

COMPATIBLE RELATIVE LEFSCHETZ FIBRATIONS ON ADMISSIBLE
RELATIVE STEIN PAIRS

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RELATIVE STEIN PAIRS**

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ABSTRACT

COMPATIBLE RELATIVE LEFSCHETZ FIBRATIONS ON ADMISSIBLE RELATIVE STEIN PAIRS

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For more than two decades it has been known that any compact Stein surface (of real dimension four) admits a compatible Lefschetz fibration over a two-disk. More recently, Giroux and Pardon have generalized this result by giving a complex geometric proof for the existence of compatible Lefschetz fibrations on Stein domains of any even dimension. As a preparatory step in proving the former, Akbulut and Özbağcı have shown that there exist infinitely many pairwise non-equivalent Lefschetz fibrations on the four-ball by using a result of Lyon constructing fibrations on the complements of (p, q) -torus links in the three- sphere. In this thesis, we first extend this result to obtain compatible Lefschetz fibrations on the six-ball whose pages are $(p, q, 2)$ -Brieskorn varieties, and then construct a compatible relative Lefschetz fibration on any Stein domain (of dimension six) which admits a certain (admissible) relative Stein pair structure. In particular, we provide a purely topological proof for the existence of Lefschetz fibrations on specific 6-dimensional Stein domains.

Keywords: Stein domain, Lefschetz Fibration, Contact Structure, Monodromy, Handle Decomposition

ÖZ

KABUL EDİLİR BAĞINTILI STEIN İKİLİLERİ ÜZERİNDE UYUMLU BAĞINTILI LEFSCHETZ LİFLEMELERİ

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Yirmi yılı aşkın bir süredir, herhangi bir kompakt Stein yüzeyinin (gerçek 4 boyutlu) iki disk üzerinde uyumlu bir Lefschetz liflemesine sahip olduğu bilinmektedir. Yakın zamanda Giroux ve Pardon, herhangi bir çift boyutlu Stein yüzeyinde uyumlu Lefschetz liflemelerinin varlığı için kompleks geometrik teknikleri kullandıkları bir kanıt sunarak bu sonucu genelleştirdiler. Akbulut ve Özbağcı, ilkinin kanıtlamak için, Lyon tarafından verilen üç küredeki (p, q) -torus linklerinin komplementi üzerinde liflemeler inşa eden sonucu kullanarak dört boyutlu diskin sonsuz sayıda çift yönlü eşdeğer olmayan Lefschetz liflemesinin var olduğunu gösterdiler. Bu tezde, önce bu sonucu, sayfaları $(p, q, 2)$ -Brieskorn varyeteleri olan altı disk için uyumlu Lefschetz liflemeleri inşa etmek için genişletildi, ve daha sonra belirli (kabul edilir) bir bağıntılı Stein çifti yapısını kabul eden herhangi bir Stein alanı (altı boyutlu) için uyumlu bir bağıntılı Lefschetz liflemeleri oluşturuldu. Özellikle, belirli 6-boyutlu Stein alanları üzerinde Lefschetz liflemelerinin varlığı için tamamen topolojik bir kanıt sunuldu.

Anahtar Kelimeler: Stein Alan, Lefschetz Liflemesi, Kontakt Yapı, Monodromi, Kulp Ayrışımı



To Arel

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CHAPTER 1

INTRODUCTION

Two centuries ago, symplectic geometry is first emerged in the Hamiltonian formulation of classical mechanics in which the phase space of specific classical systems has the structure of a symplectic manifold. A symplectic manifold is an even dimensional smooth manifold equipped with a nondegenerate closed 2-form, which is called a symplectic form. A fundamental result on symplectic manifolds, Darboux's Theorem, asserts that every $2n$ -dimensional symplectic manifold locally looks like an open subset of \mathbb{R}^{2n} with its canonical symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. It was shown by Weinstein that analogous result to the Darboux's theorem for symplectic manifolds provide standard neighborhood of special types of submanifolds of symplectic manifolds, e.g. Lagrangian submanifolds, [35].

From the mid 1980's, several breakthroughs have been done concerning the understanding and theory of symplectic manifolds. One of the most important developments is the introduction of the technique of pseudoholomorphic curves in symplectic manifolds. This technique was introduced by Gromov whose compactness theorem for the moduli spaces of pseudoholomorphic curves, enables the construction of strong invariants of symplectic manifolds, [20].

Other central geometric techniques are due to Gompf, who introduced several operations on symplectic manifolds: symplectic normal sum and cut/paste operations, [17]. These operations are used to construct new symplectic manifolds.

Donaldson has shown fundamental structural results about symplectic manifolds. Donaldson has proved that any closed symplectic manifold admits a symplectic submanifold of codimension two, whose complement is Weinstein, [8]. He has also

provided purely topological descriptions of symplectic manifolds by showing the existence of Lefschetz pencils on every closed symplectic manifold, [9]. A Lefschetz fibration is a fibration over surfaces with finitely many singular fibers. This concept was first introduced on complex manifolds by Lefschetz in order to study their topologies.

Symplectic Lefschetz fibration has also been proven effective in higher dimensions due to the contributions made by Seidel, [33]. In 2001, Akbulut and Özbağcı constructed compatible Lefschetz fibrations on Stein surfaces by using torus knots and handlebody theory, [2]. Afterwards, Plamenevskaya gave an overview of their construction, and proved that the open book induced by the Lefschetz fibration is compatible with the contact structure on the boundary of the Stein manifold, [32]. Later in 2017, Giroux and Pardon proved the existence of Lefschetz fibrations on arbitrary Stein domains, these are symplectically equivalent to Weinstein domains by [7], of all even dimensions by employing techniques from complex geometry, [16].

The main goal of this thesis is to give a topological proof for the existence of Lefschetz fibrations and to generalize (extend) the techniques and ideas of Akbulut and Özbağcı to dimension 6.

In the paper of Akbulut and Özbağcı, in order to construct Lefschetz fibrations, they used Eliashberg's characterization of compact Stein surfaces [11] and examined Stein surfaces case by case with respect to their handle decompositions. For completeness we recall their approach below:

First, consider the case when a given Stein surface X has no 1-handles. Consider a Legendrian link L with ℓ components in the contact boundary $S^3 = \partial D^4$ which is formed by the attaching circles of the Weinstein 2-handles of X . Put L into its square bridge position and apply Lyon's algorithm ([27]) to realize each component L_i of L ($i = 1, \dots, \ell$) on a Seifert surface of a (p, q) -torus link (a page of a compatible open book on (S^3, ξ_{std}^3)). Then by construction the framing $tb(L_i) - 1$ of each Weinstein 2-handle coincides with the page framing $lk(L_i, L_i^+) - 1$, and hence, each Weinstein 2-handle of X is indeed a Lefschetz 2-handle. For the general case, considering the Weinstein 1-handles of X as dotted circles, insert them into the previous case and

realize them on the Seifert surface constructed as well. Finally, observe that attaching a Stein 1-handle is equivalent to removing a disk near the boundary of each fiber and extending the monodromy with identity.

We generalize the topological construction of Akbulut and Özbağcı to real dimension six in a relative way. We first extend their result, the existence of Lefschetz fibrations on D^4 , to obtain a compatible Lefschetz fibration on D^6 whose regular fibers are $(p, q, 2)$ -Brieskorn varieties. For our purposes, we define relative structures: relative Stein pair, compatible relative Lefschetz fibrations and admissible relative Stein pair. A relative Stein pair (W, X) consists of a Stein domain W , and a Stein subdomain $X \subset W$ of codimension 2, for which there exists a Weinstein handle decomposition of W whose intersection with X produces a handle decomposition of the latter Stein subdomain. We are particularly interested in the case when the Weinstein pair is admissible, by which we mean that the attaching spheres used in the handle decomposition of W are obtained from the attaching spheres for X by doubling a certain Lagrangian disk fillings of the latter attaching spheres. Lagrangian spheres produced by doubling a Lagrangian disk filling in this manner are called suspensions. We prove that every admissible Stein pair admits a compatible relative Lefschetz fibration.

This work is organized into seven main chapters:

In Chapter 1, we give a literature review and the motivation of this work.

In Chapter 2, fundamental definitions, examples and theorems are given which appear as the keystones in the context of symplectic and contact topology. We give basic concepts: Stein and Weinstein domains, Lefschetz fibrations on Weinstein domains and their relation between contact manifolds and open book decompositions on contact manifolds. In addition, the concept of handle attachments in smooth and Weinstein category is given.

In Chapter 3, we introduce the relative structures which are the main objects of the thesis. We define *relative handle decomposition* which lead us to a *relative Stein pair*. In here, we also introduce *relative Stein diagrams* both in Gompf form and Akbulut form. The vital concept of the thesis, *compatible relative Lefschetz fibration*, has been

introduced in this chapter as well. Moreover, a subclass of relative Stein pair which is called *admissible Stein pair* has been defined and main result of this work is also stated there.

In Chapter 4, we give the basics of Brieskorn varieties and Brieskorn manifolds. In Proposition 4.0.1, we prove that D^6 carries a compatible Lefschetz fibration with regular fibers $V_\epsilon(p, q, 2)$. In the proof of Proposition 4.0.1, we introduce the technique, called *suspension* which produces a Lagrangian sphere inside the fiber $V_\epsilon(p, q, 2)$ from a Lagrangian disk filling inside D^4 . Also, we give Example 4.0.4 for Proposition 4.0.1.

Chapter 5 is devoted to proving Proposition 5.0.1. There, we prove that one can realize a relative Legendrian 2-link satisfying certain additional assumptions in some Brieskorn variety $V_\epsilon(p, q, 2)$ which can be considered as a higher dimensional version of Lyon's result. Moreover, the notion of *suspendibility* of links is defined in Definition 5.0.7 which is a technical condition for the main results.

In Chapter 6, we prove the main theorem into three cases with the help of above chapters. In Proposition 6.0.2, we prove the existence of compatible relative Lefschetz fibrations on admissible Stein pairs constructed by relative 1- and relative 3-handles only. In Proposition 6.0.3, it is proven that a subcritical admissible Stein pair admits a compatible relative Lefschetz fibration. Here the property of being subcritical refers to the six-dimensional Stein domain. In Theorem 3.0.10, we examine the general case, i.e. when an admissible Stein pair consists of relative Weinstein handles of all indices.

In the last chapter (Chapter 7) we give explicit examples for the concepts and structures introduced in the thesis.

CHAPTER 2

BACKGROUND

In this chapter basic definitions and fundamental theorems will be given. For rigorous information we refer to [7], [28], [13] and [12].

2.1 Stein and Weinstein Domains

Definition 2.1.1. A *Stein manifold* is a properly embedded complex submanifold of some \mathbb{C}^N . A properly embedded complex submanifold of a Stein manifold is called a *Stein submanifold*. A Stein (sub)manifold is necessarily an open manifold of even dimensional.

In this thesis, we are interested in only the case of a compact subset of a Stein manifold and submanifold obtained by the intersection with a generic compact ball, for which the following equivalent characterization is given:

Definition 2.1.2. A *Stein domain* (W, J, ϕ) is a compact complex manifold of real dimension $2n$ with boundary which admits a complex structure J and a J -convex generalized Morse function $\phi : W \rightarrow \mathbb{R}$. A properly embedded complex submanifold $X \subset W$ equipped with the restricted structures $(J|_X, \phi|_X)$ is called a *Stein subdomain*.

Theorem 2.1.3. [11] Any real $2n$ -dimensional Stein domain admits a proper plurisubharmonic Morse function such that all critical points are of index $\leq n$.

The outcome of this theorem is that any real $2n$ -dimensional Stein domain admits a handle decomposition whose handles have indices less than or equal to n .

To define Weinstein domains, we need the following ingredients:

Definition 2.1.4. Let W be a real $2n$ -dimensional smooth manifold equipped with a closed nondegenerate 2-form ω , (i.e. $d\omega = 0$ and $\omega^n \neq 0$). Then the pair (W, ω) is called a *symplectic manifold*.

Example 2.1.5. Consider \mathbb{R}^{2n} with coordinates $(x_1, y_1, \dots, x_n, y_n)$ and the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. The 2-form ω is symplectic, called the *standard symplectic structure* on \mathbb{R}^{2n} , and so the pair $(\mathbb{R}^{2n}, \omega)$ is a symplectic manifold.

Definition 2.1.6. A symplectic manifold (W, ω) is called *exact* if the symplectic 2-form is exact, that is, ω is the exterior derivative of a 1-form λ .

Note that an exact symplectic manifold cannot be closed, as follows by Stoke's Theorem.

Definition 2.1.7. A *Liouville domain* is a pair (W, λ) where the manifold W is compact with boundary carrying a *Liouville structure* λ . The 1-form λ is the primitive of symplectic 2-form ω and χ is the ω dual vector field of λ which is defined by the equation $\mathcal{L}_\chi \omega = \omega$. The 1-form λ is called a *Liouville form* such that $\lambda|_{\partial W}$ is a contact form on the boundary of W . Also, the orientation on the boundary of W given by $\lambda \wedge (d\lambda)^{n-1}|_{\partial W}$ matches its orientation as the boundary of the symplectic manifold (W, ω) . The vector field χ is called a *Liouville vector field* which satisfies $\mathcal{L}_\chi \omega = \omega$. The Liouville vector field χ points outwards along the contact boundary ∂W .

Definition 2.1.8. A *Weinstein domain* (W, λ, ϕ) is a compact Liouville domain (W, λ) that carries a Morse function $\phi : W \rightarrow \mathbb{R}$ such that the Liouville vector field corresponding to λ is gradient-like for ϕ . A properly embedded submanifold $X \subset W$ equipped with the restricted structures $(\lambda|_{TX}, \phi|_X)$ is called a *Weinstein subdomain* if the restricted structures satisfy the above.

Example 2.1.9. Any Stein domain can be endowed with a Weinstein structure by setting $\lambda = -d^{\mathbb{C}}\phi$ as Liouville form, $\omega = -dd^{\mathbb{C}}\phi$ as symplectic form and $\chi = \nabla_{\phi}\phi$ as Liouville vector field for a generic plurisubharmonic function ϕ and so that any Stein subdomain thus naturally becomes a Weinstein subdomain.

