ORTHOGONAL POLYNOMIALS AND STOCHASTIC INTEGRALS

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

It has been known for a long time that there is close connection between probability theory and orthogonal polynomials. The purpose of this study is to give a detailed survey of several such connections, namely between stochastic processes and orthogonal polynomials, generating functions for orthogonal polynomials and expectations. A new generating function for a class of orthogonal polynomials (Bernstein-Szegö) is presented.

Keywords: Generating functions for orthogonal polynomaials; renormalization method; stochastic processes; classical orthogonal polynomials

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ÖZ

Çok uzun zamandır olasılık teorisi ile ortogonal polinomlar arasındaki yakın ilişki bilinmektedir. Bu çalışmanın ana amacı stokastik süreçler ile ortogonal polinomlar ve ortogonal polinomların genelleştirici fonksiyonları ve beklenen değer ile ilgili detaylı bir inceleme vermektedir. Bu tezde Bernstein-Szegö polinomları için yeni bir genelleştirici fonksiyon tanıtılmaktadır.

Anahtar Kelimeler: Ortogonal polinomlar için genelleştirici fonksiyonlar; Renormalizasyon metodu; Stokastik süreç; Klasik ortogonal polinomlar.

DEDICATION

To my family, my husband and my child.

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CHAPTER I

INTRODUCTION

It has been known for a long time that there is a close connection between stochastic processes and orthogonal polynomials. The starting point here is N. Wiener paper of 1930[Wiener, N. 1930]. Many of these connections are described in a recent book of [Schoutens, W.,2000]. But there are other close relations between probability theory and orthogonal polynomials. For instance, recently in a series of papers of N. Asai, I. Kubo and H. H. Kuo [3,4,5,6,7] a new method for deriving generating functions for orthogonal polynomials was developed. It is called multiplicative renormalization method or AKK method.

The main purpose of this study is to give a detailed survey of these methods. Furthermore, a new generating function for a class of orthogonal polynomials is given.

Firstly, in Chapter 2, a short survey of the stochastic integration theory (Ito's integral) together with applications in financial mathematics is given.

The most detailed study is given in Chapter 3. After short introduction in the subject of orthogonal polynomials the generating functions method is presented with reconstruction of many details which are omitted in the original papers. Furthermore, section 3.2.7 contains a new generating function for Bernstein-Szegő polynomials. Namely, the Bernstein-Szegő polynomials $T_n(x,\rho)$ are defined as a system polynomials orthogonal in the sense that

$$\int_{-1}^{1} p_n(x)x^v(1-x^2)^{\frac{-1}{2}}\{\rho(x)\}^{-1}dx = 0, \qquad v = 0, 1, \cdot\cdot\cdot, n-1,$$

where $\rho(x)$ is a fixed polynomial positive in $[-1,1]$. Then

$$\psi(t,x) = 2\sum_{n=k}^{\infty} T_n(x,\omega)t^n = \frac{t^k(\cos\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx} - t\cos(\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx} - \arccos x))}{1-2tx+t^2}$$

where

$$\omega(x) - \rho^2(x) = \prod_{j=1}^{k}(1-a_jx)^2.$$

Chapter 4 contains a detailed survey of several results about connection between stochastic processes and orthogonal polynomials.

CHAPTER II

STOCHASTIC INTEGRALS

The material of the chapter is taken from [Steele, M. J.1979], [Protter, P.E.2004], [Schoutens,W.2000]

2.1 *Main Facts and Definitions*

Definition(Brownian Motion): A continuous-time stochastic process $\{B_t : 0 \leq t < T\}$ is called a Standard Brownian Motion on $[0, T)$ if it has the following four properties:

$i)$ $B_0 = 0$

$ii)$The increments of B_t are independent; that is, for any finite set of times $0 \leq t_1 < t_2 < \cdots < t_n < T$ the random variables

$B_{t_2} - B_{t_1}, \; B_{t_3} - B_{t_2}, \cdots, \; B_{t_n} - B_{t_{n-1}}$ are independent.

$iii)$For any $0 \leq s \leq t < T$ the increment $B_t - B_s$ has the Gaussian distribution with mean 0 and variance $t - s$

$iv)$For all ω in a set of probability one, $B_t(\omega)$ is a continuous function of t .

Definition(Standard Brownian Filtration): Let $\mathcal{B}$ denote the set of Borel sets of $[0, T]$ .We then take adopted functions $\{\mathcal{F}_t\}$ to be the standard Brownian motion filtration, and for each $t \geq 0$ we take $\mathcal{F}_t \times \mathcal{B}$ to be the smallest $\sigma - field$ that contains all of the product sets $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} \in \mathcal{F}_t$ and $\mathbf{B} \in \mathcal{B}$. $f(.,.)$ is measurable if $f(.,.) \in \mathcal{F}_t \times \mathcal{B}$ and we will say that $f(.,.)$ is adapted provided that $f(.,.) \in \mathcal{F}_t$ for each $t \in [0, T]$.

We will consider only integrands from the class $\mathcal{H}^2 = \mathcal{H}^2\,[0, T]$ that contains all measurable adapted functions f that satisfy the integrability constraint

$$E[\int_0^T f^2(\omega, t)\, dt] < \infty \ , \tag{2.1.1}$$

we should note that the expectation is actually a double integral and that $\mathcal{H}^2$ is a closed linear subspace of $L^2\,(dP \times dt)$.

If we take $f(\omega, t)$ to be the indicator of the interval $(a, b] \subset [0, T]$, then

$$I(f)(\omega) = \int_a^b dB_t = B_b - B_a \tag{2.1.2}$$

Let $\mathcal{H}_0^2$ denote the subset of $\mathcal{H}^2$ that consists of all functions of the form

$$f(\omega, t) = \sum_{i=0}^{n-1} a_i(\omega) 1(t_i < t \leq t_{i+1}) \tag{2.1.3}$$

where $a_i \in \mathcal{F}_{t_i}$, $E(a_i^2) < \infty$, and $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$.
By the definition, the Ito integral of f from (2.1.3) is equal to

$$I(f)(\omega) \ = \sum_{i=0}^{n-1} a_i(\omega)\{B_{t_{i+1}} - B_{t_i}\} \tag{2.1.4}$$

Now for any $f \in \mathcal{H}^2$ there is a sequence $\{f_n\} \subset \mathcal{H}_0^2$ such that f_n converges to f in $L^2(dP \times dt)$, hence the Ito integral $I(f)$ is taken by definition as

$$I(f) = \lim_{n \to \infty} I(f_n)$$

It might be proved that I is well defined.

Theorem 1 *(Markov's inequality) If X is any random variable and $a > 0$ then*

$$P(|X| \geq a) \leq \frac{E(|X|)}{a}.$$

Definition 2 *An adopted process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$ a.s. is a Levy Process if*

- X has increments independent of the past ; that is , $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$; and
- X has stationary increments , that is , $X_t - X_s$ has the same distribution as X_{t-s} , $0 \leq s < t < \infty$, and
- X_t is continuous in probability, that is , $lim_{t \to s} X_t = X_s$, where the limit is taken in probability.

Definition 3 *A process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$ a.s. is an intrinsic Levy process if*

- X has independent increments ; that is , $X_t - X_s$ is independent of $X_v - X_u$ if $(u, v) \cap (s, t) = \varnothing$ and
- X has stationary increments , that is , $X_t - X_s$ has the same distribution as $X_v - X_u$ if $t - s = v - u > 0$ and

- X_t is continuous in probability.

Let $\{X_t, t \geq 0\}$ be a stochastic process and $0 \leq t_1 \leq t_2$. The random variable $X_{t_2} - X_{t_1}$ is called the increment of the process X_t over the interval $[t_1, t_2]$

A stochastic process X_t is said to be a process with independent increments if the increments over nonoverlapping intervals (common end points are allowed) are stochastically independent. A process X_t is called a stationary or homogenous process if the distribution of the increment $X_{t-s} - X_t$ depends only on t, but is independent of s. A stationary process with independent increments is called a Levy process.

Definition 4 *Let X_t be a Levy process. We denote the characteristic function of the distribution of $X_{t-s} - X_t$ by*

$$\phi(u,t) \equiv E[\exp(iu(X_{t-s} - X_t))].$$

It is known that $\phi(u,t)$ is infinitely divisible (i.e., for every positive integer n, it is the nth power of some characteristic function), and that

$$\phi(u,t) = [\phi(u,1)]^t.$$

We denote by

$$X_{t-} = \lim_{s \to t, s<t} X_s, \quad t > 0$$

the left limit process and by $\Delta X_t = X_t - X_{t-}$ the jump size at time t.

The following representation is valid for the characteristic function of an infinitely divisible distribution with finite variance.

Theorem 5 *(Kolmogorov Canonical Representation) The function $\phi(\theta)$ is the characteristic function of an infinitely divisible distribution with finite second moment if, and only if, it can be written in the form*

$$\psi(\theta) = \log \phi(\theta) = ic\theta + \int_{-\infty}^{+\infty} (e^{i\theta x} - 1 - i\theta x) \frac{dK(x)}{x^2},$$

where c is a real constant and $K(y)$ is a nondecreasing and bounded function such that $K(-\infty) = 0$. The representation is unique.

For $x = 0$, the integrand $\frac{(e^{i\theta x}-1-i\theta x)}{x^2}$ is defined to be equal to $-(\frac{\theta^2}{2})$. The function ψ is often called the characteristic exponent of the Levy process.

Theorem 6 *(Levy-Khintchine formula) A function $\psi : R \to C$ is the characteristic exponent of an infinitely divisible distribution if and only if there are constants $a \in R$, $\sigma^2 \geq 0$, and a measure ν on $R \setminus \{0\}$ with $\int_{-\infty}^{+\infty}(1 \wedge x^2)\nu(dx) < \infty$ such that*

$$\psi(\theta) = ia\theta - \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{+\infty}(\exp(i\theta x) - 1 - i\theta x 1_{\{|x|<1\}})\nu(dx)$$

for every θ.

The measure ν is called the Levy measure.

If we have an infinitely divisible distribution with characteristic function $\phi(\theta)$, we define a Levy process X_t through the relations

$$exp(\psi_X(\theta)) \equiv \phi_X(\theta) \equiv E[\exp(i\theta X_1)] = \phi(\theta).$$

2.2 *Ito Formula*

Theorem 7 ***(Ito Formula):****If $f : R \to R$ has a continuous second derivative, then*

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds \tag{2.2.1}$$

Proof. We will prove it only for the case when f has compact support. Let's break $f(B_t) - f(0)$ into small pieces of the form $f(B_{t_i}) - f(B_{t_{i-1}})$. By setting $t_i = \frac{it}{n}$ for $0 \leq i \leq n$ and then the telescoping differences give representation

$$f(B_t) - f(0) = \sum_{i=1}^{n}\{f(B_{t_i}) - f(B_{t_{i-1}})\} \tag{2.2.2}$$

To make this concrete , use Taylor's formula in the remainder form which says that if f has a continuous second derivative, then for all real x and y we have

$$f(y) - f(x) = (y-x)f'(x) + \frac{1}{2}(y-x)^2 f''(x) + r(x,y) \tag{2.2.3}$$

where the remainder term $r(x, y)$ is given by

$$r(x,y) = \int_x^y (y-u)(f''(u) - f''(x))du$$

From the continuity of f'' the function

$$h(x,y) = sup_u \, |f''(u) - f''(x)|$$

$$u : |u - x| \leq |y - x|$$

$$(u-x)^2 \leq (y-x)^2$$

is uniformly continuous, bounded,and $h(x,x) = 0$ for all x. Hence

$$|r(x,y)| \leq (y-x)^2 h(x,y)$$

The telescoping sum (2.2.2) can be rewritten as a sum of three terms, A_n, B_n and C_n, where the first two terms are given by

$$A_n = \sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) \;\; andB_n = \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}})^2$$

and the third terms C_n satisfies

$$|C_n| \leq \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 h(B_{t_{i-1}}, B_{t_i}) \tag{2.2.4}$$

■

Lemma 8 *(Reimann Representation).For any continuous $f : R \to R$, if we take partition of $[0,T]$ given by $t_i = \frac{iT}{n}$ for $0 \leq i \leq n$, then we have*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) = \int_0^T f^{'}(B_s)\, dBs \tag{2.2.5}$$

where the limit is understood in the sense of convergence in probability. Because f' is continuous , we know from the Reimann representation

$$A_n \to^p \int_0^t f^{'\prime}(B_s)\, dB_s$$

When we write B_n as a centered sum

$$\frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1}) + \frac{1}{2}\sum_{i=1}^{n} f''(B_{t_{i-1}})\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\},$$

we find by the continuity of $f''(B_s(\omega))$ as a function of s that the first summand converges as an ordinary integral for all ω in a set of probability one.

$$\lim_{n\to\infty}\sum_{i=1}^{n} f''(B_{t_{i-1}})(t_i - t_{i-1}) = \int_0^t f''(B_s)\,ds$$

If we denote the second summand of B_n by $\widetilde{B}_n$, we find by the orthogonality of the summands that

$$E(\widetilde{B}_n^2) = \frac{1}{4}\sum_{i=1}^{n} E[f''(B_{t_{i-1}})^2\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})^2\}]$$

$$\leq \frac{1}{4}\,\|f''\|_\infty \sum_{i=1}^{n} E[\{(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\}^2] = \frac{t^2}{2n}\,\|f''\|_\infty^2$$

where in the last step we used the fact that $B_{t_i} - B_{t_{i-1}} \sim N(0, \frac{t}{n})$ and consequently $Var((B_{t_i} - B_{t_{i-1}})^2) = \frac{2t^2}{n^2}$. By Markov's inequality , the last bound on $E(\widetilde{B}_n^2)$ is more than we need to show $\widetilde{B}_n \to^p 0$.

To prove $C_n \to^p 0$, we apply firstly the Cauchy inequality to the summands in the bound (2.2.4) on C_n to find

$$E\,|C_n| \leq \sum_{i=1}^{n}[E(B_{t_i} - B_{t_{i-1}})^4]^{\frac{1}{2}}[Eh^2(B_{t_{i-1}}, B_{t_i})]^{\frac{1}{2}} \tag{2.2.6}$$

The first factor in the sum is easily calculated since $B_{t_i} - B_{t_{i-1}} \sim N(0, \frac{t}{n})$ gives us

$$E[(B_{t_i} - B_{t_{i-1}})^4] = \frac{3t^2}{n^2} \tag{2.2.7}$$

To estimate the second factor , we first note by the uniform continuity of h and the fact that $h(x,x) = 0$ for all x, we have for each ε>0 a $\delta = \delta(\varepsilon)$ such that $|h(x,y)| \leq \varepsilon$ for all x, y with $|x - y| \leq \delta$, so we also have

$$E[h^2(B_{t_{i-1}}, B_{t_i})\ \leq \varepsilon^2 + \|h\|_\infty^2\, P(\left|B_{t_i} - B_{t_{i-1}}\right| \geq \delta)$$

$$\leq \varepsilon^2 + \|h\|_\infty^2\,\delta^{-2}E(\left|B_{t_i} - B_{t_{i-1}}\right|^2)$$

$$= \varepsilon^2 + \|h\|_\infty^2 \delta^{-2} \frac{t}{n} \tag{2.2.8}$$

When we apply the bounds given by equations (2.2.6) and (2.2.7) to the sum in inequality (2.2.5) , we find

$$E|C_n| \le n(\frac{3t^2}{n^2})^{\frac{1}{2}} (\varepsilon^2 + \|h\|_\infty^2 \delta^{-2} \frac{t}{n})^{\frac{1}{2}}$$

and consequently

$$\limsup_{n\to\infty} \; E|C_n| \le 3^{\frac{1}{2}} t\varepsilon$$

By the arbitrariness of ε , we finally see $E|C_n| \to 0$ as $n \to \infty$, so Markov's inequality tells us that we also have $C_n \to^p 0$.

Now we have seen that for any given $t \in R^+$ the sums A_n and B_n converge in probability to the two integral terms of Ito's formula (2.2.1) , and we have seen that C_n converges in probability to zero. Hence , if we fix $t \in R^+$, we can choose a subsequence n_j such that A_{n_j}, B_{n_j} and C_{n_j} all converge with probability one , so in fact we see that Ito's formula (2.2.1) holds with probability one for each fixed $t \in R^+$ Finally,if we then apply this fact for each rational t and if we also observe that both sides of Ito's formula are continuous , then we see that there is a set Ω_0 with $P(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$ we have Ito's formula for all $t \in R^+$.

Theorem 9 *(Ito's Formula with Space and Time Variables) For any function* $f \in C^{1,2}(R^+ \times R)$, *we have the representation*

$$f(t, B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds$$

If

$$f \in C^{1,2}(R^+ \times R), X_t = f(t, B_t)$$

$$X_t = X_0 + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds \tag{2.2.9}$$

and because the three integrals use up so much of the page , it is usual to write equation (2.2.9) in the shorthand

$$dX_t = \frac{\partial f}{\partial x}(t, B_t)\, dB_t + \frac{\partial f}{\partial t}(t, B_t)\, dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)\, dt \tag{2.2.10}$$

In fact we have not given any definition of dX_t except as shorthand.

Definition 10 *Standard Brownian motion in R^d is defined to be the vector - valued process given by*

$$B_t^{\rightarrow} = (B_t^1, B_t^2, ..., B_t^d),$$

where the one-dimensional component processes $\{B_t^k : 0 \leq t < \infty\}$ are independent standard Brownian motions.

Theorem 11 *(Ito's Formula -Vector Version) If $f \in C^{1,2}(R^+ \times R^d)$ and $\overrightarrow{B_t}$ is standard Brownian motion in R^d , then*

$$df(t, \overrightarrow{B_t}) = f_t(t, \overrightarrow{B_t})dt + \nabla f(t, \overrightarrow{B_t})d\overrightarrow{B_t} + \frac{1}{2}\Delta f(t, \overrightarrow{B_t})dt$$

If X_t is a Brownian motion with general drift and variance , then we have our choice whether to write X_t as a stochastic integral or as a function of Brownian motion:

$$dX_t = \mu dt + \sigma dB_t \quad , X_0 = 0 \ or \ X_t = \mu t + \sigma B_t$$

Now we will consider Ito's formula for a function of t and X_t.

If we have $Y_t = f(t, X_t)$, then we can also write $Y_t = g(t, B_t)$ where we take g to be defined by $g(t, x) = f(t, \mu t + \sigma x)$. When we apply Ito's formula (2.2.10) to the representation $Y_t = g(t, B_t)$, we find

$$dY_t = g_t(t, B_t)dt + g_x(t, B_t)dBt + \frac{1}{2}g_{xx}(t, B_t)dt$$

and the chain rule gives

$$g_t(t, x) = f_t(t, \mu t + \sigma x) + f_x(t, \mu t + \sigma x)\mu$$

$$g_x(t, x) = f_x(t, \mu t + \sigma x)\sigma$$

$$g_{xx}(t, x) = f_{xx}(t, \mu t + \sigma x)\sigma^2$$

so in terms of f we have

$$dY_t = \{f_t(t, X_t) + \mu f_x(t, X_t)\}dt + \sigma f_x(t, X_t)dB_t + \frac{1}{2}\sigma^2 f_{xx}(t, X_t)dt$$

$$= f_t(t, X_t) + f_x(t, X_t)dXt + \frac{1}{2}\sigma^2 f_{xx}(t, X_t)dt \qquad (2.2.11)$$

Shortly the Ito's Formula (2.2.11) may be written by using of the box algebra.

TABLE (2.2.1) BOX ALGEBRA MULTIPLICATION TABLE

$\cdot$	dt	dB_t
dt	0	0
dB_t	0	dt

Definition 12 *We say that a process $\{X_t : 0 \leq t \leq T\}$ is standard provided that $\{X_t\}$ has the integral representation*

$$X_t = x_0 + \int_0^t a(w, s)ds + \int_0^t b(w, s)dB_s for 0 \leq t \leq T,$$

and where $a(.,.)$ and $b(.,.)$ are adapted , measurable processes that satisfy the integrability conditions

$$P(\int_0^t |a(w, s)|\, ds < \infty) = 1 \text{ and } P(\int_0^t |b(w, s)|^2\, ds < \infty) = 1$$

Theorem 13 *(Ito's Formula for Standard Processes) If $f \in C^{1,2}(R^+ \times R)$ and $\{X_t : 0 \leq t \leq T\}$ is a standard process with the integral representation*

$$X_t = \int_0^t a(w, s)ds + \int_0^t b(w, s)dB_s \text{ for } 0 \leq t \leq T,$$

then we have

$$f(t, X_t) = f(0, 0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)b^2(w, s)ds$$

When we look at the definition in the language of the box algebra , it tells us that for the process $Y_t = f(t, X_t)$ we have

$$dY_t = f_t(t, X_t)dt + f_x(t, X_t)dXt + \frac{1}{2}f_{xx}(t, X_t)dX_t \cdot dX_t \qquad (2.2.12)$$

We want to extend our box calculus to functions of several processes. We need to extend the box algebra multiplication table by one row and one column

TABLE (2.2.2) EXTENDED BOX ALGEBRA

$\cdot$	dt	dB_t^1	dB_t^2
dt	0	0	0
dB_t^1	0	dt	0
dB_t^2	0	0	dt

Theorem 14 *(Ito's Formula for Two Standard Processes) If $f \in C^{2,2}(R\times R)$ and both of the standard processes $\{X_t : 0 \le t \le T\}$ and $\{Y_t : 0 \le t \le T\}$ have the integral representations*

$$X_t = \int_0^t a(w,s)ds + \int_0^t b(w,s)dB_s \text{ and } Y_t = \int_0^t \alpha(w,s)ds + \int_0^t \beta(w,s)dB_s$$

then we have

$$f(X_t,Y_t) = f(0,0) + \int_0^t f_x(X_s,Y_s)dX_s + \int_0^t f_y(X_s,Y_s)dY_s$$

$$+\frac{1}{2}\int_0^t f_{xx}(X_s,Y_s)b^2(w,s)ds$$

$$+\int_0^t f_{xy}(X_s,Y_s)b(w,s)\beta(w,s)ds$$

$$+\frac{1}{2}\int_0^t f_{yy}(X_s,Y_s)\beta^2(w,s)ds.$$

In the language of the box algebra , the Ito formula for $Z_t = f(X_t,Y_t)$ can be written as

$$dZ_t = f_x(X_t,Y_t)dX_t + f_y(X_s,Y_s)dY_t$$

$$+\frac{1}{2}f_{xx}(X_s,Y_s)dX_t.dX_t$$

$$+f_{xy}(X_s,Y_s)dX_t.dY_t$$

$$+\frac{1}{2}f_{yy}(X_s,Y_s)dY_t$$

2.3 *Applications in Financial Mathematics*

2.3.1 Replication and Examples of Arbitrage

The basis of the arbitrage is that any two investments with identical payout streams must have the same price. If this were not so , we could simultaneously sell the more expensive instrument and buy the cheaper one; we would make an immediate profit and the payment stream from our purchase could be used to meet the obligations from our sale. There would be no net cash flows after the initial action so we would have secured our arbitrage profit.

2.3.2 Forward Contracts

Forward contracts provide an example of arbitrage pricing that has been honored for centuries . A typical forward contract is an agreement to buy a commodity -say 10000 ounces of gold- at time T, for a price K. If the current time is t and the current gold price is S, then in a world where there are economic agents who stand ready to borrow or lend at the continuous compound rate r, the arbitrage price F of the forward contract is given by

$$F = S - e^{-r(T-t)}K.$$

In other words , if the forward contract were to be available at a different price, one would have an arbitrage opportunity.

