AN INVESTIGATION OF RING STURCTURE OF GRAPH MAGMA ALGEBRAS

ÇİZGE MAGMA CEBİRLERİNİN HALKA YAPISININ BİR İNCELEMESİ

GÜLHAN MISRA BAYER

PROF. DR. PINAR AYDOĞDU
Supervisor

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ABSTRACT

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Gülhan Mısra Bayer
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Supervisor: Prof. Dr. Pınar Aydoğdu
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Let $G = (V, E)$ be a simple directed graph, where V is any set. By ”simple directed graph” means that there exists at most one edge from u to v for $u, v \in V$ (i.e, a directed graph can have loops but not multiple edges.) Consider, in addition, a symbol $0 \in V$ and an operation on $S = V \cup \{0\}$ via the rule: $uv = u$ if $u, v \in E$ and $uv = 0$, otherwise. The element 0 is called the annihilator element of S. This structure is called *graph magma induced by* G and is denoted by $M(G)$. $R = A[G]$ is a *graph magma algebra* if it has $\mathcal{B} = V \cup \{1\}$ as a basis and, for $u, v \in V$, $uv = u$ if $(u, v) \in E$ and $uv = 0$, otherwise.

The aim of this thesis is to investigate the ring structure of graph magma algebras generated by associative graphs and certain special ideals of these rings within the framework of the Diaz-Boils and Lopez-Permouth study. This thesis consists of four chapters. In the first part of our thesis, we give a survey of the literature on graph magma algebras. In the second chapter, we offer fundamental background information on definitions and theorems in ring theory, module theory, and graph theory.

In the third part of the thesis, we investigate graph magma algebras. It will be shown how a graph magma algebra is constructed, and the problem of when two associative graphs induce an isomorphic algebra will be characterized.

In the fourth chapter, we first determined the Jacobson radical and analyzed simple left and right modules with a single vertex for graph magma algebras induced by graphs with infinitely many non-null connected components. We examined characterization graph magma algebras with finitely many non-null connected components. These rings were identified as semiperfect rings and we examined the conditions under which semiperfect algebras can arise as graph magma algebras. Furthermore, we investigate the right and left socle and the singular ideal of graph magma algebras with finitely many non-null connected components. Lastly, we studied commutative graph magma algebras with infinitely many non-null connected components and examined the characterization of those with finitely many non-null connected components.

In the last section, we examined algebras with bases formed by the vertices of the components $N_1 \oplus K_1$ and $N_p \oplus K_1$, for this focused on upper-triangular and lower-triangular matrix algebras.

Keywords: Graph magma algebras, simple directed graphs, semiperfect rings, (semi)regular rings, semiprimary rings.

ÖZET

ÇİZGE MAGMA CEBİRLERİNİN HALKA YAPISININ BİR İNCELEMESİ

Gülhan Mısra Bayer

Yüksek Lisans, Matematik

Danışman: Prof. Dr. Pınar Aydoğdu

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Çizge magma tanımı ilk olarak [1]'de tanıtıldı, ancak Kelarev ve Sokratova, ortaya çıkan ikili işlemlere çizge cebiri adını verdiler. Bu ikili işlemlere "çizge cebirleri" terimi kullanılsa da, bu ikili işlemler birleşmeli olmayabileceği için, bildiğimiz cebirlerden farklıdır. [2]'deki yazarlar, karışıklığı önlemek için literatürde herhangi bir ek özelliği olmayan ikili işlemlere sahip bir kümeyi tanımlamak için yaygın olarak kullanılan "magma" terimini kullandılar.

Tanım [1, p. 471] V herhangi bir küme olmak üzere $G = (V, E)$ bir basit yönlü çizge olsun. G çizgesinin basit oluşu ile kastedilen, $u, v \in V$ için u'dan v'ye en fazla bir kenar olmasıdır. Dolayısıyla, kenarlar kümesi E, $V \times V$'nin bir alt kümesidir. Ek olarak, bir $0 \notin V$ sembolü ve $S = V \cup \{0\}$ üzerinde tanımlanan işlemi göz önüne alınsın. 0 elemanına S'nin *sıfırlayan elemanı* denir. Bu yapıya, G ile üretilen *çizge magma* denir ve $M(G)$ ile gösterilir.

Tanım [3, Definition 4] Aşağıda verilen tipteki çizgelere *bağlantılı birleşmeli çizgeler* adı verilir: $k, p \in \{\infty\} \cup \mathbb{Z}^+$

1. p adet köşe üzerinde tam çizge K_p (her köşe diğer her bir köşe ile bağlantılı).

2. Tek bir köşe üzerinde boş çizge N_1 (tek köşe var, kenar yok).

3. N_k ile K_p'nin dik toplamı $N_k \oplus K_p$ (K_p'nin kenarlarına ek olarak, N_k'nın her köşesi K_p'nin her bir köşesi ile bağlantılı. Bu tipteki çizgeler için N_k'nın elemanlarına kaynak köşeler, K_p'nin elemanlarına hedef eşkare köşeler denir.)

Kelarev ve Sokratova [1] çalışmasında, çizge magmalarının ne zaman birleşmeli olacağına dair aşağıdaki karakterizasyonu vermişlerdir.

Teorem [1, Proposition 4] Bir $G = (V, E)$ yönlü çizgesi için aşağıdaki ifadeler denktir:

1. Çizge magma $M(G)$ birleşmelidir.

2. Her $(x, y) \in E$ ve $z \in V$ için $(x, z) \in E$'dir ancak ve ancak $(y, z) \in E$'dir.

3. G'nin her bir bağlantılı bileşeni bağlantılı birleşmeli çizgelerden birine izomorftur.

Sonuç [3, Corollary 1] Herhangi bir yönlü çizge $G = (V, E)$ çizgesi için aşağıdaki koşullar denktir:

1. Çizge magma $M(G)$ birleşmeli ve değişmelidir;

2. G çizgesinin her bağlantılı bileşeni N_1 ya da K_1'e izomorftur.

Bu tezde, aksi belirtilmediği sürece, çizgelerimiz birleşmelidir, yani çizgelerimiz $t, p \in \mathbb{Z}^+$ için N_1, K_1, $N_t \oplus K_p$ bileşenlerinden oluşmaktadır.

Aydoğdu ve arkadaşları, [2] çalışmasında, bir birleşmeli çizge magma $M(G)$ tarafından üretilen cebiri ele aldılar ve bu cebire *çizge magma cebiri* adını verdiler. Başka bir deyişle, bir cisim üzerinde birleşme çizge magmaları tarafından oluşturulan yarıgrup cebirleri üzerinde çalıştılar. Ayrıca, çalışmalarında temel modüllerle ilişkili olarak herhangi bir cisim üzerinde çizge magmaları tarafından oluşturulan cebirleri incelediler.

Son yıllarda, sonsuz boyutlu cebirlerin uyumlu tabanlarını araştırmak için çizge magma cebirlerine haklı bir ilgi duyulmuştur (bkz. [2], [4], [5]). Uyumluluk ve bu kavramla ilintili soruları cevaplama yetisi, mümkün olduğunca basit işleme sahip cebirler üzerinde çalışmaya bağlıdır. Çizge magma cebirleri bu isteği karşılamaktadır.

Tanım [3, Definition 3] $G = (V, E)$ basit yönlü çizge ve F bir cisim olsun. $\mathcal{B} = V \cup \{1\}$ tabanına sahip ve her $u, v \in V$,

$$uv = \begin{cases} u & \text{eğer } (u,v) \in E, \\ 0 & \text{aksi halde.} \end{cases}$$

olan $A[G]$ cebirine *çizge magma cebiri* denir.
Birleşmeli G çizgesi ile belirlenen yarıgrup cebirinin monoid olduğundan emin olmak için $M(G)$ yarıgrubuna birim eleman eklenir. $M(G)$ yarıgruna birim eleman eklemesiyle ilgili endişe yoktur; çünkü bu yarıgruplar yalnıza $G = (\{u\}, \{(u,u)\})$ olduğunda bir monoiddir.

2022'de Diaz-Boils ve Lopez-Permouth iki çizgenin ne zaman izomorf çizge magma cebirleri ürettiği problemini ele almış ve bu problem, sonlu sayıda bağlantılı bileşenlere sahip çizgelerden elde edilen magmalar için çözülmüştür.

Bu tezin amacı, Diaz-Boils ve Lopez-Permouth [3] çalışması çerçevesinde, birleşmeli çizgelerin ürettiği magma cebirlerinin halka yapısının ve bu halkaların bazı özel ideallerinin incelenmesidir. Bu tez beş bölümden oluşmaktadır. Tezimizin giriş bölümünde, çizge magma cebirleri hakkında literatür taraması yapılacaktır. İkinci kısımda, çizge teorisindeki bazı tanımlar ve diğer bölümlerde gerekli olan halka ve modül teorisindeki tanım ve teoremler verilecektir. Tezin üçüncü bölümünde, çizge magma cebirleri incelenecektir. Bir çizge magma cebirinin nasıl üretildiği gösterilecek ve birleşmeli çizgelerin izomorf cebirler ürettiği durumlar ele alınacaktır.

Grup cebiri teorisindeki izomorfizma problemi, bir F cismi ve iki grup G ve K verildiğinde, $F[G]$ ve $F[K]$ grup cebirlerinin izomorf olmasının G ve K'nın izomorf olmasını gerektirip gerektirmediğidir (bkz. [6]).

Tanım [3, Definition 5] $A[G]$ ve $A[H]$ izomorf ise G ve H çizgelerine *izobariktir* denir. İzobariklik bir denklik bağıntısıdır ve $G \simeq H$ ile ifade edilir.

Tanım [7, p.3] $G_1 = (V_1, E_1)$ ve $G_2 = (V, E)$ iki basit çizge olsun. Eğer V_1 ve V_2 kümeleri arasında, $u, v \in V_1$ için $(u, v) \in G_1 \Leftrightarrow (\phi(u), \phi(v))) \in G_2$ koşulunu sağlayan bire-bir bir ϕ eşleşmesi varsa G_1 ile G_2 *çizgeleri izomorftur* denir.

Diaz-Boils ve Lopez-Permouth [3] makalesinde aşağıdakiler ispatlanmıştır:

1. Birbirine izobarik bağlantılı birleşmeli çizgelerin ayrık birleşiminden oluşan çizgeler birbirlerine izobariktir: Eğer G ve H çizgeleri sırasıyla $\bigsqcup_{i\in I} G_i$, $\bigsqcup_{i\in I} H_i$ formunda ve her $i \in I$ için G_i ve H_i ikişer ikişer izobarik bağlantılı birleşmeli çizgeler ise G, H'ye izobariktir.

2. Tam bağlantılı bileşenlerin izobraiklik sınıfları aşağıdaki formlara sahiptir:

   (a) Birden fazla köşeye sahip herhangi iki bağlantılı çizge aynı sayıda köşeye sahip olduğunda izobariktir, yani $m = p + l$ ise K_m ve $N_p \oplus K_l$ izobraiktir.

   (b) Bağlantılı bir çizge, bağlantısız bir çizgeyle izobarik olamaz.

Ayrıca değişmeli çizge magma cebirleri için,

$$A[G] \cong A[H] \iff G \cong H$$

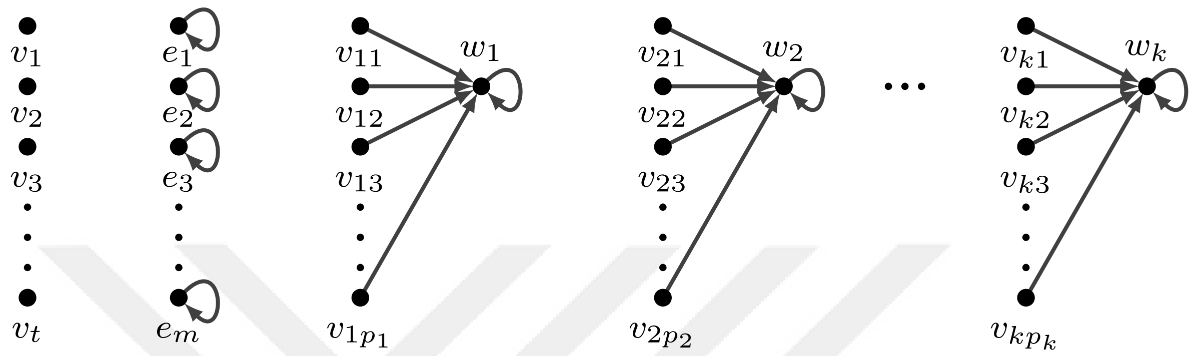
olduğunu gösterdiler.

Bu sonuçtan yola çıkarak, tezin dördüncü bölümünde; t, m, k sıfır, sonlu, ya da (sayılabilir ya da sayılamaz) sonsuz ve her bir p_j sıfırdan farklı ve (sayılabilir ya da sayılamaz) sonsuz olmak üzere,

$$G = N_t \sqcup {K_1}^{(m)} \sqcup (\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1),$$

formundaki çizgeler ele alınmıştır.

Başka bir ifadeyle, G çizgesinde, $\{v_1, v_2, \ldots, v_t\}$ izole üstel sıfır köşeleri, $\{e_1, e_2, \ldots, e_m\}$ izole eşkare köşeleri, $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ kaynak üstel sıfır köşeleri ve w_j onların hedef eşkare köşelerini temsil etmek üzere aşağıdaki şekilde gösterilen çizgeler üzerinde çalışılmıştır.



[3] makalesinde, sonlu sayıda boştan farklı bağlantılı bileşene sahip (yani, G çizgesinde m ve k sonlu) çizge magma cebirleri tam olarak karakterize edilmiş ve bu cebirler için bir ayrışım verilmiştir.

<u>**Teorem** [3, Proposition 5 and Proposition 6]</u> Boştan farklı sonlu sayıda bağlantılı bileşene sahip bir G çizgesi tarafından belirlenen çizge magma cebiri $A[G]$ olsun. $\{e_i\}_{i=1}^m$ izole eşkare köşeler, $\{w_j\}_{j=1}^k$ hedef eşkare köşeler ve $e = 1 - (\sum_{i=1}^m e_i) - (\sum_{j=1}^k w_j)$ olmak üzere;

1. $R = A[G]$ halkası birbirine izomorf olmayan projektif modüllerin toplamı

   $$Re \oplus Re_1 \oplus \cdots \oplus Re_m \oplus Rw_1 \oplus \cdots \oplus Rw_k$$

   olarak ifade edilebilir.

2. $R = A[G]$ halkası birbirine izomorf olmayan projektif modüllerin toplamı

   $$eR \oplus e_1R \oplus \cdots \oplus e_mR \oplus w_1R \oplus \cdots \oplus w_kR$$

   olarak ifade edilebilir.

m ve k sonlu olmak üzere, G çizgesi tarafından üretilen çizge magma cebirlerin yarıtam halkalar olduğu ve yarıtam çizge magma cebirinin yarıasal olduğu gösterilmiştir.

Teorem [3, Theorem 4] $R = A[G]$, boştan farklı sonlu sayıda bağlantılı bileşene sahip bir G çizgesi tarafından üretilen çizge magma cebiri olsun. Aşağıdaki ifadeler denktir:

1. G sonlu sayıda bağlantılı bileşene sahiptir,
2. R yarıtam halkadır,
3. R yarıasal halkadır,
4. R izomorfizma farkıyla sonlu sayıda basit sol (sağ) modüle sahiptir,
5. R tam halkadır.

Ayrıca, hangi yarıtam cebirlerinin çizge magma cebirleri olarak ortaya çıkabileceği belirlenmiştir.

Teorem [3, Theorem 5] R, F cismi üzerinde bir cebir olsun. O halde aşağıdaki iki koşul birbirine denktir:

(1) $R = A[G]$ olacak şekilde sonlu sayıda boş olmayan bağlantılı bileşene sahip bir G çizgesi vardır.

(2) R,

$$R = Re_0 \oplus Re_1 \oplus \cdots \oplus Re_m,$$

ayrışımına sahip yarıasal bir halkadır ve $d_i = \text{dim}Re_i$, $J = J(R)$ ve her bir $i = 0, \ldots, m$ için basit modülleri $S_i = \frac{Re_i}{Je_i}$ olacak şekilde aşağıdaki koşulları sağlar :

(a) $J^2 = 0$,

(b) her $i = 0, \ldots, m$ için, $\text{dim}S_i = 1$,

(c) her $i = 1, \ldots, m$ için , $e_iJe_i = 0$, ve $e_0Je_0 = Je_0$,

(d) her $i = 1, \ldots, m$, için eğer $Re_i \neq Je_i \neq 0$, ise $Je_i = Soc(Re_i) = [S_0]^{(d_i-1)}$.

Yarıtam bir halka üreten G çizgesi sonlu sayıda eşkare köşeye sahiptir, ancak sonlu sayıda üstel sıfır köşeye sahip olmayabilir. Eğer G sonlu sayıda üstel sıfır köşeye sahipse, o zaman aşağıdaki karakterizasyon elde edilmiştir.

Teorem [3, Theorem 6] Bir G çizgesinin ürettiği $R = A[G]$ yarıtam halkası için aşağıdaki ifadeler denktir:

1. R sağ noetherdir,
2. R sol noetherdir,
3. R sağ artindir,
4. R sol artindir,
5. R sonlu boyutludur,
6. G sonlu sayıda köşeye sahiptir.

Yarıtam halkaların karakterizasyonuna dayanarak, birleşmeli çizgeler sınıfı için tüm izomorfik magma cebirleri [3] makalesinde tam olarak karakterize edilmiştir:

Teorem [3, Theorem 7] Eğer G'nin sonlu sayıda boş olmayan bağlantılı bileşeni varsa ve G, H'ye izobraik ise, o zaman H'nin sonlu sayıda boş olmayan bağlantılı bileşeni vardır ve G ile H'nin bileşenleri arasında birbiriyle izobarik olacak şekilde birebir bir ilişki vardır.

[3] çalışmasında, belirli bir sonlu boyuta sahip kaç tane birbirine izomorf olmayan çizge magma cebirinin var olduğu belirlenmiştir:

Önerme [3, Proposition 7] Keyfi bir $n \in \mathbb{Z}^+$ için, $n + 1$ boyutunda tam olarak N tane çizge magma cebirlerinin izomorfizma sınıfı vardır, burada $N = 1 + \sum_{j \leqslant n} p(j)$ ve herhangi bir pozitif tam sayı j için $p(j)$, j parçalanış sayısını belirtir.

Sonlu sayıda boş olmayan bağlantılı bileşene sahip bir G çizgesi tarafından üretilen çizge magma cebiri R'nin her basit sol ideali sağ idealdir. Fakat her basit sağ ideali sol ideal olmak zorunda değildir. Örneğin w_jR basit bir sağ idealdir fakat sol ideal değildir. Diaz-Boils ve Lopez-Permouth [3] makalesinde her basit sağ ideali sol ideal olan R halkasının karakterizasyonu verilmiştir:

Teorem [3, Proposition 8 and Remark 4] Sonlu sayıda boş olmayan bileşene sahip bir G çizgesi tarafından üretilen $R = A[G]$ magma cebiri için aşağıdaki koşullar denktir:

1. R değişmelidir,
2. R sağ duo'dur (yani, her sağ ideali sol idealdir),
3. R halkasının her basit sağ ideali sol idealidir,
4. R sol duo'dur (yani, her sol ideali sağ idealdir).

Yukarıdaki denk koşullardan herhangi biri sağlandığında, $B = 0$ ya da

$$B = \frac{F[x_i | i \in I]}{<x_i x_j | i, j \in I>},$$

ve $C = 0$ ya da C, F cisminin dik kopyalarından oluşacak şekilde $R \cong B \oplus C$ elde edilir.

$B = 0$ durumunda çizgemiz keyfi p için $G = N_p$ ve $C = 0$ durumunda sonlu m için $G = K_1^{(m)}$ formunda olur.

Saraç ve Aydoğdu [8]'de sonlu sayıda boştan farklı bağlantılı bileşene sahip G çizgesi tarafından belirlenen çizge magma cebirinin her zaman sol (sağ) artin halka olduğunu gösterdiler. M'nin *injektiflik bölgesi* M'nin N-injektif olduğu modüllerden oluşan kümedir, yani $\mathfrak{In}^{-1}(M) = \{N \in Mod-R | M \text{ is } N\text{- injektif}\}$ şeklinde tanımlıdır (bkz. [9]). Buradan, M injektiftir ancak ve ancak $\mathfrak{In}^{-1}(M) = Mod-R$ olduğu sonucu çıkar. Her modül yarıbasit modüllere göre injektif olduğundan, $SSMod-R$, R-modüllerinin injektiflik bölgesi için bir alt sınırdır. Bir R-modül M için, M'nin injektiflik bölgesi sadece yarıbasit modüllerden oluşuyorsa, M'ye *fakir (poor) modül* denir (bkz. [10]). Saraç ve Aydoğdu, [8]'de sonlu sayıda boş olmayan bağlantılı bileşene sahip bir G çizgesi tarafından belirlenen bir $R = A[G]$ çizge magma cebiri için, R'nin üstel sıfır köşeleri tarafından üretilen basit sol alt modül içermeyen her sol R-modülün injektif ve yarıbasit olduğunu göstermişlerdir. Ayrıca, basit sol modüllerinin ya injektif ya da fakir olduğunu ispatlamışlardır.

Bu sonuçlara ek olarak, sonsuz çoklukta boş olmayan bağlantılı bileşene sahip bir çizge de dahil olmak üzere genel bir çizge magmanın ürettiği cebir için eşkare elemanlarının yapısı ve Jacobson radikali belirledik. Ayrıca, böyle bir çizge magma cebirinin hem sağ hemde sol yarı-artin halka olduğunu gördük. Bu halkanın basit sağ modüllerinin toplamı $Soc(R_R)$ ve basit sol modüllerin toplamı $Soc(_RR)$ idealleri ile sağ ve sol tekil ideallerini açık olarak ifade ettik.

Önerme : $R = A[G]$, G çizgesi tarafından belirlenen çizge magma cebiri olsun. Eğer x, R'nin eşkare elemanı ise x ya $\sum_{i\in I} e_i + \sum_{j\in J} w_j + \sum_{j\in J} \gamma_i^{(j)} v_{ji}$ ya da $1 - \sum_{i\in I} e_i - \sum_{j\in J} w_j + \sum_{j\in J} \gamma_i^{(j)} v_{ji}$ formundadır. (Burada, $\gamma_i^{(j)} \in F$, I ve J sonlu kümelerdir.)

Önerme : $R = A[G]$, G çizgesi tarafından belirlenen çizge magma cebiri olsun. $R = A[G]$'nin Jacobson radikali

$$J(R) = (\bigoplus Rv_i) \oplus (\bigoplus Rv_{ji}).$$

Dahası $J(R)^2 = 0$'dır.

Sonuç : $R = A[G]$, G çizgesi tarafından belirlenen çizge magma cebiri olsun. O zaman,

$$Soc(R_R) = \begin{cases} (\bigoplus_{j=1}^{k} \bigoplus_{i=1}^{p_j} v_{ji}R) \oplus (\bigoplus_{i=1}^{m} e_iR) \oplus (\bigoplus_{j=1}^{k} w_jR), & \text{eğer} \quad t = 0, \\ J(R) \oplus (\bigoplus_{i=1}^{m} e_iR) \oplus (\bigoplus_{j=1}^{k} w_jR), & \text{aksi durumda} \end{cases}$$

elde edilir.

Sonuç : $R = A[G]$, G çizgesi tarafından belirlenen çizge magma cebiri olsun. O zaman,

$$Soc({}_RR) = \begin{cases} Re \oplus (\bigoplus_{i=1}^{m} Re_i) \oplus (\bigoplus_{j=1}^{k} J(R)w_j), & \text{eğer } t = 0, \\ J(R)e \oplus (\bigoplus_{i=1}^{m} Re_i) \oplus (\bigoplus_{j=1}^{k} J(R)w_j), & \text{aksi durumda} \end{cases}$$

elde edilir.

Önerme : $R = A[G]$, G tarafından belirlenen çizge magma cebiri verilsin. O zaman,

$$Z_r(R) = \begin{cases} 0, & \text{eğer } t = 0 \\ \bigoplus_{i=1}^{t} v_iR, & \text{eğer } t > 0 \end{cases}$$

ve

$$Z_l(R) = \begin{cases} 0, & \text{if } t = 0 \\ J(R), & \text{if } t > 0 \end{cases}$$

olur.

Ayrıca, değişmeli bir çizge magma cebiri R için aşağıdaki sonuçları elde ettik:

1. Her asal ideal maksimaldir.
2. R, yarıartin bir halkadır.
3. $t = 0$ ise, yani izole üstel-sıfır elemanlar yoksa, R halkası düzenlidir.
4. $t \neq 0$ ise R halkası yarıdüzenlidir.
5. R halkasının, izomorfizma farkıyla, injekktif olmayan basit tekil tek modülü vardır.

Tezin son bölümünde, [3] makalesindeki örnekler incelendi. Bu makaleden yola çıkarak, üst üçgen ve alt üçgen matris cebirlerinin, sırasıyla, $N_1 \oplus K_1$ ve $N_p \oplus K_1$ bileşenlerinin köşelerinden oluşan tabanlara sahip cebirler olduğunu gözlemlendi.

Anahtar Kelimeler: Çizge magma cebirleri, basit yönlü çizgeler, yarıtam halkalar, (yarı)düzenli halkalar, yarıasıl halkalar.

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ABBREVIATIONS

F	Field
$G = (V, E)$	Graph with vertex set V and edge set G
$M(G)$	The graph magma induced by G
$A[G]$	The graph magma algebra induced by graph G
K_p	The complete graph on p vertices (every vertex is incident to every other vertex)
N_1	The null graph on a single vertex (one vertex, no edges)
$N_t \oplus K_m$	The direct sum of N_t and K_m (in addition to the edges of K_m every vertex of N_t is incident to every vertex of K_m)
$\bigsqcup$	Disjoint union
$G^{(m)} = \bigsqcup_{i=1}^{m} G$	The graph consist of m disconnected graphs G
$\bigoplus M_i$	Direct sum of R-modules M_i
$\prod M_i$	Direct product of R-modules M_i
$M^{(\Lambda)}$	$\bigoplus_{i\in\Lambda} M_i$, $M_i = M$
$ann_r(m)$	$\{r \in R \| mr = 0\}$, the right annihilator of $m \in M$ in R
$N \leq M$	A submodule N of an R-module M
$N \leq_e M$	An essential submodule N of an R-module M
$N \leq_d M$	A direct summand N of an R-module M
$N \ll M$	A small submodule N of an R-module M
$Rad(M)$	Jacobson radical of an R-module M
$J(R)$	Jacobson radical of a ring R
$Soc(M)$	$\oplus\{K \leq M \| K$ is a simple submodule of $M\} = \bigcap\{L \leq M \| L \leq_e M\}$
S_r, S_l	$Soc(R_R), Soc(_RR)$
$Z(M)$	$\{m \in M \| ann_R^r(m) \leq_e R\}$ the singular submodule of an R-module M
Z_r, Z_l	$Z_r(R_R), Z_l(_RR)$
$ker\ f$	Kernel of an R-homomorphism f
$Im\ f$	Image of an R-homomorphism f

$SSMod$-R	Class of semisimple modules
$\mathfrak{In}^{-1}(M)$	$\{N \in Mod - R \mid M$ is N- injective$\}$

1. INTRODUCTION

The definition of graph magma was originally introduced in [1], but Kelarev and Sokratova referred to the resulting binary operations as graph algebra. These graph algebras differ from traditional algebras, as the binary operations may not be associative despite the use of the term. To avoid confusion, the authors in [2] used the term ”magma,” which is commonly used in literature to describe a set with binary operations without any further properties.

Let $G = (V, E)$ be a simple directed graph, where V is any set. By ”simple directed graph” means that there exists at most one edge from u to v for $u, v \in V$ (i.e, a directed graph can have loops but not multiple edges.) Consider, in addition, a symbol $0 \in V$ and an operation on $S = V \cup \{0\}$ via the rule: $uv = u$ if $u, v \in E$ and $uv = 0$, otherwise. The element 0 is called the annihilator element of S. This structure is called *graph magma* induced by G and is denoted by $M(G)$.

Kelarev and Sokratova characterized the associative graph magmas in [1]. That is, they characterized of graphs such that $M(G)$ is a semigroup. In [2], these graphs are referred to as *associative graphs*.

For any directed graph $G = (V, E)$, the following conditions are equivalent:

1. The graph magma $M(G)$ of G is associative;
2. For all $(x, y) \in E$ and $z \in V$,

$$(x, z) \in E \Leftrightarrow (y, z) \in E;$$

3. Each connected component of G is isomorphic to the N_1, or a complete graph, or a direct sum of a null graph and a complete graph (denoted as $N_t \oplus K_p$ for $t, p \in \mathbb{Z}^+$).

