STRUCTURES OF FRACTIONAL SPACES GENERATED BY POSITIVE OPERATORS WITH PERIODIC CONDITION

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Supervisor: Prof. Dr. Allaberen ASHYRALYEV

ABSTRACT

The second order differential operator A^x defined by the formula

$$A^x u = -u_{xx}(x) + \delta u(x),\ \delta > 0$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u'(x), u''(x) \in C[0,2\pi], u(x) = u(x+2\pi),\ \int_0^{2\pi} u(x)dx = 0 \right\}$$

is considered. The Green's function of the differential operator A^x is constructed. The estimates for the Green's function are obtained. The positivity of the operator A^x in the Banach space $C\,[0,2\pi]$ is established. It is proved that for any $\alpha \in (0, \frac{1}{2})$, the norms in spaces $E_\alpha = E_\alpha(C\,[0,2\pi]\,, A^x)$ and $\overset{\circ}{C}{}^{2\alpha}\,[0,2\pi]$ are equivalent. The positivity of the operator A^x in the Hölder spaces of $\overset{\circ}{C}{}^{2\alpha}\,[0,2\pi]$ is proved. The structure of fractional spaces generated by this operator is investigated.

Keywords: Positivity of Differential Operators, Fractional Spaces, Nonlocal Boundary Conditions, Greens Function.

PERİYODİK KOŞULLU POZİTİF OPERATÖRLER TARAFINDAN ÜRETİLEN KESİRLİ UZAYLARIN YAPILARI

Fatih Sabahattin TETİKOĞLU

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ÖZ

Tanım kümesi

$$D(A^x) = \left\{ u(x) : u(x), u'(x), u''(x) \in C[0, 2\pi], u(x) = u(x + 2\pi), \quad \int_0^{2\pi} u(x)dx = 0 \right\}$$

olan ve

$$A^x u = -u_{xx}(x) + \delta u(x), \ \delta > 0$$

şeklinde tanımlanan ikinci mertebeden diferansiyel operatör ele alındı. A^x operatörünün Green fonksiyonu oluşturuldu. Bu Green fonksiyon için kestirimler alındı. $C[0, 2\pi]$ Banach uzayındaki A^x operatörünün pozitifliği gösterildi. Herhangi $\alpha \in (0, \frac{1}{2})$ için, $E_\alpha = E_\alpha(C[0, 2\pi], A^x)$ ve $\overset{\circ}{C}{}^{2\alpha}[0, 2\pi])$ uzaylarındaki normların denkli ği ispatlandı. A^x operatörünün $\overset{\circ}{C}{}^{2\alpha}[0, 2\pi], \alpha \in (0, \frac{1}{2})$ Hölder uzayındaki pozitifliği kanıtlandı. Bu operatör tarafından üretilen kesirli uzayların yapıları incelendi.

Anahtar Kelimeler: Diferansiyel Operatörlerin Pozitifliği, Kesirli Uzaylar, Lokal Olmayan Sınır Şartları, Green Fonksiyonu.

To my wife

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Various local and nonlocal boundary value problems for partial differential equations can be considered as an abstract boundary value problem for the ordinary differential equation in a Banach space with a densely defined unbounded space operator. The role played by positivity of differential and difference operators in a Banach space in the study of various properties of boundary value problems for partial differential equations,of stability of difference schemes for partial differential equations and of summation Fourier series is well-known, (Ashyralyev and Sobolevskii, 1994) , (Simmons, 1963), (Fattorini, 1985).The positivity of wider class of differential and difference operators have been studied by many researchers, (Ashyralyev and Sobolevskii, 2004), (Ashyralyev and Sobolevskii, 1984), (Ashyralyev, 2006), (Sobolevskii, 2005), (Stewart, 1980), (Solomyak, 1960), (Simirnitskii and Sobolevskii, 1981), (Danelich, 1989), (Ashyralyev and Karakaya, 1995), (Ashyralyev and Yaz, 2006), (Ashyralyev, 1995), (Ashyralyev and Kendirli, 2000), (Ashyralyev and Kendirli, 2001), (Ashyralyev and Yenial-Altay, 2005), (Ashyralyev and Yakubov, 1998), (Ashyralyev and Prenov, 2012), (Bazarov, 1989), (Alibekov, 1978), (Nalbant, 2011).

Let E be a Banach space and $A : D(A) \subset E \to E$ be a linear unbounded operator densely defined in E. We call A positive operator A in the Banach space if the operator $(\lambda + A)$ has a bounded inverse in E and for any $\lambda > 0$ and the following estimate holds:

$$\left\|(\lambda + A)^{-1}\right\|_{E\to E} \leq \frac{M}{\lambda + 1}.$$

Throughout the present work, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use $M(\alpha, \beta, ...)$ to stress the fact that the constant depends only on $M(\alpha, \beta, ...)$.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$ consisting of those $v \in E$ for which the norm,

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \left\|A(\lambda + A)^{-1}v\right\|_E + \|v\|_E$$

is finite.

Let us introduce the Banach space $C^\beta\,[0, 2\pi]$, $\beta \in (0, 1)$ of all continuous functions $\varphi(x)$ defined on $[0, 2\pi]$ and satisfying a Hölder condition for which the following norm is finite

$$\|\varphi\|_{C^\beta[0,2\pi]} = \|\varphi\|_{C[0,2\pi]} + \sup_{\substack{x,\ x+\tau\in[0,2\pi],\\ \tau\neq 0}} \frac{|\varphi(x+\tau) - \varphi(x)|}{|\tau|^\beta},$$

where $C\,[0, 2\pi]$ is the Banach space of all continuous functions $\varphi(x)$ defined on $[0, 2\pi]$ with the norm

$$\|\varphi\|_{C[0,2\pi]} = \max_{x\in[0,2\pi]} |\varphi(x)|\,.$$

Let

$$\overset{\circ}{C}{}^\beta\,[0, 2\pi] = \left\{ u(x) \in C^\beta\,[0, 2\pi] : \int_0^{2\pi} u(x)dx = 0, u(0) = u(2\pi) \right\}.$$

The aim of this thesis is to investigate the positivity of second order differential operators with periodic boundary conditions, the structure of fractional spaces generated by these operators, to establish the equivalence of the norm of these fractional spaces and Hölder spaces, and to obtain the positivity of this differential operator in Hölder spaces.

Let us briefly describe the contents of the various chapters of the thesis. It consists of four chapters.

First chapter is the introduction.

Second chapter presents two sections. In the first section, the Green's function of the second order difference operator A^x defined by

$$A^x u = -u_{xx}(x) + \delta u(x),\ \ \delta > 0$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u'(x), u''(x) \in C[0, 2\pi], u(x) = u(x + 2\pi),\ \int_0^{2\pi} u(x)dx = 0 \right\}$$

is considered.

The Green's function of the differential operator A^x is constructed. The estimates for the Green's function are obtained. The positivity of the operator A^x in the Banach space $C[0,2\pi]$ is established. It is proved that for any $\alpha \in (0, \frac{1}{2})$, the norms in spaces $E_\alpha = E_\alpha(C[0,2\pi], A^x)$ and $\overset{\circ}{C}^{2\alpha}[0,2\pi]$ are equivalent. The positivity of the operator A^x in the Hölder spaces of $\overset{\circ}{C}^{2\alpha}[0,2\pi]$ is proved. The structure of fractional spaces generated by this operator is investigated.

Third chapter consists the positivity of the second order of operator with dependent in x variable.

Fourth chapter contains conclusions.

CHAPTER 2

THE GREEN'S FUNCTION OF A^X AND POSITIVITY OF A^X IN C

2.1 THE GREEN'S FUNCTION OF A^X AND POSITIVITY OF A^X IN C

We consider the differential operator A^x defined by the formula

$$A^x u = -u_{xx}(x) + \delta u(x), \ \delta > 0 \tag{2.1}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u'(x), u''(x) \in C[0, 2\pi], u(x) = u(x + 2\pi), \ \int_0^{2\pi} u(x)dx = 0 \right\}.$$

We will study the resolvent of the operator A^x, i.e.

$$A^x u + \lambda u = \varphi \tag{2.2}$$

or

$$\begin{cases} -u''(x) + (\delta + \lambda)\, u(x) = \varphi(x), \quad 0 < x < 2\pi, \\ u(0) = u(2\pi), \ \int_0^{2\pi} u(x)dx = 0 \end{cases} \tag{2.3}$$

will be investigated. The Green's function of A^x is constructed. The estimates for the Green's function will be obtained. The positivity of the operator A^x in the Banach space $C\,[0, 2\pi]$ is established. It is proved that for any $\alpha \in (0, \frac{1}{2})$ the norms in space $E_\alpha(C\,[0, 2\pi]\,, A)$ and $C^{2\alpha}\,[0, 2\pi]$ are equivalent. The positivity of A^x in the Hölder spaces of $C^{2\alpha}\,[0, 2\pi]\,, \, \alpha \in (0, \frac{1}{2})$ is proved. The structure of fractional spaces generated by this operator will be investigated.

Lemma 2.1. Assume that $\varphi \in C\,[0, 2\pi]$ and, $\varphi(0) = \varphi(2\pi)$, $\int\limits_0^{2\pi} \varphi(x)dx = 0$. For any $\lambda \geq 0$ problem (2.3) is uniquely solvable and the following formula holds:

$$u(x) = (A^x + \lambda)^{-1} \varphi(x) = \int\limits_0^{2\pi} G(x, s; \lambda)\varphi(s)ds, \tag{2.4}$$

where

$$G(x, s; \lambda) = \frac{1}{2\sqrt{\delta + \lambda}} \frac{1}{I - e^{-\sqrt{\delta+\lambda}2\pi}} \times \begin{cases} e^{-\sqrt{\delta+\lambda}(x-s)} + e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}, & 0 \leq s \leq x, \\ e^{-\sqrt{\delta+\lambda}(s-x)} + e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}, & x \leq s \leq 2\pi. \end{cases} \tag{2.5}$$

Proof. Using the resolvent equation(2.2), we get

$$-u''(x) + (\delta + \lambda)\, u(x) = \varphi(x), \quad 0 < x < 2\pi, \quad u(0) = u(2\pi), \int\limits_0^{2\pi} u(x)dx = 0.$$

We have the formula

$$\begin{aligned} u(x) = {} & \left(I - e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1} \left\{ \left(e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(4\pi-x)}\right) u(0) \right. \\ & + \left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right) u(2\pi) - \left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right) \\ & \left. \times \left(2\sqrt{\delta + \lambda}\right)^{-1} \int\limits_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right) \varphi(s)ds \right\} \\ & + (2\sqrt{\delta + \lambda})^{-1} \int\limits_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right) \varphi(s)ds \end{aligned} \tag{2.6}$$

for the solution of the following problem

$$\begin{cases} -u''(x) + (\delta + \lambda)\, u(x) = \varphi(x), \quad 0 < x < 2\pi, \\ u(0), u(2\pi) \text{ are given.} \end{cases}$$

From this formula and $\int\limits_0^{2\pi} u(x)dx = 0, u(0) = u(2\pi)$ it follows that

$$\int_0^{2\pi} u(x)dx = \left(I - e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1} \int_0^{2\pi} \Big\{(e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(4\pi-x)})u(0)$$

$$+(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)})u(0) - (e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)})$$

$$\times(2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi} (e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)})\varphi(s)ds \Big\} dx$$

$$+(2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi}\int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right) \varphi(s)dsdx = 0.$$