The key observation is that there is an easy way to replicate the financial consequences of a forward contract. Specifically , one could buy the gold right now and borrow $e^{-r(T-t)}K$ dollars for a net cash outlay of $S - e^{-r(T-t)}K$, then at time T pay off the loan (with the accrued interest) and keep the gold. At the end of this process , one makes a payment of K dollars gets ownership of the gold, so the payout of the forward contract is perfectly replicated ,both with respect to cash and the commodity. The cash required to initiate the immediate purchase strategy is $S - e^{-r(T-t)}K$ and the cost of the forward contract is F , so the arbitrage argument tells us that these two quantities must be equal.

TABLE (2.3.1) REPLICATION OF A FORWARD CONTRACT

	Cash Paid Out(Time=t)	Commodity and Cash(Time=T)
Forward Contract	F	Gold owned, K\$ Cash paid
Replication	$S - e^{-r(T-t)}K$	Gold owned, K\$ Cash paid

2.3.3 Put-Call Parity

A European call option on a stock is the right , but not the obligation , to buy the stock for the price K at time T. The European put option is the corresponding right to sell the stock at time T at a price of K . Our second illustration of arbitrage pricing will tell us that the arbitrage price of a European put is a simple function of the price of the call , the price of the stock , and the two-way interest rate.

First, we consider a question in the geometry of puts and calls. What is the effect of buying a call and selling a put , each with the same strike K ? Some funds will flow from the initiation of this position, then we will find at time T that

If the stock price is above K , we will realize a profit of that price minus K ;

If the stock price is below K , we will realize a loss equal to K minus the stock price .

A moment's reflection will tell us this payout is exactly what we would get from a contract for the purchase of the stock at time T for a price K. Because we already know what the price of such a contract must be, we see that the price of C of the call and the price P of the put must satisfy,

$$C - P = S - e^{-r(T-t)}K \tag{2.3.3.1}$$

This relationship is often referred to as the *put-call parity formula* , and it tells us how to price the European put if we know how to price the European call, or vice versa.

2.3.4 The Binomial Arbitrage

It does not take a rocket scientist to replicate a forward contract, or to value a put in terms of a call , but the idea of replication can be pushed much further, and, before long, some of the techniques familiar to rocket scientists start to show their value . Before we come to a mathematically challenging problem, however , there is one further question that deserves serious examination - even though it requires only elementary algebra.

For the sake of argument , we first consider an absurdly simple world with one stock, one bond, and two times- time 0 and time 1 . The stock has a price of 10$ at time 0, and its price at time 1 is either equal to 5$ or 20$. The bond has a price of 5$ at time 0 and is also 5$ at time 1. People in this thought-experiment world are so kind that they borrow or lend at a zero interest rate.

Now consider a contract that pays 15$ if the stock moves to 20$ and pays nothing if the stock moves to 5$. This contract is a new security that derives its value from the value of the stock , a toy example of a derivative security. The natural question is to determine the arbitrage price of X the security.

From our earlier analysis, we know that to solve this problem we only need to find a replicating portfolio. In other words , we only need to find α and β such that the portfolio consisting of α units of the stock and β units of the bond will exactly replicate the payout of the contract . The possibility of such a replication is made perfectly clear when we consider a table that spells out what is required under the two possible contingencies - the stock goes up , or the stock goes down.

TABLE (2.3.2) REPLICATION OF A DERIVATIVE SECURITY

	Portfolio	Derivative Security
Original cost	$\alpha S + \beta B$	X
Payout if stock goes up	$20\alpha + 5\beta$	15
Payout if stock goes down	$5\alpha + 5\beta$	0

When we require that the portfolio must replicate the payout of the derivative security, we get the two equations

$$20\alpha + 5\beta = 15 \text{ and } 5\alpha + 5\beta = 0$$

We can solve these questions to find $\alpha = 1$ and $\beta = -1$, so by the purchase of one share of stock and the short sale of one bond , we produce a portfolio that perfectly replicates the derived security. This replicating portfolio requires an initial investment of five dollars to be created , so the arbitrage price of the derived security must also equal five dollars.

2.3.5 The Black-Scholes Model

We will now follow a remarkable continuous-time arbitrage argument that will lead us to the famous Black-Scholes formula for the pricing of European call options. We let S_t denote the price at time t of a stock and let β_t denote the price at time t of a bond . We then take the time dynamics of these two processes to be given by the SDEs

$$Stockmodel : dS_t = \mu S_t dt + \sigma S_t dB_t \quad Bondmodel : d\beta_t = r\beta_t dt \tag{2.3.5.1}$$

that is , we assume that the stock price is given by a geometric Brownian motion , and the bond price is given by a deterministic process with exponential growth.

For a European call option with strike price K at termination time T , the payout is given by $h(S_T) = (S_T - K)_+$. To find the arbitrage price for this security, we need to find a way to replicate this payout . The new idea is to build a dynamic portfolio where the quantities of stocks and bonds are continuously readjusted as time passes.

2.3.6 Arbitrage and Replication

If we let a_t denote the number of units of stock that we hold in the replicating portfolio at time t and let b_t denote the corresponding number of units of the bond , then the total value of the portfolio at time t is

$$V_t = a_t S_t + b_t \beta_t$$

The condition where the portfolio replicates the contingent claim at time T is simply

$$\text{terminal replication constraint} : V_T = h(S_T) \tag{2.3.6.1}$$

In the one-period model of the binomial arbitrage , we only needed to solve a simple linear system to determine the stock and bond positions of our replicating portfolio , but in the continuous-time model we face more difficult task. Because the prices of the stock and bond change continuously, we have the opportunity to continuously rebalance our portfolio that is, at each instant we may sell some of the stock to buy more bonds or vice versa. This possibility of continuous rebalancing gives us the flexibility we need to replicate the cash flow of the call option.

Because the option has no cash flow until the terminal time , the replicating portfolio must be continuously rebalanced in such a way that there is no cash flowing into or out of the portfolio until the terminal time T . This means that at all intermediate times we require that restructuring of the portfolio be self-financing in the sense that any change in the value of the portfolio value must equal the profit loss due to change in the price of the stock or the price of the bond . In terms of stochastic differentials , this requirement is given by the equation ,

$$\text{self-financing condition} : dV_t = a_t dS_t + b_t d\beta_t \tag{2.3.6.2}$$

This equation imposes a strong constraint on the possible values for a_t and b_t . When coupled with the termination constraint $V_T = h(S_T)$, the self-financing condition (2.3.6.2) turns out to be enough to determine a_t and b_t uniquely.

2.3.7 Coefficient Matching

In order to avail ourselves of the Ito calculus , we now suppose that the portfolio value V_t can be written as $V_t = f(t, S_t)$, for an appropriately smooth f . Under this hypothesis, we will then be able to use the self-financing condition to get expressions for a_t and b_t in terms of f and its derivatives. The replication identity can then be used to turn these expressions into a PDE for f . The solution of this PDE will in turn provide formulas for a_t and b_t as functions of the time and the stock price.

To provide the coefficient matching equation , we need to turn our to expressions for V_t into SDEs. First, from the self-financing condition and the models for the stock and bond, we have

$$dV_t = a_t dS_t + b_t d\beta_t = a_t\{\mu S_t dt + \sigma S_t dB_t\} + b_t\{r\beta_t dt\}$$

$$= \{a_t \mu S_t + b_t r \beta_t\}dt + a_t \sigma S_t dB_t \tag{2.3.7.1}$$

From our assumption that $V_t = f(t, S_t)$ and the Ito formula for geometric Brownian motion (or the box calculus), we then find

$$dV_t = f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t) dS_t dS_t + f_x(t, S_t) dS_t$$

$$= \{f_t(t, S_t) + \frac{1}{2} f_{xx}(t, S_t)\sigma^2 S_t^2 + f_x(t, S_t)\mu S_t\}dt + f_x(t, S_t)\sigma S_t dB_t \tag{2.3.7.2}$$

When we equate the dB_t coefficients from (2.3.7.1) and (2.3.7.2) , we find a delightfully simple expression for the size of the stock portion of our replicating portfolio:

$$a_t = f_x(t, S_t)$$

Now, to determine the size of the bond portion , we only need to equate the dt coefficients from equations (2.3.7.1) and (2.3.7.2) to find

$$\mu S_t f_x(t,S_t) + rb_t\beta_t = f_t(t,S_t) + \frac{1}{2}f_{xx}(t,S_t)\sigma^2 S_t^2 + f_x(t,S_t)\mu S_t$$

The $\mu S_t f_x(t,S_t)$ terms cancel, and we can then solve for b_t to find

$$b_t = \frac{1}{r\beta_t}\{f_t(t,S_t) + \frac{1}{2}f_{xx}(t,S_t)\sigma^2 S_t^2\} \tag{2.3.7.3}$$

Because V_t is equal both $f(t,S_t)$ and $a_t S_t + b_t\beta_t$, the values for a_t and b_t gives us a PDE for $f(t,S_t)$

$$f(t,S_t) = V_t = a_t S_t + b_t\beta_t$$

$$= f_x(t,S_t)S_t + \frac{1}{r\beta_t}\{f_t(t,S_t) + \frac{1}{2}f_{xx}(t,S_t)\sigma^2 S_t^2\}\beta_t$$

we arrive at the justly famous formula Black - Scholes PDE :

$$f_t(t,x) = -\frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - rxf_x(t,x) + rf(t,x) \tag{2.3.7.4}$$

with its terminal boundary condition

$$f(T,x) = h(x) \quad forall x \in R.$$

Theorem 15 *(A Solution Formula) If a, b and c are constants , then the initial-value problem given by*

$$v_t = av_{xx} + bv_x + cv \quad and \quad v(0,x) = \psi(x) \tag{2.3.7.5}$$

has a solution that can be written as

$$v(t,x) = \exp(-t(b^2 - 4ac)/4a - xb/2a)\int_{-\infty}^{\infty} k_t(x/\sqrt{a} - s)e^{sb/2\sqrt{a}}\psi(s\sqrt{a})\, ds \tag{2.3.7.6}$$

provided that $a > 0$ and that the initial data satisfy the exponential bound

$$|\psi(x)| \leq A\exp\left(B\,|x|^{\sigma}\right)$$

for some constants $A, B,$ *and* $\sigma < 2$.

For a European call option with strike price K at termination time T

$$h(x) = (x - K)_+ \ for\ all\ x \in \mathbb{R}$$

We have an equation with a terminal condition instead of an initial condition, first we want to make a change of time variable to reverse time , by defining a new variable

$$\tau = T - t$$

so that $\tau = 0$ corresponds to $t = T$. Now rewrite the Black-Scholes PDE as a function of τ and x .

We introduce a new variable

$$y = \log x$$

write $f(t, x)$ as $g(\tau, y)$

$$f_t = g_\tau \tau_t = -g_\tau$$

$$f_x = g_y y_x = g_y(\frac{1}{x})$$

$$f_{xx} = g_{yy}(\frac{1}{x^2}) - g_y(\frac{1}{x^2})$$

so the equation (2.3.7.4) gives a new initial-value problem for g :

$$g_\tau = \frac{1}{2}\sigma^2 g_{yy} + (r - \frac{1}{2}\sigma^2)g_y - rg \text{ and } g(0, y) = (e^y - K)_+ \tag{2.3.7.7}$$

This problem is precisely of the form

$$v_t = av_{xx} + bv_x + cv \; for \; t \in (0, T] \; and \; x \in R$$

so after identifying the coefficients

$$a = \frac{1}{2}\sigma^2, \; b = r - \frac{1}{2}\sigma^2 \text{ and } \; c = -r \tag{2.3.7.8}$$

$f(t, x) = g(\tau, y)$ is given by the product of the exponential factor

$$\exp(-\tau(b^2 - 4ac)/4a - yb/2a) \tag{2.3.7.9}$$

and the corresponding integral term of (2.3.7.6)

$$I = \int_{-\infty}^{\infty} k_\tau(y/\sqrt{a} - s)e^{sb/2\sqrt{a}}(e^{s\sqrt{a}} - K)_+ ds \tag{2.3.7.10}$$

To compute the integral, make change of variables $u = y/\sqrt{a} - s$ and restrict integration to the domain D where the integrand is nonzero,

$$D = \{u : y - u\sqrt{a} \geq \log K\} = \{u : u \leq (y - \log K)/\sqrt{a}\},$$

so

$$\begin{aligned} I &= \exp(y + yb/2a) \int_D k_\tau(u) \exp(-u(b/2\sqrt{a} + \sqrt{a})du \\ &\quad -K \; \exp(yb/2a) \int_D k_\tau(u) \exp(-ub/2\sqrt{a})du \end{aligned} \tag{2.3.7.11}$$

By the familiar completion of the square in the exponent , we can compute the general integral

$$\int_{-\infty}^{a} k_t(s)e^{-\beta s} \, ds = \; e^{\tau\beta^2}\Phi(\frac{a}{\sqrt{2t}} + \beta\sqrt{2t}) \tag{2.3.7.12}$$

Finally , since both integrals of equation (2.3.7.11) are of the same type as equation (2.3.7.12), we obtain 2.3.7.11.

Now by substitution $x = e^y$ the solution $f(t, x)$ of the Black-Scholes terminal-value problem is given by the Black-Scholes formula:

$$f(x, t) = x\Phi(\frac{\log(\frac{x}{K}) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{t}}) - Ke^{-r\tau}\Phi(\frac{\log(\frac{x}{K}) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{t}}).$$

CHAPTER III

ORTHOGONAL POLYNOMIALS

Most of this chapter is taken from [Chihara, T.S., 1978], [Asai N, Kubo I, Kuo H.H., 2004], [Delgado,M.A., Geronim,S.J., Iliev, P., Xu, Y., 2009].

3.1 *Background*

The systems of orthogonal polynomials associated with the names of Hermite, Laguerre and Jacobi (including special cases named after Chebyshev, Legendre and Gegenbauer) are unquestionably the most extensively studied and widely applied systems. These three systems are called collectively the *classical orthogonal polynomials.*

The literature on these polynomials is enormous and we will present only the most basic facts concerning them. The most thorough single account of the classical polynomials is found in the treatise of [Szegő,G. 1939].

The most part of the material for this section is taken from [Chihara,T.1978]

Definition 16 *The Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$ are the polynomials defined by the formula*

$$P_n^{(\alpha,\beta)}(x) = (-2)^n(n!)(1-x)^\alpha(1+x)^\beta \frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}] \tag{3.1.1}$$

Here α and β are parameters which, or integrability purposes, are restricted to $\alpha > -1$, $\beta > -1$. However , many of the identities and other formal properties of these polynomials remain valid under the less restrictive condition that neither α nor β is a negative integer.

(i)The Legendre polynomials $(\alpha = \beta = 0)$

$$P_n(x) = P_n^{(0,0)}(x) \tag{3.1.2}$$

(ii)The Chebyshev polynomials of the first kind $(\alpha = \beta = \frac{-1}{2})$

$$T_n(x) = 2^{2n} \binom{2n}{n}^{-1} P_n^{(-\frac{1}{2},-\frac{1}{2})}(x) \tag{3.1.3}$$

(iii)The Chebyshev polynomials of the second kind $(\alpha = \beta = \frac{1}{2})$

$$U_n(x) = 2^{2n} \binom{2n+1}{n+1}^{-1} P_n^{(\frac{1}{2},\frac{1}{2})}(x) \tag{3.1.4}$$

(iv) The Gegenbauer (or Ultraspherical) for polynomials $(\alpha = \beta)$

$$P_n^{(\lambda)}(x) = \binom{2\alpha}{\alpha}^{-1} \binom{n+2\alpha}{\alpha} P_n^{(\alpha,\alpha)}(x) \qquad \alpha = \lambda - \frac{1}{2} \neq \frac{-1}{2} \tag{3.1.5}$$

The formula (3.1.1) is usually called Rodrigues formula in the case $\alpha = \beta = 0$ while the general case is called Rodrigues' type formula.

Legendre investigated the polynomials which bear his name in 1785 while Rodrigues' formula appeared in 1816 . The general Jacobi polynomial was introduced by Jacobi in 1859.

Using Leibniz' formula for the nth derivative of a product we obtain from (3.1.1)

$$(-2)^n(n!)(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(1-x)^{n+\alpha} D^k(1+x)^{n+\beta}$$

$$= (-1)^n(1-x)^\alpha(1+x)^\beta n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k} \;)$$

(where we have written $D = \frac{d}{dx}$)

We thus obtain the explicit formula,

$$P_n^{(\alpha,\beta)}(x) = (2)^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k} \tag{3.1.6}$$

We see that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n whose leading coefficient

$$k_n = k_n(\alpha,\beta) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k} = 2^{-n}\binom{2n+\alpha+\beta}{n} \tag{3.1.7}$$

We also have

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{3.1.8}$$

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \tag{3.1.9}$$

The Laguerre polynomial , $L_n(x)$, can also be defined by a Rodrigues' type formula. Namely

$$L_n^{(\alpha)}(x) = (n!)^{-1}x^{-\alpha}e^{x}\frac{d^n}{dx^n}[x^{n-\alpha}e^{-x}] \tag{3.1.10}$$

It is customary to require that $\alpha > -1$ but most formal relations remain valid if α is not a negative integer . The case $\alpha = 0$ is the one originally studied by Laguerre although it occurred earlier in the works of Abel , Lagrange and Chebyshev. The notation,

$$L_n(x) = L_n^{(0)}(x)$$

is standard. The case of general α is due to Sonine and $L_n^{(\alpha)}(x)$ is obtained from (3.1.10) with the aid of Leibniz' formula. The result is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} \tag{3.1.11}$$

We conclude that $L_n^{(\alpha)}(x)$ is a polynomial of degree n with leading coefficient

$$k_n = k_n(\alpha) = \frac{(-1)^n}{n!} \tag{3.1.12}$$

The Hermite polynomials, $H_n(x)$ are defined by Rodrigues' type formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \tag{3.1.13}$$

Taylor's theorem yields the generating function

$$e^{2x\omega-\omega^2} = \sum_{n=0}^{\infty} H_n(x) \frac{\omega^n}{n!}$$

Expanding $e^{2x\omega}$ and $e^{-\omega^2}$ as power series in w taking the Cauchy product of the result yields

$$e^{2x\omega-\omega^2} = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k (2x)^{n-2k} \omega^n}{(n-2k)!k!}$$

In these formulas , $[x]$ denotes the largest integer not exceeding x. $H_n(x)$ is thus seen to be a polynomial of degree n whose leading coefficient is

$$k_n = 2^n \tag{3.1.14}$$

The most common alternative to the above terminology is to replace $exp(-x^2)$ by $exp(\frac{-x^2}{2})$ in (3.1.13). This yields a polynomial, $He_n(x)$, which can be expressed in terms of $H_n(x)$ by

$$He_n(x) = 2^{\frac{-n}{2}} H_n(2^{\frac{-1}{2}} x)$$

$He_n(x)$ is called the Hermite polynomial for example by [Jackson,D.,1941] and seems to be the preferred form for applications to statistics.

In each of the three cases, we have defined a sequence of polynomials by a formula of the type

$$P_n(x) = K_n^{-1}[w(x)]^{-1}D^n[\rho^n(x)w(x)], \ n = 0, 1, 2, ... \tag{3.1.15}$$

where for the Jacobi ,Laguerre and Hermite cases , respectively ;

(i) $K_n = (-2)^n n!$, $n!$, $(-1)^n$;

(ii) $\rho(x)$ is a polynomial independent of n and of degree 2,1 and 0;

(iii) $w(x)$ is positive and integrable over (a, b) where (a, b) is $(-1, 1)$, $(0, \infty)$ and $(-\infty, \infty)$, respectively;

$(iv) D^n[\rho^n(x)w(x)]$ vanishes for $x = a$ and $x = b$, $0 \leq k < n$.

(Here and in (iii) , the conditions , $\alpha > -1$, $\beta > -1$ in the Jacobi case and $\alpha > -1$ in the Laguerre case must be imposed.)

For nonnegative integers m and n, write

$$I_{mn} = \int_a^b x^m P_n(x)w(x)dx = K_n^{-1} \int_a^b x^m D^n[\rho^n(x)w(x)]dx.$$

Integrating by parts and using (iv) , we find

$$K_n I_{mn} = x^m D^{n-1}[\rho^n(x)w(x)]_a^b - m \int_a^b x^{m-1} D^{n-1}[\rho^n(x)w(x)]dx$$

$$= -mx^{m-1}D^{n-1}[\rho^n(x)w(x)]dx$$

We assume that $0 \leq m \leq n$. If the above procedure is repeated, we then obtain after m such steps ,

$$K_n I_{mn} = (-1)^m m! \int_a^b D^{n-m}[\rho^n(x)\omega(x)]dx$$

Then if $m < n$, one more integration yields

$$K_n I_{mn} = (-1)^m m! x^{m-1} \int_a^b D^{n-m-1}[\rho^n(x)\omega(x)]_a^b dx = 0$$

On the other hand , if $m = n$, we have

$$K_n I_{mn} = (-1)^n n! \int_a^b [\rho^n(x)\omega(x)]_a^b dx$$

Thus in the three cases, we have specifically,

(Jacobi)

$$I_{nn} = (2)^{-n} n! \int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{n+\beta} dx$$

$$= 2^{n+\alpha+\beta+1} B(n+\alpha+1, n+\beta+1),$$

where B denotes the beta function which can be expressed in terms of the gamma function by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)};$$

(Laguerre)

$$I_{nn} = (-1)^n \int_0^\infty x^{n+\alpha} e^{-x} dx = (-1)^n \Gamma(n+\alpha+1);$$

(Hermite)

$$n! \int_{-\infty}^{\infty} e^{-x^2} dx = n!\sqrt{\pi}.$$

Referring to (3.1.7) , (3.1.12) and (3.1.14) for the leading coefficients of each the three polynomials , we then can write the explicit orthogonality relations,

$$\int_{-1}^{1} P_m^{(\alpha,\beta)} P_n^{(\alpha,\beta)} (1-x)^\alpha (1+x)^\beta dx, \qquad \alpha > 1, \beta > 1$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{mn}, \tag{3.1.16}$$

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^\alpha e^{-x} dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}, \qquad \alpha > -1;$$

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 2^n n! \sqrt{\pi}\delta_{mn}$$

For the Hermite, Laguerre, and Jacobi polynomials, there are simple formulas for the derivatives

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x) \tag{3.1.17}$$

The formula

$$\frac{dL_n^{(\alpha)}(x)}{dx} = -L_{n+1}^{(\alpha+1)}(x) \tag{3.1.18}$$

can be derived from (3.1.11).

$$\frac{dP_n^{(\alpha,\beta)}}{dx} = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{3.1.19}$$

We note that (3.1.17) , (3.1.18) , (3.1.19) show that in all three cases, the sequence of derivatives forms another orthogonal polynomial sequence.

There are a large number of generating functions known for the classical polynomials. By "generating function " for $\{P_n(x)\}$, we mean here a function F of two variables that has a formal Taylor's expansion of the form

$$F(x,\omega) \sim \sum_{n=0}^{\infty} a_n P_n(x)\omega^n,$$

where $\{a_n\}$ is a known sequence of constants.

The classical generating function for the Jacobi polynomials is

$$2^{\alpha+\beta}R^{-1}(1-\omega+R)^{-\alpha}(1+\omega+R)^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\omega^n, \tag{3.1.20}$$

where

$$R = (1 - 2x\omega + \omega^2)^{\frac{1}{2}}.$$

This generating function was obtained originally by Jacobi.