For any directed graph $G = (V, E)$, the following conditions are equivalent:

(1) The graph magma $M(G)$ is associative and commutative.

(2) Each connected component of G is isomorphic to either N_1 or to K_1.

In this thesis, unless stated otherwise, our graphs are associative, i.e, our graphs consist of connected components N_1, K_1, $N_t \oplus K_p$ for $t, p \in \mathbb{Z}^+$.

Aydoğdu et al. introduced a *graph magma algebra* in [2]. They called a graph algebra in this work an algebra induced by an associative graph magma. In other words, they worked on semigroup algebras, which are generated by associative graph magmas over a field. Also, they studied algebras generated by graph magmas over any field in relation to fundamental modules in their work. In recent years, there has been a justified interest in graph magma algebras for investigating the amenable bases of infinite-dimensional algebras (see [2], [4], [5]). Amenability and the ability to answer questions related to this concept depend on studying algebras that are as simple operation as possible. Graph magma algebras satisfy this request.

Let $G = (V, E)$ be a simple directed graph and F be a field. $A = A[G]$ is a graph magma algebra if it has $\mathcal{B} = V \cup \{1\}$ as a basis and, for $u, v \in V$,

$$uv = \begin{cases} u & \text{if}(u, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

To ensure that the semigroup algebra induced by the associative graph G is a monoid, an identity is added to $M(G)$. There is no concern about awkwardness from attaching an identity to an semigroup $M(G)$, as the semigroups they consider are monoids only in trivial cases ($M(G)$ is a monoid only when $G = (\{u\}, \{(u, u)\})$).

Based on these two articles [1] and [2], Diaz-Boils and Lopez-Permouth [3] focused on the isomorphism problem for graph magma algebra and this problem has been resolved for graph magma algebras with finitely many non-null connected components. The aim of this thesis is to investigate the ring structure of magma algebras generated by associative graphs and some special ideals of these rings within the framework of the Diaz-Boils and Lopez-Permouth study.

The isomorphism problem in group algebra theory asks whether, given a field F and two groups G and K, the isomorphism of the group algebras $F[G]$ and $F[K]$ requires G and K to be isomorphic (see [6]). Following [3] two graphs G and H are said to be *isobraic* if they give rise to isomorphic graph algebras $A[G]$ and $A[H]$.

Let the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be given. G_1 and G_2 are said to be *isomorphic* if there is a bijection f from V_1 to V_2 such that $(x, y) \in E_1$ if and only if $(f(x), f(y)) \in E_2$ (see [7]).

They showed in [3] that, in the commutative case, two graph algebras $A[G]$ and $A[H]$ are isomorphic if and only if the graphs G and H are isomorphic.

Also, in [3]

1. It is proved that graphs that are disjoint unions of pairwise isobaric connected associative graphs are isobaric to each other: For all $i \in I$, G_i and H_i are pairwise isobraic connected associative graphs, then $G = \bigsqcup_{i\in I} G_i$ is isobraic to $H = \bigsqcup_{i\in I} H_i$.

2. It is determined the characterization of the isobraicity class of K_m for all m (finite or infinite, possibly uncountable):

   (a) The complete graph K_m is isobraic to the every graph of the form $N_k \oplus K_p$, where $k + p = m$, $p \geq 1$ and $m \geq 1$ (k, m and p may be infinite, possibly uncountable).

   (b) A connected graph cannot be isobraic to a disconnected one:

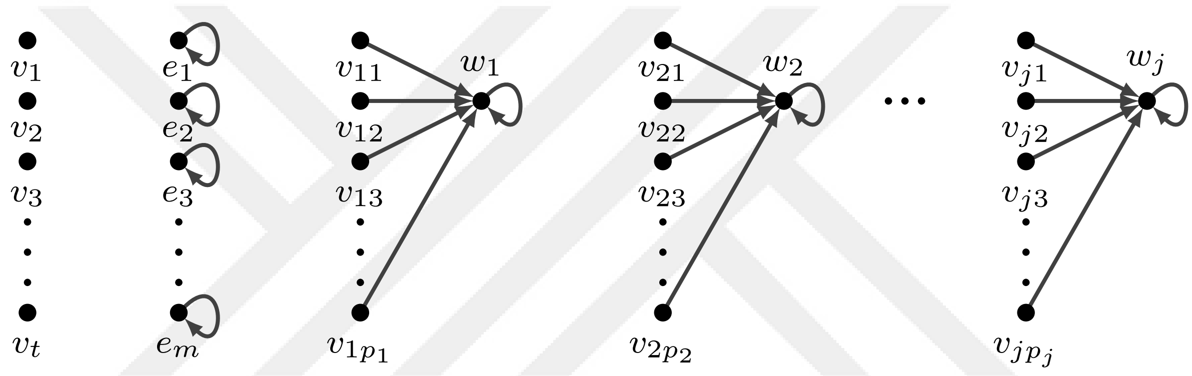
      - A graph of the form $N_1 \sqcup G$ is not isobraic to any graph having only idempotent vertices.
      - If $m < p$, then $N_m \sqcup G$ is not isobraic to $N_p \sqcup G^*$, where G is generated by idempotents.

Therefore, $\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1$ isobraic to $\bigsqcup_{j=1}^{k} K_{p_j+1}$. Thus, they considered the graphs of the following form:

$$G = N_t \sqcup {K_1}^{(m)} \sqcup (\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1)$$

where t, m, k may be zero, finite, or (countably or uncountably) infinite, and each p_i is non-zero but possibly (countably or uncountably) infinite.

We illustrate the graph G as follows:



In [3], graph magma algebras with finitely many non-null connected components are fully characterized. It is shown that such algebras are exactly semiperfect rings and semiperfect graph magma algebra is semiprimary. Furthermore, it is determined what conditions semiperfect algebras can arise as graph magma algebras. The left and right indecomposable decompositions of a semiperfect magma algebra $R = A[G]$ is given. Moreover, it is proved that a graph has finitely many vertices if and only of $R = A[G]$ is right (left) Artinian, if and only if R is right (left) Noetherian. With the following theorem, Diaz-Boils and Lopez-Permouth in [3] managed to reach their goal and fully characterized all isomorhic magma algebras for the class of associative graphs with a finite number of non-null connected components: If G has a finite number of non-null connected components and G is isobraic to H then H has a finite number of non-null connected components and there is a one-to-one correspondence between the components of G and H in such a way that the corresponding components are isobraic.

Moreover, they determined how many pairwise non-isomorphic graph magma algebras of a given finite dimension exist: For an arbitrary $n \in \mathbb{Z}^+$, there exist exactly N isomorphism classes of graph magma algebras of dimension $n+1$, where $N = 1 + \sum_{j \leqslant n} p(j)$, and, for any positive integer j, $p(j)$ denotes the number of partitions j.

Also, it is given a characterization of commutative graph magma algebras R in which the graph has finitely many non-null connected components: R is commutative if and only if R is right duo (every right ideal is a left ideal.) if and only if every simple right ideal of R is a left ideal,if and only if R is left duo (every left ideal is a right ideal.). Under any equivalent conditions, R is of the form $R \cong B \oplus C$, where B is zero or a quotient algebra

$$B = \frac{F[x_i|i \in I]}{<x_i x_j|i,j \in I>}, \text{ and}$$

C is zero or a direct sum of copies of the field F.

The *injectivity domain* of M is $\mathfrak{In}^{-1}(M)$ which consists of those modules N such that M is N-injective, i.e. $\mathfrak{In}^{-1}(M) = \{N \in Mod - R | M \text{ is } N\text{- injective}\}$. It follows that M is injective if and only if $\mathfrak{In}^{-1}(M) = Mod - R$ (see [9]). Since every module is injective relative to semisimple modules, making $SSMod - R$ a lower bound for the injectivity domains of R-modules. For an R-module M, the injectivity domain of M is the lower bound SSMod-R; such modules are called a poor module (see [10]). Saraç and Aydoğdu in [8] showed that the simple left modules of these semiperfect rings are either injective or poor and said that these algebras may have singular non-injective simple left modules. (it is known as rings of type (A) in literature) . Also they showed that graph magma algebras with finitely many non-null connected components are left (right) artinian rings.

In the fourth chapter, the ring structure of graph magma algebras is examined in detail based on articles [1], [3], and [8]. Also, the structure of idempotent elements and the Jacobson radical graph magma algebras induced by graphs with infinitely many non-null connected components are determined. We determined the right and left Socle and the singular ideal of a graph magma algebras induced by graphs with finitely many non-null connected components. Lastly, we explored commutative graph magma algebras with infinitely many non-null connected components, uncovering some properties of these algebras.

In [3], Diaz-Boils and Lopez-Permouth saw that the upper triangular and lower triangular matrix algebras are algebras with bases of the vertices of the components $N_1 \oplus K_1$ and $N_p \oplus K_1$, respectively.

2. PRELIMINARIES

In this chapter, we will offer fundamental background information that is frequently used in the following sections.

2.1 Algebra

Definition 2.1.1. [6, Definition 7.1] Let F be a commutative ring with identity. A F-algebra (or algebra over F) A is a ring A such that:

(i) $(A, +)$ is a unitary (left) F-module;

(ii) $k(ab) = (ka)b = a(kb)$ for all $k \in F$ and $a, b \in A$.

An F-algebra A which, as a ring, is a division ring, is called a division algebra.

The classical theory of algebras deals with algebras over a field F. Such an algebra is a vector space over F and hence various results of linear algebra are applicable. An algebra over a field F that is finite dimensional as a vector space over F is called *finite dimensional*. (see for instance, [6]).

Examples 2.1.2. [6, p.227, Example] *If K is a commutative ring with identity, then the polynomial ring $K[X_1, \ldots, X_n]$ and the power series ring $K[[X]]$ are K-algebras, with the respective K-module structures given in the usual way.*

Definition 2.1.3. [6, Definition 7.3] Let F be a commutative ring with identity and A, B F-algebras.

(i) A *subalgebra* of A is a subring of A that is also an F-submodule of A.

(ii) A (left, right, two-sided) *algebra ideal* of A is a (left, right, two-sided) ideal of the ring A that is also an F-submodule of A.

(iii) A *homomorphism* (respectively, *isomorphism*) of F-algebras $f : A \to B$ is a ring homomorphism (respectively, isomorphism) that is also an F-module homomorphism (respectively, isomorphism).

Definition 2.1.4. [6, Definition 5.1] Let A be an algebra over a commutative ring F with identity.

(i) A *left (algebra) A-module* is unitary left F-module M such that M is a left module over the ring A and $k(rc) = (kr)c = r(kc)$ for all $k \in F$, $r \in A$, $c \in M$.

(ii) An *A-submodule* of an A-module M is a subset of M which is itself an algebra A-module (under the operations in M).

(iii) An algebra A-module M is *simple* (or *irreducible*) if $AM \neq 0$ and M has no proper A-submodules.

(iv) A *homomorphism* $f : M \to N$of algebra A-modules is a map that is both an F-module and an A-module homomorphism.

Remark 2.1.5. [6] If A is an F-algebra the term "A-module" will always indicate an algebra A-module. Modules over the ring A will be so labeled. A right A-module N is defined analogously and satisfies $k(cr) = (kc)r = c(kr)$ for all $k \in F, r \in A, c \in N$.

Simple F-algebras, primitive F-algebras, the Jacobson radical of an F-algebra, semisimple F-algebras, etc. can be defined in the same way the corresponding concepts for rings are defined, with algebra ideals, modules, homomorphisms, etc. in place of ring ideals, modules, and homomorphisms (see [6]).

Definition 2.1.6. [6, Example] Let G be a (multiplicative) group and R a ring. Let $R(G)$ be the additive abelian group $\sum_{g \in G} R$ (one copy of R for each $g \in G$). It will be convenient to adopt a new notation for the elements of $R(G)$. An element $x = \{r_g\}_{g \in G}$ of $R(G)$ has only finitely many nonzero coordinates, say $r_{g_1}, \ldots, r_{g_n}$ ($g_i \in G$). Denote x by the **formal sum** $r_{g_1}g_1 + r_{g_2}g_2 + \ldots + r_{g_n}g_n$ or $\sum_{i=1}^{n} r_{g_i}g_i$. We also allow the possibility that some of the r_{g_i} are zero or that some g_i are repeated, so that an element of $R(G)$ may be written in formally different ways (for example, $r_1g_1 + 0g_2 = r_1g_1$ or $r_1g_1 + s_1g_2 = (r_1 + s_1)g_1$). In this notation, addition in the group $R(G)$ is given by;

$$\sum_{i=1}^{n} r_{g_i}g_i + \sum_{i=1}^{n} s_{g_i}g_i = \sum_{i=1}^{n}(r_{g_i} + s_{g_i})g_i;$$

(by inserting zero coefficients if necessary we can always assume that two formal sums involve exactly the same indices $g_1, \ldots, g_n$). Define multiplication in $R(G)$ by

$$(\sum_{i=1}^{n} r_ig_i)(\sum_{j=1}^{m} s_jh_ij) = \sum_{i=1}^{n}\sum_{j_1}^{m}(r_is_j)(g_ih_j);$$

this makes sense since there is a product defined in both $R(r_is_j)$ and $G(g_ih_j)$ and thus the expression on the right is a formal sum as desired. With these operations $R(G)$ is a ring, called the *group ring* of G over R. $R(G)$ is commutative if and only if both R and R are commutative. If R has an identity 1_R, and e is the identity element of G, then 1_Re is the identity element of $R(G)$.

Definition 2.1.7. [6, Example] Let G be a multiplicative group and F a commutative ring with identity. Then the group ring $R(G)$ is actually a F-algebra with F-module structure given by

$$k(\sum r_ig_i) = \sum(kr_i)g_i \text{ , } (k, r_i \in F; g_i \in G).$$

$F(G)$ is called the *group algebra* of G over F.

Remark 2.1.8. [6] The semigroup algebra $F(S)$, with F as a field and S as a semigroup, is defined similarly to the group algebra $F(G)$.

2.2 Decomposition of Rings

The following definitions, examples, propositions, and theorems not cited here are available in [9].

Let M_1 and M_2 be submodules of a module M. They span M in case

$$M_1 + M_2 = M;$$

and they are *independent* in case

$$M_1 \cap M_2 = 0.$$

There is a canonical R-homomorphism i from the cartesian product $M_1 \times M_2$ module to M defined via

$$i : (x_1, x_2) \to x_1 + x_2 \qquad\qquad ((x_1, x_2) \in M_1 \times M_2)$$

with image and kernel

$$Im\, i = M_1 + M_2 \qquad \text{and} \qquad Ker\, i = \{(x, -x) | x \in M_1 \cap M_2\}.$$

So i is epic iff M_1 and M_2 span M, and monic iff M_1 and M_2 are independent. If this canonical homomorphism i is an isomorphism (i.e., if M_1 and M_2 are independent and span M), then M is the (*internal*) *direct sum* of its submodules M_1 and M_2, and we write

$$M = M_1 \oplus M_2.$$

Thus, $M = M_1 \oplus M_2$ iff for each $x \in M$, there exist unique elements $x_1 \in M$ and $x_2 \in M_2$ such that

$$x = x_1 + x_2.$$

Not every submodule of a module M need appear in such a direct factorization of M. Those that do, however, are of considerable interest. A submodule M_1 of M is a *direct summand* of M in case there is a submodule M_2 of M with $M = M_1 \oplus M_2$; such an M_2 is also a direct summand, and M_1 and M_2 are *complementary direct summands* or *direct complements* of each other.

More generally, suppose that $(M_\alpha)_{\alpha\in A}$ is an indexed set of submodules of a module M. Let

$$i_\alpha : M_\alpha \to M \qquad (\alpha \in A)$$

be the corresponding inclusion maps. Generalizing our definition of direct sum of two submodules, we say that M is the (*internal*) *direct sum of its submodules* $(M_\alpha)_{\alpha\in A}$ in case the direct sum map

$$i = \oplus_A i_\alpha : \oplus_A M_\alpha \to M$$

is an isomorphism. This condition holds, namely i is an isomorphism, iff each $x \in M$ has a unique representation as a sum

$$x = \sum_A x_\alpha$$

with $x_\alpha \in M_\alpha$ zero for almost all $\alpha \in A$ (see [9]).

Corollary 2.2.1. [9, Corollary 6.11] *The module M is the internal direct sum of its submodules $(M_\alpha)_{\alpha\in A}$ if and only $(M_\alpha)_{\alpha\in A}$ is independent and spans M.*

If the submodules $(M_\alpha)_{\alpha\in A}$ of M are independent, we say that the sum $\sum_A M_\alpha$ is *direct* and write

$$\sum_A M_\alpha = \oplus_A M_\alpha.$$

Lemma 2.2.2. [9, Lemma 5.6] *Let e be an idempotent in $End(_RM)$. Then $1-e$ is an idempotent in $End(_RM)$ such that*

$$ker\ e = \{x \in M | x = x(1-e)\} = Im\ (1-e),$$

$$Im\ e = \{x \in M | x = xe\} = Ker\ e(1-e)$$

and $M = Me \oplus M(1-e)$.

Proposition 2.2.3. [9, Proposition 5.7] *If $_RM = K \oplus K'$, then there is a unique idempotent $e_K \in End(_RM)$ such that*

$$K = Me_K \qquad K' = M(1-e_K)$$

Corollary 2.2.4. [9, Corollary 5.8] *A submodule $K \leqslant M$ is a direct summand of M if and only if $K = Im\, e$ for some idempotent endomorphism e of M.*

Definition 2.2.5. [9, p.72]

1. A pair of idempotents e_1 and e_2 in a ring R are said to be *orthogonal* if

$$e_1 e_2 = 0 = e_2 e_1.$$

2. An idempotent $e \in R$ is called a *primitive* idempotent in case $e \neq 0$ and for every pair e_1, e_2 of orthogonal idempotents

$$e = e_1 + e_2 \text{ implies } e_1 = 0 \text{ or } e_2 = 0.$$

3. A finite orthogonal set of idempotents $e_1, \ldots, e_n$ in a ring R is said to be *complete* in case

$$e_1 + \cdots + e_n = 1.$$

Proposition 2.2.6. [9, Proposition 5.10] *Let M be a non-zero module. Then the following are equivalent:*

(a) M is indecomposable;

(b) 0 and 1 are the only idempotents in $End(M)$;

(c) 1 is a primitive idempotent in $End(M)$.

Corollary 2.2.7. [9, Corollary 5.11] *Let e be a non-zero idempotent endomorphism of a left module M. Then the direct summand Me of M is indecomposable if and only if e is primitive idempotent in $End(M)$.*

Corollary 2.2.8. [9, Corollary 6.20] *Let $M_1, \ldots, M_n$ be submodules of M. Then*

$$M = M_1 \oplus \cdots \oplus M_n$$

if and only if there exists a (necessarily unique) complete set $e_1, \ldots, e_n$ of orthogonal idempotents in $End(_RM)$ with

$$M_i = Me_i \qquad (i = 1, \ldots, n).$$

Proposition 2.2.9. [9, Proposition 6.21] *Let $(M_\alpha)_{\alpha \in A}$ be an indexed set of modules, let M be a module, and for each $\alpha \in A$, let $j_\alpha : M_\alpha \to M$ be a homomorphism. Then $(M(j_\alpha)_{\alpha \in A})$ is a direct sum of $(M_\alpha)_{\alpha \in A}$ if and only if there exist (necessarily unique) homomorphisms $q_\alpha : M \to M_\alpha$ ($\alpha \in A$) satisfying, for all $\alpha, \beta \in A$ and all $x \in M$,*

(i) $q_\beta j_\alpha = \delta_{\alpha\beta} 1_{M_\alpha}$,

(ii) $q_\alpha(x) = 0$ *for almost all* $\alpha \in A$,

(iii) $\sum_A j_\alpha q_\alpha(x) = x$.

Moreover, if $(M, (j_\alpha)_{\alpha \in A})$ is a direct sum of $(M_\alpha)_{\alpha \in A}$ and if $f_\alpha : M_\alpha \to N$ ($\alpha \in A$) are homomorphisms, then

$$f : x \mapsto \sum_A f_\alpha q_\alpha(x) \qquad (x \in M)$$

is the unique homomorphism $f : M \to N$ such that $f_\alpha f j_\alpha$ ($\alpha \in A$).

For each ring R, there are the three "regular" modules $_RR$, R_R, and $_RR_R$, and each has its own decomposition theory. The result of the preivous readily specializes to give us basic information about the decompositions of $_RR$ and of R_R. Since $R \cong End(_RR)$ and $R \cong End(R_R)$, we can apply 2.2.2 and 2.2.4 to characterize the direct summands of $_RR$ (and of R_R) (see [9]).

Proposition 2.2.10. [9, Proposition 7.1] *A left ideal I of a ring R is a direct summand of $_RR$ if and only if there is an idempotent $e \in R$ such that $I = Re$.*

Moreover, if $e \in R$ is an idempotent, then so is $1 - e$, and Re and $R(1 - e)$ are direct complements of each other. That is,

$$_RR = Re \oplus R(1 - e).$$

No direct decomposition of $_RR$ (or of R_R) can have infinitely many non-zero summands. Indeed, suppose that

$$R = \oplus_A I_\alpha$$

is a decomposition of $_RR$ as a direct sum of left ideals $(I_\alpha)_{\alpha \in A}$, and let $(p_\alpha)_{\alpha \in A}$ be the projection maps $p_\alpha : R \to I_\alpha$. By Proposition 2.2.9, $p_\alpha(1) \neq 0$ for at most finitely $\alpha \in A$. But

$$I_\alpha = Imp_\alpha = Rp_\alpha(1),$$

so $I_\alpha \neq 0$ for at most finitely many $\alpha \in A$. It should be noted, however, that there need be no bound on the number of non-zero terms that can appear in a direct decomposition of $_RR$ (or R_R). (Consider $\mathbb{Z}^{\mathbb{N}}$.)

By Corollary 2.2.8, we get the following extension of Proposition 2.2.10:

Proposition 2.2.11. [9, Proposition 7.2] *Let $I_1, \ldots, I_n$ be left ideals of the ring R. Then the following statements are equivalent about the left R-module $_RR$:*

(a) $R = I_1 \oplus \cdots \oplus I_n$;

(b) Each element $r \in R$ has a unique expression

$$r = r_1 + \cdots + r_n$$

with $r_i \in I_i$ ($i = 1, \ldots, n$);

(c) There exists a (necessarily unique) complete set $e_1, \ldots, e_n$ of pairwise orthogonal idempotents in R with

$$I_i = Re_i \; (i = 1, \ldots, n).$$

Note in particular, that if $e_1, \ldots, e_n$ are idempotents in R that satisfy (c), then for each $r \in R$

$$r = re_1 + \cdots + re_n.$$

In the unique expression $r = r_1 + \ldots + r_n$ from (b), we have $r_i = re_i \;\; (i = 1, \ldots, n)$. Since a result similar to Proposition 2.2.11 holds for the decompositions of the right regular module R_R, we have the following result.

Corollary 2.2.12. [9, Corollary 7.3] *If $e_1, \ldots, e_n$ is a complete set of pairwise orthogonal idempotents for the ring R, then*

$$_RR = Re_1 \oplus \cdots \oplus Re_n$$

and

$$R_R = e_1R \oplus \cdots \oplus e_nR.$$

Definition 2.2.13. [9] A left (respectively, right) ideal of R is *primitive* in case it is of the form Re (respectively eR) for some primitive idempotent $e \in R$.

Since the endomorphism ring of Re is isomorphic to eRe, we have by Proposition 2.2.6 and Corollary 2.2.7:

Corollary 2.2.14. [9, Corollary 7.4] *Let $e \in R$ be a non-zero idempotent. Then the following statements are equivalent:*

(a) e is a primitive idempotent;

(b) Re *is a primitive left ideal of* R*;*

(c) eR *is a primitive right ideal of* R*;*

(d) Re *is an indecomposable direct summand of* ${}_RR$*;*

(e) eR *is an indecomposable direct summand of* R_R*;*

(f) The ring eRe *has exactly one non-zero idempotent, namely* e*.*

Definition 2.2.15. [9] A ring R is said to be *indecomposable* in case it has no ring decompositions with more than one term.

Corollary 2.2.16. [9, Corollary 7.7] *A ring* R *is an indecomposable ring if and only if* 1 *is the only non-zero central idempotent of* R*.*

2.3 Essential and Superfluous Submodules

Definition 2.3.1. [9, p.72] A submodule K of M is *essential* or *large* in M, abbreviated

$$K \leqslant_e M,$$

in case for every submodule $L \leqslant M$,

$$K \cap L = 0 \qquad \text{implies} \qquad L = 0.$$

Dually, a submodule K of M is *superfluous* or *small* in M, abbreviated

$$K \ll M,$$

in case for every $L \leqslant M$

$$K + L = M \qquad \text{implies} \qquad L = M.$$

Proposition 2.3.2. [9, Proposition 5.16] *Let M be a module with submodules $K \leqslant N \leqslant M$ and $H \leqslant M$. Then*

(1) $K \leqslant_e M$ iff $K \leqslant_e N$ and $N \leqslant_e M$;

(2) $H \cap K \leqslant_e M$ iff $H \leqslant_e M$ and $K \leqslant_e M$.

Proposition 2.3.3. [9, Proposition 5.17] *Let M be a module with submodules $K \leqslant N \leqslant M$ and $H \leqslant M$. Then*

(1) $N \ll M$ iff $K \ll M$ and $N/K \ll M/K$;

(2) $H + K \ll M$ iff $H \ll M$ and $K \ll M$.

Proposition 2.3.4. [9, Proposition 5.20] *Suppose that $K_1 \leqslant M_1 \leqslant M$, $K_2 \leqslant M_2 \leqslant M$, and $M = M_1 \oplus M_2$. Then*

(1) $K_1 \oplus K_2 \ll M_1 \oplus M_2$ iff $K_1 \ll M_1$ and $K_2 \ll M_2$;

(2) $K_1 \oplus K_2 \leqslant_e M_1 \oplus M_2$ iff $K_1 \leqslant_e M_1$ and $K_2 \leqslant_e M_2$.

Theorem 2.3.5. [11] *If $\{M_\alpha\}_{\alpha \in I}$ is an independent family of submodules of M, and if $M_\alpha \leq_e B_\alpha \leq M$ for each α, then $\{B_\alpha\}_{\alpha \in I}$ is an independent family and $\oplus M_\alpha \leq_e \oplus B_\alpha$. Conversely, if $\oplus M_\alpha \leq_e \oplus B_\alpha$, then $M_\alpha \leq_e B_\alpha \leq M$ for each α.*

2.4 Semisimple Modules -The Socle and the Radical of a Module

Definition 2.4.1. [9] A non-zero module ${}_RT$ is *simple* in case it has no non-trivial submodules.

Proposition 2.4.2. [9, Proposition 9.1] *A left R-module T is simple if and only if $T \cong R/M$ for some maximal left ideal M of R.*

Definition 2.4.3. [9] Let $(T_\alpha)_{\alpha \in A}$ be an indexed set of simple submodules of M. If M is the direct sum of this set, then

$$M = \bigoplus_A T_\alpha$$

is a *semisimple decomposition* of M. A module M is said to be *semisimple* in case it has a semisimple decomposition.

A ring is said to be *left semisimple* if it is semisimple as a left module over itself (see [9]).

Lemma 2.4.4. [9, Lemma 9.2] *Let $(T_\alpha)_{\alpha \in A}$ be an indexed set of simple submodules of the left R-module M. If*

$$M = \sum_A T_\alpha,$$

then for each submodule K of M there is a subset $B \subseteq A$ such that $(T_\beta)_{\beta \in B}$ is independent and

$$M = K \oplus (\oplus_{\beta \in B} T_\beta).$$

Proposition 2.4.5. [9, Proposition 9.4] *Let M a semisimple left R-module with semisimple decomposition $M = \bigoplus_A T_\alpha$. If*

$$0 \to K \to M \to N \to 0$$

is an exact sequence of R-modules, then the sequence splits and both K and N are semisimple. Indeed, there is a subset $B \subseteq A$ and isomorphisms

$$N \cong \bigoplus_B T_\beta \text{ and } K \cong \bigoplus_{A \setminus B} T_\alpha.$$

All submodules and factor modules of a semisimple module are semisimple, and each submodule is a direct summand. The below proposition ultimately characterizes semisimple modules.

Theorem 2.4.6. [9, Theorem 9.6] *For a left R-module the following statements are equivalent:*

(a) M is semisimple;

(b) M is generated by simple modules;

(c) M *is the sum of some set of simple submodules;*

(d) M *is the sum of its simple submodules;*

(e) Every submodule of M *is a direct summand;*

(f) Every short exact sequence

$$0 \to K \to M \to N \to 0$$

of left R*-modules splits.*

It is clear then that if R is a division ring then every R-vector space ${}_RM$ is semisimple, for M is generated by its cyclic modules, and every non-zero cyclic R-module is simple.