Example 2.1.10. Consider $D^{2n} = \left\{ (x_1, y_1, \dots, x_n, y_n) \in \mathbb{C}^n \mid \sum_{i=1}^n x_i^2 + y_i^2 \leq 1 \right\}$

equipped with standard symplectic 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ and the Morse function $\phi(x_1, y_1, \dots, x_n, y_n) = \frac{1}{4} \sum_{i=1}^n (x_i^2 + y_i^2)$ such that Liouville vector field, radial vector field, $\chi = \frac{1}{2} \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i})$ is gradient like for ϕ . Thus, $(D^{2n}, \lambda = \frac{1}{2} \sum_{i=1}^n x_i dy_i - y_i dx_i, \phi)$ is a Weinstein domain.

Any Stein domain admits a Weinstein structure, which can be seen by choosing the J -convex function ϕ to be generic and choice of the Liouville form to be $-d^{\mathbb{C}}\phi$. The converse, that any Weinstein domain admits a Stein structure, was proven by Cieliebak and Eliashberg, [7].

2.1.1 Lefschetz Fibrations on Weinstein Domains

Ever since Donaldson showed the existence of compatible Lefschetz pencils on closed symplectic manifolds [9], the theory of Lefschetz fibrations is well-studied in several different settings. Here we merely focus on those that fiber over D^2 and whose total spaces are Weinstein domains. We refer to [9], [28] and [33] for more comprehensive definitions and facts about Lefschetz fibrations in different settings.

Definition 2.1.11. Let E be a compact $2n$ -dimensional manifold with corners equipped with an exact symplectic form $\omega = d\lambda$ such that the both faces of the boundary $\partial E = \partial_v E \cup \partial_h E$ are convex. A *compatible Lefschetz fibration* on a compact $2n$ -dimensional manifold with corners E is a holomorphic map $\pi : E^{2n} \rightarrow D^2$ for some choice of compatible complex structure on E , which satisfies the following conditions:

1. The map π has finitely many nondegenerate distinct critical values $s_1, \dots, s_k \in \text{int}(D^2)$, and there exists an unique critical point $r_j \in \pi^{-1}(s_j)$ for each $j = 1, \dots, k$, and compatible complex structure on E is integrable near the critical points,
2. Near each critical point r_j and the corresponding critical value s_j , there are local complex coordinate charts matching with the orientations of E and D^2 such that the map π locally has the form $\pi(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$,

3. The restriction of the map π to $E \setminus \pi^{-1}(\{s_1, \dots, s_k\})$ is a locally trivial fibration whose fibers are $(2n - 2)$ -dimensional exact symplectic manifolds with convex boundary for the restricted forms,
4. The restriction of the map $\pi|_{\partial_v E} : \partial_v E \rightarrow \partial D^2$ to the vertical boundary $\partial_v E = \pi^{-1}(\partial D^2)$ is a surjective smooth fiber bundle. Moreover, there is a neighborhood of the horizontal boundary $\partial_h E = \bigcup_{z \in D^2} \partial(\pi^{-1}(z))$ such that the restriction of π to this neighborhood is a product fibration $D^2 \times \mathcal{N}(\partial F)$ where F is a regular fiber of π and $\mathcal{N}(\partial F)$ is a neighborhood of ∂F . The restricted ω and λ to this neighborhood are sum of forms from the two factors. Also, $\pi|_{\partial_h E} : \partial_h E \rightarrow D^2$ are surjective fiber bundles.

The corners of E can be rounded to get a Liouville domain so that one can obtain an exact symplectic Lefschetz fibration on the Liouville domain over disk whose regular fibers are Liouville subdomains equipped with the restriction of λ . If E has a Weinstein structure endowed with a Lefschetz fibration whose fibers are $(2n - 2)$ -dimensional Weinstein subdomains, then by rounding the corners of E one can get a compatible Lefschetz fibration $\pi : W \rightarrow D^2$ on the Weinstein domain W such that each regular fiber is a Weinstein subdomain with the restricted structure.

Definition 2.1.12. The Weinstein subdomain mentioned above is called a *regular fiber* which is the preimage of a regular value under the smooth map π . The preimage of each critical value under the smooth map π is called a *singular fiber*.

Lemma\Definition 2.1.13. [33] A *vanishing path* is an embedded path $\gamma : [0, 1] \rightarrow \mathbb{C}$ that starts at a regular value and ends at a critical value. A *Lefschetz thimble* Δ_γ can be associated to each vanishing path, which is the unique embedded Lagrangian disk D^n that satisfies $\pi(\Delta_\gamma) = \gamma([0, 1])$ and $\pi(\partial\Delta_\gamma) = \gamma(0)$. The boundary L of the Lefschetz thimble is called the *vanishing cycle* of the vanishing path γ . The monodromy about any critical value of a Lefschetz fibration is described by a Dehn twist τ_L along the corresponding vanishing cycle.

Example 2.1.14. Consider real $2n$ -dimensional affine variety A_k (the so-called Milnor fiber)

$$A_k = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \dots + z_{n-1}^2 + z_n^{k+1} = 1\}$$

which can be described as k -many linear plumbings of T^*S^n . The map $\pi : A_k \rightarrow D^2$ with projection onto the last coordinate z_n is a compatible Lefschetz fibration on the affine variety A_k with regular fibers T^*S^{n-1} and monodromy $(k + 1)$ -many right handed Dehn twists along the zero section of T^*S^{n-1} .

In dimension 4, Akbulut and Özbağcı proved the following:

Theorem 2.1.15. [2] *A compact Stein surface admits infinitely many pairwise nonequivalent Lefschetz fibrations.*

In 2004, Plameveskaya improved this result by showing that the Lefschetz fibration can be taken to be symplectic, in the sense that the handle decomposition induced by the Lefschetz fibration consists of standard Weinstein handles, [32].

2.2 Contact Manifolds

Definition 2.2.1. A *contact manifold* (M^{2n-1}, ξ) is a smooth manifold M equipped with a completely non-integrable hyperplane distribution $\xi \subset TM$ that is locally of the form $\xi = \text{Ker}(\alpha)$ for a 1-form α on M that satisfies the *contact condition* $\alpha \wedge (d\alpha)^{n-1} \neq 0$. The 1-form α is called a *contact form* and the hyperplane distribution ξ is called a *contact structure*. In particular, we have a global identity $\xi = \text{Ker}(\alpha) \Leftrightarrow \xi$ is co-orientable, i.e. $\frac{TM}{\xi}$ is trivial. Observe that the same contact structure ξ also can be given by $f\alpha$ for some smooth function $f : M \rightarrow \mathbb{R} \setminus \{0\}$.

Example 2.2.2. Consider the unit sphere S^{2n-1} in \mathbb{R}^{2n} with coordinates $(x_1, y_1, \dots, x_n, y_n)$ equipped with *the standard contact structure* $\xi_{std}^{2n-1} = \text{Ker}(\alpha_{std}^{2n-1})$ which is given by the contact 1-form $\alpha_{std}^{2n-1} = \sum_{i=1}^n x_i dy_i - y_i dx_i$. Therefore the pair $(S^{2n-1}, \xi_{std}^{2n-1})$ is a contact manifold.

Definition 2.2.3. The *Reeb vector field* R induced by the contact form α is uniquely defined by the conditions:

- (i) $d\alpha(R, -) \equiv 0$,
- (ii) $\alpha(R) \equiv 1$.

Definition 2.2.4. A k -dimensional submanifold L of (M^{2n-1}, ξ) is *isotropic* if $T_p L \subset \xi_p$ for all $p \in L$, and an $(n-1)$ -dimensional isotropic submanifold of (M^{2n-1}, ξ) is said to be *Legendrian*. Necessarily, $k \leq n-1$ holds by contact condition.

If L is an isotropic submanifold, then $TL \subset (TL)^\perp$. The subbundle $(TL)^\perp$ of $\xi|_L$ is symplectically orthogonal to TL with respect to the symplectic bundle structure $d\alpha|_\xi$. Therefore, $(TL)^\perp$ is determined by the contact structure ξ , not on the defining 1-form α .

Definition 2.2.5. The *conformal symplectic normal bundle* of an isotropic submanifold L in the contact manifold M is defined to be the quotient bundle

$$CSN_M(L) := \frac{(TL)^\perp}{(TL)}$$

with the conformal symplectic structure given by $d\alpha$.

Observe that

$$TL \subset (TL)^\perp \subset \xi|_L \subset TM|_L$$

and so we get the following splitting of the normal bundle $N(L)$ of L in M which is defined to be $N(L) := \frac{TM|_L}{TL}$:

$$N(L) \cong \frac{TM|_L}{\xi|_L} \oplus \frac{\xi|_L}{(TL)^\perp} \oplus CSN_M(L).$$

If $\dim(M) = 2n-1$ and $\dim(L) = k \leq n-1$, then $\text{rank}((TL)^\perp) = 2n-2-k$, $\text{rank}(\frac{TM|_L}{\xi|_L}) = 1$, $\text{rank}(\frac{\xi|_L}{(TL)^\perp}) = k$ and $\text{rank}(CSN_M(L)) = 2(n-1-k)$.

The bundle $\frac{TM|_L}{\xi|_L}$ is generated by the restriction of Reeb vector field R of α to L and $\frac{\xi|_L}{(TL)^\perp}$ is isomorphic to the cotangent bundle T^*L as a symplectic vector bundle.

Proposition 2.2.6. [13] The bundle $\frac{\xi|_L}{(TL)^\perp}$ is isomorphic to $J(TL)$ where J is a complex bundle structure on ξ compatible with the symplectic bundle structure inherited by $d\alpha$ and the bundle $CSN_M(L)$ is isomorphic to $(TL \oplus J(TL))^\perp$ as a conformal

symplectic bundle. Therefore, we have the following splitting of the normal bundle of L in M :

$$N(L) \cong \langle R|_L \rangle \oplus J(TL) \oplus (TL \oplus J(TL))^\perp.$$

Example 2.2.7. Let (M, ξ) be the contact manifold given in Example 2.2.2. Let $L \subset M$ be the k -dimensional sphere given by the equation $x_0^2 + \dots + x_k^2 = 1$ and all other coordinates zero. The standard complex structure on \mathbb{R}^{2n} is determined by $J(\partial x_i) = \partial y_i$, $i = 0, \dots, n-1$, preserves the contact distribution ξ_{std}^{2n-1} and is compatible with $d\alpha_{std}^{2n-1}$. The Reeb vector field R of α_{std}^{2n-1} is given by

$$R = \sum_{i=0}^{n-1} (x_i \partial y_i - y_i \partial x_i),$$

which gives on L

$$R|_L = \sum_0^k x_i \partial y_i.$$

Thence, the bundle $\langle R|_L \rangle \oplus J(TL)$ is generated by $\langle \partial y_0, \dots, \partial y_k \rangle$ and $CSN_M(L)$ is generated by $\langle \partial x_{k+1}, \partial y_{k+1}, \dots, \partial x_{n-1}, \partial y_{n-1} \rangle$.

Stein (resp. Weinstein) domains carry contact structures on their boundaries induced by Stein (resp. Weinstein) structures, and the handles which construct Stein (resp. Weinstein) domains are attached along isotropic spheres.

2.2.1 Open Book Decomposition on Contact Manifolds

Existence of open book decompositions on closed and orientable 3-manifolds has been shown by Alexander, [3] and in the high dimensional case it has been shown by Lawson, [25].

Definition 2.2.8. An *open book* on a closed manifold M is a pair (B, f) where B is a codimension 2-submanifold of M with trivial normal bundle (called the *binding*) and the map $f : M - B \rightarrow S^1$ is a fibration such that $\overline{\partial f^{-1}(\theta)} = B$ for all $\theta \in S^1$. The closure of the fiber $F = \overline{f^{-1}(\theta)}$ is called the *page* over $\theta \in S^1$, and the *monodromy*

of (B, f) is the self-diffeomorphism $f : F \rightarrow F$ defined by the first return map of the flow of a lift of the vector field $\partial/\partial\theta$ to the pages.

The following relation between open books and contact structures was introduced by Giroux, [14]:

Definition 2.2.9. An open book (B, f) and the contact structure ξ on M are *compatible* if there exists a contact form α for ξ such that α is a contact form on B , $d\alpha$ defines a symplectic structure on each page, and the orientation on B induced from α agrees with the one induced from $d\alpha$ considering B as the contact boundary of a symplectic page.

In dimension 3, Giroux proved:

Theorem 2.2.10. [14] *Any contact 3-dimensional manifold admits a compatible open book decomposition.*

In high dimensions, Giroux and Mohsen proved:

Theorem 2.2.11. [15] *Any compact contact manifold admits a compatible open book decomposition with Weinstein pages.*

Any compatible Lefschetz fibration on a Stein (or Weinstein) domain induces a compatible open book decomposition on the contact boundary. The fibers over $S^1 = \partial D^2$ give the pages of the open book. The *(global) monodromy* of a Lefschetz fibration is defined to be the monodromy of the boundary open book.

2.3 Handle Attachments in Smooth and Weinstein Category

Handlebody theory is proven to be very useful in order to understand smooth manifolds in general. For comprehensive discussion, we refer to [30], [19], [1], [18] and [36].

2.3.1 Smooth Category

Definition 2.3.1. An n -dimensional k -handle is a copy of the product manifold $D^k \times D^{n-k}$ which is attached along to the boundary of an n -dimensional manifold M via an embedding $\psi : \partial D^k \times D^{n-k} \rightarrow \partial M$.

The corners which might appear after attaching handles can be smoothen in a canonical way: In dimension 2, it is straightforward, by changing an angular corner by a region below a hyperbola. In arbitrary dimensions, same process can be applied by taking the multiplication of this 2-dimensional model with the extra dimensions, [19].

Definition 2.3.2. The map ψ is called the *attaching map*, the *index* of the handle is the number k and the *attaching region* is $\partial D^k \times D^{n-k}$. The *core* of the handle is the part $D^k \times \{0\}$ and the part $\{0\} \times D^{n-k}$ of the handle is called the *cocore* of the handle. The image $\psi(\partial D^k \times \{0\})$ is called the *attaching sphere*, $\{0\} \times \partial D^{n-k}$ is called the *belt sphere*.

