (i,j)th element of $I(\theta)$ (see Rao [51]). Then, we have

$$\begin{aligned}
\sigma_{R_{n,k}}^2 = &\left(\frac{\partial R_{n,k}}{\partial\alpha_1}\right)^2 I_{11}^{-1}+\left(\frac{\partial R_{n,k}}{\partial\alpha_2}\right)^2 I_{22}^{-1}+\left(\frac{\partial R_{n,k}}{\partial\beta}\right)^2 I_{33}^{-1}+2\frac{\partial R_{n,k}}{\partial\alpha_1}\frac{\partial R_{n,k}}{\partial\alpha_2}I_{12}^{-1}\\
&+2\frac{\partial R_{n,k}}{\partial\alpha_1}\frac{\partial R_{n,k}}{\partial\beta}I_{13}^{-1}+2\frac{\partial R_{n,k}}{\partial\alpha_2}\frac{\partial R_{n,k}}{\partial\beta}I_{23}^{-1}.
\end{aligned} \tag{4.18}$$

As a remark $I(\theta)$ can be replaced by $J(\theta)$ in case of $I(\theta)$ is not obtained. Therefore, $R_{n,k} \in (\hat{R}_{n,k}^{MLE} \pm z_{\gamma/2}\hat{\sigma}_{R_{n,k}})$ introduces an asymptotic $100(1-\gamma)\%$ confidence interval of $R_{n,k}$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{\sigma}_{R_{n,k}}$ is the value at MLE of parameters.

4.1.3 Bayes Estimation of $R_{n,k}$

In this section, approximate Bayes estimates of $R_{n,k}$ are obtained when all the parameters $\alpha_1, \alpha_2, \beta$ and λ follow statistically independent gamma distributions as prior with parameters $(a_i, b_i), i = 1, 2, 3, 4$, respectively. If X be a gamma random variable with parameters (a, b_i), then its pdf is

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-xb}, \tag{4.19}$$

where $x > 0$, $a, b > 0$. Then, in this case, the joint posterior density function of α_1, α_2, β and λ is given by

$$\begin{aligned}\pi(\alpha_1, \alpha_2, \beta, \lambda|\underline{x}, \underline{y_1}, \underline{y_2}) =& \frac{L\left(\alpha_1, \alpha_2, \beta, \lambda; \underline{x}, \underline{y_1}, \underline{y_2}\right)\pi(\alpha_1)\pi(\alpha_2)\pi(\beta)\pi(\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L\left(\alpha_1, \alpha_2, \beta, \lambda; \underline{x}, \underline{y_1}, \underline{y_2}\right)\pi(\alpha_1)\pi(\alpha_2)\pi(\beta)\pi(\lambda)d\alpha_1 d\alpha_2 d\beta d\lambda} \\ =&[I(\underline{x}, \underline{y_1}, \underline{y_2})]^{-1}\alpha_1^{n_1 m+a_1-1}\alpha_2^{n_2 m+a_2-1}\beta^{m+a_3-1}\lambda^{a_4-1}e^{-\alpha_1(b_1+t_\lambda^{(1)})} \\ & e^{-\alpha_2(b_2+t_\lambda^{(2)})}e^{-\beta(b_3+t_\lambda)}e^{-\lambda b_4}e^{t_\lambda^{(1)}-t_\lambda^{(1)*}}e^{t_\lambda^{(2)}-t_\lambda^{(2)*}}e^{t_\lambda-t_\lambda^*}\end{aligned} \tag{4.20}$$

where $I(\underline{x}, \underline{y_1}, \underline{y_2})$ is the normalizing constant and shown as

$$\frac{[I(\underline{x}, \underline{y_1}, \underline{y_2})]^{-1}}{\Gamma(n_1 m+a_1)\Gamma(n_2 m+a_2)\Gamma(m+a_3)} = \int_0^\infty \frac{\lambda^{a_4-1}e^{t_\lambda^{(1)}-t_\lambda^{(1)*}}e^{t_\lambda^{(2)}-t_\lambda^{(2)*}}e^{t_\lambda-t_\lambda^*}}{(b_1+t_\lambda^{(1)})^{n_1 m+a_1}+(b_2+t_\lambda^{(2)})^{n_2 m+a_2}+(b_3+t_\lambda)^{m+a_3}}d\lambda \tag{4.21}$$

Hence, Bayes estimate of $R_{n,k}$ is given as follows

$$\hat{R}_{n,k,B} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R_{n,k}\pi\left(\alpha_1, \alpha_2, \beta, \lambda|\underline{x}, \underline{y_1}, \underline{y_2}\right)d\alpha_1 d\alpha_2 d\beta d\lambda, \tag{4.22}$$

under the SE loss function. Two approximation methods Lindley's approximation and MCMC method are applied in Bayes estimation of $R_{n,k}$ due to the multiple integrals in (4.22) are not computed analytically and difficulties in numerical computations of these integrals.

4.1.3.1 Lindley's Approximation

In our case, we have four unknown parameters as $\theta = (\alpha_1, \alpha_2, \beta, \lambda)$, and so Lindley's approximation leads to

$$\begin{aligned}\hat{R}^{Lin}_{n,k,B} &= u + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_5 + a_6) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) \\ &+ B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33}) \\ &+ D(u_1\sigma_{41} + u_2\sigma_{42} + u_3\sigma_{43})]\end{aligned} \tag{4.23}$$

evaluated at $\hat{\theta} = (\hat{\alpha_1}, \hat{\alpha_2}, \hat{\beta}, \hat{\lambda})$ where $u = u(\theta) = R_{n,k}$,

$$a_i = \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3} + \rho_4\sigma_{i4}, i = 1, 2, 3, \tag{4.24}$$

$$a_5 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},\ a_6 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}), \tag{4.25}$$

$$\begin{aligned}A = \sigma_{11}L_{111} + 2\sigma_{12}L_{121} + 2\sigma_{13}L_{131} + 2\sigma_{14}L_{141} + 2\sigma_{23}L_{231} \\ + 2\sigma_{24}L_{241} + 2\sigma_{34}L_{341} + \sigma_{22}L_{221} + \sigma_{33}L_{331} + \sigma_{44}L_{441},\end{aligned} \tag{4.26}$$

$$\begin{aligned}B = \sigma_{11}L_{112} + 2\sigma_{12}L_{122} + 2\sigma_{13}L_{132} + 2\sigma_{14}L_{142} + 2\sigma_{23}L_{232} \\ + 2\sigma_{24}L_{242} + 2\sigma_{34}L_{342} + \sigma_{22}L_{222} + \sigma_{33}L_{332} + \sigma_{44}L_{442},\end{aligned} \tag{4.27}$$

$$\begin{aligned}C = \sigma_{11}L_{113} + 2\sigma_{12}L_{123} + 2\sigma_{13}L_{133} + 2\sigma_{14}L_{143} + 2\sigma_{23}L_{233} \\ + 2\sigma_{24}L_{243} + 2\sigma_{34}L_{343} + \sigma_{22}L_{223} + \sigma_{33}L_{333} + \sigma_{44}L_{443},\end{aligned} \tag{4.28}$$

$$\begin{aligned}D = \sigma_{11}L_{114} + 2\sigma_{12}L_{124} + 2\sigma_{13}L_{134} + 2\sigma_{14}L_{144} + 2\sigma_{23}L_{234} \\ + 2\sigma_{24}L_{244} + 2\sigma_{34}L_{344} + \sigma_{22}L_{224} + \sigma_{33}L_{334} + \sigma_{44}L_{444}.\end{aligned} \tag{4.29}$$

We also have $\rho_1 = ((a_1 - 1)/\alpha_1) - b_1$, $\rho_2 = ((a_2 - 1)/\alpha_2) - b_2$, $\rho_3 = ((a_3 - 1)/\beta) - b_3$, $\rho_4 = ((a_4 - 1)/\lambda) - b_4$, and $\sigma_{ij}, i = 1,2,3,4$ terms are obtained by using L_{ij}, $i,j = 1,2,3,4$ where $L_{11} = -n_1 m/\alpha_1^2$, $L_{22} = -n_2 m/\alpha_2^2$, $L_{33} = -m/\beta^2$, $L_{14} = L_{41} = -J_{14}$, $L_{24} = L_{42} = -J_{24}$, $L_{34} = L_{43} = -J_{34}$, $L_{44} = -J_{44}$.

Moreover, $L_{111} = 2n_1 m/\alpha_1^3$, $L_{222} = 2n_2 m/\alpha_2^3$, $L_{333} = 2m/\beta^3$,

$$L_{144} = L_{441} = L_{414} = \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{\overline{F}_{0\lambda\lambda}(y_{ij_1}^{(1)};\lambda)\overline{F}_0(y_{ij_1}^{(1)};\lambda) - (\overline{F}_{0\lambda}(y_{ij_1}^{(1)};\lambda))^2}{(\overline{F}_0(y_{ij_1}^{(1)};\lambda))^2}, \quad (4.30)$$

$$L_{244} = L_{442} = L_{424} = \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{\overline{F}_{0\lambda\lambda}(y_{ij_2}^{(2)};\lambda)\overline{F}_0(y_{ij_2}^{(2)};\lambda) - (\overline{F}_{0\lambda}(y_{ij_2}^{(2)};\lambda))^2}{(\overline{F}_0(y_{ij_2}^{(2)};\lambda))^2}, \quad (4.31)$$

$$L_{344} = L_{443} = L_{434} = \sum_{i=1}^{m} \frac{\overline{F}_{0\lambda\lambda}(x_i;\lambda)(\overline{F}_0(x_i;\lambda)) - (\overline{F}_{0\lambda}(x_i;\lambda))^2}{(\overline{F}_0(x_i;\lambda))^2}, \quad (4.32)$$

$$\begin{aligned} L_{444} = &\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} C_1(y_{ij_1}^{(1)};\lambda) + \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} C_1(y_{ij_2}^{(2)};\lambda) + \sum_{i=1}^{m} C_1(x_i;\lambda) \\ &+ (\alpha_1 - 1)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} C(y_{ij_1}^{(1)};\lambda) + (\alpha_2 - 1)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} C(y_{ij_2}^{(2)};\lambda) \\ &+ (\beta - 1)\sum_{i=1}^{m} C(x_i;\lambda), \end{aligned} \quad (4.33)$$

where

$$C(x;\lambda) = \frac{\overline{F}_{0\lambda\lambda\lambda}(x;\lambda)(\overline{F}_0(x;\lambda))^2 - 3\overline{F}_0(x;\lambda)\overline{F}_{0\lambda}(x;\lambda)\overline{F}_{0\lambda\lambda}(x;\lambda) + 2(\overline{F}_{0\lambda}(x;\lambda))^3}{(\overline{F}_0(x;\lambda))^3}, \quad (4.34)$$

$$C_1(x;\lambda) = \frac{f_{0\lambda\lambda\lambda}(x;\lambda)(f_0(x;\lambda))^2 - 3f_0(x;\lambda)f_{0\lambda}(x;\lambda)f_{0\lambda\lambda}(x;\lambda) + 2(f_{0\lambda}(x;\lambda))^3}{(f_0(x;\lambda))^3}. \quad (4.35)$$

Furthermore, the derivatives of $u = R_{n,k}$ in Equation (4.1) with respect to λ are zero, and others are not given here. Hence, the constants in $\hat{R}^{Lin}_{n,k,B}$ Equation (4.23) are obtained as

$$A = \sigma_{11}L_{111} + \sigma_{44}L_{441}, \tag{4.36}$$

$$B = \sigma_{22}L_{222} + \sigma_{44}L_{442}, \tag{4.37}$$

$$C = \sigma_{33}L_{333} + \sigma_{44}L_{443}, \tag{4.38}$$

$$D = 2\sigma_{14}L_{144} + 2\sigma_{24}L_{244} + 2\sigma_{34}L_{344} + \sigma_{44}L_{444}. \tag{4.39}$$

4.1.3.2 MCMC Method

The marginal posterior density functions of α_1, α_2, β and λ are expressed as

$$\alpha_1|\lambda, \underline{x}, \underline{y_1}, \underline{y_2} \sim Gamma(n_1 m + a_1, b_1 + t_\lambda^{(1)}), \tag{4.40}$$

$$\alpha_2|\lambda, \underline{x}, \underline{y_1}, \underline{y_2} \sim Gamma(n_2 m + a_2, b_2 + t_\lambda^{(2)}), \tag{4.41}$$

$$\beta|\lambda, \underline{x}, \underline{y_1}, \underline{y_2} \sim Gamma(m + a_3, b_3 + t_\lambda), \tag{4.42}$$

and

$$\pi(\lambda|\alpha_1, \alpha_2, \beta, \underline{x}, \underline{y_1}, \underline{y_2}) \propto \lambda^{a_4-1} e^{-\alpha_1 t_\lambda^{(1)} - \alpha_2 t_\lambda^{(2)} - \beta t_\lambda - \lambda b_4} e^{t_\lambda^{(1)} - t_\lambda^{(1)*} + t_\lambda^{(2)} - t_\lambda^{(2)*} + t_\lambda - t_\lambda^*}. \tag{4.43}$$

Obviously, it can be generated samples for or α_1, α_2 and β with gamma distributions but the marginal posterior distribution of λ has not a well-known distribution. So, we are not be able to use standard methods to generate sample. In that case, if the posterior density function is log-concave and roughly symmetric, a normal distribution can be used to approximate it. In our case, the marginal posterior density of λ is log-concave function that is

$(\partial^2 ln(\pi(\lambda|\alpha_1, \alpha_2, \beta, \underline{x}, \underline{y})/\partial\lambda^2 \leq 0)$ when the baseline pdf f_0 and cdf F_0 are log-concave functions, and α_1, α_2, β, $a_4 \geq 1$. Hence, by using the M-H algorithm with the normal proposal distribution, a random sample can be generated from the marginal posterior density of λ and Tierney [52] suggested the hybrid M-H and Gibbs sampling algorithm. In our problem, we use this algorithm as follows:

Step 1: Start with initial guess $\lambda^{(0)}$.

Step 2: Set $i = 1$.

Step 3: Generate $\alpha_1^{(i)}$ from $Gamma(n_1 m + a_1, b_1 + t_\lambda^{(1)})$.

Step 4: Generate $\alpha_2^{(i)}$ from $Gamma(n_2 m + a_2, b_2 + t_\lambda^{(2)})$.

Step 5: Generate $\beta^{(i)}$ from $Gamma(m + a_3, b_3 + t_\lambda)$.

Step 6: Generate $\lambda^{(i)}$ from $\pi(\lambda|\alpha_1, \alpha_2, \beta, \underline{x}, \underline{y_1}, \underline{y_2})$ using the Metropolis-Hastings with the proposal distribution $q(\lambda) \equiv N(\lambda^{(i-1)}, 1)$.

- Let $v = \lambda^{(i-1)}$
- Generate w from the proposal distribution q.
- Let $p(v, w) = \min\left\{1, \dfrac{\pi(w|\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}, \underline{x}, \underline{y_1}, \underline{y_2})q(v)}{\pi(v|\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}, \underline{x}, \underline{y_1}, \underline{y_2})q(w)}\right\}$
- Generate u from $U(0, 1)$,
  - If $u \leq p(v, w)$, then accept the proposal and set $\lambda^{(i)} = w$;
  - Otherwise, set $\lambda^{(i)} = v$.

Step 7: Compute the $R_{n,k}^{(i)}$ at $(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)}, \lambda^{(i)})$.

Step 8: Set $i = i + 1$.

Step 9: Repeat Steps 2 through 8, N times and posterior sample $R_{n,k}^{(i)}$, $i = 1, \ldots, N$.

Then, using this sample, Bayes estimate of $R_{n,k}$ under a SE loss function is

$$\hat{R}_{n,k,B}^{MC} = \frac{1}{N - M} \sum_{i=M+1}^{N-M} R_{n,k}^{(i)}, \tag{4.44}$$

where M is the burn-in period. The HPD $100(1-\gamma)\%$ credible interval of $R_{n,k}$ is constructed by Chen and Shao's technique [53].

4.2. ESTIMATION OF $R_{n,k}$ WHEN THE SECOND PARAMETERS ARE DIFFERENT AND KNOWN

In this section, the estimation methods of the stress-strength reliability of a consecutive k-out-of-n:G system with non-identical strengths is examined when the second parameters λ_1, λ_2 and λ_3 of underlying distributions are different and known.

4.2.1 MLE of $R_{n,k}$

Let n strength variables such that first n_1 ones $Y_{j_1}^{(1)} \sim PHR(\alpha_1, \lambda_1)$, $j_1 = 1, ..., n_1$ the remaining n_2 ones $Y_{j_2}^{(2)} \sim PHR(\alpha_2, \lambda_2)$, $j_2 = n_1+1, .., n$, stress variable $X \sim PHR(\beta, \lambda_3)$ and $(\lambda_1, \lambda_2, \lambda_3)$ are known constants. Then, the stress-strength reliability $R_{n,k}$ is computed by using the same Equations in (4.1) and (4.4). The likelihood function of the observed sample for this case is given by

$$L(\alpha_1, \alpha_2, \beta; \lambda_1, \lambda_2, \lambda_3, \underline{x}, \underline{y_1}, \underline{y_2}) \propto \alpha_1^{n_1 m} \alpha_2^{n_2 m} \beta^m \exp\left[-(\alpha_1 - 1)\, t^{(1)} - (\alpha_2 - 1)\, t^{(2)} - (\beta - 1)t\right], \quad (4.45)$$

and the log-likelihood function is

$$l(\alpha_1, \alpha_2, \beta; \lambda_1, \lambda_2, \lambda_3, \underline{x}, \underline{y_1}, \underline{y_2}) \propto n_1 m \ln \alpha_1 + n_2 m \ln \alpha_2 + m \ln \beta - (\alpha_1 - 1) \\ t^{(1)} - (\alpha_2 - 1)\, t^{(2)} - (\beta - 1)t, \quad (4.46)$$

where

$$t^{(1)} \equiv t^{(1)}_{\lambda_1} = -\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \ln \overline{F}_0(y^{(1)}_{ij_1};\lambda_1)\ ,\ t^{(2)} \equiv t^{(2)}_{\lambda_2} = -\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \ln \overline{F}_0(y^{(2)}_{ij_2};\lambda_2) \quad (4.47)$$

$$\text{and } t \equiv t_{\lambda_3} = -\sum_{i=1}^{m} \ln \overline{F}_0(x_i;\lambda_3). \quad (4.48)$$

The ML estimates of α_1, α_2 and β are $\hat{\alpha}_1 = n_1 m/T^{(1)}$, $\hat{\alpha}_2 = n_2 m/T^{(2)}$ and $\hat{\beta} = m/T$. Hence, the invariance property can be applied in Equations (4.1) and (4.4) the MLE of $R_{n,k}$, $\hat{R}^{MLE}_{n,k}$.

An asymptotic confidence interval of $R_{n,k}$ is obtained by using the Fisher information matrix. In this case, $\theta = (\alpha_1, \alpha_2, \beta)$ and the elements of the Fisher information matrix are $I_{11} = n_1 m/\alpha_1^2$, $I_{22} = n_2 m/\alpha_2^2$, $I_{33} = m/\beta^2$, $I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{31} = 0$. Hence, $\hat{R}^{MLE}_{n,k}$ is asymptotically normal with mean $R_{n,k}$ and asymptotic variance (see Rao [51])

$$\sigma^2_{R_{n,k}} = \left(\frac{\partial R_{n,k}}{\partial \alpha_1}\right)^2 \frac{\alpha_1^2}{n_1 m} + \left(\frac{\partial R_{n,k}}{\partial \alpha_2}\right)^2 \frac{\alpha_2^2}{n_2 m} + \left(\frac{\partial R_{n,k}}{\partial \beta}\right)^2 \frac{\beta^2}{m}. \quad (4.49)$$

Thus, $R_{n,k} \in (\hat{R}^{MLE}_{n,k} \pm z_{\gamma/2}\hat{\sigma}_{R_{n,k}})$ introduces the the asymptotic $100(1-\gamma)\%$ confidence interval of $R_{n,k}$ where $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{\sigma}_{R_{n,k}}$ is the value at MLE of parameters.