For $\alpha = \beta$, (3.1.20) yields a generating function for the Gegenbauer polynomials. However, in this case there is simpler generating function due to Gegenbauer:

$$(1 - 2x\omega + \omega^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x)\omega^n \qquad (3.1.21)$$

In turn, (3.1.21) yields for $\lambda = \frac{1}{2}$ and 1, respectively, generating functions for the Legendre polynomials and Chebyshev polynomials of the second kind:

$$(1 - 2x\omega + \omega^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\omega^n, \qquad (3.1.22)$$

$$(1 - 2x\omega + \omega^2)^{-1} = \sum_{n=0}^{\infty} U_n(x)\omega^n, \qquad (3.1.23)$$

Using the readily verified limit

$$lim_{\lambda \to 0} \lambda^{-1} P_n^{(\lambda)}(x) = \frac{2}{n} T_n(x),$$

one can also obtain from (3.1.21) a generating function for the Chebyshev polynomials of the first kind:

$$log(1 - 2x\omega + \omega^2)^{-1} = 2 \sum_{n=1}^{\infty} n^{-1} T_n(x)\omega^n. \qquad (3.1.24)$$

However, there is the simpler algebraic generating function

$$\frac{1-x\omega}{1-2x\omega+\omega^2}=\sum_{n=0}^{\infty}T_n(x)\omega^n. \tag{3.1.25}$$

The most common generating function for the Laguerre polynomials is

$$(1-\omega)^{-\alpha-1}\exp\frac{-x\omega}{1-\omega}=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\omega^n \tag{3.1.26}$$

This can be proved by expanding the left side as a series in ω and using (3.1.11) to identify the coefficients in the resulting expansion.

For the Hermite polynomials, the standard generating function is

$$e^{2x\omega-\omega^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}\omega^n, \tag{3.1.27}$$

which follows from Taylor's theorem and (3.1.13).

We will define the monic Charlier polynomials, $C_n^{(a)}(x)$, by the generating function

$$e^{-a\omega}(1+\omega)^x=\sum_{n=0}^{\infty}C_n^{(a)}(x)\frac{\omega^n}{n!}, \qquad a\neq 0. \tag{3.1.28}$$

The explicit representation is

$$C_n^{(a)}(x)=\sum_{k=0}^{\infty}\binom{n}{k}\binom{x}{k}k!(-a)^{n-k}, \tag{3.1.29}$$

and the orthogonality relation is

$$\int_0^\infty C_m^{(\alpha)}(x)C_n^{(\alpha)}(x)d\psi^{(a)}(x) = a^n n!\delta_{mn}, \tag{3.1.30}$$

where $\psi^{(a)}$ is the step function whose jumps are

$$d\psi^{(a)}(x) = \frac{e^{-a}a^x}{x!} \ at \ x = 0, 1, 2, \cdots$$

Thus positive-definite case occurs for $a > 0$ and in this case , $d\psi^{(a)}(x)$ is the Poisson distribution of probability theory.

The recurrence formula is

$$C_{n+1}^{(\alpha)}(x) = (x - n - a)C_n^{(\alpha)}(x) - anC_{n-1}^{(\alpha)}(x). \tag{3.1.31}$$

The Charlier polynomials can be expressed in terms of Laguerre polynomials

$$C_n^{(\alpha)}(x) = n!L_n^{(x-n)}(a). \tag{3.1.32}$$

There is simple difference relation

$$\triangle C_n^{(\alpha)}(x) = nC_{n-1}^{(\alpha)}(x). \tag{3.1.33}$$

In connection with (3.1.31), this yields the second order difference equation

$$a\triangle^2 C_n^{(\alpha)}(x) - (x + 1 - a - n)\triangle C_n^{(\alpha)}(x) + nC_n^{(\alpha)}(x) = 0. \tag{3.1.34}$$

There exists also a Rodrigues type formula for Charlier polynomials,

$$C_n^{(\alpha)}(x) = (-1)^n a^{-x}\Gamma(x + 1)\Delta^n[\frac{a^x}{\Gamma(x - n + 1)}]. \tag{3.1.35}$$

3.2 *Generating Functions Method*

Most of the material for this section is taken from [Asai.N,Kubo.I,Kuo.H.2004]

Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders such that the linear span of the monomials x^n , $n \geq 0$, is dense in $L^2(\mu)$. It is well known that there exists a complete system $\{P_n\}_{n=0}^{\infty}$ of orthogonal polynomials with respect to μ such that P_n is a polynomial of degree n with leading coefficient 1 and the following recursion formula is satisfied:

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \quad n \geq 0 \tag{3.2.1}$$

where $\alpha_n \in \mathbb{R}, \omega_n \geq 0$ for $n \geq 0$ and by convention $\omega_0 = 1, P_{-1} = 0$. The numbers α_n and ω_n are called Szegő-Jacobi parameters of μ.

Define a sequence $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ by

$$\lambda_n = \omega_0 \omega_1 \cdots \omega_n, \tag{3.2.2}$$

Assume that the sequence λ satisfies the condition

$$\inf_{n \geq 0} \lambda_n^{1/n} > 0. \tag{3.2.3}$$

Let μ be a fixed probability measure on $\mathbb{R}$ satisfying conditions (3.2.3). For convenience we make the following definitions.

Definition 17 *By a pre-generating function for μ we mean a function $\varphi(t, x)$ with a power series expansion in t*

$$\varphi(t, x) = \sum_{n=0}^{\infty} g_n(x) t^n, \tag{3.2.4}$$

satisfying the following conditions:

1. $g_n(x)$ is a polynomial of degree n for each $n \geq 0$.

2. $limsup_{n\to\infty} \|g_n\|_{L^2(\mu)}^{1/n} < \infty$

We point out that conditions 1. and 2. imply the following fact:
There exists $\tau_0 > 0$ such that $E_\mu\varphi(t,.) \neq 0$ for all $|t| < \tau_0$.
To verify this fact, note that by 2.,

$$R_0 = \lim\inf_{n\to\infty} \|g_n\|_{L^2(\mu)}^{-1/n} > 0.$$

Hence the series

$$\varphi(t,x) = \sum_{n=0}^{\infty} g_n(x)t^n$$

converges in $L^2(\mu)$ for $t \in \mathbb{C}$ with $|t| < R_0$. Therefore ,

$$E_\mu \|\varphi(t,.)\| \leq \sum_{n=0}^{\infty} \|g_n\|_{L^2(\mu)} |t|^n < \infty$$

for $t \in \mathbb{C}$, $|t| < R_0$.This implies that $E_\mu[\varphi(t,.)]$ is analytic on $\{t \in \mathbb{C}; |t| < R_0\}$.
On the other hand , by 1.

$$E_\mu[\varphi(0,.)] = g_0 \neq 0.$$

Thus there exists τ_0 such that $0 < \tau_0 < R_0$ and $E_\mu[\varphi(t,.)] \neq 0$ for all $t \in \mathbb{C}$, $|t| < \tau_0$.

Definition 18 *By a generating function for μ we mean a pre-generating function $\psi(t,x)$ given by*

$$\psi(t,x) = \sum_{n=0}^{\infty} Q_n(x)t^n, \tag{3.2.5}$$

where the polynomials $Q_n, n \geq 0$,are orthogonal in $L^2(\mu)$.

Definition 19 *The multiplicative renormalization of a pre-generating function $\varphi(t,x)$ is defined to be the function*

$$\psi(t,x) = \frac{\varphi(t,x)}{E_\mu \varphi(t,.)}.$$

Note that a generating function is very different from moment generating function. A moment generating function is used to find moments of a probability measure, while a generating function is used to find a sequence of orthogonal polynomials .

Lemma 20 *Let $\varphi(t,x)$ be a pre-generating function given by*

$\varphi(t,x) = \sum_{n=0}^{\infty} g_n(x)t^n$ and its renormalization factor can be expanded as

$$C(t) = C(\varphi, \mu, t) = \frac{1}{E_\mu \varphi(t,.)} = \sum_{n=0}^{\infty} b_n t^n.$$

Let $\psi(t,x) = C(t)\varphi(t,x)$ be the multiplicative renormalization of $\varphi(t,x)$ and

$$\psi(t,x) = \sum_{n=0}^{\infty} Q_n(x)t^n.$$

Then $\psi(t,x)$ is also a pre-generating function, $Q_0(x) = 1$ and $E_\mu \psi(t,.) = 1$ for all t where $\psi(t,x)$ is defined. Moreover, for each $n \geq 0$, Q_n is a linear combination of g_0,$g_1, ..., g_n$

$$Q_n(x) = b_0 g_n + ... + b_n g_0$$

and vice versa g_n is also a linear combination of $Q_0, Q_1, ..., Q_n$.

Theorem 21 *Let $\psi(t,x)$ be the multiplicative renormalization of a pre-generating function and*

$$\psi(t,x) = \sum_{n=0}^{\infty} Q_n(x)t^n. \tag{3.2.6}$$

Then the polynomials $Q_n, n \geq 0$, are orthogonal in $L^2(\mu)$ if and only if $E_\mu \psi(t,.)\psi(s,.)$ depends only on ts. (Hence $\psi(t,x)$ is a generating function for μ.)

Proof. Suppose the polynomials $Q_n, n \geq 0$, are orthogonal. Then

$$E_\mu \psi(t,.)\psi(s,.) = \sum_{n,m=0}^{\infty} E_\mu Q_n Q_m t^n s^m$$

$$= \sum_{n,m=0}^{\infty} E_\mu Q_n^2 (ts)^n.$$

Hence $E_\mu \psi(t,.)\psi(s,.)$ depends only on ts. Conversely, suppose a double series in t and s depends only on ts, namely,

$$\sum_{n,m=0}^{\infty} a_{nm} t^n s^m = \theta(ts). \tag{3.2.7}$$

Let $n > m$. Differentiate equation (3.2.7) n times in t to get

$$\partial_t^n \sum_{n,m=0}^{\infty} a_{nm} t^n s^m = s^n \theta^{(n)}(ts). \tag{3.2.8}$$

Then differentiate equation (3.2.8) m times in s and put $t = s = 0$ to show that $a_{nm} = 0$. In case $n < m$, we first differentiate equation (3.2.7) m times in s, then n times in t, put $t = s = 0$ to show that $a_{nm} = 0$. Hence we have $a_{nm} = 0$ if $n \neq m$. Therefore, if $E_\mu \psi(t,.)\psi(s,.)$ depends only on ts, then

$$E_\mu Q_n Q_m = 0, \qquad \forall n \neq m$$

and so the polynomials $Q_n, n \geq 0$, are orthogonal in $L^2(\mu)$. ■

For a given probability measure μ on $\mathbb{R}$, Lemma 20 and Theorem 21 provide a method to derive the corresponding orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ in equation 3.2.1 which is called renormalization method.

Try a certain form $\varphi(t,x)$ of a pre-generating function and take the multiplicative renormalization $\psi(t,x) = \frac{\varphi(t,x)}{E_\mu \varphi(t,.)}$. Then use the condition in Theorem 21 to find the exact form of $\varphi(t,x)$. With this $\varphi(t,x)$ we can compute $\psi(t,x)$ and by the series expansion we can obtain the polynomials Q_n in equation (3.2.6).Let a_n be the leading coefficient of Q_n. Then the polynomials $P_n(x) = \frac{Q_n(x)}{a_n}, n \geq 0$, are those associated with μ as given in (3.2.1).

Thus a critical question is how to find an appropriate form of a pre-generating function. It is desirable to find a way to derive Szegő-Jacobi parameters from the corresponding generating function. In order to do that we rewrite the generating function in Theorem 21 as follows:

$$\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n, \tag{3.2.9}$$

where a_n is the leading coefficient of $Q_n(x)$ in $\psi(t,x) = \sum_{n=0}^{\infty} Q_n(x)t^n$ and $P_n(x) = Q_n(x)/a_n$.

Thus polynomials P_n' s are those in (3.2.1)

There are two ways to compute the Szegő-Jacobi parameters. After deriving the polynomials P_n's from $\psi(t,x)$, we can use (3.2.1) to compare the coefficients (e.g., of x^n and x^0) in both sides to find α_n and ω_n.

Another way to compute the Szegő-Jacobi parameters is a classical one. Multiply both sides of (3.2.1) by P_n and take the expectation to get

$$\alpha_n = \frac{E_\mu(xP_n^2)}{E_\mu P_n^2}. \tag{3.2.10}$$

On the other hand, multiply both sides of (3.2.1) by P_{n-1} and take the expectation to get

$$\omega_n = \tfrac{E_\mu(xP_nP_{n-1})}{E_\mu P_{n-1}^2}.$$

But from equation (3.2.1) with n being replaced by $n-1$ we get

$$\begin{gathered} E_\mu(xP_nP_{n-1}) = E_\mu(P_n(xP_{n-1})) \\ = E_\mu(P_n(P_n + \alpha_{n-1}P_{n-1} + \omega_{n-1}P_{n-2})) \\ = E_\mu P_n^2 \end{gathered}$$

Hence ω_n and λ_n are given by

$$\omega_n = \frac{E_\mu P_n^2}{E_\mu P_{n-1}^2} and \lambda_n = E_\mu P_n^2 \tag{3.2.11}$$

Theorem 22 *Let $\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x)t^n$ be a generating function for μ. Then*

$$\psi(t, \frac{x}{t}) = \sum_{n=0}^{\infty} a_n x^n, \tag{3.2.12}$$

$$E_\mu \psi(t, .)^2 = \sum_{n=0}^{\infty} a_n^2 \lambda_n t^{2n}, \tag{3.2.13}$$

$$E_\mu x\psi(t, .)^2 = \sum_{n=0}^{\infty} (a_n^2 \lambda_n t^{2n} + 2a_n a_{n-1} \lambda_n t^{2n-1}), \tag{3.2.14}$$

where $a_{-1} = 0$ by convention.

Proof. Equation (3.2.13) follows from the orthogonality of the polynomials P_n and Equation (3.2.11). To show Equation (3.2.14) note that

$$E_\mu x\psi(t, .)^2 = \sum_{n,m=0}^{\infty} a_n a_m E_\mu(xP_n P_m) t^n t^m$$

$$= \sum_{n=0}^{\infty} a_n^2 E_\mu(xP_n^2) t^{2n} + 2 \sum_{n>m} a_n a_m E_\mu(xP_n P_m) t^{n+m}. \tag{3.2.15}$$

But when $n > m$ we have

$$E_\mu(xP_n P_m) = E_\mu((P_{n+1} + \alpha_n P_n + \omega_n P_{n-1}) P_m) = \delta_{m,n-1} \omega_n E_\mu(P_{n-1}^2).$$

Therefore,

$$\sum_{n>m} a_n a_m E_\mu(xP_n P_m) t^{n+m} = \sum_{n=1}^{\infty} a_n a_{n-1} E_\mu(P_n^2) t^{2n-1}. \tag{3.2.16}$$

Hence Equations (3.2.10), (3.2.11), (3.2.15) and (3.2.16) yield equation (3.2.14).Equation (3.2.12) follows easily from the fact that $P_n(x)$ is a monic polynomial. ∎

Once we have a generating function $\psi(t, x)$ for μ, we can find the power series of $E_\mu \psi(t, .)^2$ and $E_\mu x\psi(t, .)^2$. Then by the above theorem we can find a_n and the Szegő-Jacobi parameters α_n and ω_n.

We try two types of pre-generating functions

$$\varphi(t, x) = e^{\rho(t)x} = \sum_{n=0}^{\infty} \frac{1}{n!} (\rho(t)x)^n, \tag{3.2.17}$$

$$\varphi(t,x) = (1-\rho(t)x)^c = \sum_{n=0}^{\infty} \binom{c}{n} (-1)^n (\rho(t)x)^n, \tag{3.2.18}$$

where the function $\rho(t)$ and constant c are to be derived so that the multiplicative renormalization $\psi(t,x)$ satisfies the condition that $E_\mu \psi(t,.)\psi(s,.)$ depends only on ts according to Theorem 21. These two types functions cover many classical examples of orthogonal polynomials. In fact, the first type in (3.2.17) can be applied to more general cases.

In both cases in Equation (3.2.17) and (3.2.18), $\rho(t)$ must be analytic around $t=0$, $\rho(0)=0$, and $\rho'(0) \neq 0$ in order to get a polynomial $g_n(x)$ of degree n in equation (3.2.4). For these cases, the coefficients $\{a_n\}$ in equation (3.2.9) can be obtained as follows.

$$\lim_{t\to 0} \psi(t, \frac{x}{t}) = \lim_{t\to 0} \frac{\varphi(t, \frac{x}{t})}{E_\mu \varphi(t,.)} = \lim_{t\to 0} \varphi(t, \frac{x}{t}).$$

For the case of equation (3.2.17), we have

$$\lim_{t\to 0} \varphi(t, \frac{x}{t}) = \lim_{t\to 0} e^{\rho(t)x/t} = e^{\rho'(0)x} = \sum_{n=0}^{\infty} \frac{\rho'(0)^n}{n!} x^n,$$

and hence

$$a_n = \frac{\rho'(0)^n}{n!}. \tag{3.2.19}$$

For the case of Equation (3.2.18)

$$\lim_{t\to 0} \varphi(t, \frac{x}{t}) = (1-\rho'(0)x)^c = \sum_{n=0}^{\infty} \binom{c}{n} (-\rho'(0))^n x^n$$

and hence

$$a_n = \binom{c}{n}(-\rho'(0))^n x^n. \tag{3.2.20}$$

Definition 23 *A probability measure μ on $\mathbb{R}$ is said to be of exponential type if there exists a constant $0 < a < \infty$ such that $\int_{\mathbb{R}} e^{a|x|} d\mu(x) < \infty$.*

For an exponential type probability measure μ, define its Laplace transform by

$$l(r) = \int_{\mathbb{R}} e^{rx} d\mu(x), \quad |r| \le a$$

.

Theorem 24 *Let μ be an exponential type probability measure on $\mathbb{R}$. Let l be its Laplace transform and $g(r) = l'(r)/l(r)$. Suppose $\rho(t)$ has a power series expansion near 0 such that $\rho(0) = 0$, $\rho'(0) = 1$ and satisfies the following equation*

$$g(\rho(t) + \rho(s))(t\rho'(t) - s\rho'(s)) = g(\rho(t))t\rho'(t) - g(\rho(s))s\rho'(s). \tag{3.2.21}$$

Then the multiplicative renormalization of $e^{\rho(t)x}$

$$\psi(t, x) = \frac{e^{\rho(t)x}}{E_\mu e^{\rho(t)x}} = \frac{e^{\rho(t)x}}{l(\rho(t))}$$

is a generating function for μ.

Note:Let $t, s > 0$ and put $\rho(t) = \theta(\log t)$.Then Equation (3.2.21) can be reduced to an equation for $\theta(r), r < -K,$ (K is a positive constant)

$$g(\theta(r) + \theta(u))(\theta'(r) + \theta'(u)) = g(\theta(r))\theta'(r) - g(\theta(u))\theta'(u)$$

and

$$lim_{r\to-\infty}\theta(r) = 0, lim_{r\to-\infty}e^{-r}\theta'(r) = 1.$$

Proof. Let $\rho(t)$ be a function with a power series expansion near 0 and $\rho(0) = 0$, $\rho'(0) = 1$.Then it is easy to see that function $\varphi(t, x) = e^{\rho(t)x}$ is a pre-generating function. Note that $E_\mu\varphi(t, .) = l(\rho(t))$ and so the multiplicative renormalization of $\varphi(t, x)$ is given

$$\psi(t, x) = \frac{e^{\rho(t)x}}{l(\rho(t))}.$$

Then for any t, s, we have

$$E_\mu\psi(t, .)\psi(s, .) = \frac{l(\rho(t) + \rho(s))}{l(\rho(t))l(\rho(s))}. \tag{3.2.22}$$

Observe that $E_\mu\psi(t, .)\psi(s, .)$ in Equation (3.2.22) depends on ts only if and only if after substituting $s = \frac{r}{t}$ the following function is independent of t.

$$E_\mu\psi(t, .)\psi(\frac{r}{t}, .) = \frac{l(\rho(t) + \rho(\frac{r}{t}))}{l(\rho(t))l(\rho(\frac{r}{t}))}.$$

By independence of t we have,

$$\frac{\partial}{\partial t}\log\frac{l(\rho(t) + \rho(\frac{r}{t}))}{l(\rho(t))l(\rho(\frac{r}{t}))} = 0,$$

which is equivalent to

$$g(\rho(t) + \rho(\frac{r}{t}))(\rho'(t) - \rho'(\frac{r}{t})\frac{r}{t^2}) - g(\rho(t))\rho'(t) - g(\rho(\frac{r}{t}))\rho'(\frac{r}{t})\frac{r}{t^2} = 0,$$

where $g = \frac{l'}{l}$, the logarithmic derivative of l. Putting $\frac{r}{t} = s$ back,

$$g(\rho(t) + \rho(s))(\rho'(t) - \rho'(s)\frac{s}{t}) = g(\rho(t))\rho'(t) - g(\rho(s))\rho'(s)\frac{s}{t}$$

or

$$g(\rho(t) + \rho(s))(\frac{t\rho'(t) - \rho'(s)s}{t}) = \frac{tg(\rho(t))\rho'(t) + g(\rho(s))\rho'(s)s}{t}$$

If we cancel t which is at the denominator we can see this equation is equivalent to equation (3.2.22) in the theorem. Thus if $\rho(t)$ satisfies equation (3.2.21) then $E_\mu\psi(t, .)\psi(s, .)$ depends on ts and so $\psi(t, x)$ is generating function. ■

In this section we will use multiplicative renormalization method to find several generating functions and derive the corresponding orthogonal polynomials together with Szegő-Jacobi parameters. In addition, we will verify that our orthogonal polynomials derived from generating functions are indeed the classical ones. By this technique we first find a generating function then use it to derive orthogonal polynomials and other quantities.

3.2.1 Gaussian Measure and Hermite Polynomials

Let μ be the Gaussian measure with mean 0 and variance σ^2

$$d\mu(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}x^2} dx.$$

Try the type of pre-generating function $\varphi(t,x) = e^{\rho(t)x}$ in equation (3.2.17). It is easily checked that $E_\mu \varphi(t,.) = e^{\frac{1}{2}\sigma^2\rho(t)^2}$ and so the multiplicative renormalization of φ is given by

$$\psi(t,x) = e^{\rho(t)x - \frac{1}{2}\sigma^2\rho(t)^2}.$$

We can find $\rho(t)$ by using Theorem 21 as follows. The Laplace transform of μ and its logaritmic derivative are given by $l(r) = e^{\frac{1}{2}\sigma^2 r^2}$ and $g(r) = \sigma^2 r$, respectively. We can find the Laplace transform of μ as follows

$$\int_0^\infty e^{-rx} d\mu(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty e^{-rx} e^{-\frac{1}{2\sigma^2}x^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty e^{-\frac{1}{2\sigma^2}(x^2+2\sigma^2 rx+\sigma^4 r^2)} e^{\frac{1}{2}\sigma^2 r^2} dx = e^{\frac{\sigma^2 r^2}{2}}.$$

By using Theorem 24

$$\sigma^2(\rho(t)+\rho(s))(t\rho'(t) - s\rho'(s)) = \sigma^2\rho(t)t\rho'(t) - \sigma^2\rho(s)s\rho'(s)$$

if we cancel σ^2 from both sides of the equation we have

$$(\rho(t)+\rho(s))(t\rho'(t) - s\rho'(s)) = \rho(t)t\rho'(t) - \rho(s)s\rho'(s)$$

If we distribute $\rho(t)+\rho(s)$ to parenthesis we have

$$(\rho(t)+\rho(s))t\rho'(t) - s\rho'(s)(\rho(t)+\rho(s)) = \rho(t)t\rho'(t) - \rho(s)s\rho'(s)$$

and distribute $t\rho'(t)$ and $s\rho'(s)$ to the parenthesis

$$\rho(t)t\rho'(t) + \rho(s)t\rho'(t) - s\rho'(s)\rho(t) - s\rho'(s)\rho(s) = \rho(t)t\rho'(t) - \rho(s)s\rho'(s)$$

we have

$$t\rho(s)\rho'(t) = s\rho'(s)\rho(t)$$

and $\frac{t\rho'(t)}{\rho(t)} = \frac{s\rho'(s)}{\rho(s)} = c$, a constant. Thus $\rho(t) = c_1 t^c$. Choose $c_1 = c = 1$ to get $\rho(t) = t$ and we have the following pre-generating and generating functions:

$$\varphi(t,x) = e^{tx}, \quad \psi(t,x) = e^{tx-\frac{1}{2}\sigma^2 t^2}.$$

To derive the orthogonal polynomials, note that

$$e^{tx}e^{-\frac{1}{2}\sigma^2 t^2} = (\sum_{n=0}^{\infty} \frac{1}{n!} t^n x^n)(\sum_{m=0}^{\infty} \frac{(-\sigma^2)^m}{m! 2^m} t^{2m})$$

$$\sum_{n=0}^{\infty} (\sum_{k=0}^{[\frac{n}{2}]} \frac{(-\sigma^2)^k}{(n-2k)! k! 2^k} x^{n-2k}) t^n.$$

Therefore, we have

$$\psi(t,x) = e^{tx-\frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n, \tag{3.2.1.1}$$

where the polynomial $P_n(x)$ is defined by

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{(n-2k)! k! 2^k} (-\sigma^2)^k x^{n-2k}. \tag{3.2.1.2}$$

To find the Szegő-Jacobi parameters, first note that $P_n(x)$ is even or odd when n is even or odd, respectively. Hence $P_n(x)^2$ is even for any n and so $E_\mu P_n(x)^2 = 0$. Hence by Equation (3.2.10) we have

$$\alpha_n = 0, \qquad n \geq 0.$$

(In fact, it is well-known that μ is symmetric if and only if $\alpha_n = 0$ for all $n \geq 0$.)