A Morse function can be always found for a manifold M that induces a handle decomposition so that handles are attached in the increasing order in terms of indices to the boundary of the previous one, [29], [30]. With this process, we decompose M into a *handlebody*. For a closed and connected n -dimensional manifold, it can be given such a structure with a unique index 0-handle and unique index n -handle.

Tubular Neighborhood Theorem states that an embedding $\psi : \partial D^k \times D^{n-k} \rightarrow \partial M$ can be obtained from an embedding $\psi_0 : \partial D^k \times \{0\} \rightarrow \partial M$ together with an identification η of the normal bundle of the image of ψ_0 with $\partial D^k \times \mathbb{R}^{n-k}$, [21]. The attaching map ψ is determined by this data, providing $\psi|_{\partial D^k \times \{0\}} = \psi_0$ and $d\psi|_{\{0\} \times T_0 D^{n-k}} = \eta^{-1}$, up to isotopy. Therefore, handle attachments are specified by the following informations:

- An embedding $\psi_0 : S^{k-1} \rightarrow \partial M$ with trivial normal bundle,
- A normal framing η of $\psi_0(S^{k-1})$.

In order to understand framings of an embedded sphere S^{k-1} in ∂M with trivial normal bundle, we fix a framing η_0 . By comparing any other framing η with η_0 , we get

an element of $GL(n - k)$ at each point of the embedded sphere S^{k-1} . Since the topological space $GL(n - k)$ deformation retracts onto $O(n - k)$ by the Gram-Schmidt process, we get an element of $\pi_{k-1}(GL(n - k)) \cong \pi_{k-1}(O(n - k))$. Therefore, the framings of an embedded sphere S^{k-1} is identified with $\pi_{k-1}(O(n - k))$.

- Example 2.3.3.**
1. An n -dimensional index 0-handle is attached to ∂M along $\partial D^0 \times D^n = \emptyset$. Thus, attaching an n -dimensional 0-handle is taking a disjoint union with n -dimensional ball D^n .
 2. An n -dimensional 1-handle is attached to ∂M along $\partial D^1 \times D^{n-1}$. The attaching sphere is $\partial D^1 \times \{0\}$ which is a pair of points. For $n \geq 2$, there exist only two framings on $\psi(\partial D^1 \times \{0\})$ since $\pi_0(O(n - 1)) = \mathbb{Z}_2$. The consequence of the framings will be result in two manifolds which can be distinguished by their orientability.
 3. An n -dimensional 2-handle is attached to ∂M along $\partial D^2 \times D^{n-2}$. For $n = 4$, the framings are parametrized by $\pi_1(O(2)) = \mathbb{Z}$ and the framings are parametrized by $\pi_1(O(n - 2)) = \mathbb{Z}_2$ for $n > 4$.
 4. A 6-dimensional index 3-handle has a unique framing since $\pi_2(O(3)) = 0$.
- By the work of Laudenbach and Poenaru, closed 4-manifolds are determined by their index 1- and 2-handles since 4-dimensional index 3-handles are attached uniquely, [26].

Another way of attaching a 4-dimensional 1-handle is via the so-called carving method due to Akbulut, [1]. The process of carving is defined as follows:

If the attaching sphere $\psi(S^{k-1} \times \{0\})$ of an n -dimensional k -handle of M , which is obtained by attaching the k -handle along the n -dimensional manifold N , bounds a disk in the boundary of N , then M can be constructed by excising an open tubular neighborhood of a suitable embedded disk $(D^{n-k-1}, \partial D^{n-k-1}) \subset (N, \partial N)$ from the boundary of N . We find the $(n - k - 1)$ -disk as follows: The attached k -handle can be cancelled by the n -dimensional $(k + 1)$ -handle and after attaching the n -dimensional $(k + 1)$ -handle we obtain the manifold N . If we revoke the $(k + 1)$ -handle attachment by removing the $(n - k - 1)$ -dimensional cocore disk of the $(k + 1)$ -handle, we obtain the manifold M . This cocore is the sought $(n - k - 1)$ -dimensional disk. Specifically,

for a connected 4-dimensional manifold M , a 1-handle attachment corresponds to carve out properly embedded disk D^2 from the unique 4-dimensional 0-handle.

2.3.2 Weinstein Category

The following description of a symplectic handlebody is due to Weinstein, [36]. We refer a symplectic handlebody as a Weinstein handlebody.

Consider \mathbb{R}^{2n} with coordinates $(x_1, y_1, \dots, x_n, y_n)$ equipped with the standard symplectic 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ and the following Liouville vector field for each $k = 0, 1, \dots, n$

$$\chi_k = \sum_{i=1}^{n-k} \left(\frac{x_i}{2} \frac{\partial}{\partial x_i} + \frac{y_i}{2} \frac{\partial}{\partial y_i} \right) + \sum_{i=n-k+1}^n \left(2x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i} \right)$$

which is the positive gradient vector field of ω -convex Morse function

$$\phi_k = \sum_{i=1}^{n-k} \left(\frac{1}{4} x_i^2 + \frac{1}{4} y_i^2 \right) + \sum_{i=n-k+1}^n \left(x_i^2 - \frac{1}{2} y_i^2 \right).$$

We obtain the following 1-form α_k by contracting ω with the vector field χ_k :

$$\alpha_k = \iota_{\chi_k} \omega = \sum_{i=1}^{n-k} \left(\frac{1}{2} x_i dy_i - \frac{1}{2} y_i dx_i \right) + \sum_{i=n-k+1}^n \left(2x_i dy_i + y_i dx_i \right).$$

Computing Lie derivative of ω with χ_k , we observe that χ_k is a Liouville vector field:

$$\mathcal{L}_{\chi_k} \omega = d(\iota_{\chi_k} \omega) + \iota_{\chi_k} (d\omega) = d(\alpha_k) + 0 = \omega.$$

Consider the stable manifold $E_-^k = \{x_1 = \dots = x_n = y_1 = \dots = y_{n-k} = 0\}$, and contact type hypersurface $M_- = \phi_k^{-1}(-1)$. Notice that pull back of 1-form α_k on the stable manifold E_-^k vanishes, and therefore the ascending sphere $S^{k-1} = E_-^k \cap M_-$ is an isotropic submanifold in the contact manifold $(M_-, \alpha_k|_{TM_-})$. In a similar fashion, consider the unstable manifold $E_+^{2n-k} = \{y_{n-k+1} = \dots = y_n = 0\}$ and the contact

type hypersurface $M_+ = \phi_k^{-1}(1)$. The intersection of the unstable manifold with the hypersurface is the descending sphere $S^{2n-k-1} = E_+^{2n-k} \cap M_+$ which is a submanifold of $(M_+, \alpha_k|_{TM_+})$.

Definition 2.3.4. A *Weinstein handle* in \mathbb{R}^{2n} is a region bounded by $\phi_k^{-1}(-1)$ and a region of the form $\{\|\mathbf{x}\| \leq \rho(\mathbf{y})\}$.

In [36], it is shown that one can choose the handle in such a way that it is transverse to the Liouville vector field χ_k .

The following is proven by Weinstein, [36].

Theorem 2.3.5. [36] *Let Y be an isotropic sphere in the contact manifold M with a trivialization of $CSN_M(Y)$ and \widetilde{M} be the manifold obtained from M by performing a surgery on Y with the canonical trivialization of its normal bundle induced by the trivialization of $CSN_M(Y)$. Then the elementary cobordism from M to \widetilde{M} obtained by a Weinstein handle attachment on the boundary component $M \times [0, 1]$ along a neighborhood of Y carries a symplectic structure and a Liouville vector field which is transverse to M and \widetilde{M} . The contact structure on \widetilde{M} coincides with the one of M away from a small neighborhood of the spheres where the surgery takes place.*

CHAPTER 3

RELATIVE STRUCTURES

In [16], Giroux and Pardon constructed Lefschetz fibrations on Stein (and so Weinstein) domains by using complex geometric techniques and the close relationship between Stein and Weinstein structures given in [7]. In this thesis, we give a topological proof of this result in dimension six for certain cases. To this end, we introduce the following relative structure.

Definition 3.0.1. A pair (W^6, X^4) is a *relative Stein pair* if W is a Stein domain, X is a Stein subdomain of W , and such that the induced pair (W, X) of symplectic manifolds admits a Weinstein structure in which the handles are relative (*relative handle decomposition*) in the following sense:

(0) The unique 0-handle D^4 of X is properly embedded in the unique 0-handle D^6 of W .

1. The (*relative*) 1-handles of W and X coincide in the following way:

They have the same number of 1-handles and the cocore of each 1-handle of X is properly embedded in the cocore of the corresponding 1-handle of W .

2. Each (*relative*) 2-handle attachment of W corresponds to a 2-handle attachment of X in the following way:

The attaching circle K of each 2-handle of W is an embedded Legendrian knot in the contact boundary ∂X_1 so that the framing of the 2-handle of X

is $tb(K) - 1$ where $X_1 = D^4 \cup 1$ -handles. Also, the cocore of each 2-handle of X is properly embedded in the cocore of the corresponding 2-handle of W .

3. Each (*relative*) 3-handle attachment of W corresponds to a 2-handle attachment of X in the following way:

An equator of the attaching sphere of each 3-handle of W is the attaching circle of the corresponding 2-handle of X . This equator is properly embedded in $\partial X_2 \cap \partial W_2$ where $X_2 = X_1 \cup 2$ -handles, $W_2 = W_1 \cup 2$ -handles, and $W_1 = D^6 \cup 1$ -handles.

The main idea of this definition is the following: While we create the Stein domain W handle by handle, the Stein domain X is evolving as a subdomain of W in the meantime. The Liouville vector field of W is hence tangent to X , with all critical points contained in X , and the index of the critical point in X being less than or equal to the index of the critical point in W . Moreover, the advantage of the conditions in Definition 3.0.1 is that “*admissible Stein pairs*” (see Definition 3.0.8), which form a subclass of relative Stein pairs, can be described by means of special diagrams which we call *relative Stein diagrams*. Such diagrams are, indeed, relative versions of standard Stein diagrams of Stein surfaces introduced by Gompf in [18].

Definition 3.0.2. A *relative Stein diagram* of a relative Stein pair (W, X) in the *standard form* (see Figure 3.1) is a diagram defined in the following way:

- The (common) r 1-handles of W and X are shown by r pairs of horizontal balls.
- There is a collection of Legendrian (isotropic in the boundary of the 6-dimensional manifold) horizontal distinguished dashed line segments corresponding to each pair of ball. A dashed line segment is drawn whenever the isotropic attaching circle of an index 2-handle of W crosses over the corresponding 1-handle.
- There is a collection of Legendrian (isotropic) horizontal distinguished solid line

segments corresponding to each pair of ball. A solid line segment is drawn whenever the isotropic equatorial circle of the Legendrian attaching sphere of an index 3-handle of W crosses over the corresponding 1-handle.

- There is a front projection of a framed *Legendrian (isotropic) tangle* (i.e., disjoint union of framed dashed and solid Legendrian (isotropic) knots and arcs) with endpoints meeting the distinguished dashed and solid segments so that they together form the isotropic attaching circles and the equatorial circles of the Legendrian attaching spheres of index 2- and 3-handles of W , respectively.

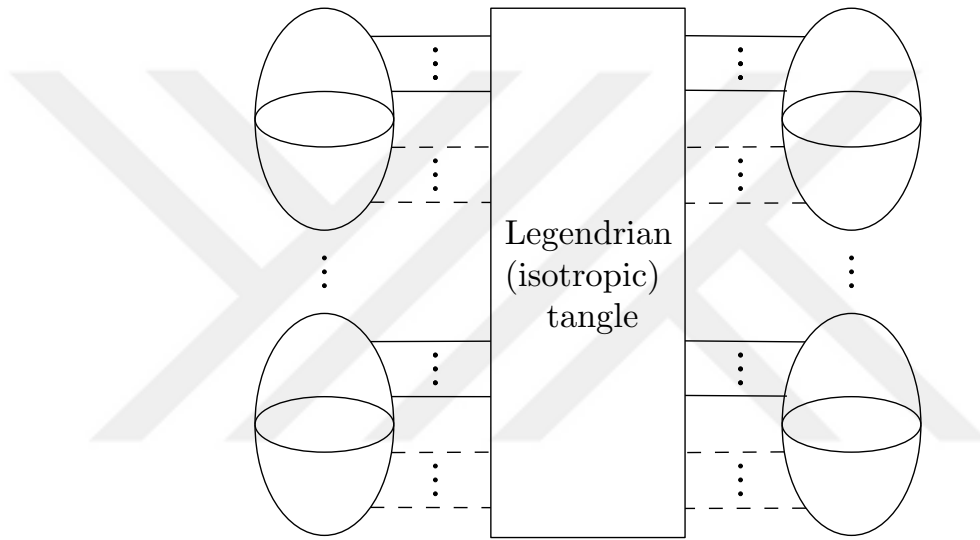


Figure 3.1: A relative Stein diagram of (W, X) in the standard (Gompf) form.

Remark 3.0.3. It will be explained (at the beginning of Chapter 6) that any admissible Stein pair (W, X) can be represented by a relative Stein diagram where the framing of each dashed/solid Legendrian (isotropic) attaching knot is one less than its Thurston-Bennequin framing in the contact boundary of $X_1 = D^4 \cup 1$ -handles, and hence, a relative Stein diagram in the standard form of the pair (W, X) is, indeed, a standard Stein diagram of the Stein surface X . Moreover, by representing 1-handles by dotted circles (Akbulut convention), (W, X) can be also described by a *relative Stein diagram in dotted form* as depicted in Figure 3.2.