Then

$$(I - e^{-\sqrt{\delta+\lambda}4\pi})^{-1} \Big\{ \frac{1}{\sqrt{\delta+\lambda}} \Big(-e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(4\pi-x)} + e^{-\sqrt{\delta+\lambda}(2\pi-x)}$$

$$+e^{-\sqrt{\delta+\lambda}(2\pi+x)}\Big)\Big]_0^{2\pi} u(0) - \left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} + e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right)\Big]_0^{2\pi}$$

$$\times(2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right) \varphi(s)ds \Big\}$$

$$+(2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi}\int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)dsdx = 0,$$

where

$$\frac{1}{\sqrt{\delta+\lambda}}\, 2\left(I - e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1}\left(I - e^{-\sqrt{\delta+\lambda}2\pi}\right)^2 u(0) =$$

$$\frac{1}{\sqrt{\delta+\lambda}}\left(I - e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1}\left(I - e^{-\sqrt{\delta+\lambda}2\pi}\right)^2$$

$$\times(2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right) \varphi(s)ds$$

$$-(2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi}\int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)dsdx$$

or

$$u(0) = \left(4\sqrt{\delta+\lambda}\right)^{-1} \int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right) \varphi(s)ds - (4\sqrt{\delta+\lambda})^{-1}$$

$$\times\left(I + e^{-\sqrt{\delta+\lambda}2\pi}\right)\left(I - e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1} \int_0^{2\pi}\int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)dsdx.$$

Let

$$B=\int_0^{2\pi}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}|x-s|}-e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)dsdx$$

$$=\int_0^{2\pi}\left(\int_0^{2\pi}e^{-\sqrt{\delta+\lambda}|x-s|}\,dx\right)\varphi(s)ds-\int_0^{2\pi}\left(\int_0^{2\pi}e^{-\sqrt{\delta+\lambda}(x+s)}dx\right)\varphi(s)ds$$

$$=\int_0^{2\pi}B_1\varphi(s)ds-\int_0^{2\pi}B_2\varphi(s)ds,$$

where

$$B_1=\int_0^{2\pi}e^{-\sqrt{\delta+\lambda}|x-s|}\,dx \quad\text{and}\quad B_2=\int_0^{2\pi}e^{-\sqrt{\delta+\lambda}(x+s)}dx.$$

Let $0<s<2\pi$. Then

$$B_1=\int_0^{2\pi}e^{-\sqrt{\delta+\lambda}|x-s|}\,dx=\int_0^{s}e^{-\sqrt{\delta+\lambda}(s-x)}\,dx+\int_s^{2\pi}e^{-\sqrt{\delta+\lambda}(x-s)}dx$$

$$=\frac{1}{\sqrt{\delta+\lambda}}e^{-\sqrt{\delta+\lambda}(s-x)}\Big]_0^s-\frac{1}{\sqrt{\delta+\lambda}}e^{-\sqrt{\delta+\lambda}(x-s)}\Big]_s^{2\pi}$$

$$=\frac{1}{\sqrt{\delta+\lambda}}\left[1-e^{-\sqrt{\delta+\lambda}s}-\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)}-1\right)\right]$$

$$=\frac{1}{\sqrt{\delta+\lambda}}\left[2-e^{-\sqrt{\delta+\lambda}s}-e^{-\sqrt{\delta+\lambda}(2\pi-s)}\right],$$

$$B_2=\int_0^{2\pi}e^{-\sqrt{\delta+\lambda}(x+s)}dx$$

$$=-\frac{1}{\sqrt{\delta+\lambda}}e^{-\sqrt{\delta+\lambda}(x+s)}\Big]_0^{2\pi}$$

$$=-\frac{1}{\sqrt{\delta+\lambda}}\left(e^{-\sqrt{\delta+\lambda}(2\pi+s)}-e^{-\sqrt{\delta+\lambda}s}\right).$$

Then

$$B=\int_0^{2\pi}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}|x-s|}-e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)dsdx$$

$$=\int_0^{2\pi}\frac{1}{\sqrt{\delta+\lambda}}\left(2-e^{-\sqrt{\delta+\lambda}s}-e^{-\sqrt{\delta+\lambda}(2\pi-s)}\right)\varphi(s)ds$$

$$-\int_0^{2\pi}-\frac{1}{\sqrt{\delta+\lambda}}\left(e^{-\sqrt{\delta+\lambda}(2\pi+s)}-e^{-\sqrt{\delta+\lambda}s}\right)\varphi(s)ds$$

$$=\frac{1}{\sqrt{\delta+\lambda}}\int_0^{2\pi}\left(2-2e^{-\sqrt{\delta+\lambda}s}-e^{-\sqrt{\delta+\lambda}(2\pi-s)}+e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right)\varphi(s)ds.$$

Using this formula, we can write

$$u(0)=\left(I-e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1}\left(4\sqrt{\delta+\lambda}\right)^{-1}\left\{\left(I-e^{-\sqrt{\delta+\lambda}2\pi}\right)\right.$$

$$\times\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right)\varphi(s)ds-\left(I+e^{-\sqrt{\delta+\lambda}2\pi}\right)$$

$$\left.\times\left[\int_0^{2\pi}\left(2-2e^{-\sqrt{\delta+\lambda}s}-e^{-\sqrt{\delta+\lambda}(2\pi-s)}+e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right)\varphi(s)ds\right]\right\}$$

$$=\left(I-e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1}\left(4\sqrt{\delta+\lambda}\right)^{-1}\left\{\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)}\right.\right.$$

$$-e^{-\sqrt{\delta+\lambda}(2\pi+s)}-e^{-\sqrt{\delta+\lambda}(4\pi-s)}+e^{-\sqrt{\delta+\lambda}(4\pi+s)}-2+2e^{-\sqrt{\delta+\lambda}s}$$

$$+e^{-\sqrt{\delta+\lambda}(2\pi-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+s)}+e^{-\sqrt{\delta+\lambda}s}-2e^{-\sqrt{\delta+\lambda}2\pi}$$

$$\left.\left.+2e^{-\sqrt{\delta+\lambda}(2\pi+s)}+e^{-\sqrt{\delta+\lambda}(4\pi-s)}-e^{-\sqrt{\delta+\lambda}(4\pi+s)}\right)\varphi(s)ds\right\}$$

$$=\left(I-e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1}\left(4\sqrt{\delta+\lambda}\right)^{-1}\left\{\int_0^{2\pi}\left(2e^{-\sqrt{\delta+\lambda}(2\pi-s)}\right.\right.$$

$$\left.\left.+2e^{-\sqrt{\delta+\lambda}s}-2e^{-\sqrt{\delta+\lambda}2\pi}-2\right)\varphi(s)ds\right\}$$

or

$$u(0)=\left(I-e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1}\left(2\sqrt{\delta+\lambda}\right)^{-1}\left[\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)}\right.\right.$$

$$\left.\left.+e^{-\sqrt{\delta+\lambda}s}-e^{-\sqrt{\delta+\lambda}2\pi}-1\right)\varphi(s)ds\right].$$

By using (2.6), we get

$$u(x)=\left(I-e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1}\left\{\left(e^{-\sqrt{\delta+\lambda}x}-e^{-\sqrt{\delta+\lambda}(4\pi-x)}+e^{-\sqrt{\delta+\lambda}(2\pi-x)}\right.\right.$$

$$\left.-e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right)u(0)-\left(e^{-\sqrt{\delta+\lambda}(2\pi-x)}-e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right)$$

$$\left.\times\left(2\sqrt{\delta+\lambda}\right)^{-1}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right)\varphi(s)ds\right\}$$

$$+\left(2\sqrt{\delta+\lambda}\right)^{-1}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}|x-s|}-e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)ds$$

$$= \left(I - e^{-\sqrt{\delta+\lambda}4\,\pi}\right)^{-1}\left(2\sqrt{\delta+\lambda}\right)^{-1}\left\{\left(e^{-\sqrt{\delta+\lambda}x} + e^{-\sqrt{\delta+\lambda}(2\pi-x)}\right)\right.$$

$$\left.\times\left[\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} + e^{-\sqrt{\delta+\lambda}s} - e^{-\sqrt{\delta+\lambda}2\pi} - 1\right)\varphi(s)ds\right]\right\}$$

$$-\left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right)\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right)\varphi(s)ds\Bigg\}$$

$$+\left(2\sqrt{\delta+\lambda}\right)^{-1}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)ds$$

$$= \left(I - e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1}\left(2\sqrt{\delta+\lambda}\right)^{-1}\left\{\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}\right.\right.$$

$$+e^{-\sqrt{\delta+\lambda}(x+s)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)} - e^{-\sqrt{\delta+\lambda}x} + e^{-\sqrt{\delta+\lambda}(4\pi-x-s)}$$

$$\left.+e^{-\sqrt{\delta+\lambda}(2\pi-x+s)} - e^{-\sqrt{\delta+\lambda}(4\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi-x)}\right)\varphi(s)ds$$

$$-\int_0^{2\pi} e^{-\sqrt{\delta+\lambda}(4\pi-x-s)} - e^{-\sqrt{\delta+\lambda}(\,4\pi-x+s)} - e^{-\sqrt{\delta+\lambda}(4\pi+x-s)}$$

$$\left.+e^{-\sqrt{\delta+\lambda}(\,4\pi+x+s)}\varphi(s)ds - \left(I - e^{-\sqrt{\delta+\lambda}4\pi}\right)\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)ds\right\}$$

$$+\left(2\sqrt{1+\lambda}\right)^{-1}\int_0^{2\pi} e^{-\sqrt{1+\lambda}|x-s|}\varphi(s)ds$$

or

$$u(x) = \int_0^{2\pi}\frac{1}{2\sqrt{\delta+\lambda}}\left[\frac{1}{I - e^{-\sqrt{\delta+\lambda}2\,\pi}}\left(e^{-\sqrt{\delta+\lambda}(2\pi+x-s)} + e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}\right.\right.$$

$$\left.\left.-e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}x}\right) + e^{-\sqrt{\delta+\lambda}|x-s|}\right]\varphi(s)ds.$$

Then,

$$G(x,s;\lambda) = \frac{1}{2\sqrt{\delta+\lambda}}\left[e^{-\sqrt{\delta+\lambda}|x-s|} + \frac{1}{I - e^{-\sqrt{\delta+\lambda}2\pi}}\right.$$

$$\left.\times\left(e^{-\sqrt{\delta+\lambda}(2\pi+x-s)} + e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}\right)\right]$$

or

$$G(x,s;\lambda) = \frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{I-e^{-\sqrt{\delta+\lambda}2\pi}} \times \begin{cases} e^{-\sqrt{\delta+\lambda}(x-s)} + e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}, 0 \le s \le x, \\ \\ e^{-\sqrt{\delta+\lambda}(s-x)} + e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}, x \le s \le 2\pi. \end{cases}$$

Lemma 2.1. is proved.

Note that, the following pointwise estimate for $G(x,s\,;\lambda)$ holds:

$$|G(x,s\,;\lambda)| \le \frac{M}{\sqrt{\delta+\lambda}} \begin{cases} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}, \ 0 \le s \le x-\pi, \\ e^{-\sqrt{\delta+\lambda}(x-s)}, \ x-\pi \le s \le x, \\ e^{-\sqrt{\delta+\lambda}(s-x)}, \ x \le s \le \pi+x, \\ e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}, \ \pi+x \le s \le 2\pi. \end{cases} \tag{2.7}$$

Lemma 2.2. The following estimate holds:

$$\int_0^{2\pi} |G(x,s;\lambda)|ds \le \frac{1}{\delta+\lambda}. \tag{2.8}$$

Proof. Using (2.5) and the triangle inequality, we get

$$\int_0^{2\pi} |G(x,s\,;\lambda)|ds \le \frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{I-e^{-\sqrt{\delta+\lambda}2\pi}}\left\{\int_0^x \left(e^{-\sqrt{\delta+\lambda}(x-s)} + e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}\right)ds\right.$$

$$\left.+\int_x^{2\,\pi}\left(e^{-\sqrt{\delta+\lambda}(s-x)} + e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}\right)ds\right\}$$

$$= \frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{I-e^{-\sqrt{\delta+\lambda}2\pi}}\left\{\frac{1}{\sqrt{\delta+\lambda}}\left(e^{-\sqrt{\delta+\lambda}(x-s)} - e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}\right)\right]_0^x$$

$$\left.+\frac{1}{\sqrt{\delta+\lambda}}\left(-e^{-\sqrt{\delta+\lambda}(s-x)} + e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}\right)\right]_x^{2\pi}\right\}$$

$$= \frac{1}{2\,(\delta+\lambda)}\frac{1}{\left(I-e^{-\sqrt{\delta+\lambda}2\pi}\right)}2\left(I-e^{-\sqrt{\delta+\lambda}2\pi}\right) = \frac{1}{\delta+\lambda}.$$

Lemma 2.2. is proved.