To find ω_n we first check that $E_\mu \psi(t,x)^2 = e^{\sigma^2 t^2}$ and so

$$E_\mu \psi(t,x)^2 = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!} t^{2n}.$$

Compare this equation to equation (3.2.13) with $a_n = \frac{1}{n!}$ by equation (3.2.1.1) to get

$$\lambda_n = \sigma^{2n} n!,$$

which satisfies the condition in equation (3.2.3). Therefore we get

$$\omega_n = \sigma^2 n, \qquad n \geq 1. \qquad (\omega_0 = 1).$$

Finally we show that the polynomials defined by (3.2.1.2) are the classical Hermite polynomials with parameter σ^2.

Theorem 25 *Let P_n be the polynomial defined by Equation (3.2.1.2). Then*

$$P_n(x) = (-\sigma^2)^n e^{\frac{x^2}{2\sigma^2}} D_x^n e^{-\frac{x^2}{2\sigma^2}}.$$

Proof. From equation (3.2.1.1) we have

$$e^{tx - \frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n.$$

By completing the square of the exponent in t we can rewrite this equation as

$$e^{\frac{x^2}{2\sigma^2}} f(x - \sigma^2 t) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n$$

where

$$f(u) = e^{\frac{-1}{2\sigma^2} u^2}.$$

Note that

$$D_t^n f(x - \sigma^2 t) = f^{(n)}(x - \sigma^2 t)(-\sigma^2)^n.$$

Hence if we differentiate both sides of Equation (3.2.1.3) n-times in t and then let $t = 0$ then we get

$$P_n(x) = e^{\frac{x^2}{2\sigma^2}} f^{(n)}(x)(-\sigma^2)^n = (-\sigma^2)^n e^{\frac{x^2}{2\sigma^2}} D_x^n e^{-\frac{x^2}{2\sigma^2}}.$$

For the case $\sigma = 1$ the polynomial $P_n(x)$ coincides with the Hermite polynomial $He_n(x)$ given in section 3.1 ■

3.2.2 Poisson Measure and Charlier Polynomials

Let μ be the Poisson measure with parameter $\lambda > 0$

$$\mu(\{k\}) = e^{-\lambda}\tfrac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

Try the type of pre-generating function $\varphi(t, x) = e^{\rho(t)x}$ in equation (3.2.17). It is easily checked that $E_\mu\varphi(t, .) = \exp(\lambda(e^{\rho(t)} - 1))$. Then we can use Theorem 21 to derive that $e^{\rho(t)} = 1 + t$. Thus the multiplicative renormalization of $\varphi(t, x)$

$$\psi(t, x) = e^{-\lambda t}(1 + t)^x \tag{3.2.2.1}$$

is a generating function for μ. To derive the corresponding orthogonal polynomials we need to use the binomial series

$$(1 + t)^x = \sum_{n=0}^{\infty} \frac{p_{x,n}}{n!} t^n,$$

where $p_{x,0} = 1$ be convention and $p_{x,n} = x(x - 1) \cdots (x - n + 1)$ for $n \geq 1$. Hence we have

$$\begin{aligned} e^{-\lambda t}(1 + t)^x &= (\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} t^n)(\sum_{m=0}^{\infty} \frac{p_{x,m}}{m!} t^m) \\ &= \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty} \frac{(-\lambda)^{n-k}}{(n-k)!k!} p_{x,k})t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(\sum_{k=0}^{\infty} \binom{n}{k} (-\lambda)^{n-k} p_{x,k})t^n \end{aligned}$$

Therefore, we have

$$\psi(t, x) = e^{-\lambda t}(1 + t)^x = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(x) t^n, \tag{3.2.2.2}$$

where the polynomial $P_n(x)$ is defined by

$$P_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} (-\lambda)^{n-k} p_{x,k}. \qquad (3.2.2.3)$$

One way to find the Szegő-Jacobi parameters is to use Equation (3.2.14) with $a_n = \frac{1}{n!}$ in view of Equation (3.2.2.3). First we can easily compute that

$$E_\mu \psi(t,.)^2 = \lambda(1+t)^2 e^{\lambda t^2}.$$

Therefore, by Equation (3.2.14) ,

$$\lambda(1+t)^2 e^{\lambda t^2} = \sum_{n=0}^{\infty} (\frac{1}{(n!)^2} \lambda_n \alpha_n t^{2n} + 2\frac{1}{n!(n-1)!} \lambda_n t^{2n-1}).$$

By comparing the coefficients of t^{2n} and t^{2n-1} we get

$$\lambda_n \alpha_n = \lambda^n (\lambda + n) n!, \qquad \lambda_n = \lambda^n n!.$$

Hence by (3.2.11) the Szegő-Jacobi parameters are given by

$$\alpha_n = \lambda + n, \quad n \geq 0,$$

$$\omega_n = \lambda n, \quad n \geq 1, \quad (\omega_0 = 1).$$

Moreover the parameter λ_n satisfies the condition Equation (3.2.3).

Now we will show that the polynomials defined by Equation (3.2.2.3) are the classical Charlier polynomials with parameter λ.

Theorem 26 *Let P_n be the polynomials defined by equation (3.2.2.3). Then*

$$C_n(x;\lambda) = (-1)^n \lambda^{-x} \Gamma(x+1) \Delta_x^n (\frac{\lambda^x}{\Gamma(x-n+1)}).$$

where Δ is the difference operator $\Delta f(x) = f(x+1) - f(x)$ and $\Gamma(\cdot)$ is the Gamma function.

Proof. Let

$$\psi(t,x) = e^{-\lambda t}(1+t)^x.$$

Then

$$\partial_t \psi(t,x) = e^{-\lambda t}(1+t)^{x-1}(x-\lambda(1+t)). \tag{3.2.2.4}$$

On the other hand, we can easily check that

$$\Delta_x(\frac{(\lambda(1+t))^x}{\Gamma(x)}) = -\frac{(\lambda(1+t))^x}{\Gamma(x+1)}(x-\lambda(1+t)). \tag{3.2.2.5}$$

Cancel out the common last factor in equations (3.2.2.4) and (3.2.2.5) to get

$$\partial_t \psi(t,x) = -e^{-\lambda t}(1+t)^{x-1}\frac{\Gamma(x+1)}{(\lambda(1+t))^x}\Delta_x(\frac{(\lambda(1+t))^x}{\Gamma(x)})$$

$$= -e^{-\lambda t}\lambda^{-x}(1+t)^{x-1}\Gamma(x+1)\Delta_x(\frac{(\lambda(1+t))^x}{\Gamma(x)})$$

Bring the factor $e^{-\lambda t}(1+t)^{-1}$ inside the operator Δ_x to get

$$\partial_t \psi(t,x) = -\lambda^{-x}\Gamma(x+1)\Delta_x(\frac{\lambda^x}{\Gamma(x)}e^{-\lambda t}(1+t)^{x-1})$$

$$= -\lambda^{-x}\Gamma(x+1)\Delta_x(\frac{\lambda^x}{\Gamma(x)}\psi(t,x-1)).$$

Then apply induction to show that for any n we have

$$\partial_t^n \psi(t,x) = (-1)^n\lambda^{-x}\Gamma(x+1)\Delta_x^n(\frac{\lambda^x}{\Gamma(x-n+1)}\psi(t,x-n)).$$

Finally put $t=0$ to get

$$P_n(x) = \partial_t^n \psi(t,x)\mid_{t=0} = (-1)^n\lambda^{-x}\Gamma(x+1)\Delta_x^n(\frac{\lambda^x}{\Gamma(x-n+1)}).$$

■

3.2.3 Gamma Distribution and Laguerre Polynomials

Let μ be the Gamma distribution with parameter $\alpha > 0$

$$d\mu(x) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx, \qquad x > 0.$$

Try the type of pre-generating function of $\varphi(t,x) = e^{\rho(t)x}$ in Equation (3.2.17). The Laplace transform l and multiplicative renormalization $\psi(t,x)$ of μ are given by

$$l(r) = \frac{1}{(1-r)^{\alpha}}, \tag{3.2.3.1}$$

$$\psi(t,x) = \frac{e^{\rho(t)x}}{l(\rho(t))}. \tag{3.2.3.2}$$

We can use Theorem 21 to derive the function $\rho(t)$.On the other hand, we can apply Theorem 25 to derive $\rho(t)$ as follows. From equation (3.2.3.1) we see that the logarithmic derivative of l is given by

$$g(r) = \frac{l'(r)}{l(r)} = \frac{\alpha}{1-r}$$

and so equation (3.2.21) in Theorem 24 becomes

$$\frac{t\rho'(t) - s\rho'(s)}{1-\rho(t)-\rho(s)} = \frac{t\rho'(t)}{1-\rho(t)} - \frac{s\rho'(s)}{1-\rho(s)}.$$

By letting $\xi(t) = 1-\rho(t)$ we see that this equation is equivalent to

$$t\xi'(t)(\xi(s)^2 - \xi(s)) - s\xi'(s)(\xi(t)^2 - \xi(t)) = 0$$

Therefore, we have

$$\frac{t\xi'(t)}{(\xi(t)^2 - \xi(t)} = \frac{s\xi'(s)}{(\xi(s)^2 - \xi(s))} = c,$$

$$t\xi'(t) = c(\xi(t)^2 - \xi(t)$$

$$t\xi'(t) + c\xi(t) = c\xi(t)^2$$

$$y = \xi^{-1}(t)$$

$$y\prime = \frac{-1}{\xi(t)^2}\xi'(t)$$

$$-ty\prime + cy = c$$

$$y\prime = \frac{c}{t}y$$

$$\frac{dy}{y} = c\frac{dt}{t}$$

$$ln\,|y| = c\ln|t| + \ln c_1$$

$$y = c_1 t^c$$

$$y(t) = c_1(t)t^c$$

$$-tc_1'(t)t^c = c$$

$$c\prime_1(t) = -\frac{c}{t^{c+1}}$$

$$\xi(t) = \frac{1}{1 - c_1 t^c}.$$

Hence $\rho(t)$ is given by

$$\rho(t) = -\frac{c_1 t^c}{1 - c_1 t^c}.$$

Since $\rho(0) = 0$ and $\rho'(0) = 1$ as required in Theorem 24, we get $\rho(t) = \frac{t}{1+t}$. Hence the resulting generating function for μ in Equation (3.2.3.2) is given by

$$\psi(t, x) = (1 + x)^{-\alpha} e^{\frac{tx}{1+t}}.$$

To derive the corresponding orthogonal polynomials, first note that

$$e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n (1 + t)^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n \sum_{k=0}^{\infty} \binom{-n}{k} t^k$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{\infty} \binom{-n}{k} t^{n+k}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=n}^{\infty} \binom{-n}{m-n} t^m$$

Next, change the order of summation to get

$$e^{\frac{tx}{1+t}} = \sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} \frac{x^n}{n!} \binom{-n}{m-n}] t^m.$$

Therefore,

$$(1+t)^{-\alpha} e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \binom{-\alpha}{n} t^n \sum_{m=0}^{\infty} [\sum_{k=0}^{m} \frac{x^k}{k!} \binom{-k}{m-k}] t^m$$

$$= \sum_{n=0}^{\infty} [\sum_{j=0}^{n} \binom{-\alpha}{n-j} \sum_{k=0}^{j} \frac{x^k}{k!} \binom{-k}{j-k}] t^n.$$

In the double summation inside [...] change the order of summation to get

$$(1+t)^{-\alpha} e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} [\sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=k}^{n} \binom{-\alpha}{n-j} \binom{-k}{j-k}] t^n$$

$$= \sum_{n=0}^{\infty} [\sum_{k=0}^{n} \frac{x^k}{k!} \sum_{m=0}^{n-k} \binom{-\alpha}{n-k-m} \binom{-k}{m}] t^n.$$

In summation over m we apply the formula

$$\sum_{m=0}^{j} \binom{a}{j-m} \binom{b}{m} = \binom{a+b}{j}$$

and get the following equality

$$(1+t)^{-\alpha}e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty}[\sum_{k=0}^{n}\frac{1}{k!}\binom{-\alpha-k}{n-k}x^k]t^n.$$

Therefore we have shown that

$$\psi(t,x) = (1+t)^{-\alpha}e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty}\frac{1}{n!}P_n(x)t^n, \qquad (3.2.3.3)$$

where the polynomial $P_n(x)$ is defined by

$$P_n(x) = \sum_{k=0}^{n}\frac{n!}{k!}\binom{-\alpha-k}{n-k}x^k. \qquad (3.2.3.4)$$

Next we find the Szegő-Jacobi parameters.

Firstly we find the following expectation

$$E_\mu\psi(t,.)^2 = \int_{-\infty}^{\infty}\psi(t,x)^2 d\mu(x)$$

$$= \int_{0}^{\infty}\frac{1}{\Gamma(x)}x^{\alpha-1}e^{-x}(1+t)^{-2\alpha}e^{\frac{2tx}{1+t}}dx$$

$$= \frac{(1+t)^{-2\alpha}}{\Gamma(x)}\int_{0}^{\infty}x^{\alpha-1}e^{-x}e^{\frac{2tx}{1+t}}dx$$

$$= \frac{(1+t)^{-2\alpha}}{\Gamma(x)}\int_{0}^{\infty}x^{\alpha-1}e^{-(1-\frac{2t}{1+t})x}dx$$

The substitution $y = (1-\frac{2t}{1+t})x$
gives

$$E_\mu\psi(t,.)^2 = \frac{(1+t)^{-2\alpha}}{\Gamma(x)(1-\frac{2t}{1+t})^{\alpha}}\int_{0}^{\infty}y^{\alpha-1}e^{-y}dy$$

$$= \frac{(1+t)^{-\alpha}}{(1-t)^{\alpha}} = (1-t^2)^{-\alpha}$$

$$= (1-t^2)^{-\alpha}$$

and we have the power series expansion

$$E_\mu \psi(t,.)^2 = \sum_{n=0}^{\infty} (-1)^n \binom{-\alpha}{n} t^{2n}.$$

Hence use Equation (3.2.13) in Theorem 22 with $a_n = \frac{1}{n!}$ by Equation (3.2.2.1) to get

$$\lambda_n = (n!)^2 (-1)^n \binom{-\alpha}{n} = \frac{n!\Gamma(\alpha+n)}{\Gamma(\alpha)}, \ n \geq 1,$$

which satisfies the condition Equation (3.1.3). Therefore , by Equation (3.2.11) we have

$$\omega_n = n(\alpha + n - 1), \ n \geq 1, \ (\omega_0 = 1).$$

On the other hand it can be easily checked that

$$E_\mu^x \psi(t,.)^2 = \int_0^\infty \frac{1}{\Gamma(x)} x^\alpha e^{-x} (1+t)^{-2\alpha} e^{\frac{2tx}{1+t}} dx$$

$$= \alpha \frac{(1+t)^{-2\alpha}}{(1 - \frac{2t}{1+t})^{\alpha+1}}$$

$$= \alpha \frac{(1+t)^{-\alpha}(1+t)}{(1-t)^{\alpha+1}}$$

$$= \alpha (1+t)^{-\alpha+1} (1-t)^{-\alpha-1}.$$

Thus we have the power series expansion

$$E_\mu^x \psi(t,.)^2 = \alpha (1-t^2)^{-\alpha} \frac{1+t}{1-t}$$

$$= \alpha [\sum_{n=0}^{\infty} (-1)^n \binom{-\alpha}{n} t^{2n}] (1 + 2 \sum_{m=1}^{\infty} t^m),$$

whose coefficients for t^{2n} is given by

$$\alpha(-1)^n\binom{-\alpha}{n}+2\alpha[(-1)^{n-1}\binom{-\alpha}{n-1}+(-1)^{n-2}\binom{-\alpha}{n-2}+\cdots+(-1)^0\binom{-\alpha}{0}].$$

By using formula

$$(-1)^0\binom{-\alpha}{0}+(-1)^1\binom{-\alpha}{1}+\cdots+(-1)^{n-1}\binom{-\alpha}{n-1}$$

$$=(-1)^{n-1}\binom{-\alpha-1}{n-1}$$

Therefore, by Equation (3.2.11)

$$\lambda_n\alpha_n=(n!)^2\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!}(\alpha+2n)=\frac{(n!)\Gamma(n+\alpha)}{\Gamma(\alpha)}(\alpha+2n).$$

Hence by Equation (3.2.7) we have

$$\alpha_n=\alpha+2n,n\geq 0.$$

Finally we show that the polynomials defined in Equation (3.2.3.4) are the classical Laguerre polynomials up to a constant multiple.

Theorem 27 *Let $P_n(x)$ be the polynomial defined by Equation (3.2.3.4) . Then*

$$P_n(x)=(-1)^n n!L_n^{(\alpha)}(x),$$

where $L_n^{(\alpha)}(x)$ is the classical Laguerre polynomial defined by

$$L_n^{(\alpha)}(x)=\frac{1}{n!}x^{-\alpha+1}e^xD_x^n(x^{\alpha+n-1}e^{-x}).$$

Proof. Since we will use the parameter α in the proof, we denote the generating function $\psi(t,x)$ in equation 11 by $\psi_\alpha(t,x)$, namely, let

$$\psi_\alpha(t,x)=(1+t)^{-\alpha}e^{\frac{tx}{1+t}}.$$

Differentiate $\psi_\alpha(t,x)$ in t to get

$$\partial_t \psi_\alpha(t,x) = (1+t)^{-\alpha-1} e^{\frac{tx}{1+t}}(-\alpha + \frac{x}{1+t}). \tag{3.2.3.5}$$

On the other hand, we have

$$D_x(-x^\alpha e^{-\frac{x}{1+t}}) = -x^{\alpha-1} e^{-\frac{x}{1+t}}(-\alpha + \frac{x}{1+t}). \tag{3.2.3.6}$$

Cancel out common last factor in Equations (3.2.3.5) and (3.2.3.6) to get

$$\partial_t \psi_\alpha(t,x) = -x^{\alpha+1} e^x (1+t)^{-\alpha-1} D_x(x^\alpha e^{-\frac{x}{1+t}})$$

$$= -x^{\alpha+1} e^x D_x(x^\alpha (1+t)^{-\alpha-1} e^{-\frac{x}{1+t}})$$

$$= -x^{\alpha+1} e^x D_x(x^\alpha e^{-x} \psi_{\alpha+1}(t,x)).$$

Inductively we have for any $n \geq 1$

$$\partial_t \psi_\alpha(t,x) = (-1)^n x^{-\alpha+1} e^x D_x^n(x^{\alpha+n-1} e^{-x} \psi_{\alpha+n}(t,x)).$$

Put $t = 0$ to get

$$P_n(x) = \partial_t \psi_\alpha(t,x) \mid_{t=0} = (-1)^n x^{-\alpha+1} e^x D_x^n(x^{\alpha+n-1} e^{-x}).$$

Hence we conclude that

$$P_n(x) = (-1)^n n! L_n^{(\alpha)}(x).$$

■

3.2.4 Uniform Distribution and Legendre Polynomials

Let μ be the uniform distribution on the interval [-1,1]

$$d\mu(x) = \frac{1}{2} dx, \qquad -1 \leq x \leq -1$$

Try the type of pre-generating function in Equation (3.2.18) with$c = -1/2$

$$\varphi(t,x) = \frac{1}{\sqrt{1-\rho(t)x}}.$$

The expectation and the multiplicative renormalization of $\varphi(t,x)$ are given by

$$E_\mu\varphi(t,.) = \frac{\sqrt{1+\rho(t)} - \sqrt{1-\rho(t)}}{\rho(t)},$$

$$\psi(t,x) = \frac{\rho(t)}{\sqrt{1+\rho(t)} - \sqrt{1-\rho(t)}}\frac{1}{\sqrt{1-\rho(t)x}}. \tag{3.2.4.1}$$

Consider small $t, s > 0$ so that $\rho(t), \rho(s) > 0$. We can calculate expectation as follows,

Firstly we use the following integration formula for calculation of the expectation.

$$\int \frac{dx}{\sqrt{1-\alpha x}\sqrt{1-\beta x}} = \frac{2}{\sqrt{\alpha\beta}}\ln(\sqrt{\beta(1-\alpha x)} - \sqrt{\alpha(1-\beta x)}) + c$$

$$E_\mu\psi(t,.)\psi(s,.) =$$

$$= [\frac{\rho(t)}{\sqrt{1+\rho(t)} - \sqrt{1-\rho(t)}}\frac{\rho(s)}{\sqrt{1+\rho(s)} - \sqrt{1-\rho(s)}}] \quad \int_{-1}^{1}\frac{1}{2}\frac{dx}{\sqrt{1-\rho(t)x}\sqrt{1-\rho(s)x}}$$

$$= \frac{1}{2}\frac{2}{\sqrt{\rho(t)\rho(s)}}[\frac{\rho(t)}{\sqrt{1+\rho(t)} - \sqrt{1-\rho(t)}}\frac{\rho(s)}{\sqrt{1+\rho(s)} - \sqrt{1-\rho(s)}}]$$

$$\ln(\sqrt{\rho(s)(1-\rho(t)x)} + \sqrt{\rho(t)(1-\rho(s)x)}) \mid_{-1}^{1}$$

$$= \frac{1}{\sqrt{\rho(t)\rho(s)}}[\frac{\rho(t)}{\sqrt{1+\rho(t)} - \sqrt{1-\rho(t)}}\frac{\rho(s)}{\sqrt{1+\rho(s)} - \sqrt{1-\rho(s)}}]$$

$$\times[\ln(\sqrt{\rho(s)(1-\rho(t))} - \sqrt{\rho(t)(1-\rho(s))}) - \ln(\sqrt{\rho(s)(1+\rho(t))} - \sqrt{\rho(t)(1+\rho(s))})]$$

$$= \quad [\frac{\sqrt{\rho(t)}}{\sqrt{1+\rho(t)} - \sqrt{1-\rho(t)}}\frac{\sqrt{\rho(s)}}{\sqrt{1+\rho(s)} - \sqrt{1-\rho(s)}}] \tag{3.2.4.2}$$

$$\ln\frac{\sqrt{\rho(s)}\sqrt{(1-\rho(t))} - \sqrt{\rho(t)}\sqrt{(1-\rho(s))}}{\sqrt{\rho(s)}\sqrt{(1+\rho(t))} - \sqrt{\rho(t)}\sqrt{(1+\rho(s))}}$$