Remark 3.0.4. In a relative Stein diagram (in dotted form) of (W, X) , by replacing dotted unknots with $(+1)$ -framed unknots one obtains a “*relative contact surgery*

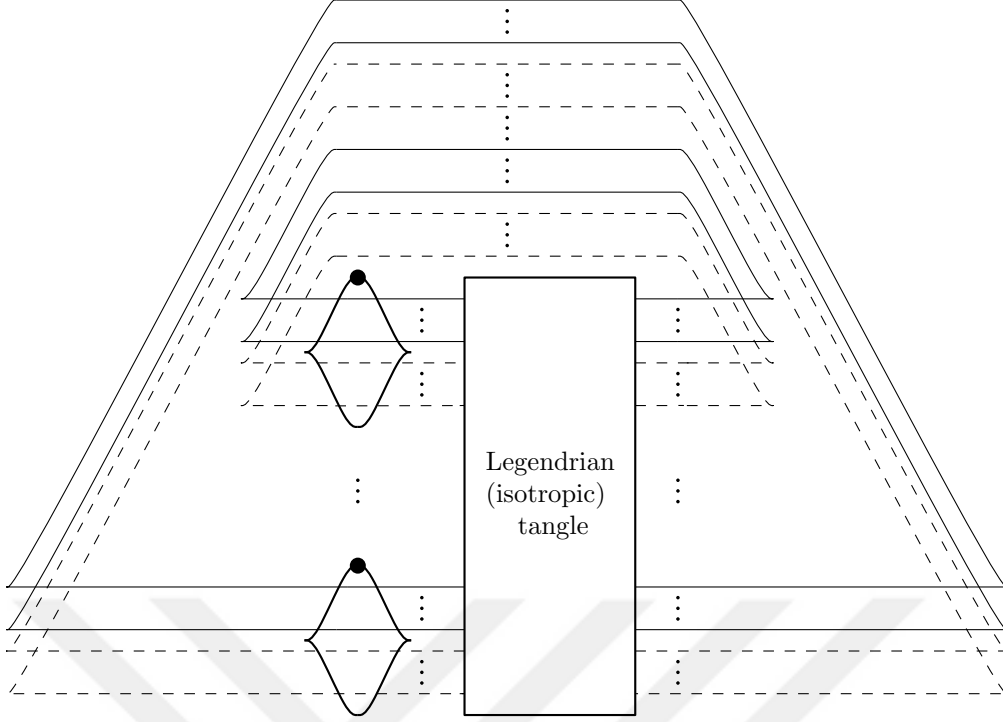


Figure 3.2: A relative Stein diagram in the dotted (Akbulut) form.

diagram” for the boundary *relative contact pair* $(\partial W, \partial X)$ (also see [5]). In particular, the empty relative Stein diagram describes the relative Stein pair (D^6, D^4) and the corresponding empty relative contact surgery diagram describes the boundary relative (*standard*) contact pair (S^5, S^3) equipped with $(\xi_{std}^5, \xi_{std}^3)$. In the case when W has no index 2- and 3-handles (i.e., when there are no dashed and solid knots and arcs), the relative Stein diagram (consisting of r dotted unknots only) describes the relative Stein pair $(\natural_r S^1 \times D^5, \natural_r S^1 \times D^3)$ where \natural denotes the boundary connected sum operation. In that case the corresponding relative contact surgery diagram (consisting of r $(+1)$ -framed unknots only) describes the relative contact pair $(\#_r S^1 \times S^4, \#_r S^1 \times S^2)$ equipped with their *standard* (Stein fillable) *contact structures* (η_r^5, η_r^3) .

Next, we define a class of Legendrian 2-links (also see [5]) below where we make use of the conventions $(\#_0 S^1 \times S^4, \#_0 S^1 \times S^2) = (S^5, S^3)$ and $(\eta_0^5, \eta_0^3) = (\xi_{std}^5, \xi_{std}^3)$.

Definition 3.0.5. Let $r \geq 0$ be any fixed integer. A *relative Legendrian 2-link* (L, K) in the relative contact pair $(\#_r S^1 \times S^4, \#_r S^1 \times S^2)$ equipped with (η_r^5, η_r^3) is a link of Legendrian 2-spheres L in $(\#_r S^1 \times S^4, \eta_r^5)$ whose isotropic equatorial link K is a

Legendrian 1-link in $(\#_r S^1 \times S^2, \eta_r^3) \subset (\#_r S^1 \times S^4, \eta_r^5)$.

Remark 3.0.6. By drawing its Legendrian (isotropic) equatorial 1-link K , any relative Legendrian 2-link (L, K) can be depicted in the relative contact surgery diagram of $(\#_r S^1 \times S^4, \#_r S^1 \times S^2)$ mentioned in Remark 3.0.4. (Just introduce a suitably drawn (solid) knot to the diagram which realizes the front projection of each component of K .)

In order to state our main result (see Theorem 3.0.10 below), one also needs two other notions introduced below. First, for a given relative Stein pair (W^6, X^4) , one can consider a Lefschetz fibration on W consistent with the one on X as follows:

Definition 3.0.7. A *compatible relative Lefschetz fibration* on a relative Stein pair (W^6, X^4) is a pair (π_W, π_X) where $\pi_W : W \rightarrow D^2$, $\pi_X : X \rightarrow D^2$ are compatible Lefschetz fibrations with regular fibers F_W^4 and F_X^2 , respectively, such that

1. F_X is a properly embedded subdomain of F_W ,
2. The set of critical values of π_W is contained in the set of critical values of π_X .

The main interest of the thesis is the following subclass of (relative) Stein pairs:

Definition 3.0.8. A pair (W, X) is called an *admissible Stein pair* if one has

$$W = \widetilde{W} \cup (\sqcup_{j=1}^{\ell} H_j) \quad \text{and} \quad X = \widetilde{X} \cup (\sqcup_{j=1}^{\ell} h_j)$$

where \widetilde{W} is a subcritical Stein 6-domain which splits as $\widetilde{W} = \widetilde{X} \times D^2$, and for each $j = 1, \dots, \ell$ the pair (H_j, h_j) is a relative 3-handle attached along a *suspendible* Legendrian 2-link.

Remark 3.0.9. *Suspendibility* of 2-links will be introduced in Definition 5.0.5 and Definition 5.0.7. For an admissible Stein pair (W, X) , its subcritical part $(\widetilde{W}, \widetilde{X})$ satisfies the conditions (0), (1) and (2) of Definition 3.0.1 (see the proof of Proposition 6.0.3 or Remark 5.0.6). Also by definition of admissibility, the condition (3) of Definition 3.0.1 is obviously satisfied. Hence, after attaching relative 3-handles to the relative Stein pair $(\widetilde{W}, \widetilde{X})$ we obtain a new relative Stein pair (W, X) . Therefore, admissible Stein pairs, indeed, form a special subclass of relative Stein pairs.

Finally, we state our main result:

Theorem 3.0.10. *Every admissible Stein pair (W^6, X^4) admits a compatible relative Lefschetz fibration (π_W, π_X) .*

Proof of Theorem 3.0.10 will be given in Chapter 6.



CHAPTER 4

BRIESKORN VARIETIES AS SEIFERT FILLINGS

Higher dimensional analogues of torus knots are Brieskorn manifolds. The *Brieskorn varieties* are affine varieties of the form

$$V_\epsilon(a_0, \dots, a_n) = \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{j=0}^n z_j^{a_j} = \epsilon \right\}.$$

The links of singular Brieskorn varieties at 0, that is, the sets of the form

$$\Sigma(a_0, \dots, a_n) = \left\{ (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{j=0}^n |z_j|^2 = 1 \quad \text{and} \quad \sum_{j=0}^n z_j^{a_j} = 0 \right\},$$

are called *Brieskorn manifolds*. Equivalently, Brieskorn manifolds $\Sigma(a_0, \dots, a_n)$ arise as the intersection $V_\epsilon(a_0, \dots, a_n) \cap S^{2n+1}$. On a Brieskorn manifold $\Sigma(a_0, \dots, a_n)$, an open book decomposition can be explicitly constructed as follows: Consider the map

$$\Psi : \Sigma(a_0, \dots, a_n) \rightarrow \mathbb{C}, \quad (z_0, \dots, z_n) \mapsto z_n.$$

Normalization of this map gives an open book with binding $B = \{z_n = 0\} = \Sigma(a_0, \dots, a_{n-1})$ and page $V_\epsilon(a_0, \dots, a_{n-1})$. (See [23] and [24] for more details.) In particular, on the Brieskorn manifold $\Sigma(p, q, 2, 1)$ (which is diffeomorphic to S^5), one can construct an open book by projecting onto the last coordinate. Thus, we obtain an open book of S^5 with binding $\Sigma(p, q, 2)$ and pages $V_\epsilon(p, q, 2)$.

In S^3 , every link bounds an orientable surface which is called a *Seifert surface*. By definition, the binding of an open book decomposition of S^3 is a fibered link and each page is a Seifert surface of the link. By adapting this terminology to fibered submanifolds of S^5 , the Brieskorn variety $V_\epsilon(a_0, \dots, a_{n-1})$ will be called a *Seifert*

filling of the Brieskorn manifold $\Sigma(a_0, \dots, a_{n-1})$. Note that $V_\epsilon(p, q, 2)$ is a Seifert filling of $\Sigma(p, q, 2)$.

In [24], the monodromy of an open book which arises from a fibered Brieskorn manifold has been studied. In particular, the monodromy of the open book on $S^5 = \Sigma(p, q, 2, 1)$ (discussed above) is the product of right-handed Dehn twists along certain 2-spheres embedded in the Seifert filling $V_\epsilon(p, q, 2)$. One can realize this open book as the boundary open book induced from a Lefschetz fibration on D^6 . More precisely, we have

Proposition 4.0.1. *For each (p, q) , there exists a compatible Lefschetz fibration on D^6 whose regular fibers are $V_\epsilon(p, q, 2)$. Moreover, the monodromy consists of the product of $(p - 1)(q - 1)$ right-handed Dehn twists along embedded Lagrangian 2-spheres in $V_\epsilon(p, q, 2)$ intersecting each other in an explicit pattern. In particular, the binding of the induced open book on S^5 is $\Sigma(p, q, 2)$.*

Remark 4.0.2. The explicit intersection pattern of the vanishing cycles (i.e., Lagrangian 2-spheres in the above statement) are, in fact, the generalization of the explicit pattern described for compatible Lefschetz fibrations on D^4 given in [2]. As we will explain in the proof of the proposition, such patterns are described by plumbing diagrams as we depict in Figure 4.1 for three particular cases.

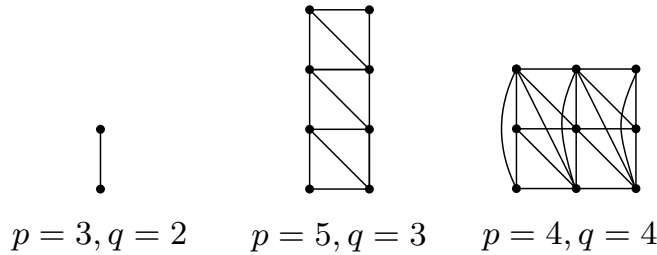


Figure 4.1: Intersection patterns for some particular cases.

Proof of Proposition 4.0.1. We begin with a compatible Lefschetz fibration on D^4 whose monodromy is determined by the torus link $\Sigma(p, q)$, and then extend it to a compatible Lefschetz fibration on D^6 whose monodromy is determined by the Brieskorn manifold $\Sigma(p, q, 2)$. Denote by (z_0, z_1) the complex coordinates on \mathbb{C}^2 .

The restriction of the map

$$\pi : \mathbb{C}^2 \rightarrow \mathbb{C}, \quad \pi(z_0, z_1) = z_0^p + z_1^q + M_\delta(z_0, z_1)$$

defines a compatible Lefschetz fibration on $D^4 \subset \mathbb{C}^2$. Here, $M_\delta(z_0, z_1)$ is a morsification of the polynomial $z_0^p + z_1^q$ where $\delta = (\delta_0, \delta_1)$ for sufficiently small δ_0, δ_1 . (Note that the Hessian of the polynomial $z_0^p + z_1^q$ is singular if $p > 2$ or $q > 2$ or both. However, Lefschetz fibrations have nondegenerate critical points by definition, so one needs to morsify the polynomial $z_0^p + z_1^q$.) We will consider the morsification $M_\delta(z_0, z_1) = -\delta_0 z_0 - \delta_1 z_1$ so that for δ_0 and δ_1 sufficiently small, one can easily check that the above map $\pi : D^4 \rightarrow D^2 \subset \mathbb{C}$ has nondegenerate critical points.

The map π has exactly $(p-1)(q-1)$ many nondegenerate distinct critical points, and is injective on the set of its critical points, and so, there are $(p-1)(q-1)$ many vanishing cycles one for each critical value of π . For a regular value $0 < \epsilon \in \mathbb{R} \subset \mathbb{C}$, the regular fiber $F = \pi^{-1}(\epsilon)$ is a Seifert surface $V_\epsilon(p, q)$ of the (p, q) -torus link $\Sigma(p, q)$. The monodromy $\varphi_{p,q}$ of the Lefschetz fibration is the product of $(p-1)(q-1)$ right-handed Dehn twists along Lagrangian circles [2].

Next we will consider the “suspension” $\tilde{\pi} : D^6 \rightarrow D^2$ of π . This is not a customary use of the word “suspension”, but it is the best option to give the idea presented here. However, its customary meaning will be referred when vanishing cycles are considered.

By considering the coordinates (z_0, z_1, z_2) on \mathbb{C}^3 , we extend π to obtain Lefschetz fibration

$$\tilde{\pi} : D^6 \rightarrow D^2, \quad \tilde{\pi}(z_0, z_1, z_2) = z_0^p + z_1^q + z_2^2 + M_\delta(z_0, z_1).$$

The map $\tilde{\pi}$ has nondegenerate distinct critical points. To see this, note that $\tilde{\pi}$ is obtained by “suspending” π by adding the term z_2^2 , and so the z_0 and z_1 coordinates of the critical points of the map $\tilde{\pi}$ are the same as those of π , and their z_2 coordinates are zero. Therefore, the critical points of $\tilde{\pi}$ are distinct. Also, one can easily check that the determinant of the Hessian of $\tilde{\pi}$ is nonzero at each critical point.

The map $\tilde{\pi}$ is injective on the set of its $(p-1)(q-1)$ nondegenerate distinct critical points, so it has $(p-1)(q-1)$ vanishing cycles corresponding to its critical values.