Theorem 2.1. The operator $(\lambda + A^x)$ has a bounded in $C\,[0, 2\pi]$ inverse for any $\lambda \geq 0$ and the following estimate holds:

$$\left\|(A^x + \lambda)^{-1}\right\|_{C[0,2\pi]\to C[0,2\pi]} \leq \frac{1}{\delta + \lambda}. \tag{2.9}$$

Proof. Using the formula (2.4) and triangle inequality, we get

$$|u(x)| \leq \int_0^{2\pi} |G\,(x, s\,; \lambda)|\, ds \max_{0\leq s\leq 2\pi} |\varphi(s)| \tag{2.10}$$

for any $x \in [0, 2\pi]$ .Using estimate (2.10), we get

$$\max_{x\in[0,2\pi]} |u(x)| \leq \max_{x\in[0,2\pi]} \int_0^{2\pi} |G\,(x, s\,; \lambda)|\, ds\, \|\varphi\|_{C[0,2\pi]}\,.$$

Then, we have that

$$\left\|(A^x + \lambda)^{-1}\,\varphi\right\|_{C[0,2\pi]} \leq \frac{1}{\delta + \lambda}\, \|\varphi\|_{C[0,2\pi]}\,.$$

From that it follows estimate (2.9). Theorem 2.1 is proved.

2.2 THE STRUCTURES OF FRACTIONAL SPACES GENERATED BY A^X AND POSITIVITY OF A^X IN HÖLDER SPACES

Clearly, the operator commutes A^x and its resolvent $(A^x + \lambda)^{-1}$. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(C\,[0, 2\pi]\,, A^x)$, we get

$$\left\|(A^x + \lambda)^{-1}\right\|_{E_\alpha\to E_\alpha} \leq \left\|(A^x + \lambda)^{-1}\right\|_{C[0,2\pi]\to C[0,2\pi]}\,.$$

Thus, from Theorem 2.1 it follows that A^x is a positive operator in the fractional spaces $E_\alpha(C\,[0, 2\pi]\,, A^x)$. Moreover, we have the following result.

Theorem 2.2. For $\alpha \in (0, \frac{1}{2})$, the norms of the spaces $E_\alpha(C\,[0, 2\pi]\,, A^x)$ and the Hölder space $\overset{\circ}{C}{}^{2\alpha}\,[0, 2\pi]$ are equivalent. Here

$$\overset{\circ}{C}{}^{2\alpha}\,[0, 2\pi] = \left\{ u(x) \in C^{2\alpha}\,[0, 2\pi] : \int_0^{2\pi} u(x)dx = 0, u(0) = u(2\pi) \right\}.$$

Proof. For any $\lambda \geq 0$ we have the obvious equality

$$A^x(A^x+\lambda)^{-1}\varphi(x) = (A^x+\lambda-\lambda)\,(A^x+\lambda)^{-1}\varphi(x) = \varphi(x) - \lambda(A^x+\lambda)^{-1}\varphi(x).$$

By formula (2.4), we can write

$$A^x(A^x+\lambda)^{-1}\varphi(x) = \varphi(x) - \lambda\int_0^{2\pi} G\,(x,s\,;\lambda)\;\;\varphi(s)ds \tag{2.11}$$

$$= \frac{\delta+\lambda}{\delta+\lambda}\varphi(x) - \lambda\int_0^{2\pi} G\,(x,s\,;\lambda)\;\;\varphi(s)ds$$

$$= \frac{\delta}{\delta+\lambda}\varphi(x) + \frac{\lambda}{\delta+\lambda}\varphi(x) - \lambda\int_0^{2\pi} G\,(x,s\,;\lambda)\,\varphi(s)ds$$

and

$$A^x(A^x+\lambda)^{-1}\varphi(x) = \frac{\delta}{\delta+\lambda}\varphi(x) + \lambda\int_0^{2\pi} G\,(x,s;\lambda)\,(\varphi(x)-\varphi(s))\,ds. \tag{2.12}$$

Then,

$$\begin{aligned}\lambda^\alpha A^x(A^x+\lambda)^{-1}\varphi(x) &= \frac{\delta\lambda^\alpha}{\delta+\lambda}\varphi(x) + \lambda^{\alpha+1}\int_0^{2\pi} G\,(x,s;\lambda)\,(\varphi(x)-\varphi(s))\,ds \\ &= P_1(x)+P_2(x),\end{aligned}$$

where

$$\begin{aligned}P_1(x) &= \frac{\delta\lambda^\alpha}{\delta+\lambda}\varphi(x), \\ P_2(x) &= \lambda^{\alpha+1}\int_0^{2\pi} G\,(x,s;\lambda)\,(\varphi(x)-\varphi(s))\,ds.\end{aligned}$$

Using the definition of norm of space $C^{2\alpha}\,[0,2\pi]$ norm and $\frac{\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda} \leq 1$, we can write

$$|P_1(x)| \leq \frac{\delta\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda}\,|\varphi(x)| \leq \delta \max_{0\leq x\leq 2\pi}|\varphi(x)| \leq \delta\,\|\varphi\|_{C^{2\alpha}[0,2\pi]}$$

for any $x \in [0,2\pi]$. Then,

$$\max_{0\leq x\leq 2\pi}|P_1(x)| \leq \delta\,\|\varphi\|_{C^{2\alpha}[0,2\pi]}$$

or

$$\|P_1\|_{C[0,2\pi]} \leq \delta\,\|\varphi\|_{C^{2\alpha}[0,2\pi]}\,. \tag{2.13}$$

Then, using estimate(2.7), we get

$$|P_2(x)| \le \lambda^{\alpha+1} \int\limits_0^{2\pi} |G\,(x,s;\lambda)| \;\; |\varphi(x)-\varphi(s)|\,ds$$

$$\le \frac{M\;\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \left(\int\limits_0^{x-\pi} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)} \;\; |\varphi(x)-\varphi(s)|\,ds \right.$$

$$+ \int\limits_{x-\pi}^{x} e^{-\sqrt{\delta+\lambda}(x-s)} \;\; |\varphi(x)-\varphi(s)|\,ds$$

$$+ \int\limits_{x}^{x+\pi} e^{-\sqrt{\delta+\lambda}(s-x)} \;\; |\varphi(x)-\varphi(s)|\,ds$$

$$\left. + \int\limits_{x+\pi}^{2\pi} e^{-\sqrt{\delta+\lambda}(2\pi+x-s)} \;\; |\varphi(x)-\varphi(s)|\,ds \right)$$

$$= P_{21}(x)+P_{22}(x)+P_{23}(x)+P_{24}(x),$$

where

$$P_{21}(x) = \frac{M\;\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_0^{x-\pi} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)} \;\; |\varphi(x)-\varphi(s)|\,ds,$$

$$P_{22}(x) = \frac{M\;\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_{x-\pi}^{x} e^{-\sqrt{\delta+\lambda}(x-s)} \;\; |\varphi(x)-\varphi(s)|\,ds,$$

$$P_{23}(x) = \frac{M\;\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_{x}^{x+\pi} e^{-\sqrt{\delta+\lambda}(s-x)} \;\; |\varphi(x)-\varphi(s)|\,ds,$$

$$P_{24}(x) = \frac{M\;\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_{x+\pi}^{2\pi} e^{-\sqrt{\delta+\lambda}(2\pi+x-s)} \;\; |\varphi(x)-\varphi(s)|\,ds.$$

Clearly, using the condition $\varphi(s)=\varphi(s+2\pi)$, $P_{21}(x)$ can be rewritten as

$$P_{21}(x) = \frac{M\;\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_0^{x-\pi} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)} \;\; |\varphi(x)-\varphi(2\pi+s)|\,ds$$

$$\le \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\;\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_0^{x-\pi} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)} \;\; (x-2\pi-s)^{2\alpha}\,ds.$$

Using the following substitution $\sqrt{\delta+\lambda}(2\pi - x + s) = p$, we get

$$P_{21}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_{0}^{\infty} e^{-p}\ \frac{p^{2\alpha}}{(\delta+\lambda)^{\frac{2\alpha}{2}}} \frac{dp}{\sqrt{\delta+\lambda}}$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\left(\delta+\lambda\right)^{\alpha+1}} \left(\int\limits_{0}^{\infty} e^{-p}\ p^{2\alpha} dp \right)$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\left(\delta+\lambda\right)^{\alpha+1}} \Gamma\left(2\alpha+1\right)$$

for any $x \in [0, 2\pi]$. Then,

$$\max_{x\in[0,2\pi]} P_{21}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M}{\alpha} \Gamma\left(2\alpha+1\right).$$

Let us estimate $P_{22}(x)$.

$$P_{22}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_{x-\pi}^{x} e^{-\sqrt{\delta+\lambda}(x-s)}\ (x-s)^{2\alpha}\, ds.$$

Using the following substitution $\sqrt{\delta+\lambda}(x - s) = p$, we get

$$P_{22}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \left(-\int\limits_{\infty}^{0} e^{-p}\ \frac{p^{2\alpha}}{(\delta+\lambda)^{\frac{2\alpha}{2}}} \frac{dp}{\sqrt{\delta+\lambda}} \right)$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\left(\delta+\lambda\right)^{\alpha+1}} \left(\int\limits_{0}^{\infty} e^{-p}\ p^{2\alpha} dp \right)$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\left(\delta+\lambda\right)^{\alpha+1}} \Gamma\left(2\alpha+1\right)$$

for any $x \in [0, 2\pi]$. Then,

$$\max_{x\in[0,2\pi]} P_{22}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M}{\alpha} \Gamma\left(2\alpha+1\right).$$

Let us estimate $P_{23}(x)$.

$$P_{23}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_{x}^{x+\pi} e^{-\sqrt{\delta+\lambda}(s-x)}\ (s-x)^{2\alpha}\ ds.$$

Using the following substitution $\sqrt{\delta+\lambda}(s-x)=p$, we get

$$P_{23}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_0^\infty e^{-p} \frac{p^{2\alpha}}{(\delta+\lambda)^{\frac{2\alpha}{2}}} \frac{dp}{\sqrt{\delta+\lambda}}$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\,(\delta+\lambda)^{\alpha+1}} \left(\int\limits_0^\infty e^{-p}\, p^{2\alpha} dp \right)$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\,(\delta+\lambda)^{\alpha+1}} \Gamma\left(2\alpha+1\right)$$

for any $x \in [0,2\pi]$. Then,

$$\max_{x\in[0,2\pi]} P_{23}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M}{\alpha} \Gamma\left(2\alpha+1\right).$$

Let us estimate for $P_{24}(x)$.

$$P_{24}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int\limits_{x+\pi}^{2\pi} e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}\ (2\pi+x-s)^{2\alpha}\, ds$$

Using the following substitution $\sqrt{\delta+\lambda}(2\pi-x+s)=p$, we get

$$P_{24}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \left(-\int\limits_\infty^0 e^{-p} \frac{p^{2\alpha}}{(\delta+\lambda)^{\frac{2\alpha}{2}}} \frac{dp}{\sqrt{\delta+\lambda}} \right)$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\,(\delta+\lambda)^{\alpha+1}} \left(\int\limits_0^\infty e^{-p}\, p^{2\alpha} dp \right)$$

$$\leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M\ \lambda^{\alpha+1}}{2\,(\delta+\lambda)^{\alpha+1}} \Gamma\left(2\alpha+1\right)$$

for any $x \in [0,2\pi]$. Then,

$$\max_{x\in[0,2\pi]} P_{21}(x) \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M}{\alpha} \Gamma\left(2\alpha+1\right).$$