4.2.2 UMVUE of $R_{n,k}$

In our assumptions, the UMVUE of $R_{n,k}$, say $\hat{R}^{U}_{n,k}$, is derived by using the linear property of UMVUE. Since the stress-strength reliability $R_{n,k}$ in Equation (4.1) is a linear combination of $g(a,b,c,d)$ in Equation (4.4), it is needed to find the UMVUE of

$$\psi(\alpha_1,\alpha_2,\beta) = \frac{\beta}{a_1\alpha_1 + a_2\alpha_2 + \beta}, \tag{4.50}$$

$a_1, a_2 \in \mathbb{R}$ for $\hat{R}^U_{n,k}$.

$\mathbf{T}^* = (T, T^{(1)}, T^{(2)})$ is a complete sufficient statistics for $(\alpha_1,\alpha_2,\beta)$ from the likelihood function in Equation (4.45). $T, T^{(1)}, T^{(2)}$ statistics have gamma distributions with parameters (m,β), $(n_1 m, \alpha_1)$ and $(n_2 m, \alpha_2)$, respectively. Let

$$\phi(T_1, T_1^{(1)}, T_1^{(2)}) = \begin{cases} 1 & , \ T_1^{(1)} > a_1 T_1 \text{ and } T_1^{(2)} > a_2 T_1 \\ 0 & , \ \text{otherwise} \end{cases}, \tag{4.51}$$

where $T_1^{(1)} = -\ln \overline{F}_0(Y_{11}^{(1)})$, $T_1^{(2)} = -\ln \overline{F}_0(Y_{11}^{(2)})$ and $T_1 = -\ln \overline{F}_0(X_1)$. It is easy to see that $T_1, T_1^{(1)}$ and $T_1^{(2)}$ have exponential distributions with means $1/\beta$, $1/\alpha_1$ and $1/\alpha_2$, respectively. Hence, the statistic $\phi(T_1, T_1^{(1)}, T_1^{(2)})$ is an unbiased estimate of $\psi(\alpha_1,\alpha_2,\beta)$. Based on the Rao-Blackwell Theorem and Lehmann-Scheffé Theorem, the UMVUE of $\psi(\alpha_1,\alpha_2,\beta)$ is given by

$$\begin{aligned} \hat{\psi}_U(\alpha_1,\alpha_2,\beta) &= E(\phi(T_1, T_1^{(1)}, T_1^{(2)}|\mathbf{T}^*) \\ &= P(T_1^{(1)} > a_1 T_1, T_1^{(2)} > a_2 T_1 | \mathbf{T}^*). \end{aligned} \tag{4.52}$$

Let $U^{(1)} = T_1^{(1)}/T^{(1)}$, $U^{(2)} = T_1^{(2)}/T^{(2)}$, $U = T_1/T$ and $V_1 = T/T^{(1)}$, $V_2 = T/T^{(2)}$. Then, we have

$$\hat{\psi}_U(\alpha_1,\alpha_2,\beta) = P(U^{(1)} > a_1 V_1 U, U^{(2)} > a_2 V_2 U | \mathbf{T}^*). \tag{4.53}$$

It can be easily obtained that $U^{(1)}$, $U^{(2)}$, U, V_1 and V_2 have beta distributions. Therefore,

$U^{(1)}$, $U^{(2)}$ and U are ancillary statistics, and independent of $\mathbf{T}^*$ using Basu's Theorem. Then, we obtain

$$f_{U,U^{(1)},U^{(2)}}(u,u^{(1)},u^{(2)}|\mathbf{T}^*=\mathbf{t}^*) = (m-1)(n_1m-1)(n_2m-1)(1-u)^{m-2} \\ (1-u_1)^{n_1m-1}(1-u_2)^{n_2m-1}, \tag{4.54}$$

$0<u<1, 0<u_1<1, 0<u_2<1$. Hence, we have

$$\hat{\psi}_U(\alpha_1,\alpha_2,\beta) = \begin{cases} I(1) & \text{, if } a_1V_1 \le 1 \text{ and } a_2V_2 \le 1 \\ I(1/a_2V_2) & \text{, if } a_1V_1 \le 1 \text{ and } a_2V_2 > 1 \\ I(1/a_1V_1) & \text{, if } a_1V_1 > 1 \text{ and } a_2V_2 \le 1 \\ I\left(\min\{1/a_1V_1, 1/a_2V_2\}\right) & \text{, if } a_1V_1 > 1 \text{ and } a_2V_2 > 1 \end{cases}, \tag{4.55}$$

where

$$I(r) = \int_0^r (m-1)(1-a_1V_1u)^{n_1m-1}(1-a_2V_2u)^{n_2m-1}(1-u)^{m-2}du. \tag{4.56}$$

Thus, $\hat{R}^U_{n,k}$ is obtained by using (4.55) in Equations (4.1) and (4.4). Moreover, integrals in $\hat{\psi}_U(\alpha_1,\alpha_2,\beta)$ can be computed analytically as follows:

$$I(1) = \sum_{c=0}^{n_1m-1}\sum_{d=0}^{n_2m-1}(-1)^{c+d}(a_1V_1)^c(a_2V_2)^d\frac{\binom{n_1m-1}{c}\binom{n_2m-1}{d}}{\binom{c+d+m-1}{c+d}}, \tag{4.57}$$

$$I(1/a_1V_1) = \frac{m-1}{n_1m}\sum_{d=0}^{n_2m-1}\sum_{e=0}^{m-2}(-1)^{d+e}\frac{(a_2V_2)^d}{(a_1V_1)^{d+e+1}}\frac{\binom{n_2m-1}{d}\binom{m-2}{e}}{\binom{d+e+n_1m}{d+e}} \tag{4.58}$$

$$I(1/a_2V_2) = \frac{m-1}{n_2 m} \sum_{c=0}^{n_1 m-1} \sum_{e=0}^{m-2} (-1)^{c+e} \frac{(a_1V_1)^c}{(a_2V_2)^{c+e+1}} \frac{\binom{n_1m-1}{c}\binom{m-2}{e}}{\binom{c+e+n_2m}{c+e}} \tag{4.59}$$

$$\begin{aligned} I\left(\min\left\{1/a_1V_1, 1/a_2V_2\right\}\right) = (m-1) \sum_{c=0}^{n_1m-1} \sum_{d=0}^{n_2m-1} \sum_{e=0}^{m-2} (-1)^{c+d+e} \binom{n_1m-1}{c} \binom{n_2m-1}{d} \binom{m-2}{e} \\ (a_1V_1)^c (a_2V_2)^d \frac{\left(\min\left\{1/a_1V_1, 1/a_2V_2\right\}\right)^{c+d+e+1}}{c+d+e+1} \end{aligned} \tag{4.60}$$

4.2.3 Bayes Estimation of $R_{n,k}$

In this Bayesian section, exact and approximate Bayes estimates of $R_{n,k}$ are obtained when the parameters α_1, α_2 and β follow statistically independent gamma distributions as prior with parameters (a_i, b_i), $i = 1, 2, 3$, respectively. The joint posterior density function of α_1, α_2 and β is given by

$$\begin{aligned} \pi(\alpha_1, \alpha_2, \beta \left| \underline{x}, \underline{y_1}, \underline{y_2} \right.) = & \frac{(b_1 + t^{(1)})^{n_1 m + a_1} (b_2 + t^{(2)})^{n_2 m + a_2} (b_3 + t)^{m + a3}}{\Gamma(n_1 m + a_1)\Gamma(n_2 m + a_2)\Gamma(m + a_3)} \alpha_1^{n_1 m + a_1 - 1} \\ & \alpha_2^{n_2 m + a_2 - 1} \beta^{m + a_3 - 1} e^{-\alpha_1 (b_1 + t^{(1)}) - \alpha_2 (b_2 + t^{(2)}) - \beta(b_3 + t)} \end{aligned} \tag{4.61}$$

The marginal posterior densities of α_1, α_2 and β have gamma distributions with parameters $(n_1m + a_1, b_1 + t^{(1)})$, $(n_2m + a_2, b_2 + t^{(2)})$ and $(m + a_3, b_3 + t)$, respectively. In this case, we can obtain the exact Bayes estimate of $R_{n,k}$. The Bayes estimate of $R_{n,k}$, say $\widehat{R}_{n,k,B}$, is the mean of the joint posterior density function in Equation (4.61) under the SE loss function. Since the stress-strength reliability $R_{n,k}$ is evaluated by using $g(a, b, c, d)$ function in Equations (4.1) and (4.4), we have

$$\widehat{R}_{n,k,B} = \int_0^\infty \int_0^\infty \int_0^\infty R_{n,k}\, \pi(\alpha_1, \alpha_2, \beta \left| \underline{x}, \underline{y_1}, \underline{y_2} \right.) \, d\alpha_1 d\alpha_2 d\beta \tag{4.62}$$

$$= \sum_{j=k}^{n} \int_0^\infty \int_0^\infty \int_0^\infty p(j,k)\, \pi(\alpha_1, \alpha_2, \beta \left| \underline{x}, \underline{y_1}, \underline{y_2} \right.) \, d\alpha_1 d\alpha_2 d\beta. \tag{4.63}$$

Then, $\widehat{R}_{n,k,B}$ is evaluated by using the following integral $Q(a,b,c,d)$ due to the function $p(j,k)$ depends on $g(a,b,c,d)$

$$\begin{aligned} Q(a,b,c,d) &= \int_0^\infty \int_0^\infty \int_0^\infty g(a,b,c,d)\, \pi(\alpha_1,\alpha_2,\beta \left| \underline{x},\underline{y_1},\underline{y_2}\right)\, d\alpha_1 d\alpha_2 d\beta \\ &= \sum_{i=0}^{c}\sum_{j=0}^{d}(-1)^{i+j}\binom{c}{i}\binom{d}{j}\int_0^\infty \int_0^\infty \int_0^\infty \frac{\beta\, \pi(\alpha_1,\alpha_2,\beta \left| \underline{x},\underline{y_1},\underline{y_2}\right)}{\alpha_1(a+i)+\alpha_2(b+j)+\beta} d\alpha_1 d\alpha_2 d\beta. \end{aligned} \tag{4.64}$$

We need to use different transformations based on the values of a and b in analytic derivation of $Q(a,b,c,d)$ integral. In this case, the function $p(j,k)$ in Equation (4.2) includes seven different $g(a,b,c,d)$ function. Using the values of a and b in these cases, the integrand in (4.64) will be one of the following three forms.

$Case\ (i) : a = 0, b \neq 0$

$$\begin{aligned} Q_1 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\beta\, \pi(\alpha_1,\alpha_2,\beta \left| \underline{x},\underline{y_1},\underline{y_2}\right)}{\alpha_2 b^* + \beta} d\alpha_1 d\alpha_2 d\beta \\ &= \frac{(1-z_1)^{r_3}}{B(r_2,r_3)}\int_0^1 u_1^{r_3}(1-u_1)^{r_2-1}(1-u_1 z_1)^{-(r_2+r_3)} du_1 \end{aligned} \tag{4.65}$$

where $b^* = b + j$, $r_2 = n_2 m + a_2$, $r_3 = m + a_3$, $z_1 = 1 - \left\{(b_3+t)b^*/\left(b_2 + t^{(2)}\right)\right\}$ and $B(.,.)$ is the beta function.

$Case\ (ii) : a \neq 0, b = 0$

$$\begin{aligned} Q_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\beta\, \pi(\alpha_1,\alpha_2,\beta \left| \underline{x},\underline{y_1},\underline{y_2}\right)}{\alpha_1 a^* + \beta} d\alpha_1 d\alpha_2 d\beta \\ &= \frac{(1-z_2)^{r_3}}{B(r_1,r_3)}\int_0^1 u_1^{r_3}(1-u_1)^{r_1-1}(1-u_1 z_2)^{-(r_1+r_3)} du_1 \end{aligned} \tag{4.66}$$

where $a^* = a + i$, $r_1 = n_1 m + a_1$ and $z_2 = 1 - \left\{(b_3+t)a^*/\left(b_1 + t^{(1)}\right)\right\}$.

$Case\ (iii) : a \neq 0, b \neq 0$

$$
\begin{aligned}
Q_3 &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\beta\, \pi(\alpha_1, \alpha_2, \beta \,|\underline{x}, \underline{y_1}, \underline{y_2})}{\alpha_1 a^* + \alpha_2 b^* + \beta} d\alpha_1 d\alpha_2 d\beta \\
&= \frac{\Gamma(r_1 + r_2 + r_3)(1 - v_1)^{r_1}(1 - v_2)^{r_2}}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)} \\
&\int_0^1 \int_0^{1-u_2} u_1^{r_1 - 1} u_2^{r_2 - 1}(1 - u_1 - u_2)^{r_3}(1 - v_1 u_1 - v_2 u_2)^{-(r_1 + r_2 + r_3)} du_1 du_2
\end{aligned}
\tag{4.67}
$$

where $v_1 = 1 - \left\{b_1 + t^{(1)}/(b_3 + t)a^*\right\}$ and $v_2 = 1 - \left\{b_2 + t^{(2)}/(b_3 + t)b^*\right\}$. Therefore, $\widehat{R}_{n,k,B}$ is computed analytically by using Q_1, Q_2 and Q_3 integrals in Equation (4.63).

In derivation process of Q_1, Q_2 and Q_3 integrals, one-one-one transformation is used. How this transformation is applied is given in detail as follows:

For Q_1: Let we define a one-to-one transformation $u_1 = \beta/(\alpha_2 b^* + \beta)$ and $u_2 = \alpha_2 b^* + \beta$ for $b^* \neq 0$. Then, $0 < u_1 < 1$, $0 < u_2 < \infty$ and the Jacobian of (u_1, u_2) is $J(u_1, u_2) = u_2/b^*$. Hence, we have

$$
\begin{aligned}
Q_1 &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\beta\, \pi(\alpha_1, \alpha_2, \beta \,|\underline{x}, \underline{y_1}, \underline{y_2})}{\alpha_2 b^* + \beta} d\alpha_1 d\alpha_2 d\beta \\
&= \frac{(b_2 + t^{(2)})^{r_2}(b_3 + t)^{r_3}}{\Gamma(r_2)\Gamma(r_3)\,(b^*)^{r_2}} \int_0^1 {}_0^{\infty} u_1^{r_3}(1 - u_1)^{r_2 - 1} u^{r_2 + r_3 - 1} \\
&\exp\left\{-u_2\left[\frac{(1 - u_1)(b_2 + t^{(2)})}{b^*} + u_1(b_3 + t)\right]\right\} du_2 du_1 \\
&= \frac{(1 - z_1)^{r_3}}{B(r_2, r_3)} \int_0^1 u_1^{r_3}(1 - u_1)^{r_2 - 1}(1 - u_1 z_1)^{-(r_2 + r_3)} du_1
\end{aligned}
\tag{4.68}
$$

where $r_2 = n_2 m + a_2$, $r_3 = m + a_3$ and $z_1 = 1 - \left\{(b_3 + t)b^*/\left(b_2 + t^{(2)}\right)\right\}$.

For Q_2: Similar to Q_1 integral, when we apply the transformation $u_1 = \beta/(\alpha_1 a^* + \beta)$ and

$u_2 = \alpha_1 a^* + \beta$ for $a^* \neq 0$, we obtain

$$\begin{aligned} Q_2 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\beta\, \pi(\alpha_1, \alpha_2, \beta \,|\underline{x}, \underline{y_1}, \underline{y_2})}{\alpha_1 a^* + \beta} d\alpha_1 d\alpha_2 d\beta \\ &= \frac{(1-z_2)^{r_3}}{B(r_1, r_3)} \int_0^1 u_1^{r_3}(1-u_1)^{r_1-1}(1-u_1 z_2)^{-(r_1+r_3)} du_1 \end{aligned} \tag{4.69}$$

where $r_1 = n_1 m + a_1$ and $z_2 = 1 - \left\{(b_3 + t)a^* / \left(b_1 + t^{(1)}\right)\right\}$.

For Q_3: Using the following one-to-one transformation

$$u_1 = \frac{\alpha_1 a^*}{\alpha_1 a^* + \alpha_2 b^* + \beta},\ u_2 = \frac{\alpha_2 b^*}{\alpha_1 a^* + \alpha_2 b^* + \beta},\ u_3 = \alpha_1 a^* + \alpha_2 b^* + \beta, \tag{4.70}$$

for $a^* \neq 0$ and $b^* \neq 0$, we have $0 < u_1 + u_2 < 1$, $0 < u_3 < \infty$ and the Jacobian of (u_1, u_2, u_3) is $J(u_1, u_2, u_3) = u_3^2/(a^* b^*)$. Then,

$$\begin{aligned} Q_3 &= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\beta\, \pi(\alpha_1, \alpha_2, \beta \,|\underline{x}, \underline{y_1}, \underline{y_2})}{\alpha_1 a^* + \alpha_2 b^* + \beta} d\alpha_1 d\alpha_2 d\beta \\ &= \frac{(b_1 + t^{(1)})^{r_1}(b_2 + t^{(2)})^{r_2}(b_3 + t)^{r_3}}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)(a^*)^{r_1}(b^*)^{r_2}} \int_0^1 \int_0^{1-u_2} \int_0^\infty u_1^{r_1-1} u_2^{r_2-1} u_3^{r_1+r_2+r_3-1}(1-u_1-u_2)^{r_3} \\ &\exp\left(-u_3\left(\frac{u_1(b_1+t^{(1)})}{a+i} + \frac{u_2(b_2+t^{(2)})}{b+j} + (1-u_1-u_2)(b_3+t)\right)\right) du_3 du_1 du_2 \\ &= \frac{\Gamma(r_1+r_2+r_3)(b_1 + t^{(1)})^{r_1}(b_2 + t^{(2)})^{r_2}(b_3 + t)^{r_3}}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)(a^*)^{r_1}(b^*)^{r_2}} \\ &\left\{\int_0^1 \int_0^{1-u_2} \frac{u_1^{r_1-1} u_2^{r_2-1}(1-u_1-u_2)^{r_3}}{\left(\frac{u_1(b_1+t^{(1)})}{a+i} + \frac{u_2(b_2+t^{(2)})}{b+j} + (1-u_1-u_2)(b_3+t)\right)^{(r_1+r_2+r_3)}} du_1 du_2\right\} \\ &= \frac{\Gamma(r_1+r_2+r_3)(1-v_1)^{r_1}(1-v_2)^{r_2}}{\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)} \\ &\int_0^1 \int_0^{1-u_2} u_1^{r_1-1} u_2^{r_2-1}(1-u_1-u_2)^{r_3}(1-v_1u_1-v_2u_2)^{-(r_1+r_2+r_3)} du_1 du_2 \end{aligned} \tag{4.71}$$

where $v_1 = 1 - \left\{b_1 + t^{(1)}/(b_3+t)a^*\right\}$ and $v_2 = 1 - \left\{b_2 + t^{(2)}/(b_3+t)b^*\right\}$.