The regular fiber $\tilde{F} = \tilde{\pi}^{-1}(\epsilon)$ of $\tilde{\pi}$ is the Brieskorn variety $V_\epsilon(p, q, 2)$. Note that π and $\tilde{\pi}$ have the same morsification term and if we intersect the fiber \tilde{F} with $z_2 = 0$ plane, we obtain the fiber F . Also, we have $F \setminus \partial F \subset \tilde{F} \setminus \partial \tilde{F}$ and $\partial F = \partial \tilde{F} \cap F$ which means that the regular fibers of π is properly embedded in those of $\tilde{\pi}$. Moreover, as discussed above the critical points of $\tilde{\pi}$ are the same as those of π (with additional z_2 -coordinates). To understand the vanishing cycles of $\tilde{\pi}$, one can investigate the Lefschetz fibration

$$\Pi : \tilde{F} \rightarrow D^2, \quad \Pi(z_0, z_1, z_2) = z_2$$

on the fiber \tilde{F} . For each critical point of π , there are exactly two critical points of Π . So, for each vanishing cycle in $\pi^{-1}(0)$, there are two vanishing cycles in the fiber of Π . Since these two vanishing cycles coincide, one can form a matching sphere in \tilde{F} by joining the corresponding two critical points of Π . These matching spheres form a basis of vanishing cycles in \tilde{F} for the fibration $\tilde{\pi}$. They can be taken to intersect in the fiber $\tilde{\pi}^{-1}(0)$ according to the same graph as the vanishing cycles in $\pi^{-1}(0)$, (see [33]). In conclusion, the vanishing cycles in the fiber \tilde{F} of $\tilde{\pi}$ are these Lagrangian spheres which we call the suspensions of those in the fiber F of π . Hence, introducing a z_2^2 term extends (“suspends”) the Lefschetz fibration π on D^4 to the Lefschetz fibration $\tilde{\pi}$ on D^6 , and the monodromy $\tilde{\varphi}_{p,q}$ of $\tilde{\pi}$ is the monodromy of the fibered Brieskorn manifold $\Sigma(p, q, 2) \subset S^5$ which is the product of $(p-1)(q-1)$ right-handed Dehn twists along Lagrangian spheres which are the suspensions of the corresponding vanishing cycles of π . In particular, note that the binding $\Sigma(p, q)$ of the induced open book on S^3 is properly embedded in the binding $\Sigma(p, q, 2)$ of the induced open book on S^5 .

Finally, it is known (see [2]) that the $(p-1)(q-1)$ vanishing cycles of π (sitting on F) intersects each other according to a certain intersection pattern which can be described by a plumbing diagram as depicted (for several cases) in Figure 4.1. (A bold dot is drawn for each vanishing cycle and a line segment (or an arc) is drawn connecting two dots for each common point of the corresponding cycles.) Since the vanishing cycles of $\tilde{\pi}$ (sitting on \tilde{F}) are the suspensions of those of π , they intersect each other in the same way (and at the same points) that their equators intersect each other. Hence, the vanishing cycles of $\tilde{\pi}$ (there are exactly $(p-1)(q-1)$ of them)

intersect each other according to the same intersection pattern described by the same plumbing diagram. \square

Remark 4.0.3. As we realized above, the vanishing cycles (circles) of π on D^4 are the equators of the vanishing cycles (2-spheres) of $\tilde{\pi}$ on D^6 . Indeed, the above proof verifies that $(\tilde{\pi}, \pi)$ is a compatible relative Lefschetz fibration on the (trivial) Stein pair (D^6, D^4) equipped with their standard Stein structures. We will write the monodromy of $(\tilde{\pi}, \pi)$ as the pair $(\tilde{\varphi}_{p,q}, \varphi_{p,q})$.

Example 4.0.4. For $p = 3, q = 2$, consider the map $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$ defined by

$$\pi(z_0, z_1) = z_0^3 - \frac{z_0}{3^5} + z_1^2$$

where we take $M_\delta(z_0, z_1) = -\frac{z_0}{3^5}$ ($\delta_1 = 0$). One can check π restricts to a compatible Lefschetz fibration $\pi : D^4 \rightarrow D^2$ as follows: By (simultaneously) solving the equations

$$\frac{\partial \pi}{\partial z_0} = 0, \quad \frac{\partial \pi}{\partial z_1} = 0,$$

we see that $(-1/27, 0)$ and $(1/27, 0)$ are the critical points of π . By substituting these into π , we get the corresponding critical values -0.00020322105 and -0.00010161052 , respectively.

Then one needs to calculate the determinant of the Hessian matrix at the critical points. The Hessian matrix is given by (see [10] for a complete discussion):

$$\begin{bmatrix} \frac{\partial^2 \pi}{\partial z_0 \partial z_0} & \frac{\partial^2 \pi}{\partial z_0 \partial z_1} \\ \frac{\partial^2 \pi}{\partial z_1 \partial z_0} & \frac{\partial^2 \pi}{\partial z_1 \partial z_1} \end{bmatrix} = \begin{bmatrix} 6z_0 & 0 \\ 0 & 2 \end{bmatrix}$$

which has nonzero determinants at $(\pm 1/27, 0)$. Therefore, π has two nondegenerate critical points and two vanishing cycles corresponding to the critical values above. The regular fiber $F = \pi^{-1}(1) = \{z_0^3 - \frac{z_0}{3^5} + z_1^2 = 1\}$ is the plumbing of two left-handed Hopf bands $V_1(3, 2)$ (2-dimensional A_2 -Milnor fiber). The monodromy of the Lefschetz fibration is the product of two right-handed Dehn twists along the red and blue curves which is given in Figure 4.2.

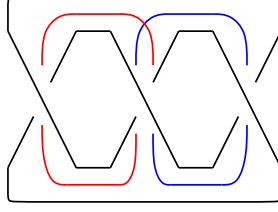


Figure 4.2: The regular fiber and the monodromy curves of the torus knot $\Sigma(3, 2)$ (trefoil).

Now, we extend (suspend) π to a compatible Lefschetz fibration $\tilde{\pi}$ on D^6 which is given by $\tilde{\pi}(z_0, z_1, z_2) = z_0^3 - \frac{z_0}{3^5} + z_1^2 + z_2^2$. Solving the equations

$$\frac{\partial \tilde{\pi}}{\partial z_0} = 0, \quad \frac{\partial \tilde{\pi}}{\partial z_1} = 0, \quad \frac{\partial \tilde{\pi}}{\partial z_2} = 0$$

(simultaneously), we see that $(-1/27, 0, 0)$ and $(1/27, 0, 0)$ are the critical points of $\tilde{\pi}$, and corresponding critical values (coinciding with those of π) are -0.00020322105 and -0.00010161052 , respectively. The Hessian matrix of $\tilde{\pi}$ is

$$\begin{bmatrix} \frac{\partial^2 \tilde{\pi}}{\partial z_0 \partial z_0} & \frac{\partial^2 \tilde{\pi}}{\partial z_0 \partial z_1} & \frac{\partial^2 \tilde{\pi}}{\partial z_0 \partial z_2} \\ \frac{\partial^2 \tilde{\pi}}{\partial z_1 \partial z_0} & \frac{\partial^2 \tilde{\pi}}{\partial z_1 \partial z_1} & \frac{\partial^2 \tilde{\pi}}{\partial z_1 \partial z_2} \\ \frac{\partial^2 \tilde{\pi}}{\partial z_2 \partial z_0} & \frac{\partial^2 \tilde{\pi}}{\partial z_2 \partial z_1} & \frac{\partial^2 \tilde{\pi}}{\partial z_2 \partial z_2} \end{bmatrix} = \begin{bmatrix} 6z_0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which has nonzero determinants at $(\pm 1/27, 0, 0)$, and so the map $\tilde{\pi}$ has two nondegenerate critical points and two vanishing cycles corresponding to the critical values. The regular fiber $\tilde{F} = \tilde{\pi}^{-1}(1) = \{z_0^3 - \frac{z_0}{3^5} + z_1^2 + z_2^2 = 1\}$ is the Brieskorn variety $V_1(3, 2, 2)$ (4-dimensional A_2 -Milnor fiber) into which the fiber $F = V_1(3, 2)$ of π is properly embedded. Finally, we note that the intersection pattern of the vanishing cycles of both π and $\tilde{\pi}$ is given in Figure 4.1-(a).

Remark 4.0.5. For any $n \geq 2$, one can generalize the above construction to obtain a compatible Lefschetz fibration on D^{2n} equipped with its standard Stein structure: For

a pair (p, q) , consider

$$\pi : \mathbb{C}^n \rightarrow \mathbb{C}, \quad \pi(z_0, z_1, z_2, \dots, z_{n-1}) = z_0^p + z_1^q + \sum_{i=2}^{n-1} z_i^2 + M_\delta(z_0, z_1)$$

where $M_\delta(z_0, z_1)$ is a suitable morsification term so that all critical points are nondegenerate. Then the restriction map $\pi : D^{2n} \rightarrow D^2$ defines a compatible Lefschetz fibration on some D^{2n} over a disk $D^2 \subset \mathbb{C}$. The regular fibers are diffeomorphic to the Brieskorn variety $V_\epsilon^{2n-2}(p, q, 2, \dots, 2)$, and the monodromy consists of the product of $(p-1)(q-1)$ many right-handed Dehn twists. In particular, the binding of the induced open book is the Brieskorn manifold $\Sigma^{2n-3}(p, q, 2, \dots, 2)$.





CHAPTER 5

SUSPENDED LEGENDRIAN 2-LINKS IN THE COMPLEMENT OF BRIESKORN MANIFOLDS

In [32], Plamenevskaya extended the result of Lyon [27]: For any Legendrian 1-link K in (S^3, ξ_{std}^3) , there exists a torus link $\Sigma(p, q)$ such that K is embedded in the interior of a Seifert surface of $\Sigma(p, q)$ which arises as the page of a compatible open book decomposition of S^3 . In this chapter, we adapt these results for relative Legendrian 2-links in (S^5, ξ_{std}^5) . More precisely,

Proposition 5.0.1. *Let $K = \sqcup_{j=1}^{\ell} K_j$ be a Legendrian 1-link in the standard contact manifold $(S^3, \xi_{std}^3) \subset (S^5, \xi_{std}^5)$ such that each K_j is the boundary of a Lagrangian 2-disk D^2 in $D^4 \subset \mathbb{C}^2$ which is cylindrical near the contact boundary. Then there exists a corresponding Legendrian 2-knot L_j in (S^5, ξ_{std}^5) for each K_j and a pair of Brieskorn manifolds $(\Sigma(p, q, 2), \Sigma(p, q))$ contactly embedded in (S^5, S^3) such that $\Sigma(p, q)$ is contactly embedded in $\Sigma(p, q, 2)$, and for each L_j and its equator $K_j \subset K$, the following are satisfied:*

- (i) L_j is embedded in $V_\epsilon(p, q, 2) \setminus \partial(V_\epsilon(p, q, 2))$ where $V_\epsilon(p, q, 2) \subset S^5$ is a Seifert filling of $\Sigma(p, q, 2)$.
- (ii) K_j is embedded in $\text{int}(V_\epsilon(p, q))$ where $V_\epsilon(p, q) \subset S^3$ is a Seifert surface of $\Sigma(p, q)$.
- (iii) The page framing of L_j in $V_\epsilon(p, q, 2)$ is equal to its contact framing in (S^5, ξ_{std}^5) .
- (iv) The page framing of K_j in $V_\epsilon(p, q)$ is equal to its contact framing in (S^3, ξ_{std}^3) .
- (v) $V_\epsilon(p, q)$ is a properly embedded Weinstein subdomain of $V_\epsilon(p, q, 2)$.

Proof. By [32], we may assume there exists a torus knot $\Sigma(p, q)$ such that the equatorial Legendrian 1-link $K \subset (S^3, \xi_{std}^3)$ is embedded in the interior of a Seifert surface $F := V_\epsilon(p, q)$ of $\Sigma(p, q)$. Also, since F is the page of a compatible open book (with binding $\Sigma(p, q)$) on (S^3, ξ_{std}^3) , the framing of each component of K in F is equal to its contact framing in (S^3, ξ_{std}^3) (see [2]). Therefore, (ii) and (iv) follow. By assumption, for each $j = 1, \dots, \ell$, there exists an embedded Lagrangian disk D_j in $D^4 \subset \mathbb{C}^2$ such that the boundary is the embedded Legendrian K_j in S^3 . In what follows, we construct the Lagrangian 2-knot L_j in the Brieskorn variety $V_\epsilon(p, q, 2)$ by lifting the disk D_j under a double branched cover. We will also make use of the fact that $V_\epsilon(p, q, 2)$ is obtained by suspending the Seifert surface $F = V_\epsilon(p, q)$ of K as observed in the proof of Proposition 4.0.1. Consider the map

$$f : V_\epsilon(p, q, 2) \rightarrow \mathbb{C}^2, \quad f(z_0, z_1, z_2) = (z_0, z_1)$$

where (z_0, z_1, z_2) are complex coordinates on \mathbb{C}^3 . This map is a two-fold cover branched along the fiber $\pi^{-1}(0)$ in \mathbb{C}^2 where π is the Lefschetz fibration given in the proof of Proposition 4.0.1. For every point except on the fiber $\pi^{-1}(0)$, there are two preimage points of f which are related by changing the sign on the z_2 -coordinate. A disk that is not intersecting the fiber $\pi^{-1}(0)$ lifts to two disjoint embedded disks in $V_\epsilon(p, q, 2)$. Since f is a holomorphic covering map away from $\pi^{-1}(0)$, we may assume that f is a symplectomorphism so that the two lifted disks are Lagrangian. If the disk has boundary equal to K_j in $\pi^{-1}(0)$ and a neighborhood of the boundary of the disk projects to a line $[0, \epsilon)$ in \mathbb{C} under π , then the two lifts of D_j can be glued to form a smooth Lagrangian sphere in $V_\epsilon(p, q, 2)$ whose equator projects to $(-\epsilon, \epsilon)$ under the Lefschetz fibration Π on $V_\epsilon(p, q, 2)$ also given in the proof of Proposition 4.0.1. Note that, if you take $(z^2 \circ \Pi)$ or $(\Pi)^2$ is equal to $\pi \circ f$. Therefore, (i) follows.

As explained in the proof of Proposition 4.0.1, $\tilde{F} = V_\epsilon(p, q, 2)$ and F are the regular fibers of a compatible relative Lefschetz fibration on the Stein pair (D^6, D^4) , so F is a Weinstein subdomain of \tilde{F} as claimed in (v). In particular, \tilde{F} is the page of the (induced) compatible open book on (S^5, ξ_{std}^5) , and so its boundary (the binding) $\Sigma(p, q, 2)$ is a contact submanifold of (S^5, ξ_{std}^5) and the binding $\Sigma(p, q)$ of the (induced) compatible open book on (S^3, ξ_{std}^3) is contactly embedded in $\Sigma(p, q, 2)$.