Lemma 2.4.7. [9] *Let* M *be a left* R*-module. The following statements are equivalent:*

(a) M *is semisimple;*

(b) For every $K \leq M$ *and every* R*-homomorphism* $f : K \to H$ *there is an extension* $\bar{f} : M \to H$ *of* f*.*

Each module M has a (unique) largest semisimple submodule known as the *socle* of M, denoted $Soc(M)$.

Definition 2.4.8. [9, Proposition 9.7] If M is a left R-module, then

$$\begin{aligned} Soc(M) &= \sum\{K \leq M | K \text{ is minimal in } M\} \\ &= \bigcap\{L \leq M | L \text{ is essential in } M\}. \end{aligned}$$

Corollary 2.4.9. [9] $Soc({}_RR)$ *is an ideal of* R*.*

Corollary 2.4.10. [9] *Let* M *is a left* R*-module.* M *is semisimple iff* $M = Soc(M)$*.*

Proposition 2.4.11. [9, Proposition 9.8] *Let* M *and* N *be left* R*-modules and let* $f : M \to N$ *be an* R*-homomorphism. Then*

$$f(SocM) \leq SocN.$$

Corollary 2.4.12. [9, Corollary 9.9] *Let M be a module and let $K \leq M$. Then*

$$Soc(K) = K \cap Soc(M).$$

In particular,

$$Soc(Soc(M)) = Soc(M).$$

The socle $Soc(M)$ of M is the largest submodule of M contained in every essential submodule of M. However, $Soc(M)$ is not necessarily essential in M, and non-zero modules can indeed have a zero socle. Nevertheless, we do have:

Corollary 2.4.13. [9, Corollary 9.10] *Let M be a left R-module. Then $SocM \leq_e M$ if and only if every non-zero submodule of M contains a minimal submodule.*

Definition 2.4.14. [12] The *socle series* or (or *Loewy series*) of an R-module M is the ascending chain of submodules

$$0 = S_0(M) \subset S_1(M) \subset \ldots \subset S_\alpha(M) \subset S_{\alpha+1} \subset \ldots,$$

where, for each ordinal $\alpha \geq 0$, $S_{\alpha+1}/S_\alpha(M)$ is the socle of the module $M/S_\alpha(M)$. M is semiartinian if and only if $S_p(M) = M$ for some ordinal $p \geq 0$. A ring R is called left *semi-artinian* if the left R-module ${}_RR$ is semi-artinian.

Definition 2.4.15. [12] The ring R is said to be a *semiartinian* if every non-zero left (right) R-module has a non-zero socle.

Definition 2.4.16. [9, Proposition 9.13] Let M be a left R-module. Then

$$\begin{aligned} Rad(M) &= \bigcap\{L \leq M | L \text{ is maximal in } M\} \\ &= \sum\{K \leq M | K \text{ is superfluous in } M\}. \end{aligned}$$

Corollary 2.4.17. [9] *$Rad({}_RR)$ is an ideal of R.*

Proposition 2.4.18. [9, Proposition 9.14] *Let M and N be left R-modules and let $f : M \to N$ be an R-homomorphism. Then*

$$f(RadM) \leq RadN.$$

However, even if f is an epimorphism, $f(RadM)$ need not be the radical of N.

Proposition 2.4.19. [9, Proposition 9.15] *If $f : M \to N$ is an epimorphism and if $ker f \leq RadM$, then $RadN = f(RadM)$. In particular,*

$$Rad(M/RadM) = 0.$$

The radical of M is the smallest submodule containing all superfluous submodules, but it is not necessarily superfluous itself.

Proposition 2.4.20. [9, Proposition 9.18] *If every proper submodule of M is contained in a maximal submodule of M, then $RadM$ is the unique largest superfluous submodule of M.*

Proposition 2.4.21. [9, Proposition 9.19] *If $(M_\alpha)_{\alpha \in A}$ is an indexed set of submodules of M with $M = \oplus_A M_\alpha$, then*

$$SocM = \oplus_A SocM_\alpha \text{ and } RadM = \oplus_A RadM_\alpha.$$

2.5 The Jacobson Radical

In this subsection, the Jacobson radical is defined and its various characterizations are presented.

Definition 2.5.1. [9] An element $x \in R$ is *left quasi-regular* in case $1 - x$ has a left inverse in R. Similarly, $x \in R$ is *right quasi-regular (quasi-regular)* in case $1 - x$ has a right (two-sided) inverse in R. A subset of R is *left quasi-regular* (etc.) in case each element of R is left quasi-regular (etc.).

Every nilpotent element $x \in R$ is left quasi-regular: if $x^n = 0$, then $(1+x+\cdots+x^{n-1})(1-x) = 1$.

Proposition 2.5.2. [9, Proposition 15.2] *For a left ideal I of R the following statements are equivalent:*

(a) *I is left quasi-regular;*

(b) *I is quasi-regular;*

(c) *I is superfluous in R.*

The following theorem provides multiple characterizations of $J(R)$.

Theorem 2.5.3. [9, Theorem 15.3] *For a ring R each of the following subsets of R is equal to the radical $J(R)$ of R:*

(J_1) *The intersection of all maximal left (right) ideals of R;*

(J_2) *The intersection of all left (right) primitive ideals of R;*

(J_3) $\{x \in R | rxs$ *is quasi-regular for all* $r, s \in R\}$;

(J_4) $\{x \in R | rx$ *is quasi-regular for all* $r \in R\}$;

(J_5) $\{x \in R | xs$ *is quasi-regular for all* $s \in R\}$;

(J_6) *The union of all the quasi-regular left (right) ideals of R;*

(J_7) *The union of all the quasi-regular ideals of R;*

(J_8) *The unique largest superfluous left (right) ideal of R.*

Moreover, $(J_3), (J_4), (J_5), (J_6)$ and (J_7) also decribe the radical $J(R)$ if "quasi-regular" is replaced by "left quasi-regular" or by "right quasi-regular".

Corollary 2.5.4. [9, Corollary 15.4] *If R is a ring, then*

$$Rad(_RR) = J(R) = Rad(R_R).$$

Corollary 2.5.5. [9, Corollary 15.5] *If R is a ring, then $J(R)$ is the annihilator in R of the class of simple left (right) R-modules.*

Corollary 2.5.6. [9, Corollary 15.8] *If R and S are rings and if $\phi : R \to S$ is a surjective ring homomorphism, then $\phi(J(R)) \subseteq J(S)$. Moreover, if $ker\ \phi \subseteq J(R)$, then $\phi(J(R)) = J(S)$. In particular,*

$$J(R/J(R)) = 0.$$

Corollary 2.5.7. [9, Corollary 15.9] *If R is the ring direct sum of ideals $R_1, R_2, \ldots, R_n$ then*

$$J(R) = J(R_1) + J(R_2) + \cdots + J(R_n).$$

Lemma 2.5.8. [9, Exercise 5.18] *Let I be a nilpotent left ideal of R. For each left R-module M, $IM \ll M$.*

Every nilpotent element in R is left quasi-regular, as noted previously. Therefore, $J(R)$ contains every nilpotent left ideal of R. Recall that an ideal (left, right, or two-sided) is *nil* in case each of its elements is nilpotent. Thus, more generally,

Corollary 2.5.9. [9, Corollary 15.10] *If R is a ring, then every nil left, right, or two-sided ideal of R is left quasi-regular, whence every nil left right or two-sided ideal of R is contained in $J(R)$.*

Corollary 2.5.10. [9] *R is a ring, then $J(R)$ contains no non-zero idempotent.*

Proposition 2.5.11. [9, Proposition 17.18] *Let e and f be idempotent elements in a ring R. Then the following are equivalent:*

(a) $Re \cong Rf$;

(b) $Re/J(R)e \cong Rf/J(R)f$*;*

(c) $eR/eJ(R) \cong fR/fJ(R)$*;*

(d) $eR \cong fR$*.*

2.6 Local Rings

Definition 2.6.1. [9] A ring R is said to be *local* in case for each pair $a, b \in R$ if $a + b$ is invertible, then either a or b is invertible.

Lemma 2.6.2. [9] *If R is local, then 0 and 1 are its only idempotents.*

The radical provides a key characterization of this important class of rings.

Proposition 2.6.3. [9, Proposition 15.15] *For a ring R is the following statements are equivalent:*

(a) R *is a local ring;*

(b) R *has a unique maximal left ideal;*

(c) $J(R)$ *is a maximal left ideal;*

(d) The set of elements of R without left inverse is closed under addition;

(e) $J(R) = \{x \in R | Rx \neq R\}$*;*

(f) $R/J(R)$ *is a division ring;*

(g) $J(R) = \{x \in R | x \text{ is not invertible}\}$*;*

(h) If $x \in R$ then either x or $1 - x$ is invertible.

Proposition 2.6.4. [9, Corollary 17.20] *The following statements about an idempotent e in a ring R are equivalent:*

(a) $Re/J(R)e$ *is simple;*

(b) Je *is the unique maximal submodule of* Re*;*

(c) eRe *is a local ring;*

(d) $eJ(R)$ *is the unique maximal submodule of* eR*;*

(e) $eR/eJ(R)$ *is simple.*

2.7 Chain Conditions

Definition 2.7.1. [6, Definition 1.1] A module A is said to satisfy *the ascending chain condition (ACC)* on submodules (or to be *Noetherian*) if for every chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ of submodules of A, there is an integer n such that $A_i = A_n$ for all $i \geq n$.

A module B is said to satisfing *the descending chain condition (DCC)* on submodules (or to be *Artinian*) if for every chain $B_1 \supset B_2 \supset B_3 \supset \ldots$ of submodules of B there is an integer m such that $B_i = B_m$ for all $i \geq m$.

Proposition 2.7.2. [9, Propposition 10.9] *For a module* A *the following statements are equivalent:*

(a) A *is noetherian;*

(b) Every non-empty set of submodules of A *has a maximal element;*

(c) Every submodule of A *is finitely generated.*

Proposition 2.7.3. [9, Proposition 10.10] *For a module* B *the following statements are equivalent:*

(a) B *is artinian;*

(b) Every factor module of B *is finitely cogenerated;*

(c) Every non-empty set of submodules of B has a minimal element.

Corollary 2.7.4. [9, Corollary 10.11] *Let M be a non-zero module.*

(1) If M is artinian, then M has a simple submodule; in fact, $SocM$ is an essential submodule;

(2) If M is noetherian, then M has a maximal submodule; in fact, $RadM$ is a superfluous submodule.

Proposition 2.7.5. [9, Proposition 10.12] *Let*

$$0 \to K \to M \to N \to 0$$

be an exact sequence of left R-modules. Then M is artinian (noetherian) if and only both K and N are artinian (noetherian).

Proposition 2.7.6. [9, Proposition 10.15] *The following statements are equivalent for each module M:*

(a) $RadM = 0$ and M is artinian;

(b) $RadM = 0$ and M is finitely cogenerated;

(c) M is semisimple and finitely generated;

(d) M is semisimple and noetherian;

(e) M is the direct sum of a finite set of simple modules.

Corollary 2.7.7. [9, Corollary 10.16] *For a semisimple module M, the following statements are equivalent:*

(a) M is artinian;

(b) M is noetherian;

(c) M *is finitely generated;*

(d) M *is finitely cogenerated.*

If a ring R is viewed as a left (or right) module over itself, its submodules correspond exactly to the left (or right) ideals of R. Therefore, it is common to refer to chain conditions on left or right ideals instead of submodules (see [6]).

Definition 2.7.8. [6, Definition 1.2] A ring R is left (right) Noetherian if R satisfies the ascending chain condition on left (right) ideals. R is said to be Noetherian if R is both left and right Noetherian.

A ring R is left (right) Artinian if R satisfies the descending chain condition on left (right) ideals. R is said to be Artinian if R is both left and right Artinian (see [6]).

Proposition 2.7.9. [9, Proposition 10.18] *For each ring* R *the following statements are equivalent;*

(a) R *is left artinian;*

(b) Every finitely generated left R*-module is artinian.*

Proposition 2.7.10. [9, Proposition 10.19] *For each ring* R *the following statements are equivalent;*

(a) R *is left noetherian;*

(b) Every finitely generated left R*-module is noetherian.*

Proposition 2.7.11. [9, Theorem 15.16] *Let* R *be left artinian. Then* R *is semisimple if and only if* $J(R) = 0$*. In particular,* $R/J(R)$ *is semisimple.*

Proposition 2.7.12. [9, Proposition 15.17] *For a ring* R *with radical* $J(R)$ *the following statements are equivalent;*

(a) $R/J(R)$ *is semisimple;*

(b) $R/J(R)$ *is left artinian;*

(c) *Every product of simple left* R*-modules is semisimple;*

(d) *Every product of semisimple left* R*-modules is semisimple;*

(e) *For every left* R*-module* M, $SocM = \boldsymbol{r}_M(J(R))$.

Corollary 2.7.13. [9, Corollary 15.18] *Let* R *be a ring with radical* $J = J(R)$*. Then for every left* R*-module* M*,*

$$JM \leqslant RadM.$$

If R *is semisimple modulo its radical, then for every left* R*-module* M*,*

$$JM = RadM$$

and M/JM *is semisimple.*

If R is left artinian, then its radical $J(R)$ is the unique smallest ideal such that R is semisimple. Additionally, for artinian rings, $J(R)$ can be characterized as the unique largest nilpotent ideal.

Theorem 2.7.14. [9, Theorem 15.19] *If* R *is a left artinian ring, then its radical* $J(R)$ *is the unique largest nilpotent left, right, or two-sided ideal in* R*.*

Proposition 2.7.15. [9] *Every left (right) Artinian ring with identity is left (right) Noetherian.*

Theorem 2.7.16 (Hopkins). [9, Theorem 15.20] *Let* R *be a ring with* $J = J(R)$*. Then* R *is left artinian if and only if* R *is left noetherian,* J *is nilpotent, and* R/J *is semisimple.*

Corollary 2.7.17. [9, Corollary 15.21] *Let* R *be left artinian. If* M *is a left* R*-module, then*

$$SocM = \boldsymbol{r}_M(J(R)) \leq_e M \quad \text{and} \quad RadM = J(R)M \ll M.$$

Moreover, for M *the following statements are equivalent:*

(a) M *is finitely generated;*

(b) M *is noetherian;*

(c) M *has a composition series;*

(d) M *is artinian;*

(e) $M/J(R)M$ *is finitely generated.*

Theorem 2.7.18. [The Krull-Schmidt Theorem] *Let* M *be a non-zero module of finite lenght. Then* M *has a finite indecomposable decomposition*

$$M = M_1 \oplus \cdots \oplus M_n$$

such that for every indecomposable decomposition

$$M = N_1 \oplus \cdots \oplus N_k,$$

$n = k$ *and there is a permutation* σ *of* $\{1, \ldots, n\}$ *such that*

$$M_{\sigma(i)} \cong N_i \ (i = 1, \ldots, n),$$

and for each $1 \leq l \leq n$*,*

$$M = M_{\sigma(1)} \oplus \cdots \oplus M_{\sigma(l)} \oplus N_{l+1} \oplus \cdots \oplus N_n.$$

In fact the decomposition $M = M_1 \oplus \cdots \oplus M_n$ *complements direct summands.*

2.8 Projective and Injective Modules

Definition 2.8.1. [6, Theorem 2.1] Let R be a ring with identitiy. The following conditions on a unitary R-module F are equivalent:

(i) F has a nonempty basis;

(ii) F is the internal direct sum of a family of cyclic R-modules, each of which is isomorphic as a left R-module to R;

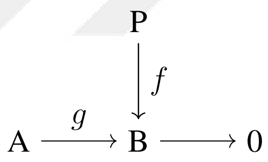
(iii) F is R-module isomorphic to a direct sum of copies of the left R-module R.

A unitary module F over a ring R with identity, which satisfies the equivalent conditions of above conditions, is called a *free* R-module (see [6]).

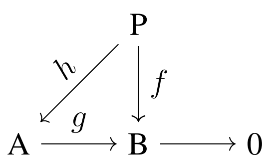
Remark 2.8.2. [6] If R be a ring with identitiy then R is free module over itself.

Lemma 2.8.3. [6] *Every (unitary) module A over a ring R (with identity) is the homomorphic image of a free R-module F.*

Definition 2.8.4. [6, Definition 3.1] A module P over a ring R is said to be *projective* if given any diagram of R-module homomorphisms



with bottom row exact (that is, g an epimorphism), there exists an R-module homomorphism $h : P \to A$ such that the diagram



is commutative (that is, $gh = f$).

Theorem 2.8.5. [6, Theorem 3.2] *Every free module F over a ring R with identity is projective.*

Corollary 2.8.6. [6, Corollary 3.3] *Every module A over a ring R is the homomorphic image of a projective R-module.*

Proposition 2.8.7. [9, Proposition 17.2] *The following statements about a left R-module P are equivalent:*

(a) P is projective;

(b) Every epimorphsim ${}_RM \to {}_RP \to 0$ splits;

(c) P is isomorphic to a direct summand of a free left R-module.

Corollary 2.8.8. [6] *Let R be a ring with identity. If $e^2 = e$ is idempotent element of R, then Re is projective left R-module.*

Proposition 2.8.9. [6, Proposition 3.5] *Let R be a ring. A direct sum of R-modules $\sum_{i\in I} P_i$ is projective if and only if each P_i is projective.*

Definition 2.8.10. [9] A pair (P, p) is a projective cover of the module ${}_RM$ in case P is a projective left R-module and

$$\mathrm{P} \xrightarrow{p} \mathrm{M} \longrightarrow 0$$

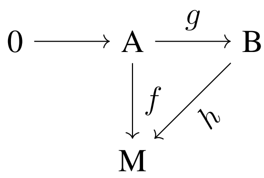
is a superfluous epimorphism ($ker\ p \ll P$).

Definition 2.8.11. [9] Let M and N be left R-modules. M is called N-*injective* (or *injective relative* to N) if and only for each submodule ${}_RK \leqslant {}_RN$ every R- homomorphism $h : K \to M$ can be extended to an R- homomorphism $\bar{h} : N \to M$ (i.e., every $h : K \to M$ factors through the natural monomorphism $i_K : K \to N$). An R-module M is called an *injective module* if it is N-injective for every R-module N.

By the definition, An R-module M is injective if and only if it is injective to every module in Mod-R. That is,

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{A} & \xrightarrow{g} & \mathrm{B} \\ & & \downarrow{\scriptstyle f} & & \\ & & \mathrm{M} & & \end{array}$$

with top row exact (that is, g a monomorphism), there exists an R-module homomorphism $h : B \to M$ such that the diagram



is commutative (that is, $hg = f$).

Proposition 2.8.12. [9, Proposition 3.7] *A direct product of R- modules $\prod_{i\in I} J_i$ is injective if and only if J_i is injective for every $i \in I$.*

Lemma 2.8.13. [6, Lemma 3.8.] *Let R be a ring with identity. A unitary R-module M is injective if and only if for every left ideal L of R, any R-module homomorphism $L \to M$ may be extended to an R-module homomorphism $R \to M$.*

Proposition 2.8.14. [6, Proposition 3.12] *Every unitary module A over a ring R with identity may be embedded in an injective R-module.*

Theorem 2.8.15. [9] *If I is an injective module and K is a direct summand of I, then K is also an injective module.*

Proposition 2.8.16. [9, Proposition 18.7] *A left R-module E is injective if and only if every monomorphism*

$$0 \to {}_R E \to {}_R M$$

splits.

This raises the question of how far a given module can be considered injective. Towards this end, we utilize the notion of injectivity domains as defined in [9].

Definition 2.8.17. [9] The *injectivity domain* of M is $\mathfrak{In}^{-1}(M)$ which consist of those modules N such that M is N-injective,i.e., $\mathfrak{In}^{-1}(M) = \{N \in Mod - R | M \text{ is } N\text{-injective}\}$.

Corollary 2.8.18. [9] *M is injective if and only if $\mathfrak{In}^{-1}(M) = Mod - R$.*

Proposition 2.8.19. [10] *Every module is injective relative to semisimple modules.*

The injectivity domain of every R-module include the class of all semisimple modules , denoted $SSMod - R$, making $SSMod - R$ a lower bound for the injectivity domains of R-modules.

Proposition 2.8.20. [10, Proposition 3.1] $\bigcap_{M \in Mod-R} \mathfrak{In}^{-1}(M) = SSMod - R$

Definition 2.8.21. [10] For an R-module M, the injectivity domain of M is the lower bound SSMod-R; thus, M is called a poor module.

Definition 2.8.22. [13] Let the R-module E can be an extension of the R-module A. Then E is said to be an *essential extension* of A if for every non-zero submodule E' of E, $E' \cap A \neq 0$. This is equivalent to the condition that, for every non-zero element e of E, there exists $r \in R$ such that re is a non-zero element of A.

Theorem 2.8.23. [13, Theorem 2.17] *Let E be an R-module. Then the following statments are equivalent:*

(a) E is injective;

(b) E has no proper essential extension.

Theorem 2.8.24. [13, Theorem 2.21] *Let A be an R-module. Then there exists an R-module E satisfying the following equivalent conditions:*

(a) E is an essential injective extension of A;

(b) E is a maximal essential extension of A;

(c) E is a minimal injective extension of A. Moreover, if E_1, E_2 are both essential injective extensions of A, then there is an isomorphism $\Phi : E_1 \to E_2$.

Definition 2.8.25. [13] Let A be an R-module. An R-module E satisfying conditions of Thorem 2.8.24 is called *injective envelope* of A, we use the symbol $E(A)$ to denote an injective envelope of A.

Corollary 2.8.26. [13] *R-module A is injective if and only if $E(A) = A$.*

2.9 Semiperfect and Perfect Rings

Definition 2.9.1. [9, p.301] Let I be an ideal in a ring R and let $g + I$ be an idempotent element of R/I. We say that this idempotent can be *lifted (to e) modulo I* in case there is an idempotent $e \in R$ such that $g + I = e + I$. We say that *idempotents lift modulo I* in case every idempotent in R/I can be lifted to an idempotent in R.

Proposition 2.9.2. [9, Proposition 27.4] *Let R be a ring and let I be an ideal of R with $I \subseteq J(R)$. Then the following are equivalent:*

(a) Idempotents lift modulo I;

(b) Every (complete) finite orthogonal set of idempotents in $R/J(R)$ lifts to a (complete) orthogonal set of idempotents in R.

Proposition 2.9.3. [9, Proposition 27.1] *If I is a nil ideal in a ring R then idempotents lift modulo I.*

Proposition 2.9.4. [9, Proposition 27.4] *Let R be a ring and let I be an ideal of R with $I \subseteq J(R)$. Then the following are equivalent:*

(a) Idempotents lift modulo I;

(b) Every direct summand of the left R-module $R/J(R)$ has a projective cover;

(c) Every (complete) finite orthogonal set of idempotents $R/J(R)$ lifts to a (complete) orthogonal set of idempotents in R.

Definition 2.9.5. [9, p.303] A ring R is called *semiperfect* in case $R/J(R)$ is semisimple and idempotents lift module $J(R)$.

Theorem 2.9.6. [9, Theorem 27.6] *For a ring R the following statements are equivalent:*

(a) R is semiperfect;

(b) R *has a complete orthogonal set* $e_1, \ldots, e_n$ *of idempotents with each* e_iRe_i *a local ring.*

Corollary 2.9.7. [9, Corollary 27.7] *Let* $e_1, \ldots, e_n \in R$ *be non-zero orthogonal idempotents with* $1 = e_1 + \cdots + e_n$*. Then* R *is semiperfect if and only if each* e_iRe_i *is semiperfect.*

Proposition 2.9.8. [9, Proposition 27.10] *Let* R *be semiperfect with* $J = J(R)$*. Then every complete set of orthogonal primitive idempotents for* R *contains a basic set. Moreover, for pairwise orthogonal primitive idempotens* $e_1, \ldots, e_m \in R$ *the following are equivalent:*

(a) $e_1, \ldots, e_m$ *is a basic set of primitive idempotents for* R*;*

(b) $Re_1, \ldots, Re_m$ *is an irredundant set of representatives of the indecomposable projective left* R*-modules;*

(c) $Re_1/Je_1, \ldots, Re_m/Je_m$ *is an irredundant set of represantatives of the simple left* R*-modules.*

Definition 2.9.9. [9] A ring R is *left perfect (right perfect)* in case each of its left (right) modules has a projective cover.

Definition 2.9.10. *[9]* A subset of I a ring R is *left* T*-nilpotent* in case for every sequence $a_1, a_2, \ldots$ in I there is an n such that

$$a_1 \ldots a_n = 0.$$

The subset I is *right* T*-nilpotent* in case for each $a_1, a_2, \ldots$ in I

$$a_n \ldots a_1 = 0$$

for some n.

Lemma 2.9.11. [6] *Every left or right* T*-nilpotent ideal is nil.*

The following theorem characterizes perfect rings.

Theorem 2.9.12. [Bass][9, Theorem 28.4] *Let R be a ring with radical $J = J(R)$. Then the following statements are equivalent:*

(a) R is left perfect;

(b) R/J is semisimple and J is left T-nilpotent;

(c) R/J is semisimple and every non-zero left R-module contains a maximal submodule;

(d) Every flat left R-mdoule is projective;

(e) R satisfies the minimum condition for principal right ideals;

(f) R contains no infinite orthogonal set of idempotents and every non-zero right R-module contains a minimal submodule.

Corollary 2.9.13. [9, Corollary 28.7] *If R is perfect, then so is every factor ring of R.*

Definition 2.9.14. [9] A ring R, with Jacobson radical $J(R) = J$ is said to be *semiprimary* if R/J is semisimple and J is nilpotent.

Corollary 2.9.15. [9] *Every semiprimary ring is perfect, and every perfect ring is semiperfect.*

2.10 The Singular Submodule

The following definitions are from [12] and [11].

Definition 2.10.1. [12] Let A be left R-module. The set $ann_R(A) = \{r \in R | rA = 0\}$ is the *annihilator* of left R-module A in R.

Definition 2.10.2. [12] For any $a \in A$, the set $ann_l(a) = \{r \in R | ra = 0\}$ is the left annihilator of a in R, and it is a left ideal of R.

Definition 2.10.3. [12, Definition 7.1] Let A be a left module over a ring R. An element $a \in A$ is said to be a *singular element* of A if the left ideal $ann_l(a) \leq_e {}_RR$. The set of all singular elements of A is denoted by $Z(A)$.

Equivalent to the above definition;

Definition 2.10.4. [11] For a left R-module A, the singular submodule of A is the set

$$Z(A) = \{a \in A | Ia = 0 \text{ for some } I \leq_e R\}$$

Lemma 2.10.5. [12, Lemma 7.2] *(1) $Z(A)$ is a submodule, called the singular submodule of A.*

(2) $Soc(R_R)Z(A) = 0$.

(3) If $f : A \to B$ is any R- homomorphism, then $f(Z(A)) \subseteq Z(B)$.

(4) If $A \subset B$, then $Z(A) = A \cap Z(B)$.

Corollary 2.10.6. [11] *The following statements are satisfied :*

(1) $Z({}_RR)$ is an ideal in R, called the left singular ideal of R. (The right singular ideal is similarly defined to be $Z(R_R)$.) .

(2) If $R \neq 0$, then $Z({}_RR)$.

(3) $Z({}_RR)$ contains no nonzero idempotents.

Definition 2.10.7. [12, Definition 7.5] We say that ${}_RA$ is a *singular* (resp. *nonsingular*) module if $Z(A) = A$ (resp.$Z(A) = 0$). In particular, we say that R is a left nonsingular ring if $Z({}_RR) = 0$. Right nonsingular rings are defined similarly, and " *nonsingular ring*" shall mean a ring that is both right and left nonsingular.

Remark 2.10.8. [11] If $A \leqslant_e B$, then B/A is singular. The converse of this can easily fail; for example, let $B = \mathbf{Z}/2\mathbf{Z}$ and $A = 0$. B/A is singular, yet A is not an essential submodule of B. The following two propositions state the conditions required for the converse to hold true:

Proposition 2.10.9. [11] *Let B be nonsingular, and let $A \leq B$. Then B/A is singular if and only if $A \leq_e B$.*

Proposition 2.10.10. [11] *Let P be the projective module and $X \leqslant P$. P/X is singular if and only if $X \leqslant_e P$. In particular, $P = 0$ if P is projective and singular.*

Examples 2.10.11. [12, Examples 7.6]

(1) If R is a simple ring, $R \neq 0$. Then $Z({}_RR) \neq R$. Therefore, $Z({}_RR)$ must be zero. Consequently, any simple ring is nonsingular.

(2) ${}_{\mathbb{Z}}\mathbb{Z}$ is non singular, because every non-zero ideal of $\mathbb{Z}$ is essentail submodule.