Moreover, Q_1 and Q_2 integrals can be written by using the hypergeometric function. The integral representation of the hypergeometric function is

$$ {}_2F_1(\alpha,\beta;\gamma,z) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt, \tag{4.72} $$

when $|z| < 1$, $Re(\gamma) > 0$ and $Re(\beta) > 0$ (see Equation 9.111 in Gradshteyn and Ryzhik [54]). Then, Q_1 and Q_2 can be also given as

$$ Q_1 = \begin{cases} \dfrac{r_3(1-z_1)^{r_3}}{r_2+r_3} \, {}_2F_1\left(r_2+r_3, r_3+1; r_2+r_3+1, z_1\right), & , \ |z_1| < 1 \\ \dfrac{r_3}{(r_2+r_3)(1-z_1)^{r_2}} \, {}_2F_1\left(r_2+r_3, r_2; r_2+r_3+1, \dfrac{z_1}{z_1-1}\right) & , \ z_1 < -1 \end{cases}, \tag{4.73} $$

and

$$ Q_2 = \begin{cases} \dfrac{r_3(1-z_2)^{r_3}}{r_1+r_3} \, {}_2F_1\left(r_1+r_3, r_3+1; r_1+r_3+1, z_2\right), & , \ |z_2| < 1 \\ \dfrac{r_3}{(r_1+r_3)(1-z_2)^{r_1}} \, {}_2F_1\left(r_1+r_3, r_1; r_1+r_3+1, \dfrac{z_2}{z_2-1}\right) & , \ z_2 < -1 \end{cases}. \tag{4.74} $$

Furthermore, Q_3 integral can be also written by using the hypergeometric functions of two variables following the idea of Rasethuntsa and Nadar [36]. The first kind double integrals of the Euler Type is given by

$$ F_1(\alpha,\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \int_0^1 \int_0^{1-v} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha} du dv, \tag{4.75} $$

when $Re\left(\beta\right) > 0$, $Re\left(\beta^{'}\right) > 0$, $Re(\gamma - \beta - \beta^{'}) > 0$, $|x| < 1$, $|y| < 1$ (see Equation 9.184(1) in Gradshteyn and Ryzhik [54]). Then, Q_3 can be given as

$$Q_3 = \frac{r_3}{r_1 + r_2 + r_3} \begin{cases} (1-v_1)^{r_1}(1-v_2)^{r_2} F_1\left(\sum_{i=1}^{3} r_i, r_1, r_2, 1 + \sum_{i=1}^{3} r_i; v_1, v_2\right) & \text{, if } |v_1| < 1, |v_2| < 1 \\ (1-v_2)^{r_2 - r_1} F_1\left(1, r_1, r_2, 1 + \sum_{i=1}^{3} r_i; \dfrac{v_1}{v_1 - 1}, \dfrac{v_2}{v_2 - 1}\right) & \text{, if } v_1 < -1, v_2 < -1 \\ (1-v_1) F_1\left(1, r_3 + 1, r_2, 1 + \sum_{i=1}^{3} r_i; v_1, \dfrac{v_1 - v_2}{1 - v_2}\right) & \text{, if } |v_1| < 1, v_2 < -1 \\ (1-v_2) F_1\left(1, r_1, r_3 + 1, 1 + \sum_{i=1}^{3} r_i; \dfrac{v_2 - v_1}{1 - v_1}, v_2\right) & \text{, if } v_1 < -1, |v_2| < 1 \end{cases} . \tag{4.76}$$

4.2.3.1 Lindley's Approximation

In our three parameter case $\theta = (\alpha_1, \alpha_2, \beta)$, Lindley's approximation leads to

$$\begin{aligned} \hat{R}_{n,k,B}^{Lin} = u + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) + \frac{1}{2}[A(u_1\sigma_{11} + u_2\sigma_{12} + u_3\sigma_{13}) \\ + B(u_1\sigma_{21} + u_2\sigma_{22} + u_3\sigma_{23}) + C(u_1\sigma_{31} + u_2\sigma_{32} + u_3\sigma_{33})] \end{aligned} \tag{4.77}$$

where $u(\alpha_1, \alpha_2, \beta) = R_{n,k}$, $u_i = \partial u / \partial \square_i$, $i = 1, 2, 3$, $A = \sigma_{11} L_{111}$, $B = \sigma_{22} L_{222}$, $C = \sigma_{33} L_{333}$, $L_{111} = 2n_1 m / \alpha_1^3$, $L_{222} = 2n_2 m / \alpha_2^3$, $L_{333} = 2m / \beta^3$,

$$a_i = \rho_1 \sigma_{i1} + \rho_2 \sigma_{i2} + \rho_3 \sigma_{i3}, i = 1, 2, 3, \tag{4.78}$$

$$a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},\ a_5 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33}), \tag{4.79}$$

$$\rho_1 = \frac{a_1 - 1}{\alpha_1} - b_1, \rho_2 = \frac{a_2 - 1}{\alpha_2} - b_2, \rho_3 = \frac{a_3 - 1}{\beta} - b_3. \tag{4.80}$$

4.2.3.2 MCMC Method

For this case, since we know the marginal posterior densities of α_1, α_2 and β, Gibbs sampling algorithm is used to compute the Bayes estimate of $R_{n,k}$. The following algorithm is used for the generation of random samples

Step 1: Set $i = 1$.

Step 2: Generate $\alpha_1^{(i)}$ from $Gamma(n_1 m + a_1, b_1 + t^{(1)})$.

Step 3: Generate $\alpha_2^{(i)}$ from $Gamma(n_2 m + a_2, b_2 + t^{(2)})$.

Step 4: Generate $\beta^{(i)}$ from $Gamma(m + a_3, b_3 + t)$.

Step 5: Compute the $R_{n,k}^{(i)}$ at $(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta^{(i)})$

Step 6: Set $i = i + 1$.

Step 7: Repeat Steps 2 through 6, N times and obtain the posterior sample $R_{n,k}^{(i)}$, $i = 1, ..., N$.

Then, the Bayes estimate of $R_{n,k}$ under the SE loss function is computed by

$$\widehat{R}_{n,k,B}^{MC} = \frac{1}{N-M} \sum_{i=M+1}^{N-M} R_{n,k}^{(i)}, \tag{4.81}$$

where M is the burn-in period. The HPD $100(1-\gamma)\%$ credible interval of $R_{n,k}$ is obtained by Chen and Shao's technique [53].

4.3. EXAMPLE

Kumaraswamy and Burr Type XII distributions from the PHR family are considered as examples in the case of λ unknown, and Weibull distribution is considered for λ known case. In this context, the following derivations are obtained for each case.

When λ is unknown, Kumaraswamy distributions with parameters (α_1, λ) and (α_2, λ) are used to generate strength variables, and stress variables are generated from Kumaraswamy distribution with parameters (β, λ). Then, the survival functions are $\overline{F}_0(y_{j_1}^{(1)}; \lambda) = (1 - y_{j_1}^{\lambda})$, $\overline{F}_0(y_{j_2}^{(2)}; \lambda) = (1 - y_{j_2}^{\lambda})$ and $\overline{F}_0(x; \lambda) = (1 - x^{\lambda})$, respectively. So,

$$t_{\lambda}^{(1)} = -\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \ln(1 - y_{ij_1}^{\lambda}),\ t_{\lambda}^{(2)} = -\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \ln(1 - y_{ij_2}^{\lambda}),\ t_{\lambda} = -\sum_{i=1}^{m} \ln(1 - x_i^{\lambda}), \quad (4.82)$$

$$t_{\lambda}^{(1)*} = -\left(n_1 m \ln \lambda + (\lambda - 1) \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \ln y_{ij_1} \right), \quad (4.83)$$

$$t_{\lambda}^{(2)*} = -\left(n_2 m \ln \lambda + (\lambda - 1) \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \ln y_{ij_2} \right), \quad (4.84)$$

and

$$t_{\lambda}^{*} = -\left(m \ln \lambda + (\lambda - 1) \sum_{i=1}^{m} \ln x_i \right). \quad (4.85)$$

The MLE of λ, $\hat{\lambda}$ is obtained from the solution of the nonlinear equation given below

$$\begin{aligned} &\frac{m(n_1 + n_2 + 1)}{\lambda} + \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{\ln y_{ij_1}}{1 - y_{ij_1}^{\lambda}} + \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{\ln y_{ij_2}}{1 - y_{ij_2}^{\lambda}} + \sum_{i=1}^{m} \frac{\ln x_i}{1 - x_i^{\lambda}} - \\ &\frac{n_1 m}{T_{\lambda}^{(1)}} \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{y_{ij_1}^{\lambda} \ln y_{ij_1}}{1 - y_{ij_1}^{\lambda}} - \frac{n_2 m}{T_{\lambda}^{(2)}} \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{y_{ij_2}^{\lambda} \ln y_{ij_2}}{1 - y_{ij_2}^{\lambda}} - \frac{m}{T_{\lambda}} \sum_{i=1}^{m} \frac{x_i^{\lambda} \ln x_i}{1 - x_i^{\lambda}} = 0. \end{aligned} \quad (4.86)$$

Some elements of observed information matrix are derived as

$$J_{14} = J_{41} = \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{y_{ij_1}^{\lambda} \ln y_{ij_1}}{1 - y_{ij_1}^{\lambda}},\ J_{24} = J_{42} = \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{y_{ij_2}^{\lambda} \ln y_{ij_2}}{1 - y_{ij_2}^{\lambda}}, \quad (4.87)$$

$$J_{34} = J_{43} = \sum_{i=1}^{m} \frac{x_i^{\lambda} \ln x_i}{1 - x_i^{\lambda}} \text{ and} \quad (4.88)$$

$$J_{44} = \frac{m(n_1+n_2+1)}{\lambda^2} + (\alpha_1-1)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\frac{y_{ij_1}^{\lambda}(\ln y_{ij_1})^2}{(1-y_{ij_1}^{\lambda})^2} + (\alpha_2-1)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\frac{y_{ij_2}^{\lambda}(\ln y_{ij_2})^2}{(1-y_{ij_2}^{\lambda})^2}$$
$$+ (\beta-1)\sum_{i=1}^{m}\frac{x_i^{\lambda}(\ln x_i)^2}{(1-x_i^{\lambda})^2}. \tag{4.89}$$

Some elements for Bayes estimation of $R_{n,k}$ using Lindley's approximation are obtained as

$$L_{144} = -\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\frac{y_{ij_1}^{\lambda}(\ln y_{ij_1})^2}{(1-y_{ij_1}^{\lambda})^2},\ L_{244} = -\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\frac{y_{ij_2}^{\lambda}(\ln y_{ij_2})^2}{(1-y_{ij_2}^{\lambda})^2}, \tag{4.90}$$

$$L_{344} = -\sum_{i=1}^{m}\frac{x_i^{\lambda}(\ln x_i)^2}{(1-x_i^{\lambda})^2} \text{ and} \tag{4.91}$$

$$L_{444} = \frac{2m(n_1+n_2+1)}{\lambda^3} - (\alpha_1-1)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}C(y_{ij_1};\lambda) - (\alpha_2-1)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}C(y_{ij_2};\lambda)$$
$$- (\beta-1)\sum_{i=1}^{m}C(x_i;\lambda), \tag{4.92}$$

$$C(x;\lambda) = \frac{-x^{\lambda}(\ln x)^3(1+x^{\lambda})}{(1-x^{\lambda})^3}. \tag{4.93}$$

The marginal posterior density of λ is derived as

$$\pi(\lambda|\alpha_1,\alpha_2,\beta,\underline{x},\underline{y_1},\underline{y_2}) \propto \lambda^{m(n_1+n_2+1)+a_4-1}e^{-(\alpha_1-1)t_\lambda^{(1)}}e^{-(\alpha_2-1)t_\lambda^{(2)}}e^{-(\beta-1)t_\lambda}$$
$$\exp\left\{\lambda\left(-b_4+\sum_{i=1}^{m}\ln x_i+\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\ln y_{ij_1}+\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\ln y_{ij_2}\right)\right\}, \tag{4.94}$$

for implementation of MCMC method.

When the strength and stress variables are assumed generating from Burr Type XII distribution with parameters (α_1,λ), (α_2,λ) and (β,λ), then the baseline survival functions are $\overline{F}_0(y_{j_1}^{(1)};\lambda) = 1/\left(1+y_{j_1}^{\lambda}\right)$, $\overline{F}_0(y_{j_2}^{(1)};\lambda) = 1/\left(1+y_{j_2}^{\lambda}\right)$ and $\overline{F}_0(x;\lambda) = 1/\left(1+x^{\lambda}\right)$, respectively. We have

$$t_\lambda^{(1)} = \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \ln(1+y_{ij_1}^\lambda),\ t_\lambda^{(2)} = \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \ln(1+y_{ij_2}^\lambda),\ t_\lambda = \sum_{i=1}^{m} \ln(1+x_i^\lambda), \qquad (4.95)$$

$$t_\lambda^{(1)*} = -\left(n_1 m \ln\lambda + (\lambda-1)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \ln y_{ij_1} - 2\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \ln(1+y_{ij_1}^\lambda) \right), \qquad (4.96)$$

$$t_\lambda^{(2)*} = -\left(n_2 m \ln\lambda + (\lambda-1)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \ln y_{ij_2} - 2\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \ln(1+y_{ij_2}^\lambda) \right), \qquad (4.97)$$

and

$$t_\lambda^{*} = -\left(m \ln\lambda + (\lambda-1)\sum_{i=1}^{m} \ln x_i - 2\sum_{i=1}^{m} \ln(1+x_i^\lambda) \right). \qquad (4.98)$$

Some elements of observed information matrix are derived as

$$J_{14} = J_{41} = \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{y_{ij_1}^\lambda \ln y_{ij_1}}{1+y_{ij_1}^\lambda}, J_{24} = J_{42} = \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{y_{ij_2}^\lambda \ln y_{ij_2}}{1+y_{ij_2}^\lambda}, J_{34} = J_{43} = \sum_{i=1}^{m} \frac{x_i^\lambda \ln x_i}{1+x_i^\lambda}, \qquad (4.99)$$

$$J_{44} = \frac{m(n_1+n_2+1)}{\lambda^2} + (\alpha_1+1)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{y_{ij_1}^\lambda (\ln y_{ij_1})^2}{(1+y_{ij_1}^\lambda)^2} + (\alpha_2+1)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{y_{ij_2}^\lambda (\ln y_{ij_2})^2}{(1+y_{ij_2}^\lambda)^2} + (\beta+1)\sum_{i=1}^{m} \frac{x_i^\lambda (\ln x_i)^2}{(1+x_i^\lambda)^2}. \qquad (4.100)$$

Some elements for Bayes estimation of $R_{n,k}$ using Lindley's approximation are obtained as

$$L_{144} = -\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} \frac{y_{ij_1}^\lambda (\ln y_{ij_1})^2}{(1+y_{ij_1}^\lambda)^2}, L_{244} = -\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} \frac{y_{ij_2}^\lambda (\ln y_{ij_2})^2}{(1+y_{ij_2}^\lambda)^2},\ L_{344} = -\sum_{i=1}^{m} \frac{x_i^\lambda (\ln x_i)^2}{(1+x_i^\lambda)^2}, \qquad (4.101)$$

$$L_{444} = \frac{2m(n_1+n_2+1)}{\lambda^3} - (\alpha_1+1)\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} C(y_{ij_1};\lambda) - (\alpha_2+1)\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} C(y_{ij_2};\lambda) - (\beta+1)\sum_{i=1}^{m} C(x_i;\lambda) \qquad (4.102)$$

where

$$C(x;\lambda) = \frac{-x^{\lambda}(\ln x)^3(1-x^{\lambda})}{(1+x^{\lambda})^3}. \tag{4.103}$$

The marginal posterior density of λ for this case is obtained as

$$\pi(\lambda|\alpha_1,\alpha_2,\beta,\underline{x},\underline{y_1},\underline{y_2}) \propto \lambda^{m(n_1+n_2+1)+a_4-1} e^{-(\alpha_1+1)t_\lambda^{(1)}} e^{-(\alpha_2+1)t_\lambda^{(2)}} e^{-(\beta+1)t_\lambda} \exp\left\{\lambda\left(-b_4+\sum_{i=1}^{m}\ln x_i+\sum_{i=1}^{m}\sum_{j_1=1}^{n_1}\ln y_{ij_1}+\sum_{i=1}^{m}\sum_{j_2=1}^{n_2}\ln y_{ij_2}\right)\right\}, \tag{4.104}$$

When the strength and stress variables are assumed generating from Weibull distributions with parameters (α_1,λ_1), (α_2,λ_2), (β,λ_3) with known common parameters $(\lambda_1,\lambda_2,\lambda_3)$, the baseline survival functions are $\overline{F_0}(y_{j_1}^{(1)};\lambda_1) = e^{-y_{j_1}^{\lambda_1}}$, $\overline{F_0}(y_{j_2}^{(2)};\lambda_2) = e^{-y_{j_2}^{\lambda_2}}$ and $\overline{F_0}(x;\lambda_3) = e^{-x^{\lambda_3}}$. We have

$$t^{(1)} \equiv t_{\lambda_1}^{(1)} = \sum_{i=1}^{m}\sum_{j_1=1}^{n_1} y_{ij_1}^{\lambda_1},\ t^{(2)} \equiv t_{\lambda_2}^{(2)} = \sum_{i=1}^{m}\sum_{j_2=1}^{n_2} y_{ij_2}^{\lambda_2},\ t \equiv t_{\lambda_3} = \sum_{i=1}^{m} x_i^{\lambda_3}, \tag{4.105}$$

and the ML estimates of α_1,α_2 and β are given by

$$\hat{\alpha_1} = \frac{n_1 m}{\sum_{i=1}^{m}\sum_{j_1=1}^{n_1} y_{ij_1}^{\lambda_1}},\ \hat{\alpha_2} = \frac{n_2 m}{\sum_{i=1}^{m}\sum_{j_2=1}^{n_2} y_{ij_2}^{\lambda_2}},\ \hat{\beta} = \frac{m}{\sum_{i=1}^{m} x_i^{\lambda_3}}. \tag{4.106}$$

4.4. SIMULATION STUDY

In this section, a Monte Carlo simulation study is executed for the comparison of the derived estimates of $R_{n,k}$ for both cases: i. the second parameters of underlying distributions are common and unknown; ii. the second parameters of underlying distributions are different and known.

In the point estimation procedure, MSE, ERs and biases are used for the comparisons in ML and Bayes estimates, respectively. When θ is estimated by $\hat{\theta}$, the ER of θ is given by

$$ER(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta_i)^2, \tag{4.107}$$

under the SE loss function. The performances of asymptotic confidence and credible intervals are considered by their ALs and CPs. All simulations study results are obtained by using statistical software R [46] based on 2500 replications. Point and interval estimates of $R_{n,k}$ are given under different scenarios in tables. In the estimates tables, the first, second, and third rows represent the estimate of $R_{n,k}$, its bias, and MSE (or ER) values, respectively.

4.4.1 When the Second Parameters are Common and Unknown

In this subsection, Monte Carlo simulation study is executed for the comparison of ML and Bayes estimates of $R_{n,k}$ when the second parameters of underlying distributions are common (λ) and unknown. Numerical results based on point and interval estimation are presented for Kumaraswamy and Burr Type XII distributions from the PHR family with different sample sizes $m = 10, 20, 30, 40$. The true values of the parameters are taken as $(\alpha_1, \alpha_2, \beta, \lambda) = (0.75, 1.5, 8, 2.5)$ and $(1.5, 3, 15, 5)$, respectively.

Point estimates of $R_{n,k}$ and corresponding interval estimates are presented in Tables 4.1-4.4, for Kumaraswamy and Burr Type XII distributions. The consecutive k-out of-$n : G$ system is considered for the different combinations of (k, n_1, n_2) under the condition of $2k \geq n$. Bayes estimates are obtained by using both Lindley's approximation and MCMC method based on the following informative and non-informative priors: Prior 1: $(a_1, b_1) = (0.75, 1)$, $(a_2, b_2) = (1.5, 1)$, $(a_3, b_3) = (8, 1)$, $(a_4, b_4) = (2.5, 1)$, Prior 2: $(a_1, b_1) = (3, 2)$, $(a_2, b_2) = (6, 2)$, $(a_3, b_3) = (30, 2)$, $(a_4, b_4) = (10, 2)$, and non-informative priors $(a_i, b_i) = (0.0001, 0.0001)$, $i = 1, 2, 3, 4$. In the MCMC case, two MCMC chains are generated, and 10000 iterations for each one. In each chain, we discard the first 5000 results as burn-in period which reduces the effect of the starting distribution. Then, we obtain MCMC

Bayes estimate using every 5^{th} sampled values in chains in thinning procedure.