Then, we can write

$$\max_{x\in[0,2\pi]} |P_2(x)| \leq \max_{x\in[0,2\pi]} P_{21}(x) + \max_{x\in[0,2\pi]} P_{22}(x) + \max_{x\in[0,2\pi]} P_{23}(x) + \max_{x\in[0,2\pi]} P_{24}(x)$$

or

$$\|P_2\|_{C[0,2\pi]} \leq \|\varphi\|_{C^{2\alpha}[0,2\pi]} \frac{M(\delta)}{\alpha} \Gamma\left(2\alpha+1\right). \tag{2.14}$$

Using estimate (2.13) and (2.14), we get

$$\max_{x\in[0,2\pi]} \left|\lambda^\alpha A^x(\lambda+A^x)^{-1}\varphi(x)\right| \ \leq \frac{M(\delta)}{\alpha} \|\varphi\|_{C^{2\alpha}[0,2\pi]} + \frac{M(\delta)}{\alpha} \Gamma\left(2\alpha+1\right) \|\varphi\|_{C^{2\alpha}[0,2\pi]}$$

for any $\lambda \geq 0$. Thus,

$$\|\varphi\|_{E_\alpha(C[0,2\pi],\ A^x)} \leq \frac{M(\delta)}{\alpha}\left[1+\Gamma\left(2\alpha+1\right)\right]\|\varphi\|_{C^{2\alpha}[0,2\pi]}.$$

Now, let us prove the opposite inequality. For any positive operator A^x in the Banach space, we can write

$$I = \int_0^\infty A^x(\lambda + A^x)^{-2} d\lambda,$$

we have I is the identity operator. From this relation and formula (2.4), it follows that

$$\varphi(x) = \int_0^\infty (A^x+\lambda)^{-1} A^x (A^x+\lambda)^{-1}\varphi(x)\ d\lambda$$

$$= \int_0^\infty \int_0^{2\pi} G(x,s;\lambda)\, A^x(A^x+\lambda)^{-1}\varphi(s)\, ds d\lambda.$$

Consequently,

$$\varphi(x_1) - \varphi(x_2) = \int_0^\infty \int_0^{2\pi} \left(G(x_1,s;\lambda) - G(x_2,s;\lambda)\right) A^x(A^x+\lambda)^{-1}\varphi(s) ds d\lambda$$

$$= \int_0^\infty \lambda^{-\alpha} \int_0^{2\pi} \left(G(x_1,s;\lambda) - G(x_2,s;\lambda)\right) \lambda^\alpha A^x(A^x+\lambda)^{-1}\varphi(s) ds d\lambda$$

whence

$$|\varphi(x_1) - \varphi(x_2)| \leq \left(\int_0^\infty \lambda^{-\alpha} \int_0^{2\pi} |G(x_1,s;\lambda) - G(x_2,s;\lambda)|\, ds d\lambda\right) \|\varphi\|_{E_\alpha(C^{2\alpha}[0,2\pi],A)}.$$

Let

$$T = |x_1 - x_2|^{-2\alpha}\left(\int_0^\infty \lambda^{-\alpha} \int_0^{2\pi} |G(x_1,s;\lambda) - G(x_2,s;\lambda)|\ ds d\lambda\right).$$

Then for any $x_1, x_2 \in [0,2\pi]$ such that $x_2 \geq x_1$, we have that

$$\frac{|\varphi(x_1) - \varphi(x_2)|}{|x_1 - x_2|^{2\alpha}} \leq T\ \ \|\varphi\|_{E_\alpha(C^{2\alpha}[0,2\pi],A^x)}.$$

Now, we will prove that

$$T \leq \frac{M(\delta)}{2\alpha\left(1-2\alpha\right)}. \tag{2.15}$$

Note that

$$G(x_1,s;\lambda)-G(x_2,s;\lambda)=\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}$$

$$\times\begin{cases} e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}, 0\le s\le x_1-\pi,\\ e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}, x_1-\pi\le s\le x_2-\pi,\\ e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}, x_2-\pi\le s\le x_1,\\ e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}, x_1\le s\le \frac{x_1+x_2}{2},\\ e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}, \frac{x_1+x_2}{2}\le s\le x_2,\\ e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}, x_2\le s\le x_1+\pi,\\ e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}, x_1+\pi\le s\le x_2+\pi,\\ e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}, x_2+\pi\le s\le 2\pi.\end{cases}$$

Then

$$T=|x_1-x_2|^{-2\alpha}\int_0^\infty\lambda^{-\alpha}$$

$$\times\Bigg[\int_0^{x_1-\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\right|ds$$

$$+\int_{x_1-\pi}^{x_2-\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\right|ds$$

$$+\int_{x_2-\pi}^{x_1}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}\right|ds$$

$$+\int_{x_1}^{\frac{x_1+x_2}{2}}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}\right|ds$$

$$+\int_{\frac{x_1+x_2}{2}}^{x_2}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}\right|ds$$

$$+\int_{x_2}^{x_1+\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}\right|ds$$

$$+\int_{x_1+\pi}^{x_2+\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}\right|ds$$

$$+\int_{x_2+\pi}^{2\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}\right|ds\Bigg]d\lambda$$

$$=T_1+T_2+T_3+T_4+T_5+T_6+T_7+T_8,$$

where

$$\begin{aligned}
T_1 &= |x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_0^{x_1-\pi}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\right|\,ds\Bigg)\,d\lambda,
\end{aligned}$$

$$\begin{aligned}
T_2 &= |x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_{x_1-\pi}^{x_2-\pi}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\right|\,ds\Bigg)\,d\lambda,
\end{aligned}$$

$$\begin{aligned}
T_3 &= |x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_{x_2-\pi}^{x_1}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}\right|ds\Bigg)\,d\lambda,
\end{aligned}$$

$$\begin{aligned}
T_4 &= |x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_{x_1}^{\frac{x_1+x_2}{2}}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}\right|ds\Bigg)\,d\lambda,
\end{aligned}$$

$$\begin{aligned}
T_5 &= |\,x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_{\frac{x_1+x_2}{2}}^{x_2}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}\right|ds\Bigg)\,d\lambda,
\end{aligned}$$

$$\begin{aligned}
T_6 &= |\,x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_{x_2}^{x_1+\pi}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}\right|ds\Bigg)\,d\lambda,
\end{aligned}$$

$$\begin{aligned}
T_7 &= |\,x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_{x_1+\pi}^{x_2+\pi}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}\right|ds\Bigg)\,d\lambda,
\end{aligned}$$

$$\begin{aligned}
T_8 &= |x_1-x_2|^{-2\alpha}\int\limits_0^\infty \lambda^{-\alpha}\int\limits_{x_2+\pi}^{2\pi}\Bigg(\frac{1}{2\sqrt{\delta+\lambda}}\,\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\\
&\quad\times\left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}\right|ds\Bigg)\,d\lambda.
\end{aligned}$$

Let us estimate the expression T_1. For $0 \le s \le x_1 - \pi$, using

$$\left| e^{-\sqrt{\delta+\lambda}(2\pi - x_1 + s)} - e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)} \right| \le 2e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)},$$
$$\left| e^{-\sqrt{\delta+\lambda}(2\pi - x_1 + s)} - e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)} \right| \le \sqrt{\delta+\lambda}\, 2e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)} (x_2 - x_1),$$

we can write

$$T_1 \le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \left(\int_0^{x_1-\pi} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \right.$$
$$\left. \times \left(\sqrt{\delta+\lambda}\, 2e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)} (x_2 - x_1)\, ds \right) \right) d\lambda$$

$$= (x_2 - x_1)^{-2\alpha+1} \int_0^\infty \lambda^{-\alpha} \left(\int_0^{x_1-\pi} \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)}\, ds \right) d\lambda$$

$$= M_1 \int_0^\infty \lambda^{-\alpha} \left(\frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \frac{1}{\sqrt{\delta+\lambda}} e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)} \Big]_0^{x_1-\pi} \right) d\lambda$$

$$= M_1 \int_0^\infty \lambda^{-\alpha} \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \frac{1}{\sqrt{\delta+\lambda}} \left(e^{-\sqrt{\delta+\lambda}(2\pi - x_2)} - e^{-\sqrt{\delta+\lambda}(\pi - x_2 + x_1)} \right) d\lambda$$

$$= 2M_1 \int_0^\infty \lambda^{-\alpha} \frac{e^{-\sqrt{\delta+\lambda}(\pi - x_2 + x_1)} \left(1 - e^{-\sqrt{\delta+\lambda}(\pi - x_1)} \right)}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \frac{d\lambda}{2\sqrt{\delta+\lambda}}$$

$$\le M_2 \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} e^{-\sqrt{\delta+\lambda}(\pi - x_2 + x_1)}\, d\lambda \le M_2 \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}(\pi - x_2 + x_1)}\, d\lambda.$$

Using the following substitutions $\sqrt{\lambda} = \tau$ and $\pi - x_2 + x_1 = a$, we get

$$T_1 \le M_2 \int_0^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau = M_2 \left(\int_0^1 \tau^{-2\alpha} e^{-\tau a}\, d\tau + \int_1^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau \right)$$

$$\le M_2 \left(\int_0^1 \tau^{-2\alpha} d\tau + \int_1^\infty e^{-\tau a}\, d\tau \right) = M_2 \left(\frac{1}{1 - 2\alpha} + \frac{1}{a} e^{-a} \right). \tag{2.16}$$

Using formula (2.16), we get

$$T_1 \le \frac{M_2(\delta)}{2\alpha(1 - 2\alpha)}. \tag{2.17}$$

Now, let us estimate the expression T_2. Using

$$\Delta_1 = \int_{x_1-\pi}^{x_2-\pi} \left| e^{-\sqrt{\delta+\lambda}(x_1 - s)} - e^{-\sqrt{\delta+\lambda}(2\pi - x_2 + s)} \right| ds$$

and the substitution

$$\frac{x_1 - s - \pi}{x_2 - x_1} = -y.$$

So, we get

$$x_1 - s = \pi - (x_2 - x_1)\, y,$$
$$2\pi - x_2 + s = \pi - (x_2 - x_1)\,(1 - y),$$

we can rewrite

$$\Delta_1 = (x_2 - x_1) \int_0^1 \left(e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1)y)} - e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1)(1-y))} \right) dy$$

$$= (x_2 - x_1) \int_0^1 \int_{1-y}^{y} d\left\{ e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z} \right\} dy$$

$$= (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_0^1 \int_{1-y}^{y} e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z}\, dz dy$$

$$\leq (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_0^1 e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z}\, dz.$$

Then,

$$T_2 \leq |x_1 - x_2|^{-2\alpha} \left(\int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \right.$$

$$\left. \times (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_0^1 e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z}\, dz \right) d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int_0^\infty (x_2 - x_1)^2 \lambda^{-\alpha} \frac{1}{2\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)}$$

$$\times \left(\frac{1}{\sqrt{\delta+\lambda}\,(x_2 - x_1)} e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z} \Big]_0^1 \right) d\lambda$$

$$= (x_2 - x_1)^{1-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)} \frac{1}{2\sqrt{\delta+\lambda}}$$
$$\times \left(e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))} - e^{-\sqrt{\delta+\lambda}\pi} \right) d\lambda$$

$$= (x_2 - x_1)^{1-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{e^{-\sqrt{\delta+\lambda}(\pi-x_2+x_1)} \left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right)}{\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)} \frac{d\lambda}{2\sqrt{\delta+\lambda}}$$

$$\leq M_3 \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} e^{-\sqrt{\delta+\lambda}(\pi - x_2 + x_1)}\, d\lambda \leq M_3 \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}(\pi - x_2 + x_1)}\, d\lambda.$$