Finally, (iii) follows from Lemma 5.0.2 below which applies to all (odd) dimensions.

This concludes the proof of Proposition 5.0.1. \square

Lemma 5.0.2. *Let L be any Legendrian n -sphere on a page F^{2n} of a compatible open book on a contact manifold $(M^{2n+1}, \xi = \text{Ker}(\alpha))$. Then the page framing of L coincides with its contact framing.*

Proof. From compatibility we may assume that the Reeb vector field R of α (defined by the equations $\alpha(R) = 1$ and $\iota_R d\alpha = 0$) are transverse to both ξ and each page of the open book, and also that $d\alpha$ restricts to a symplectic form on both ξ and on F . Let J (resp. \tilde{J}) be an almost complex structure on the symplectic bundle ξ (resp. on the (exact) symplectic manifold F) compatible with $d\alpha|_\xi$ (resp. $d\alpha|_F$). Note at each point $p \in L$, we have $T_p L = \xi(p) \cap T_p F$. Observe $Jv \in \xi(p)/T_p L$ for every $v \in T_p L$ (otherwise $d\alpha|_\xi$ would not be symplectic). Therefore, the bundle $J(TL) \subset \xi|_L$ is a subbundle of the normal bundle $N(L)$ of L in M . $N(L)$ can be identified (by Legendrian neighborhood theorem) with $T^*L \oplus \epsilon$ where ϵ corresponds to the summand generated by the restriction $R|_L$ of the Reeb field. Since L is an embedded n -sphere, we obtain

$$N(L) = J(TL) \oplus \langle R|_L \rangle \cong T^*L \oplus \epsilon \cong T^*S^n \oplus \epsilon \cong S^n \times \mathbb{R}^{n+1}.$$

That is, a compatible almost complex structure J on $(\xi, d\alpha|_\xi)$ determines a trivialization $\Xi_J : S^n \times \mathbb{R}^{n+1} \rightarrow N(L) = J(TL) \oplus \langle R|_L \rangle$ which is known as the *contact framing* of L .

On the other hand, being Legendrian on a page of a compatible open book, L is Lagrangian in $(F, d\alpha|_F)$. So, at each $p \in L$, $\tilde{J}v \in T_p F/T_p L$ for every $v \in T_p L$ (as $d\alpha|_{TL} = 0$). Thus, the bundle $\tilde{J}(TL) \subset TF|_L$ is the normal bundle of L in F which can be identified (by Lagrangian neighborhood theorem) with T^*L . Similarly, considering ϵ as the bundle (of rank 1) generated by $R|_L$ (which is transverse to F), we obtain

$$N(L) = \tilde{J}(TL) \oplus \langle R|_L \rangle \cong T^*L \oplus \epsilon \cong T^*S^n \oplus \epsilon \cong S^n \times \mathbb{R}^{n+1}.$$

That is, a compatible almost complex structure \tilde{J} on $(F, d\alpha|_F)$ determines a trivialization $\Xi_{\tilde{J}} : S^n \times \mathbb{R}^{n+1} \rightarrow N(L) = \tilde{J}(TL) \oplus \langle R|_L \rangle$ which is known as the *page framing* of L .

It remains to show that the maps $\Xi_J, \Xi_{\tilde{J}} : S^n \times \mathbb{R}^{n+1} \rightarrow N(L) \subset M$ are isotopic as maps into M . Let N be a closed neighborhood of L in the page F . Consider the symplectic bundles $E_0 = \xi|_N \rightarrow N$, $E_1 = TF|_N \rightarrow N$ whose symplectic forms both descend from $d\alpha$. Compatibility condition is equivalent to the fact that, via a suitable isotopy through oriented $2n$ -plane fields on the (compact) subset $N \subset F$, the fibers of E_0 can be made arbitrarily close to the corresponding fibers of E_1 . Thus, there exists a homotopy of oriented $2n$ -plane bundles, say $E_t \rightarrow N$ ($t \in [0, 1]$), and its restriction $E_t|_L \rightarrow L$ defines a homotopy of oriented $2n$ -plane bundles between $\xi|_L \rightarrow L$ and $TF|_L \rightarrow L$. Since each bundle is oriented and $d\alpha$ is a volume form on fibers when $t = 0, 1$, it is a volume form on fibers of $E_t|_L \rightarrow L$ for all $t \in [0, 1]$, and so is of full rank which means that the Reeb field R of α is transverse to the fibers of $E_t|_L \rightarrow L$ for all t . Also since $TL = \xi|_L \cap TF|_L$, we may assume that TL is fixed during this homotopy. Thus, we have a homotopy of oriented $2n$ -plane bundles $J_t \rightarrow L$ ($t \in [0, 1]$) between the normal summands $J_0 = J(TL)$ and $J_1 = \tilde{J}(TL)$ of $\xi|_L$ and $TF|_L$, respectively. Using (following) J_t and the transversality of R to the fibers, we conclude that there exists an isotopy

$$\Xi_t : S^n \times \mathbb{R}^{n+1} \rightarrow N(L) = J_t \oplus \langle R|_L \rangle, \quad t \in [0, 1]$$

between $\Xi_0 = \Xi_J$ and $\Xi_1 = \Xi_{\tilde{J}}$ as claimed. \square

Remark 5.0.3. Note that, using the Reeb flow each component L_j constructed in Proposition 5.0.1 can be put in a different page $V_1(p, q, 2)$.

Definition 5.0.4. A relative link (L, K) in (S^5, S^3) will be called a *suspended 2-link based on K* if its each component L_j of L sits in a different page as constructed in Proposition 5.0.1.

Clearly, every suspended Legendrian 2-link is a relative Legendrian 2-link by its construction. However, whether every relative Legendrian 2-link in (S^5, S^3) is Legendrian isotopic to a suspended one is not clear in general. We introduce:

Definition 5.0.5. A relative Legendrian 2-link (L, K) in the standard contact pair (S^5, S^3) is said to be *suspendible* if it is Legendrian isotopic to a suspended Legendrian 2-link based on some Legendrian 1-link K' (possibly different than K) in the standard contact S^3 .

Remark 5.0.6. For the purpose of the thesis, one needs to define suspendibility of a relative Legendrian 2-link (L, K) embedded in the contact boundary of a subcritical Stein 6-domain. If \widetilde{W} is such a domain, then it can be written as a product $\widetilde{W} = \widetilde{X} \times D^2$ for some Stein 4-domain \widetilde{X} , and the pair $(\widetilde{W}, \widetilde{X})$ is a relative Stein pair because it clearly satisfies the conditions (0)-(2) of Definition 3.0.1. (The Weinstein handles of \widetilde{W} can be taken as the thickening of those of \widetilde{X}). Note that using a given Weinstein handle decomposition of \widetilde{X} , $(\widetilde{W}, \widetilde{X})$ can be described by a relative Stein diagram in the dotted form containing only dotted unknots (1-handles of \widetilde{X} , and so \widetilde{W}) and the framed dashed knots (2-handles of \widetilde{X} , and so \widetilde{W}). One should note that such a diagram is not unique since one can start with a different Weinstein handle decomposition of \widetilde{X} (still defining the same Stein structure on \widetilde{X} (and so on \widetilde{W}) up to Stein deformation). So, let $RSD_{\mathcal{H}}(\widetilde{X} \times D^2)$ be the relative Stein diagram (in the dotted form) of $\widetilde{W} = \widetilde{X} \times D^2$ obtained from a given Weinstein handle decomposition \mathcal{H} of \widetilde{X} . Recall that, in a diagram $RSD_{\mathcal{H}}(\widetilde{X} \times D^2)$, replacing dotted unknots with $(+1)$ -framed unknots, we obtain a relative contact surgery diagram, denoted by $CSD_{\mathcal{H}}(\partial(\widetilde{X} \times D^2))$, describing the contact boundary $\partial(\widetilde{X} \times D^2)$. Therefore, one can realize (L, K) in this surgery diagram by drawing the components of K as solid knots. Forgetting framings on the $(+1)$ -framed unknots and the framed dashed knots in the diagram, we obtain a Legendrian 1-link $K_{\mathcal{H}}$ (in the standard contact S^3) which consists of the above (unframed) unknots and (unframed) dashed knots. Note that after forgetting all the framings in the diagram $CSD_{\mathcal{H}}(\partial(\widetilde{X} \times D^2))$, the link (L, K) becomes a relative Legendrian 2-link in the standard contact pair (S^5, S^3) .

Under the setting and notation given in Remark 5.0.6, we define

Definition 5.0.7. A relative Legendrian 2-link (L, K) in the contact boundary $\partial(\widetilde{X} \times D^2)$ of a subcritical Stein 6-domain $\widetilde{W} = \widetilde{X} \times D^2$ is said to be *suspendible* if there exists a Weinstein handle decomposition \mathcal{H} of \widetilde{X} such that there exists a Legendrian isotopy in the complement $S^5 \setminus K_{\mathcal{H}}$ from (L, K) to a suspended Legendrian 2-link based on some Legendrian 1-link K' (possibly different than K) in the complement $S^3 \setminus K_{\mathcal{H}}$.



CHAPTER 6

PROOF OF THEOREM 3.0.10

We will first explain that any admissible Stein pair can be described by a relative Stein diagram as promised in Chapter 3. Then we will prove the theorem in three cases/steps: Proposition 6.0.2 (lack of 2-handles), Proposition 6.0.3 (lack of 3-handles), and the general case.

Let (W^6, X^4) be an admissible Stein pair. By definition they admit a relative handle decomposition such that the link L (along which the 3-handles of W are attached) and its equatorial link K (along which the corresponding 2-handles of X are attached) together form a relative Legendrian 2-link (L, K) in the contact boundary of the Stein 6-domain

$$W_2 = D^4 \cup \{1\text{-handles of } W\} \cup \{2\text{-handles of } W\}$$

which splits as $W_2 = \tilde{X} \times D^2$ where \tilde{X} is a Stein 4-domain. By assumption $(L, K) \subset \partial(\tilde{X} \times D^2)$ is suspendible, therefore, there exists a Weinstein handle decomposition \mathcal{H} of \tilde{X} such that we can isotope (L, K) (through Legendrian 2-links in $S^5 \setminus K_{\mathcal{H}}$) to some suspended Legendrian 2-link. Assuming Legendrian isotopy taking (L, K) to a suspended Legendrian 2-link has been already performed, we may assume that (L, K) is suspended.

In the diagram $RSD_{\mathcal{H}}(\tilde{X} \times D^2)$, suppose the 2-handles of \tilde{X} (and so \tilde{W} , and hence W) are attached along an isotropic (Legendrian) 1-link M . The 1-link M and the suspended 2-link (L, K) lie in the relative contact pair $(\#_r S^1 \times S^4, \#_r S^1 \times S^2)$ (equipped with (η_r^5, η_r^3)) which is the relative boundary of the relative Stein pair $(\natural_r S^1 \times D^5, \natural_r S^1 \times D^3)$ where r is the number of 1-handles of \tilde{X} (and so \tilde{W} , and

hence W). Thus, by first introducing r dotted unknots and then drawing the components of M (as dashed knots) and the components of K (as solid knots), one can form a relative Stein diagram of (W, X) as depicted in Figure 3.2.

One should note that the resulting relative Stein diagram describes the admissible Stein pair (W, X) uniquely up to Stein deformation. This is because of the fact that the construction (given in Proposition 5.0.1) of a suspended Legendrian 2-link depends only on the base knot, and that the suspended link (L, K) is Legendrian isotopic to the original relative Legendrian 2-link along which the relative 3-handles of W are attached. Hence, we have shown that

Lemma 6.0.1. *Every admissible Stein pair can be described via some relative Stein diagram. Conversely, every relative Stein diagram determines an admissible Stein pair which is unique up to Stein deformation.*

Proposition 6.0.2. *Every admissible Stein pair without relative 2-handles admits a compatible relative Lefschetz fibration.*

Proof. The relative Stein diagram of an admissible Stein pair (W, X) without relative 2-handles consists of r dotted unknots U_1, \dots, U_r (representing relative 1-handles) and the solid knots drawn for each component of K (the base of the suspended Legendrian 2-link (L, K) along which the relative 3-handles are attached) as discussed above. (Here, in particular, it is assumed that this relative Stein diagram of (W, X) is obtained from a diagram $RSD_{\mathcal{H}}(\tilde{X} \times D^2)$ for a suitable Weinstein handle decomposition \mathcal{H} .) First, we forget the dots on U_i 's and consider the entire diagram as a relative 2-link in the relative standard contact pair (S^5, S^3) . Then, by Proposition 5.0.1, there exists a pair of Brieskorn manifolds $(\Sigma(p, q, 2), \Sigma(p, q))$ in (S^5, S^3) such that (L, K) is properly embedded on their Seifert fillings $(V_\epsilon(p, q, 2), V_\epsilon(p, q))$ as Legendrian. Note that each U_i is also realized on $V_\epsilon(p, q) \subset V_\epsilon(p, q, 2)$. Moreover, by Proposition 4.0.1, there exist Lefschetz fibrations on D^6 and D^4 with regular fibers $F_{D^6} = V_\epsilon(p, q, 2)$ and $F_{D^4} = V_\epsilon(p, q)$, respectively, as depicted in Figure 6.1-(a). Recall (from Remark 4.0.3) that the monodromies of these fibrations are denoted by $\tilde{\varphi}_{p,q}$ and $\varphi_{p,q}$, respectively, which are the product of $(p-1)(q-1)$ many right-handed Dehn twists along explicit spheres.