In order for $E_\mu\psi(t,.)\psi(s,.)$ to be a function of ts the quantity inside $[\cdots]$ must be a function of ts. Hence

$$\frac{\sqrt{\rho(t)}}{\sqrt{1+\rho(t)}-\sqrt{1-\rho(t)}} = at^b.$$

which can be easily solved for $\rho(t)$ to be

$$\rho(t) = \frac{4a^2t^{2b}}{1+4a^4t^{4b}}.$$

Choose $a = \frac{1}{\sqrt{2}}, b = \frac{1}{2}$ to get

$$\rho(t) = \frac{2t}{1+t^2}. \tag{3.2.4.3}$$

Now with this $\rho(t)$, we can check that the ln factor Equation (3.2.4.2) is indeed a function of ts. Thus we can conclude from Theorem 21 that for the choice $\rho(t)$ in Equation (3.2.4.3) the corresponding function from Equation (3.2.4.1), namely,

$$\psi(t,x) = \frac{1}{\sqrt{1-2tx+t^2}} \tag{3.2.4.4}$$

is a generating function for the uniform measure on $[-1,1]$. To derive the power series expansion of $\psi(t,x)$ we first use the binomial series to get

$$\psi(t,x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n (2tx-t^2)^n.$$

and apply the binomial theorem to expand $(2tx-t^2)^n$. Then observe that the powers in each expansion have the pattern

$$\{0\}, \{1,2\}, \{2,3,4\}, \{3,4,5,6\}, \cdots, \{n, n+1, n+2\}, \cdots$$

Therefore, the coefficient of t^n in the series expansion of $\psi(t,x)$ is given by

$$\sum_{n=0}^{\infty}(-1)^n 2^{n-2k}\binom{-\frac{1}{2}}{n-k}\binom{n-k}{n}x^{n-2k}.$$

which is a polynomial of degree n in x with the leading coefficient

$$(-1)^n 2^n\binom{-\frac{1}{2}}{n} = \frac{(2n-1)!!}{n!},$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ and by convention $(-1)!! = 1$. Thus we have obtained the power series expansion of the function in Equation (3.2.4.4)

$$\psi(t,x) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!}P_n(x)t^n, \tag{3.2.4.5}$$

where $P_n(x)$ is defined by

$$P_n(x) = \frac{n!}{(2n-1)!!}\sum_{k=0}^{[n/2]}(-1)^n 2^{n-2k}\binom{-\frac{1}{2}}{n-k}\binom{n-k}{k}x^{n-2k}. \tag{3.2.4.6}$$

To find the Szegő-Jacobi parameters, first note that the uniform measure μ on $[-1,1]$ is symmetric and so

$$\alpha_n = 0, \qquad n \geq 0.$$

Next we can easily evaluate

$$E_\mu\psi(t,x)^2 = \frac{1}{2t}(\log(1+t) - \log(1-t))$$

and so we have the series expansion

$$E_\mu\psi(t,x)^2 = \sum_{n=0}^{\infty}\frac{1}{2n+1}t^{2n}.$$

Thus Equation (3.2.13) in Theorem 22 with $a_n = (2n-1)!!/n!$ in view of Equation (3.2.4.5) we get

$$(\frac{(2n-1)!!}{n!})^2\lambda_n = \frac{1}{2n+1},$$

which yields that

$$\lambda_n = \frac{1}{2n+1}\frac{(n!)^2}{((2n-1)!!)^2} \qquad n \geq 1$$

satisfying the condition in Equation (3.2.3). Then by Equation (3.2.11) we have

$$\omega_n = \frac{n^2}{4n^2-1}, \quad n \geq 1, \quad (\omega_0 = 1).$$

Finally we show that the polynomials defined in Equation (3.2.4.6) are the classical Legendre polynomials up to a constant multiple.

Theorem 28 *Let $P_n(x)$ be the polynomial defined by Equation (3.2.4.6). Then*

$$P_n(x) = \frac{n!}{(2n-1)!!}L_n(x),$$

where

$$L_n(x) = \frac{1}{2^n n!}D_x^n(x^2-1)^n.$$

Proof. The coefficient of x^{n-2k} in the summation of Equation (3.2.4.6)

$$(-1)^n 2^{n-2k}\binom{-\frac{1}{2}}{n-k}\binom{n-k}{n}$$

can be simplified to

$$(-1)^k\frac{1}{2^n n!}\binom{n}{k}(2n-2k)(2n-2k-1)\cdots(n-2k+1).$$

Hence $P_n(x)$ can be rewritten as

$$P_n(x) = \frac{n!}{(2n-1)!!}\frac{1}{2^n n!}\times\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}(2n-2k)(2n-2k-1)\cdots(n-2k+1)x^{n-2k}.$$

Now, observe that

$$(2n-2k)(2n-2k-1)\cdots(n-2k+1)x^{n-2k} = D_x x^{2n-2k}$$

.Therefore

$$P_n(x) = \frac{n!}{(2n-1)!!}\frac{1}{2^n n!} \times \sum_{k=0}^{[n/2]}(-1)^k \binom{n}{k} D_x^n x^{2n-2k}$$

$$= \frac{n!}{(2n-1)!!}\frac{1}{2^n n!} \times D_x^n[\sum_{k=0}^{[n/2]}(-1)^k \binom{n}{k} x^{2n-2k}].$$

Note that $D_x^n x^{2n-2k} = 0$ for any $[n/2] < k \leq n$.Hence we have

$$P_n(x) = \frac{n!}{(2n-1)!!}\frac{1}{2^n n!} \times D_x^n[\sum_{k=0}^{n}(-1)^k \binom{n}{k} (x^2)^{n-k}]$$

$$= \frac{n!}{(2n-1)!!}\frac{1}{2^n n!} \times D_x^n(x^2-1)^n$$

$$= \frac{n!}{(2n-1)!!} L_n(x).$$

■

3.2.5 Arcsine Distribution and Chebyshev Polynomials of the First Kind

Let μ be the arcsine distribution given by

$$d\mu(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}dx, \qquad |x| < 1.$$

Try the type of pre-generating function in Equation (3.2.18) with $c = -1$.

$$\varphi(t,x) = \frac{1}{1-\rho(t)x} \tag{3.2.5.1}$$

The expectation of $\varphi(t,x)$ is calculated as follows:

$$E_\mu\varphi(t,.) = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}(1-\rho(t)x)}dx$$

$$= \frac{1}{-\rho(t)\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}(x-\frac{1}{\rho(t)})}$$

$$= \frac{1}{-\rho(t)\pi}\int_{-1}^{1}\frac{1}{\sqrt{1+x}\sqrt{1-x}(x-\frac{1}{\rho(t)})}$$

$$\frac{1}{-\rho(t)\pi}.\frac{1}{\sqrt{-(\frac{1}{\rho(t)}+1)(-\frac{1}{\rho(t)}+1)}}.\arcsin(\frac{(-\frac{1}{\rho(t)}+1)(x+1)+(\frac{1}{\rho(t)}+1)(-x+1)}{2\left|x-\frac{1}{\rho(t)}\right|})\ |_{-1}^{1}$$

$$= \frac{1}{-\rho(t)\pi}.\pi.\frac{|\rho(t)|}{\sqrt{1-\rho^2(t)}}$$

$$E_\mu\varphi(t,.) = \frac{1}{\sqrt{1-\rho(t)^2}}$$

and so the multiplicative renormalization of $\varphi(t,x)$ is given by

$$\psi(t,x) = \sqrt{1-\rho(t)^2}\frac{1}{1-\rho(t)x}. \tag{3.2.5.2}$$

we know compute multiplication of expectation as follows:

$$E_\mu\psi(t,.)\psi(s,.) = \sqrt{1-\rho(t)^2}\sqrt{1-\rho(s)^2}\frac{1}{\pi}\int_{-1}^{1}\frac{1}{1-\rho(t)x}\frac{1}{1-\rho(s)x}\frac{1}{\sqrt{1-x^2}}dx,$$

here we have to use partial fractions method,

$$\frac{A}{1-\rho(t)x}+\frac{B}{1-\rho(s)x} = \frac{1}{(1-\rho(t)x)(1-\rho(s)x)}$$

therefore we have ,

$$A = \frac{-\rho(t)}{\rho(s)-\rho(t)}, B = \frac{\rho(s)}{\rho(s)-\rho(t)}$$

$$\begin{aligned} = \quad & \sqrt{1-\rho(t)^2}\sqrt{1-\rho(s)^2}\frac{1}{\pi} \\ & [\int_{-1}^{1}\frac{\rho(t)}{\rho(t)-\rho(s)}\frac{1}{1-\rho(t)x}\frac{1}{\sqrt{1-x^2}}dx \\ & -\int_{-1}^{1}\frac{\rho(s)}{\rho(t)-\rho(s)}\frac{1}{1-\rho(s)x}\frac{1}{\sqrt{1-x^2}}dx] \end{aligned}$$

with help of the previous integration

$$= \sqrt{1-\rho(t)^2}\sqrt{1-\rho(s)^2}[\frac{\rho(t)}{\rho(t)-\rho(s)}\frac{1}{\sqrt{1-\rho(t)^2}} - \frac{\rho(s)}{\rho(t)-\rho(s)}\frac{1}{\sqrt{1-\rho(s)^2}}]$$

$$E_\mu\psi(t,.)\psi(s,.) = \frac{\rho(t)\sqrt{1-\rho(s)^2} - \rho(s)\sqrt{1-\rho(t)^2}}{\rho(t)-\rho(s)}. \qquad (3.2.5.3)$$

In order to find a function $\rho(t)$ so that $E_\mu\psi(t,.)\psi(s,.)$ depends on only ts, let

$$\theta(t) = \frac{1+\sqrt{1-\rho(t)^2}}{1-\sqrt{1-\rho(t)^2}}, \rho(t) = \frac{2\sqrt{\theta(t)}}{1+\theta(t)}$$

Then equation (3.2.5.3) becomes

$$E_\mu\psi(t,.)\psi(s,.) = \frac{\sqrt{\theta(t)\theta(s)}+1}{\sqrt{\theta(t)\theta(s)}-1}.$$

Hence $\theta(t)$ is given by

$$\theta(t) = at^b$$

and so have

$$\rho(t) = \frac{2\sqrt{a}t^{b/2}}{1+at^b}.$$

Choose $a = 1$ and $b = 2$ to get

$$\rho(t) = \frac{2t}{1+t^2}.$$

Thus the resulting function from Equation (3.2.5.2)

$$\psi(t,x) = \frac{1-t^2}{1-2tx+t^2} \qquad (3.2.5.4)$$

is a generating function for the arcsine distribution μ. To derive the power series expansion of $\psi(t,x)$,

$$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty}(2tx-t^2)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(2tx)^{n-k}(-1)^k t^{2k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}t^{n+k}2^{-n-k}x^{n-k}(-1)^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=m}^{2n}\binom{n}{m-n}t^m 2^{2n-m}x^{2n-m}(-1)^{m-n}$$

$$\sum_{m=0}^{\infty}t^m \sum_{n=\lceil \frac{m}{2}\rceil}^{m}\binom{n}{m-n}2^{2n-m}x^{2n-m}(-1)^{m-n}$$

$$\begin{cases} m-\lceil \frac{m}{2}\rceil = 2p-p=p, \text{if } m=2p \\ m-\lceil \frac{m}{2}\rceil = 2p+1-(p+1)=p, \text{ if } m=2p+1 \end{cases}$$

therefore we can write $m-\lceil \frac{m}{2}\rceil = [\frac{m}{2}]$.

if $k=m-n$ we have

$$\sum_{m=0}^{\infty}t^m \sum_{k=0}^{[m/2]}\binom{m-k}{k}2^{m-2k}x^{m-2k}(-1)^k$$

$$\frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty}[\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}2^{n-2k}x^{n-2k}]t^n. \tag{3.2.5.5}$$

With this equality we can easily derive the power series expansion of the function $\psi(t,x)$ in Equation (3.2.5.4).

$$\psi(t,x) = \frac{1-t^2}{1-2tx+t^2} = \sum_{n=0}^{\infty}2^n P_n(x)t^n. \tag{3.2.5.6}$$

where $P_n(x)$ is defined by

$$P_n(x) = x^n + \frac{1}{2^n}\sum_{k=1}^{[n/2]}(-1)^k\binom{n-k-1}{k-1}2^{n-2k}x^{n-2k}. \tag{3.2.5.7}$$

Next we derive the Szegő-Jacobi parameters. Since μ is symmetric, we have

$$\alpha_n = 0, \quad n \geq 0.$$

Expectation can be calculated as follows:

$$E_\mu \psi(t,.)^2 = \frac{1}{\pi}\int_{-1}^{1} \frac{(1-t^2)^2}{(1-2tx+t^2)^2} \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{(1-t^2)^2}{\pi}\int_{-1}^{1} \frac{dx}{(1-2tx+t^2)^2\sqrt{1-x^2}}$$

$$= \frac{(1-t^2)^2}{\pi}\int_{0}^{\pi} \frac{\sin\theta d\theta}{(1-2t\cos\theta+t^2)^2\sin\theta}$$

if $2\pi - \theta = \xi$

$$= \frac{(1-t^2)^2}{\pi}\int_{\pi}^{2\pi} \frac{d\xi}{(1-2t\cos\xi+t^2)^2}$$

$$= \frac{(1-t^2)^2}{2\pi}\int_{0}^{2\pi} \frac{d\theta}{(1-2t\cos\theta+t^2)^2}$$

by substitution of $e^{i\theta} = z$ we have

$$= \frac{(1-t^2)^2}{2\pi}\int_{|z|=1} \frac{dz}{iz(1-tz-\frac{t}{z}+t^2)^2}$$

$$= \frac{(1-t^2)^2}{2\pi i}\int_{|z|=1} \frac{zdz}{(z-tz^2-t+t^2z)^2}$$

$$= \frac{(1-t^2)^2}{2\pi i}\operatorname{Re}s(\frac{z}{z-tz^2-t+t^2z})$$

$$\begin{aligned} z - tz^2 - t + t^2 z &= 0 \\ -tz^2 + z(1+t^2) - t &= 0 \end{aligned}$$

$$z_{1,2} = \frac{-(1+t^2) \pm \sqrt{(1+t^2)^2 - 4t^2}}{-2t}$$

$$z_1 = \frac{-(1+t^2) + \sqrt{(1+t^2)^2 - 4t^2}}{-2t} = t$$

$$z_2 = \frac{-(1+t^2) - \sqrt{(1+t^2)^2 - 4t^2}}{-2t} = \frac{1}{t}$$

$$= (1-t^2)^2 \operatorname{Re} s(\frac{z}{t^2(z-1)^2(z-\frac{1}{t})^2}, t)$$

$$= (1-t^2)^2 \operatorname{Re} s(\frac{z}{(z-t)^2(1-2t)^2}, t)$$

$$= (1-t^2)^2 (\frac{z}{(1-zt)^2})' \mid_{z=t}$$

$$= (1-t^2)^2 (\frac{(1-2t)^2 + tz2(1-zt)}{(1-zt)^4}) \mid_{z=t}$$

$$= (1-t^2)^2 (\frac{1+tz}{(1-zt)^3}) \mid_{z=t}$$

$$E_\mu \psi(t,.)^2 = \frac{1+t^2}{1-t^2}$$

and so we have the power series expansion

$$E_\mu \psi(t,.)^2 = 1 + 2\sum_{n=1}^{\infty} t^{2n}.$$

Therefore, by Equation (3.2.13) with $a_n = 2^n$ in view of Equation (3.2.5.6)

$$\lambda_n = 2^{1-2n}, \quad n \geq 1 \quad and \quad \lambda_0 = 1,$$

which satisfy the condition in Equation (3.2.3). Recall that $P_0 = 1$ and so by Equation (3.2.11) we get

$$\omega_n = \begin{cases} 1 & if \ n = 0; \\ \frac{1}{2} & if \ n = 1; \\ \frac{1}{4} & if \ n \geq 2. \end{cases}$$

Now we show that the polynomials defined in Equation (3.2.5.7) are the classical Chebyshev polynomials of the first kind up to a constant multiple .

Theorem 29 *Let $P_n(x)$ be the polynomial defined by Equation (3.2.5.7). Then*

$$P_n(x) = \frac{1}{2^{n-1}} T_n(x), \quad n \geq 1, \quad (P_0 = 1),$$

where $T_n(x)$ is the classical Chebyshev polynomial of the first kind defined by

$$T_n(x) = \cos(n \arccos x), \quad n \geq 0.$$

Proof. It is well-known that $cos(n\theta)$ is a polynomial of $cos\theta$ as given by

$$\cos(n\theta) = 2^{n-1} \cos^n \theta + \sum_{k=1}^{[n/2]} (-1)^k \frac{n}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \cos^{n-2k} \theta, \; n \geq 1.$$

Let $x = cos\theta$ and divide both sides by 2^{n-1} to get

$$\frac{1}{2^{n-1}} \cos(n \arccos x) = x^n + \frac{1}{2^n} \sum_{k=1}^{[n/2]} (-1)^k \binom{n-k-1}{k-1} 2^{n-2k} x^{n-2k}.$$

Thus from the definition of $P_n(x)$ in Equation (3.2.5.7) we see that for

$$n \geq 1, P_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x) = \frac{1}{2^{n-1}} T_n(x).$$

■

3.2.6 Semi-circle Distribution and Chebyshev Polynomials of the Second Kind

Let μ be the semi-circle distribution given by

$$d\mu(x) = \tfrac{2}{\pi}\sqrt{1-x^2}dx, \quad |x| < 1.$$

Try the type of pre-generating function in Equation (3.2.18) with $c = -1$.

$$\varphi(t, x) = \frac{1}{1 - \rho(t)x}.$$

The expectation of $\varphi(t, x)$ can be checked to be

$$E_\mu \varphi(t, .) = \frac{2}{1 + \sqrt{1 - \rho(t)^2}}$$

and so the multiplicative renormalization of $\varphi(t,x)$ is given by

$$\psi(t,x) = \frac{1+\sqrt{1-\rho(t)^2}}{2}\frac{1}{1-\rho(t)x}. \tag{3.2.6.1}$$

Direct computation shows that

$$E_\mu\psi(t,.)\psi(s,.) = \frac{1}{2} + \frac{1}{2}\frac{\rho(t)\sqrt{1-\rho(s)^2} - \rho(s)\sqrt{1-\rho(t)^2}}{\rho(t)-\rho(s)}.$$

Thus in view of Equation (3.2.5.3) we get

$$\rho(t) = \frac{2t}{1+t^2}.$$

and resulting function from Equation (3.2.6.1)

$$\psi(t,x) = \frac{1}{1-2tx+t^2}$$

is a generating function for the semi-circle distribution μ. Its power series expansion is already given in Equation (3.2.5.5). Hence we have

$$\psi(t,x) = \frac{1}{1-2tx+t^2} = \sum_{n=0}^{\infty} 2^n P_n(x) t^n, \tag{3.2.6.2}$$

where $P_n(x)$ is defined by

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]}(-1)^k \binom{n-k}{k} 2^{n-2k} x^{n-2k}. \tag{3.2.6.3}$$

To find the Szegő-Jacobi parameters, first we have

$$\alpha_n = 0, \qquad n \geq 0$$

since measure μ is symmetric. On the other hand, it is easy to check that

$$E_\mu \psi(t,.)^2 = \frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n}.$$

Hence by Equation (3.2.18) with $a_n = 2^n$ in view of Equation (3.2.6.2)

$$\lambda_n = \frac{1}{4^n}, \qquad n \geq 0,$$

and so by Equation (3.2.11) we get

$$\omega_n = \frac{1}{4}, \quad n \geq 0.$$

Obviously, the condition in Equation (3.2.3) is satisfied.

Next we show that the polynomials defined in Equation (3.2.6.3) are the classical Chebyshev polynomials of the second kind up to a constant multiple.

Theorem 30 *Let $P_n(x)$ be the polynomial defined by Equation (3.2.6.3). Then*

$$P_n(x) = \frac{1}{2^n} U_n(x), \quad n \geq 0,$$

where $U_n(x)$ is the classical Chebyshev polynomial of the second kind defined by

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)}, \quad n \geq 0. \tag{3.2.6.4}$$

Proof. It is well-known that $\sin[(n+1)\theta]/\sin\theta$ is a polynomial of $\cos\theta$ as given by

$$\frac{\sin[(n+1)\theta]}{\sin\theta} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} 2^{n-2k} \cos^{n-2k}\theta, \quad n \geq 0.$$

Let $x = \cos\theta$ to get the function $U_n(x)$ in Equation (3.2.6.4)

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)} = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} 2^{n-2k} x^{n-2k}. \tag{3.2.6.5}$$

By comparing Equations (3.2.6.3) and (3.2.6.5) we see that

$$P_n(x) = \frac{1}{2^n} U_n(x), \ n \geq 0.$$

■

3.3 Bernstein-Szegő Polynomials

Let $\rho(x)$ be a polynomial of precise degree l and positive in [-1,1]. Then the orthonormal polynomials $p_n(x)$, which are associated with the weight functions

$$\omega(x) = \begin{cases} (1-x^2)^{-i}\{\rho(x)\}^{-1}, \\ (1-x^2)^{i}\{\rho(x)\}^{-1}, \\ (\frac{1-x}{1+x})^{i}\{\rho(x)\}^{-1} \end{cases} \tag{3.3.1}$$

are called Berstein-Szegő polynomials.

Theorem 31 *Let $g(\theta)$ be the trigonometric polynomial with real coefficients which is non-negative for all real values of θ. Then there exists a polynomial $\rho(z)$ of the same degree as $g(\theta)$ such that $g(\theta) = |\rho(z)|^2$, where $z = e^{i\theta}$. Conversely, if $z = e^{i\theta}$, the expression $|\rho(z)|^2$ always represents a non-negative trigonometric polynomial in θ of the same degree as the polynomial $\rho(z)$.*

Theorem 32 *Let $g(\theta)$ satisfy the condition of the previous theorem and $g(\theta) \neq 0$. Then a representation $g(\theta) = \left|h(e^{i\theta})\right|^2$ exists such that $h(x)$ is a polynomial of the same degree as $g(\theta)$, with $h(z) \neq 0$ in $|z| < 1$, and $h(z) > 0$. This polynomial is uniquely determined. If $g(\theta)$ is a cosine polynomial, $h(z)$ is a polynomial with real coefficients.*

Theorem 33 *Let $\rho(x)$ be a polynomial of precise degree l and positive in $[-1,1]$. Let $\rho(\cos\theta) = \left|h(e^{i\theta})\right|^2$ be the normalized representation of $\rho(\cos\theta)$ in the sense of the previous theorem. Writing $h(e^{i\theta}) = c(\theta) + is(\theta)$, $c(\theta)$ and $s(\theta)$ real, we have the following formulas:*

$$\begin{aligned} p_n(\cos\theta) &= (2/\pi)^{\frac{1}{2}}\Re\{e^{in\theta}\overline{h(e^{i\theta})}\} \\ &= (2/\pi)^{\frac{1}{2}}\{c(\theta)\cos n\theta + s(\theta)\sin n\theta\}, \\ w(x) &= (1-x^2)^{-\frac{1}{2}}\{\rho(x)\}^{-1}, \quad l < 2n; \end{aligned} \tag{3.3.2}$$

$$\begin{aligned} p_n(\cos\theta) &= (2/\pi)^{\frac{1}{2}}(\sin\theta)^{-1}\Im\{e^{i(n+1)\theta}\overline{h(e^{i\theta})}\} \\ &= (2/\pi)^{\frac{1}{2}}\{c(\theta)\frac{\sin(n+1)\theta}{\sin\theta} - s(\theta)\frac{\cos(n+1)\theta}{\sin\theta}\}, \\ \omega(x) &= (1-x^2)^{-\frac{1}{2}}\{\rho(x)\}^{-1}, \ l < 2(n+1); \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} p_n(\cos\theta) &= \pi^{-\frac{1}{2}}(\sin(\theta/2))^{-1}\Im\{e^{i(n+\frac{1}{2})\theta}\overline{h(e^{i\theta})}\} \\ &= \pi^{-\frac{1}{2}}\{c(\theta)\frac{\sin(n+\frac{1}{2})\theta}{\sin(\theta/2)} - s(\theta)\frac{\cos(n+\frac{1}{2})\theta}{\sin(\theta/2)}\}, \\ \omega(x) &= (\frac{1-x}{1+x})^{\frac{1}{2}}\{\rho(x)\}^{-1}, \ l < 2n+1. \end{aligned} \tag{3.3.4}$$

These formulas must be modified for $l = 2n$, $1 = 2(n+1)$, and $l = 2n+1$, respectively, by multiplying the right-hand member of (3.3.2) *by* $(1 + h_1/h_0)^{-\frac{1}{2}}$, *and those of* (3.3.3) *and* (3.3.4) $(1 - h_1/h_0)^{-\frac{1}{2}}$, *where* $h_0 = h(0)$ *and* h_1 *is the coefficient of* z^l *in* $h(z)$.