Using the following substitutions $\sqrt{\lambda} = \tau$ and $\pi - x_2 + x_1 = a$, we get

$$T_2 \leq M_3 \int\limits_0^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau = M_2 \left(\int\limits_0^1 \tau^{-2\alpha} e^{-\tau a}\, d\tau + \int\limits_1^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau \right)$$

$$= 2M_3 \left(\int\limits_0^1 \tau^{-2\alpha} d\tau + \int\limits_1^\infty e^{-\tau a}\, d\tau \right) = M_4 \left(\frac{1}{1-2\alpha} + \frac{1}{a} e^{-a} \right). \tag{2.18}$$

Using formula (2.18), we get

$$T_2 \leq \frac{M_4(\delta)}{2\alpha\,(1-2\alpha)}. \tag{2.19}$$

Now, let us estimate the expression T_3. We have that

$$T_3 = |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \frac{\left(1 - e^{-\sqrt{\delta+\lambda}(x_2 - x_1)}\right)}{\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)}$$

$$\times \int\limits_{x_2-\pi}^{x_1} e^{-\sqrt{\delta+\lambda}(x_1 - s)}\, ds d\lambda$$

$$= \quad |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\,(\delta+\lambda)}$$

$$\times \frac{\left(1 - e^{-\sqrt{\delta+\lambda}(x_2 - x_1)}\right)\left(1 - e^{-\sqrt{\delta+\lambda}(\pi - x_2 + x_1)}\right)}{\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)} d\lambda$$

$$\leq M_5\, |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\,(\delta+\lambda)} \left(1 - e^{-\sqrt{\delta+\lambda}(x_2 - x_1)}\right) d\lambda$$

$$\leq M_5\, |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda} \left(1 - e^{-\sqrt{\lambda}(x_2 - x_1)}\right) d\lambda.$$

Using the following substitution $\sqrt{\lambda}\,(x_2 - x_1) = \tau$, we get

$$T_3 \leq M_5\, |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \frac{1 - e^{-\tau}}{2\left(\frac{\tau^2}{(x_2 - x_1)^2}\right)^{1+\alpha}} \frac{2\tau d\tau}{(x_2 - x_1)^2}$$

$$= M_5 \int\limits_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+2\alpha}} d\tau = M_5 \int\limits_0^\infty \frac{1}{\tau^{1+2\alpha}} \int\limits_0^\tau e^{-s}\, ds d\tau.$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$T_3 \le M_5 \int\limits_0^\infty e^{-s} \int\limits_s^\infty \frac{1}{\tau^{1+2\alpha}} d\tau ds$$

$$= \frac{M_5}{2\alpha} \int\limits_0^\infty e^{-s} \left(\frac{1}{\tau^{2\alpha}} \Big]_s^\infty \right) ds = \frac{M_5}{2\alpha} \int\limits_0^\infty e^{-s}\, s^{-2\alpha} ds$$

$$= \frac{M_5}{2\alpha} \Gamma(2\alpha). \tag{2.20}$$

Using formula (2.20), we get

$$T_3 \le \frac{M_5(\delta)}{2\alpha(1-2\alpha)}. \tag{2.21}$$

Let us estimate the expression T_4. We have that

$$T_4 = |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \Bigg(\frac{1}{2\sqrt{\delta+\lambda}} \, \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}}$$

$$\times \frac{1}{\sqrt{\delta+\lambda}} \left| -e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(x_2-s)} \right| \Bigg) \Bigg]_{x_1}^{\frac{x_1+x_2}{2}} d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \Bigg(\frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \frac{1}{2\,(\delta+\lambda)}$$

$$\times \left| 1 + e^{-\sqrt{\delta+\lambda}(x_2-x_1)} - e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)} - e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)} \right| \Bigg) d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\,(\delta+\lambda)} \frac{\left(1 - e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)}\right)^2}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} d\lambda$$

$$\le |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\,(\delta+\lambda)} \left(1 - e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)}\right) d\lambda$$

$$\le |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda} \left(1 - e^{-\sqrt{\lambda}\left(\frac{x_2-x_1}{2}\right)}\right) d\lambda.$$

Using the following substitution $\sqrt{\lambda}\left(\frac{x_2-x_1}{2}\right) = \tau$, we get

$$T_4 \le |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \frac{1 - e^{-\tau}}{2\left(\frac{4\tau^2}{(x_2-x_1)^2}\right)^{1+\alpha}} \frac{8\tau d\tau}{(x_2-x_1)^2}$$

$$= \frac{M_6}{4^\alpha} \int\limits_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+2\alpha}} d\tau = \frac{M_6}{4^\alpha} \int\limits_0^\infty \frac{1}{\tau^{1+2\alpha}} \left(\int\limits_0^\tau e^{-s}\, ds \right) d\tau.$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$T_4 \le \frac{M_6}{4^\alpha} \int_0^\infty e^{-s} \int_s^\infty \frac{1}{\tau^{1+2\alpha}} d\tau ds$$

$$= \frac{M_6}{4^\alpha 2\alpha} \int_0^\infty e^{-s\ -s} \left(\frac{1}{\tau^{2\alpha}} \Big]_s^\infty \right) ds = \frac{M_6}{4^\alpha 2\alpha} \int_0^\infty e^{-s}\ s^{-2\alpha} ds$$

$$= \frac{M_6}{4^\alpha \alpha} \Gamma(2\alpha) = \frac{M_6}{4^\alpha 2\alpha(1-2\alpha)}. \tag{2.22}$$

Using formula (2.22), we get

$$T_4 \le \frac{M_6(\delta)}{2\alpha(1-2\alpha)}. \tag{2.23}$$

Let us estimate the expression T_5. We have that

$$T_5 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \left(\frac{1}{2\sqrt{\delta+\lambda}}\ \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \right.$$

$$\left. \times \frac{1}{\sqrt{\delta+\lambda}} \left| -e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(x_2-s)} \right| \right) \Big]_{\frac{x_1+x_2}{2}}^{x_2} d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \left(\frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \frac{1}{2(\delta+\lambda)} \right.$$

$$\left. \times \left| e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)} + e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)} - e^{-\sqrt{\delta+\lambda}(x_2-x_1)} - 1 \right| \right) d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha}\ \frac{1}{2(\delta+\lambda)} \frac{\left(1 - e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)}\right)^2}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} d\lambda$$

$$\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)} \left(1 - e^{-\sqrt{\delta+\lambda}\left(\frac{x_2-x_1}{2}\right)}\right) d\lambda$$

$$\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda} \left(1 - e^{-\sqrt{\lambda}\left(\frac{x_2-x_1}{2}\right)}\right) d\lambda.$$

Using the following substitution $\sqrt{\lambda}\left(\frac{x_2-x_1}{2}\right) = \tau$, we get

$$T_5 \le |x_1 - x_2|^{-2\alpha} \int_0^\infty \frac{1 - e^{-\tau}}{2\left(\frac{4\tau^2}{(x_2-x_1)^2}\right)^{1+\alpha}} \frac{8\tau d\tau}{(x_2-x_1)^2}$$

$$= \frac{M_7}{4^\alpha} \int_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+2\alpha}} d\tau = \frac{M_7}{4^\alpha} \int_0^\infty \frac{1}{\tau^{1+2\alpha}} \left(\int_0^\tau e^{-s}\ ds \right) d\tau.$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$T_5 \le \frac{M_7}{4^\alpha} \int\limits_0^\infty e^{-s} \int\limits_s^\infty \frac{1}{\tau^{1+2\alpha}} d\tau ds$$

$$= \frac{M_7}{4^\alpha 2\alpha} \int\limits_0^\infty e^{-s} \left(\frac{1}{\tau^{2\alpha}} \Big]_s^\infty \right) ds = \frac{M_7}{4^\alpha 2\alpha} \int\limits_0^\infty e^{-s} s^{-2\alpha} ds$$

$$= \frac{M_7}{4^\alpha \alpha} \Gamma(2\alpha) = \frac{M_7}{4^\alpha 2\alpha(1-2\alpha)}. \tag{2.24}$$

Using formula (2.24), we get

$$T_5 \le \frac{M_7(\delta)}{2\alpha(1-2\alpha)}. \tag{2.25}$$

Let us estimate the expression T_6. We have that

$$T_6 = |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \int\limits_{x_2}^{x_1+\pi} \Bigg(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}$$

$$\times e^{-\sqrt{\delta+\lambda}(s-x_2)} \left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right) ds \Bigg) d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \frac{\left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right)}{\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)}$$

$$\times \int\limits_{x_2}^{x_1+\pi} e^{-\sqrt{\delta+\lambda}(s-x_2)} ds d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)} \frac{\left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right)}{\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)}$$

$$\times \left[-e^{-\sqrt{\delta+\lambda}(s-x_2)} \right]_{x_2}^{x_1+\pi} d\lambda$$

$$= |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)}$$

$$\times \frac{\left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right)\left(1 - e^{-\sqrt{\delta+\lambda}(\pi-x_2+x_1)}\right)}{\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)} d\lambda$$

$$\le |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)} \left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right) d\lambda$$

$$\le |x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda} \left(1 - e^{-\sqrt{\lambda}(x_2-x_1)}\right) d\lambda.$$

Using the following substitution $\sqrt{\lambda}\,(x_2 - x_1) = \tau$ we get

$$T_6 \le M_8 \,|\; x_1 - x_2|^{-2\alpha} \int\limits_0^{\infty} \frac{1 - e^{-\tau}}{2\left(\frac{\tau^2}{(x_2-x_1)^2}\right)^{1+\alpha}} \frac{2\tau d\tau}{(x_2 - x_1)^2}$$

$$= M_8 \int\limits_0^{\infty} \frac{1 - e^{-\tau}}{\tau^{1+2\alpha}} d\tau = M_8 \int\limits_0^{\infty} \frac{1}{\tau^{1+2\alpha}} \int\limits_0^{\tau} e^{-s}\, ds d\tau.$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$T_6 \le 2M_8 \int\limits_0^{\infty} e^{-s} \int\limits_s^{\infty} \frac{1}{\tau^{1+2\alpha}} d\tau ds$$

$$= 2M_8 \int\limits_0^{\infty} e^{-s} \left(\frac{1}{2\alpha\ \tau^{2\alpha}} \Bigg]_s^{\infty} \right) ds$$

$$= \frac{M_9}{2\alpha} \int\limits_0^{\infty} e^{-s}\, s^{-2\alpha} ds = \frac{M_9}{2\alpha} \Gamma(2\alpha). \tag{2.26}$$

Using formula (2.26), we get

$$T_6 \le \frac{M_9(\delta)}{2\alpha(1 - 2\alpha)}. \tag{2.27}$$

Let us estimate the expression T_7. Using

$$\Delta_2 = \int\limits_{x_1+\pi}^{x_2+\pi} \left| e^{-\sqrt{\delta+\lambda}(2\pi + x_1 - s)} - e^{-\sqrt{\delta+\lambda}(s - x_2)} \right| ds$$

and the substitution

$$\frac{\pi + x_2 - s}{x_2 - x_1} = y,$$

we get

$$s - x_2 = \pi - (x_2 - x_1)\, y,$$
$$2\pi + x_1 - s = \pi - (x_2 - x_1)\,(1 - y).$$

Then

$$\Delta_2 = \int\limits_1^{0} -(x_2 - x_1) \left(e^{-\sqrt{\delta+\lambda}(\pi - (x_2 - x_1)(1-y))} - e^{-\sqrt{\delta+\lambda}(\pi - (x_2 - x_1)y)} \right) dy$$

$$= (x_2 - x_1) \int\limits_1^{0} \int\limits_{1-y}^{y} d\left\{ e^{-\sqrt{\delta+\lambda}(\pi - (x_2 - x_1))z} \right\} dy$$

$$= (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_1^0 \int_{1-y}^{y} e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z}\, dzdy$$

$$\leq (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_0^1 \left| -e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z} \right| dz.$$