(3) Let $A \subseteq B$ be R-modules. If B is nonsingular, so is A, and the converse holds if $A \leqslant_e B$.

(4) An R-module ${}_RM$ is singular iff there exist R-modules $A \leqslant_e B$ such that $M \cong B/A$.

(5) Any semisimple ring is nonsingular. More precisely, a ring R is semisimple iff every left R-module ${}_RM$ is nonsingular.

Proposition 2.10.12. [11] *The following statements are satisfied:*

(a) The class of all nonsingular left R-modules is closed under submodules, direct products, essential extensions, and module extensions.

(b) The class of all singular left R-modules is closed under submodules, factor modules, and direct sums.

Proposition 2.10.13. [11, Proposition 1.24] *If A is any simple left R-module, then A is either singular or projective, but not both.*

Corollary 2.10.14. [11, Corollary 1.25] *Every nonsingular semisimple left R-module is projective.*

Lemma 2.10.15. [12, Lemma 7.8] *If R has non-zero nilpotent elements, then R is left (and also right) nonsingular.*

Lemma 2.10.16. [12, Lemma 7.11] *Let x be a central nilpotent element in a ring R. Then $x \in Z({}_RR)$ (and by symmetry, $x \in Z(R_R)$ as well).*

Proposition 2.10.17. [12, Proposition 7.13] *For any ring R, let $S = Soc({}_RR)$ be its right socle. Then $Z({}_RR) \subseteq ann_r(S)$. If S is an essential submodule, equality holds.*

2.11 Graphs

The following definitions are from [3] and [7].

Definition 2.11.1. [7] A graph is a pair $G = (V, E)$ of sets such that $E \subseteq V \times V$. The elements of V are the *vertices* of the graph G, the elements of E are its *edges*.

The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices from an edge [7].

Definition 2.11.2. [7, p.4] If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G, written as $G' \subseteq G$. If $G' \subseteq G$ and $G' \neq G$, then G' is a *proper subgraph* of G.

Definition 2.11.3. Let $G = (V, E), G' = (V', E'), G'' = (V'', E'')$ be graphs If $V'', V' \subset V$ and $E', E'' \subset E$, then $G' \cup G'' = (V' \cup V'', E' \cup E'')$ is a subgraph of G. If $G' \cap G'' = \emptyset$ then then this graphs are called *discrete (disconnect)* and denoted $G' \sqcup G''$.

Definition 2.11.4. [7, p.10] A graph G is called *connected* if it is non-empty and any two of its vertices are linked. A maximal connected subgraph of a graph G is called a *connected component* of G.

Definition 2.11.5. [7, p.27] A *directed graph* is a graph that is made up of a set of vertices connected by directed edges.

Definition 2.11.6. [7, p.28] Let $G = (V, E)$ be a directed graph. If there are edges between the same two vertices then such edges are called *multiple edges*. If $(v, v) \in E$ for some $v \in V$, then the edge (v, v) is called a *loop*.

In this thesis, all graphs are assumed to contain no multiple edges but may include loops.

Definition 2.11.7. [3, Definition 4] Let $G = (V, E)$ be graph and for $k, p \in \{\infty\} \cup \mathbb{Z}^+$ $|V| = k$,

1. If $E = \emptyset$, graph G is called the null graph and denoted N_k

2. If $E = V\text{x}V$, i.e., for all $u, v \in V$, $(u, v) \in E$ then graph G is called the complete graph and denoted K_k.

Definition 2.11.8. [4] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be directed graphs. The sum $G_1 + G_2$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup E_{1,2}$, where $E_{1,2} = \{(x, y)|x \in V_1, y \in V_2\}$. If G_1 and G_2 have no common vertices, then we say the sum is direct, and we denote it by $G_1 \oplus G_2$.

Remark 2.11.9. [2] According to the definition, Since for $k, p \in \{\infty\} \cup \mathbb{Z}^+$ the null graph N_k and the complete graph K_p have no common vertices, then sum of N_k and K_p is direct and denoted as $N_k \oplus K_p$. We say that the elements of N_k are the *source* vertices and the elements of K_p are the *target* vertices.

Definition 2.11.10. [7, p.3] Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, G_1 and G_2 are *isomorphic* if there is a bijection f from V_1 to V_2 such that $(x, y) \in E_1$ if and only if $(f(x), f(y)) \in E_2$.

3. GRAPH MAGMA ALGEBRAS

In this section, we investigate graph magma algebras. First, we show how a graph magma algebra is constructed. The commutative case is examined. It is shown that two commutative graph magma algebras are isomorphic if and only if the graphs are isomorphic. But this need not be true for non-commutative case. When two graphs give rise to isomorphic non-commutative graph magma algebras is discussed in the last subsection.

Note that, $G = (V, E)$ always denotes a directed graph with a vertex set V and an edge set E.

3.1 The Construction of Graph Magma Algebras

By a graph magma algebra, we mean a *contracted semigroup algebra* induced by associative graphs which will be defined below. This definition depends on the notion of semigroup algebras. Since we work on associative algebras with identitiy, we actually mean the semigroup algebra of the monoid obtained by attaching an identity to the underlined semigroup.

A graph magma induced by a directed graph without multiple edges, but possibly with loops, is defined in [1] as follows:

Definition 3.1.1. [1, p. 471] Let V be an arbitrary set and $G = (V, E)$ be a directed graph. Consider a symbol $0 \notin V$, called the *annihilator element* of $S = V \cup \{0\}$. Then the *graph magma* $M(G)$ of the graph G is the set S together with the multiplication defined by

$$u.v = \begin{cases} u, & \text{if } (u, v) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Now some necessary and sufficient conditions for a graph magma to be associative are given.

Proposition 3.1.2. [1, Proposition 4] *For any directed graph $G = (V, E)$, the following conditions are equivalent:*

(1) The graph magma $M(G)$ is associative.

(2) For all $(x, y) \in E$ and $z \in V$, then $(x, z) \in E$ if and only if $(y, z) \in E$.

(3) Each connected component of G is isomorphic to the null graph, a complete graph, or a direct sum of a null graph and a complete graph. (i.e., G consists of connected components of the form N_1, K_m, $N_p \oplus K_l$, where $m, p, l \in \mathbb{Z}^+$.)

Proof. $(1) \Rightarrow (2)$: Let $(x, y) \in E$ and $z \in V$. If $(x, z) \in E$, then $(xy)z = x$. Since $M(G)$ is associative, $x(yz) = x$. This shows that $yz \neq 0$. Thus, $(y, z) \in E$. If $(y, z) \in E$, then $x(yz) = x$. Since $M(G)$ is associative, $(xy)z = xz = x$. This implies that $(x, z) \in E$.

$(2) \Rightarrow (1)$: Let $x, y, z \in V$. If $(x, y) \notin E$, then $(xy)z = 0 = x(yz)$. If $(x, y) \in E$, then by (2), $(x, z) \in E$ if and only if $(y, z) \in E$. Thus, $(xy)z = x(yz)$.

$(2) \Rightarrow (3)$: Let X be a connected component of G. If X includes only one vertex, then it must be either N_1 or K_1. Now, suppose X has at least two vertices. Let K be the subgraph consisting of vertices v for which there exists at least one $u \in V$ such that $(u, v) \in E$. Pick any vertex $v \in V_K$. Then there exists $u \in V_C$ such that $(u, v) \in E$. By (2), we have $(v, v) \in E$. Therefore, all vertices in K have loops. If $u \in V_K$, then by (2) we obtain $(v, u) \in E$, because $(u, u) \in E$. Thus, K is a complete graph. If $u \notin V_K$, then $(u, u) \notin E$. By (2), it follows that $(v, u) \notin E$. Thus, $V_C \backslash V_K$ represent the null graph. Consequently, if $V_K = V_C$ then C is a complete graph; otherwise, C is a direct sum of a complete graph and a null graph.

$(3) \Rightarrow (2)$: Let $(x, y) \in E$ and $z \in V$. Then x and y are vertices of the same connected component, i.e., x, y are vertices of either K_m or $N_p \oplus K_l$. So y is a vertex of a complete subgraph in both cases. If $(x, z) \in E$, then similarly, z is a vertex of a complete subgraph. Since y and z belong to the same complete subgraph, $(y, z) \in E$. Similarly, it can be seen that if $(y, z) \in E$, then $(x, z) \in E$. □

The following result is an immediate consequence of Proposition 3.1.2.

Corollary 3.1.3. [1, Corollary 5] *If directed graph G is associative, then for all $x, y, z \in V$, the following conditions hold:*

(1) $(x, x), (x, y) \in E \implies (y, x) \in E$

(2) $(x, y) \in E \implies (y, y) \in E$

From this point on, we assume that all graphs are associative.

Corollary 3.1.4. [3, Corollary 1] *For any directed graph $G = (V, E)$, the following conditions are equivalent:*

(1) The graph magma $M(G)$ is associative and commutative.

(2) Each connected component of G is isomorphic to either N_1 or to K_1.

Proof. $(1) \Rightarrow (2)$: Let $M(G)$ be associative and commutative. Then by proposition 3.1.2 G is consist of connected component of the form N_1, K_m, $N_p \oplus K_l$, where $m, p, l \in \mathbb{Z}^+$. Since $M(G)$ is commutative, $(u, v) \in E$ if and only if $(v, u) \in E$ for all $u, v \in V$. Therefore G is not contain component of the form $N_p \oplus K_l$.
$(2) \Rightarrow (1)$: It is clear by Proposition 3.1.2 □

Definition 3.1.5. [3, Definition 4] For $k, p \in \{\infty\} \cup \mathbb{Z}^+$ $|V| = k$, the graphs N_1, K_p and $N_k \oplus K_p$ are referred to as connected associative graphs.

Now we give two alternative definitions of graph magma algebras. The first one depends on the definition of a contracted semigroup algebra.

Definition 3.1.6. [3, Definition 2] Let F be a field and S be a monoid which contains an annihilator element $0 \in S$. The *contracted semigroup algebra* over F induced by S is the quotient of the semigroup algebra $F[S]$ modulo the ideal consisting of scalar multiples of $0 \in S$.

Definition 3.1.7. [3, Definition 2] Let F be a field and G be an associative graph. The contracted semigroup algebra induced by $M(G)$ is the *graph magma algebra* over the field F induced by the graph G, and it is denoted by $A[G]$.

Another definition for graph magma algebras can be given by considering that they are algebras having a basis $\mathcal{B} = \{1\} \cup V$ with the operation defined in Definition 3.1.1.

Definition 3.1.8. [3, Definition 3] Let $G = (V, E)$ be a simple directed graph and F be a field. $A = A[G]$ is said to be a *graph magma algebra* if it has a $\mathcal{B} = \{1\} \cup V$ such that for all $u, v \in V$,

$$u.v = \begin{cases} u, & \text{if } (u,v) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.1.9. Let $v \in G$, where G is an associative graph. If $(v, v) \notin E$, then v is a nilpotent element in $A[G]$. If $(v, v) \in E$, then v is an idempotent element in $A[G]$. Hence, every vertex in graph is either a nilpotent element or an idempotent element in the algebra $A[G]$. We refer to the nilpotent and idempotent elements from a connected component of the form $N_p \oplus K_1$ as *source nilpotents* and *target idempotents* , respectively. We refer to the nilpotent and idempotent elements from a component of the form N_t and $K_1^{(m)}$ as *isolated nilpotents* and *isolated idempotents*, respectively.

3.2 Isobraicity of Graphs

In this subsection, we will investigate which graphs give rise to isomorphic graph magma algebras.

Definition 3.2.1. [3, Definition 5] Two graphs G and H are said to be *isobraic* if and only if $A[G]$ and $A[H]$ are isomorphic algebras.

Notice that isobraicity of graphs is an equivalence relation. When G and H are isobraic graphs, we will indicate it by $G \sim H$. The equivalence class of a graph G will be denoted by $[G]$. In order to work on this relation, we first give an alternative approach to the graph magma algebras which is equivalent to the previous ones.

Definition 3.2.2. [3, Definition 6] Let A be an algebra and V be a subset of A.

(1) V is called a *set of vertices* if for every $u, v \in V$, $uv \in \{u, 0\}$.

(2) A set of vertices V is called a *spanning set of vertices* if $1 \notin < V >$, and $V \cup \{0\}$ spans A as a vector space.

(3) A spanning set of vertices V is a *base of vertices* if $B = \{1\} \sqcup V$ is a basis for A.

(4) Let V be a base of vertices. Then the graph $G = (V, E)$, where $(u, v) \in E$ if and only if $uv = u$, is called *the graph induced by base of vertices*. Notice that the graph G induces an algebra isomorphic to A.

(5) If two bases of vertices induce isomorphic graphs, then we say that bases of vertices themselves are isomorphic (as graphs).

Remark 3.2.3. Do not confuse the base of the vertices and the spaning set of the vertices with the expressions we know from linear algebra. Either kind is not really a spanning set of A; one must add the element 1 to V to obtain a spanning set or a basis.

The following result can be obtained by Definition 3.2.2 directly.

Proposition 3.2.4. [3, Proposition 1] *The following statements hold:*

(1) A is a graph magma algebra if and only if A has a base of vertices.

(2) The elements of a base of vertices are either idempotent or nilpotent. Nilpotent elements are either isolated vertices or source elements in the connected components of the form $N_p \oplus K_1$, where $p \in \{\infty\} \cup \mathbb{Z}^+$. Idempotent elements are isolated loops or target vertices of a direct sum.

(3) Every base of vertices is a spanning set of vertices.

(4) Since every spanning set contains a basis, an algebra A is a graph magma algebra if and only if it has a spanning set of vertices.

The next proposition highlights the importance of the perspective constructed by the bases of vertices.

Proposition 3.2.5. [3, Proposition 2] *Let G and H be two graphs.*

(1) G and H are isobraic if and only if there exists a basis of vertices in $A[G]$ whose induced graph are isomorphic to H.

(2) G and H are isobraic if and only if an algebra A has two bases of vertices V_G and V_H which induce graphs isomorphic, respectively, to G and H.

3.3 Isobraic Graphs For Commutative Graph Magma Algebras

In the commutative case, two graphs are isobraic if and only if the graphs are isomorphic. Before proving this result, we characterize nilpotent and idempotent elements in a commutative graph magma algebra.

Lemma 3.3.1. [3, Lemma 1] *Let $G = (V, E)$ be a directed commutative graph and $A[G]$ be the graph magma algebra induced by the graph G. Let $n, w \in A[G]$.*

(1) n is nilpotent if and only if n is an element of the subspace spanned by the set $\{v \in V|(v, v) \notin E\}$.

(2) If w is an idempotent, then it is an element of the subspace spanned by the set $\{w \in V|(w, w) \in E\} \cup \{1\}$.

Proof. Since $G = (V, E)$ is a commutative graph, then by Corollary 3.1.4, the graph G must be of the form $N_k \sqcup K_1^{(m)}$. Let $\{v_i\}_{i=1}^{k}$ and $\{w_j\}_{j=1}^{m}$ be the nilpotent and idempotent elements of V, respectively. Take $n, w \in A[G]$. Then $n = \alpha + \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j$ and $w = \beta + \sum_{i=1}^{k} \gamma_i v_i + \sum_{j=1}^{m} \lambda_j w_j$, where $\alpha, \beta, \alpha_i, \beta_j, \gamma_i, \lambda_j \in F$.

(1) The necessity is obvious. For sufficiency, assume that n is a nilpotent element of $A[G]$. Then $n^2 = 0$. It follows from the equation $(\alpha + \sum_{i=1}^{k} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j)^2 = 0$ that $\alpha^2 = 0$,

$2\alpha\beta_1 + \beta_1^2 = 0, \ldots, 2\alpha\beta_m + \beta_m^2 = 0$. Hence, one gets that $\alpha = 0$ and $\beta_i = 0$ for all $i = 1, \ldots, m$. Thus, n can be written as a linear combination of nilpotent elements in V.

(2) Let w be an idempotent element in $A[G]$. Then

$$\begin{aligned} w^2 &= (\beta + \sum_{i=1}^{k} \gamma_i v_i + \sum_{j=1}^{m} \lambda_j w_j)^2 \\ &= \beta^2 + 2\beta \sum_{i=1}^{k} \gamma_i v_i + 2\beta \sum_{j=1}^{m} \lambda_j w_j + \sum_{j=1}^{m} \lambda_j^2 w_j \\ &= \beta^2 + \sum_{i=1}^{k} 2\beta\gamma_i v_i + \sum_{j=1}^{m} (2\beta\lambda_j + \lambda_j^2) w_j. \end{aligned}$$

Since $w^2 = w$, one obtains that $\beta^2 = \beta$, $2\beta\gamma_i = \gamma_i$ and $2\beta\lambda_j + \lambda_j^2 = \lambda_j$ for all i, j. Hence, $\beta = 0$ or $\beta = 1$. Then $\gamma_i = 0$ for all i, and $2\beta\lambda_j + \lambda_j^2 = \lambda_j$ for all j. It follows that

$$w = \beta + \sum_{j=1}^{m} \lambda_j w_j$$

which shows that w is a linear combination of idempotents of V and the unit. $\square$

Remark 3.3.2. Let w be an idempotent vertex in the commutative graph G. $1 + w$ is not an idempotent element in $A[G]$ unless the characteristic of the field F is 2.

Lemma 3.3.3. [3, Lemma 1] *Consider the commutative graph magma algebra $A[G]$ induced by the graph $G = N_k \sqcup K_1^{(m)}$. If V and V^* are two bases of $A[G]$, then V and V^* have the same number of idempotents and nilpotents.*

Proof. Let $\{v_i\}_{i=1}^k$ and $\{w_j\}_{j=1}^m$ be the nilpotent and idempotent elements of V, respectively. By Lemma 3.3.1, any nilpotent element of V^* is contained in the subspace generated by the nilpotent elements of V. Therefore, V^* has at most k nilpotent elements. Reversing the roles of V and V^*, one gets both bases have the same number of nilpotent elements.

Again by Lemma 3.3.1, any idempotent element of V^* is contained the subspace generated by $\{1\} \sqcup \{w_j\}_{j=1}^m$. Since 1 is not contained in the subspace spanned by V^*, it is a proper subspace of $< \{1\} \sqcup \{w_j\}_{j=1}^m >$. Then V^* has at most m idempotent elements. Reversing the roles of V and V^*, one gets that they have the same number of idempotent elements. $\square$

As a direct consequence of Lemma 3.3.1, the following characterization is obtained.

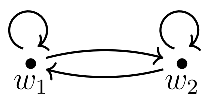
Proposition 3.3.4. [3, Lemma 1] *Let $A[G]$ and $A[H]$ be two commutative algebras. Then $A[G] \cong A[H]$ if and only if the graphs G and H are isomorphic.*

3.4 Isobraic Graphs For Non-Commutative Graph Magma Algebras

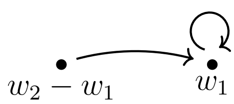
Proposition 3.3.4 is not true for non-commutative graph magma algebras, i.e., there are non-isomorphic graphs which induce isomorphic graph magma algebras. This will be illustrated by the following examples.

Example 3.4.1. [3, Example 4] The isobraicity class $[K_2]$ of K_2 contains $N_1 \oplus K_1$.

K_2 :



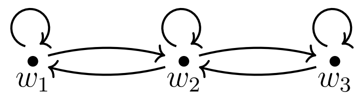
$N_1 \oplus K_1$:



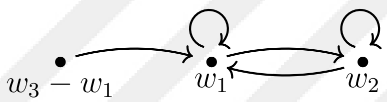
We have that $< w_1, w_2, 1 >= A[K_2]$. Obviously, $< w_1, w_2, 1 >=< w_1, w_2 - w_1, 1 >$. Since $(w_2 - w_1)^2 = 0$, $(w_2 - w_1)w_1 = w_2 - w_1$, $w_1(w_2 - w_1) = 0$ and $w_1^2 = w_1$, the second graph represents $N_1 \oplus K_1$. Therefore, K_2 and $N_1 \oplus K_1$ generate the same algebra, although the graphs are not isomorphic.

Example 3.4.2. [3, Example 4] The isobraicity class $[K_3]$ of K_3 contains $N_1 \oplus K_2$ and $N_2 \oplus K_1$.

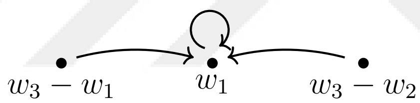
K_3:



$N_1 \oplus K_2$:



$N_2 \oplus K_1$



We have that $< \{w_1, w_2, w_3, 1\} >= A[K_3] =< \{1, w_1, w_2, w_3 - w_1\} >=< \{1, w_1, w_3 - w_1, w_3 - w_2\}$. The relations give that the second graph represents $N_1 \oplus K_2$ and the third one represents $N_2 \oplus K_1$.

Now this observation will be generalized in the next proposition. Notice that since K_1 and N_1 are not isobraic, in order to determine the isobraicty class of a connected graph, we assume that the graph has more than one vertex.

Proposition 3.4.3. [3, Proposition 3] *The complete graph K_m is isobraic to the every graph of the form $N_k \oplus K_p$, where $k + p = m$, $p \geq 1$ and $m \geq 1$ (k, m and p may be infinite, possibly uncountable).*

Proof. Suppose that V is the basis of vertices for $A[K_m]$. Then it consists of the vertices of K_m. Consider a partition $X \sqcup Y$ for V such that $|X| = k$ and $|Y| = p$. Pick a fixed element $y \in Y$. Define $Z = \{x - y | x \in X\}$. Then the set $Z \sqcup Y$ is a basis of vertices isomorphic to $N_k \oplus K_p$. □

Corollary 3.4.4. [3, Proposition 3] *If G and H are two connected graphs with more than one vertex and have the same number of vertices, then they are isobraic.*

Proposition 3.4.5. [3, Proposition 4] If two graphs G and H are, respectively, of the forms

$$\bigsqcup_{i \in I} G_i \text{ and } \bigsqcup_{i \in I} H_i$$

and, for all $i \in I$, G_i and H_i are pairwise isobraic connected associative graphs, then G is isobraic to H.

Proof. Let V_i and V_i^* be bases of vertices of G_i and H_i for all $i \in I$, respectively. Since $< V_i \cup \{1\} > \cong < V_i^* \cup \{1\} >$ for all $i \in I$ by hypothesis, and $V = \sqcup_{i \in I} V_i$, $V^* = \sqcup_{i \in I} V_i^*$ are bases of vertices of G and H,

$$\begin{aligned} A[G] &= < \sqcup_{i \in I} V_i \cup \{1\} > \\ &= (\bigoplus_{i \in I} < V_i >) \oplus F \cong (\bigoplus_{i \in I} < V_i^* >) \oplus F \\ &= A[H] \end{aligned}$$

This completes the proof. □

To complete the characterization of the isobraicity class of K_m for all m (finite or infinite, possibly uncountable), one must show that a connected graph cannot be isobraic to a disconnected one. First we need the following lemmas.

Lemma 3.4.6. [3, Lemma 2] *Every quotient A/I of graph magma algebra A with $I \neq A$ is itself a graph magma algebra.*

Proof. Assume that A is a graph magma algebra with basis of vertices V and I is an ideal of A. Let be $x+I, y+I \in \overline{V}$. Since V is basis of vertices of A, for $x, y \in V$, $xy \in \{x, 0\}$, $1 \notin < V >$ and $V \cup \{1\}$ spans A. Thus, $(x+I)(y+I) = xy+I \in \{x+I, I\}$, $1+I \notin < \overline{V} >$ and $< \overline{V} \cup \{1+I\} >= A/I$. By Definition 3.2.2, $\overline{V}$ is basis of vertices for A/I. A/I is a graph magma algebra by Proposition 3.2.4. □

Lemma 3.4.7. [3, Lemma 3] *For any basis of vertices $V = G \sqcup H$ of a graph magma algebra A, $< G >$, the subspace of A spanned by G, is an ideal of A.*

Proof. Let $x, y \in < G >$ and $r \in A$. Then $x = \sum_{i=1}^{n} \alpha_{g_i} g_i$, $y = \sum_{i=1}^{t} \beta_{g_i} g_i$ and $r = \alpha_0 + \sum_{i=1}^{m} \lambda_{g_i} g_i + \sum_{j=1}^{k} \gamma_{h_j} h_j$, where g_i's are vertices of G and h_j's are vertices of H. Then $x+y = \sum(\alpha_{g_i} + \beta_{g_i}) g_i \in < G >$. Since G and H are disjoint, $gh = hg = 0$ for all elements $g \in G$ and $h \in H$. Therefore,

$$
\begin{aligned}
rx &= (\alpha_0 + \sum_{i=1}^{m} \lambda_{g_i} g_i + \sum_{j=1}^{k} \gamma_{h_j} h_j)(\sum_{i=1}^{n} \alpha_{g_i} g_i) \\
&= \sum_{i=1}^{n} \alpha_0 \alpha_{g_i} g_i + \sum_{i=1}^{m} \sum_{i=1}^{n} \lambda_{g_i} \alpha_{g_i} g_i g_i
\end{aligned}
$$

and

$$
\begin{aligned}
xr &= (\sum_{i=1}^{n} \alpha_{g_i} g_i)(\alpha_0 + \sum_{i=1}^{m} \lambda_{g_i} g_i + \sum_{j=1}^{k} \gamma_{h_j} h_j) \\
&= \sum_{i=1}^{n} \alpha_0 \alpha_{g_i} g_i + \sum_{i=1}^{m} \sum_{i=1}^{n} \lambda_{g_i} \alpha_{g_i} g_i g_i
\end{aligned}
$$

Since $g_i g_i \in \{g_i, 0\}$, one gets that $rx \in < G >$ and $xr \in < G >$. Consequently, $< G >$ is an ideal of A. □

Theorem 3.4.8. [3, Theorem 3] *The following statements hold:*

1. *(1) A graph of the form $N_1 \sqcup G$ is not isobraic to any graph having only idempotent vertices.*
2. *(2) If $m < p$, then $N_m \sqcup G$ is not isobraic to $N_p \sqcup G^*$, where G is generated by idempotents.*

Proof. (1) Let e be an idempotent element in $A[N_1 \sqcup G]$. Then $e = \alpha + \beta v_1 + x$, where $x \in< G >$, $v_1 \in N_1$ and $\alpha, \beta \in F$. It follows that $e^2 = \alpha^2 + 2\alpha\beta v_1 + y$, where $y \in< G >$. Since e is an idempotent, one obtains $\alpha^2 = \alpha$ and $2\alpha\beta = \beta$. Therefore, $\alpha \in \{0, 1\}$. In both cases, one gets that $\beta = 0$. This implies that $e = \alpha + x$, where $x \in< G >$. Consequently, $e \in< G \sqcup \{1\} >$. Since $< G \sqcup \{1\} >$ is a proper subspace of $A[N_1 \sqcup G]$, any graph having only idempotent vertices does not span $A[N_1 \sqcup G]$.

(2) Let $m < p$ and G be a graph which is generated by idempotents and G^* be an arbitrary graph. Assume, on the contrary, that $N_m \sqcup G$ and $N_p \sqcup G^*$ are isobraic. Then $A[N_m \sqcup G] \cong A[N_p \sqcup G^*]$. Let X and Y be graphs that are isomorphic to $N_m \sqcup G$ and $N_p \sqcup G^*$, respectively. Hence, X is of the form $N_X \sqcup G_X$ and Y is of the form $N_Y \sqcup G^*_Y$ such that $N_X \simeq N_m$, $N_Y \simeq N_p$, $G_X \simeq G$ and $G^*_Y \simeq G^*$. It follows by assumption that $A[X] \cong A[Y]$. Therefore, $\frac{A[X]}{<N_x>} \cong \frac{A[Y]}{<N_x>}$, and by Lemma 3.4.7, $\frac{A[X]}{<N_x>}$ and $\frac{A[Y]}{<N_x>}$ are graph magma algebras . Then $\{g+ < N_X > | g \in G_X \sqcup \{1\}\}$ is a basis of vertices of $\frac{A[X]}{<N_x>}$ and $\{a+ < N_X > | \, a \in N_Y \sqcup G^*_Y \sqcup \{1\}\}$ is a spanning set of vertices of $\frac{A[Y]}{<N_x>}$. Since $m < p$, at least one of the elements $a+ < N_m >$ with $a \in N_p$ is non-zero. Therefore, this spanning set of vertices must contain a basis of vertices of the form $N_1 \sqcup H$. But this contradicts to the statement (1). □

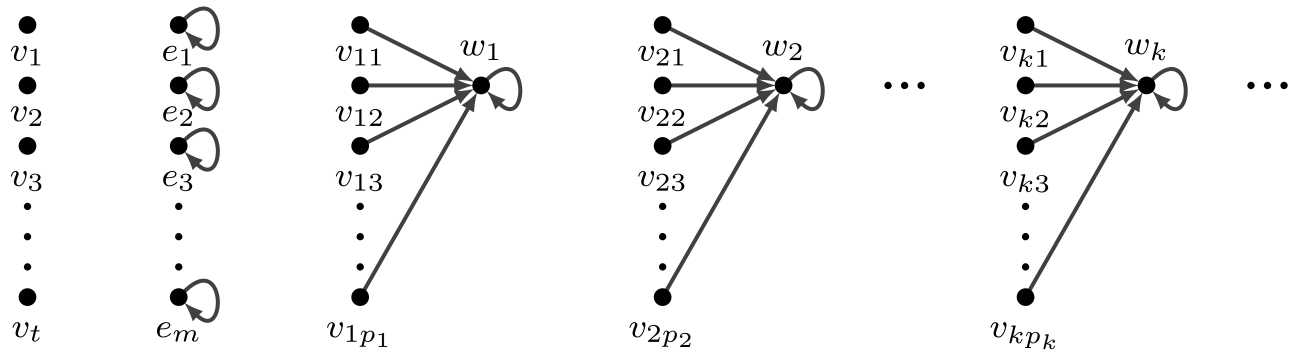
4. RING STRUCTURE OF GRAPH MAGMA ALGEBRAS

Throughout, G is a simple directed graph, $M(G)$ is an associative graph magma, and $R = A[G]$ is a magma algebra generated by the graph G. We know that from Proposition 3.4.3, K_m isobraic to $N_k \oplus K_1$, where $m = k+1$. Therefore, $\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1$ isobraic to $\bigsqcup_{j=1}^{k} K_{p_j+1}$ by Proposition 3.4.5. Thus, we will consider the graph, which is of the form

$$G = N_t \sqcup K_1{}^{(m)} \sqcup (\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1),$$

where t, m, k may be zero, finite, or (countably or uncountably) infinite, and each p_i is non-zero but possibly (countably or uncountably) infinite.