First we observe that the right-hand members of (3.3.2), (3.3.3), (3.3.4) are cosine polynomials with the highest terms

$$(2/\pi)^{\frac{1}{2}} h_0 \cos n\theta, \qquad (2/\pi)^{\frac{1}{2}} h_0 \frac{\sin(n+1)\theta}{\sin\theta}, \qquad \pi^{-\frac{1}{2}} \frac{\sin(n+\frac{1}{2})\theta}{\sin(\theta/2)}, \tag{3.3.5}$$

respectively. In first of these expressions, if, $l = 2n > 0$, h_0 must be replaced by $h_0 + h_1$; in the second and last, if $l = 2(n+1)$ and $l = 2n+1$, respectively, we have $h_0 - h_1$ in place of h_0.

We give the proof of (3.3.2). First we show that

$$\int_{-1}^{1} p_n(x) x^v (1-x^2)^{\frac{-1}{2}} \{\rho(x)\}^{-1} dx = 0, \qquad v = 0, 1, \cdots, n-1,$$

or, what amounts to the same thing,

$$\int_{0}^{\pi} p_n(\cos\theta) \cos v\theta \{\rho(\cos\theta)\}^{-1} d\theta = 0, \qquad v = 0, 1, \cdots, n-1,$$

Now,

$$\frac{(2/\pi)^{\frac{1}{2}}}{2} \Re \left\{ \int_{0}^{\pi} e^{in\theta} \overline{h(e^{in\theta})} (e^{in\theta} + e^{-in\theta}) \left|\left| h(e^{i\theta}) \right|^{-2} d\theta \right\} =$$

$$= \frac{(2/\pi)^{\frac{1}{2}}}{4} \Re \left\{ \int_{-\pi}^{\pi} \frac{e^{i(n+v)\theta} + e^{i(n-v)\theta}}{h(e^{i\theta})} d\theta \right\} = \frac{(2/\pi)^{\frac{1}{2}}}{4} \Re \left\{ \frac{1}{i} \int_{|z|=1} \frac{z^{n+v} + z^{n-v}}{zh(z)} dz \right\} = 0,$$

since the function $z^{n+v} + z^{n-v} \{zh(z)\}^{-1}$ is regular for $|z| \leq 1$. Furthermore,

$$\int_{-1}^{1} \{p_n(x)\}^2 (1-x^2)^{\frac{-1}{2}} \{\rho(x)\}^{-1} dx = \int_{0}^{\pi} \{p_n(\cos\theta)\}^2 \{\rho(\cos\theta)\}^{-1} d\theta$$

$$= \int_{0}^{\pi} \{p_n(\cos\theta)\} (2/\pi)^{\frac{1}{2}} h_0 \cos n\theta \{\rho(\cos\theta)\}^{-1} d\theta$$

$$= \frac{1}{4} (2/\pi)^{\frac{1}{2}} h_0 (2/\pi)^{\frac{1}{2}} \Re \{ \frac{1}{i} \int_{|z|=1} \frac{z^{2n}+1}{zh(z)} dz \} = \frac{1}{4} (2/\pi) h_0 (2\pi/h_0) = 1.$$

The proofs of (3.3.3) and (3.3.4) are similar. In place of $\cos v\theta$ we use $sin(v+1)\theta/\sin\theta$ and $\sin(v+\frac{1}{2})\theta/\sin(\frac{\theta}{2})$, respectively. The modifications necessary for $l=2n$, $l=2(n+1)$, and $l=2n+1$, in (3.3.2), (3.3.3), and (3.3.4), respectively, are also obvious. Finally, we notice that (3.3.2) arises from (3.3.4), (3.3.4) from (3.3.3), and (3.3.2) from (3.3.3) be replacing $\rho(x)$ by $(1-x)\rho(x)$, $(1+x)\rho(x)$, and $(1-x^2)\rho(x)$, respectively.

Given polynomial

$$\omega(x) = (1-\frac{x}{a_1})(1-\frac{x}{a_2})\cdots(1-\frac{x}{a_2q}),$$

which is positive in $[-1,1]$. Let

$$x = \frac{1}{2}(v+\frac{1}{v}) \qquad (|v| \le 1),$$

$$a_k = \frac{1}{2}(c_k+\frac{1}{c_k}) \ \ (|c_k|<1, k=1,2,\cdots,2q),$$

$$\Omega(v) = \prod_{k=1}^{2q}\sqrt{v-c_k},$$

$$\mathcal{L}_m = \begin{cases} \frac{1}{2^{m-1}}\prod_{k=1}^{2q}\sqrt{1+c_k^2} \quad (m>q), \\ \frac{1}{2^{q-1}}\frac{1}{1+c_1c_2\cdots c_{2q}}\prod_{k=1}^{2q}\sqrt{1+c_k^2} \quad (m=q). \end{cases}$$

Then,

$$T_m(x;\omega) = \frac{\mathcal{L}_m}{2}\{v^{2q-m}\frac{\Omega(\frac{1}{v})}{\Omega(v)} + \frac{\Omega(v)}{\Omega(\frac{1}{v})}\}\sqrt{\omega(x)},$$

where an integer number $m \ge q$, is a monic polynomial of degree m.

Theorem 34 *[Bernstein]*

$$a)\min_{A_k}\max_{-1\le x\le 1}\frac{|x^m+A_1x^{m-1}+\cdots+A_m|}{\sqrt{\omega(x)}} = \mathcal{L}_m, \text{ and the extremial polynomial is } T_m(x;\omega),$$

$$b)\int_{-1}^{1}T_m(x;\omega)\frac{x^k}{\omega(x)\sqrt{1-x^2}}dx = \begin{cases} 0 \ (k=0,1,2,\cdots,m-1), \\ \pi\mathcal{L}_m\mathcal{L}_{m+1} \ (k=m) \ (m=q,q+1,\cdots), \end{cases}$$

$$c)\prod_{k=1}^{2q}(1+c_k^2) = e^{-\frac{1}{\pi}\int_{-1}^{1}\frac{\ln\omega(t)}{\sqrt{1-t^2}}dt}$$

Proof. a) We'll use the Chebyshev alternation theorem. [Achieser N.I.,1992]. So it is sufficient to study variation of the argument.

$$v^{2q-m} = \frac{\Omega(\frac{1}{v})}{\Omega(v)},$$

when v runs along upper half of the unit circle. Proof of b follows by relation:

$$\int_{-1}^{1} T_m(x;\omega)\frac{x^m}{\omega(x)\sqrt{1-x^2}}dx =$$

$$= \frac{\mathcal{L}_m}{2^{k+1}}\prod_{r=1}^{2q}\sqrt{1+c_r^2}\int_0^{\pi}\{v^{2q-m}\frac{\Omega(\frac{1}{v})}{\Omega(v)} + v^{m-2q}\frac{\Omega(v)}{\Omega(\frac{1}{v})}\} \times (v+\frac{1}{v})^k\frac{dv}{\Omega(v)\,\Omega(\frac{1}{v})}$$

$$= \frac{\mathcal{L}_m\mathcal{L}_{m+1}}{2^{k-m+1}}\int_{|v|=1} v^{m-2q}\frac{(v+\frac{1}{v})^k}{\Omega(\frac{1}{v})^2}\frac{dv}{iv} \qquad v = e^{iv}$$

by simple application of their residues theorem. ■

Theorem 35

$$a) U_m(x;\omega) = \mathcal{L}_{m+1}\{v^{2q-m-1}\frac{\Omega(\frac{1}{v})}{\Omega(v)} - v^{m+1-2q}\frac{\Omega(v)}{\Omega(\frac{1}{v})}\}\frac{\sqrt{\omega(x)}}{\frac{1}{v}-v},$$

where integer $m \geq q$ *is a monic polynomial of degree* m.

$$b)\int_{-1}^{1} U_m(x;\omega)\frac{x^k\sqrt{1-x^2}}{\omega(x)}dx = \begin{cases} 0 \ (k = 0,1,2,\cdots,m-1), \\ \pi\mathcal{L}_{m+1}^2 \ (k=m) \ (m = q, q+1,\cdots), \end{cases}$$

$$c) \min_{A_k}\int_{-1}^{1}\left|(\left|x^m + A_1x^{m-1} + \cdots + A_m\right|\frac{\sqrt{1-x^2}}{\sqrt{\omega(x)}})\right|^p\frac{dx}{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1}\left|U_m(x;\omega)\frac{\sqrt{1-x^2}}{\sqrt{\omega(x)}})\right|^p\frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)}\mathcal{L}_{m+1}^p \ (m = q, q+1, \cdots) \textit{ for any number } p \geq 1.$$

Proof. b) is proven similarly to b from the former theorem.

c) For p=1, follows from b.

We'll prove c) for $p = 1$. Since for $v = e^{iv}$

$$v^{m+1-2q}\frac{\Omega(v)}{\Omega(\frac{1}{v})} = e^{i\Phi},$$

where Φ is real, and hence

$$U_m(x;\omega) = \mathcal{L}_{m+1}\sqrt{\omega(x)}\frac{\sin\Phi}{\sin v} \qquad (-1 \le x \le 1), \tag{1}$$

then,

$$signU_m(x;\omega) = sign\sin\Phi = \frac{4}{\pi}\sum_{r=0}^{\infty}\frac{\sin(2r+1)\Phi}{2r+1}.$$

From the other side

$$\sin(2r+1)\Phi = \frac{1}{2i}\{[v^{m+1-2q}\frac{\Omega(v)}{\Omega(\frac{1}{v})}]^{2r+1} - [v^{2q-m-1}\frac{\Omega(\frac{1}{v})}{\Omega(v)}]^{2r+1}\} =$$

$$= \frac{1}{2i}\frac{1}{\mathcal{L}_{m+1}^{2r+1}}U_{(2r+1)m}(x;\omega^{r+1})\frac{v-\frac{1}{v}}{[\sqrt{\omega(x)}]^{2r+1}}$$

and consequently by the property b

$$\begin{aligned}
&\int_{-1}^{1}\sin(2r+1)\Phi\frac{x^k dx}{\sqrt{\omega(x)}} \\
&= \frac{1}{2i}\frac{1}{\mathcal{L}_{m+1}^{2r+1}}U_{(2r+1)m}(x;\omega^{r+1})x^k[\omega(x)]^r\frac{\sqrt{1-x^2}dx}{[\omega(x)]^{2r+1}} = 0 \\
k &= 0,1,2,\cdots,m-1;
\end{aligned} \tag{2}$$

Hence,

$$\int_{-1}^{1} signU_m(x;\omega)\frac{x^k dx}{\sqrt{\omega(x)}} = 0 \quad k = 0,1,2,\cdots,m-1 \tag{3'}$$

and generally

$$\int_{-1}^{1}\cos 2n\Phi signU_m(x;\omega)\frac{x^k dx}{\sqrt{\omega(x)}} = 0 \;\; k = 0,1,2,\cdots,m-1; n = 0,1,2,\cdots \tag{3}$$

After obtaining (3') further considerations are similar to the proof of A. Markov theorem. Now we'll give the proof c) any $p > 1$. For $p > 1$ the extremial polynomial is unique. Since the space L_p is strictly convex. [DeVore R.A., Lorentz G.G., 1993]. Hence by elementary property of extremum, it is sufficient to prove for any $p > 1$ the following relations ■

Proof.

$$\int_{-1}^{1}\left|\frac{U_m(x;\omega)\sqrt{1-x^2}}{\sqrt{\omega(x)}}\right| signU_m(x;\omega)\frac{x^k dx}{\sqrt{\omega(x)}} = 0 \;\; k = 0,1,2,\cdots,m-1. \tag{4}$$

But for any $\sigma > 0$ the expansion (5) is uniformly convergent.

$$|\sin\Phi|^p = a_0^{(p)} + a_2^{(p)}\cos 2\Phi + a_4^{(p)}\cos 4\Phi + \cdots \tag{5}$$

Hence (4) follows by (3). Now it remains to find the value of the integral

$$\mu_p = \int_{-1}^{1} \left| U_m(x;\omega) \frac{\sqrt{1-x^2}}{\sqrt{\omega(x)}} \right|^p \frac{dx}{\sqrt{1-x^2}}$$

For that reason observe that because of (1) and (5).

$$\mu_p = \mathcal{L}_{m+1}^p \int_{-1}^{1} a_0^{(p)} \frac{dx}{\sqrt{1-x^2}} = \pi a_0^{(p)} \mathcal{L}_{m+1}^p,$$

since

$$\begin{aligned} \cos 2r\Phi &= \frac{1}{2}\{[v^{m+1-2q}\frac{\Omega(v)}{\Omega(\frac{1}{v})}]^{2r} + [v^{2q-m-1}\frac{\Omega(\frac{1}{v})}{\Omega(v)}]^{2r}\} = \\ &= const \frac{T_{2r(m+1)}(x;\omega^{2r})}{[\omega(x)]^r}, \end{aligned}$$

and then

$$\int_{-1}^{1} \cos 2r\Phi \frac{dx}{\sqrt{1-x^2}} = 0 \qquad (r = 1, 2, \cdots).$$

From the outside it follows from directly by (5) that

$$a_0^{(p)} = \frac{1}{\pi} \int_0^{\pi} \sin^p \Phi d\Phi$$

so

$$\mu_p = \mathcal{L}_{m+1}^p \int_0^{\pi} \sin^p \Phi d\Phi = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)} \mathcal{L}_{m+1}^p.$$

Now for the case

$$\omega(x) = \prod_{j=1}^{k} (1 - a_j x)^2$$

$T_m(x;\omega)$ might be written also in the form

$$cos((m-x)arccosx + \sum_{j=1}^{k} \arccos \frac{x-a_j}{1-a_j x})\omega(x)$$

what can be seen easily by observing the alternation property of the rational function and

$$R_n(x) = cos((m-x)arccosx + \sum_{j=1}^{k} \arccos \frac{x-a_j}{1-a_j x})\omega(x).$$

■

Theorem 36 *The following formula holds*

$$\psi(t,x) = 2\sum_{n=k}^{\infty} T_n(x,\omega)t^n$$

$$= \frac{t^k(\cos\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx} - t\cos(\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx} - \arccos x))}{1-2tx+t^2}\prod_{j=1}^{k}(1-a_jx).$$

Proof. Consider

$$2\sum_{n=k}^{\infty} R_n(x)t^n =$$

$$\sum_{n=k}^{\infty}(e^{i(n-k)arccosx+\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}} + e^{-i(n-k)arccosx+\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}})t^n =$$

$$e^{i\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}}t^k\sum_{m=0}^{\infty}e^{im\arccos x}t^m + e^{-i\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}}t^k\sum_{m=0}^{\infty}e^{-im\arccos x}t^m =$$

$$\frac{t^k e^{i\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}}}{1-te^{i\arccos x}} + \frac{t^k e^{-i\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}}}{1-te^{-i\arccos x}}$$

$$= \frac{t^k(e^{i\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}}(1-te^{-i\arccos x}) + e^{-i\sum_{j=1}^{k}\arccos\frac{x-a_j}{1-a_jx}}(1-te^{i\arccos x}))}{1-2tx+t^2}$$

Now observe that the generating function for the orthogonal Bernstein-Szegő polynomial with$n \geq k$ is given by

$$\psi(t,x) = f(t,x)\prod_{j=1}^{k}(1-a_jx).$$

where

$$f(t,x) = 2\sum_{n=k}^{\infty} R_n(x)t^n$$

$$= 2\sum_{n=k}^{\infty}\frac{P_n(x)}{\prod_{j=1}^{k}(1-a_jx)}t^n$$

$$= \frac{2}{\prod_{j=1}^{k}(1-a_jx)}\sum_{n=k}^{\infty}P_n(x)t^n.$$

■

Concerning first k polynomials of Bernstein-Szegő we'll give the following result from paper of [Delgado M.A., Geronimo S.J., Iliev P., Xu Y.]

Proposition 37 *Assume that* $h(z) = 1+h_1(z)+\cdots+h_N(z)^N$ *is a stable polynomial of degree* N *with real coefficients. Let* q_k *be defined by*

$$q_k(x) = \sum_{i=0}^{N} h_i U_{k-i}(x), \ \ k \geq 0, \tag{3.3.6}$$

where $U_n(x) = -U_{-n-2}(x)$ *for* $n < 0$*. For* $0 \leq k \leq \lceil \frac{N-2}{2} \rceil - 1$*, there are constants* $h_k^{'}$ *such that*

$$\hat{q}_k(x) := q_k(x) + h_{k+1}^{'} q_{k+1}(x) + \cdots + h_{N-k+2}^{'} q_{N-k-2}(x)$$

is a polynomial of degree k *and orthogonal to every polynomial of degree less than* k *with respect to*

$$d\mu(x) = \frac{\sqrt{1-x^2}dx}{|h(z)|^2}, x = (z+1/z)/2,$$

where $h_{N-k+2}^{'} = h_N, h_{N-k+1}^{'} = h_{N-1} - h_N h_1$*, and the other* $h_j^{'}$ *can be deduced inductively from* h_j*.*

Lemma 38 *Let* $N \in \mathbb{N}$ *be a fixed positive integer. For any* $i = 0, \cdots, N$*, let* $h_i(y)$ *be polynomials in* y *with real coefficients of degree at most* $\frac{N}{2} - \left|\frac{N}{2} - i\right|$*, with* $h_0(y) = 1$*, such that*

$$h(z,y) = \sum_{i=0}^{N} h_i(y) z^i, \tag{3.3.7}$$

is a stable polynomial in z *for all* $-1 \leq y \leq 1$*, i.e.* $h(z,y) \neq 0$ *for any* $|z| \leq 1$*. Define*

$$q_k(x,y) = \sum_{i=0}^{N} h_i(y) U_{k-i}(x), \tag{3.3.8}$$

where $U_n(x)$ *is the nth Chebyshev polynomial of the second kind. Here if* $n < 0$*, the Chebyshev polynomial is understood as* $U_n(x) = -U_{-n-2}(x)$*. Then* $q_k(x,y)$*, is a polynomial in two-variables which is orthogonal to every polynomial in* x *of degree less than* k *with respect to*

$$d\mu(x) = \frac{\sqrt{1-x^2}dx}{|h(z,y)|^2}, x = (z+1/z)/2.$$

Moreover, for $k \geq \lceil \frac{N-2}{2} \rceil, q_k(x,y)$ *is a polynomial of total degree* k *and*

$$\int_{-1}^{1} q_k^2(x,y) d\mu_y(x) = \begin{cases} \frac{\pi}{2} & if \ \ k > \lceil \frac{N-2}{2} \rceil, \\ \frac{\pi}{2}(1-h_N) & if \ \ 2k+2 = N. \end{cases}$$

Proof. (Proposition) Previous lemma shows that q_k is orthogonal to all polynomials of degree at most $k-1$. Thus, it follows readily that for $0 \leq k \leq \lceil \frac{N-2}{2} \rceil - 1$, $\hat{q}_k(x)$ is orthogonal to all polynomials of degree at most $k-1$. We now prove that we can choose constants $h_j^{'}$ such that $\hat{q}_k$ is of degree k. Using $U_n(x) = -U_{-n-2}(x)$ for $n < 0$, we can write

$$q_k(x) = -\sum_{i=k+1}^{N-k-2} h_{k+i+2} U_i(x) + \sum_{i=0}^{k} (h_{k-i} - h_{k+i+2}) U_i(x),$$

$$k \leq \lceil \frac{N-2}{2} \rceil - 1 \tag{3.3.9}$$

in which the first sum contains terms that have degree $> k$. In particular, $degq_k \leq N-k-2$ for $k \leq \lceil \frac{N-2}{2} \rceil - 1$. Furthermore, for $k+1 \leq j \leq N-1$ we can write, by () and the fact that $U_n(x) = -U_{-n-2}(x)$ for $n < 0$.

$$q_j(x) = +\sum_{i=k+1}^{j} h_{j-i} U_i(x) + \sum_{i=0}^{k} h_{j-1} U_i(x) - \sum_{i=0}^{N-j-2} h_{j+i+2} U_i(x),$$

where again the first sum contains terms that have degree $> k$ in x. Since $h_0 = 1$, it follows readily that

$$q_k(x) + h_N q_{N-k-2}(x) = -\sum_{i=k+1}^{N-k-3} (h_{k+i+2} - h_N h_{N-k-2}) U_i(x) + \cdots,$$

where only terms whose degree $\geq k+1$ are given explicitly in the right hand side. Thus the right-hand side of the above expression has degree $N-k-3$. Continuing, we add $(h_{N-1} - h_N h_1) q_{N-k-3}$ to eliminate U_{N-k-3}. Proceeding in this way , we keep adding terms until the right-hand side contains only terms of degree $\leq k$. This proves that $\hat{q}_k$ is indeed a polynomial of degree k. ■

CHAPTER IV

ORTHOGONAL POLYNOMIALS IN STOCHASTIC THEORY

The material of the chapter is taken from [Schoutens, W., 2000],[Plucinska, A., 1998].

The equation

$$f(z)exp(xu(z) = \sum_{m=0}^{\infty} Q_m(x)\frac{z^m}{m!} \tag{4.1}$$

generates a family of polynomials $\{Q_m(x), m \geq 0\}$ when both functions $u(z)$ and $f(z)$ can be expanded in a formal power series and if $u(0) = 0$, $u'(0) \neq 0$, and $f(0) \neq 0$. The polynomials $Q_m(x)$ so defined are of exact degree m and are called Sheffer polynomials. Any set of such polynomials is called a Sheffer set since the first treatment of such polynomials was started by Sheffer [Sheffer, I.M ,1937],[Sheffer, I.M ,1939].

Define τ as the inverse function of u , so that $\tau(u(z)) = z$. Then τ also can be expanded formally in a power series with $\tau(0) = 0$ and $\tau'(0) \neq 0$. Let introduce an additional time parameter $t \geq 0$ into the polynomials defined

$$f(z)exp(xu(z)) = \sum_{m=0}^{\infty} Q_m(x)\frac{z^m}{m!}$$

by replacing the function $f(z)$ by

Definition 39 *A polynomial set $\{Q_m(x), m \geq 0, t \geq 0\}$ is called a Levy-Sheffer system if it is defined by a generating function of the form*

$$(f(z))^t.exp(xu(z) = \sum_{m=0}^{\infty} Q_m(x,t)\frac{z^m}{m!} \tag{4.2}$$

where

1. $f(z)$ and $u(z)$ are analytic in a neighborhood of $z = 0$,

2. $u(0) = 0, f(0) = 1$, and $u'(0) \neq 0$, and

3. $\frac{1}{f(\tau(i\theta))}$ is an infinitely divisible characteristic function.