From Tables 4.1 and 4.3, it is observed that MSE, ERs and biases of all estimates generally decrease when the sample size increases which shows the consistency of estimators. Also, the proposed Bayes estimators based on informative priors have more effective performance on system reliability than ML estimators. In Table 4.1 Bayes estimate using Lindley's approximation outperforms the MCMC results in terms of ERs except for $m = 10$ under informative priors while these estimators show similar performances under non-informative priors. In Table 4.3, Bayes estimate using MCMC method outperform Lindley's approximation except for $m = 40$ under informative priors, while these estimators show similar results under non-informative priors. In general, Bayes estimates based on non-informative priors have similar performance in terms of ER and MSEs when compared to ML estimate results. Also, these estimates and their error values are getting closer to each other as the sample size increases.

From Tables 4.2 and 4.4, the ALs of all intervals decrease with the increase in sample sizes, as expected, and the CPs of all intervals are satisfactory. The HPD credible intervals based on informative priors have the smallest AL while HPD credible intervals based on non-informative priors are similar to the asymptotic confidence intervals. Therefore, the HPD credible intervals may be preferred if the prior information is available or not.

Table 4.1. Estimates of $R_{n,k}$ when $(\alpha_1, \alpha_2, \beta, \lambda) = (0.75, 1.5, 8, 2.5)$ for Kumaraswamy distribution

				Bayes (Prior 1)		*Bayes* (Non-inf. prior)					*Bayes* (Prior 1)		*Bayes* (Non-inf. prior)	
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	(k, n_1, n_2)	$R_{n,k}$	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(3,5,1)	0.93334	10	0.92933	0.89991	0.91870	0.91534	0.91516	(3,1,5)	0.71708	0.71710	0.64456	0.69578	0.70073	0.70181
			-0.00401	-0.03343	-0.01464	-0.01800	-0.01818			0.00002	-0.07252	-0.02130	-0.01635	-0.01526
			0.00158	0.00177	0.00072	0.00217	0.00217			0.00552	0.01022	0.00211	0.00571	0.00582
		20	0.93224	0.92028	0.92544	0.92508	0.92497			0.71615	0.69384	0.70481	0.70780	0.70793
			-0.00110	-0.01306	-0.00790	-0.00826	-0.00837			-0.00093	-0.02324	-0.01227	-0.00928	-0.00914
			0.00076	0.00035	0.00041	0.00090	0.00091			0.00295	0.00125	0.00147	0.00301	0.00301
		30	0.93346	0.92621	0.92849	0.92866	0.92859			0.71923	0.70672	0.71120	0.71362	0.71367
			0.00012	-0.00713	-0.00485	-0.00468	-0.00475			0.00216	-0.01036	-0.00588	-0.00346	-0.00341
			0.00050	0.00025	0.00030	0.00055	0.00055			0.00196	0.00085	0.00110	0.00195	0.00196
		40	0.93368	0.92849	0.92978	0.93006	0.93002			0.72055	0.71169	0.71412	0.71632	0.71594
			0.00034	-0.00485	-0.00356	-0.00328	-0.00332			0.00347	-0.00538	-0.00296	-0.00076	-0.00114
			0.00037	0.00021	0.00024	0.00040	0.00040			0.00142	0.00072	0.00087	0.00141	0.00139

Table 4.1 Continued

				Bayes (Prior 1)		*Bayes* (Non-inf. prior)					*Bayes* (Prior 1)		*Bayes* (Non-inf. prior)	
(k,n_1,n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	(k,n_1,n_2)	$R_{n,k}$	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(3,3,3)	0.90399	10	0.89787	0.86095	0.88557	0.88280	0.88303	(4,1,5)	0.63283	0.63575	0.54837	0.61155	0.62003	0.62184
			-0.00612	-0.04304	-0.01841	-0.02119	-0.02096			0.00292	-0.08446	-0.02127	-0.01280	-0.01099
			0.00261	0.00297	0.00115	0.00325	0.00324			0.00748	0.01477	0.00270	0.00729	0.00762
		20	0.90064	0.88740	0.89338	0.89288	0.89286			0.63481	0.60842	0.62197	0.62680	0.62704
			-0.00335	-0.01659	-0.01061	-0.01111	-0.01113			0.00198	-0.02440	-0.01086	-0.00603	-0.00579
			0.00130	0.00059	0.00070	0.00147	0.00147			0.00393	0.00147	0.00180	0.00386	0.00388
		30	0.90363	0.89528	0.89804	0.89843	0.89844			0.63680	0.62244	0.62788	0.63141	0.63149
			-0.00036	-0.00871	-0.00595	-0.00555	-0.00555			0.00398	-0.01039	-0.00495	-0.00142	-0.00134
			0.00082	0.00040	0.00049	0.00088	0.00088			0.00255	0.00101	0.00137	0.00250	0.00250
		40	0.90509	0.89897	0.90057	0.90119	0.90117			0.63886	0.62861	0.63166	0.63478	0.63424
			0.00110	-0.00502	-0.00342	-0.00280	-0.00282			0.00603	-0.00422	-0.00117	0.00195	0.00142
			0.00064	0.00035	0.00041	0.00066	0.00066			0.00194	0.00092	0.00114	0.00189	0.00187

Table 4.1 Continued

				$Bayes$ (Prior 1)		$Bayes$ (Non-inf. prior)					$Bayes$ (Prior 1)		$Bayes$ (Non-inf. prior)	
(k,n_1,n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	(k,n_1,n_2)	$R_{n,k}$	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(3,3,3)	0.90399	10	0.89787	0.86095	0.88557	0.88280	0.88303	(4,1,5)	0.63283	0.63575	0.54837	0.61155	0.62003	0.62184
			-0.00612	-0.04304	-0.01841	-0.02119	-0.02096			0.00292	-0.08446	-0.02127	-0.01280	-0.01099
			0.00261	0.00297	0.00115	0.00325	0.00324			0.00748	0.01477	0.00270	0.00729	0.00762
		20	0.90064	0.88740	0.89338	0.89288	0.89286			0.63481	0.60842	0.62197	0.62680	0.62704
			-0.00335	-0.01659	-0.01061	-0.01111	-0.01113			0.00198	-0.02440	-0.01086	-0.00603	-0.00579
			0.00130	0.00059	0.00070	0.00147	0.00147			0.00393	0.00147	0.00180	0.00386	0.00388
		30	0.90363	0.89528	0.89804	0.89843	0.89844			0.63680	0.62244	0.62788	0.63141	0.63149
			-0.00036	-0.00871	-0.00595	-0.00555	-0.00555			0.00398	-0.01039	-0.00495	-0.00142	-0.00134
			0.00082	0.00040	0.00049	0.00088	0.00088			0.00255	0.00101	0.00137	0.00250	0.00250
		40	0.90509	0.89897	0.90057	0.90119	0.90117			0.63886	0.62861	0.63166	0.63478	0.63424
			0.00110	-0.00502	-0.00342	-0.00280	-0.00282			0.00603	-0.00422	-0.00117	0.00195	0.00142
			0.00064	0.00035	0.00041	0.00066	0.00066			0.00194	0.00092	0.00114	0.00189	0.00187

Table 4.2. Average confidence/credible lengths (ACL) and coverage probabilities (CP) of $R_{n,k}$ for Kumaraswamy distribution when $(\alpha_1, \alpha_2, \beta, \lambda) = (0.75, 1.5, 8, 2.5)$

			ACI		HPD (Prior 1)		HPD (Non-inf.)	
m	(k, n_1, n_2)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP
10	(3,5,1)	0.93334	0.15389	0.8780	0.11682	0.9932	0.16312	0.9364
20			0.10802	0.9044	0.08963	0.9896	0.11068	0.9388
30			0.08776	0.9164	0.07599	0.9780	0.08876	0.9408
40			0.07610	0.9296	0.06755	0.9728	0.07656	0.9460
10	(3,1,5)	0.71708	0.30300	0.9356	0.22536	0.9920	0.30150	0.9532
20			0.21654	0.9448	0.17743	0.9800	0.21481	0.9444
30			0.17655	0.9380	0.15167	0.9752	0.17488	0.9456
40			0.15282	0.9452	0.13498	0.9800	0.15133	0.9538
10	(3,4,2)	0.92098	0.17370	0.8844	0.13207	0.9924	0.18096	0.9400
20			0.12123	0.9016	0.10063	0.9880	0.12321	0.9368
30			0.09909	0.9200	0.08556	0.9860	0.09974	0.9484
40			0.08550	0.9240	0.07595	0.9716	0.08562	0.9352
10	(3,2,4)	0.80435	0.25890	0.9112	0.19555	0.9884	0.26113	0.9376
20			0.18461	0.9328	0.15254	0.9844	0.18451	0.9416
30			0.15083	0.9428	0.13007	0.9788	0.15027	0.9584
40			0.13075	0.9400	0.11585	0.9696	0.12995	0.9408
10	(3,3,3)	0.90399	0.19860	0.8912	0.14976	0.9924	0.20343	0.9436
20			0.14085	0.9288	0.11608	0.9864	0.14173	0.9448
30			0.11351	0.9268	0.09796	0.9840	0.11356	0.9424
40			0.09758	0.9196	0.08660	0.9672	0.09735	0.9356
10	(4,1,5)	0.63283	0.34103	0.9312	0.25010	0.9836	0.33397	0.9348
20			0.24410	0.9356	0.19819	0.9800	0.24024	0.9388
30			0.19999	0.9416	0.17069	0.9764	0.19694	0.9468
40			0.17320	0.9424	0.15214	0.9724	0.17068	0.9472

Table 4.3. Estimates of $R_{n,k}$ when $(\alpha_1, \alpha_2, \beta, \lambda) = (1.5, 3, 15, 5)$ for Burr Type XII distribution

				Bayes (Prior 2)		*Bayes* (Non-inf. prior)					*Bayes* (Prior 2)		*Bayes* (Non-inf. prior)	
m	(k, n_1, n_2)	$R_{n,k}$	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	(k, n_1, n_2)	$R_{n,k}$	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
10	(4,6,1)	0.85119	0.85376	0.73332	0.84221	0.83804	0.83859	(4,1,6)	0.61748	0.63058	0.53110	0.60924	0.61638	0.61788
			0.00257	-0.11787	-0.00898	-0.01315	-0.01260			0.01311	-0.08638	-0.00824	-0.00109	0.00040
			0.00361	0.04199	0.00059	0.00402	0.00400			0.00770	0.05441	0.00103	0.00723	0.00738
20			0.85248	0.82672	0.84611	0.84433	0.84443			0.62269	0.57544	0.61247	0.61553	0.61590
			0.00128	-0.02447	-0.00508	-0.00686	-0.00676			0.00521	-0.04203	-0.00500	-0.00195	-0.00158
			0.00192	0.00282	0.00040	0.00203	0.00202			0.00405	0.00844	0.00083	0.00393	0.00394
30			0.85351	0.83803	0.84787	0.84802	0.84806			0.62225	0.59812	0.61439	0.61744	0.61757
			0.00232	-0.01316	-0.00332	-0.00317	-0.00313			0.00477	-0.01935	-0.00309	-0.00003	0.00010
			0.00127	0.00048	0.00034	0.00130	0.00130			0.00263	0.00127	0.00072	0.00257	0.00257
40			0.85292	0.84319	0.84855	0.84877	0.84879			0.61909	0.60672	0.61417	0.61549	0.61560
			0.00172	-0.00800	-0.00264	-0.00242	-0.00240			0.00161	-0.01076	-0.00330	-0.00199	-0.00188
			0.00092	0.00016	0.00031	0.00094	0.00094			0.00190	0.00038	0.00065	0.00188	0.00188

Table 4.3 Continued

				$Bayes$ (Prior 2)		$Bayes$ (Non-inf. prior)					$Bayes$ (Prior 2)		$Bayes$ (Non-inf. prior)	
m	(k, n_1, n_2)	$R_{n,k}$	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	(k, n_1, n_2)	$R_{n,k}$	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
10	(4,5,2)	0.84034	0.84319	0.72489	0.83208	0.82732	0.82789	(4,2,5)	0.68692	0.69512	0.59084	0.67804	0.67990	0.68123
			0.00285	-0.11545	-0.00825	-0.01302	-0.01244			0.00820	-0.09608	-0.00888	-0.00702	-0.00569
			0.00389	0.04472	0.00060	0.00425	0.00422			0.00706	0.06095	0.00104	0.00688	0.00698
20			0.84200	0.81424	0.83501	0.83378	0.83390			0.69134	0.64913	0.68180	0.68355	0.68378
			0.00167	-0.02610	-0.00533	-0.00656	-0.00643			0.00442	-0.03779	-0.00512	-0.00337	-0.00314
			0.00194	0.00309	0.00041	0.00204	0.00203			0.00348	0.00652	0.00075	0.00343	0.00343
30			0.84201	0.82677	0.83655	0.83647	0.83654			0.68991	0.66921	0.68316	0.68468	0.68480
			0.00168	-0.01357	-0.00378	-0.00387	-0.00380			0.00299	-0.01771	-0.00376	-0.00224	-0.00212
			0.00134	0.00052	0.00037	0.00138	0.00138			0.00232	0.00106	0.00064	0.00230	0.00230
40			0.84053	0.83206	0.83691	0.83635	0.83637			0.68779	0.67662	0.68330	0.68386	0.68396
			0.00020	-0.00828	-0.00342	-0.00399	-0.00397			0.00087	-0.01030	-0.00362	-0.00306	-0.00296
			0.00107	0.00019	0.00035	0.00110	0.00110			0.00177	0.00036	0.00061	0.00176	0.00176

Table 4.3 Continued

				Bayes (Prior 2)		*Bayes* (Non-inf. prior)					*Bayes* (Prior 2)		*Bayes* (Non-inf. prior)	
m	(k, n_1, n_2)	$R_{n,k}$	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	(k, n_1, n_2)	$R_{n,k}$	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
10	(4,4,3)	0.82540	0.83063	0.71884	0.81684	0.81483	0.81571	(4,3,4)	0.76535	0.77065	0.64755	0.75678	0.75465	0.75553
			0.00523	-0.10656	-0.00856	-0.01057	-0.00969			0.00529	-0.11780	-0.00858	-0.01070	-0.00982
			0.00463	0.05667	0.00068	0.00486	0.00486			0.00580	0.06365	0.00084	0.00588	0.00587
20			0.82825	0.79736	0.82073	0.82003	0.82022			0.76944	0.73141	0.76034	0.76120	0.76141
			0.00285	-0.02803	-0.00467	-0.00537	-0.00518			0.00409	-0.03395	-0.00501	-0.00416	-0.00394
			0.00233	0.00368	0.00048	0.00239	0.00238			0.00287	0.00498	0.00061	0.00287	0.00286
30			0.82824	0.81109	0.82221	0.82269	0.82275			0.76539	0.74946	0.76027	0.75984	0.75998
			0.00284	-0.01430	-0.00319	-0.00270	-0.00265			0.00004	-0.01589	-0.00508	-0.00551	-0.00538
			0.00156	0.00062	0.00042	0.00158	0.00157			0.00205	0.00079	0.00058	0.00208	0.00208
40			0.82650	0.81696	0.82243	0.82231	0.82234			0.76554	0.75524	0.76136	0.76136	0.76140
			0.00110	-0.00843	-0.00297	-0.00308	-0.00305			0.00019	-0.01011	-0.00399	-0.00399	-0.00395
			0.00121	0.00021	0.00040	0.00123	0.00123			0.00159	0.00030	0.00055	0.00160	0.00161

Table 4.4. Average confidence/credible lengths (ACL) and coverage probabilities (CP) of $R_{n,k}$ for Burr Type II distribution when $(\alpha_1, \alpha_2, \beta, \lambda) = (1.5, 3, 15, 5)$

			ACI		HPD (Prior 2)		HPD (Non-inf.)	
m	(k, n_1, n_2)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP
10	(4,6,1)	0.85119	0.23070	0.8820	0.12707	0.9956	0.23723	0.9300
20			0.16690	0.9108	0.10688	0.9968	0.16839	0.9340
30			0.13647	0.9212	0.09574	0.9920	0.13677	0.9324
40			0.11897	0.9232	0.08828	0.9896	0.11882	0.9400
10	(4,1,6)	0.61748	0.33635	0.9188	0.18427	0.9948	0.33155	0.9332
20			0.24210	0.9272	0.15551	0.9940	0.23879	0.9368
30			0.19875	0.9424	0.13949	0.9920	0.19609	0.9468
40			0.17292	0.9480	0.12847	0.9872	0.17060	0.9448
10	(4,5,2)	0.84034	0.24350	0.8948	0.13359	0.9944	0.24856	0.9376
20			0.17620	0.9312	0.11303	0.9960	0.17702	0.9452
30			0.14461	0.9280	0.10143	0.9948	0.14450	0.9448
40			0.12630	0.9312	0.09352	0.9892	0.12573	0.9356
10	(4,2,5)	0.68692	0.32023	0.9088	0.17684	0.9936	0.31788	0.9308
20			0.23091	0.9280	0.14914	0.9916	0.22866	0.9372
30			0.18976	0.9380	0.13368	0.9912	0.18770	0.9476
40			0.16514	0.9436	0.12305	0.9856	0.16329	0.9420
10	(4,4,3)	0.82540	0.25767	0.8744	0.14282	0.9984	0.26054	0.9288
20			0.18757	0.9144	0.12060	0.9944	0.18757	0.9356
30			0.15412	0.9192	0.10838	0.9964	0.15356	0.9408
40			0.13475	0.9348	0.09997	0.9892	0.13387	0.9396
10	(4,3,4)	0.76535	0.29258	0.9012	0.16112	0.9936	0.29323	0.9352
20			0.21119	0.9288	0.13633	0.9916	0.21018	0.9424
30			0.17492	0.9300	0.12285	0.9912	0.17363	0.9396
40			0.15174	0.9320	0.11293	0.9840	0.15030	0.9388

We also compare the performances of Bayes estimates using Lindley's approximation (based on informative and non-informative priors) and MLE by plots. Figures 4.1 and 4.2 present MSE and ER of estimates for Kumaraswamy and Burr Type XII distributions with sample sizes $m = 50, 75, 100$ and 125. In these figures, the parameters are chosen as $\alpha_{1_i} = 12 - (2i/10)$, $\alpha_{2_i} = 6 - (i/10)$, $\beta_i = 2 + (2i/10)$ and $\lambda_i = 1 + (i/10)$, $i = 1, ..., 58$. Figure 4.1 represents the estimate results of $R_{9,5}$ for Kumaraswamy distribution which takes the values 0.06813 to 0.96244 when the values of system are taken as $(k, n_1, n_2) = (5, 6, 3)$. Figure 4.2 represents the estimate results of $R_{8,4}$ for Burr Type XII distribution which takes the values 0.09671 to 0.99025 when the system values are taken as $(k, n_1, n_2) = (4, 5, 3)$. The following procedure is used to draw the plots:

Step 1: For given $(\alpha_1, \alpha_2, \beta, \lambda)$, $R_{9,5}$ $(R_{8,4})$ is computed.

Step 2: For given m, samples from Kumaraswamy (Burr Type XII) distribution are generated for the strength and the stress variables.

Step 3: Estimates of $R_{9,5}$ $(R_{8,4})$ are evaluated.