Let us estimate the expression

$$T_7 \leq |x_1 - x_2|^{-2\alpha} \int_0^\infty \left(\lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \right.$$

$$\left. \times (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_0^1 -e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z}\, dz \right) d\lambda$$

$$= \ (x_1 - x_2)^{-2\alpha} \int_0^\infty \lambda^{-\alpha} (x_2 - x_1)^2 \frac{1}{2} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}$$

$$\times \left[\frac{1}{\sqrt{\delta+\lambda}\,(x_2 - x_1)} \left(-e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z} \right) \right]_0^1 d\lambda$$

$$= (x_1 - x_2)^{-2\alpha+1} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}$$

$$\times \left| e^{-\sqrt{\delta+\lambda}\pi} - e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))} \right| d\lambda$$

$$= M_{10} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}$$

$$\times \left| -e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))} \right| \left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)} \right) d\lambda$$

$$\leq M_{10} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \left(e^{-\sqrt{\delta+\lambda}(\pi-x_2+x_1)} \right) d\lambda$$

$$\leq M_{10} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\lambda}} \left(e^{-\sqrt{\lambda}(\pi-x_2+x_1)} \right) d\lambda.$$

Using the following substitutions $\sqrt{\lambda} = \tau$ and $(\pi - x_2 + x_1) = a$, we get

$$T_7 \leq M_{10} \int_0^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau = M_{10} \left(\int_0^1 \tau^{-2\alpha} e^{-\tau a}\, d\tau + \int_1^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau \right)$$

$$= 2M_{10} \left(\int_0^1 \tau^{-2\alpha} d\tau + \int_1^\infty e^{-\tau a}\, d\tau \right) = M_{11} \left(\frac{1}{1-2\alpha} + \frac{1}{a} e^{-a} \right). \qquad (2.28)$$

Using formula (2.28), we get

$$T_7 \leq \frac{M_{11}(\delta)}{2\alpha(1-2\alpha)}. \qquad (2.29)$$

Let us estimate the expression T_8. For $x_2+\pi \leq s \leq 2\pi$, using

$$\begin{aligned} \left| e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)} \right| &\leq 2e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}, \\ \left| e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)} \right| &\leq \sqrt{\delta+\lambda}\, 2e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} (x_2-x_1), \end{aligned}$$

we can write

$$T_8 \leq |\, x_1 - x_2|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \int\limits_{x_2+\pi}^{2\pi} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \right.$$

$$\left. \times \sqrt{\delta+\lambda} 2\, e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} (x_2-x_1)\, ds \right) d\lambda$$

$$= (x_2-x_1)^{-2\alpha+1} \int\limits_0^\infty \lambda^{-\alpha} \int\limits_{x_2+\pi}^{2\pi} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \left(e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} \right) ds d\lambda$$

$$= M_{12} \int\limits_0^\infty \lambda^{-\alpha} \left[\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \frac{1}{\sqrt{\delta+\lambda}} e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} \right]_{x_2+\pi}^{2\pi} d\lambda$$

$$= M_{12} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \frac{1}{\sqrt{\delta+\lambda}} \left(e^{-\sqrt{\delta+\lambda}x_1} - e^{-\sqrt{\delta+\lambda}(\pi-x_2+x_1)} \right) d\lambda$$

$$= M_{12} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{\sqrt{\delta+\lambda}} \frac{e^{-\sqrt{\delta+\lambda}x_1} \left(1-e^{-\sqrt{\delta+\lambda}(\pi-x_2)}\right)}{1-e^{-\sqrt{\delta+\lambda}2\pi}} d\lambda$$

$$\leq M_{12} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{\sqrt{\delta+\lambda}} e^{-\sqrt{\delta+\lambda}x_1}\, d\lambda \leq 2M_{12} \int\limits_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}x_1}\, d\lambda.$$

Using the following substitutions $\sqrt{\lambda} = \tau$ and $x_1 = a$, we get

$$T_8 \leq M_{13} \int\limits_0^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau = M_{13} \left(\int\limits_0^1 \tau^{-2\alpha} e^{-\tau a}\, d\tau + \int\limits_1^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau \right)$$

$$= M_{13} \left(\int\limits_0^1 \tau^{-2\alpha} d\tau + \int\limits_1^\infty e^{-\tau a}\, d\tau \right) = M_{13} \left(\frac{1}{1-2\alpha} + \frac{1}{a} e^{-a} \right). \qquad (2.30)$$

Using formula (2.30), we get

$$T_8 \leq \frac{M_{13}(\delta)}{2\alpha\,(1-2\alpha)}. \qquad (2.31)$$

Applying the triangle inequality and estimates (2.15),(2.17),(2.19),(2.21),(2.23),(2.25),(2.27),(2.29),(2.31), we get

$$T \leq \frac{M(\delta)}{2\alpha(1-2\alpha)}.$$

So (2.15) is proved. Thus, for any $x_1, x_2 \in [0, 2\pi]$ we have that

$$|x_1 - x_2|^{-2\alpha} |\varphi(x_1) - \varphi(x_2)| \leq \frac{M(\delta)}{2\alpha(1-2\alpha)} \|\varphi\|_{E_\alpha(C[0,2\pi],A)}.$$

This means that the following inequality holds:

$$\|\varphi\|_{C^{2\alpha}[0.2\pi]} \leq \frac{M(\delta)}{2\alpha(1-2\alpha)} \|\varphi\|_{E_\alpha(C[0,2\pi],A).}$$

So, Theorem 2.2. is proved

Since the A^x is positive operator in the fractional spaces $E_\alpha(C[0, 2\pi], A^x)$, from the result of Theorem 2.2 it follows also it is positive operator in the Hölder space $\overset{\circ}{C}{}^{2\alpha}[0, 2\pi]$. Namely,

Theorem 2.3. The operator $(A^x + \lambda)$ has a bounded in $\overset{\circ}{C}{}^{2\alpha}[0, 2\pi]$ inverse for any $\lambda \geq 0$ and the following estimate holds:

$$\left\|(A^x + \lambda)^{-1}\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,2\pi] \to \overset{\circ}{C}{}^{2\alpha}[0,2\pi]} \leq \frac{M(\delta)}{2\alpha(1-2\alpha)} \frac{M_{14}}{\delta + \lambda}.$$

CHAPTER 3

THE POSITIVITY OF THE DIFFERENTIAL OPERATOR WITH VARIABLE COEFFICIENTS

We consider the differential operator A^x defined by the formula

$$A^x u = -a(x)u_{xx}(x) + \delta u(x), \quad \delta > 0 \tag{3.1}$$

with domain

$$\begin{aligned} D(A^x) \;\; = \;\; & \{u(x) : u(x), u'(x), u''(x) \in C[0, 2\pi] \;, \\ & u(x) = u(x + 2\pi), \int_0^{2\pi} u(x)dx = 0\} \;. \end{aligned} \tag{3.2}$$

Here $a(x) \geq a > 0$ is continuously differentiable function defined on $[0, 2\pi]$.We will study the resolvent of the operator A^x, i.e.

$$A^x u(x) + \lambda u(x) = \varphi(x), \qquad 0 < x < 2\pi \tag{3.3}$$

or

$$\begin{cases} -a(x)u''(x) + (\delta + \lambda)\, u(x) = \varphi(x), \quad 0 < x < 2\pi, \\ \\ u(0) = u(2\pi), \; \int\limits_0^{2\pi} u(x)dx = 0. \end{cases} \tag{3.4}$$

We will rewrite it in the following form

$$\begin{cases} -u''(x) + k(x)u(x) = f(x), \quad 0 < x < 2\pi, \\ \\ u(0) = u(2\pi), \; \int\limits_0^{2\pi} u(x)dx = 0, \end{cases} \tag{3.5}$$

where $k(x) = \frac{(\delta+\lambda)}{a(x)}$ and $f(x) = \frac{\varphi(x)}{a(x)}$. The Green's function of A^x is constructed. The estimates for the Green's function will be obtained. The positivity of the operator A^x in the Banach space $C\,[0, 2\pi]$ is established. It is proved that for any $\alpha \in (0, \frac{1}{2})$

the norms in space $E_\alpha(C[0,2\pi], A)$ and $\overset{\circ}{C}{}^{2\alpha}[0,2\pi]$ are equivalent. The positivity of A^x in the Hölder spaces of $\overset{\circ}{C}{}^{2\alpha}[0,2\pi]$, $\alpha \in (0, \frac{1}{2})$ is proved. The structure of fractional spaces generated by this operator will be investigated.

Lemma 3.1. Assume that $f \in [0,2\pi], f(0) = f(2\pi)$ and $\int_0^{2\pi} f(z)dz = 0$. For any $\lambda \geq 0$ problem (3.5) is uniquely solvable and the following formula holds:

$$u(x) = (A^x + \lambda)^{-1} \varphi(x) = \int_0^{2\pi} G(x,z;\lambda)\ f(z)dz. \tag{3.6}$$

Here

$$G(x,z;\lambda) = \begin{cases} N(x)\left\{\int_0^{2\pi}\int_0^{x} K(x,s)\,K(2\pi,s)\,dsdx \int_0^{z} K^2(z,s)\,ds \right. \\ -\int_z^{2\pi}\int_0^{z} K(x,s)\,K(z,s)\,dsdx \\ \left. -\int_0^{z}\int_0^{x} K(x,s)\,K(z,s)\,dsdx \right\} \\ -Q\int_0^{x} K(x,s)\,K(2\pi,s)\,ds \int_0^{z} K^2(z,s)\,ds \\ +\int_0^{z} K(x,s)\,K(z,s)\,ds\ , \quad 0 \leq z \leq x, \\ \\ N(x)\left\{\int_0^{2\pi}\int_0^{x} K(x,s)\,K(2\pi,s)\,dsdx \int_0^{z} K^2(z,s)\,ds \right. \\ -\int_z^{2\pi}\int_0^{z} K(x,s)\,K(z,s)\,dsdx \\ \left. -\int_0^{z}\int_0^{x} K(x,s)\,K(z,s)\,dsdx \right\} \\ +\int_0^{x} K(x,s)\,K(z,s)\,ds, \quad x \leq z \leq 2\pi, \end{cases} \tag{3.7}$$

where

$$K(x,s) = e^{-\int_s^x b(\lambda)d\lambda}, \quad Q = \left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1},$$

$$L(x) = \int_x^{2\pi} K^2(2\pi,s)\,dsK(x,0) + \int_0^{x} K(x,s)\,K(2\pi,s)\,ds,$$

$$S(x) = L(x)\left(\int_0^{2\pi} L(x)dx\right)^{-1},$$

$$N(x) = Q\ S(x).$$

Here $b(x)$ is a solution of the equation

$$b^2(x) - b'(x) = a(x) \tag{3.8}$$

and the following estimate holds:

$$b(x) \geq b_0 > 0, \ 0 \leq x \leq 2\pi. \tag{3.9}$$

Proof. Using relation (3.8), we can obviously write the boundary-value problem (3.5) in equivalent form as a system of first-order linear differential equations

$$\begin{cases} u'(x) + b(x)u(x) = v(x), \ u(0) = \varphi, \ u(2\pi) = \psi \\ \\ -v'(x) + b(x)v(x) = f(x). \end{cases}$$

Applying the Cauchy formula, we get

$$\begin{cases} u(x) = e^{-\int\limits_0^x b(\lambda)d\lambda}\varphi + \int\limits_0^x e^{-\int\limits_s^x b(\lambda)d\lambda} v(s)ds, \\ \\ v(x) = e^{-\int\limits_x^{2\pi} b(\lambda)d\lambda} v\left(2\pi\right) + \int\limits_x^{2\pi} e^{-\int\limits_x^z b(\lambda)d\lambda} f(z)dz. \end{cases}$$

Using the definition of the function $K\left(x,s\right)$,we get

$$\begin{cases} u(x) = K\left(x,0\right)\varphi + \int\limits_0^x K\left(x,s\right)v(s)ds, \\ \\ v(x) = K\left(2\pi,x\right)v\left(2\pi\right) + \int\limits_x^{2\pi} K\left(z,x\right)f(z)dz. \end{cases}$$