In this section, unless otherwise stated, $\{v_1, v_2, \dots, v_t\}$ denote the isolated nilpotent vertices of N_t, $\{e_1, e_2, \dots, e_m\}$ represent isolated idempotent vertices of $K_1^{(m)}$, and for each $j = 1, \dots, k$, $\{v_{j1}, v_{j2}, \dots, v_{jp_j}\}$ represent of source nilpotent vertices of component $N_{p_j} \oplus K_1$, with w_j representing its target idempotent. We may illustrate the graph G as follows:



4.1 The Jacobson Radical of Graph Magma Algebras

In this subsection, the structure of the idempotent elements in the graph magma algebra and the Jacobson radical has been established. The Jacobson radical $J(R)$ contains all left nilpotent ideals, but $J(R)$ itself need not be a nilpotent ideal. However, in the graph magma algebra, the Jacobson radical is a nilpotent ideal generated by nilpotent elements with the nilpotency index 2.

Proposition 4.1.1. *If x is an idempotent vertex of $R = A[G]$, then x is either in the form $\sum_{i\in I} e_i + \sum_{j\in J} w_j + \sum_{j\in J} \gamma_i^{(j)} v_{ji}$ or $1 - \sum_{i\in I} e_i - \sum_{j\in J} w_j + \sum_{j\in J} \gamma_i^{(j)} v_{ji}$, where $\gamma_i^{(j)} \in F$, I and J are finite sets.*

Proof. Let

$$x = \alpha + \sum_{\text{finite}} \alpha_i v_i + \sum_{\text{finite}} \beta_i e_i + \sum_{\text{finite}} \gamma_i^{(j)} v_{ji} + \sum_{\text{finite}} \lambda_j w_j$$

be an idempotent in $R = A[G]$, where $\alpha, \alpha_i, \beta_i, \gamma_i^{(j)} \in F$. Then

$$x^2 = \alpha x + \alpha \sum_{\text{finite}} \alpha_i v_i + \alpha \sum_{\text{finite}} \beta_i e_i + \sum_{\text{finite}} \beta_i^2 e_i + \alpha \sum_{\text{finite}} \gamma_i^{(j)} v_{ji} + \sum_{\text{finite}} \gamma_i^{(j)} v_{ji} + \alpha \sum_{\text{finite}} \lambda_j w_j + \sum_{\text{finite}} \lambda_j^2 w_j.$$

Since $x^2 = x$, one obtains that $\alpha = \alpha^2, \alpha_i = 2\alpha\alpha_i, \beta_i = 2\alpha\beta_i + \beta_i^2, \gamma_i^{(j)} = 2\alpha\gamma_i^{(j)} + \lambda_j\gamma_i^{(j)}$ and $\lambda_j = 2\alpha\lambda_j + \lambda_j^2$. Hence, $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$, then $\alpha_i = 0, \beta_i = 0$ or $\beta_i = 1$, $\lambda_j = 0$ or $\lambda_j = 1$. $\gamma_i^{(j)} = 0$ in case $\lambda_j = 0$. It follows that

$$x = \sum_{i\in I} e_i + \sum_{j\in J} w_j + \sum_{j\in J} \gamma_i^{(j)} v_{ji},$$

where $I \subseteq \{1, 2, \ldots\}$, $J \subseteq \{1, 2, \ldots\}$, and sums over empty sets are taken as zero.

If $\alpha = 1$, then $\alpha_i = 0, \beta_i = 0$ or $\beta_i = -1$, $\lambda_j = 0$ or $\lambda_j = -1$. Again $\gamma_i^{(j)} = 0$, in case $\lambda_j = 0$. It follows that

$$x = 1 - \sum_{i\in I} e_i - \sum_{j\in J} w_j + \sum_{j\in J} \gamma_i^{(j)} v_{ji},$$

where $I \subseteq \{1, 2, \ldots\}$, $J \subseteq \{1, 2, \ldots\}$, and sums over empty sets are taken as zero. □

Corollary 4.1.2. *Let x and y be idempotent elements in $R = A[G]$.*

(1) $x = \sum_{i\in I_1} e_i + \sum_{j\in J_1} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji}$ *and* $y = \sum_{i\in I_2} e_i + \sum_{j\in J_2} w_j + \sum_{j\in J_2} \theta_i^{(j)} v_{ji}$ *are orthogonal idempotents if and only if* $I_1 \cap I_2 = \emptyset$ *and* $J_1 \cap J_2 = \emptyset$.

(2) If $x = \sum_{i\in I_1} e_i + \sum_{j\in J_1} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji}$ *and* $y = 1 - \sum_{i\in I_2} e_i - \sum_{j\in J_2} w_j + \sum_{j\in J_2} \theta_i^{(j)} v_{ji}$ *then*

(i) $xy = 0$ *if and only if* $I_1 \subseteq I_2$ *and* $J_1 \subseteq J_2$.

(ii) $yx = 0$ *if and only if* $I_1 \subseteq I_2$, $J_1 \subseteq J_2$, *and* $\gamma_i^{(j)} + \theta_i^{(j)} = 0$ *for all* $1 \leq i \leq p_j$ *and for all* $j \in J_1$.

Proof. (1) Let $x = \sum_{i\in I_1} e_i + \sum_{j\in J_1} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji}$ and $y = \sum_{i\in I_2} e_i + \sum_{j\in J_2} w_j + \sum_{j\in J_2} \theta_i^{(j)} v_{ji}$. Then

$$xy = \sum_{i\in I_1\cap I_2} e_i + \sum_{j\in J_1\cap J_2} w_j + \sum_{j\in J_1\cap J_2} \gamma_i^{(j)} v_{ji},$$

and so $xy = 0$ if and only if $I_1 \cap I_2 = \emptyset$ and $J_1 \cap J_2 = \emptyset$ if and only if $yx = 0$.

(2) Let $x = \sum_{i\in I_1} e_i + \sum_{j\in J_1} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji}$ and $y = 1 - \sum_{i\in I_2} e_i - \sum_{j\in J_2} w_j + \sum_{j\in J_2} \theta_i^{(j)} v_{ji}$.

(i) We obtain the product

$$xy = \sum_{i\in I_1} e_i - \sum_{i\in I_1\cap I_2} e_i + \sum_{j\in J_1} w_j - \sum_{j\in J_1\cap J_2} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji} - \sum_{j\in J_1\cap J_2} \gamma_i^{(j)} v_{ji},$$

and so $xy = 0$ if and only if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$.

(ii) We obtain the product

$$yx = \sum_{i\in I_1} e_i - \sum_{i\in I_1\cap I_2} e_i + \sum_{j\in J_1} w_j - \sum_{j\in J_1\cap J_2} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji} + \sum_{j\in J_1\cap J_2} \theta_i^{(j)} v_{ji}.$$

It follows that $yx = 0$ if and only if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$, and $\gamma_i^{(j)} + \theta_i^{(j)} = 0$ for all $1 \leqslant i \leqslant p_j$ and $j \in J_1$.

□

The following result shows that the Jacobson radical $J(R)$ of $R = A[G]$ is nilpotent.

Proposition 4.1.3. *The Jacobson radical of $R = A[G]$ is*

$$J(R) = (\bigoplus Fv_i) \oplus (\bigoplus Fv_{ji}),$$

where $\{v_1, v_2, \ldots, v_t\}$ is the isolated nilpotent vertices and $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ is the source nilpotent vertices of component $N_{p_j} \oplus K_1$. Moreover, $J(R)^2 = 0$.

Proof. Let

$$x = \alpha + \sum_{\text{finite}} \alpha_i v_i + \sum_{\text{finite}} \beta_i e_i + \sum_{\text{finite}} \gamma_i^{(j)} v_{ji} + \sum_{\text{finite}} \lambda_j w_j \in J(R).$$

If $\alpha \neq 0$, then $1 - \alpha^{-1}x$ is a unit in R (by Theorem 2.5.3), which is impossible since $1 - \alpha^{-1}x$ has zero constant term. Then $\alpha = 0$. If $\beta_j \neq 0$ for some j, then $1 - \beta_j^{-1} e_j x = 1 - e_j$ is a unit, but this is a contradiction since $1 - e_j$ is an idempotent and $e_j \neq 1$. Then $\beta_j = 0$ for every j. Similarly, $\gamma_i = 0$ for every i. It follows that $J(R) \subseteq (\bigoplus Fv_i) \oplus (\bigoplus Fv_{ji})$. For the reverse inclusion, pick $y \in (\bigoplus Fv_i) \oplus (\bigoplus Fv_{ji})$. We will now show that $1 - ry$ is a unit for each $r \in R$. Let $r \in R$. Since $xv = 0$ for all vertex x and for all nilpotent vertex v, $ry = \alpha y$, where $\alpha \in F$. Hence $1 - ry = 1 - \alpha y$, and so $(1 - \alpha y)(1 + \alpha y) = 1$. It follows that $1 - ry = 1 - \alpha y$ is a unit. Consequently, $y \in J(R)$. This completes the proof. □

4.2 Simple Modules of Graph Magma Algebras

In this subsection, we deal with simple left and right modules over a graph magma algebra. Note that by the *dimension* of an F-algebra, we always mean its dimension as an F-vector space.

The following statements provide information about the cyclic left and right submodules generated by a vertex:

Lemma 4.2.1. [3, Lemma 5] *Let V be the basis of vertices of $R = A[G]$ and $v, w \in V$.*

(1) If v and w are nilpotent (either coming from a copy of N_1 or a source vertex from a copy of $N_p \bigoplus K_1$), then the subspaces $Rv = Fv$ and $Rw = Fw$ are simple left modules of dimension 1. Additionally, the left modules Rv and Rw are isomorphic and the left ideals Rv and Rw are nilpotent with nilpotency 2.

(2) If v is a source in a component of the form $N_p \oplus K_1$ with target w, then $Rv \subset Rw =< N_p \oplus K_1 >$, where $N_p \oplus K_1$ has $p + 1$ elements as a basis in the subspace of R.

(3) If v, w are isolated idempotent vertices from distinct connected components in the form of K_1, then $Rv = Fv$ and $Rw = Fw$ are non-isomorphic projective simple left modules of dimension 1.

(4) If w is an idempotent, then wRw is a local ring.

(5) If v and w are distinct target idempotents, then $Rv/J(R)v$ and $Rw/J(R)w$ are non-isomorphic simple left modules of dimension 1.

Proof. (1) Let v be a nilpotent vertex. Then, for all $x \in V$, we have $(x, v) \notin E$. A typical element r of R is of the form $\alpha + \{$an F-linear combination of vertices from $G\}$. Thus, $rv = \alpha v$.

(2) Let v be a source nilpotent in the component $N_p \oplus K_1$, and let w be its target idempotent. Since $vw = v$ (i.e. there is an edge from v to w), we have $rv = r(vw)$ (clearly, $Rv \neq Rw$ because $w \notin Rv$). Thus, $Rv \subset Rw$. Now we will show that $Rw =< N_p \oplus K_1 >$. Pick $a \in< N_p \oplus K_1 >$. Then $a = \sum_{i \in I} \alpha_i v_i + \beta w$, where $\alpha_i, \beta \in F$, $I \subseteq \{1, \ldots, p\}$ is finite and v_i's are source nilpotents. Since $Rv \subset Rw$, we have that $a \in Rw$. Conversely, if $a \in Rw$, then there exists an element $r \in R$ such that $a = rw$. Write $r = \alpha + \{$an F-linear combination of vertices from $N_p \oplus K_1\} + \{$an F-linear combination of vertices from the other connected component of $G\}$. Since $xw = w$ for all vertex x in $N_p \oplus K_1$ and $yw = 0$ for all vertices y from other connected components of G, we obtain that $a = rw = \alpha w + \{$an F-linear combination of vertices from $N_p \oplus K_1\}$. Therefore, $a \in< N_p \oplus K_1 >$.

(3) Let v and w be isolated idempotents and $r \in R$. Write $r = \alpha + \beta v + \gamma w +$ {an F-linear combination of the remaining vertices from G}, where $\alpha, \beta \in F$. Since $(x, v) \notin E$ for each $x \in V - \{v\}$ and $(y, w) \notin E$ for each $y \in V - \{w\}$, we have $rv = \alpha v + \beta v = (\alpha + \beta)v$ and $rw = \alpha w + \gamma w = (\alpha + \beta)w$. This shows that $Rv = Fv$ and $Rw = Fw$, indicating that Rv and Rw are simple left modules. Since w and v are idempotent elements, $Rv = Fv$ and $Rw = Fw$ are projective by Corollary 2.8.8. Now, we will show that Rv and Rw are non-isomorphic. Assume, to the contrary, that there is an isomorphism $f : Rv \to Rw$ such that $f(v) = \gamma w$, where $\gamma \in F$. For $0 \neq r = v$, $f(v) = f(v^2) = vf(v) = v\gamma w = \gamma vw = 0$, which leads to a contradiction.

(4) As seen in condition (2) and (3), if w is idempotent vertex, then either $Rw = Fw$ or $Rw =< N_p \oplus K_1 >$. If $Rw = Fw$, then the proof is straightforward. Assume that $Rw =< N_p \oplus K_1 >$. Hence, $wRw = w < N_p \oplus K_1 >$. We will now show that $w < N_p \oplus K_1 >= Fw$. Let $x \in w < N_p \oplus K_1 >$. Then there exists $y \in< N_p \oplus K_1 >$ such that $x = wy$. A typical element y in $< N_p \oplus K_1 >$ is of the form $y = \alpha + \sum_{i \in I} \alpha_i v_i + \beta w$, where $\alpha_i, \beta \in F$, $I \subseteq \{1, \ldots, p\}$ is finite and v_i's are source nilpotents and w is its target idempotent. Since $(w, v_i) \notin E$ for all $i \in I$, $x = wy = w(\alpha + \sum_{i \in I} \alpha_i v_i + \beta w) = w\alpha + w \sum_{i \in I} \alpha_i v_i + w\beta w = w\alpha + \sum_{i \in I} \alpha_i w v_i + \beta ww = w\alpha + \beta w = (\alpha + \beta)w$. Thus, $w < N_p \oplus K_1 >\subseteq wF$. Now pick $x \in wF$. Then $x = w\alpha$, for some $\alpha \in F$. Since w is an idempotent, we have that $x = w\alpha w$, which gives that $wF \subseteq w < N_P \oplus K_1 >$. Hence, in either case, wRw is isomorphic to F.

(5) Let v be a target idempotent element of $< N_p \oplus K_1 >$ and $J(R)$ be the Jacobson radical of R. Then, by (2), $Rv =< N_p \oplus K_1 >$. Since $J(R)v = Fv_i$, where v_i is a source nilpotent element of $N_p \oplus K_1$, it follows that $Rv/J(R)v$ is a simple left module. Assume that v and w are distinct target idempotents. If there is an isomorphism f between $Rv/J(R)v$ and $Rw/J(R)w$ such that $f(v + J(R)v) = \alpha w + J(R)w$, then for $0 \neq r = v$, we have that $f(v(v + J(R)v)) = vf(v + J(R)v) = v(\alpha w + J(R)w) = \alpha vw + J(R)w = 0$. But $f(v(v + J(R)v)) = f(v^2 + J(R)v) = f(v + J(R)v)$. Hence, $0 \neq v \in \ker f$, a contradiction. □

Corollary 4.2.2. [3, Proposition 5] *Let V be the basis of vertices of $R = A[G]$ and $v, w \in V$.*

(1) *If w is an idempotent vertex, then Rw is an indecomposable left R-module.*

(2) *If w is the target idempotent of the component of the form $N_p \oplus K_1$, then $Soc(Rw) = \bigoplus_{i=1}^{p} Rv_i$, where v_i's are source nilpotent vertex of w.*

Proof. (1) If w is an idempotent vertex, then wRw is a local ring, by Lemma 4.2.1. Consequently, Rw is indecomposable by Corollary 2.2.14 and Lemma 2.6.2

(2) Rv_i is a simple submodule of Rw by Lemma 4.2.1. Then $\bigoplus_{i=1}^{p} Rv_i$ is a submodule of $Soc(Rw)$. Since $Soc(Rw)$ is a proper submodule of Rw, the result follows from a dimension consideration. □

Corollary 4.2.3. *Let w be the target idempotent of the component of the form $N_p \oplus K_1$. Then $Soc(Rw)$ is essential submodule of Rw.*

Proof. By Corollary 4.2.2(2), $Rw/Soc(Rw)$ is a simple submodule of dimension 1. According to Proposition 2.10.13, $Rw/Soc(Rw)$ is either projective or singular. If $Rw/Soc(Rw)$ were projective, $Soc(Rw)$ would have to be the direct summand of Rw by Proposition 2.8.7, which would contradict the fact that Rw is indecomposable. Thus, $Rw/Soc(Rw)$ is singular. □

Lemma 4.2.4. *Let V be the basis of vertices of $R = A[G]$ and $v, w \in V$.*

(1) *If v and w are nilpotent (either a copy of N_1 or a source vertex from a copy of $N_p \oplus K_1$), then $vR = vF$ and $wR = wF$ are non-isomorphic simple right modules of dimension* 1*. Additionally, the right ideals vR and wR are nilpotent with nilpotency index 2.*

(2) *If v and w are idempotent (i.e., coming from a copy of K_1 or a target vertex from a copy of $N_p \oplus K_1$), then $vR = vF$ and $wR = wF$ are subspaces of R with dimension* 1*, making them simple right modules. Moreover, the right modules vR and wR are non-isomorphic (*see [3, Proof of Proposition 6]).

(3) If v is a source in a component of the form $N_p \bigoplus K_1$ and w is its target, then vR is isomorphic to wR.

Proof. (1) Let v be a nilpotent vertex in N_1, w be a nilpotent vertex in the component of the form $N_p \oplus K_1$, and let w' be its target vertex. An arbitrary element r of R can be expressed as $r = \alpha + \beta w' + \{$an F-linear combination of vertices from $G\}$. Therefore, $vr = \alpha v$ and $wr = w\alpha + w\beta = w(\alpha + \beta)$, confirming our first claim. Now suppose that there is an isomorphism $f : wR \to vR$ with $f(w) = \gamma v$, where $\gamma \in F$. Then for $0 \neq r = w'$, we have $f(ww') = f(w)w' = \gamma vw' = 0$. On the other hand, $f(ww') = f(w)$. Then $v = f(w) = 0$, a contradiction.

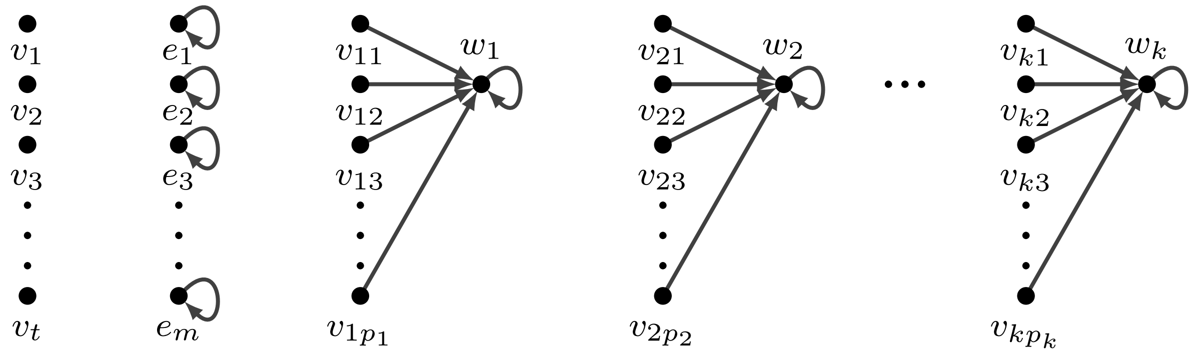
(2) Let v be an isolated idempotent vertex, and w be the target idempotent in the component $N_p \oplus K_1$. An arbitrary element r of R can be expressed as $r = \alpha + \beta v + \gamma w + \{$an F-linear combination of the remaining vertices from $G\}$. Then we have $vr = v(\alpha + \beta)$ and $wr = w(\alpha + \gamma)$. This shows that vR and wR are simple right R-module. If $f : wR \to vR$ is an isomorphism with $f(w) = v\alpha$, where $\alpha \in F$, then for $0 \neq r = w$, we have $f(ww) = f(w)w = v\alpha w = \alpha vw = 0$ and $f(ww) = f(w)$. Thus $0 \neq w \in \ker f$, which leads to contradiction.

(3) Let v be a source and w be the target of the component $N_p \oplus K_1$. From conditions (1) and (2), $wr = w(\alpha + \beta)$ and $vr = v(\alpha + \beta)$. Then we can define a homomorphism f from vR to wR such that $f(v) = w$. It is clear that f is an isomorphism. □

From this point on, we will assume that all graphs have a finite number of non-null connected components, specifically of the form :

$$G = N_t \sqcup {K_1}^{(m)} \sqcup (\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1),$$

where t can be zero, finite, or (countably or uncountably) infinite, each p_j is non-zero and possibly (countably or uncountably) infinite, and both m and k must be finite. We can illustrate this situation as follows:



Throughout, unless stated otherwise, e denotes $1 - (e_1 + \cdots + e_m) - (w_1 + \cdots + w_k)$, where $\{e_1, e_2, \ldots, e_m\}$ represent isolated idempotent vertices of $K_1^{(m)}$, and $\{w_1, w_2, \ldots, w_k\}$ represent k target idempotent vertices of $\sqcup_{j=1}^{k} N_{p_j} \oplus K_1$. $\{v_1, v_2, \ldots, v_t\}$ denote the isolated nilpotent vertices, and $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ represent of source nilpotent vertices of component $N_{p_j} \oplus K_1$.

Remark 4.2.5. A typical element of R is a linear combination of 1 and the elements of the given basis V. If we replace 1 with e, we obtain a different basis for R. Thus, a typical element r of R is a F-linear combination of e and elements of V.

The element e is the idempotent of the ring R. The following statements provide information about the cyclic left and right submodules generated by the idempotent element e:

Lemma 4.2.6. [3, Proposition 5] *Consider the graph magma algebra $R = A[G]$ induced by the graph G which has finitely many non-null components. Then the following statements hold:*

(1) Re is an indecomposable left R-module spanned by $\{e\} \cup \{v_i | i = 1, \ldots, t\}$, where v_i is nilpotent vertex coming from N_t. Additionally, $Re = eRe$.

(2) If there is no isolated nilpotent vertex in the graph, then $Soc(Re) = Re$; otherwise, we have that $Soc(Re) = \bigoplus_{i=1}^{t} Rv_i$.

Proof. Let $r \in R$. Then

$$r = \alpha e + \sum_{i=1}^{t} \alpha_i v_i + \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{j=1}^{k} \lambda_j w_j.$$

(1) Since $e_i e = v_{ji} e = w_j e = 0$ and $v_i e = v_i$, we find

$$re = \alpha e + \sum_{i=1}^{t} \alpha_i v_i.$$

Thus, $Re \subseteq < \{e\} \cup \{v_i | i = 1, \dots, t\} >$. The reverse inclusion is evident because $v_i e = v_i$. Take an idempotent element x in Re to show that Re is indecomposable. Then $x = \alpha e + \sum_{i=1}^{t} \alpha_i v_i$. This leads to the equation $\alpha^2 e^2 + \sum_{i=1}^{t} 2\alpha\alpha_i v_i = \alpha e + \sum_{i=1}^{t} \alpha_i v_i$. Consequently, we obtain $\alpha^2 = \alpha$, $2\alpha\alpha_i = \alpha_i$ for $1 \leqslant i \leqslant t$. If $\alpha^2 = \alpha$, then α must be 0 or 1. In both cases, it follows that $\alpha_i = 0$ for all $1 \leqslant i \leqslant t$. Thus, the only idempotents in Re are e or 0, confirming that Re is indecomposable by Corollay 2.2.16.

(2) If there is no isolated nilpotent vertex in the graph, $Re = Fe$ is a simple submodule of R with dimension 1, implying $Soc(Re) = Re$. Otherwise, since Rv_i is a simple submodule of Re, $\bigoplus_{i=1}^{t} Rv_i$ is a submodule of $Soc(Re)$. As $Soc(Re)$ is a proper submodule of Re, the result follows from dimensional consideration.

□

Corollary 4.2.7. *Let $R = A[G]$ be the graph magma algebra induced by the graph G which has finitely many non-null components. Then $Soc(Re)$ is an essential submodule of Re.*

Proof. According to Proposition 2.10.13, $Re/Soc(Re)$ is either projective or singular. If $Re/Soc(Re)$ were projective, $Soc(Re)$ would have to be the direct summand of Re, which would contradict the fact that Re is indecomposable by Proposition 2.8.7 and $Soc(Re) \neq 0$. Thus, $Re/Soc(Re)$ is singular. □

Lemma 4.2.8. *Let $R = A[G]$ be the graph magma algebra induced by the graph G which has finitely many non-null components. Then the following statements hold:*

(1) eR is an indecomposable right R-module spanned by $\{e\} \cup N$, where N is the set of all nilpotent vertices in the graph.

(2) If there is no nilpotent vertex in the graph, then $Soc(eR) = eR$. Otherwise, $Soc(eR) = \bigoplus_{v \in N} vR$.

Proof. Let $r \in R$. Write

$$r = \alpha e + \sum_{i=1}^{t} \alpha_i v_i + \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{j=1}^{k} \lambda_j w_j.$$

(1) Since $ee_i = ew_j = 0$, $ev_i = v_i$, and $v_{ji} = v_{ji}$, we have

$$re = \alpha e + \sum_{i=1}^{t} \alpha_i v_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji}.$$

Hence, $eR \subseteq < \{e\} \cup \{v | v \in N\} >$, where N is set of all nilpotent vertices. The reverse inclusion holds because $ve = v$. We will show that eR is indecomposable. Let $x = \alpha e + \sum_{i=1}^{t} \alpha_i v_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji}$ be an idempotent in eR. Then $x^2 = \alpha^2 e + 2\alpha \sum_{i=1}^{t} \alpha_i v_i + 2\alpha \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji}$. Since $x^2 = x$, we have $\alpha^2 = \alpha$, $2\alpha\alpha_i = \alpha_i$, for all $i = 1, \ldots, t$ and $2\alpha\gamma_i^{(j)} = \gamma_i^{(j)}$ for all $j = 1, \ldots, k$ and $i = 1, \ldots, p_j$. It follows that $\alpha = 0$ or $\alpha = 1$. This implies that $\alpha_i = 0$ for all $i = 1, \ldots, t$ and $\gamma_i^{(j)} = 0$ for all $j = 1, \ldots, k$ and $i = 1, \ldots, p_j$. Therefore, $x = e$ or $x = 1$. Hence, eR is an indecomposable right R-module by Corollary 2.2.16.