Step 4: Steps 2-3 are repeated $N = 2500$ times, the MSE or ER for estimates of $R_{n,k}$ are calculated as by using $\sum_{i=1}^{N}(\widehat{R}_{n,k}^{(i)} - R_{n,k})^2/N$.

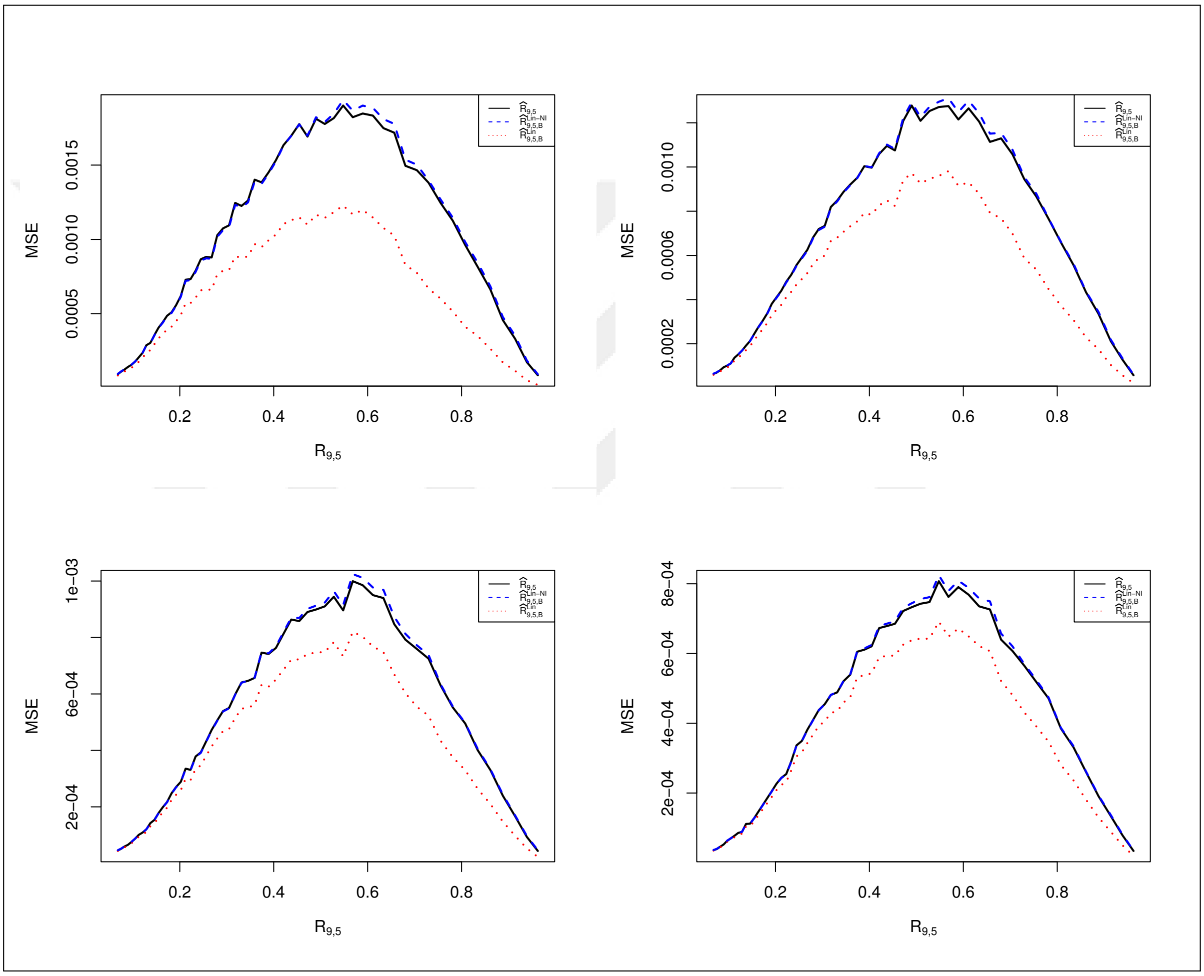

Figure 4.1. MSE (or ERs) of the estimates when $m = 50, 75, 100$ and 125 for Kumaraswamy distribution

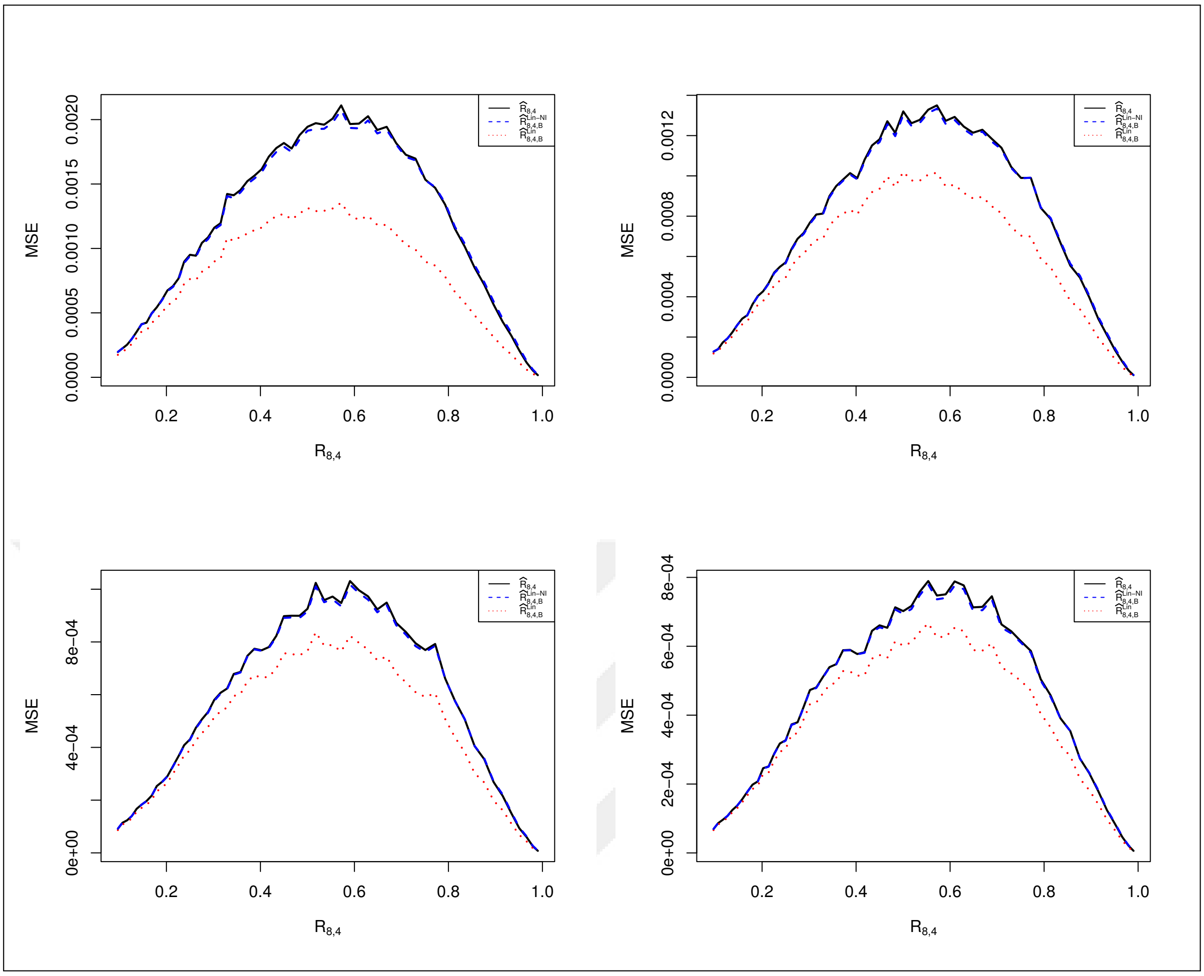

Figure 4.2. MSE (or ERs) of the estimates when $m = 50, 75, 100$ and 125 for Burr Type XII distribution

From Figures 4.1 and 4.2, it is observed that the error values of all estimators are getting bigger when $R_{n,k}$ is getting closer to 0.5 and getting smaller when $R_{n,k}$ is getting closer to the extreme values. The smallest error is obtained by the Bayes estimate based on informative prior. Moreover, ML estimate and Bayes estimate based on non-informative prior show similar performances and are close to each other around extreme values. These observations show that the results obtained from figures are similar to Tables 4.1 and 4.3.

4.4.2 When the Second Parameters are Different and Known

In this subsection, Monte Carlo simulation study is executed for the comparison of ML, UMVU and Bayes estimates of $R_{n,k}$ when the second parameters parameters of underlying distributions $(\lambda_1, \lambda_2, \lambda_3)$ are known. Point and interval estimation results are presented for one-parameter Weibull distribution with different sample sizes.

In Tables 4.5-4.7, ML, UMVU, exact Bayes and two approximate Bayes estimates of $R_{n,k}$ are listed. In the Bayesian inference, the following informative priors are considered with the non-informative prior: Prior 3 : $(a_1, b_1) = (1.25, 1)$, $(a_2, b_2) = (3, 1)$, $(a_3, b_3) = (7, 1)$, Prior 4: $(a_1, b_1) = (2, 1)$, $(a_2, b_2) = (0.75, 1)$, $(a_3, b_3) = (12, 1)$. Bayesian estimates based on MCMC method are computed by using Gibbs sampling with 4000 iterations. In addition, the asymptotic confidence and HPD intervals corresponding to obtained point estimates are given in Table 4.8.

From Tables 4.5- 4.7, MSE, ERs and biases decrease with the increase in sample sizes, as expected. It is observed that ML and UMVU estimates show similar results, and they are getting close to each other as the sample size increases. Bayes estimators based on informative priors show better performance than these classical estimates while Bayes estimates based on non-informative priors give similar results to them. In these tables, Bayes estimates using Lindley's approximation based on informative priors show the best performance in terms of error values. Moreover, it is also observed that approximate Bayes estimates and their corresponding ER values are generally close to the exact Bayes estimate values. From Table 4.8, asymptotic confidence interval of $R_{n,k}$ has wider length than the HPD credible intervals. HPD credible intervals based on informative priors provide the smallest AL, and the coverage probabilities of all intervals are quite satisfactory.

Table 4.5. Estimates of $R_{n,k}$ for Weibull distribution when $(\alpha_1, \alpha_2, \beta) = (1.25, 3, 7)$ and $(\lambda_1, \lambda_2, \lambda_3) = (6, 9, 3)$

					$Bayes$ (Prior 3)			$Bayes$ (Non-inf. prior)		
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(2,3,1)	0.88474	10	0.87913	0.8418	0.87298	0.86829	0.87297	0.80163	0.86682	0.86657
			-0.00561	-0.00056	-0.01176	-0.01645	-0.01177	-0.08311	-0.01793	-0.01817
			0.00284	0.00288	0.00148	0.00114	0.00148	0.01015	0.00332	0.00330
		20	0.88304	0.88565	0.87811	0.87648	0.87822	0.84391	0.87669	0.87661
			-0.00170	0.00090	-0.00663	-0.00827	-0.00652	-0.04083	-0.00806	-0.00813
			0.00136	0.00137	0.00092	0.00084	0.00092	0.00319	0.00147	0.00147
		30	0.88340	0.88514	0.87975	0.87918	0.88000	0.85743	0.87912	0.87911
			-0.00134	0.00040	-0.00499	-0.00556	-0.00474	-0.02732	-0.00562	-0.00563
			0.00094	0.00095	0.00072	0.00068	0.00072	0.00178	0.00100	0.00100
		40	0.88381	0.88511	0.88063	0.88059	0.88109	0.86425	0.88058	0.88054
			-0.00094	0.00037	-0.00411	-0.00415	-0.00366	-0.02050	-0.00416	-0.00420
			0.00072	0.00072	0.00060	0.00056	0.00058	0.00120	0.00075	0.00075

Table 4.5 Continued

					Bayes (Prior 3)			*Bayes* (Non-inf. prior)		
(k,n_1,n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(2,2,2)	0.84630	10	0.84208	0.84546	0.83583	0.82290	0.83584	0.83038	0.82839	0.83040
			-0.00422	-0.00084	-0.01047	-0.02340	-0.01045	-0.01592	-0.01790	-0.01590
			0.00418	0.00441	0.00203	0.00170	0.00203	0.00445	0.00461	0.00446
		20	0.84387	0.84561	0.83963	0.83572	0.83960	0.83780	0.83692	0.83779
			-0.00242	-0.00069	-0.00667	-0.01058	-0.00670	-0.00849	-0.00938	-0.00851
			0.00201	0.00206	0.00130	0.00114	0.00130	0.00209	0.00212	0.00209
		30	0.84513	0.84630	0.84192	0.83971	0.84176	0.84105	0.84048	0.84102
			-0.00117	0.00000	-0.00438	-0.00658	-0.00454	-0.00525	-0.00582	-0.00528
			0.00139	0.00142	0.00104	0.00097	0.00103	0.00143	0.00144	0.00143
		40	0.84599	0.84688	0.84330	0.84197	0.84325	0.84310	0.84249	0.84289
			-0.00031	0.00058	-0.00300	-0.00432	-0.00304	-0.00320	-0.00381	-0.00341
			0.00102	0.00103	0.00082	0.00076	0.00080	0.00105	0.00104	0.00103

Table 4.5 Continued

					$Bayes$ (Prior 3)			$Bayes$ (Non-inf. prior)		
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{U}_{n,k}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(3,1,3)	0.52293	10	0.52948	0.52297	0.51859	0.50352	0.51860	0.52063	0.51974	0.52065
			0.00655	0.00004	-0.00434	-0.01941	-0.00433	-0.00230	-0.00319	-0.00228
			0.00849	0.00905	0.00372	0.00262	0.00372	0.00798	0.00801	0.00798
		20	0.52735	0.52403	0.52164	0.51815	0.52161	0.52281	0.52252	0.52281
			0.00442	0.00110	-0.00129	-0.00478	-0.00132	-0.00013	-0.00041	-0.00012
			0.00410	0.00423	0.00248	0.00209	0.00248	0.00396	0.00397	0.00396
		30	0.52555	0.52331	0.52192	0.52043	0.52191	0.52250	0.52234	0.52248
			0.00262	0.00038	-0.00101	-0.00250	-0.00102	-0.00043	-0.00059	-0.00046
			0.00281	0.00287	0.00197	0.00183	0.00197	0.00275	0.00275	0.00274
		40	0.52379	0.52210	0.52119	0.52041	0.52124	0.52146	0.52140	0.52153
			0.00086	-0.00084	-0.00174	-0.00252	-0.00169	-0.00147	-0.00153	-0.00140
			0.00212	0.00216	0.00162	0.00156	0.00162	0.00209	0.00210	0.00209

Table 4.6. Estimates of $R_{n,k}$ for Weibull distribution when $(\alpha_1, \alpha_2, \beta) = (1.25, 3, 7)$ and $(\lambda_1, \lambda_2, \lambda_3) = (2, 3, 6)$

					$Bayes$ (Prior 3)			$Bayes$ (Non-inf. prior)		
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(3,5,1)	0.83773	10	0.83471	0.83702	0.82553	0.81859	0.82555	0.66787	0.81983	0.81971
			-0.00302	-0.00071	-0.01220	-0.01915	-0.01218	-0.16986	-0.01790	-0.01802
			0.00414	0.00442	0.00199	0.00142	0.00199	0.03304	0.00462	0.00459
		20	0.83526	0.83643	0.82952	0.82740	0.82952	0.75058	0.82757	0.82753
			-0.00247	-0.00131	-0.00821	-0.01033	-0.00821	-0.08715	-0.01016	-0.01020
			0.00217	0.00224	0.00142	0.00126	0.00142	0.00990	0.00231	0.00231
		25	0.83652	0.83747	0.83149	0.83009	0.83154	0.76889	0.83034	0.83034
			-0.00121	-0.00026	-0.00624	-0.00764	-0.00619	-0.06884	-0.00739	-0.00739
			0.00172	0.00177	0.00120	0.00110	0.00120	0.00657	0.00181	0.00180
		30	0.83908	0.83990	0.82252	0.83317	0.83429	0.76050	0.83382	0.83376
			0.00135	0.00217	-0.01521	-0.00456	-0.00344	-0.07723	-0.00391	-0.00397
			0.00134	0.00137	0.00085	0.00089	0.00096	0.00673	0.00137	0.00137

Table 4.6 Continued

					Bayes (Prior 3)			*Bayes* (Non-inf. prior)		
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(3,4,2)	0.80952	10	0.81028	0.81134	0.79993	0.79112	0.80000	0.72111	0.79593	0.79590
			0.00076	0.00182	-0.00958	-0.01839	-0.00952	-0.08840	-0.01359	-0.01362
			0.00478	0.00516	0.00218	0.00153	0.00219	0.01252	0.00504	0.00501
		20	0.80994	0.81046	0.80371	0.80133	0.80383	0.76445	0.80251	0.80249
			0.00042	0.00094	-0.00581	-0.00819	-0.00569	-0.04507	-0.00701	-0.00702
			0.00247	0.00257	0.00156	0.00135	0.00156	0.00456	0.00254	0.00253
		25	0.81042	0.81084	0.80483	0.80346	0.80515	0.77380	0.80443	0.80441
			0.00090	0.00132	-0.00468	-0.00606	-0.00437	-0.03572	-0.00508	-0.00510
			0.00192	0.00198	0.00131	0.00119	0.00131	0.00326	0.00196	0.00196
		30	0.80856	0.80889	0.80039	0.80317	0.80431	0.76833	0.80355	0.80353
			-0.00096	-0.00063	-0.00913	-0.00634	-0.00521	-0.04119	-0.00597	-0.00598
			0.00168	0.00172	0.00109	0.00113	0.00121	0.00291	0.00172	0.00172

Table 4.6 Continued

					Bayes (Prior 3)			*Bayes* (Non-inf. prior)		
(k,n_1,n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{U}_{n,k}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(3,3,3)	0.77339	10	0.77561	0.77471	0.76543	0.75262	0.76539	0.76229	0.76180	0.76234
			0.00222	0.00132	-0.00796	-0.02077	-0.00800	-0.01110	-0.01159	-0.01105
			0.00588	0.00640	0.00259	0.00183	0.00260	0.00590	0.00597	0.00590
		20	0.77618	0.77572	0.76986	0.76632	0.76984	0.76924	0.76910	0.76931
			0.00279	0.00233	-0.00353	-0.00707	-0.00355	-0.00415	-0.00429	-0.00408
			0.00301	0.00314	0.00184	0.00159	0.00184	0.00300	0.00301	0.00300
		25	0.77314	0.77272	0.76843	0.76629	0.76845	0.76754	0.76745	0.76759
			-0.00025	-0.00067	-0.00496	-0.00710	-0.00494	-0.00585	-0.00594	-0.00580
			0.00235	0.00244	0.00159	0.00146	0.00158	0.00238	0.00238	0.00237
		30	0.77453	0.77419	0.76990	0.76870	0.77028	0.76672	0.76978	0.76988
			0.00114	0.00080	-0.00349	-0.00469	-0.00311	-0.00667	-0.00361	-0.00351
			0.00207	0.00213	0.00144	0.00137	0.00146	0.00187	0.00208	0.00207

Table 4.7. Estimates of $R_{n,k}$ for Weibull distribution when $(\alpha_1, \alpha_2, \beta) = (2, 0.75, 12)$ and $(\lambda_1, \lambda_2, \lambda_3) = (4, 2, 8)$