The quantity t can be considered to be a positive parameter ; as such the function $Q_m(x,t)$ will also be a polynomial in t.

If Condition 3. is satisfied, then there is a Levy-process $\{X_t, t \geq 0\}$ defined by the function

$$\phi(\theta) = \phi_X(\theta) = \frac{1}{f(\tau(i\theta))} \tag{4.3}$$

through the characteristic function. From the Kolmogorov representation theorem , the latter can be equivalently phrased in terms of the pair (c, K).

The basic link between the polynomials and the corresponding Levy processes is the following martingale equality

$$E[Q_m(X_t,t)|X_s] = Q_m(X_s,s), 0 \leq s \leq t, m \geq 0 \tag{4.4}$$

Indeed , taking generating functions , we find on the left hand side of (4.4)

$$\sum_{m=0}^{\infty} E\{Q_m(X_t,t)|X_s\}\frac{z^m}{m!}$$

$$= E\{\sum_{m=0}^{\infty} Q_m(X_t,t)\frac{z^m}{m!}|X_s\}$$

$$= E\{(f(z))^t.exp(u(z)X_t)|X_s\}$$

$$= (f(z))^t.exp(u(z)X_s)E\{exp(u(z)(X_t - X_s))|X_s\}.$$

For the right side of (4.4) we immediately find

$$\sum_{m=0}^{\infty} Q_m(X_s, s)\frac{z^m}{m!} = (f(z))^s.exp(u(z)X_s)$$

Combination of both expressions leads to the relationship

$$E\{exp(u(z)(X_t - X_s))|X_s\} = (f(z))^{s-t}$$

If we compare this relationship with equation determining the Levy process

$$E\{exp(u(z)(X_t - X_s))|X_s\} = (\phi(\theta))^{t-s}.$$

then we realize that (4.4) will be satisfied if and only if (4.3) holds.

Example 40 *(The Laguerre polynomials) The following generating function of a version of the Laguerre polynomials is well known.*

$$\sum_{m=0}^{\infty} L_m^{(\alpha-m)}(y)w^m = (1+w)^{\alpha}\exp(-yw).$$

We identify the ingredients of this example.

$$\begin{cases} u(z) = -z \\ f(z) = (1+z)^{\alpha} \\ \phi(\theta) = (1-i\theta)^{-\alpha} \end{cases}$$

The function $\phi(\theta)$ resembles the characteristic function of the infinitely divisible Gamma distribution. Let $\{G_t, t \geq 0\}$ be the Gamma process with $(G_1 = G)$,

$$E[exp(i\theta G_t)] = exp(t\psi_G(\theta)),$$

where $\psi_G(\theta) = \log\phi(\theta) = -\log(1-i\theta)$. Hence in Levy process $\psi_X(\theta) = i\theta A + C\psi_Y(B\theta)$ take $A = 0, B = 1$, and $C = \alpha$ so that $X_t = G_{\alpha t}$. We derive the martingale property by putting $Q_m(x,t) = L_m^{(\alpha t - m)}(x)$ so that

$$E[L_m^{(\alpha t-m)}(G_{\alpha t})|G_{\alpha s}] = L_m^{(\alpha s-m)}(G_{\alpha s}).$$

Example 41 *The Actuarial Polynomials.*

The Actuarial polynomials are determined by the generating function

$$\sum_{m=0}^{\infty} g_m^{(\lambda)}(y)\frac{\omega^m}{m!} = \exp(\lambda\omega + y(1-e^{\omega})),$$

where $\lambda > 0$. We identify the ingredients of this example.

$$\begin{cases} u(z) = 1 - \exp(z), \\ f(z) = \exp(\lambda z), \\ \phi(z) = (1 - i\theta)^{-\lambda} \end{cases}$$

Indeed, it easily follows that $\tau(z) = \log(1-z)$. As before we can put $X_t t = G_{\lambda t}$. With the identification $Q_m(x,t) = g_m^{(\lambda)}(x)$ we arrive at the martingale property

$$E[g_m^{(\lambda)}(G_{\lambda t})|G_{\lambda s}] = g_m^{(\lambda t)}(G_{\lambda s}).$$

This martingale relation seems to be new.

Sheffer Sets and Orthogonality

Meixner Set of Orthogonal Polynomials If as set of polynomials is defined as (4.1) , some extra conditions have to be satisfied to make these polynomials orthogonal. In his historic paper , Meixner determined all set of orthogonal polynomials that also satisfy the generating function relation (4.1) . As we need some of the ingredients of Meixner's approach, let us briefly sketch the construction.

Put $D = \frac{d}{dx}$ for the differential operator with respect to x. Relation (4.1) implies that

$$\tau(D)Q_m(x) = mQ_{m-1}(x), \qquad m \geq 0$$

This equation in turn leads to the relation

$$\tau(D)(xQ_m(x)) = \tau'(D)Q_m(x) + mxQ_{m-1}(x), \qquad m \geq 0$$

By Favard's theorem , the monic set $\{Q_m(x), m \geq 0\}$ will be orthogonal if and only if the polynomials satisfy a three-term recurrence relation

$$Q_{m+1}(x) = (x + l_{m+1})Q_m(x) + k_{m+1}Q_{m-1}(x), \tag{4.5}$$

where the numbers l_m are real and $k_m < 0, m \geq 2$. Apply $\tau(D)$ to (4.5). Subtract from this relation (4.5) for Q_m after multiplying it by m. We obtain

$$(1-\tau'(D))Q_m(x) = (l_{m-1}-l_m)mQ_{m-1}(x)+\left(\frac{k_{m+1}}{m} - \frac{k_m}{m-1}\right)m(m-1)Q_{m-2}(x).$$

If we shift m to $m+1$ in this equation and then again $\tau(D)$, we find

$$(1-\tau'(D))Q_m(x) = (l_{m+2}-l_{m+1})mQ_{m-1}(x)+\left(\frac{k_{m+2}}{m+1} - \frac{k_{m+1}}{m}\right)m(m-1)Q_{m-2}(x).$$

Comparing the last two formulas , we obtain the following relations

$$l_{m+1} - l_m = \lambda, \quad m \geq 1,$$

$$\frac{k_{m+1}}{m} - \frac{k_m}{m-1} = \kappa, \ m \geq 2,$$

$$(1 - \tau'(D))Q_m(x) = \lambda\tau(D)Q_m(x) + \kappa\tau^2(D)Q_m(x), \quad m \geq 0 \tag{4.6}$$

and (4.5) becomes

$$Q_{m+1}(x) = (x + l_1 + m\lambda)Q_m(x) + m(k_2 + (m-1)\kappa)Q_{m-1}(x), \quad m \geq 0 \tag{4.7}$$

where $k_2 < 0$ and $\kappa \leq 0$. From (4.6) follows

$$\tau'(y) = 1 - \lambda\tau(y) - \kappa\tau^2(y). \tag{4.8}$$

Furthermore we obtain from (4.7), using

$$f(z) = \sum_{m=0}^{\infty} Q_m(x)\frac{z^m}{m!},$$

the following relation for $f(z)$,

$$\frac{f'(z)}{f(z)} = \frac{k_2 z + l_1}{1 - \lambda z - \kappa z^2}.$$

We define two quantities α and β by the equation

$$1 - \lambda z - \kappa z^2 = (1 - \alpha z)(1 - \beta z),$$

where $\alpha\beta \geq 0$. With these quantities we can rewrite the equation for f and obtain from (4.8) a differential equation for the function $u(z)$.

$$u'(z) = \frac{1}{(1 - \alpha z)(1 - \beta z)}, \qquad \frac{f'(z)}{f(z)} = \frac{l + kz}{(1 - \alpha z)(1 - \beta z)}, \tag{4.9}$$

where $l \in R$ and $k \leq 0$ are constants. The solution of these equations is standard. We obtain

$$\begin{cases} \frac{1}{\alpha-\beta}\log(\frac{1-\beta z}{1-\alpha z}) & if \quad \alpha \neq \beta \\ \frac{z}{1-\alpha z} & if \quad \alpha = \beta \end{cases}$$

Note that the value for $\beta = \alpha$ is the limiting expression from the case $\beta \neq \alpha$ when $\beta \to \alpha$.

The explicit form of f is a bit more complicated.

$$\log f(z) = \begin{cases} \frac{-(k+\alpha l)\log(1-\alpha z)}{\alpha(\alpha-\beta)} + \frac{(k+\beta l)\log(1-\beta z)}{\beta(\alpha-\beta)}, & 0 \neq \alpha \neq \beta \neq 0 \\ \frac{k\log(1-\alpha z)}{\alpha^2} + \frac{k+\alpha l}{\alpha}\frac{z}{1-\alpha z}, & \alpha = \beta \neq 0 \\ \frac{-(k+\alpha l)\log(1-\alpha z)}{\alpha^2} - \frac{kz}{\alpha}, & \alpha \neq \beta = 0 \\ \frac{k}{2}z^2 + lz, & \alpha = \beta = 0 \end{cases}$$

Again the last three forms are the obvious limiting cases of the first form. Also the function τ can be obtained explicitly , giving

$$\tau(v) = \begin{cases} \frac{\exp(\beta v)-\exp(\alpha v)}{\beta\exp(\beta v)-\alpha\exp(\alpha v)} & if\ \alpha \neq \beta \\ \frac{v}{1+\alpha v} & if\ \alpha = \beta \end{cases}$$

For each choice of the allowed pairs (α, β) we obtain a Meixner set of orthogonal polynomials. Their explicit expression turns up after we have introduced the martingale context.

Levy-Meixner System Let us apply the above to equation (4.2). We call any resulting system a Levy-Meixner system. Each one of these systems is therefore a Levy-Sheffer system but with orthogonal polynomials. Since the explicit form of the functions f and τ is known, we can identify the ingredients in the Kolmogorov Canonical representation. This then automatically determines the underlying process.

Before embarking on the different subcases, we try to get as far as possible with general quantities. The differential equation for ϕ follows from that for f .

From $z = \tau(v)$ and $-log f(z) = log\phi(-iu(z))$ we easily find that

$$-\frac{f'(z)}{f(z)} = \frac{\phi'(-iu(z))}{\phi(-iu(z))}(-i)u'(z)$$

and hence by the equations (4.9) ,

$$\frac{\phi'(-iu(z))}{\phi(-iu(z))} = -i(l + kz).$$

Changing back to the argument θ we find

$$\frac{\phi'(\theta)}{\phi(\theta)} = -i(l + k\tau(i\theta)).$$

The last equation allows us to identify the quantities k and l.

Put $\theta = 0$; then $\phi'(\theta) = iE[X_1] = -il$.

A further derivation similarly yields that $k = -Var[X_1] < 0$.

Henceforth, we write

$l = -\mu, \quad k = -\sigma^2.$

We solve the differential equation for $\phi(\theta)$. We easily find

$$log\phi(\theta) = i\mu\theta + \sigma^2 \int_0^{i\theta} \tau(z)dz.$$

The identification of c and K in the Kolmogorov representation is done as follows. By taking derivatives in Kolmogorov representation at $\theta = 0$ we see that $ic = \phi'(\theta) = iE[X_1] = -il$, and hence $c = \mu$.

Taking another derivative we get the equation

$$\int_{-\infty}^{\infty} \exp(i\theta x)dK(x) = \sigma^2\tau'(i\theta),$$

which determines K uniquely. The result of the subsequent calculations are rather easy and lead to the explicit form

$$\int_{-\infty}^{\infty} \exp(i\theta x)dK(x) = \begin{cases} \frac{\sigma^2(\alpha-\beta)\exp(i(\alpha+\beta)\theta)}{(\beta\exp(i\beta\theta)-\alpha\exp(i\alpha\theta))^2} & if\ \alpha \neq \beta \\ (\frac{\sigma}{1+i\alpha\theta})^2 & if\ \alpha = \beta \end{cases} \tag{4.10}$$

We later verify that function $\int_{-\infty}^{\infty} \exp(i\theta x)d(K(x)/K(\infty))$ is indeed a characteristic function for all possible values of α and β where $\alpha\beta \geq 0$.

To simplify the further analysis and without loss of generality , we make the following choice.

$$l = c = 0,\ k = -K(\infty) = -\sigma^2 = -1. \tag{4.11}$$

From the identification we then get

$$\psi(\phi) = \log\phi(\theta) = \begin{cases} \frac{i\theta(\alpha+\beta)+\log((\alpha-\beta)/(\alpha e^{i\alpha\theta}-\beta e^{i\beta\theta}))}{\alpha\beta}, & \text{if } 0 \neq \alpha \neq \beta \neq 0 \\ \frac{i\theta}{\alpha} - \frac{\log(1+i\alpha\theta)}{\alpha^2}, & \text{if } \alpha = \beta \neq 0 \\ \frac{i\theta}{\alpha} - \frac{(1-\exp(-i\alpha\theta))}{\alpha^2}, & \text{if } \alpha \neq \beta = 0 \\ -\frac{\theta^2}{2}, & \text{if } \alpha = \beta = 0 \end{cases} \tag{4.12}$$

The Levy-Meixner Systems Our general approach now is to link all Meixner polynomials to a unique Levy process. The departing form of the polynomials is (4.2), while for the Levy process we take Kolmogorov representation and (4.12); i.e.,

$$\log E[e^{i\theta X_t}] = t\psi(\theta) = t\log\phi_X(\theta) = t\int_{-\infty}^{\infty}(e^{i\theta x} - 1 - i\theta x)\frac{dK(x)}{x^2},$$

where K is a probability measure. The two ingredients are linked by Equation (4.1.3), $\psi(\theta) = -\log f(\tau(i\theta))$.

The measure of orthogonality $\Psi_t(x)$, is also the distribution function of our Levy process X_t.Indeed , by taking generating functions in

$$\int_{-\infty}^{\infty} Q_m(x,t)Q_n(x,t)d\Psi_t(x) = \delta_{mn}c_m(t)$$

and setting $n = 0$ we have

$$\int_{-\infty}^{\infty}(f(z))^t.exp(xu(z)d\Psi_t(x) = c_0 = 1$$

Putting $u(z) = i\theta$ so that $z = \tau(i\theta)$ finally gives

$$\int_{-\infty}^{\infty}\exp(i\theta x)d\Psi_t(x) = (\frac{1}{f(\tau(i\theta))})^t = E[\exp(i\theta X_t)].$$

We also use the unit step distribution at the origin

$$I(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Brownian Motion-Hermite This is the case where $\alpha = \beta = 0$ and it is by far the easiest. The fundamental quantities are

$$\begin{cases} u(z) = z \\ f(z) = \exp(-z^2/2) \\ \psi(\theta) = -\theta^2/2 \end{cases}$$

The first two quantities gives us

$$\sum_{m=0}^{\infty} Q_m(x;t)\frac{z^m}{m!} = \exp(zx - tz^2/2),$$

while the last quantity tells us that we are working with a Brownian motion $\{B_t, t \geq 0\}$ as the appropriate Levy process. Also $K(x) = I(x)$ is easily derived.

The generating function of the Hermite polynomials , $H_m(y)$, is given by

$$\sum_{m=0}^{\infty} H_m(y)\frac{\omega^m}{m!} = \exp(2y\omega - \omega^2).$$

Identification of the two expressions requires

$$y = \frac{x}{\sqrt{2t}}, \quad \omega = z\sqrt{\frac{t}{2}}.$$

We therefore find that for each $m \in N$, $\{(t/2)^{m/2} H_m(B_t/\sqrt{2t})\}$ is a martingale ; i.e., for $0 \leq s < t$.

Poisson Process-Charlier This is the case $\alpha \neq \beta = 0$ and the fundamental quantities are

$$\begin{cases} u(z) = -\frac{1}{a}\log(1 - az), \\ f(z) = (1 - \alpha z)^{1/\alpha^2}\exp(z/\alpha), \\ \psi(\theta) = i\frac{\theta}{\alpha} - \frac{1}{\alpha^2}(1 - \exp(-i\alpha\theta)). \end{cases}$$

The first two quantities give us

$$\sum_{m=0}^{\infty} Q_m(x;t)\frac{z^m}{m!} = \exp(zt/\alpha)(1-\alpha z)^{(t-\alpha x)/\alpha^2}$$

Moreover (4.2.2.1) tells us that

$$\int_{-\infty}^{\infty} \exp(i\theta x) dK(x) = \exp(-i\alpha\theta)$$

and hence that

$K(x) = I(x+\alpha)$. From the form of $\psi(\theta)$ it seems natural to search for a connection with Poisson process $\{N_t, t \geq 0\}$ for which $\psi_N(\theta) = \exp(i\theta) - 1$.From

$$\psi_X(\theta) = i\theta A + C\psi_Y(B\theta)$$

we derive that

$$A = \alpha^{-1}, \quad B = -\alpha \qquad C = \alpha^{-2}$$

so that

$$X_t = \frac{t}{\alpha} - \alpha N_{t/\alpha^2}.$$

The Charlier polynomials are defined for a>0 by their generating function

$$\sum_{m=0}^{\infty} C_m(y;a)\frac{\omega^m}{m!} = e^{\omega}(1-\frac{\omega}{a})^y.$$

Identification of them with the polynomials (4.1.2) requires

$$\omega = \frac{zt}{\alpha}, \qquad a = \frac{t}{\alpha^2}, \quad y = \frac{t-\alpha z}{\alpha^2}$$

and hence

$$Q_m(x;t) = C_m(\frac{t-\alpha x}{\alpha^2}, \frac{t}{\alpha^2})(\frac{t}{\alpha})^m.$$

Replacing x by $X(t)$ we derive by

$$\psi_X(\theta) = i\theta A + C\psi_Y(B\theta)$$

the martingale property

$$E[C_m(N_{t/\alpha^2}, \frac{t}{\alpha^2})|N_{s/\alpha^2}] = (\frac{s}{t})^m C_m(N_{s/\alpha^2}, \frac{s}{\alpha^2}).$$

Gamma Process-Laguerre Now we look the case where $\alpha = \beta \neq 0$. The primary quantities are

$$\begin{cases} u(z) = \frac{z}{1-\alpha z}, \\ f(z) = (1-\alpha z)^{-\frac{1}{\alpha^2}} \exp(-\frac{z}{\alpha(1-\alpha z)}), \\ \psi(\theta) = i\frac{\theta}{\alpha} - \frac{1}{\alpha^2}\log(1+i\alpha\theta). \end{cases}$$

The first two ingredients lead to the generating function of the polynomials

$$\sum_{m=0}^{\infty} Q_m(x;t)\frac{z^m}{m!} = \exp(\frac{z(\alpha x - t)}{\alpha(1-\alpha z)})(1-\alpha z)^{-\frac{t}{\alpha^2}}.$$

Furthermore

$$\int_{-\infty}^{\infty} \exp(i\theta x)dK(x) = (1+i\alpha\theta)^{-2}$$

easily yields

$$K(x) = \int_{-\infty}^{x/\alpha\wedge 0} |y|\exp(y)dy$$

From the form of $\psi(\theta)$ we are led to a Gamma process $\{G_t, t \geq 0\}$ which is determined through the expression $\psi_G(\theta) = -\log(1-i\theta)$. The requested identification leads to the same expressions as for the Poisson case. Hence again

$$X_t = \frac{t}{\alpha} - \alpha G_{t/\alpha^2}.$$

The obvious set of orthogonal polynomials is now provided by the Laguerre polynomials defined for all α through the generating function

$$\sum_{m=0}^{\infty} L_m^{(\alpha)}(y)\omega^m = (1-\omega)^{-\alpha-1}\exp(\frac{y\omega}{\omega-1}).$$

The appropriate identification with the basic polynomials requires

$$\omega = \alpha z, \quad a = \frac{t}{\alpha^2} - 1, \quad y = \frac{t - \alpha x}{\alpha^2}.$$

and hence

$$Q_m(x;t) = m!\alpha^m L_m^{(-1+t/\alpha^2)}(\frac{t-\alpha x}{\alpha^2}).$$

Replace x be X_t to derive the martingale property

$$E[L_m^{(-1+t/\alpha^2)}(G_{t/\alpha^2})|G_{s/\alpha^2}] = L_m^{(-1+s/\alpha^2)}(G_{s/\alpha^2}).$$

4.1 *Stochastic Integration with respect to Brownian Motion and Hermite Polynomials*

The most studied stochastic case is integration with respect to Brownian motion $\{B_t, t \geq 0\}$, where B_t has a normal distribution $N(0,t)$. The notion of multiple stochastic integration for this process was first introduced by Norbert Wiener. It is well known that Ito integration theory with respect to standard Brownian motion, the Hermite polynomials play the role of the p_n. [Ito,K.,1951]. We have

Theorem 42

$$\int_o^t \tilde{H}_n(B_s;s)dB_s = \frac{\tilde{H}_{n+1}(B_t;t)}{n+1}, \tag{4.1.1}$$

where

$$\tilde{H}_n(x;t) = (t/2)^{n/2} H_n(x/\sqrt{2t}) \tag{4.1.2}$$

is the monic Hermite polynomial with parameter t.

Proof. According to Ito formula

$$\frac{\tilde{H}_{n+1}(B_t;t)}{n+1} = \frac{\tilde{H}_{n+1}(B_t;t)}{n+1} \mid_{t=0} + \int_0^t \frac{\partial}{\partial s}\frac{\tilde{H}_{n+1}(B_s;s)}{n+1} ds$$

$$+ \int_0^t \frac{\partial}{\partial B_s}\frac{\tilde{H}_{n+1}(B_s;s)}{n+1} dB_s + \frac{1}{2}\int_0^t \frac{\partial}{\partial B_s^2}\frac{\tilde{H}_{n+1}(B_s;s)}{n+1} ds \tag{4.1.3}$$

$$\frac{\tilde{H}_{n+1}(B_t;t)}{n+1} \mid_{t=0} = \lim_{t\to 0} \frac{1}{n+1}\left(\frac{t}{2}\right)^{(n+1)/2} H_{n+1}(\frac{B_t}{\sqrt{2t}}) =$$

$$= \lim_{t\to 0} \frac{1}{n+1}\left(\frac{t}{2}\right)^{(n+1)/2} H_{n+1}(a_{n+1}\frac{B_t^{n+1}}{(2t)^{(n+1)/2}} + \cdots) = 0$$

because of continuity of Brownian motion almost sure and $B_0 = 0$.

$$\frac{\partial}{\partial x}\frac{\tilde{H}_{n+1}(x;t)}{n+1} = \frac{1}{n+1}(\frac{t}{2})^{(n+1)/2}(-1)^{n+1}\frac{\partial}{\partial x}[e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+1)}] =$$

$$= \frac{1}{n+1}(\frac{t}{2})^{(n+1)/2}(-1)^{n+1}[\frac{x}{t}e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+1)} + \frac{1}{\sqrt{2t}}e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+2)}] =$$

$$= \frac{1}{n+1}\frac{x}{\sqrt{2t}}(\frac{t}{2})^{n/2}(-1)^{n+1}e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+1)} - \frac{1}{n+1}(-1)^{n+2}\frac{1}{2}(\frac{t}{2})^{n/2}e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+2)}$$

$$= (\frac{t}{2})^{n/2}\frac{1}{2(n+1)}[\frac{2x}{\sqrt{2t}}H_{n+1}(\frac{x}{\sqrt{2t}}) - H_{n+2}(\frac{x}{\sqrt{2t}})] = \tilde{H}_n(x;t).$$

Here we used the recurrence identity

$$H_n^{''}(x) - 2xH_n^{'}(x) + 2nH_n(x) = 0$$

it follows from

$$\frac{\partial^2}{\partial x^2}\frac{\tilde{H}_{n+1}(x;t)}{n+1} = n\tilde{H}_{n-1}(x;t).$$

$$\frac{\partial}{\partial t}\frac{\tilde{H}_{n+1}(x;t)}{n+1} = \frac{(-1)^{n+1}}{n+1}\frac{\partial}{\partial t}[(\frac{t}{2})^{(n+1)/2}e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+1)}] =$$

$$= \frac{(-1)^{n+1}}{n+1}[\frac{n+1}{4}(\frac{t}{2})^{(n-1)/2}e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+1)} + (\frac{t}{2})^{(n+1)/2}(-\frac{x^2}{2t})e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+1)} +$$

$$+(\frac{t}{2})^{(n+1)/2}e^{\frac{x^2}{2t}}(e^{-\frac{x^2}{2t}})^{(n+2)}(-\frac{x}{2t\sqrt{2t}})] = -\frac{n}{2}\tilde{H}_{n-1}(x;t).$$

Comparing the expression just obtained with the result of

$$P_n(x) = c_n\frac{1}{h(x)}[h(x)B^n(x)]^{(n)}, n = 0, 1, 2, \cdots,$$

and taking this into account formula (4.2.3) we'll finally obtain (4.1.1) and (4.1.2).