(2) If there is no nilpotent vertex in the graph G, then $eR = eF$ is simple, which implies that $Soc(eR) = eR$. Otherwise, vR is a simple submodule of eR. Therefore, $\bigoplus_{v \in N} vR$ is a submodule of $Soc(eR)$. Since $Soc(eR)$ is a proper submodule of eR and $\dim eR = \dim < N > +1$, the result follows by a dimension consideration. □

Corollary 4.2.9. *Let $R = A[G]$ be the graph magma algebra induced by the graph G which has finitely many non-null connected components. Then $Soc(eR)$ is an essential submodule of eR.*

Proof. From above (1) and (2), $eR/Soc(eR) \cong Fe$, indicating that $eR/Soc(eR)$ is a simple submodule. Therefore, according to Proposition 2.10.13, $eR/Soc(eR)$ is either projective or singular. If $eR/Soc(eR)$ were projective, $Soc(eR)$ would have to be the direct summand of eR by Proposition 2.8.7, which would contradict the fact that eR is indecomposable and $Soc(eR) \neq 0$. Thus, $eR/Soc(eR)$ must be singular. □

Lemma 4.2.10. [8, Theorem 2.11] *Every simple submodule of Re is isomorphic to $Re/J(R)e$.*

Proof. It follows from Lemma 4.2.6 all 1-dimensional subspaces generated by isolated nilpotent vertices v_i are simple submodules of Re. We can define a homomorphism $f : Rv_i \to Re/J(R)e$ such that $f(v_i) = \alpha e + J(R)e$. Since $rv = \alpha v_i$ and $r(e + J(R)e) = \alpha e + J(R)e$ for any $r \in R$, we get that f is an isomorphism. □

Corollary 4.2.11. [8, Theorem 2.11] *Every simple submodule of Rw_j is isomorphic to $Re/J(R)e$, where w_j is a target idempotent.*

Proof. Rv_{ji} is a simple submodule of Rw_j for all $i = 1, \ldots, p_j$, and all left submodules generated by nilpotent vertices are isomorphic, as established in Lemma 4.2.1. Thus, the proof is complete based on Lemma 4.2.10. □

Lemma 4.2.12. *Every simple submodule of eR is isomorphic to either $eR/eJ(R)$ or w_jR for each $j = 1, \ldots, k$, where w_j is a target idempotent.*

Proof. All 1-dimensional subspaces generated by nilpotent vertices are simple submodules of eR by Lemma 4.2.8. By Lemma 4.2.4, we have $v_{ji}R \cong w_jR$ for all $j = 1, \ldots, k$, where v_{ji}'s are source nilpotent vertices of w_j . Since $v_ir = v_i\alpha$ and $(e + eJ(R))r = e\alpha + eJ(R)$ for any $r \in R$, one obtains that $v_iR \cong eR/eJ(R)$. □

Corollary 4.2.13. *Let $R = A[G]$ be a graph magma algebra generated by the graph G with finitely many non-null connected components, and $J = J(R)$ be the Jacobson radical of $R = A[G]$. The following statements hold, where $\{v_1, v_2, \ldots, v_t\}$ is set of isolated nilpotent* *vertices,* $\{e_1, e_2, \ldots, e_m\}$ *is set of isolated idempotent vertices of* $K_1^{(m)}$*,* $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ *is set of source nilpotent vertices of component* $N_{p_j} \oplus K_1$ *and* w_j *is its target the idempotent and* $e = 1 - \sum_{i=1}^{m} e_i - \sum_{j=1}^{k} w_j$*.*

(1) $Je_i = e_iJ = 0$ *for all* i*. Therefore,* $e_iJe_i = 0$*.*

(2) $Jw_j = \bigoplus_{i=1}^{p_j} Rv_{ji} = Soc(Rw_j) = [\frac{Re}{Je}]^{(d_j-1)}$*, where* $d_j = dimRw_j$ *and* $w_jJ = 0$ *for all* j*. Therefore,* $w_jJw_j = 0$*.*

(3) $Je = \bigoplus_{i=1}^{t} Rv_i$ *and* $eJ = J$*. Therefore,* $eJe = Je$*.*

Proof. (1) Since e_i is an isolated idempotent and J is generated by nilpotent vertices, we have $e_iJ = Je_i = 0$.
(2) $J(R) = (\bigoplus_{i=1}^{t} Rv_i) \oplus (\bigoplus_{j=1}^{k} \bigoplus_{i=1}^{p_j} Rv_{ji})$ and $d_j = dimRw_j = p_j + 1$. Since v_{ji} are source nilpotent vertices of the component $N_{p_j} \oplus K_1$ for all $i \in \{1, \ldots, p_j\}$ and w_j its target idempotent, we have $v_iw_j = 0$ for all $i = 1, \ldots, t$, $v_{ji}w_j = v_{ji}$ for all $i \in \{1, \ldots, p_j\}$ and $w_jv = 0$ for all v nilpotent vertices. Thus, we conclude that $Jw_j = \bigoplus_{i=1}^{p_j} Rv_{ji}$, $w_jJ = 0$ and $w_jJw_j = 0$. From Corollary 4.2.2, it follows that $Soc(Rw_j) = \bigoplus_{i=1}^{p_j} Rv_{ji} = \bigoplus_{d_j-1} Rv_{ji}$. Noting that $Rv_{ji} \cong Re/Je$, we get $\bigoplus_{d_i-1} Rv_{ji} \cong \bigoplus_{d_j-1} Re/Je$.

(3) Since $v_ie = v_i$, $v_{ji}e = 0$, and $ev_i = ev_{ji}$, we have $Je = \bigoplus_{i=1}^{t} Rv_i$. Additionally, since $ev_i = v_i$ and $ev_{ji} = v_{ji}$ we conclude that $eJ = J$. Thus, $eJe = Je$. □

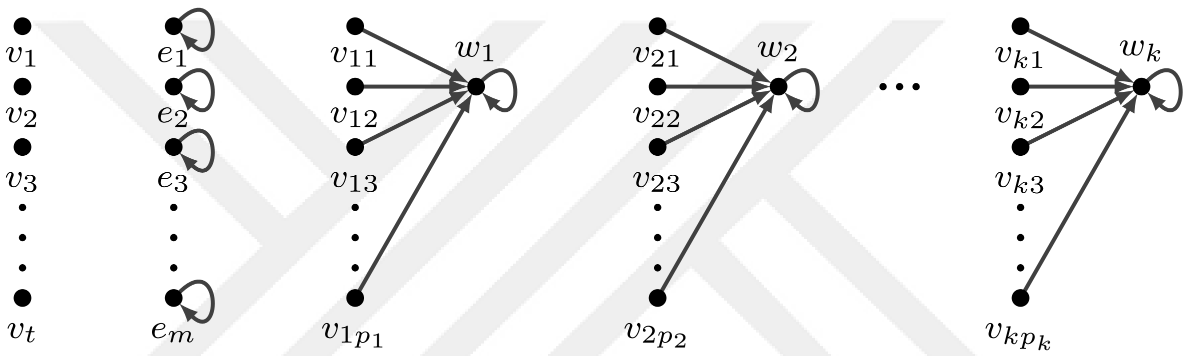
4.3 A Characterization of Semiperfect Rings

In [3], graph magma algebras with finitely many non-null connected components are fully characterized. It is shown that such algebras are exactly semiperfect rings. Moreover, it is proved that a graph has finitely many vertices if and only of $R = A[G]$ is right (left) Artinian, if and only if R is right (left) Noetherian.

We still assume that all graphs have a finite number of non-null connected components, specifically of the form

$$G : N_t \sqcup {K_1}^{(m)} \sqcup (\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1),$$

where t can be zero, finite, or (countably or uncountably) infinite, each p_j is non-zero and possibly (countably or uncountably) infinite, and both m and k must be finite. We can illustrate the graph G as follows:



Throughout, e denotes $1 - (e_1 + \cdots + e_m) - (w_1 + \cdots + w_k)$, where $\{e_1, e_2, \ldots, e_m\}$ represent isolated idempotent vertices of $K_1^{(m)}$, and $\{w_1, w_2, \ldots, w_k\}$ represent k target idempotent vertices of $\sqcup_{j=1}^{k} N_{p_j} \oplus K_1$. $\{v_1, v_2, \ldots, v_t\}$ denote the isolated nilpotent vertices and $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ represent of source nilpotent vertices of component $N_{p_j} \oplus K_1$.

A typical element of R is a linear combination of 1 and the elements of the given basis. If we replace 1 with e, we obtain a different basis for R. Thus, a typical element r of R is an F-linear combination of e and elements of V.

Since G has finitely many non-null connected components, it has a finite number of idempotent vertex. $e, e_1, e_2, \ldots, e_m, w_1, w_2, \ldots, w_k$ is a complete set of pairwise orthogonal idempotents for the ring $R = A[G]$. By Corollary 2.2.12, both R_R and ${}_RR$ have indecomposable decompositions.

Proposition 4.3.1. [3, Proposition 5] *Let R be the graph magma algebra induced by the graph G. Then $R = A[G]$ is direct sum of $m+k+1$ mutually non-isomorphic indecomposable* *projective left modules* $Re \oplus Re_1 \oplus \cdots \oplus Re_m \oplus Rw_1 \oplus \cdots \oplus Rw_k$*, where* $\{e_1, e_2, \ldots, e_m\}$ *is set of isolated idempotent vertices of* $K_1^{(m)}$*,* $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ *is set of source nilpotent vertices of component* $N_{p_j} \oplus K_1$ *and* w_j *is its target the idempotent and* $e = 1 - \sum_{i=1}^{m} e_i - \sum_{j=1}^{k} w_j$*.*

Proof. Take $r \in R$. Then, by remark 4.2.5, we can write r as a

$$\alpha e + \sum_{i=1}^{t} \alpha_i v_i + \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{j=1}^{k} \lambda_j w_j,$$

where $\alpha, \alpha_i, \beta_i, \gamma_i^{(j)}, \lambda_j \in F$. According to Lemma 4.2.1(2) and Lemma 4.2.6, $\sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} \in Rw_j$ for all $j = \{1, \ldots, k\}$ and $\sum_{i=1}^{t} \alpha_i v_i \in Re$. Thus,

$$r \in Re + Re_1 + \cdots + Re_m + Rw_1 + \cdots + Rw_k.$$

Since $Re =< \{e\} \cup \{v_i | i = 1, \ldots, t\} >$, $Re_i = Fe_i$ for all $i = \{1, \ldots, m\}$ and $Rw_j =< N_{p_j} \oplus K_1 >$ for all $j = \{1, \ldots, k\}$ by Lemma 4.2.1 and Lemma 4.2.6, the decomposition $R = Re \oplus Re_1 \oplus \cdots \oplus Re_m \oplus Rw_1 \oplus \cdots \oplus Rw_k$ holds. From Lemma 4.2.2 and Lemma 4.2.6, we conclude that each component is indecomposable, completing the proof. □

Proposition 4.3.2. [3, Proposition 6] *Let* R *be the graph magma algebra induced by the graph* G*. Then* $R = A[G]$ *is direct sum of* $m+k+1$ *mutually non-isomorphic indecomposable projective right modules* $eR \oplus e_1R \oplus \cdots \oplus e_mR \oplus w_1R \oplus \cdots \oplus w_kR$*, where* $\{e_1, e_2, \ldots, e_m\}$ *is set of isolated idempotent vertices of* $K_1^{(m)}$*,* $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ *is set of source nilpotent vertices of component* $N_{p_j} \oplus K_1$ *and* w_j *is its target the idempotent and* $e = 1 - \sum_{i=1}^{m} e_i - \sum_{j=1}^{k} w_j$*.*

Proof. Take $r \in R$, then by remark 4.2.5 we can write r as

$$\alpha e + \sum_{i=1}^{t} \alpha_i v_i + \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{j=1}^{k} \lambda_j w_j,$$

where $\alpha, \alpha_i, \beta_i, \gamma_i^{(j)}, \lambda_j \in F$. By Lemma 4.2.4, and Lemma 4.2.8 $\sum_{i=1}^{t} \alpha_i v_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} \in eR$, it follows that

$$r \in eR + e_1R + \cdots + e_mR + w_1R + \cdots + w_kR.$$

Since $e_iR = e_iF$, $w_jR = w_jF$, $eR =< \{e\} \cup \{v|v \in N\} >$, where N is the set of all nilpotent vertices, $R = eR \oplus e_1R \oplus \cdots \oplus e_mR \oplus w_1R \oplus \cdots \oplus w_kR$ holds. From Lemma 4.2.8 and Lemma 4.2.4, we see that each component in the decomposition is indecomposable, completing the proof. □

Proposition 4.3.3. [3, Proposition 5] *Let $R = A[G]$ be the graph magma algebra induced by the graph G. There is a simple left module S such that $Soc(Re) \cong S^t$ and $Soc(Rw_j) \cong S^{p_j}$. Then the left socle of R is isomorphic to $S^{(t+p_1+\cdots+p_k)} \oplus Re_1 \oplus \cdots \oplus Re_m$.*

Proof. If the graph G has no nilpotent vertices, i.e., $t = p_1 = \cdots = p_k = 0$, then every cyclic left R-module generated by a vertex and by e is simple, according to Lemma 4.2.1 and Lemma 4.2.6. Thus, there exist a simple $S = Re$ left module, leading to the conclusion that $Soc(R) \cong S^{(t+0+\cdots+0)} \oplus Re_1 \oplus \cdots \oplus Re_m$. If there is at least one nilpotent vertex, then $Soc(Re) = \bigoplus_{i=1}^{t} Rv_i$ and $Soc(Rw_j) = \bigoplus_{i=1}^{p_j} Rv_{ji}$. Denote $S = Rv$ for a nilpotent vertex v. Since every cyclic left submodule generated by a nilpotent vertex is isomorphic to each other by Lemma 4.2.1(1), we have $Soc(Re) \cong S^t$ and $Soc(Rw_j) \cong S^{p_j}$. Therefore, $Soc(R) \cong S^{(t+p_1+\cdots+p_k)} \oplus Re_1 \oplus \cdots \oplus Re_m$. □

Corollary 4.3.4. *Let $R = A[G]$ be the graph magma algebra induced by the graph G. Then*

$$Soc({}_RR) = \begin{cases} Re \oplus (\bigoplus_{i=1}^{m} Re_i) \oplus (\bigoplus_{j=1}^{k} J(R)w_j), & \textit{if } t = 0 \\ J(R)e \oplus (\bigoplus_{i=1}^{m} Re_i) \oplus (\bigoplus_{j=1}^{k} J(R)w_j), & \textit{if } t \neq 0 \end{cases}$$

Proof. By Proposition 4.3.1, we have

$$R = Re \oplus (\bigoplus_{i=1}^{m} Re_i) \oplus (\bigoplus_{j=1}^{k} Rw_j).$$

Consequently, applying Proposition 2.4.21, we find that

$$Soc({}_RR) = Soc(Re) \oplus (\bigoplus_{i=1}^{m} Soc(Re_i)) \oplus (\bigoplus_{j=1}^{k} Soc(Rw_j))$$

If $t = 0$, then $Soc(Re) = Re$; otherwise, $Soc(Re) = \oplus_{i=1}^{t} Rv_i = J(R)e$ according to Lemma 4.2.6 and by Corollary 4.2.13. Whether t is zero or not, we have that $Soc(Re_i) = Re_i$ by Lemma 4.2.1, and that $Soc(Rw_j) = \bigoplus_{i=1}^{p_j} Rv_{ji} = J(R)w_j$ by Corollary 4.2.13. Thus, the conclusion follows. □

The socle $Soc({}_RR)$ of ${}_RR$ is the largest submodule of ${}_RR$ contained in every essential submodule of ${}_RR$. For a graph magma algebra $R = A[G]$ with finitely many non-null connected components, $Soc({}_RR)$ is an essential submodule of ${}_RR$.

Corollary 4.3.5. *Let $R = A[G]$ be the graph magma algebra induced by the graph G. Then $Soc({}_RR)$ is an essential submodule of ${}_RR$.*

Proof. $Soc(Re) \leqslant_e Re$ by Lemma 4.2.6 and $Soc(Rw_j) \leqslant_e Rw_j$ by Corollary 4.2.3. Additionally, since $Soc(\oplus_{i=1}^{m} Re_i) = \oplus_{i=1}^{m} Re_i$, $Soc(\oplus_{i=1}^{m} Re_i) \leq_e \oplus_{i=1}^{m} Re_i$. Therefore, $Soc({}_RR) = Soc(Re) \oplus (\bigoplus_{i=1}^{m} Soc(Re_i)) \oplus (\bigoplus_{j=1}^{k} Soc(Rw_j))$ is an essential submodule of ${}_RR$ by Proposition 2.3.4. □

Corollary 4.3.6. *Let $R = A[G]$ be the graph magma algebra induced by the graph G. Then*

$$Soc(R_R) = \begin{cases} (\bigoplus_{j=1}^{k} \bigoplus_{i=1}^{p_j} v_{ji}R) \oplus (\bigoplus_{i=1}^{m} e_iR) \oplus (\bigoplus_{j=1}^{k} w_jR), & \textit{if} \quad t = 0 \\ J(R) \oplus (\bigoplus_{i=1}^{m} e_iR) \oplus (\bigoplus_{j=1}^{k} w_jR), & \textit{otherwise} \end{cases}$$

Proof. As seen in Proposition 4.3.1,

$$R = eR \oplus (\bigoplus_{i=1}^{m} e_iR) \oplus (\bigoplus_{j=1}^{k} w_jR).$$

Consequently, by Proposition 2.4.21, we have

$$Soc(R) = Soc(eR) \oplus (\bigoplus_{i=1}^{m} Soc(e_iR)) \oplus (\bigoplus_{j=1}^{k} Soc(w_jR)).$$

Since e_iR and w_jR are simple right R-modules, $Soc(e_iR) = e_iR$ and $Soc(w_jR) = w_jR$ for all $i = \{1, \dots, m\}$ and $j = \{1, \dots, k\}$. We know that $Soc(eR) = (\bigoplus_{i=1}^{t} v_iR) \oplus (\bigoplus_{j=1}^{k} \bigoplus_{i=1}^{p_j} v_{ji}R) = eJ(R) = J(R)$ by Lemma 4.2.8 and Corollary 4.2.13. When $t = 0$, we have that $Soc(eR) = \bigoplus_{j=1}^{k} \bigoplus_{i=1}^{p_j} v_{ji}R$. □

Corollary 4.3.7. *Let $R = A[G]$ be the graph magma algebra induced by the graph G. Then $Soc(R_R)$ is an essential submodule of R_R.*

Proof. $Soc(eR) \leqslant_e eR$ by Lemma 4.2.8 . Since $Soc(\oplus_{i=1}^{m} e_iR) = \oplus_{i=1}^{m} e_iR$ and $Soc(\oplus_{j=1}^{k} w_jR) = \oplus_{j=1}^{k} w_jR$, we have $Soc(\oplus_{i=1}^{m} e_iR) \leqslant_e \oplus_{i=1}^{m} e_iR$ and $Soc(\oplus_{j=1}^{k} w_jR) \leqslant_e \oplus_{j=1}^{k} w_jR$. Thus, $Soc(R_R) = Soc(eR) \oplus (Soc(\bigoplus_{i=1}^{m} e_iR)) \oplus Soc(\bigoplus_{j=1}^{k} w_jR)$ is an essential submodule of R_R by Proposition 2.3.4. □

We know that every semiprimary ring is a semiperfect ring, but the converse need not be true. However, the result below shows that semiperfect graph magma algebras are semiprimary.

Theorem 4.3.8. [3, Theorem 4] *Let $R = A[G]$ be the graph magma algebra induced by the graph G. Then the following statements are equivalent:*

(1) G has finitely many non-null connected components.

(2) R is a semiperfect ring.

(3) R is a semiprimary ring.

(4) R has only finitely many simple left modules up to isomorphism.

(5) R is left perfect.

Proof. $(1) \Rightarrow (2)$: Assume that G has finitely many non-null connected components. Then G contains finitely many idempotent vertices $\{e_1, \ldots, e_m, w_1, \ldots, w_k\}$. The set $\{e, e_1, \ldots, e_m, w_1, \ldots, w_k\}$ forms a complete orthogonal set of idempotents, where $e = 1 - \sum_{i=1}^{m} e_i - \sum_{i=1}^{k} w_i$. By Lemma 4.2.1 and Lemma 4.2.6, the rings e_iRe_i, w_jRw_j and eRe are local for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$. Therefore, by Theorem 4.3.8, R is semiperfect.

$(2) \Rightarrow (3)$: $R/J(R)$ is a semisimple ring because $R = A[G]$ is a semiperfect ring, and by Proposition 4.1.3, $J(R)$ is nilpotent. Therefore, R is semiprimary.

$(3) \Rightarrow (4)$: It follows from the fact that semilocal rings (i.e., rings with $R/J(R)$ semisimple) have finitely many simple left modules.

$(4) \Rightarrow (1)$: Assume, on the contrary, that G has infinite (countable or uncountable) non-null connected components. This implies that G contains infinitely many distinct idempotent vertices. Therefore, by Lemma 4.2.1, R has an infinitely many non-isomorphic simple left R-modules, which is a contradiction.

Now, since $(2) \Leftrightarrow (3)$, we have that $(5) \Leftrightarrow (3)$. □

The next theorem precisely states when the semiperfect algebras can arise as graph magma algebras.

Theorem 4.3.9. [3, Theorem 5] *Let R be an algebra over the field F. Then the following two conditions are equivalent:*

(1) There exists a graph G with finitely many non-null connected components such that $R = A[G]$, the graph magma algebra induced by G.

(2) R is a basic semiprimary ring with an indecomposable decomposition

$$R = Re_0 \oplus Re_1 \oplus \cdots \oplus Re_m,$$

and the following properties hold, where $d_i = dimRe_i$, $J = J(R)$ *and the simple modules* $S_i = \frac{Re_i}{Je_i}$ *for each* $i = 0, \ldots, m$ *:*

(a) $J^2 = 0$,

(b) for all $i = 0, \ldots, m$, $dimS_i = 1$,

(c) for all $i = 1, \ldots, m$, $e_iJe_i = 0$, *whereas* $e_0Je_0 = Je_0$, *and,*

(d) for all $i = 1, \ldots, m$, *if* $Re_i \neq Je_i \neq 0$, *then* $Je_i = Soc(Re_i) = [S_0]^{(d_i-1)}$.

Proof. (1) $\Rightarrow$ (2) It follows from Theorem 4.3.8, Corollary 4.2.13 and Lemma 4.2.1.

(2) $\Rightarrow$ (1) Let Re_i be a simple left R-module for all $i = 0, \ldots, m$. Since the set $\{e_0, e_1, \ldots e_m\}$ consists of pairwise orthogonal idempotents such that $e_0 + e_1 + \cdots + e_m = 1$, $\{e_1, \ldots, e_m\}$ a set of vertices and $\{e_0\} \sqcup \{e_1, \ldots, e_m\}$ is the basis of vertices. Hence, the bases of vertices correspond to a graph with m copies of K_1, completing the proof. Assume that for some $j \in \{1, \ldots, m\}$, Re_j and Re_0 are not simple. Consider the set $V = \{e_i | Re_i \text{ is simple}\} \sqcup \{e_j, x | Re_j \text{ is not simple}, x \in \mathcal{B}_{Je_j}, \text{for some } j \in \{1, \ldots, m\}\} \sqcup \{y | y \in \mathcal{B}_{Je_0}\}$. Given $e_0Je_0 = Je_0$, $e_jJe_j = 0$ and $J^2 = 0$, both Je_0 and Je_i are nilpotent ideals and also nil. It follows that their bases consist of nilpotent elements. Since $e_jJe_j = 0$ and e_j is an idempotent, we have $xe_j = x$ and $e_jx = 0$, for $x \in \mathcal{B}_{Je_j}$. Therefore, $\{e_j, x | x \in \mathcal{B}_{Je_j}\}$ corresponds to the graph $N_{d_j-1} \oplus K_1$, where $d_j - 1 = dim\mathcal{B}_{Je_j}$. Hence, $\{e_j, x | Re_j \text{ is not simple and } x \in \mathcal{B}_{Je_j}\}$ corresponds to the graph $\bigsqcup(N_{d_j-1} \oplus K_1)$. Additionally, since $\{e_0, e_1, \ldots, e_m\}$ consists of pairwise orthogonal idempotents, we have $yr = ry = 0$ for all $r \in V$ and $y \in \{y | y \in \mathcal{B}_{Je_0}\}$. Therefore, the basis $\{y | y \in \mathcal{B}_{Je_0}\}$ corresponds to the graph N_{d_0-1}, where $d_0 - 1 = dimJe_0$. Dimension considerations confirm that V is the spaning set of vertices and $V \sqcup \{e_0\}$ forms the basis of vertices. □

The graph G that generates a semiperfect ring has finitely many idempotent vertices, but it may not have finitely many nilpotent vertices. If G does contain finitely many nilpotent vertices, then we have the following characterization.

Theorem 4.3.10. [3, Theorem 6] *The following conditions are equivalent for a semiperfect graph magma algebra* $R = A[G]$.

(1) R *is right noetherian.*

(2) R *is left noetherian.*

(3) R *is right artinian.*

(4) R *is left artinian.*

(5) R *is finite dimensional.*

(6) G *has finitely many vertices.*

Proof. Suppose that G has n vertices. Then the dimension of R is $n+1$. Note that since R is semiperfect, there are finitely many idempotent vertices.

$(2) \Rightarrow (5)$: Since the left R-module R is noetherian, every submodule of R is finitely generated, according to Lemma 2.7.2. Therefore, the Jacobson radical $J(R)$ is also finitely generated. This implies that there are finite number of nilpotent vertices by Proposition 4.1.3. This leads to the conclusion that R is finite-dimensional.

$(5) \Rightarrow (4), (2)$: If R is finite dimensional, then we conclude that $Soc({}_RR)$ is artinian (noetherian) as well as $R/Soc({}_RR)$ by Corollary 4.3.4. Hence, R is left artinian (noetherian) by Proposition 2.7.5.

$(4) \Rightarrow (2)$: It follows from Proposition 2.7.15.

$(1) \Rightarrow (5)$: Since right module R is noetherian, every submodule of R is finitely generated. Thus, $J(R)$ is finitely generated. This implies that there are only finitely many nilpotent vertices. Consequently, R is finite-dimensional.

$(5) \Rightarrow (3), (1)$: If R is finite dimensional, then we conclude that $Soc(R_R)$ is artinian (noetherian) as well as $R/Soc(R_R)$ by Corollary 4.3.6. Hence, R is right artinian (noetherian) by Proposition 2.7.5.

$(3) \Rightarrow (1)$: It follows from Proposition 2.7.15.

$(5) \Leftrightarrow (6)$: The equivalence clearly holds because the dimension of R is $n+1$. □

Remark 4.3.11. If G has a component of the form $N_p \oplus K_1$ where p is countably or uncountably infinite, then $A[G]$ is neither right nor left Noetherian.

Lemma 4.3.12. [3, Lemma 4] *Consider that graph* $G = K_{m_1} \sqcup K_{m_2} \sqcup \cdots \sqcup K_{m_k}$ *such that* $m_j \geq 2$*, for all* $j \in \{1, \ldots, k\}$*. Then the codimension of the socle* $Soc(A)$ *of the graph magma algebra* $A = A[G]$ *equals* k*.*

Proof. We will show that the dimension of $\frac{A}{SocA}$ is k. For $1 \leq j \leq k$, let $B_j = \{v_{j1}, \ldots, v_{jm_j}\}$ be set of vertices of K_{m_j}. Then $\{1\} \cup \{\cup_{j=1}^{k} B_j\}$ is a basis for A. Define $S_j = \{v_{j1}\} \cup \{v_{jk} - v_{j(k-1)} | k = 2, \ldots, m_j\}$. It follows that S_j is an alternative basis for the subspace $< B_j >$ for all j. Hence, $S = \{1\} \cup \{\cup_{j=1}^{k} S_j\}$ is another basis for A.