					$Bayes$ (Prior 4)			$Bayes$ (Non-inf. prior)		
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{U}_{n,k}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(4,1,4)	0.83258	10	0.83070	0.83200	0.82477	0.80273	0.82478	0.81916	0.80532	0.81910
			-0.00188	-0.00058	-0.00781	-0.02985	-0.00780	-0.01342	-0.02726	-0.01347
			0.00270	0.00276	0.00104	0.00862	0.00104	0.00303	0.00407	0.00303
		20	0.83307	0.83374	0.82857	0.81842	0.82859	0.82722	0.82171	0.82730
			0.00049	0.00116	-0.00401	-0.01416	-0.00399	-0.00536	-0.01087	-0.00528
			0.00134	0.00136	0.00071	0.00057	0.00070	0.00141	0.00151	0.00141
		25	0.83265	0.83318	0.82888	0.82196	0.82891	0.82783	0.82364	0.82800
			0.00007	0.00060	-0.00370	-0.01062	-0.00366	-0.00475	-0.00894	-0.00458
			0.00101	0.00102	0.00063	0.00034	0.00059	0.00106	0.00113	0.00105
		30	0.83188	0.83232	0.82889	0.82748	0.82884	0.82768	0.82803	0.82799
			-0.00070	-0.00026	-0.00369	-0.00510	-0.00374	-0.00490	-0.00455	-0.00459
			0.00084	0.00085	0.00062	0.00045	0.00053	0.00090	0.00088	0.00088

Table 4.7 Continued

					Bayes (Prior 4)			*Bayes* (Non-inf. prior)		
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(4,2,3)	0.76664	10	0.76382	0.76647	0.75635	0.74313	0.75645	0.75112	0.75123	0.75113
			-0.00282	-0.00017	-0.01029	-0.02351	-0.01019	-0.01552	-0.01541	-0.01552
			0.00422	0.00435	0.00153	0.00147	0.00151	0.00455	0.00456	0.00455
		20	0.76563	0.76696	0.75948	0.75693	0.76066	0.75852	0.75918	0.75916
			-0.00101	0.00032	-0.00716	-0.00972	-0.00598	-0.00812	-0.00746	-0.00748
			0.00197	0.00200	0.00110	0.00072	0.00101	0.00209	0.00205	0.00205
		25	0.76523	0.76629	0.75933	0.75871	0.76114	0.75870	0.76005	0.76004
			-0.00141	-0.00035	-0.00731	-0.00793	-0.00550	-0.00794	-0.00659	-0.00661
			0.00164	0.00166	0.00135	0.00072	0.00093	0.00183	0.00170	0.00170
		30	0.76676	0.76765	0.76333	0.76122	0.76301	0.76135	0.76244	0.76241
			0.00012	0.00101	-0.00331	-0.00542	-0.00363	-0.00529	-0.00420	-0.00423
			0.00133	0.00135	0.00146	0.00066	0.00081	0.00168	0.00136	0.00136

Table 4.7 Continued

					Bayes (Prior 4)			*Bayes* (Non-inf. prior)		
(k,n_1,n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{U}_{n,k}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(4,4,1)	0.66169	10	0.66343	0.66003	0.65406	0.63575	0.65402	0.65071	0.65084	0.65094
			0.00174	-0.00166	-0.00762	-0.02594	-0.00767	-0.01097	-0.01085	-0.01074
			0.00584	0.00623	0.00220	0.00232	0.00213	0.00585	0.00586	0.00584
		20	0.66370	0.66199	0.66073	0.65257	0.65765	0.65947	0.65727	0.65730
			0.00202	0.00030	-0.00096	-0.00912	-0.00403	-0.00222	-0.00442	-0.00439
			0.00309	0.00319	0.00224	0.00109	0.00158	0.00354	0.00308	0.00308
		25	0.66365	0.66228	0.66153	0.65546	0.65865	0.66125	0.65848	0.65848
			0.00197	0.00059	-0.00016	-0.00623	-0.00304	-0.00044	-0.00321	-0.00321
			0.00244	0.00250	0.00330	0.00101	0.00134	0.00354	0.00242	0.00243
		30	0.66389	0.66275	0.65882	0.65694	0.65932	0.66083	0.65958	0.65958
			0.00221	0.00106	-0.00287	-0.00475	-0.00237	-0.00086	-0.00211	-0.00210
			0.00216	0.00220	0.00576	0.00105	0.00129	0.00513	0.00215	0.00215

Table 4.8. ACL and CP of $R_{n,k}$ for Weibull distribution when $(\alpha_1, \alpha_2, \beta) = (1.25, 3, 7)$ and $(2, 0.75, 12)$

			ACI		HPD (Prior 3 or 4)		HPD (Non-inf. prior)	
m	(k, n_1, n_2)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP
10	(2,3,1)	0.88474	0.20544	0.8980	0.16887	0.9832	0.20867	0.9416
20			0.14498	0.9308	0.12932	0.9728	0.14561	0.9452
30			0.11882	0.9332	0.10928	0.9644	0.11883	0.9448
40			0.10299	0.9352	0.09642	0.9552	0.10287	0.9384
10	(2,2,2)	0.84630	0.30142	0.9620	0.20102	0.9796	0.24504	0.9316
20			0.21376	0.9712	0.15618	0.9720	0.17536	0.9392
30			0.17393	0.9744	0.13198	0.9620	0.14320	0.9436
40			0.15094	0.9744	0.11647	0.9588	0.12403	0.9412
10	(3,5,1)	0.83773	0.24973	0.9140	0.20442	0.9848	0.25187	0.9360
20			0.17971	0.9232	0.15898	0.9660	0.17981	0.9408
25			0.16054	0.9208	0.14493	0.9664	0.16037	0.9424
30			0.14564	0.9392	0.13358	0.9716	0.14553	0.9495
10	(3,4,2)	0.80952	0.26954	0.9016	0.22020	0.9840	0.26968	0.9348
20			0.19479	0.9332	0.17209	0.9748	0.19403	0.9448
25			0.17475	0.9284	0.15755	0.9668	0.17389	0.9412
30			0.16084	0.9336	0.14687	0.9632	0.16010	0.9440
10	(3,3,3)	0.77339	0.30503	0.9280	0.24121	0.9852	0.29337	0.9376
20			0.21838	0.9364	0.18808	0.9704	0.21164	0.9408
25			0.19688	0.9412	0.17340	0.9656	0.19164	0.9376
30			0.17940	0.9412	0.16046	0.9660	0.17478	0.9400
10	(3,1,3)	0.52293	0.33973	0.9220	0.28297	0.9768	0.33910	0.9336
20			0.24138	0.9332	0.22070	0.9676	0.24649	0.9440
30			0.19732	0.9296	0.18753	0.9616	0.20289	0.9408
40			0.17097	0.9292	0.16608	0.9536	0.17667	0.9372

Table 4.8 Continued

			ACI		HPD (Prior 3 or 4)		HPD (Non-inf. prior)	
m	(k,n_1,n_2)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP
10	(4,1,4)	0.83258	0.49942	0.9960	0.14992	0.9840	0.20358	0.9396
20			0.33116	0.9984	0.11756	0.9748	0.14117	0.9416
25			0.29477	0.9992	0.10829	0.9796	0.12643	0.9520
30			0.11504	0.9389	0.10113	0.9756	0.11559	0.9433
10	(4,2,3)	0.76664	0.24776	0.9172	0.18570	0.9868	0.25077	0.9384
20			0.17642	0.9412	0.14715	0.9856	0.17694	0.9536
25			0.15828	0.9408	0.13561	0.9760	0.15832	0.9464
30			0.14406	0.9400	0.12605	0.9708	0.14399	0.9492
10	(4,4,1)	0.66169	0.30386	0.9292	0.22826	0.9868	0.30201	0.9476
20			0.21746	0.9400	0.18130	0.9756	0.21614	0.9440
25			0.19509	0.9432	0.16724	0.9796	0.19393	0.9536
30			0.17824	0.9364	0.15585	0.9704	0.17710	0.9428

In our simulation studies, we encounter some difficulties in the evaluation of the exact Bayes estimate of $R_{n,k}$ for the large values of m, n and k. It is seen that the integral in the exact Bayes estimate of $R_{n,k}$ can create some problems for some values of constants in there. Moreover, we observed that performance of two approximate Bayes estimates: Lindley's approximation and MCMC algorithm with Gibbs sampling have almost same with the exact Bayes estimate in previous tables. Therefore, we listed ML, UMVU and two approximate Bayes estimates results in Tables 4.9 - 4.11 for large values of m, n and k. Moreover, the asymptotic confidence and HPD intervals of $R_{n,k}$ for these estimates are given in Table 4.12. In these tables, the true values of the parameters are taken as $(\alpha_1, \alpha_2, \beta) = (0.75, 2, 20)$ and $(\lambda_1, \lambda_2, \lambda_3) = (6, 9, 3)$. Prior 5: $(a_1, b_1) = (0.75, 1)$, $(a_2, b_2) = (2, 1)$, $(a_3, b_3) = (20, 1)$ is used in Bayesian case as an informative prior. From Tables 4.9 - 4.11, it is observed that estimates of $R_{n,k}$ provide similar performances as in the Tables 4.5- 4.7. Also, intervals results in Table 4.12 show similar performances as in the Table 4.8.

Table 4.9. Estimates of $R_{n,k}$ for Weibull distribution when $(k,n)=(5,9)$ and $(k,n)=(5,10)$

					$Bayes$ (Prior 5)		$Bayes$ (Non-inf. prior)	
(k,n_1,n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{U}_{n,k}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(5,6,3)	0.93166	25	0.93114	0.93193	0.92786	0.92928	0.92771	0.92761
			-0.00051	0.00028	-0.00379	-0.00238	-0.00395	-0.00405
			0.00031	0.00030	0.00007	0.00013	0.00035	0.00035
		50	0.93151	0.93191	0.92982	0.93029	0.92979	0.92975
			-0.00015	0.00025	-0.00183	-0.00137	-0.00187	-0.00190
			0.00016	0.00015	0.00008	0.00009	0.00016	0.00016
		75	0.93138	0.93164	0.93028	0.93051	0.93023	0.93021
			-0.00028	-0.00001	-0.00138	-0.00115	-0.00143	-0.00144
			0.00011	0.00011	0.00007	0.00007	0.00011	0.00011
(5,4,6)	0.87442	25	0.87372	0.87015	0.86862	0.87097	0.86908	0.86899
			-0.00070	-0.00427	-0.00580	-0.00345	-0.00534	-0.00543
			0.00074	0.00080	0.00015	0.00030	0.00080	0.00080
		50	0.87428	0.87048	0.87178	0.87252	0.87194	0.87192
			-0.00014	-0.00394	-0.00263	-0.00190	-0.00248	-0.00250
			0.00037	0.00041	0.00017	0.00022	0.00039	0.00039
		75	0.87503	0.87117	0.87324	0.87363	0.87348	0.87346
			0.00061	-0.00325	-0.00118	-0.00079	-0.00094	-0.00096
			0.00025	0.00027	0.00015	0.00017	0.00025	0.00025

Table 4.9 Continued

					$Bayes$ (Prior 5)		$Bayes$ (Non-inf. prior)	
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(5,3,6)	0.81516	25	0.81533	0.81123	0.80869	0.81190	0.81020	0.81016
			0.00017	-0.00393	-0.00647	-0.00326	-0.00496	-0.00500
			0.00119	0.00128	0.00024	0.00048	0.00124	0.00124
		50	0.81539	0.81133	0.81230	0.81324	0.81280	0.81279
			0.00023	-0.00383	-0.00286	-0.00191	-0.00236	-0.00237
			0.00062	0.00066	0.00028	0.00036	0.00063	0.00063
		75	0.81491	0.81085	0.81302	0.81344	0.81318	0.81316
			-0.00025	-0.00431	-0.00214	-0.00172	-0.00198	-0.00200
			0.00040	0.00044	0.00024	0.00027	0.00041	0.00041
(5,2,8)	0.76122	25	0.76146	0.74534	0.75352	0.75743	0.75601	0.75598
			0.00024	-0.01588	-0.00770	-0.00379	-0.00521	-0.00524
			0.00154	0.00203	0.00029	0.00061	0.00159	0.00159
		50	0.76179	0.74612	0.75840	0.75946	0.75905	0.75905
			0.00057	-0.01510	-0.00282	-0.00175	-0.00217	-0.00217
			0.00078	0.00113	0.00035	0.00044	0.00079	0.00080
		75	0.76147	0.74592	0.75933	0.75982	0.75964	0.75965
			0.00025	-0.01530	-0.00189	-0.00140	-0.00158	-0.00157
			0.00050	0.00080	0.00030	0.00033	0.00050	0.00050

Table 4.10. Estimates of $R_{n,k}$ for Weibull distribution when $(k, n) = (7, 13)$

					$Bayes$ (Prior 5)		$Bayes$ (Non-inf. prior)	
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(7,9,4)	0.91090	25	0.91033	0.91097	0.90619	0.90799	0.90598	0.90587
			-0.00056	0.00008	-0.00471	-0.00291	-0.00492	-0.00503
			0.00052	0.00052	0.00009	0.00020	0.00057	0.00057
		50	0.91057	0.91089	0.90842	0.90901	0.90837	0.90834
			-0.00033	-0.00001	-0.00248	-0.00188	-0.00253	-0.00255
			0.00026	0.00026	0.00012	0.00015	0.00027	0.00027
		75	0.91057	0.91078	0.90913	0.90942	0.90910	0.90909
			-0.00033	-0.00012	-0.00176	-0.00147	-0.00180	-0.00181
			0.00017	0.00017	0.00010	0.00011	0.00017	0.00017
(7,6,7)	0.84820	25	0.84847	0.84861	0.84167	0.84482	0.84309	0.84302
			0.00028	0.00041	-0.00653	-0.00338	-0.00510	-0.00518
			0.00104	0.00106	0.00018	0.00040	0.00110	0.00110
		50	0.84732	0.84738	0.84454	0.84536	0.84459	0.84457
			-0.00088	-0.00082	-0.00366	-0.00283	-0.00361	-0.00362
			0.00054	0.00054	0.00024	0.00031	0.00056	0.00056
		75	0.84819	0.84824	0.84622	0.84665	0.84639	0.84637
			-0.00001	0.00004	-0.00197	-0.00155	-0.00181	-0.00183
			0.00036	0.00036	0.00021	0.00024	0.00036	0.00036

Table 4.10 Continued

					Bayes (Prior 5)		*Bayes* (Non-inf. prior)	
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(7,5,8)	0.79524	25	0.79541	0.79211	0.78815	0.79166	0.78974	0.78970
			0.00018	-0.00312	-0.00708	-0.00357	-0.00549	-0.00553
			0.00142	0.00149	0.00024	0.00054	0.00148	0.00148
		50	0.79483	0.79177	0.79172	0.79265	0.79197	0.79198
			-0.00040	-0.00347	-0.00352	-0.00258	-0.00327	-0.00326
			0.00067	0.00071	0.00030	0.00038	0.00069	0.00069
		75	0.79566	0.79269	0.79349	0.79398	0.79375	0.79375
			0.00043	-0.00254	-0.00174	-0.00126	-0.00149	-0.00149
			0.00047	0.00049	0.00028	0.00031	0.00048	0.00048
(7,4,9)	0.74678	25	0.74725	0.73958	0.73944	0.74334	0.74147	0.74146
			0.00047	-0.00721	-0.00734	-0.00344	-0.00531	-0.00533
			0.00170	0.00186	0.00027	0.00064	0.00174	0.00174
		50	0.74754	0.74049	0.74367	0.74488	0.74463	0.74465
			0.00076	-0.00630	-0.00311	-0.00191	-0.00215	-0.00213
			0.00083	0.00092	0.00035	0.00046	0.00084	0.00084
		75	0.74784	0.74101	0.74538	0.74594	0.74594	0.74590
			0.00106	-0.00578	-0.00140	-0.00084	-0.00084	-0.00089
			0.00060	0.00066	0.00035	0.00039	0.00060	0.00060

Table 4.11. Estimates of $R_{n,k}$ for Weibull distribution when $(k,n)=(8,15)$

					$Bayes$ (Prior 5.)		$Bayes$ (Non-inf. prior)	
(k,n_1,n_2)	$R_{n,k}$	m	$\widehat{R}^{MLE}_{n,k}$	$\widehat{R}^{U}_{n,k}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$	$\widehat{R}^{Lin}_{n,k,B}$	$\widehat{R}^{MC}_{n,k,B}$
(8,12,3)	0.90968	25	0.90880	0.90940	0.90479	0.90650	0.90429	0.90418
			-0.00088	-0.00029	-0.00490	-0.00319	-0.00540	-0.00550
			0.00051	0.00051	0.00009	0.00020	0.00057	0.00057
		50	0.90956	0.90987	0.90724	0.90788	0.90730	0.90727
			-0.00012	0.00018	-0.00244	-0.00180	-0.00238	-0.00241
			0.00025	0.00026	0.00011	0.00014	0.00027	0.00027
		75	0.90936	0.90956	0.90790	0.90820	0.90785	0.90784
			-0.00032	-0.00012	-0.00178	-0.00148	-0.00184	-0.00185
			0.00018	0.00018	0.00011	0.00012	0.00019	0.00019
(8,11,4)	0.90336	25	0.90228	0.90283	0.89817	0.89997	0.89758	0.89748
			-0.00108	-0.00053	-0.00518	-0.00339	-0.00577	-0.00588
			0.00057	0.00058	0.00010	0.00022	0.00064	0.00064
		50	0.90322	0.90350	0.90086	0.90153	0.90086	0.90085
			-0.00014	0.00014	-0.00249	-0.00183	-0.00249	-0.00251
			0.00029	0.00029	0.00013	0.00017	0.00031	0.00031
		75	0.90306	0.90325	0.90152	0.90184	0.90149	0.90147
			-0.00029	-0.00011	-0.00184	-0.00151	-0.00187	-0.00188
			0.00020	0.00020	0.00012	0.00013	0.00020	0.00020
(8,6,9)	0.78419	25	0.78652	0.78347	0.77707	0.78157	0.78065	0.78063
			0.00233	-0.00072	-0.00354	-0.00262	-0.00354	-0.00356
			0.00151	0.00157	0.00022	0.00055	0.00155	0.00155
		50	0.78525	0.78253	0.78139	0.78257	0.78229	0.78228
			0.00106	-0.00166	-0.00280	-0.00162	-0.00190	-0.00191
			0.00079	0.00082	0.00034	0.00044	0.00080	0.00080
		75	0.78480	0.78219	0.78242	0.78294	0.78281	0.78281
			0.00061	-0.00200	-0.00177	-0.00125	-0.00138	-0.00138
			0.00050	0.00052	0.00029	0.00033	0.00051	0.00051

Table 4.11 Continued

					$Bayes$ (Prior 5.)		$Bayes$ (Non-inf. prior)	
(k, n_1, n_2)	$R_{n,k}$	m	$\widehat{R}_{n,k}^{MLE}$	$\widehat{R}_{n,k}^{U}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$	$\widehat{R}_{n,k,B}^{Lin}$	$\widehat{R}_{n,k,B}^{MC}$
(8,5,10)	0.73817	25	0.73984	0.73301	0.73070	0.73520	0.73392	0.73387
			0.00167	-0.00516	-0.00748	-0.00297	-0.00425	-0.00430
			0.00179	0.00191	0.00027	0.00066	0.00182	0.00182
		50	0.73888	0.73269	0.73501	0.73621	0.73589	0.73587
			0.00071	-0.00549	-0.00317	-0.00196	-0.00228	-0.00231
			0.00091	0.00098	0.00039	0.00050	0.00092	0.00092
		75	0.73909	0.73315	0.73664	0.73719	0.73716	0.73709
			0.00091	-0.00502	-0.00153	-0.00098	-0.00101	-0.00108
			0.00060	0.00064	0.00034	0.00039	0.00060	0.00060
(8,3,12)	0.65673	25	0.65949	0.64371	0.64824	0.65399	0.65402	0.65375
			0.00276	-0.01301	-0.00849	-0.00274	-0.00271	-0.00298
			0.00237	0.00272	0.00034	0.00085	0.00235	0.00236
		50	0.65778	0.64321	0.65378	0.65508	0.65488	0.65489
			0.00105	-0.01352	-0.00294	-0.00164	-0.00184	-0.00184
			0.00109	0.00135	0.00045	0.00059	0.00109	0.00109
		75	0.65855	0.64442	0.65584	0.65646	0.65661	0.65662
			0.00182	-0.01231	-0.00089	-0.00027	-0.00012	-0.00010
			0.00075	0.00094	0.00042	0.00048	0.00074	0.00074