Next, generalizing the idea used for D^4 in [2] to D^6 , we will extend the above compatible relative Lefschetz fibration on (D^6, D^4) to a compatible relative Lefschetz fibration on the relative Stein pair $(D^6 \cup 1\text{-handles}, D^4 \cup 1\text{-handles})$ as follows:

First, isotope each U_i near the binding $\Sigma(p, q)$ of the induced open book on the boundary $S^3 = \partial D^4$ as in [2] so that it becomes transverse to each page F_{D^4} . Once this is achieved, U_i will also transversally intersect (near the binding $\Sigma(p, q, 2)$) each page F_{D^6} of the induced open book on the boundary $S^5 = \partial D^6$ (thanks to the relative Lefschetz fibration structure). For each $i = 1, \dots, r$, attaching a 1-handle to D^4 corresponds to first pushing the interior of the Lagrangian spanning disk of U_i into D^4 and then removing a tubular neighborhood of this resulting disk. This is equivalent to puncture each fiber F_{D^4} . Since each fiber F_{D^4} is properly embedded in the corresponding fiber F_{D^6} , this will correspond to carving a small neatly embedded Lagrangian 2-disk in the fiber F_{D^6} . On the other hand, by the condition (1) in Definition 3.0.1, the cocore of each 1-handle of X is properly embedded in the cocore of the corresponding 1-handle of W . Therefore, simultaneously carving these small neatly embedded Lagrangian 2-disks from the fibers F_{D^6} corresponds to attaching the corresponding 6-dimensional relative 1-handle of W to D^6 . In other words, attaching each relative 1-handle of (W, X) to the Stein pair (D^6, D^4) is equivalent to carving a small Lagrangian disk pair (D^2, D^0) from each fiber pair $(F_{D^6}, F_{D^4}) = (V_\epsilon(p, q, 2), V_\epsilon(p, q))$. This is depicted in Figure 6.1-(b) where a single relative 1-handle is considered (for simplicity), and the ‘‘punctured’’ fibers (i.e., new fibers after carving disks) are denoted by $\overset{\circ}{F}_{D^6}$ and $\overset{\circ}{F}_{D^4}$.

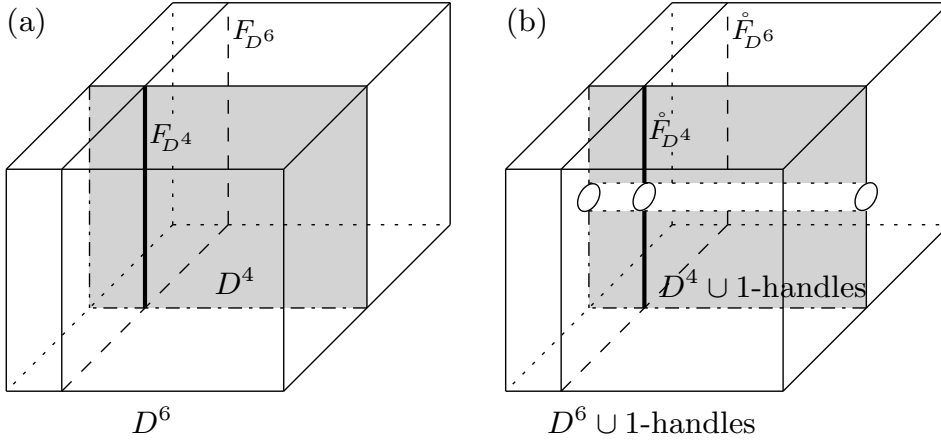


Figure 6.1: (a) Relative fibers (F_{D^6}, F_{D^4}) of the Stein Pair (D^6, D^4) , (b) Punctured relative fibers $(F_{D^6}^\circ, F_{D^4}^\circ)$ of the Stein Pair $(D^6 \cup 1\text{-handles}, D^4 \cup 1\text{-handles})$.

After attaching all relative 1-handles of (W, X) to (D^6, D^4) , we get a new compatible relative Lefschetz fibration on $(D^6 \cup 1\text{-handles}, D^4 \cup 1\text{-handles}) = (\natural_r S^1 \times D^5, \natural_r S^1 \times D^3)$ whose regular fibers are r times “punctured” $(V_\epsilon(p, q, 2), V_\epsilon(p, q))$ near boundaries, which we denote by

$$(F_{D^6}^\circ, F_{D^4}^\circ) = (V_\epsilon(p, q, 2), V_\epsilon(p, q)),$$

and whose monodromy is obtained by extending $\tilde{\varphi}_{p,q}$ and $\varphi_{p,q}$ (over the 1-handles) by identity.

Note that the suspended Legendrian 2-link (L, K) still sits on a relative page of the compatible relative open book on the boundary contact pair $(\#_r S^1 \times S^4, \#_r S^1 \times S^2)$ so that the framing conditions are still satisfied for the components of both L and K (as stated in Proposition 5.0.1).

Now assume that (L, K) has ℓ components, say (L_j, K_j) , $j = 1, \dots, \ell$. So, for each j , we attach a relative 3-handle (H_j, h_j) to $(\natural_r S^1 \times D^5, \natural_r S^1 \times D^3)$ along the relative Legendrian 2-knot (L_j, K_j) which is properly embedded in $(F_{D^6}^\circ, F_{D^4}^\circ)$. Each H_j (resp. h_j) is attached along a Legendrian L_j (resp. K_j) which lies on a Weinstein (so symplectic) page $F_{D^6}^\circ$ (resp. $F_{D^4}^\circ$) of the boundary (compatible) open book so that

the contact framing of L_j (resp. K_j) coincides with its page framing on \mathring{F}_{D^6} (resp. on \mathring{F}_{D^4}). Furthermore, recall that any Legendrian submanifold sitting on a page of a compatible open book is Lagrangian. Thus, we conclude that H_j (resp. h_j) is a six (resp. four) dimensional Lefschetz handle. Consequently, attaching each relative 3-handle corresponds to attaching a “relative” Lefschetz handle. So, attaching all such handles constructs a compatible relative Lefschetz fibration, say (π_W, π_X) , on the admissible Stein pair (W, X) . Note that the fibers of (π_W, π_X) are still $(\mathring{F}_{D^6}, \mathring{F}_{D^4})$. On the other hand, the monodromy of (π_W, π_X) is the pair

$$(\varphi_W, \varphi_X) = (\tilde{\varphi}_{p,q} \cdot \tau_{L_1} \tau_{L_2} \cdots \tau_{L_\ell}, \varphi_{p,q} \cdot \tau_{K_1} \tau_{K_2} \cdots \tau_{K_\ell})$$

where τ_{L_j} (resp. τ_{K_j}) is the right-handed Dehn twist along L_j (resp. K_j).

□

Proposition 6.0.3. *On every subcritical Stein 6-domain (no handles of critical index) there exists a relative Stein pair structure which admits a nontrivial compatible relative Lefschetz fibration with regular fibers $\mathring{V}_\epsilon(p, q, 2) \cup \bigcup_{j=1}^{2k} \tilde{h}_j$ for some p, q , where each \tilde{h}_j is a 4-dimensional 2-handle.*

Proof. Let \tilde{W} be a subcritical Stein 6-domain. By the result of Cieliebak [6], there is a splitting $\tilde{W} = \tilde{X} \times D^2$ where \tilde{X} is a Stein 4-domain. Starting from a Weinstein handle decomposition \mathcal{H} of \tilde{X} , one obtains a compatible Lefschetz fibration on \tilde{X} with fibers r times “punctured” $V_\epsilon(p, q)$ for some p, q (where r is the number of 1-handles of \tilde{X} in \mathcal{H}), and whose monodromy is that of the torus link $\Sigma(p, q)$ composed with right-handed Dehn twists corresponding to the Stein (Lefschetz) 2-handles of \tilde{X} in \mathcal{H} , as constructed in [2]. So, we have a (Lefschetz) fibration map $\pi_{\mathcal{H}} : \tilde{X} \rightarrow D^2$ which has finitely many nondegenerate critical points.

We can extend $\pi_{\mathcal{H}}$ to a fibration map on $\tilde{X} \times D^2$ by using the “suspension” technique, as in Proposition 4.0.1. More precisely, consider the following map

$$\tilde{\pi}_{\mathcal{H}} : \tilde{X} \times D^2 \rightarrow D^2, \quad (p, z_2) \mapsto \pi_{\mathcal{H}}(p) + z_2^2.$$

Critical points of $\tilde{\pi}_{\mathcal{H}}$ are given by the local equations

$$\frac{\partial \tilde{\pi}_{\mathcal{H}}}{\partial z_0} = \frac{\partial \pi_{\mathcal{H}}}{\partial z_0} = 0, \quad \frac{\partial \tilde{\pi}_{\mathcal{H}}}{\partial z_1} = \frac{\partial \pi_{\mathcal{H}}}{\partial z_1} = 0, \quad \frac{\partial \tilde{\pi}_{\mathcal{H}}}{\partial z_2} = 0$$

where (z_0, z_1) are the local complex coordinates around a critical point in \tilde{X} . So, the critical points of $\tilde{\pi}_{\mathcal{H}}$ are the same as those of $\pi_{\mathcal{H}}$ with the additional zero z_2 -coordinate. Also, if we check the Hessian of $\tilde{\pi}_{\mathcal{H}}$ at each critical point, we see that it is nondegenerate because the corresponding Hessian of $\pi_{\mathcal{H}}$ is nondegenerate. Each regular fiber of $\tilde{\pi}_{\mathcal{H}}$ can be seen as double branched cover of \tilde{X} which is branched over $\pi_{\mathcal{H}}^{-1}(c)$ where c is a regular value of $\tilde{\pi}_{\mathcal{H}}$. With this point of view, the effect of attaching k -many 4-dimensional 2-handles to the fibers of $\tilde{\pi}_{\mathcal{H}}$ will result in attaching $2k$ -many 2-handles to r -times punctured $V_{\epsilon}(p, q, 2)$. The monodromy of $\tilde{\pi}_{\mathcal{H}}$ is the product of right-handed Dehn twists along the spheres obtained by taking the suspensions of the corresponding vanishing cycles of $\pi_{\mathcal{H}}$. Therefore, $\tilde{\pi}_{\mathcal{H}}$ is a compatible Lefschetz fibration map on \tilde{W} with nontrivial monodromy.

Moreover, the pair (\tilde{W}, \tilde{X}) is a relative Stein pair since it fulfills the conditions of Definition 3.0.1. To see this, observe that Stein domains \tilde{W} and \tilde{X} both share the same handle decomposition. Indeed, the cocores of index 1- and 2-handles of \tilde{X} are properly embedded in those of the corresponding index 1 and 2-handles of \tilde{W} (as the latter are just the thickenings of the former). Note that, the condition (3) is obviously satisfied since there are no index 3-handles. Hence, we conclude that the pair $(\tilde{\pi}_{\mathcal{H}}, \pi_{\mathcal{H}})$ is a compatible relative Lefschetz fibration on the relative (admissible indeed) subcritical Stein pair (\tilde{W}, \tilde{X}) .

□

Finally, we discuss the most general case as it has been stated in Theorem 3.0.10.

Proof of Theorem 3.0.10. Let (W, X) be any admissible Stein pair. Then we have

$$W = \tilde{W} \cup (\sqcup_{j=1}^{\ell} H_j) \quad \text{and} \quad X = \tilde{X} \cup (\sqcup_{j=1}^{\ell} h_j)$$

where \tilde{W} is the subcritical part of W which split as $\tilde{W} = \tilde{X} \times D^2$ for the Stein 4-domain \tilde{X} , and (H_j, h_j) are the relative 3-handles of W attached along a relative

Legendrian 2-link (L, K) . By Proposition 6.0.3, there exists a compatible relative Lefschetz fibration $(\tilde{\pi}_{\mathcal{H}}, \pi_{\mathcal{H}})$ on the subcritical part (\tilde{W}, \tilde{X}) of (W, X) for any Weinstein handle decomposition \mathcal{H} of \tilde{X} . We choose \mathcal{H} in such a way that the link (L, K) is Legendrian isotopic (in $S^5 \setminus K_{\mathcal{H}}$) to a suspended Legendrian 2-link. Such a decomposition \mathcal{H} exists by admissibility (i.e., by suspendibility of (L, K) , see Definition 5.0.7). Hence, assuming the above Legendrian isotopy has been already performed, we may assume (L, K) is a suspended Legendrian 2-link sitting (as Lagrangian) on a relative page $(\tilde{F}_{\mathcal{H}}, F_{\mathcal{H}})$ of the relative compatible open book (induced by $(\tilde{\pi}_{\mathcal{H}}, \pi_{\mathcal{H}})$) on the boundary relative contact pair $(\partial\tilde{W}, \partial\tilde{X})$. Note that by Lemma 5.0.2 the contact and the page framing of each component (L_j, K_j) of (L, K) coincide.

To finish the proof, one just needs to proceed as in the last part of the proof of Proposition 6.0.2. More precisely, we have observed that each relative 3-handle (H_j, h_j) attached along (L_j, K_j) is a relative Lefschetz handle, i.e., $(\tilde{\pi}_{\mathcal{H}}, \pi_{\mathcal{H}})$ extends over each (H_j, h_j) . Thus, attaching all (H_j, h_j) 's, we get a compatible relative Lefschetz fibration, say (π_W, π_X) , on the admissible Stein pair (W, X) with relative regular fibers $(\tilde{F}_{\mathcal{H}}, F_{\mathcal{H}})$. If $(\tilde{\varphi}_{\mathcal{H}}, \varphi_{\mathcal{H}})$ denotes the monodromy of $(\tilde{\pi}_{\mathcal{H}}, \pi_{\mathcal{H}})$, then the monodromy of (π_W, π_X) is the pair

$$(\varphi_W, \varphi_X) = (\tilde{\varphi}_{\mathcal{H}} \cdot \tau_{L_1} \tau_{L_2} \cdots \tau_{L_\ell}, \varphi_{\mathcal{H}} \cdot \tau_{K_1} \tau_{K_2} \cdots \tau_{K_\ell})$$

where τ_{L_j} (resp, τ_{K_j}) denotes the right-handed Dehn twist along L_j (resp. K_j) as before. \square



CHAPTER 7

EXAMPLES

Example 7.0.1. Let $D(T^*S^n)$ denote the standard closed unit disk bundle over the n -sphere. The pair $(D(T^*S^3), D(T^*S^2)) \subset (\mathbb{C}^4, \mathbb{C}^3)$ equipped with their *standard* complex structures (inherited from complex spaces) is an admissible relative Stein pair: It can be obtained by attaching a relative 3-handle to the trivial Stein pair (D^6, D^4) along a relative Legendrian 2-unknot $(L, K) \subset (S^5, S^3)$ embedded in the simplest possible way. So, it admits a relative Weinstein handle decomposition with no relative 1- and 2-handles, and clearly we may assume (L, K) to be suspended. Therefore, following the steps in Proposition 6.0.2 (we take $p = q = 2$), we construct a compatible relative Lefschetz fibration on the admissible relative Stein pair $(D(T^*S^3), D(T^*S^2))$ with a relative (regular) fiber $(V_1(2, 2, 2), V_1(2, 2)) \cong (D(T^*S^2), D(T^*S^1))$ and the monodromy

$$(\tilde{\varphi}_{2,2}, \varphi_{2,2}) = (\tau_{S^2}, \tau_{S^1})$$

where the right-handed Dehn twists are along the zero sections S^2 and S^1 on the fibers $D(T^*S^2)$ and $D(T^*S^1)$, respectively.