From these formulas and the condition $u(2\pi) = \psi$ it follows that

$$u(2\pi) = K\left(2\pi,0\right)\varphi + \int\limits_0^{2\pi} K^2\left(2\pi,s\right)dsv(2\pi)$$

$$+\int\limits_0^{2\pi} K\left(2\pi,s\right)\int\limits_s^{2\pi} K\left(z,s\right)f(z)dzds.$$

Since $\int\limits_0^{2\pi} e^{-2\int\limits_s^{2\pi} b(\lambda)d\lambda} ds \neq 0$, then

$$\begin{aligned} v(2\pi) &= \left(\int\limits_0^{2\pi} K^2\left(2\pi,s\right)ds\right)^{-1}\left[\psi - K\left(2\pi,0\right)\varphi\right. \\ &\left. - \int\limits_0^{2\pi}\left(\int\limits_s^{2\pi} K\left(2\pi,s\right)K\left(z,s\right)ds\right)f(z)dz\right]. \end{aligned}$$

So, we have that

$$v(x) = K(2\pi, x)\left(\int_0^{2\pi} K^2(2\pi, s)\,ds\right)^{-1} \left[u(2\pi) - K(2\pi, 0)\varphi \right.$$

$$\left. - \int_0^{2\pi}\left(\int_s^{2\pi} K(2\pi, s) K(z, s)\,ds\right) f(z)dz\right] + \int_x^{2\pi} K(z, x) f(z)dz$$

and

$$u(x) = K(x, 0)\varphi + \int_0^x K(x, s)\left[K(2\pi, s)\left(\int_0^{2\pi} K^2(2\pi, \tau)\,d\tau\right)^{-1}\right.$$

$$\times \left[u(2\pi) - T(2\pi, 0)\varphi - \int_0^{2\pi}\left(\int_\tau^{2\pi} K(2\pi, \tau) K(z, \tau)\,d\tau\right) f(z)dz\right]$$

$$\left. + \int_s^{2\pi} K(z, s) f(z)dz\right] ds.$$

By an interchange of the order of integration, we obtain

$$u(x) = \left(\int_0^{2\pi} K^2(2\pi, s)\,ds\right)^{-1} \left[\left\{\int_x^{2\pi} K^2(2\pi, s)\,ds\; K(x, 0)\, u(0)\right.\right. \tag{3.10}$$

$$\left. + \int_0^x K(x, s) K(2\pi, s)\,ds\; u(2\pi)\right\}$$

$$\left. - \int_0^x K(x, s) K(2\pi, s)\,ds \int_0^{2\pi}\left\{\int_0^z K^2(z, s)\,ds\right\} f(z)dz\right]$$

$$+ \int_0^x \left\{\int_0^z K(x, s) K(z, s)\,ds\right\} f(z)dz$$

$$+ \int_x^{2\pi} \left\{\int_0^x K(x, s) K(z, s)\,ds\right\} f(z)dz.$$

From this formula and $\int_0^{2\pi} u(x)dx = 0$, $u(0) = u(2\pi)$ it follows that

$$\int_0^{2\pi} u(x)dx = \left(\int_0^{2\pi} K^2(2\pi, s)\,ds\right)^{-1} \int_0^{2\pi} \left[\left\{\int_x^{2\pi} K^2(2\pi, s)\,ds\; K(x, 0)\right.\right.$$

$$+\int_0^x K(x,s)\,K(2\pi,s)\,ds\Bigg\}u(2\pi)$$

$$-\int_0^x K(x,s)\,K(2\pi,s)\,ds\int_0^{2\pi}\int_0^z K^2(z,s)\,dsf(z)dz\Bigg]\,dx$$

$$+\int_0^{2\pi}\int_0^x\left\{\int_0^z K(x,s)\,K(z,s)\,ds\right\}f(z)dzdx$$

$$+\int_0^{2\pi}\int_x^{2\pi}\left\{\int_0^x K(x,s)\,K(z,s)\,ds\right\}f(z)dzdx=0.$$

Then

$$\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}\int_0^{2\pi}\Bigg\{\int_x^{2\pi} K^2(2\pi,s)\,ds\;K(x,0)$$

$$+\int_0^x K(x,s)\,K(2\pi,s)\,ds\quad\Bigg\}\,dx\;u(2\pi)$$

$$-\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}\int_0^{2\pi}\int_0^x K(x,s)\,K(2\pi,s)\,ds$$

$$\times\int_0^{2\pi}\left\{\int_0^z K^2(z,s)\,ds\right\}f(z)dzdx$$

$$\times\int_0^{2\pi}\left\{\int_0^z K^2(z,s)\,ds\right\}f(z)dzdx$$

$$+\int_0^{2\pi}\int_x^{2\pi}\left\{\int_0^x K(x,s)\,K(z,s)\,ds\right\}f(z)dzdx=0,$$

where

$$\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}\int_0^{2\pi}\Bigg\{\int_x^{2\pi} K^2(2\pi,s)\,ds\;T(x,0)$$

$$+\int_0^x K(x,s)\,K(2\pi,s)\,ds\Bigg\}\,dx\;u(2\pi)$$

$$=\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}\int_0^{2\pi}\int_0^x K(x,s)\,K(2\pi,s)\,ds$$

$$\times \int_0^{2\pi}\int_0^{z} K^2(z,s)\,ds f(z)dzdx$$

$$-\int_0^{2\pi}\int_0^{x}\left\{\int_0^{z} K(x,s)\,K(z,s)\,ds\right\} f(z)dzdx$$

$$-\int_0^{2\pi}\int_x^{2\pi}\left\{\int_0^{x} K(x,s)\,K(z,s)\,ds\right\} f(z)dzdx.$$

or

$$u(2\pi)=\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)\left(\int_0^{2\pi}\ \left\{\int_x^{2\pi} K^2(2\pi,s)\,ds\ K(x,0)\right.\right.$$

$$\left.\left.+\int_0^{x} K(x,s)\,K(2\pi,s)\,ds\right\} dx\right)^{-1}\left[\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}\right.$$

$$\times \int_0^{2\pi}\int_0^{x} K(x,s)\,K(2\pi,s)\,ds\int_0^{2\pi}\int_0^{z} K^2(z,s)\,ds f(z)dzdx$$

$$-\int_0^{2\pi}\left\{\int_0^{x}\int_0^{z} K(x,s)\,K(z,s)\,ds\right\} f(z)dzdx$$

$$\left.-\int_0^{2\pi}\int_x^{2\pi}\left\{\int_0^{x} K(x,s)\,K(z,s)\,ds\right\} f(z)dzdx\right].$$

By using (3.10), we get

$$u(x)=\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}\left[\left\{\int_x^{2\pi} K^2(2\pi,s)\,dsK(x,0)\right.\right.$$

$$\left.+\int_0^{x} K(x,s)\,K(2\pi,s)\,ds\right\}\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)$$

$$\times\left(\int_0^{2\pi}\left\{\int_x^{2\pi} K^2(2\pi,s)\,ds\ K(x,0)+\int_0^{x} K(x,s)\,K(2\pi,s)\,ds\right\} dx\right)^{-1}$$

$$\times\left\{\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}\int_0^{2\pi}\left\{\int_0^{x} K(x,s)\,K(2\pi,s)\,ds\int_0^{2\pi}\int_0^{z} K^2(z,s)\,ds f(z)dz\right\} dx\right.$$

$$-\int_0^{2\pi}\int_0^{x}\left\{\int_0^{z} K(x,s)\,K(z,s)\,ds\right\} f(z)dzdx$$

$$-\int_0^{2\pi}\int_x^{2\pi}\left\{\int_0^x K(x,s)\,K(z,s)\,ds\right\}f(z)dzdx\right\}$$

$$\left.-\int_0^x K(x,s)\,K(2\pi,s)\,ds\int_0^{2\pi}\int_0^z K^2(z,s)\,dsf(z)dz\right]$$

$$+\int_0^x\int_0^z K(x,s)\,K(z,s)\,dsf(z)dz+\int_x^{2\pi}\int_0^x K(x,s)\,K(z,s)\,dsf(z)dz.$$

Let

$$Q=\left(\int_0^{2\pi} K^2(2\pi,s)\,ds\right)^{-1}$$

and

$$L(x)=\int_x^{2\pi} K^2(2\pi,s)\,dsK(x,0)+\int_0^x K(x,s)\,K(2\pi,s)\,ds$$

Then

$$u(x)=Q\left[L(x)Q^{-1}\left(\int_0^{2\pi} L(x)dx\right)^{-1}\left\{Q\int_0^{2\pi}\int_0^x K(x,s)\,K(2\pi,s)\,ds\right.\right.$$

$$\times\int_0^{2\pi}\int_0^z K^2(z,s)\,dsf(z)dzdx-\int_0^{2\pi}\int_0^x\left\{\int_0^z K(x,s)\,K(z,s)\,ds\right\}f(z)dzdx$$

$$-\int_0^{2\pi}\int_x^{2\pi}\left\{\int_0^x K(x,s)\,K(z,s)\,ds\right\}f(z)dzdx\right\}$$

$$\left.-\int_0^x K(x,s)\,K(2\pi,s)\,ds\int_0^{2\pi}\int_0^z K^2(z,s)\,dsf(z)dz\right]$$

$$+\int_0^x\int_0^z K(x,s)\,K(z,s)\,dsf(z)dz+\int_x^{2\pi}\int_0^x K(x,s)\,K(z,s)\,dsf(z)dz.$$

Let

$$S(x)=L(x)\left(\int_0^{2\pi} L(x)dx\right)^{-1},$$

and

$$N(x)=Q\;S(x).$$

Then

$$u(x) = N(x)\left[\int_0^{2\pi}\left\{\int_0^{2\pi}\int_0^{x} K(x,s)\,K(2\pi,s)\,dsdx\int_0^{z} K^2(z,s)\,ds\right\} f(z)dz\right.$$

$$-\int_0^{2\pi}\left\{\int_z^{2\pi}\int_0^{z} K(x,s)\,K(z,s)\,dsdx\right\} f(z)dz$$

$$\left.-\int_0^{2\pi}\left\{\int_0^{z}\int_0^{x} K(x,s)\,K(z,s)\,dsdx\right\} f(z)dz\right]$$

$$-\int_0^{2\pi}\left\{Q\int_0^{x} K(x,s)\,K(2\pi,s)\,ds\int_0^{z} K^2(z,s)\,ds\right\} f(z)dz$$

$$+\int_0^{x}\left\{\int_0^{z} K(x,s)\,K(z,s)\,ds\right\} f(z)dz+\int_x^{2\pi}\left\{\int_0^{x} K(x,s)\,K(z,s)\,ds\right\} f(z)dz$$

or

$$\begin{aligned} u(x) &= N(x)\int_0^{2\pi}\left\{\int_0^{2\pi}\int_0^{x} K(x,s)\,K(2\pi,s)\,dsdx\int_0^{z} K^2(z,s)\,ds\right. \\ &\left.-\int_z^{2\pi}\int_0^{z} K(x,s)\,K(z,s)\,dsdx-\int_0^{z}\int_0^{x} K(x,s)\,K(z,s)\,dsdx\right\} f(z)dz \end{aligned}$$

$$-\int_0^{2\pi}\left\{Q\int_0^{x} K(x,s)\,K(2\pi,s)\,ds\int_0^{z} K^2(z,s)\,ds\right\} f(z)dz$$

$$+\int_0^{x}\left\{\int_0^{z} K(x,s)\,K(z,s)\,ds\right\} f(z)dz+\int_x^{2\pi}\left\{\int_0^{x} K(x,s)\,K(z,s)\,ds\right\} f(z)dz$$

$$=\int_0^{2\pi} G(x,z,\lambda)\,f(z)dz,$$

where

$$G(x,z\,;\lambda)=\begin{cases} N(x)\left\{\int\limits_0^{2\pi}\int\limits_0^{x}K\left(x,s\right)K\left(2\pi,s\right)dsdx\int\limits_0^{z}K^2\left(z,s\right)ds\right. \\ \left.-\int\limits_z^{2\pi}\int\limits_0^{z}K\left(x,s\right)K\left(z,s\right)dsdx-\int\limits_0^{z}\int\limits_0^{x}K\left(x,s\right)K\left(z,s\right)dsdx\right\} \\ -Q\int\limits_0^{x}K\left(x,s\right)K\left(2\pi,s\right)ds\int\limits_0^{z}K^2\left(z,s\right)ds \\ +\int\limits_0^{z}K\left(x,s\right)K\left(z,s\right)ds, \quad 0\le z\le x, \\ \\ N(x)\left\{\int\limits_0^{2\pi}\int\limits_0^{x}K\left(x,s\right)K\left(2\pi,s\right)dsdx\int\limits_0^{z}K^2\left(z,s\right)ds\right. \\ \left.-\int\limits_z^{2\pi}\int\limits_0^{z}K\left(x,s\right)K\left(z,s\right)dsdx-\int\limits_0^{z}\int\limits_0^{x}K\left(x,s\right)K\left(z,s\right)dsdx\right\} \\ +\int\limits_0^{x}K\left(x,s\right)K\left(z,s\right)ds, \quad x\le z\le 2\pi, \end{cases}$$

So, Lemma 3.1 is proved.