Table 4.12. ACL and CP of $R_{n,k}$ for Weibull distribution when $(\alpha_1, \alpha_2, \beta) = (0.75, 2, 20)$ and $(\lambda_1, \lambda_2, \lambda_3) = (6, 9, 3)$

			ACI		HPD (Prior 5)		HPD (Non-inf. prior)	
m	(k, n_1, n_2)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP
25	(5,6,3)	0.93166	0.06917	0.9276	0.05431	0.9848	0.07083	0.9464
50			0.04885	0.9344	0.04234	0.9736	0.04931	0.9408
75			0.04000	0.9396	0.03607	0.9672	0.04011	0.9456
25	(5,4,6)	0.87442	0.11940	0.9608	0.08317	0.9868	0.10795	0.9464
50			0.08444	0.9664	0.06502	0.9744	0.07557	0.9472
75			0.06880	0.9636	0.05516	0.9656	0.06124	0.9448
25	(5,3,6)	0.81516	0.14534	0.9560	0.10497	0.9840	0.13473	0.9456
50			0.10301	0.9520	0.08213	0.9672	0.09484	0.9304
75			0.08436	0.964	0.07000	0.966	0.07754	0.9480
25	(5,2,8)	0.76122	0.17316	0.9600	0.11944	0.9852	0.15374	0.9416
50			0.12268	0.9708	0.09363	0.9728	0.10849	0.9436
75			0.10034	0.9716	0.07983	0.9700	0.08858	0.9464
25	(7,9,4)	0.91090	0.08895	0.9348	0.06900	0.9892	0.09072	0.9460
50			0.06307	0.9416	0.05428	0.9740	0.06349	0.9460
75			0.05159	0.9468	0.04632	0.9736	0.05166	0.9504
25	(7,6,7)	0.84820	0.13019	0.9364	0.09840	0.9908	0.12810	0.9392
50			0.09300	0.9452	0.07785	0.9772	0.09077	0.9420
75			0.07576	0.9544	0.06615	0.9704	0.07371	0.9516
25	(7,5,8)	0.79524	0.15142	0.9444	0.11393	0.9864	0.14779	0.9456
50			0.10774	0.9600	0.08988	0.9808	0.10459	0.9576
75			0.08787	0.9488	0.07653	0.9676	0.08514	0.9444
25	(7,4,9)	0.74678	0.16789	0.9504	0.12579	0.9844	0.16289	0.9456
50			0.1191	0.9572	0.0991	0.9812	0.1151	0.9512
75			0.09727	0.9484	0.08449	0.9672	0.09392	0.9420

Table 4.12 Continued

			ACI		HPD (Prior 5)		HPD (Non-inf. prior)	
m	(k, n_1, n_2)	$R_{n,k}$	ACL	CP	ACL	CP	ACL	CP
25	(8,12,3)	0.90968	0.09047	0.9368	0.06991	0.9884	0.09227	0.9500
50			0.06381	0.9456	0.05487	0.9784	0.06425	0.9504
75			0.05228	0.9480	0.04686	0.9720	0.05242	0.9504
25	(8,11,4)	0.90336	0.09615	0.9340	0.07417	0.9912	0.09790	0.9516
50			0.06776	0.9344	0.05822	0.9764	0.06815	0.9496
75			0.05550	0.9484	0.04972	0.9700	0.05559	0.9516
25	(8,6,9)	0.78419	0.15526	0.9372	0.11825	0.9864	0.15342	0.9428
50			0.11070	0.9372	0.09357	0.9716	0.10895	0.9396
75			0.09065	0.9492	0.07998	0.9676	0.08905	0.9460
25	(8,5,10)	0.73817	0.16947	0.9432	0.12871	0.9880	0.16704	0.9440
50			0.12059	0.9516	0.10184	0.9764	0.11845	0.9484
75			0.09853	0.9520	0.08694	0.9684	0.09678	0.9484
25	(8,3,12)	0.65673	0.18839	0.9420	0.14218	0.9844	0.18395	0.9356
50			0.13400	0.9500	0.11253	0.9744	0.13103	0.9476
75			0.10949	0.9508	0.09610	0.9696	0.10695	0.9472

Furthermore, we also consider comparison of the performances of Bayes estimates using Lindley's approximation (based on informative and non-informative priors) and classical estimates (MLE and UMVUE) by plots. MSE and ER values of estimates are plotted in Figure 4.3 for $(k, n_1, n_2) = (7, 8, 6)$ and sample sizes $m = 50, 75, 100, 125$. Parameters are chosen as $\alpha_{1_i} = 0.3 + (2i/10)$, $\alpha_{2_i} = 0.6 + (i/10)$, $\beta_i = 19 - (2i/10)$, $i = 1, ..., 80$ and $(\lambda_1, \lambda_2, \lambda_3) = (6, 9, 3)$ for $R_{14,7}$ which takes the values 0.06108 to 0.96595. These plots are created by using the same algorithm in previous plots in Figures 4.1 and 4.2.

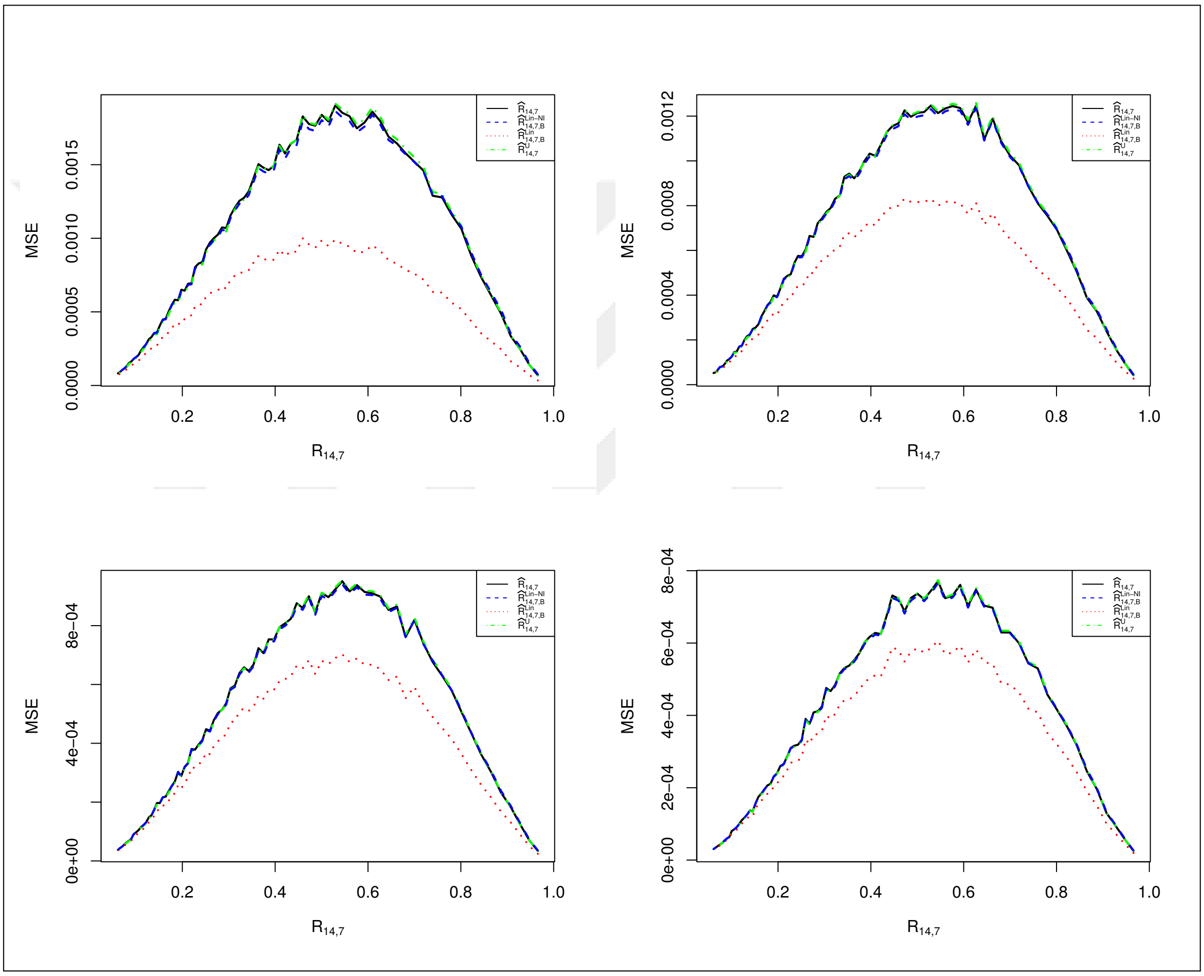

Figure 4.3. MSE (or ERs) of the estimates when $m = 50, 75, 100$ and 125 for Weibull distribution

From Figure 4.3, ML, UMVU and Bayes estimates (based on informative and non-informative prior) have greater errors when $R_{n,k}$ is around 0.5 and have smaller errors when $R_{n,k}$ is close to the extreme values. Bayes estimate based on informative prior has the smallest error while Bayes estimate based on non-informative prior, ML and UMVU estimates have similar results. Hence, it can be concluded that the results of Tables 4.5 - 4.7 and 4.9 - 4.11 are compatible with the figure.

4.5. REAL DATA ANALYSIS

Wind energy is an important a renewable energy source. It is used as a significant renewable energy source in electricity production in some countries. However, usage and modeling of renewable energy sources have been paying attention to the governments as well as researchers when we consider the problems in the consumption of fossil fuels such as global warming and climate change.

Getting reliable information about the wind capacity of the candidate region is vital for planning the installation of wind power plants. Hence, a comparison of the wind energy potentials of the candidate locations becomes important. In wind energy studies, two-parameter Weibull and Rayleigh distributions are most commonly used to model the wind speed distributions, see Akgül and Şenoğlu [55] and its references.

In this section, we use the NASA's POWER (Prediction Of Worldwide Energy Resource) data source. The POWER project aims to improve current renewable energy data set and to create new data sets from new satellite systems.

We consider a kind of comparison of the wind speed data of two districts on the Aegean coast of Turkey for our real data analysis in the stress-strength model. In our knowledge, this is the first study considering wind speed data from NASA's POWER source for the stress-strength reliability analysis.

Since the Aegean coast of Turkey has high wind energy potential, we consider NASA's POWER satellite data for the Fethiye and Datça stations instead of the real meteorological station data. These stations geographical information are the same as in Akgül and Şenoğlu

[55] study and locations are given in Figure 4.4. Data from NASA's POWER source can be taken from directly by using its data access viewer (https://power.larc.nasa.gov/data-access-viewer/) or using the *nasapower* (see Sparks [56, 57]) package in R.

Figure 4.4. Locations of Fethiye and Datça stations

Wind speed observations (m/s) at the 10 m height on an hourly basis in February 2019 are used in this study. It is assumed that Fethiye stations data from 7 am to 8 pm for each day of February 2019 are considered the first type of strength data $\mathbf{Y}_1$, and data from 8 pm to 7 am are considered the second type of strength data $\mathbf{Y}_2$. The daily average wind speed data of Datça station is considered as stress data $\mathbf{X}$. Then, we have $n_1 = 13$, $n_2 = 11$, $m = 28$ for our model.

In this structure, we can construct the following scenario for comparison of the wind energy potential of Fethiye and Datça districts. Since the hourly data is used for each day, we have the consecutive k-out-of-$24 : G$ system based on these stress and strength data sets. Wind energy investors want to decide which district is more suitable for the installation of wind power plants. If the stress-strength reliability of the consecutive k-out-of-$24 : G$ system for $k = 12, 13, 14, 15$ is greater than 0.50, they will consider that the wind energy potential of Fethiye can be more desirable with regard to Datça for a detailed feasibility study. Moreover, if the system reliability is less than 0.50, they will think of the reverse result of the aforementioned scenario.

Descriptive statistics for the interested wind speed data of $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{X}$ are given in Table 4.13. We check whether stress data set $\mathbf{X}$ and strength data sets $\mathbf{Y}_1$ and $\mathbf{Y}_2$ come from the interested underlying distribution of PHR family or not. Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Cramer-von Mises (C-VM) tests are applied for the goodness-of-fit by using the *stats* package in R. The test statistics and corresponding $p-$values (given in brackets) are computed based on the MLEs of the unknown parameters and presented in Table 4.14. Moreover, the root mean square (RMSE) and coefficient of determination (R^2) measures are used to determine which distribution provides the best fit to the considered wind speed data. RMSE and R^2 are evaluated by using the following formulas

$$RMSE = \left[\frac{1}{n} \sum_{i=1}^{n} \left(\widehat{F}_i(X_{(i)}) - \frac{i}{n+1} \right)^2 \right]^{1/2}, \tag{4.108}$$

$$R^2 = 1 - \frac{\sum_{i=1}^{n} \left(\widehat{F}_i(X_{(i)}) - \frac{i}{n+1} \right)^2}{\sum_{i=1}^{n} \left(\widehat{F}_i(X_{(i)}) - \overline{\widehat{F}}_i(X_{(i)}) \right)^2}, \tag{4.109}$$

where $\widehat{F}_i$ is the estimated cdf for the ith ordered observation, $\overline{\widehat{F}}_i(X_{(i)}) = (1/n) \sum_{i=1}^{n} \widehat{F}_i(X_{(i)})$, n is the sample size. It is known that lower values of $RMSE$ indicate better fit while the higher values of R^2 demonstrate better modeling. These measures are commonly used in the wind speed data analysis see Quarda et al. [58]. Furthermore, fitting performance of some distributions from the PHR family are visualized with the histograms of data and fitted density plots in Figure 4.5. From Table 4.14, it is observed that two-parameter Weibull distribution provides a good fit than other considered distributions for all data sets.

Table 4.13. Descriptive statistics of wind speed data (m/s)

Data	Mean	St.Dev.	Skewness	Kurtosis	Max	Size
$\mathbf{Y}_1$	3.09104	1.91416	1.64642	7.05420	11.46	364
$\mathbf{Y}_2$	2.57471	1.78901	1.91245	8.61324	11.71	308
$\mathbf{X}$	6.1579	2.63014	0.84610	4.09356	13.53	28

Table 4.14. MLEs of the parameters, goodness-of-fit test, $RMSE$ and R^2 values for $\mathbf{Y}_1, \mathbf{Y}_2$ and **X** data sets

		Distributions			
Data	MLE/Criteria	Weibull	Rayleigh	Burr Type XII	Kumaraswamy
$\mathbf{Y}_1$	$\hat{\alpha}_1$	0.12811	0.07571	0.27533	4.76345
	$\hat{\lambda}$	1.66086	-	3.62302	1.45716
	$K-S$	0.05607 (**0.2024**)	0.09418 (0.00314)	0.2206 (0)	0.08063 (0.01784)
	$A-D$	2.749 (**0.0368**)	4.80 (0.00358)	30.418 (0)	6.1915 (0.0008)
	$C-VM$	0.32878 (**0.1124**)	0.86489 (0.00509)	6.1338 (0)	0.8448 (0.0057)
	$RMSE$	**0.02939**	0.04868	0.12933	0.04763
	R^2	**0.99616**	0.98654	0.92476	0.98823
$\mathbf{Y}_2$	$\hat{\alpha}_2$	0.16696	0.10184	0.33357	6.68398
	$\hat{\lambda}$	1.66086	-	3.62302	1.45716
	$K-S$	0.08067 (**0.03632**)	0.12856 (0)	0.14673 (0)	0.10408 (0.0026)
	$A-D$	2.2514 (**0.06714**)	10.527 (0)	12.26 (0)	3.8855 (0.0099)
	$C-VM$	0.35813 (**0.09342**)	1.8913 (0)	2.4286 (0)	0.62564 (0.01929)
	$RMSE$	**0.03376**	0.14763	0.08842	0.04461
	R^2	**0.99442**	0.63735	0.96596	0.98970
X	$\hat{\beta}$	0.04463	0.02243	0.15958	2.53260
	$\hat{\lambda}$	1.66086	-	3.62302	1.45716
	$K-S$	0.17682 (0.308)	0.1357 (**0.6323**)	0.40227 (0.0001)	0.18885 (0.2565)
	$A-D$	1.6473 (0.1452)	0.81499 (**0.4692**)	7.2629 (0.0003)	1.7643 (0.1246)
	$C-VM$	0.28784 (0.1462)	0.13179 (**0.4524**)	1.5008 (0.0001)	0.31243 (0.1244)
	$RMSE$	0.09390	**0.06149**	0.22296	0.10024
	R^2	0.96166	**0.98371**	0.74391	0.95726

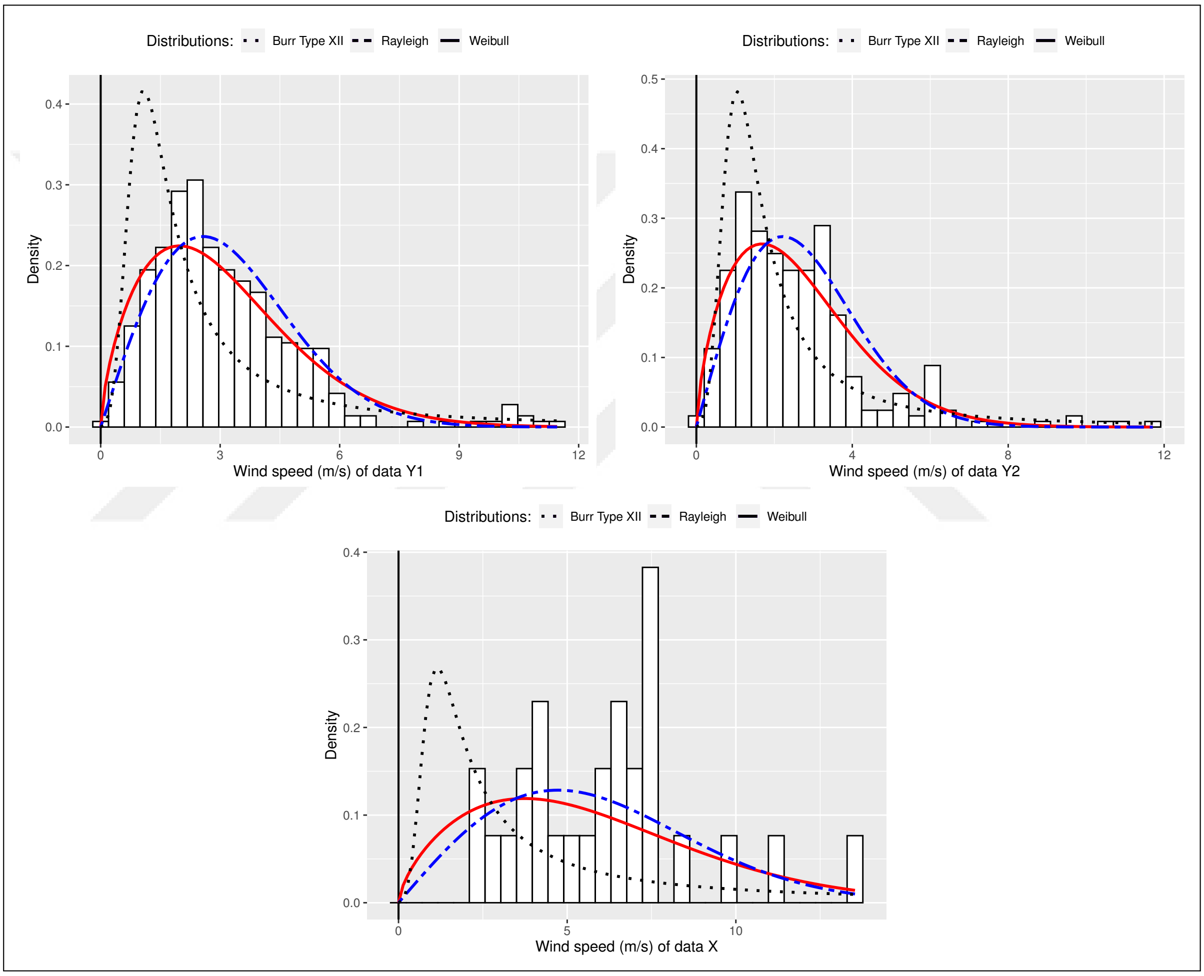

Figure 4.5. The histograms and fitted densities for the wind speed data

The ML and Bayes estimates of $R_{24,k}$, $k = 12, 13, 14, 15$ along with 95% asymptotic confidence and HPD credible intervals are presented in Tables 4.15 and 4.16. Bayes estimates are computed based on the informative priors Prior 6: $a_i = b_i = 1$, $i = 1, 2, 3, 4$, Prior 7: $a_i = b_i = 2, i = 1, 2, 3, 4$, Prior 8: $(a_1, b_1) = (2.61487, 0.84595)$, $(a_2, b_2) = (2.07798, 0.80707)$, $(a_3, b_3) = (5.68464, 0.92315)$, $(a_4, b_4) = (2.20060, 0.73684)$ which is obtained by using moment estimates of Gamma distribution, and non-informative prior $a_i = b_i = 0.0001$, $i = 1, 2, 3, 4$.