■

Note that the monic Hermite polynomials $\tilde{H}_n(x;t)$ are orthogonal with respect to the normal distribution $N(0,t)$, the distribution of B_t. Note also that because the stochastic integrals are martingales, we recover the results of section 4.1.3;1,i.e.,$\{\tilde{H}_n(B_t;t)\}$ are martingales.

The generating function of the monic Hermite polynomials $\tilde{H}_n(x;t)$ is given by

$$\sum_{n=0}^{\infty}\tilde{H}_n(x;t)\frac{z^n}{n!} = \exp(-tz^2/2 + zx). \quad (4.1.4)$$

Using the generating function (4.1.4) one can easily see that the role of the exponential function is now taken by the function

$$Y(B_t;t) = \sum_{n=0}^{\infty} \frac{\tilde{H}_n(B_t;t)}{n!} = \exp(-t/2 + B_t),$$

because we have,

$$\int_0^t Y(B_s, s) dB_s = Y(B_t, t) - Y(B_0, 0) = Y(B_t, t) - 1.$$

The transformation $Y(B_t, t)$ of Brownian motion is sometimes called geometric Brownian motion or the stochastic exponent of the Brownian motion. It plays an important role in the celebrated Black-Scholes option pricing model.

4.2 *Stochastic Integration with Respect to the Poisson Process and Charlier Polynomials*

Let $(T_n)_{n\geq 0}$ be a strictly increasing sequence of positive random variables. We always take $T_0 = 0$ a.s. Recall that indicator function $1_{\{t\geq T_n\}}$ is defined as

$$1_{\{t\geq T_n\}} = \begin{cases} 1, \text{if } t \geq T_n(\omega) \\ 0, \text{ if } \ t < T_n(\omega) \end{cases}$$

Definition 43 *The process $(N = N_t)_{0\leq t\leq\infty}$ defined by*

$$N_t = \sum_{n\geq 1} 1_{\{t\geq T_n\}}$$

with values in $\mathbb{N} \cup \{\infty\}$ where $\mathbb{N} = \{0, 1, 2, ...\}$ is called the counting process associated to the sequence $(T_n)_{n\geq 1}$.

If we set $T = sup_n T_n$, then

$$[T_n, \infty) = \{N \geq n\} = \{(t, \omega) : N_t(\omega) \geq n\}$$

as well as

$$[T_n, T_{n+1}) = \{N = n\},\ and\ [T, \infty) = \{N = \infty\}.$$

The random variable T is the explosion time of N. If $T = \infty$ a.s, then N is a counting process without explosions. For $T = \infty$, note that for $0 \leq s < t < \infty$ we have

$$N_t - N_s = \sum_{n \geq 1} 1_{\{s < T_n \leq t\}} \tag{4.2.1}$$

.

The increment $N_t - N_s$ counts the number of random times T_n that occur between the fixed times s and t.

As we defined a countly process it is not necessarily adapted to the filtration F.

Definition 44 *An adapted counting process N is a Poisson process if*

1. for any $s, t, 0 \leq s < t < \infty, N_t - N_s$ is independent of $\mathcal{F}_{s;}$
2. for any $s, t, v, u, 0 \leq s < t < \infty, 0 \leq v < u < \infty, t - s = v - u$, then the distribution of $N_t - N_s$ is the same as that of $N_v - N_u$.

Properties 1. and 2. are known respectively as increments independent of the past, and stationary increments.

Theorem 45 *Let N be a Poisson process. Then*

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!},\ n = 0, 1, 2, ...,\ for\ some\ \lambda \geq 0.$$

That is, N_t has the Poisson distribution with parameter λt. Moreover , N is continuos in probability and does not have explosions. (N is continuos in probability means that for $t > 0, lim_{u \to t} N_t = N_t$ where the limit is taken in probability)[Çinlar, E.,1975]

Definition 46 *The parameter λ associated to a Poisson process by Theorem 45 is called the intensity, or arrival rate, of the process.*

Corollary 47 *A Poisson process N with intensity λ satisfies*

$$E\{N_t\} = \lambda t$$

$$Variance(N_t) = Var(N_t) = \lambda t.$$

Theorem 48 *Let N be a Poisson process with intensity λ. Then $N_t - \lambda t$ and $(N_t - \lambda t)^2 - \lambda t$ are martingales.*

Proof. Since λt is non-random , the process $N_t - \lambda t$ has mean zero and independent increments. Therefore

$$E\{N_t - \lambda t - (N_t - \lambda t) \mid F_s\}$$

$$= E\{N_t - \lambda t - (N_t - \lambda t)\} = 0, for 0 \leq s < t < \infty.$$

The analogous statement holds for $(N_t - \lambda t)^2 - \lambda t$. ■

Definition 49 *Let H be a stochastic process. The natural filtration of H, denoted $\mathbb{F}^0 = (\mathcal{F}_t^0)_{0 \leq t < \infty}$, is defined by $\mathcal{F}_t^0 = \sigma\{H_s; s \leq t\}$. That is ,$\mathcal{F}_t^0$ is the smallest filtration that makes H adapted.*

Definition 50 *Let $A = (A_t)_{t \geq 0}$ be a cadlag process. A is an increasing process if the paths of $A : t \to A_t(\omega)$ are non-decreasing for almost all ω. A is called a finite variation process (FV) if almost all of the paths of A are of finite variation on each compact interval of $\mathbb{R}_+$.*

Let A be an increasing process. Fix an ω such that $t \to A_t(\omega)$ is right continuos and non-decreasing. This function induces a measure $\mu_t(\omega, ds)$ on $\mathbb{R}_+$. If f is a bounded , Borel function on $\mathbb{R}_+$,

then

$$\int_0^t f(s)\mu_A(\omega, ds)$$

is well-defined for each $t > 0$.

We denote this integral by

$$\int_0^t f(s) dA_s(\omega).$$

If $F_s = F(s, \omega)$ is bounded and jointly measurable, we can define , $\omega - by - \omega$, the integral

$$I(t,\omega) = \int_0^t F(s)dA_s(\omega).$$

I is right continuos in t and jointly measurable.

Proceeding analogously for A an FV process (except that the induced measure $\mu_n(\omega, ds)$ can have negative measure; that is, it is assigned measure), we can define a jointly measurable integral

$$I(t,\omega) = \int_0^t F(s,\omega)dA_s(\omega)$$

for F bounded and jointly measurable.

Let A_n be an FV process.

We define

$$|A|_t = \sup_{n\geq 1} \sum_{k=1}^{2^n} \left| A_{\frac{tk}{2^n}} - A_{\frac{t(k-1)}{2^n}} \right|. \quad (*)$$

Then $|A|_t < \infty$ a.s , and it is an increasing process.

Definition 51 *For A an FV process, the total variation process, $|A| = (|A|_t)_{t\geq 0}$, is the increasing process defined in * above .*

When the integrand process H has continuos paths , the Stieltjes integral $\int_0^t H_s dA_s$ is also known as the Reimann-Stieltjes integral (for fixed ω). In this case we can define the integral as the limit of approximating sums. Such a result is proved in elementary text books on Real Analysis.[Athreya K.B,Lahiri S.N,2006]

Theorem 52 *Let A be an FV process and let H be a jointly measurable process such that a.s. $s \longmapsto H(s,\omega)$ is continous. Let π_n be a sequence of finite random partitions of $[0,t]$ with $lim_{n\to\infty} mesh(\pi_n) = 0$. Then for $T_k \leq > S_k \leq T_{k+1}$,*

$$lim_{n\to\infty} \sum_{T_k, T_{k+1}\in \pi_n} H_{S_k}(A_{T_{k+1}} - A_{T_k}) = \int_0^t H_s dA_s \ a.s.$$

Theorem 53 *(Change of variables) Let A be an FV process with continuos paths , and let f be such that its derivative f' exists and is continuos. Then $(f(A_t))_{t\geq 0}$ is an FV process and*

$$f(A_t) - f(A_0) = \int_0^t f'(A_s)dA_s.$$

Example 54 *Let N be a Poisson process with parameter λ. Then $M_t = N_t - \lambda t$ the compensated Poisson process , is a martingale, as well as an FV process. For a bounded (say), jointly measurable process H, we have*

$$I_t = \int_0^t H_s dM_s = \int_0^t H_s d(N_s - \lambda s) = \int_0^t H_s dN_s - \lambda \int_0^t H_s ds = \sum_{n=1}^{\infty} H_{T_n} 1_{\{t \geq T_n\}} - \lambda \int_0^t H_s ds$$

where $(T_n)_{n\geq 1}$ are the arrival times of the Poisson process N.Now we suppose the process H is bounded , adapted, and has continuos sample paths. For $0 \leq s < t < \infty$, we then have

$$E\{I_t - I_s \mid F_s\} \Longrightarrow E\{\int_s^t H_u dM_u \mid F_s\} = E\{\lim_{n\to\infty} \sum_{t_k, t_{k+1} \in \pi_n} H_{t_k}(M_{t_{k+1}} - M_{t_k}) \mid F_s\}$$

$$= \lim_{n\to\infty} \sum_{t_k, t_{k+1} \in \pi_n} E\{E\{H_{t_k}(M_{t_{k+1}} - M_{t_k}) \mid F_{t_k}\} \mid F_s\} = 0$$

The interchange of limits can be justified by the Dominated Convergence Theorem. We conclude that the integral process I is a martingale. This fact, that the stochastic Stieltjes integral of an adapted, bounded, continuos process with respect to a martingale is again a martingale, is true in much greater generality.

The orthogonal polynomials with respect to Poisson distribution $P(t)$ are the Charlier polynomials and they have the generating function

$$\sum_{n=0}^{\infty} C_n(x;t)\frac{z^n}{n!} = e^z(1 - \frac{z}{t})^x.$$

We have the following theorem.

Theorem 55 $\int_{(0,t]} \tilde{C}_n(N_{s-}; s)dM_s = \frac{\tilde{C}_n(N_{t-};t)}{n+1}$, *with $\tilde{C}_n(X;t)$, the monic Charlier polynomial of degree n.*

Proof. We verify this problem by direct calculation. Using the generating functions of the Charlier polynomials, one can easily see that it is sufficient to prove

$$\int_0^t Y(N_{s-}, s, m)dM_s = \frac{Y(N_t, t, m) - 1}{\omega},$$

where

$$Y(X, t, m) = \exp(-t\omega)(1+\omega)^X = \sum_{m=0}^{\infty} \tilde{C}_n(X; t)\frac{\omega^m}{m!}.$$

Define τ_i the time of the ith jump in the Poisson process $\{N_t, t \geq 0\}$. For convenience we set $\tau_0 = 0$. Note that

$$N_{\tau_i -} = lim_{s\to\tau_i, s<\tau_i} \quad N_s = i-1, \quad i \geq 1.$$

So we have

$$\int_0^t Y(N_{s-}, s, m)dM_s$$

$$= \int_0^t e^{-s\omega}(1+\omega)^{N_{s-}}dN_s - \int_0^t e^{-s\omega}(1+\omega)^{N_{s-}}ds$$

$$= \sum_{i=1}^{N_t} e^{-\tau_i\omega}(1+\omega)^{N\tau_i -} - \sum_{i=1}^{N_t} \int_{(\tau_{i-1}, \tau_i]} e^{-s\omega}(1+\omega)^{N_{s-}}ds - \int_{(\tau_{N_t}, t]} e^{-s\omega}(1+\omega)^{N_{s-}}ds$$

$$= \sum_{i=1}^{N_t} e^{-\tau_i\omega}(1+\omega)^{i-1} - \sum_{i=1}^{N_t} \int_{(\tau_{i-1}, \tau_i]} e^{-s\omega}(1+\omega)^{i-1}ds - \int_{(\tau_{N_t}, t]} e^{-s\omega}(1+\omega)^{N_t}ds$$

$$= \sum_{i=1}^{N_t} e^{-\tau_i\omega}(1+\omega)^{i-1} - \sum_{i=1}^{N_t}(1+\omega)^{i-1}(\frac{e^{-\tau_i\omega} - e^{-\tau_{i-1}\omega}}{\omega}) - (1+\omega)^{N_t}(\frac{e^{-t\omega} - e^{-\tau_{N_t}\omega}}{\omega})$$

$$= \frac{e^{-t\omega}(1+\omega)^{N_t} - 1}{\omega}$$

$$= \frac{Y(N_t, t, m) - 1}{\omega}.$$

This proves the theorem. ■

The interpretation is that the monic Charlier polynomials are the counters for Ito's integral of the usual power

$$M_n^s = (N_s - s)^n = (\tilde{C}_1(N_s; s))^n, n \geq 0,$$

for the compensated Poisson process $\{M_t, t \geq 0\}$. The theorem also implies the results of Poisson Process Charlier; i.e.,the monic Charlier Polynomials $\{\tilde{C}_n(N_t; t)\}$ are martingales.

4.3 *Stochastic Characterization of Hermite Polynomials*

4.3.1 Introduction and Formulation of the Result

Let H_n be an Hermite polynomial (HP) of degree n defined by the formula

$$\frac{d^n}{dx^n}\exp(-\frac{x^2}{2}) = (-1)^n \exp(-\frac{x^2}{2})H_n(x).$$

Let $(W_t, t \geq 0)$ be a Wiener process and

$$\tilde{H}_n(W,t) = t^{\frac{n}{2}} H_n(t^{-\frac{1}{2}}W_t). \tag{4.3.1}$$

Characterization of Hermite polynomials on the class of orthogonal polynomials is the subject of many papers.

In this section, we give a characterization of Hermite polynomials by considering them as functions of a Wiener process will be given. The result belongs to Plucinska.[Plucinska, 1998]

We introduce the following notations: $E(\cdot)$ is the mean value ,$E(\cdot \mid W_t)$ is the conditional mean value with respect to W_t.We normalize W_t, i.e., we put

$$U_t = t^{-\frac{1}{2}}W_t.$$

Then $E(U_t) = 0, E(U_t^2) = 1$.

The aim of the this section is to prove the following theorem.

Theorem 56 *Let $h_n : \mathbb{R}^1 \not\subseteq\mapsto \mathbb{R}^1$ be an analytic function for every natural n. If for every path of the process $(W_t, t \geq 0)$*

$$E[h_n(U_t \mid W_s)] = (\frac{s}{t})^{\frac{n}{2}} h_n(U_s) \ for \ s < t, \tag{4.3.2}$$

then $h_n = c_n H_n$,where c_n are some constants, i.e., h_n is an Hermite poloynomial up to a constant factor. If, moreover,

$$h_n(1) = H_n(1), \tag{4.3.3}$$

then $h_n = H_n$, i.e., h_n is an Hermite polynomial($H_n(1)$ is given by (4.3.4)).

4.3.2 Auxiliary Results

By the definition of HP, we have

$$H_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r n!}{r!(n-2r)!2^r} x^{n-2r}. \tag{4.3.4}$$

By the properties of a Wiener process , we get

$$E(W_t^r \mid W_s) = \sum_{l=0}^{[r/2]} \frac{r!}{(r-2l)!l!2^l} W_s^{r-2l}(t-s)^l. \tag{4.3.5}$$

4.3.3 Proof of Theorem 56

Let us expand h_n in a Maclaurin series; this series is evidently uniformly convergent. Then, by (4.3.2) and (4.3.4) for $s < t$, we get

$$s^{n/2} h_n(U_s) = s^{n/2} \sum_{r=0}^{\infty} \frac{1}{r!} h_n^{(r)}(0) U_s^r = \sum_{r=0}^{\infty} \frac{1}{r!} h_n^{(r)}(0) W_s^{(n-1)/2} = t^{n/2} E[h_n(U_t) \mid W_s]$$

$$= t^{n/2} E[(\sum_{r=0}^{\infty} \frac{1}{r!} h_n^{(r)}(0) U_t^r) \mid W_s] = \sum_{r=0}^{\infty} \frac{1}{r!} h_n^{(r)}(0) t^{(n-1)/2} \sum_{l=0}^{[r/2]} \frac{r!}{(r-2l)!l!2^l} W_s^{r-2l}(t-s)^l$$

$$= \sum_{r=0}^{\infty} W_s^{2r} \sum_{l=r}^{\infty} h_n^{(2l)}(0) \frac{1}{(2r)!(l-r)!2^{l-r}} (t-s)^{l-r} t^{(n-2l)/2}$$

$$+ \sum_{r=1}^{\infty} W_s^{2r-1} \sum_{l=r}^{\infty} h_n^{(2l-1)}(0) \frac{1}{(2r)!(l-r)!2^{l-r}} (t-s)^{l-r} t^{(n-2l+1)/2} \tag{4.3.6}$$

The right-hand side and the left-hand side of (4.3.6) are series in W_s. We compare the coefficients of these series; in the obtained relations we compare the coefficients of $s^\alpha t^\beta$. It is easily follows from this comparison that

$$h_n^{(r)}(0) = 0 \; for \; r > n.$$

Therefore, $h_n(x)$ is a polynomial of degree n. Moreover, the coefficients of all the terms on the left side which the powers if s are not integer must vanish. In other words, the coefficients of these terms in which the power $(n-r)/2$ of x is not integer must vanish. Therefore, for even n

$$h_n^{(2r-1)}(0) = 0, \; r = 1, \cdots, n/2, \tag{4.3.7}$$

for odd n

$$h_n^{(2r)}(0) = 0, \; r = 0, \cdots, (n-1)/2. \tag{4.3.8}$$

Let us put

$$a_{nr} = \frac{1}{r!} h_n^{(r)}(0), \quad x = U_s = s^{-1/2} W_s, \quad y = t/s.$$

Now we shall consider the following two cases:

1.$n = 2m(even)$
2.$n = 2m + 1(odd)$.

1.Taking into account (4.3.6)-(4.3.8) and changing the order of summation, we have

$$\sum_{r=0}^{m} a_{2m,2r} x^{2r} = \sum_{r=0}^{m} a_{2m,2r} y^{m-r} \sum_{l=0}^{r} \frac{(2r)!}{(2r-2l)!l!2^l} x^{2r-2l} (y-l)^l$$

$$= \sum_{l=0}^{m} x^{2l} \sum_{r=l}^{m} a_{2m,2r} \frac{(2r)!}{(2l)!(r-l)!2^l} (y-l)^{r-l} y^{m-r}. \tag{4.3.9}$$

Formula (4.3.9) must be satisfied for every x. Thus, all the coefficients of x^{2t} must be equal to zero. In other words, for $l = 0, 1, 2, \cdots, m$,

$$\sum_{r=l}^{m} a_{2m,2r} \frac{(2r)!}{(2l)!(r-l)!2^l} (y-l)^{r-l} y^{m-r} - a_{2m,2l} = 0. \tag{4.3.10}$$

It follows from (4.3.10) for $l = m$ that

$$a_{2m,2m} = a_{2m,2m}.$$

Let $a_{2m,2m}$ be a parameter. Thus (4.3.10) is a system of a linear equations in m unknown $a_{2m,2r}$, where $r = 0, 1, \cdots, m-1$. The determinant of system (4.3.10) is

$$(y^m - 1)(y^{m-1} - 1) \cdots (y - 1) \neq 0 as y \neq 1.$$

Therefore, there is a unique solution of (4.3.10). This solution can be found by the method of induction. From the first equation of system (4.3.10), i.e., for $l = m - 1$, we get

$$a_{2m,2m-2} = -m(2m-1)a_{2m,2m}.$$

We suppose that the successive solutions of (4.3.10) for $r = m, m-1, \cdots, l+1$ are

$$a_{2m,2r} = (-1)^{m-r}\frac{(2m)!}{(2r)!(m-r)!2^{m-r}}a_{2m,2m}. \tag{4.3.11}$$

We are going to show that

$$a_{2m,2l} = (-1)^{m-l}\frac{(2m)!}{(2l)!(r-l)!2^{m-l}}a_{2m,2m}. \tag{4.3.12}$$

By virtue of (4.3.11),(4.3.10) has the following form:

$$\sum_{r=l}^{m} a_{2m,2r}\frac{(2r)!}{(2l)!(r-l)!2^l}(y-1)^{r-l}y^{m-r} - a_{2m,2l}$$

$$= \sum_{r=l+1}^{m}(-1)^{m-l}\frac{(2m)!}{(2l)!(m-r)!2r}\frac{(2r)!(y-1)^{r-l}y^{m-r}}{(2l)!(r-l)!2^l}a_{2m,2m} + (y^{m-l}-1)a_{2m,2l}$$

$$= \frac{(2m)!}{(2l)!2^{m-l}}a_{2m,2m}\sum_{r=l+1}^{m}\frac{(y-1)^{r-l}(-y)^{m-r}}{(m-r)!(r-l)!} + (y^{m-l}-1)a_{2m,2l}$$

$$= (1 - y^{m-l})(\frac{(-1)^{m-l}(2m)!}{(2l)!2^{m-l}}a_{2m,2m} - a_{2m,2l}) = 0. \tag{4.3.13}$$

Formula (4.3.12) follows from immediately from (4.3.13). Taking into account (4.3.11) and (4.3.12), we get

$$h_{2m}(x) = \sum_{r=0}^{m}\frac{(-1)^{m-r}(2m)!}{(2r)!(m-r)!2^{m-r}}x^{2r}a_{2m,2m} = \sum_{r=0}^{m}\frac{(-1)^r(2m)!}{(r)!(2m-2r)!2^r}x^{2m-2r}a_{2m,2m}. \tag{4.3.14}$$

Therefore, (4.3.14) has the form (4.3.4) up to a constant.

2. For n odd, the reasoning is analogous and we shall not repeat it. After similar transformations for $n = 2m + 1$ we get

$$h_{2m+1}(x) = \sum_{r=0}^{m} \frac{(-1)^r (2m+1)!}{(r)!(2m+1-2r)!2^r} x^{2m+1-2r} a_{2m+1,2m+1.} \tag{4.3.15}$$

For optional natural n, (4.3.14) and (4.3.15) can be jointly written in the following form:

$$h_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r (n)!}{(r)!(n-2r)!2^r} x^{n-2r} a_{n,n.} \tag{4.3.16}$$

Thus, by write of (4.3.4) and (4.3.16), h_n is an Hermite polynomial up to a constant factor. If, moreover, (4.3.3) holds, then it is evident that $a_{n,n} = 1$ for every natural n. Theorem 56 is thus proved.

CHAPTER V

CONCLUSION

This thesis is devoted to connections between probability theory an orthogonal polynomials. The following were studied.

- Stochastic integration with applications in financial mathematics (review),
- Theory of orthogonal polynomials (review),
- Generating functions for orthogonal polynomials (review and original),
- Stochastic integrals and orthogonal polynomials (review).

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