Let $x \in A$. Then $x = \alpha + \sum_{i=1}^{m_1} \alpha_{1i} v_{1i} + \cdots + \sum_{i=1}^{m_k} \alpha_{ki} v_{ki}$, where α_{ji}'s are scalars for all $j = 1, \ldots, k$. Moreover, one observes that

$$\begin{aligned} x &= \alpha + \sum_{i=1}^{m_1} \alpha_{1i} v_{1i} + \cdots + \sum_{i=1}^{m_k} \alpha_{ki} v_{ki} + \sum_{j=1}^{k} \alpha v_{j1} - \sum_{j=1}^{k} \alpha v_{j1} \\ &= \alpha - \sum_{j=1}^{k} \alpha v_{j1} + \alpha v_{11} + \sum_{i=2}^{m_1} \alpha_{1i} v_{1i} + \cdots + \alpha v_{k1} + \sum_{i=2}^{m_k} \alpha_{ki} v_{ki} \end{aligned}$$

By the fact that for all $j \in \{1, \ldots, k\}$ and $v \in B_j$, $vv_{j1} = v$, one gets that

$$\begin{aligned} x &= \alpha(1 - \sum_{j=1}^{k} v_{j1}) + (\alpha + \sum_{i=1}^{m_1} \alpha_{1i} v_{1i} + \cdots + \sum_{i=1}^{m_k} \alpha_{ki} v_{ki}) v_{11} \\ &\quad + \cdots + (\alpha + \sum_{i=1}^{m_1} \alpha_{1i} v_{1i} + \cdots + \sum_{i=1}^{m_k} \alpha_{ki} v_{ki}) v_{k1}. \end{aligned}$$

Hence, $x = \alpha(1 - \sum_{j=1}^{k} v_{j1}) + xv_{11} + \cdots + xv_{k1}$. Also, $x(1 - \sum_{j=1}^{k} v_{j1}) = \alpha(1 - \sum_{j=1}^{k} v_{j1})$ implies that $x = x(1 - \sum_{j=1}^{k} v_{j1}) + xv_{11} + \cdots + xv_{k1}$. Therefore,

$$A = A(1 - \sum_{j=1}^{k} v_{j1}) \oplus Av_{11} \oplus \cdots \oplus Av_{k1}$$

Then

$$Soc(A) = Soc(A(1 - \sum_{j=1}^{k} v_{j1})) \oplus Soc(Av_{11}) \oplus \cdots \oplus Soc(Av_{k1})$$

Now consider $\mathcal{E}_j = \{v_{jk} - v_{j(k-1)} | k = 2, \ldots, m_j\}$ for all $j = 1, \ldots, k$. The equality $Soc(Av_{j1}) =< \mathcal{E}_j >$ will be shown to complete the proof. Pick $x \in A$. Then

$$\begin{aligned} xv_{j1} &= (\alpha + \sum_{i=1}^{m_1} \alpha_{1i} v_{1i} + \cdots + \sum_{i=1}^{m_k} \alpha_{ki} v_{ki}) v_{j1} \\ &= \alpha v_{j1} + \sum_{i=1}^{m_j} \alpha_{ji} v_{ji}. \end{aligned}$$

It follows that $Av_{j1} =< B_j >$. If $(v_{jt} - v_{j(t-1)}) \in< B_j >$ and $x \in A$, then one will get

$$\begin{aligned} x(v_{jt} - v_{j(t-1)}) &= \alpha + \sum_{a=1}^{m_j} \alpha_{1a} v_{1a} + \cdots + \sum_{b=1}^{m_j} \alpha_{tb} v_{tb} \\ &= \alpha(v_{jt} - v_{j(t-1)}) \end{aligned}$$

Hence, $A(v_{jk} - v_{jk-1}) = F(v_{jk} - v_{jk-1})$. Then $A(v_{jk} - v_{jk-1})$ is a simple submodule of $< B_j >$. It follows that $A(v_{j2} - v_{j1}) \oplus \cdots \oplus A(v_{jm_j} - v_{jm_j-1}) \subseteq Soc < Bj >\subset< B_j >$. Therefore, we have $Soc(< B_j >) = A(v_{j2} - v_{j1}) \oplus \cdots \oplus A(v_{jm_j} - v_{jm_j-1})$ by a dimension consideration. Hence, $Soc(< B_j >) = Soc(A_{j1}) =< \mathcal{E}_j >$. Since $A(1 - \sum_{j=1}^{k} v_{j1}) = F(1 - \sum_{j=1}^{k} v_{j1})$, one obtains that $A(1 - \sum_{j=1}^{k} v_{j1})$ is a simple module. Thus,

$$\begin{aligned} Soc(A) &= Soc(A(1 - \sum_{j=1}^{k} v_{j1})) + Soc(Av_{11}) + \cdots + Soc(Av_{k1}) \\ &= (A(1 - \sum_{j=1}^{k} v_{j1}) \oplus < S_1 > \oplus < S_2 > \oplus \cdots \oplus < S_k > . \end{aligned}$$

Consequently,

$$\frac{A}{SocA} \simeq \frac{Av_{11}}{< S_1 >} \oplus \cdots \oplus \frac{Av_{k1}}{< S_k >}$$

□

4.4 Isomorphism Classes of Finite Dimensional Graph Magma Algebras

Recall from the previous section that the converse of Proposition 3.4.5 is not true in general. Based on the characterization of semiperfect rings, Theorem 4.4.1 states that the converse of Proposition 3.4.5 is true for the graphs with finitely many non-null connected components and fully characterized all isomorphic graph magma algebras for the class of associative graphs with finitely many non-null connected components. In Proposition 4.4.2, it is determined how many pairwise non-isomorphic graph magma algebras of a given finite dimension exist.

Theorem 4.4.1. [3, Theorem 7] *If G has a finite number of non-null connected components and G is isobraic to H, then H has a finite number of non-null connected components, and there is a one-to-one correspondence between the components of G and H in such a way that the corresponding components are isobraic.*

Proof. Let G and H be in the form $N_p \sqcup (\sqcup_{i=1}^{m}(N_{p_i} \oplus K_1))$ and $N_q \sqcup (\sqcup_{j\in J} N_{q_j} \oplus K_1)$, where m finite. It follows from Theorem 4.3.8 that $R = A[G]$ is semiperfect. Since $A[H] \cong R$ by hypothesis, $A[H]$ is semiperfect, too. Hence, J is finite.

Let $|J| = k$. By Proposition 4.3.1, $R = A[G]$ is decomposed as a sum of $m + 1$ indecomposable projective left modules $Re \oplus Re_1 \oplus \cdots \oplus Re_m$, where e_i is idempotent vertex for all $i = 1, \ldots, m$ and $e = 1 - \sum_{i=1}^{m} e_i$. Likewise, $A[H]$ is decomposed as a sum of $k+1$ indecomposable projective left modules $Rf \oplus Rf_1 \oplus \cdots \oplus Rf_k$, where $f = 1 - \sum_{i=1}^{k} f_i$. Since $A[G] \simeq A[H]$, $A[G]$ can also be decomposed into $Rf \oplus Rf_1 \oplus \cdots \oplus Rf_k$. Thus, we have $m + 1 = k + 1$ (the number of indecomposable projective summands in a semiperfect modules is invariant). According to the Krull-Schmidt Theorem, there exists a permutation that provides isomorphisms between direct summands of the two distinct decompositions.

If $J(R) = 0$, then there are no nilpotent vertices by Proposition 4.1.3. Therefore, for $1 \leqslant i \leqslant m$, $p_i = 0$ and $p = 0$. According to the hypothesis, each q_i and $q = 0$ must also be zero. Thus, G and H must be in the form ${K_1}^{(m)}$ and ${K_1}^{(k)}$, respectively. Therefore, the result follows.

Now assume that $J(R) \neq 0$. Proposition 4.1.3 together with Lemma 4.2.10 gives that $Re/J(R)e$ and $Rf/J(R)f$ can be embedded into $J(R)$. Hence, $Re/J(R)e \cong Rf/J(R)f$. Thus, $Re \cong Rf$ by Proposition 2.5.11. Therefore, dimRf = dimRe, indicating that $p = q$. For the remaining projective indecomposable summands in the two decompositions of R, there is a σ permutation such that $Re_i \cong Rf_{\sigma(i)}$ where $i = \{1, \dots, m\}$. Without loss of generality, assume that $Re_i \cong Rf_i$ for $1 \leqslant i \leqslant m$. Consequently, dim$Soc(Re_i)$ = dim$Soc(Rf_i)$, implying $p_i = q_i$. Thus the result follows by Proposition 3.4.5 and Corollary 3.4.4. □

We can calculate, with the help of the following proposition, the exact number of finite $n+1$-dimensional graph magma algebras. To achieve this, we will consider the graph in the form $N_{n-j} \sqcup (\bigsqcup_{t=1}^{k} K_{i_t})$, where $j \leqslant n$ and $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ is a partition of j.

Proposition 4.4.2. [3, Proposition 7] *For an arbitrary $n \in \mathbb{Z}^+$, there exist exactly N isomorphism classes of graph magma algebras of dimension $n+1$, where $N = 1 + \sum_{j \leqslant n} p(j)$, and, for any positive integer j, $p(j)$ denotes the number of partitions j.*

Proof. Let $j \leqslant n$ and $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ be a partition of j and consider the graph

$$G = N_{n-j} \sqcup \bigsqcup_{t=1}^{k} K_{i_t}.$$

The algebra $R = A[G]$ has dimension $n+1$, and the graph magma algebra of same dimension can be obtained this way, with the exception of the graph magma algebra generated by N_n whose absence explains the need to add the 1 in the formula for N. No two partitions produce the same algebra, as shown by the Proposition 4.4.1. □

Remark 4.4.3. For an arbitrary $n \in \mathbb{Z}^+$ and for $j \leqslant n$, a partition of j gives number of graphs having $n - j$ isolated copies of N_1.

Remark 4.4.4. Recall, by Proposition 3.4.3, that K_m is isobaric to $N_k \oplus K_l$, where $k+l=m$. Therefore, $A[K_m]$ and $A[N_k \oplus K_l]$ have the same isomorphism classes.

We explore Proposition 4.4.2 by providing an example. We will calculate the number of isomorphism classes of 4-dimensional graph magma algebras.

Example 4.4.5. [3, Example 7] *We will examine the number of isomorphism classes of* 4*-dimensional graph magma algebras, so* $n=3$*. Then there are* $N=1+p(1)+p(2)+p(3)=7$ *isomorphism classes of graph magma algebras of dimension* 4 *:*

(1) We find that $p(3)=3$ *different types of graphs have zero isolated copies of* N_1*:*

K_3*,* $K_2 \sqcup K_1$*, and* $K_1 \sqcup K_1 \sqcup K_1$*, corresponding to the partition of* 3 *as* 3*,* $2+1$*, and* $1+1+1$*, respectively.*

(a) $G=K_3$*, then* $A[G] \cong A[N_2 \oplus K_1]$*. Thus,* $A[G]=Re \oplus Rw_1$*, where* w_1 *is the target idempotent vertex of* $N_2 \oplus K_1$ *and* $e=1-w_1$*. By Corollary 4.2.2, we have* $Soc(Rw_1)=Rv_{11} \oplus Rv_{21}$*, where* v_{11}, v_{21} *are source nilpotents of* $N_2 \oplus K_1$*. Additionally,* $Soc(Rw_1) \cong (Re/J(R)e)^2$ *by Corollary 4.2.11. Since there are no isolated nilpotent vertices,* $Re=Fe$ *is simple by Corollary 4.2.6 and by Corollary 4.2.13* $J(R)e=0$*. Thus,* $Soc(Rw_1) \cong (Re)^2$*.*

(b) $G=K_2 \sqcup K_1$*, then* $R=A[K_2 \sqcup K_1] \cong A[(N_1 \oplus K_1) \sqcup K_1]$*. Thus,* $R=Re \oplus Re_1 \oplus Rw_1$*, where* w_1 *is the target idempotent of* $N_1 \oplus K_1$*,* e_1 *is the isolated idempotent vertex, and* $e=1-w_1-e_1$*. By Corollary 4.2.2, we have* $Soc(Rw_1)=Rv_{11}$*, where* v_{11} *is source nilpotent of* w_1*.* Re_1 *is simple submodules by Lemma 4.2.1. Since there are no isolated nilpotent vertices,* Re *is simple by Lemma 4.2.6 and Corollary 4.2.13* $Je=0$*. Therefore,* $Soc(Rw_1) \cong Re$ *by Lemma 4.2.10.*

(c) $G=K_1 \sqcup K_1 \sqcup K_1$ *then* $R=A[G]$ *is commutative ring with indecomposable decomposition* $R=Re \oplus Re_1 \oplus Re_2 \oplus Re_3$*, where* e_1, e_2, e_3 *are isolated idempotent vertices of* $K_1^{(3)}$ *and* $e=1-\sum_{i=1}^{3} e_i$*.* Re_1, Re_2, Re_3 *are non-isomorphic simple submodules by Lemma 4.2.1(3).*

(2) We obtain $p(2) = 2$ (partitions of 2 are $2, 1+1$) types of graphs having one isolated copy of N_1. These graphs are $N_1 \sqcup K_2$, $N_1 \sqcup K_1 \sqcup K_1$.

(a) $G = N_1 \sqcup K_2$, then $R = A[G] \cong A[N_1 \sqcup (N_1 \oplus K_1)]$. Thus, $A[G] = Re \oplus Rw_1$, where w_1 is the target idempotent vertex of $N_1 \oplus K_1$ and $e = 1 - w_1$. $Soc(Re) = Rv_1$, where v_1 is the isolated nilpotent vertex by Lemma 4.2.6 and $Soc(Rw_1) = Rv_{11}$, where v_{11} is the source nilpotent of w_1 by Corollary 4.2.2. Hence, $Soc(Rw_1) \cong Soc(Re)$ by Lemma 4.2.1(1) and $Soc(Rw_1) \cong Soc(Re) \cong Re/J(R)e$ by Lemma 4.2.10.

(b) $G = N_1 \sqcup K_1 \sqcup K_1$ then $R = A[G]$ is a commutative ring with indecomposable decomposition $R = Re \oplus Re_1 \oplus Re_2$, where e_1, e_2 are isolated idempotent vertices of $K_1^{(2)}$ and $e = 1 - e_1 - e_2$. $Soc(Re) = Rv_1$, where v_1 is isolated nilpotent vertex. Thus, $Soc(Re) \cong Re/J(R)e$ by by Lemma 4.2.10 and Re_1, Re_2 are non-isomorphic simple submodules by Lemma 4.2.1(3).

(3) We find that $p(1) = 1$ only one type of graph has two isolated copies of N_1 :

$G = N_2 \sqcup K_1$. Then $R = A[G] = Re \oplus Re_1$, where e_1 is the isolated idempotent of K_1 and $e = 1 - e_1$. $Soc(Re_1) = Re_1$ by Lemma 4.2.1 and $Soc(Re) = Rv_1 \oplus Rv_2$, where v_1, v_2 are vertices of N_2, by Lemma 4.2.6 and $Rv_1 \cong Rv_2 \cong Re/J(R)e$ by Lemma 4.2.10. Thus $Soc(Re) \cong (Re/J(R)e)^2$.

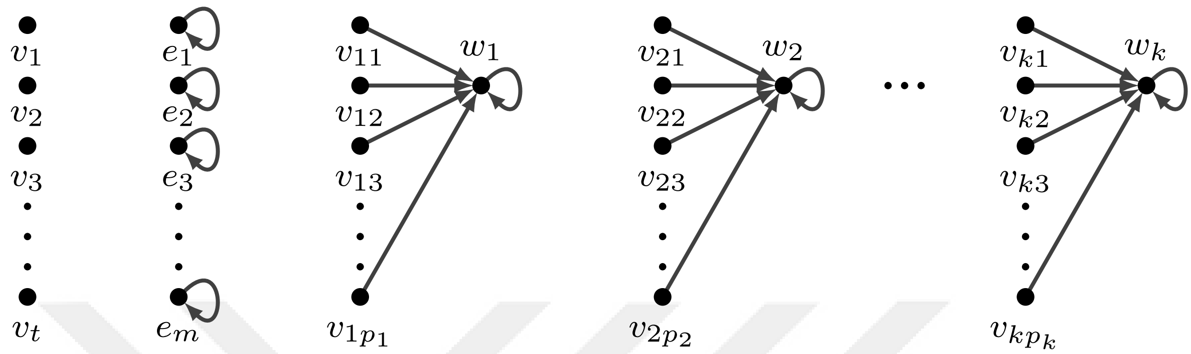
(4) The graph N_3 corresponds to the case where all vertices are isolated. Then $J(R) = Soc(R) \cong (R/J(R))^3$.

4.5 Some Further Results

We still assume that all graphs have a finite number of non-null connected components, specifically of the form

$$G : N_t \sqcup {K_1}^{(m)} \sqcup (\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1),$$

where t can be zero, finite, or (countably or uncountably) infinite, each p_j is non-zero and possibly (countably or uncountably) infinite, and both m and k must be finite. We may illustrate the graph G as follows:



As above, e denotes $1-(e_1+\cdots+e_m)-(w_1+\cdots+w_k)$, where $\{e_1, e_2, \ldots, e_m\}$ represent isolated idempotent vertices of $K_1^{(m)}$, and $\{w_1, w_2, \ldots, w_k\}$ represent k target idempotent vertices of $\bigsqcup_{j=1}^{k} N_{p_j} \oplus K_1$. $\{v_1, v_2, \ldots, v_t\}$ denote the isolated nilpotent vertices and $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ represent source nilpotent vertices of component $N_{p_j} \oplus K_1$.

In [8] it was proved that simple left modules of semiperfect graph magma algebras are either injective or poor. In addition, it was determined that these algebras may have singular, non-injective, simple modules. Furthermore, it was proved that R is always left (right) semiartinian.

In this subsection, we will go through these results. Also, we will determine the left (right) singular ideal of R. In the previous section, we showed that $Soc({}_RR) \leqslant_e {}_RR$ and $Soc(R_R) \leqslant_e R_R$ (see Corollary 4.3.5 and Corollary 4.3.7). Consequently, we can define $Z_l(R) = ann_r(Soc({}_RR))$ and $Z_r(R) = ann_l(Soc(R_R))$ by Proposition 2.10.17.

Proposition 4.5.1. *Let $R = A[G]$ be graph magma algebra generated by the graph G. Then*

$$Z_r(R) = \begin{cases} 0, & if\, t = 0 \\ \bigoplus_{i=1}^{t} v_i R, & if\, t > 0 \end{cases}$$

Proof. <u>Case I :</u> $t = 0$.

Let $x = \alpha + \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{j=1}^{k} \lambda_j w_j \in Z_r(R) = ann_l(Soc(R_R))$. Then $xv_{ji} = xe_i = xw_j = 0$ by Corollary 4.3.6. This leads to $xv_{ji} = \alpha v_{ji} = 0$, $xw_j = (\alpha + \lambda_j)w_j + \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} = 0$ for all $j = \{1, \ldots, k\}$, $i = \{1, \ldots, p_j\}$ and $xe_i = (\alpha + \beta_i)e_i = 0$ for all $i = \{1, \ldots, m\}$. Thus, we conclude that $\alpha = \gamma_i^{(j)} = \lambda_j = 0$ for all $j = \{1, \ldots, k\}$, $i = \{1, \ldots, p_j\}$ and $\beta_i = 0$ for all $i = \{1, \ldots, m\}$, which implies that x must be zero.

<u>Case II:</u> $t > 0$.

Let $x = \alpha + \sum_{i=1}^{t} \alpha_i v_i \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{j=1}^{k} \lambda_j w_j \in Z_r(R) = ann_l(Soc(R_R)$. Then $xv_i = xv_{ji} = xe_i = xw_j = 0$ by Corollary 4.3.6. This leads to $xv_i = \alpha v_i = 0$ for all $i = \{1, \ldots, t\}$, $xv_{ji} = \alpha v_{ji} = 0$, $xw_j = (\alpha + \lambda_j)w_j + \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} = 0$ for all $j = \{1, \ldots, k\}$, $i = \{1, \ldots, p_j\}$, and $xe_i = (\alpha + \beta_i)e_i = 0$ for all $i = \{1, \ldots, m\}$. Thus, we obtain $\alpha = \gamma_i^{(j)} = \lambda_j = 0$ for all $j = \{1, \ldots, k\}$, $i = \{1, \ldots, p_j\}$ and $\beta_i = 0$ for all $i = \{1, \ldots, m\}$. Hence, $x \in \bigoplus_{i=1}^{t} v_i R$. On the other hand, every element of $\bigoplus_{i=1}^{t} v_i R$ annihilates $Soc(R_R)$. Consequently, $Z_r(R) = \bigoplus_{i=1}^{t} v_i R$. □

Proposition 4.5.2. *Let $R = A[G]$ be magma algebra generated by the graph G. Then*

$$Z_l(R) = \begin{cases} 0, & if\, t = 0 \\ J(R), & if\, t > 0 \end{cases}$$

Proof. <u>Case I:</u> $t = 0$.

Let $x = \alpha + \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{j=1}^{k} \lambda_j w_j \in Z_l(R) = ann_r(Soc({}_R R))$. Then $ex = e_i x = v_{ji} x = 0$ by Corollary 4.3.4. Therefore, we have the following equations:

$$\begin{aligned} ex = \alpha - \alpha \sum_{i=1}^{m} e_i - \alpha \sum_{j=1}^{k} w_j + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} &= 0 \\ e_i x = e_i(\alpha + \beta_i) &= 0 \\ v_{ji} x = v_{ji}(\alpha + \lambda_i) &= 0. \end{aligned}$$

From this, it follows that $\alpha = \gamma_i^{(j)} = \lambda_j = 0$ for all $j = \{1, \ldots, k\}$, $i = \{1, \ldots, p_j\}$ and $\beta_i = 0$ for all $i = \{1, \ldots, m\}$. Thus, we conclude that $Z_l(R) = 0$.

Case II: $t > 0$.

Let $x = \alpha + \sum_{i=1}^{t} \alpha_i v_i \sum_{i=1}^{m} \beta_i e_i + \sum_{j=1}^{k} \sum_{i=1}^{p_j} \gamma_i^{(j)} v_{ji} + \sum_{i=1}^{k} \lambda_i w_i \in Z_l(R) = ann_r(Soc({}_R R))$. Then $e_i x = v_{ji} x = v_i x = 0$ by Corollary 4.3.4. Then

$$\begin{aligned} v_{ji} x = v_{ji}\alpha + v_{ji}\lambda_j &= 0, \\ e_i x = e_i \alpha + e_i \beta_i &= 0, \\ v_i x = v_i \alpha &= 0. \end{aligned}$$

This implies that $\lambda_j = -\alpha$, $\beta_i = -\alpha$ and $\alpha = 0$. Consequently, $x \in \bigoplus_{i=1}^{t} Rv_i \oplus \bigoplus_{j=1}^{k} \bigoplus_{i=1}^{p_j} Rv_{ji} = J(R)$. Each element in $J(R)$ annihilates e_i, v_{ji}, and v_i. Thus, $Z_l(R) = J(R)$.

□

Lemma 4.5.3. [8, Theorem 2.11] *Every left R-module that contains no simple submodules isomorphic to $Re/J(R)e$ must be injective and semisimple.*

Proof. Let U be a left R-module that contains no simple submodules isomorphic to $Re/J(R)e$ (i.e., there is no submodule is generated by a nilpotent vertex). Let $f : I \to U$ be a nonzero R-homomorphism, where I is a left ideal of R. We may choose I to be essential in ${}_R R$. Hence, $I \supseteq Soc({}_R R)$. $J(R)^2 = 0$ implies that $Soc({}_R R) \supseteq J(R)$ by Proposition 2.7.12. Thus, we have $I \supseteq Soc({}_R R) \supseteq J(R)$, and so $J(R) \subseteq Soc({}_R R) \cap I = Soc(I)$. Therefore, $f(J(R)) \subseteq f(Soc(I)) \subseteq Soc(U)$ by Proposition 2.4.11. Since $J(R)$ is the direct sum of simple submodules of R isomorphic to $Re/J(R)e$ (i.e, $J(R)$ is the direct sum of simple submodules generated by nilpotent vertices), $f(J(R))$ must be zero by assumption. Therefore, $J(R)$ must be contained in $ker f$. It follows that f induces an R-homomorphism $I/J(R) \to U$, which extends to a R-homomorphism $g : R/J(R) \to U$ since $R/J(R)$ is semisimple. Now, the composition $R \to R/J(R) \to U$, where π denotes the canonical projection, provides an extension of f. Hence, U is injective. As submodules of U do not contain any simple submodules isomorphic to $Re/J(R)e$, likewise, we show that every submodule of U is injective. Thus, every submodule of U is a direct summand by Proposition 2.8.16. Consequently, U is semisimple. □

Lemma 4.5.4. *Every right R-module containing no isomorphic copies of simple submodules generated by a nilpotent vertex must be injective and semisimple.*

Proof. Let U be a right R-module with no simple submodules generated by nilpotent elements. Let $f : I \to U$ be a nonzero R-homomorphism, where I is a right ideal of R. We may choose I to be essential in R_R. Hence, $I \supseteq Soc(R_R)$ by the definition of Socle. $J(R)^2 = 0$ implies $Soc(R_R) \supseteq J(R)$ according to Proposition 2.7.12. Thus, we have $I \supseteq Soc(R_R) \supseteq J(R)$, and so $J(R) \subseteq Soc(R_R) \cap I = Soc(I)$. Therefore, $f(J(R)) \subseteq f(Soc(I)) \subseteq Soc(U)$ by Proposition 2.4.11. Thus, $f(J(R))$ must be zero, by assumption. Therefore, $J(R) \subseteq ker f$. It follows that f induces an R-homomorphism $I/J(R) \to U$, which extends to an R-homomorphism $g : R/J(R) \to U$ since $R/J(R)$ is semisimple. Now, the composition $R \to R/J(R) \to U$, where π denotes the canonical projection, provides an extension of f. Hence, U is injective. As submodules of U do not contain any simple submodules generated by nilpotent vertex, likewise we show that every submodules of U is injective. Thus, every submodule of U is a direct summand. Consequently, U is semisimple. □

Corollary 4.5.5. *Every right R-module containing no simple submodules isomorphic to $eR/eJ(R)$, $w_1R, \ldots, w_kR$, where w_j is target idempotent for all $j = \{1, \ldots, k\}$, must be injective and semisimple.*

The ring R is said to be a *semiartinian* if every non-zero left (right) R-module has a non-zero socle (see [12]). We actually see that every non-zero left (right) R-module contains a non-zero socle in Lemma 4.5.3 (in Lemma 4.5.4).

Corollary 4.5.6. [8, Remark 2.12] *The graph magma algebra $R = A[G]$ is always left semiartinian.*

Proof. Let M be a non-zero left R-module. Assume that M does not contain any simple submodule isomorphic to $Re/J(R)e$. Then M is semisimple, implying that $Soc(M) =$

$M \neq 0$. If M contains a simple submodule isomorphic to $Re/J(R)e$, then clearly $Soc(M) \neq 0$. □

Corollary 4.5.7. *The graph magma algebra $R = A[G]$ is always right semiartinian.*

Proof. Let M be a non-zero right R-module. Assume that M does not contain any simple submodule isomorphic to $eR/eJ(R)$ and $w_1R, \ldots, w_kR$, where w_j is target idempotent for all $j = \{1, \ldots, k\}$. Then M is semisimple, implying that $Soc(M) = M \neq 0$. If M contains a simple submodule isomorphic to $Re/J(R)e$ or $w_1R, \ldots, w_kR$, where w_j is target idempotent for all $j = \{1, \ldots, k\}$, then clearly $Soc(M) \neq 0$. □

Recall that the *injectivity domain* of M is $\mathfrak{In}^{-1}(M)$ which consist of those modules N such that M is N-injective, i.e. $\mathfrak{In}^{-1}(M) = \{N \in Mod - R | M$ is N- injective$\}$ (see [9]). An R-module M is *poor*, in case the injectivity domain of M is $SSMod$-R (see [10]).

Lemma 4.5.8. [8, Theorem 2.11] *Let $R = A[G]$ be a graph magma algebra induced by the graph G. Then $Re/J(R)e$ is a poor left R-module.*

Proof. Let $Re/J(R)e$ be injective relative to R/I for some proper left ideal I of R. We will show that R/I is semisimple, which requires proving that $J(R) \subseteq I$ by Lemma 2.4.5. Assume, on the contrary, that $J(R)$ is not contained in I. Since $Soc(J(R)) = J(R) = J(R)e \oplus J(R)w_1 \oplus \cdots \oplus J(R)w_k$, there exits a simple left ideal V of R within in $J(R)e$ or $J(R)w_j$ for some $j = 1, \ldots, k$ such that $V \nsubseteq I$. Thus, $V \leqslant Re$ or $V \leqslant Rwj$ for some $j = 1, \ldots, k$. Therefore $V \cong Re/J(R)e$. Since $V \cong Re/J(R)e$, $V \cong (I \oplus V)/I$, there exists a nonzero R-homomorphism $(I \oplus V)/I \to E(Re/J(R)e)$, that extends to a homomorphism $h : R/I \to E(Re/J(R)e)$. Given that $Re/J(R)e$ is (R/I)-injective, R-homomorphism $(I \oplus V)/I \to E(Re/Je)$ extends to a homomorphism $g : R/I \to Re/J(R)e$. Thus, we have $h = ig$ with the inclusion map $i : Re/J(R)e \to E(Re/J(R)e)$, leading to $Imh = Img \subseteq Re/J(R)e$. Since $Re/J(R)e$ is simple, it follows that $Imh = Re/J(R)e$. Let K/I be the kernel of h. Then $(R/I)/(K/I) \cong Re/J(R)e \cong V$ and $(R/I)/(K/I) \cong R/K$. Thus, K is maximal left ideal of R that does not contain V, implying $R = K \oplus V$. Since V

is a direct summand of R, it is also a direct summand of Re or Rw_j for some $j = 1, \ldots, k$. However, in both cases give us contradiction since these modules are indecomposable. □

Theorem 4.5.9. [8, Theorem 2.11] *Simple left modules of the graph magma algebra $R = A[G]$ are either injective or poor.*

Proof. Note that $\{Re/J(R)e, Re_1, \ldots, Re_m, Rw_1/J(R)w_1, \ldots, Rw_k/J(R)w_k\}$ is complete set of isomorphism classes of simple left R-modules. By Lemma 4.5.3 and Lemma 4.5.8, we see that $Re_1, \ldots, Re_m, Rw_1/J(R)w_1, \ldots, Rw_k/J(R)w_k$ are all injective and $Re/J(R)e$ is poor. □

A ring R is called a left (right) SI-ring if every singular left (right) R-module is injective (see [9]).