It is concluded that Bayes estimates based on non-informative prior and the MLE of $R_{24,k}$ are very close to each other as in the previous tables. Moreover, both approximate Bayes estimates of $R_{n,k}$ are very similar. Bayes estimates of $R_{24,k}$ based on Prior 8 are greater than that of other estimates, and the corresponding HPD credible interval has the largest length with respect to other intervals. It is observed that since all the estimates of $R_{n,k}$ for $k = 12, 13, 14, 15$ values are less than 0.50, Datça district should be paid attention to for more investigations of wind energy power plant investment based on the considered scenario.

Table 4.15. Estimates of $R_{24,k}$ for different distributions

Distributions		Prior 6		Prior 7		Prior 8		Non-inf. prior	
	$\widehat{R}_{24,k}^{MLE}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$
$k = 12$									
Weibull	0.04839	0.05022	0.04994	0.05194	0.05141	0.05849	0.05743	0.04849	0.04852
Rayleigh	0.04090	0.04225	0.04225	0.04352	0.04340	0.04871	0.04879	0.04097	0.04094
Burr Type XII	0.08067	0.08291	0.08280	0.08510	0.08525	0.09537	0.09479	0.08072	0.08072
Kumaraswamy	0.07060	0.06840	0.06807	0.06612	0.06600	0.07916	0.07852	0.07068	0.07062
$k = 13$									
Weibull	0.04190	0.04349	0.04309	0.04499	0.04448	0.05067	0.04987	0.04199	0.04191
Rayleigh	0.03542	0.03659	0.03666	0.03770	0.03754	0.04221	0.04212	0.03548	0.03545
Burr Type XII	0.06993	0.07188	0.07184	0.07379	0.07367	0.08275	0.08226	0.06997	0.06987
Kumaraswamy	0.06132	0.05939	0.05928	0.05739	0.05728	0.06880	0.06820	0.06140	0.06127

Table 4.15 Continued

Distributions		Prior 6		Prior 7		Prior 8		Non-inf. prior	
	$\widehat{R}_{24,k}^{MLE}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$	$\widehat{R}_{24,k,B}^{Lin}$	$\widehat{R}_{24,k,B}^{MC}$
$k = 14$									
Weibull	0.03620	0.03757	0.03722	0.03887	0.03840	0.04380	0.04289	0.03627	0.03629
Rayleigh	0.03055	0.03155	0.03153	0.03251	0.03256	0.03641	0.03644	0.03059	0.03052
Burr Type XII	0.06070	0.06240	0.06236	0.06407	0.06403	0.07189	0.07170	0.06073	0.06074
Kumaraswamy	0.05290	0.05125	0.05103	0.04954	0.04948	0.05939	0.05892	0.05296	0.05297
$k = 15$									
Weibull	0.03160	0.03280	0.03262	0.03394	0.03354	0.03826	0.03757	0.03166	0.03163
Rayleigh	0.02663	0.02750	0.02753	0.02834	0.02825	0.03175	0.03165	0.02666	0.02677
Burr Type XII	0.05320	0.05469	0.05457	0.05616	0.05611	0.06305	0.06270	0.05322	0.05336
Kumaraswamy	0.04614	0.04471	0.04454	0.04323	0.04311	0.05183	0.05136	0.04619	0.04613

Table 4.16. ACL and CP values for the interval estimates of $R_{24,k}$ for different distributions

Distributions	ACI		HPD (Prior 6)		HPD (Prior 7)		HPD (Prior 8)		HPD (Non-inf.)	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
$k = 12$										
Weibull	0.03061	0.06616	0.03248	0.06802	0.03445	0.06982	0.03782	0.07581	0.03242	0.06682
Rayleigh	0.02591	0.05590	0.02787	0.05783	0.02905	0.05952	0.03339	0.06531	0.02657	0.05618
Burr Type XII	0.05204	0.10931	0.05492	0.11119	0.05761	0.11575	0.06441	0.12387	0.05285	0.10939
Kumaraswamy	0.04528	0.09591	0.04509	0.09279	0.04434	0.09000	0.05437	0.10540	0.04640	0.09569
$k = 13$										
Weibull	0.02645	0.05735	0.02814	0.05863	0.02948	0.06030	0.03363	0.06692	0.02723	0.05739
Rayleigh	0.02239	0.04845	0.02427	0.05034	0.02432	0.05082	0.02805	0.05581	0.02302	0.04894
Burr Type XII	0.04494	0.09491	0.04826	0.09789	0.04921	0.09895	0.05662	0.10991	0.04599	0.09551
Kumaraswamy	0.03920	0.08343	0.03878	0.07980	0.03835	0.07773	0.04721	0.09087	0.04021	0.08371

Table 4.16 Continued

Distributions	ACI		HPD (Prior 6)		HPD (Prior 7)		HPD (Prior 8)		HPD (Non-inf.)	
	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
$k = 14$										
Weibull	0.02281	0.04959	0.02473	0.05077	0.02517	0.05244	0.02901	0.05748	0.02344	0.05000
Rayleigh	0.01929	0.04181	0.02070	0.04314	0.02145	0.04395	0.02550	0.04904	0.02032	0.04248
Burr Type XII	0.03890	0.08250	0.04140	0.08340	0.04279	0.08562	0.04827	0.09430	0.04010	0.08310
Kumaraswamy	0.03373	0.07206	0.03357	0.06923	0.03342	0.06742	0.03972	0.07845	0.03435	0.07223
$k = 15$										
Weibull	0.01988	0.04333	0.02149	0.04489	0.02212	0.04539	0.02538	0.05017	0.02081	0.04351
Rayleigh	0.01679	0.03647	0.01791	0.03804	0.01858	0.03889	0.02119	0.04223	0.01687	0.03661
Burr Type XII	0.03400	0.07239	0.03667	0.07461	0.03703	0.07533	0.04307	0.08308	0.03414	0.07230
Kumaraswamy	0.02935	0.06292	0.02874	0.06107	0.02871	0.05847	0.03435	0.06788	0.02989	0.06259

4.6. CONCLUSIONS

In this chapter, we studied reliability estimation of a consecutive k-out of-n system with non-identical strength components. We derived all point and interval estimations of this reliability assuming both stress and strength components follow the proportional hazard rate model. Frequentist estimation methods and Bayesian approach with two approximate methods Lindley's approximation and MCMC method were utilized when the second parameters of underlying distributions are unknown and known.

We performed a Monte Carlo simulation study to compare the performance of proposed estimators. From the results of the simulation study, we concluded that the performances of Bayes estimators based on informative priors were better than other estimators while ML and Bayes estimators based on non-informative priors show similar performances. The implementation of proposed models was made by using the wind speed data sets from NASA's POWER project source.

5. CONCLUSIONS

In this dissertation, the estimation of stress-strength reliability of a consecutive k-out of-n system is considered in the cases of the strength components are identical and non-identical. All point and interval estimations of this reliability are derived under frequentist and Bayesian approaches when both stress and strength components follow the proportional hazard rate model. The performance of proposed estimators is compared with a simulation study and implementation of proposed models is presented by real data analysis. Detailed simulation results for each case are presented at the end of the chapters 3 and 4. We can state that this study has novelty with respect to the considered system structure as well as real data application of the obtained estimates.

For future studies, this study can be extended in different ways such as by changing the underlying distributions or data types, assuming two or more change points in strength and/or stress variables.

REFERENCES

1. Kumar UD, Crocker J, Chitra T, Saranga H. *Reliability and six sigma*. Boston: Springer US; 2006.

2. Birnbaum ZW. On a use of Mann-Whitney statistics. *In: Proceeding of 3rd Berkeley Symposium Mathematica Statistics and Probability*. University of California Press Berkeley; 1956. p. 13–17.

3. Birnbaum Z, McCarty R. A distribution-free upper confidence bounds for $Pr(Y < X)$ based on independent samples of X and Y. *The Annals of Mathematical Statistics*. 1958;29(2):558–562.

4. Kotz S, Lumelskii Y, Pensky M. *The Stress-Strength Model and its Generalizations: Theory and Applications*. Singapore: World Scientific; 2003.

5. Kundu D, Gupta RD. Estimation of $P(Y < X)$ generalized exponential distribution. *Metrika*. 2005;61:291–308.

6. Nadar M, Kızılaslan F, Papadopoulos A. Classical and Bayesian estimation of $P(Y < X)$ for Kumaraswamy's distribution.

7. Basirat M, Baratpour S, Ahmadi J. Statistical inferences for stress–strength in the proportional hazard models based on progressive type-II censored samples. *Journal of Statistical Computation and Simulation*. 2015;85(3):431–449.

8. Basirat M, Baratpour S, Ahmadi J. On estimation of stress–strength parameter using record values from proportional hazard rate models. *Communications in Statistics - Theory and Methods*. 2016;45(19):5787–5801.

9. Çetinkaya C, Genç A. Stress–strength reliability estimation under the standard two-sided power distribution. *Applied Mathematical Modelling*. 2019;65:72–88.

10. Akgül FG, Şenoğlu B. Inferences for stress–strength reliability of Burr type X dis- tributions based on ranked set sampling. *Communications in Statistics-Simulation and Computation*. 2020.

11. Basikhasteh M, Lak F, Afshari M. Bayesian estimation of stress–strength reliability for two- parameter bathtub-shaped lifetime distribution based on maximum ranked set sampling with unequal samples. *Journal of Statistical Computation and Simulation*. 2020;90(16):2975–2990.

12. Biswas A, Chakraborty S, Mukherjee M. On estimation of stress–strength reliability with log-Lindley distribution. *Journal of Statistical Computation and Simulation*. 2021;91(1):128–150.

13. Pak A, Raqab MZ, Mahmoudi MR, Band SS, Mosavi A. Estimation of stress-strength reliability $R = P(X > Y)$ based on Weibull record data in the presence of inter-record times. *Alexandria Engineering Journal*. 2021.

14. Bhattacharyya G, Johnson RA. Estimation of reliability in multicomponent stress–strength model. *Journal of the American Statistical Association*. 1974;69(348):966–970.

15. Rao GS, Aslam M, Arif OH. Estimation of reliability in multicomponent stress–strength based on two parameter exponentiated Weibull distribution. *Communications in Statistics-Theory and Methods*. 2017;46(15):7495–7502.

16. Kızılaslan F. Classical and Bayesian estimation of reliability in a multicomponent stress-strength model based on the proportional reversed hazard rate mode. *Mathematics and Computers in Simulation*. 2017;136:36–62.

17. Kızılaslan F. Classical and Bayesian estimation of reliability in a multicomponent stress-strength model based on a general class of inverse exponentiated distributions. *Statistical Papers*. 2017;59:1161–1192.

18. Akgül FG. Reliability estimation in multicomponent stress–strength model for Topp-Leone distribution. *Journal of Statistical Computation and Simulation*. 2019;89(15):2914–2929.

19. Kayal T, Tripathi YM, Dey S, Wu SJ. On estimating the reliability in a multicomponent stress-strength model based on Chen distribution. *Communications in Statistics-Theory and Methods*. 2020;49(10):2429–2447.

20. Ahmadi K, Ghafouri S. Reliability estimation in a multicomponent stress-strength model under generalized half-normal distribution based on progressive type-II censoring. *Journal of statistical computation and simulation*. 2019;89(13):2505–2548.

21. Kotb MS, Raqab MZ. Estimation of reliability for multi-component stress-strength model based on modified Weibull distribution. *Statistical Papers*. 2021;62(6):2763–2797.

22. Jha MK, Dey S, Alotaibi RM, Tripathi YM. Reliability estimation of a multicomponent stress-strength model for unit Gompertz distribution under progressive Type II censoring. *Quality and Reliability Engineering International*. 2020;36(3):965–987.

23. Maurya RK, Tripathi YM. Reliability estimation in a multicomponent stress-strength model for Burr XII distribution under progressive censoring. *Brazilian Journal of Probability and Statistics*. 2020;4(2):345–369.

24. Akgül FG. Classical and Bayesian estimation of multicomponent stress-strength reliability for exponentiated Pareto distribution. *Soft Computing*. 2021;25:9185–9197.

25. Wang L, Zhang C, Tripathi YM, Dey S, Wu SJ. Reliability analysis of Weibull multicomponent system with stress-dependent parameters from accelerated life data. *Quality and reliability engineering international*. 2021;37(6):2603–2621.

26. Kuo W, Zuo MJ. *Optimal Reliability Modeling,Principles and Applications*. New York: Wiley; 2003.

27. Kontoleon JM. Reliability Determination of a r-Successive-out-of-n:F System. *IEEE Transactions on Reliability*. 1980;29(5):437–437.

28. Chiang DT, Niu SC. Reliability of Consecutive-k-out-of-n:F System. *IEEE Transactions on Reliability*. 1981;30(1):87–898.

29. Chang GJ, Cu L, Hwang FK. *Reliabilities of consecutive-k systems*. Dordrecht, The Netherlands: Kluwer Academic Publishers; 2000.

30. Eryılmaz S. Review of recent advances in reliability of consecutive k-out-of-n and related systems. *Journal of Risk and Reliability*. 2010;224(3):225–237.

31. Zhu X, Boushaba M, Boulahia A, Zhao X. A linear m-consecutive-k-out-of-n system with sparse d of non-homogeneous Markov-dependent components. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*. 2019;233(3):328–337.

32. Dui H, Sib S, Yam CM Richard. Importance measures for optimal structure in linear consecutive-k-out-of-n systems. *Reliability Engineering System Safety*. 2018;169:339–350.

33. Li M, Hu L, Peng R, Bai Z. Reliability modeling for repairable circular consecutive-k-out-of-n: F systems with retrial feature. *Reliability Engineering System Safety*. 2021;216.

34. Eryılmaz S. Consecutive k-out-of-$n : G$ system in stress-strength setup. *Communications in Statistics -Simulation and Computation*. 2008;37(3):579–589.

35. Akıcı F. Reliability analysis of consecutive k systems in a stress-strength setup. Master's thesis, Graduate School of Natural and Applied Sciences of İzmir University of Economics; 2010.

36. Rasethuntsa TR, Nadar M. Stress-strength reliability of a non-identical-component-strengths system based on upper record values from the family of Kumaraswamy generalized distributions. *Statistics*. 2018;52(3):684–716.

37. Ali A, Khaliq S, Ali Z, Dey S. Reliability estimation of s-out-of-k system for non-identical stress-strength components. *Life Cycle Reliability and Safety Engineering*. 2018;7(1):33–41.

38. Çetinkaya C. Reliability estimation of a stress-strength model with non-identical component strengths under generalized progressive hybrid censoring scheme. *Statistics*. 2021;55(2):250–275.

39. Çetinkaya C. Reliability estimation of the stress-strength model with non-identical jointly type-II censored Weibull component strengths. *Journal of statistical computation and simulation*. 2021;91(14):2917–2936.

40. Kohansal A, Fernández AJ, Pérez-Gonz Multi-component stress-strength parameter estimation of a non-identical-component strengths system under the adaptive hybrid progressive censoring samples. *Statistics*. 2021;0(0):1–38. Available from: https://doi.org/10.1080/02331888.2021.1985499.

41. Robert C, Casella G. *Monte Carlo Statistical Methods*. New York: Spring-Verlag Press; 1999.

42. Gelman A, Carlin JB, Stern HS, Rubin DB. *Bayesian Data Analysis*. Chapman Hall, 2nd Edition; 2003.

43. Geman S, Geman D. Stochastic relaxation, gibbs distributions, and the bayesian restoration of images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 1984;6(6):721–741.

44. Metropolis N, Rosenbluth AW, Rosenblunth MN, Teller AH, Teller E. Equations of state calculations by fast computing machines. *Journal of Chemical Physics*. 1953;21:1087–1092.

45. Hastings WK. Monte Carlo sampling methods using Markov chains and their Applications. *Biometrika*. 1970;57(1):97–109.

46. R Core Team. R: A Language and Environment for Statistical Computing. Vienna, Austria; 2021. Available from: https://www.R-project.org/.

47. Faraway J, Marsaglia G, Marsaglia J, Baddeley A. goftest: Classical goodness-of-fit tests for univariate distributions.; 2019. R package version 1.2-2. Available from: https://cran.r-project.org/web/packages/goftest/goftest.pdf.

48. Lyu MR. *Handbook of software reliability engineering*. USA: McGraw-Hill; 1996.

49. Eryılmaz S, Demir S. Success runs in a sequence of exchangeable binary trials. *Journal of Statistical Planning and Inference*. 2007;137(9):2954–2963.

50. Lawless J. *Statistical models and methods for lifetime data*. New Jersey: Wiley; 2003.

51. Rao CR. *Linear statistical inference and its applications*. New York: Wiley; 1965.

52. Tierney L. Markov chains for exploring posterior distributions. *The Annals of Statistics*. 1994;22(4):1701–1728.

53. Chen MH, Shao QM. Monte Carlo estimation of Bayesian credible and HPD intervals. *The Annals of Statistics*. 1999;8(1):69–92.

54. Gradshteyn IS, Ryzhik IM. *Table of integrals, series and products*. Boston: Academic Press; 1994.

55. Akgül FG, Şenoğlu B. Comparison of wind speed distributions: a case study for Aegean coast of Turkey. *Energy Sources, Part A: Recovery, Utilization, and Environmental Effects*. 2019;0(0):1–18. Available from: https://doi.org/10.1080/15567036.2019.1663309.

56. Sparks AH. nasapower: A NASA POWER Global Meteorology, Surface Solar Energy and Climatology Data Client for R. *Journal of Open Source Software*. 2018;3(30):1035. Available from: https://doi.org/10.21105/joss.01035.

57. Sparks AH. nasapower; 2021. R package version 4.0.0. Available from: https://cran.r-project.org/web/packages/nasapower/nasapower.pdf.

58. Quarda TBMJ, Charron C, Chebana F. Review of criteria for the selection of probability distributions for wind speed data and introduction of the moment and L-moment ratio diagram methods, with a case study. *Energy Conversion and Management*. 2016;124:247–265. Available from: https://www.sciencedirect.com/science/article/pii/S0196890416305829.