Applying the triangle inequality, formula (3.7) and estimate (3.9), we get

$$|G(x,z\,;\lambda)|\le\frac{M}{\sqrt{\delta+\lambda}}\begin{cases} e^{-\sqrt{\delta+\lambda}(2\pi-x+z)}\,, & 0\le z\le x-\pi, \\ e^{-\sqrt{\delta+\lambda}(x-z)}\,, & x-\pi\le z\le x, \\ e^{-\sqrt{\delta+\lambda}(z-x)}\,, & x\le z\le \pi+x, \\ e^{-\sqrt{\delta+\lambda}(2\pi+x-z)}\,, & \pi+x\le z\le 2\pi. \end{cases} \tag{3.11}$$

Note that it is true for $a(x)=$ constant. In the same manner for the constant coefficients we can study structure of E_α and positivity of A^x in $C^{2\alpha}\left[0,2\pi\right]$ under the condition (3.11). Actually,

Theorem 3.1. The operator $(\lambda+A^x)$ has a bounded in $C\left[0,2\pi\right]$ inverse for any $\lambda\ge 0$ and the following estimate holds:

$$\left\|(A^x+\lambda)^{-1}\right\|_{C[0,2\pi]\to C[0,2\pi]}\le\frac{1}{\delta+\lambda}. \tag{3.12}$$

Clearly, the operator commutes A^x and its resolvent $\left(A^x+\lambda\right)^{-1}$. By the definition of the norm in the fractional space $E_\alpha=E_\alpha(C\left[0,2\pi\right],A^x)$, we get

$$\left\|(A^x+\lambda)^{-1}\right\|_{E_\alpha\to E_\alpha}\le\left\|(A^x+\lambda)^{-1}\right\|_{C[0,2\pi]\to C[0,2\pi]}.$$

Thus, from Theorem 2.1 it follows that A^x is a positive operator in the fractional spaces $E_\alpha(C\left[0,2\pi\right],A^x)$. Moreover, we have the following result.

Theorem 3.2. For $\alpha\in(0,\frac{1}{2})$, the norms of the spaces $E_\alpha(C\left[0,2\pi\right],A^x)$ and the Hölder space $\overset{\circ}{C}{}^{2\alpha}\left[0,2\pi\right]$ are equivalent.

Proof. For any $\lambda \geq 0$ using formula (3.6), we get

$$\lambda^{\alpha} A^{x}(A^{x}+\lambda)^{-1}\varphi(x) = \frac{\delta\lambda^{\alpha}}{\delta+\lambda}\varphi(x) + \lambda^{\alpha+1}\int_{0}^{2\pi} G\left(x,s;\lambda\right)\left(\varphi(x)-\varphi(s)\right)ds$$

Applying this formula and formula (3.7), we obtain

$$\|\varphi\|_{E_{\alpha}(C[0,2\pi],\ A^{x})} \leq \frac{M(\delta)}{\alpha}\left[1+\Gamma\left(2\alpha+1\right)\right]\|\varphi\|_{C^{2\alpha}[0,2\pi]}.$$

It is well-known that for any positive operator A^{x} in the Banach space, we can write formula

$$\varphi\left(x\right) = \int_{0}^{\infty} A^{x}(\lambda+A^{x})^{-2}\varphi\left(x\right)d\lambda.$$

From this relation and formula (3.6), it follows that

$$\varphi\left(x\right) = \int_{0}^{\infty}(A^{x}+\lambda)^{-1}A^{x}(A^{x}+\lambda)^{-1}\varphi\left(x\right)\ d\lambda$$

$$= \int_{0}^{\infty}\int_{0}^{2\pi} G\left(x,s;\lambda\right)A^{x}(A^{x}+\lambda)^{-1}\varphi\left(s\right)dsd\lambda.$$

Consequently,

$$\varphi\left(x_{1}\right)-\varphi\left(x_{2}\right) = \int_{0}^{\infty}\int_{0}^{2\pi}\left(G\left(x_{1},s;\lambda\right)-G\left(x_{2},s;\lambda\right)\right)A^{x}(A^{x}+\lambda)^{-1}\varphi(s)dsd\lambda$$

$$= \int_{0}^{\infty}\lambda^{-\alpha}\int_{0}^{2\pi}\left(G\left(x_{1},s;\lambda\right)-G\left(x_{2},s;\lambda\right)\right)\lambda^{\alpha}A^{x}(A^{x}+\lambda)^{-1}\varphi(s)dsd\lambda.$$

Using these formulas, (3.6) and the definition of norm os space $E_{\alpha}(C[0,2\pi],A^{x})$,we get

$$\|\varphi\|_{C^{2\alpha}[0.2\pi]} \leq \frac{M(\delta)}{2\alpha(1-2\alpha)}\|\varphi\|_{E_{\alpha}(C[0,2\pi],A^{x}).}$$

So, Theorem 3.2. is proved.

From positivity of A^{x} in positive operator in E_{α} and Theorem 3.2 it follows the positivity of A^{x} on $\overset{\circ}{C}{}^{2\alpha}\left[0,2\pi\right]$.

Theorem 3.3. The operator $(A^{x}+\lambda)$ has a bounded in $\overset{\circ}{C}{}^{2\alpha}\left[0,2\pi\right]$ inverse for any $\lambda \geq 0$ and the following estimate holds:

$$\left\|(A^{x}+\lambda)^{-1}\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,2\pi]\to\overset{\circ}{C}{}^{2\alpha}[0,2\pi]} \leq \frac{M(\delta)}{2\alpha(1-2\alpha)}\frac{M_{14}}{\delta+\lambda}.$$

CHAPTER 4

CONCLUSION

This thesis is devoted to study the positivity of second order differential operators with periodic boundary conditions. The following original results are obtained:

- The Green's function of the second order differential operator with periodic boundary conditions is constructed.
- The fractional spaces generated by the second order differential operator with two nonlocal boundary conditions are constructed.
- The positivity of the second order differential operator with periodic boundary conditions in Hölder spaces is obtained.
- The equivalence of the norm of these fractional spaces and Hölder spaces is established.
- The method of frozen coefficients enables to prove the positivity of the second order differential operator with periodic boundary conditions in the Banach space is established.

REFERENCES

Alibekov, K., *Investigations in C and L_p of difference schemes of higher order accuracy for approximate solution of multidimensional parabolic boundary value problems*, Ph.D. Thesis, Voronezh State University, Voronezh, pp 134, 1978.

Ashyralyev, A., "Well-posed solvability of the boundary value problem for the difference Equations of the elliptic type", *Nonlinear Analysis .Theory. Methods, and Applications*, Vol. 24, No.2, pp. 251–256, 1995.

Ashyralyev, A., "Fractional spaces generated by the positive differential and difference operators in a Banach Space", *ISMME. Springer*, Vol. 83, pp. 10–19, 2006.

Ashyralyev, A. and I. Karakaya, "The Structure of fractional spaces generated by the positive operator", *In: Ashralyev, Ch.(ed) Abstracts of Conference of Young Scientists*, Vol. , pp. Turkmen Agricultural University, Ashgabat, 1995.

Ashyralyev, A. and B. Kendirli, "Positivity in of One-Dimensional Difference Operators with Nonlocal Boundary Conditions", *Some Problems of Applied Mathematics.Fatih University.Istanbul,Turkey*, Vol. , pp. 45–60, 2000.

Ashyralyev, A. and B. Kendirli, "Positivity in Hölder Norms of One-Dimensional Difference Operators with Nonlocal Boundary Conditions", *Heron Press and Technical University of Sofia*, Vol. , pp. 134–137, 2001.

Ashyralyev, A. and R. Prenov, "The Hyperbolic System of Equations with Nonlocal Boundary Conditions", *of Applied and Engineering Mathematics*, Vol. 1, p. 2, 2012.

Ashyralyev, A. and P. Sobolevskii (editors), *Positive operators and the fractional spaces.The methodical instructions for the students of engineering grups*, Offset laboratory VSU, Voronezh, 1984.

Ashyralyev, A. and P. Sobolevskii (editors), *Well-Posedness of Parabolic Difference Equations, Operator*, Theory: Advances and Applications, vol. 69,Birkhuser Verlag, Basel,Boston,Berlin, 1994.

Ashyralyev, A. and P. Sobolevskii (editors), *New Difference Schemes for Partial Differential Equations*, Operator Theory: Advances and Applications, vol.148,Birkhuser, Verlag,, Basel,Boston,Berlin, 2004.

Ashyralyev, A. and A. Yakubov, "The structures of fractional spaces generating by the transport operator", *in: Modeling the Processes in Exploration of Gas Deposits and Applied Problems of Theoretical Gas Hydrodynamics*, Vol. , pp. 162–172 Ilim, Ashgabat, (Russian),1998.

Ashyralyev, A. and N. Yaz, "On Structure of Fractional Spaces Generated by Positivity Operators with the Nonlocal Boundary Conditions", *PCDDEA, Books from Hindawi Publishing Corporation*, Vol. , pp. 91–102, 2006.

Ashyralyev, A. and N. Yenial-Altay, "Positivity of difference operators generated by the nonlocal boundary conditions", *Dynamical Systems and Applications, Proceedings*, Vol. , pp. 113–135, 2005.

Bazarov, M., "On the structure of fractional spaces, Proceeding of the XXVII All-Union Scientific Student Conference", *The Student and Scientific-Technological Progress,(Novosibirsk, 1989),Novosibirsk. Gos. Univ., Novosibirsk*, Vol. , pp. 3–7, (Russian),1989.

Danelich, S., "Positive difference operators in R_{h1}", *Voronezh. Gosud Univ.1987 Deposited VINITI 3.18, no.1936-B87*, Vol. , pp. 13,, (Russian),1989.

Fattorini, H., *Second Order Linear Differential Equations in Banach Spaces*, Elsevier Sciense Publishing Company, North-Holland, 1985.

Nalbant, N., *The structure of fractional spaces generated by positive operators and its applications*, M.S. Thesis, Fatih University, 2011.

Simirnitskii, Y. and P. Sobolevskii, "Positivity of multidimensional difference operators in the C-norm", *Uspekhi Matematicheskikh Nauk 36, no.4*, Vol. , pp. 202–203, (Russian) 1981.

Simmons, G. F., *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1963.

Sobolevskii, P., "A New Method of Summation of Fourier Series Converging in C-Norm", *Semigroup Forum*, Vol. 71, pp. 289–300, 2005.

Solomyak, M., "Evaluation of norm of the resolvent of elliptic operators in L_p*-spaces*", *Uspekhi Matematicheskikh Nauk 15*, Vol. 96, pp. 141–148, (Russian),1960.

Stewart, H., "Generation of analytic semigroups by strongly elliptic operators under general boundary conditions", *Transactions of the American Mathematical Society 259, no. 1*, Vol. , pp. 299–310, 1980.