The relative Stein diagram of $(D(T^*S^3), D(T^*S^2))$ used in the above construction is given in Figure 7.1.

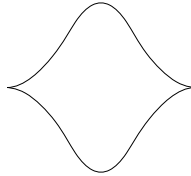


Figure 7.1: A relative Stein diagram for the standard $(D(T^*S^3), D(T^*S^2))$.

Indeed, one can generalize the above construction: If we consider k -many linear

plumbings of the pair $(D(T^*S^3), D(T^*S^2))$, we get the pair $(A_k^6, A_k^4) \cong (V_1(k+1, 2, 2, 2), V_1(k+1, 2, 2))$, which is also an admissible relative Stein pair (with their *standard* complex structures inherited from complex spaces in which they live). It admits a relative Stein diagram as in Figure 7.2 where a relative 3-handle is attached along the suspended 2-unknot (L_i, K_i) for each $i = 1, \dots, k$. (No relative 1- and 2-handles again.) Then one easily construct a compatible relative Lefschetz fibration on the admissible relative Stein pair (A_k^6, A_k^4) with the (relative) fiber $(V_1(2k, 2k, 2), V_1(2k, 2k))$ and the monodromy $(\tilde{\varphi}_{2k, 2k} \cdot \tau_{L_1} \cdots \tau_{L_k}, \varphi_{2k, 2k} \cdot \tau_{K_1} \cdots \tau_{K_k})$ where τ_{L_i}, τ_{K_i} are the right-handed Dehn twists along the zero sections of the pair $(D(T^*S^3), D(T^*S^2))$ plumbed in the i th place.

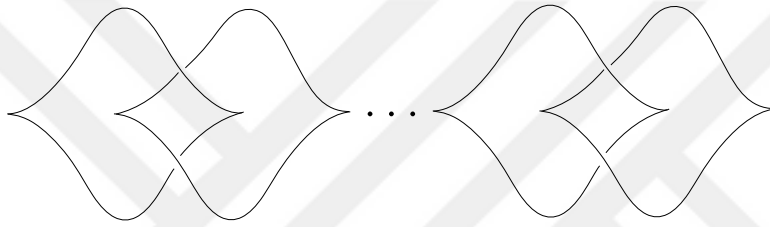


Figure 7.2: A relative Stein diagram for the standard (A_k^6, A_k^4) .

Example 7.0.2. Consider the admissible relative Stein pair (W, X) determined uniquely (up to Stein deformation) by the relative Stein diagram in Figure 7.3-(a). So the six-dimensional Stein domain W consists of (Stein) handles of each index (up to 3): One 0-handle, two 1-handles, one 2-handle (attached along the isotropic knot K drawn dashed) and one 3-handle (attached along the suspended (relative) Legendrian 2-knot \tilde{L} based on \tilde{K} drawn (bold) solid). The four-dimensional Stein subdomain X is described by the Stein diagram in Figure 7.3-(b) which is, indeed, the closed neighborhood of a fishtail fiber in an elliptic fibration (given in Figure 7.4). The subcritical part of (W, X) is the admissible Stein pair (\tilde{W}, \tilde{X}) which splits as $(\tilde{X} \times D^2, \tilde{X})$ where \tilde{X} is the D^2 -bundle over T^2 with Euler number 0 (i.e., $\tilde{X} \cong D^2 \times T^2$, so $\tilde{W} \cong D^4 \times T^2$). For a suitable choice of (p, q) , we can construct (as in the proof of Proposition 6.0.3) a compatible relative Lefschetz fibration on the subcritical part (\tilde{W}, \tilde{X}) with fibers $\left(\mathring{V}_\epsilon(p, q, 2) \bigcup_{j=1}^2 h_j, \mathring{V}_\epsilon(p, q) \right)$, where $(\mathring{V}_\epsilon(p, q, 2), \mathring{V}_\epsilon(p, q))$ is twice “punctured” Brieskorn varieties and each h_j is a 2-handle, (containing both (\tilde{L}, \tilde{K})

and (L, K) where L is the suspension of K and the monodromy

$$(\tilde{\varphi}_{p,q} \cdot \tau_L, \varphi_{p,q} \cdot \tau_K)$$

where τ_K, τ_L are the right-handed Dehn twists along K and L , respectively.

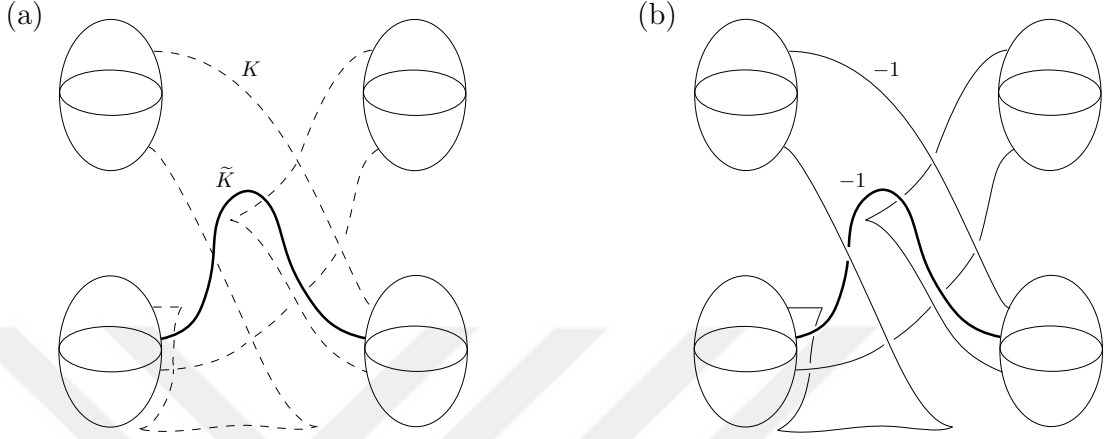


Figure 7.3: (a) A relative Stein diagram for (W, X) , (b) The Stein 4-subdomain X (the closed neighborhood of a fishtail fiber in an elliptic fibration).

The admissible relative Stein pair (W, X) is obtained by attaching relative Stein 3-handle to (\tilde{W}, \tilde{X}) along (\tilde{L}, \tilde{K}) over which the compatible relative Lefschetz fibration structure on (\tilde{W}, \tilde{X}) extends. More precisely, (W, X) also admits a compatible relative Lefschetz fibration with fibers $\left(\mathring{V}_\epsilon(p, q, 2) \bigcup_{j=1}^2 h_j, \mathring{V}_\epsilon(p, q) \right)$ and with the monodromy

$$(\tilde{\varphi}_{p,q} \cdot \tau_L \cdot \tau_{\tilde{L}}, \varphi_{p,q} \cdot \tau_K \cdot \tau_{\tilde{K}})$$

where $\tau_{\tilde{K}}, \tau_{\tilde{L}}$ are the right-handed Dehn twists along \tilde{K} and \tilde{L} , respectively.

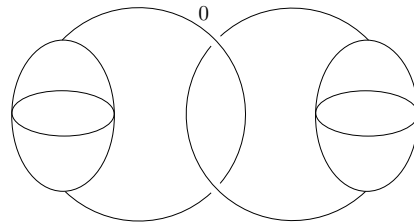


Figure 7.4: The closed neighborhood of a fishtail fiber in an elliptic fibration.



REFERENCES

- [1] S. Akbulut, 4-Manifolds, Oxford University Press, (2016).
- [2] S. Akbulut, B. Ozbagci, Lefschetz Fibrations on Compact Stein Surfaces, *Geom. Topol.* 5, (2001).
- [3] J. W. Alexander, Note on Riemann spaces, *Bulletin of the American Mathematical Society*, 26, 370–372, (1920).
- [4] J. W. Alexander, A Lemma on Systems of Knotted Curves, *J. Differential Geom.* 53, no. 2, 205-236, (1993).
- [5] M. F. Arikan, I. O. Taspinar, Compatible Relative Open Books on Relative Contact Pairs via Generalized Square Bridge Diagrams, *Acta Mathematica Hungarica* 172, 80-118, (2024).
- [6] K. Cieliebak, Subcritical Stein manifolds are split, *Arxiv*, (2002).
- [7] K. Cieliebak , Y. Eliashberg, From Stein to Weinstein and back, *Symplectic geometry of affine complex manifolds*, American Mathematical Society Colloquium Publications, 59. American Mathematical Society, Providence, RI, (2012).
- [8] S. K. Donaldson, Symplectic Submanifolds and Almost Complex Geometry, *Journal of Differential Geometry*, 44, 666–705, (1996).
- [9] S. K. Donaldson, Lefschetz Pencils on Symplectic Manifolds, *Journal of Differential Geometry* 53, 205–236, (1999).
- [10] W. Ebeling, *Functions of Several Complex Variables and Their Singularities*, American Mathematical Society, Providence, RI, (2007).
- [11] Y. Eliashberg, Topological Characterization of Stein Manifolds of Dimension > 2 , *Internat. J. Math.*, 1, (1990).

- [12] J. Etnyre, Lectures on open book decompositions and contact structures, Floer homology, gauge theory, and low-dimensional topology, 103–141, Clay Math. Proc., 5, Amer. Math. Soc., Providence, RI, (2006).
- [13] H. Geiges, An Introduction to Contact Topology, Cambridge University Press, Cambridge, (2008).
- [14] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, in Proceedings of the International Congress of Mathematicians, Vol. II, pp. 405–414, Higher Ed. Press, Beijing, MR 2004c:53144, (2002).
- [15] E. Giroux, J.-P. Mohsen, Structures de contact et fibrations symplectiques au-dessus du cercle, lecture notes.
- [16] E. Giroux, J. Pardon, Existence of Lefschetz fibrations on Stein and Weinstein domains, *Geom. Topol.* 21, (2017).
- [17] R. E. Gompf, A New Construction of Symplectic Manifolds, *Ann. of Math.* 142, 527–595, (1995).
- [18] R. E. Gompf, Handlebody Construction of Stein Surfaces, *Ann. of Math.* 148, 619–693, (1998).
- [19] R. E. Gompf, A.I. Stipsicz, 4-Manifolds and Kirby Calculus, Graduate Studies in Mathematics, vol. 20, American Math. Society, Providence, (1999).
- [20] M. Gromov, Pseudo Holomorphic Curves in Symplectic Manifolds, *Inventiones mathematicae* 82, 2, 307–347, (1985).
- [21] V. Guillemin, A. Pollack, Differential Topology, Prentice-Hall, (1974).
- [22] A. Kas, On the Handlebody Decomposition Associated to a Lefschetz Fibration, *Pacific Journal of Mathematics*, 89, (1980).
- [23] O. van Koert, K. Niederkrüger, Open Book Decompositions for Contact Structures on Brieskorn Manifolds, *Proceedings of the American Mathematical Society*, Vol. 133, No. 12, (2005).
- [24] M. Kwon, O. van Koert, Brieskorn manifolds in contact topology, *Bull. Lond. Math. Soc.*, 48(2):173–241, (2016).

- [25] T. Lawson, Open Book Decompositions for Odd Dimensional Manifolds, *Topology* 17, no.2, 189-192, (1978).
- [26] F. Laudenbach, V. Poenaru, A Note on 4-dimensional Handlebodies, *Bull. Math. Soc. France* 100, 337–344, (1972).
- [27] H. C. Lyon, Torus Knots in the Complement of Links and Surfaces, *Michigan Math. Jour.* 27, (1980).
- [28] D. McDuff, D. Salamon, Introduction to Symplectic Topology , Oxford Math. Monographs, (1995).
- [29] J. Morse, Morse Theory, Ann. Math. Studies 51, Princeton University Press, (1963).
- [30] J. Morse, Lectures on the h-Cobordism Theorem, notes by L. Siebenmann and J. Sondow, Princeton University Press, (1965).
- [31] J. Morse, Singular Points of Complex Hypersurfaces, Ann. Math. Studies 76, Princeton University Press, (1974).
- [32] O. Plamenevskaya, Contact Structures with Distinct Heegaard Floer Invariants, *Math. Res. Lett.* 11, 547-561, (2004).
- [33] P. Seidel, Fukaya Categories and Picard-Lefschetz Theory, Zur. Lect. Adv. Math., EuropeanMath.Soc., Zürich, (2008).
- [34] W. Thurston, H. Winkelnkemper, On the Existence of Contact Forms, *Proc. Amer. Math. Soc.* 52, 345–347, (1975).
- [35] A. Weinstein, Symplectic Manifolds and Their Lagrangian Submanifolds, *Advances in Mathematics* 6-(3), 329–346, (1971).
- [36] A. Weinstein, Contact Surgery and Symplectic Handlebodies, *Hokkaido Math. J.* 20 no.2, 241-251, (1991).



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EVENTS AND ACTIVITIES

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- METU, Department of Mathematics Dec 19, 2022
'On the Existence of Lefschetz Fibrations on Certain Stein Domains'
- Dicle University-5th Workshop of Association for Turkish Women in Maths May 5-7, 2018
'Open Book Decompositions and Contact Structures'

Some Events Participated

- Institut Fourier Low Dimensional Topology Summer 2024
Summer School, Grenoble, France.
- 28.-29. Gökova Geometry/Topology Conferences Spring 2023-2024
- Gatech Topology Summer School (Online) Summer 2021
- Gökova Geometry/Topology Institute Summer 2019
Summer Workshop, Turkey.

SCHOLARSHIPS AND PROJECTS

- TÜBİTAK 2211-A, PhD Scholarship 2020-2024
- Erasmus+ Staff Mobility for Training, Uppsala University Dec 9-17, 2023
Host: Georgios Dimitroglou Rizell
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REFERENCES

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