Proposition 4.5.10. [8, Remark 2.12] *Let $R = A[G]$ be graph magma algebra generated by the graph G.*

(i) If $t = 0$, then

(a) The unique non-injective simple left $A[G]$-module is non-singular (or equivalently, projective).

(b) The ring R is left SI.

(ii) If $t \neq 0$, then the unique non-injective simple left R-module is singular.

Proof. (i) (a) According to the Theorem 4.5.9, the unique non-injective simple left R-module is $Re/J(R)e$. $Z_l(Re) = 0$, due to $Z_l(R) = 0$ by Proposition 4.5.2 and $J(R)e = 0$ by Lemma 4.2.13 when $t = 0$. This implies that, $Re/J(R)e$ is non-singular.

(b) Let M be a singular left R-module. Then every submodule of M is also singular by Lemma 2.2.11(3). According to Proposition 2.10.13, any simple left R-module is either singular or projective. Thus, M contains no projective submodules, i.e., it does not contain a simple module isomorphic to $Re/J(R)e$ by (a). Consequently, M is injective by Theorem 4.5.3. This concludes the proof.

(ii) $Re/J(R)e$ is either projective or singular by Proposition 2.10.13. If $Re/J(R)e$ were projective, then $J(R)e$ was the direct summand of Re, which gives us contradiction with the Re being indecomposable and $Re \neq 0$. Therefore, $Re/J(R)e$ must be singular.

□

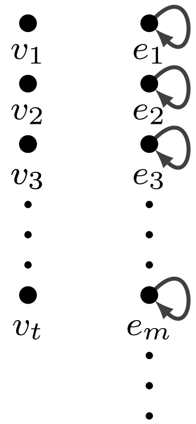
4.6 Commutative Graph Magma Algebras

Recall that if a graph magma algebra $A[G]$ is commutative, then by Corollary 3.3.3, G must be of the form $N_t \sqcup K_1^{(m)}$, where t, m are may be zero, finite, or (countably or uncountably) infinite.

Now, we will consider the graph

$$G = N_t \sqcup K_1^{(\infty)}.$$

If $v_1, \ldots, v_t$ are isolated nilpotent vertices and $e_1, \ldots, e_m, \ldots$ are isolated idempotent vertices, then we may illustrate this graph as follows:



A basis of this graph magma algebra $R = A[G]$ over F is $\{1, v_1, \ldots, v_t, e_1, \ldots, e_m, \ldots\}$. Therefore, $Re_j = Fe_j$ is a simple projective module for all j, and $Rv_i = Fv_i$ is a simple module for all i by Lemma 4.2.10. Thus, by Proposition 4.1.3, we obtain the following corollary.

Corollary 4.6.1. *Let $R = A[G]$ be the graph magma algebra induced by $G = N_t \sqcup K_1^{(\infty)}$. Then, $J(R) \subseteq Soc(R)$.*

Theorem 4.6.2. *Every prime ideal of $R = A[G]$ is maximal.*

Proof. Let P be a prime ideal of R. Assume that $e_j \notin P$ for some j. Since $e_j(1-e_j) = 0$, we have $1 - e_j \in P$. Then $e_i = e_i(1-e_j) \in P$ for $i \neq j$. Also, since $J(R)^2 = 0$, we have $J(R) \subseteq P$. Define $I_j = Span\{e_1, \ldots, e_{j-1}, 1-e_j, e_{j+1}, \ldots\}$. Hence, $I_j + J(R) \subseteq P$. For any $r \in R$,

$$r + P = (\alpha + \sum_{i=1}^{t} \alpha_i v_i + \sum_{j=1}^{m} \beta_j e_j) + P,$$

where $\alpha, \alpha_i, \beta_j \in F$. Since $J(R) + I_j \subseteq P$, we obtain $r + P = (\alpha + \beta_j e_j) + P = (\alpha + \beta_j e_j + \beta_j - \beta_j) + P = \alpha + \beta_j + (1-e_j)\beta_j + P = (\alpha + \beta_j) + P$. This shows that $\frac{R}{J(R)+I_j} \cong F$, indicating that $J(R) + I_j$ is a maximal ideal of R. Hence, $P = J(R) + I_j$.

Next, assume that $e_i \in P$ for all i. Consider $I = Span\{e_1, e_2, \ldots\}$. Again, we find $J(R) + I \subseteq P$ and $\frac{R}{J(R)+I} \cong F$. Thus, we conclude that $P = J(R) + I$ is a maximal ideal. □

Remark 4.6.3. According to above proof, prime (also maximal) ideals of R are either of the form $P_j = J(R) + I_j = (1-e_j)R = ann(e_j)$ or $P_0 = J(R) + I$, where $I_j = Span\{e_1, \ldots, e_{j-1}, 1-e_j, e_{j+1}, \ldots\}$ and $I = Span\{e_1, e_2, \ldots\}$.

The *socle series* or (or *Loewy series*) of an R-module M is the ascending chain of submodules

$$0 = S_0(M) \subset S_1(M) \subset \cdots \subset S_\alpha(M) \subset S_{\alpha+1}(M) \subset \cdots,$$

where, for each ordinal $\alpha \geq 0$, $S_{\alpha+1}(M)/S_\alpha(M)$ is the socle of the module $M/S_\alpha(M)$. M is semiartinian if and only if $S_p(M) = M$ for some ordinal $p \geq 0$. A ring R is called left *semiartinian* if the left R-module ${}_RR$ is semiartinian (see [14]).

Proposition 4.6.4. *R is a semiartinian ring.*

Proof. Since $J(R) \subseteq Soc(R)$ and $I = Span\{e_1, e_2, \ldots\} \subseteq Soc(R)$, it follows that $P_0 = J(R) + I \subseteq Soc(R)$. Given that P_0 is maximal and $Soc(R) \neq R$, we conclude that $P_0 = Soc(R)$. Therefore, socle series of an R-module R,

$$0 = S_0(R) \subset S_1(R) = Soc(R) \subset S_2(R) \subset \cdots \subset R$$

Since $S_1(R) = Soc(R)$ is maximal ideal, $S_2(R) = R$. Therefore R is semiartinian. □

One of the most important examples of a ring of fractions of a commutative ring R is given when the multiplicatively closed subset is taken to be $R \backslash P$ for some prime ideal P of R. We denote the ring $S^{-1}R$, where $S = R \backslash P$, by R_P, and call it the *localization of R at P.* The ring R_P (the localization of R at P) has the unique maximal ideal

$$PR_P = \{\alpha \in R_P : \alpha = \frac{r}{s} \text{ for some } r \in P, s \in R \backslash P\}$$

Note that $PR_P = 0$ if and only if for all $r \in P$, $tr = 0$ for some $t \in R \backslash P$ (see [13]).

In the commutative setting, R is a regular ring if and only if all localizations at every prime ideal of R are field if and only if all simple modules of R is injective (see [15]).

Theorem 4.6.5. *If $t = 0$ (i.e., there is no isolated nilpotent vertex), then R is a regular ring. In other words, every simple R-module is injective.*

Proof. Take P_j prime ideal for j. Since $e_j(1 - e_j) = 0$ and for $i \neq j$, we have $e_j e_i = 0$, maximal ideal of localization R_{P_j} is $P_j R_{P_j} = 0$. Thus, R_{P_j} is a field. For the prime ideal P_0, $P_0 R_{P_0} = 0$ which implies that R_{P_0} is a field. In a commutative ring, the fact that all localizations are field means that the ring R is regular. Again, in commutative rings, regularity is equivalent to all simple modules being injective. □

R is semiregular if and only if $R/J(R)$ is regular, and idempotents can be lifted modulo $J(R)$ (see [9]).

Theorem 4.6.6. *If $t \neq 0$, then R is a semiregular ring.*

Proof. For any prime ideal P_j, since $e_j v_i = 0$ for all i, it follows that $P_j R_{P_j} = 0$. For the P_0 prime ideal, $\overline{P_0}\ \overline{R}_{\overline{P_0}} = 0$, where $\overline{R} = \frac{R}{J(R)}$. Thus, $\overline{R}$ is a regular ring in both cases. Additionally, Proposition 4.1.1 and Proposition 4.1.3 show that idempotent elements lift modulo $J(R)$. Therefore, R is a semiregular ring. □

Proposition 4.6.7. *The ring R has a non-injective simple singular module up to isomorphism.*

Proof. The proof of Theorem 4.6.2 shows that every simple submodule of R is either $\frac{R}{P_j}$ or $\frac{R}{P_0}$. Moreover, it is known that in a commutative ring, $\frac{R}{P}$ is injective if and only if R_P is field. Therefore, when $t \neq 0$, $\frac{R}{P_j}$ is injective, while $\frac{R}{P_0}$ is not. If $\frac{R}{P_0} \cong e_i R$ for any i, then P_0 would be a direct summand of R, contradicting that P_0 is not finitely generated. Thus, for all i, $\frac{R}{P_0} \cong v_i R$ is singular. □

Remark 4.6.8. $S = \bigoplus_{i=1}^{\infty} e_i R$ is not injective. If a nonzero homomorphism $f : R \to S$ could extend the identity homomorphism $i : S \to S$ then we would have $f(1)e_i = f(e_i) = e_i$ for all i. However, there is no such element $f(1) \in S$.

In the remainder of this subsection, we will now consider the graph G is in the form

$$N_t \sqcup K_1^{(m)},$$

where m is finite and t may be zero, finite, or (countably or uncountably) infinite. Here, the set $\{v_1, v_2, \ldots, v_t\}$ denotes the isolated nilpotent vertices and $\{e_1, e_2, \ldots, e_m\}$ represents the set of isolated idempotent vertices of $K_1^{(m)}$.

The following proposition gives a characterization of commutative graph magma algebras in which the graph has finitely many non-null connected components.

Proposition 4.6.9. [3, Proposition 8 and Remark 4] *The following conditions for the graph magma algebra $A[G]$ are equivalent:*

(1) R is commutative,

(2) R is right duo,

(3) Every simple right ideal of R is a left ideal,

(4) R is left duo.

Under any of the above equivalent conditions, A is of the form $A \cong B \oplus C$, where B is zero or a quotient algebra

$$B = \frac{F[x_i | i \in I]}{<x_i x_j | i, j \in I>}, \text{ and}$$

C is zero or a direct sum of copies of the field F. The case when $B = 0$ corresponds to the graph $G = N_p$ (with p arbitrary) and the case when $C = 0$ corresponds to the graph $G = K_1^{(m)}$ (with finite m).

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ and $(1) \Rightarrow (4)$ are obvious.

$(3) \Rightarrow (1)$: Suppose that every simple right ideal of R is a left ideal. R has the left decomposition $R = Re \oplus Re_1 \oplus \cdots \oplus Re_m$ and the right decomposition $R = eR \oplus e_1R \oplus \cdots \oplus e_mR$, by assumption. We complete the proof by showing that G consists of isolated nilpotents and isolated idempotents, i.e., we show that Re_i is a simple left ideal of dimension 1 for all $i = 1 \ldots m$. Since e_iR is a simple right ideal for all $i = 1, \ldots, m$, e_iR is also a left ideal. Therefore, $R(e_iR) \subset e_iR$. Thus, $R(e_iR) = R(e_iF) = Re_i \subset e_iR$. Hence, $Re_i = e_iR$ by the dimension consideration. Consequently, Re_i is a simple left ideal of dimension 1. It follows that the non-null connected components of G are of the form K_1. Thus, R is commutative.

$(4) \Rightarrow (1)$: We will prove that Re_i is a simple left ideal for each $i = 1, \ldots, m$. In other words, we will show that there is no source nilpotents in the graph. R has the left decomposition $R = Re \oplus Re_1 \oplus \cdots \oplus Re_m$, and the right decomposition $R = eR \oplus e_1R \oplus \cdots \oplus e_mR$ by assumption. Given that R is a left duo ring, Re is a right ideal as well. Thus, $(Re)R \subset Re$. As $Re = eRe$, we have $(Re)R = eR$, indicating $eR \subset Re$. It follows that there is no source nilpotents in the graph, yielding our claim. Hence, the non-null connected components of G are of the form K_1. Thus, R is commutative.

Now assume that $R = A[G]$ satisfies one of the equivalent conditions above. Then G is of the form $N_k \sqcup K_1^{(m)}$. Let $\{n_i | i \in I\} = N_p$, where $|I| = p$. Consider the algebra $F[x_i | i \in I] \oplus F \oplus \cdots \oplus F$ (m copies of F). We can define an epimorphism ψ from $F[x_i | i \in I] \oplus F \oplus \cdots \oplus F$ to R by mapping the variables x_i ($i \in I$) to n_i and each unit vector $e_i F^m$ to the idempotent element with the same name in $K_1^{(m)}$. Thus, $\ker \psi =< x_i x_j | i, j \in I >$, and the result can be deduced from the third isomorphism theorem. □

Every simple left ideal of the graph magma algebra R generated by a graph G with finitely many nonzero connected components is a right ideal; however, not all simple right ideals are left ideals.

Example 4.6.10. *For all $j \in \{1, \ldots, k\}$, $w_j R$ is a simple right ideal that is not a left ideal, where w_j is target idempotent vertex of component $N_{p_j} \oplus K_1$:*

Since $w_j R = w_j F$ is a simple right R-module by Lemma 4.2.4(2) and $Rw_j =< N_{p_j} \oplus K_1 >$ by Lemma 4.2.1(2), we have that $R(w_j R) = R(w_j F) = Rw_j \nsubseteq w_j R$.

The above proposition characterizes the ring R in which every simple right ideal is also a left ideal.

5. EXAMPLES

In this section of the thesis, examples are presented from the paper [3] by Diaz-Boils and Lopez-Permouth, focusing on Example 1 and Example 6.

5.1 Algebras only with the components of the form $N_1 \oplus K_1$

In this subsection, we will examine algebras with bases consisting of the vertices of the component $N_1 \oplus K_1$. For this, we will consider upper-triangular matrices algebras. Note that $e_{ij} = e_{i,j}$ denotes the matrix with a 1 in the j-th column of the i-th row and 0s everywhere else. Recall that A is a graph magma algebra if and only if it has a spanning set of vertices. If $V \subset A$ is basis of vertices, the graph $G = (V, E)$ where $(u, v) \in V$ if and only if $uv = u$, is called the graph induced by the base of vertices. Note that the graph G induces an algebra isomorphic to A.

(1) The algebra $A = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = T_2(F)$, consist of upper-triangular two-by-two matrices is a graph magma algebra: Take $V = \{e_{12}, e_{22}\} \subseteq A$.

- V is a set of vertices for A: $e_{12}e_{12} = e_{22}e_{12} = 0$, $e_{22}e_{22} = e_{22}$ and $e_{12}e_{22} = e_{12}$.
- V is the spanning set of vertices for $A : 1_A \notin < V >$, and $V \sqcup \{1_A\}$ spans A as an F-vector space.
- V is a base of vertices: $\mathcal{B} = \{1\} \sqcup \{e_{12}, e_{22}\}$ is a basis for A.
- $G = (V, E) = (\{e_{12}, e_{22}\}, \{(e_{12}, e_{22}), (e_{22}, e_{22})\}$ is the graph induced by the base of vertices V.
- G induces an algebra isomorphic to A.

The algebra A is isomorphic to $A[N_1 \oplus K_1]$:

- $G = (V, E)$ is isomorphic to a graph $(N_1 \oplus K_1)$: Let the vertex set of $(N_1 \oplus K_1)$ be $V' = \{v_1, w_1\}$, where v_1 is the source nilpotent vertex and

w_1 is the target idempotent vertex and E' be edge set. Define a one-to-one correspondence $\phi : V \to V'$ such that $\phi(e_{12}) = v_1$, and $\phi(e_{22}) = w_1$. Since $(\phi(e_{12}), \phi(e_{22})) = (v_1, w_1) \in E'$, $(\phi(e_{12}), \phi(e_{22})) = (v_1, w_1) \in E'$, ϕ is an edge-preserving bijection.

- Consequently, $A \cong A[N_1 \oplus K_1]$. Also $A \cong A[K_2]$ by Proposition 3.4.3.

Since G has only finitely many non-null connected components, A has both right and left indecomposable decompositions by Proposition 4.3.1 : Let $e = 1 - e_{22}$. Then $Ae = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$ and $eA = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$.

- A is the direct sum of 2 non-isomorphic indecomposable projective left modules:

$$A = Ae \oplus Ae_{22} = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}.$$

- A is the direct sum of 2 non-isomorphic indecomposable projective right modules by Proposition 4.3.2 :

$$A = eA \oplus e_{22}A = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}.$$

(2) The subalgebra $A = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix} = T_3(F)$ is upper-triangular 3×3 matrices is a graph magma algebra: Let $V = \{e_{12}, e_{22}, e_{13}, e_{33}\} \subseteq A$.

- V is a set of vertices for A : For $i = \{2, 3\}$, $e_{ii}e_{ii} = e_{ii}$, $e_{1i}e_{ii} = e_{1i}$, $e_{ii}e_{1i} = 0$, $e_{1i}e_{1i} = 0$.
- V is a spaning set of vertices for A: $1_A \notin < V >$, and $V \sqcup \{1_A\}$ spans A as an F-vector space.
- V is a base of vertices: $\mathcal{B} = \{1\} \sqcup \{e_{12}, e_{22}, e_{13}, e_{33}\}$ is a basis for A.

- $G = (V, E) = (\{e_{12}, e_{22}, e_{13}, e_{33}\}, \{(e_{12}, e_{22}), (e_{22}, e_{22}), (e_{13}, e_{33}), (e_{33}, e_{33})\}$ is the graph induced by the base of vertices.
- G induces an algebra isomorphic to A.

The algebra A is isomorphic to $A[(N_1 \oplus K_1) \sqcup (N_1 \oplus K_1)]$:

- Let the vertex set of $(N_1 \oplus K_1) \sqcup (N_1 \oplus K_1)$ be $V' = \{v_1, v_2, w_1, w_2\}$, with v_1, v_2 are source nilpotent vertes and w_1, w_2 are target idempotent vertices. The mapping $\phi : V \to V'$ is one-to-one correspondence defined by $\phi(e_{12}) = v_1$, $\phi(e_{13}) = v_2$,$\phi(e_{22}) = w_1$, and $\phi(e_{33}) = w_2$, making ϕ is an edge-preserving bijection.
- Therefore $A \cong A[(N_1 \oplus K_1) \sqcup (N_1 \oplus K_1)]$, and also $A \cong A[(K_2) \sqcup K_2)]$.

Since G has only finitely many non-null connected components, A has both right and left indecomposable decompositions: Let $e = 1 - e_{22} - e_{33}$. Then $Ae = \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $eA = \begin{pmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

- A is a direct sum of 3 non-isomorphic indecomposable projective left modules by Proposition 4.3.1

$$A = Ae \oplus Ae_{22} \oplus Ae_{33} = \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix}.$$

- A is a direct sum of 3 non-isomorphic indecomposable right modules by Proposition 4.3.2

$$A = eA \oplus e_{22}A \oplus e_{33}A = \begin{pmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix}.$$

(3) In general, for all $n \in \mathbb{Z}^+$,

$$\mathbf{A} = \begin{pmatrix} F & F & F & \dots & F \\ 0 & F & 0 & \dots & 0 \\ 0 & 0 & F & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & \dots & F \end{pmatrix} \subset T_n(F)$$

is a graph magma algebra: Let $V = \{e_{1i}, e_{ii} | 2 \le i \le n\} \subseteq A$.

- V is a set of vertices for A: For all $i = \{2, \dots, n\}$, $e_{ii}e_{ii} = e_{ii}$, $e_{1i}e_{ii} = e_{1i}$, $e_{ii}e_{1i} = 0$, $e_{1i}e_{1i} = 0$.
- V is a spaning set of vertices: $1_A \notin < V >$, and $V \sqcup \{1_A\}$ spans A as an F-vector space.
- V is a base of vertices: $\mathcal{B} = \{1\} \sqcup \{e_{1i}, e_{ii} | 2 \le i \le n\}$ is a basis for A.
- $G = (V, E) = (\{e_{1i}, e_{ii} | 2 \le i \le n\}, \{(e_{1i}, e_{ii}), (e_{ii}, e_{ii}) | 2 \le i \le n\})$ is the graph induced by the base of vertices.
- G induces an algebra isomorphic to A.

The algebra A is isomorphic to $A[(N_1 \oplus K_1)^n]$, where $(N_1 \oplus K_1)^n$ denotes a graph with n connected.

- G isomorphic to the graph with n connected components each isomorphic to $N_1 \oplus K_1$, denoted as $(N_1 \oplus K_1)^n$: By the above observation, each $(\{e_{1i}, e_{ii}\}, \{(e_{1i}, e_{ii}), (e_{ii}, e_{ii})\})$ connected component of G is isomorphic to $N_1 \oplus K_1$. Then $G = (V, E)$ isomorphic to $(N_1 \oplus K_1)^n$.
- Consequently, $A \cong A[(N_1 \oplus K_1)^n]$.

Since G has only finitely many non-null connected components, A has a right and left indecomposable decomposition: Let $e = 1 - \sum_{i=2}^{n} e_{ii}$. Then

$$Ae = \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad eA = \begin{pmatrix} F & F & F & \dots & F \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix},$$

for $2 \leq i \leq n$, Ae_{ii} is all zero except for the i columns and $e_{ii}A$ is all zero except possibly for the (i,i) entry.

- A is a direct sum of n non-isomorphic indecomposable projective left modules

$$A = Ae \oplus (\bigoplus_{i=2}^{n} Ae_{ii})$$

and

- It is a direct sum of n non-isomorphic indecomposable right modules

$$A = eA \oplus (\bigoplus_{i=2}^{n} e_{ii}A).$$

(4) An analogous infinite-dimensional example is an algebra of $\omega \times \omega$ square matrices with nonzero entries only on entries in the first row (finitely many) and on the main diagonal, where the sequence $\{a_{ii}\}$ becomes eventually constant. A typical element of A is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{22} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & a_{nn} & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

A is a graph magma algebra with basis of vertices $V = \{e_{1i}, e_{ii} | i \geq 2\}$ and edges $E = \{(e_{ii}e_{ii}), (e_{1i}, e_{ii}) | j \geq 2\}$. $A = A[G]$ where G is a graph consisting of countably infinite connected components, each isomorphic to $N_1 \oplus K_1$. Since there are no finite components, A has no left and right indecomposable decomposition.

5.2 Algebras only with the component of the form $N_p \oplus K_1$

(1) The algebra $A = \begin{pmatrix} F & 0 \\ F & F \end{pmatrix} = L_2(F)$ of lower-triangular 2×2 matrices is a graph magma algebra: Let $V = \{e_{21}, e_{11}\} \subset A$.

- V is a set of vertices for A: $e_{11}e_{11} = e_{11}$, $e_{11}e_{21} = 0$, $e_{21}e_{11} = e_{21}$ and $e_{21}e_{21} = 0$
- V is the spanning set of vertices for A: $1_A \notin < V >$, and $V \sqcup \{1_A\}$ spans A as an F-vector space.
- V is a base of vertices : $\mathcal{B} = \{1\} \sqcup \{e_{21}, e_{11}\}$ is a basis for A.
- $G = (V, E) = (\{e_{21}, e_{11}\}, \{(e_{21}, e_{11}), (e_{11}, e_{11})\})$ is the graph induced by the base of vertices
- G induce an algebra isomorphic to A.

The algebra A is isomorphic to $A[N_1 \oplus K_1]$.

- $G = (V, E)$ is isomorphic to the graph $N_1 \oplus K_1$: Let the vertex set of $(N_1 \oplus K_1)$ be $V' = \{v_1, w_1\}$, where v_1 is the source nilpotent vertex and w_1 is the target idempotent vertex, and E' be the edge set. Define a one-to-one correspondence $\phi : V \to V'$ such that $\phi(e_{21}) = v_1$ and $\phi(e_{11}) = w_1$. Since $(\phi(e_{21}), \phi(e_{11})) = (v_1, w_1) \in E'$, and $(\phi(e_{11}), \phi(e_{11})) = (w_1, w_1) \in E'$, ϕ is an edge-preserving bijection.
- Thus, $A \cong A[N_1 \oplus K_1]$.

Since G has only finitely many non-null connected components, A has a right and left indecomposable decomposition: Let $e = 1 - e_{11}$. Then A is the direct sum of 2 non-isomorphic indecomposable projective left modules

$$A = Ae \oplus Ae_{11} = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \oplus \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix},$$

and its right decomposition

$$A = eA \oplus e_{11}A = \begin{pmatrix} 0 & 0 \\ F & F \end{pmatrix} \oplus \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$$

(2) $A = \bigsqcup_{a \in F} \begin{pmatrix} F & 0 & 0 \\ F & a & 0 \\ F & 0 & a \end{pmatrix}$ is a graph magma algebra with the basis of vertices V and edges $E = \{(e_{i1}, e_{11}) | i = 1, 2, 3\}$: Take $V = \{e_{11}, e_{21}, e_{31}\} \subset A$.

- V is a set of vertices for A: For all $i = \{1, 2, 3\}$, $e_{i1}e_{11} = e_{i1}$; for $i = \{2, 3\}$, $e_{i1}e_{i1} = 0$, $e_{11}e_{i1} = 0$.
- V is the spanning set of vertices for A: $1_A \notin < V >$, and $V \sqcup \{1_A\}$ spans A as an F-vector space.
- V is a base of vertices : $\mathcal{B} = \{1\} \cup \{1, e_{11}, e_{21}, e_{31}\}$ is a basis for A.
- $G = (V, E) = (\{e_{11}, e_{21}, e_{31}\}, \{(e_{21}, e_{11}), (e_{31}, e_{11}), (e_{11}, e_{11})\})$ is the graph induced by the base of vertices.
- G induces an algebra isomorphic to A.

The algebra A is isomorphic to $A[N_2 \oplus K_1]$:

- $G = (V, E)$ is isomorphic to the graph $N_2 \oplus K_1$: Let the vertex set of $(N_2 \oplus K_1)$ be $V' = \{v_1, v_2, w_1\}$, where v_1 and v_2 are source nilpotent vertices and w_1 is a target idempotent vertex and E' be edge set. The one-to-one correspondence $\phi : V \to V'$ is defined as $\phi(e_{21}) = v_1$, $\phi(e_{31}) = v_2$, and $\phi(e_{11}) = w_1$. Since $(\phi(e_{21}), \phi(e_{11})) = (v_1, w_1) \in E'$, $(\phi(e_{31}), \phi(e_{11})) = (v_2, w_1) \in E'$, and $(\phi(e_{11}), \phi(e_{11})) = (w_1, w_1) \in E'$ the map ϕ is an edge-preserving bijection.

- Thus, $A \cong A[N_2 \oplus K_1]$.

Since G has only finitely many non-null connected components, A has a right and left indecomposable decomposition: Let $e = 1 - e_{11}$.

- Then A is the direct sum of 2 non-isomorphic indecomposable projective left modules:
$$Ae_{11} \oplus Ae = \begin{pmatrix} F & 0 & 0 \\ F & 0 & 0 \\ F & 0 & 0 \end{pmatrix} \oplus \{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} | a \in F\},$$
and
- A is the direct sum of 2 non-isomorphic indecomposable projective right modules:
$$e_{11}A \oplus eA = \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \bigsqcup_{a \in F} \begin{pmatrix} 0 & 0 & 0 \\ F & a & 0 \\ F & 0 & a \end{pmatrix}.$$

(3) In general, for all $n \in \mathbf{Z}^+$, the subalgebra

$$\mathbf{A} = \bigsqcup_{a \in F} \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ F & a & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ F & 0 & \dots & \dots & a \end{pmatrix}$$

of $L_n(F)$ is a graph magma algebra with the basis of vertices $V = \{e_{i1} | 1 \leq i \leq n\}$ and edges $E = \{(e_{i1}, e_{11}) | 1 \leq i \leq n\}$: Take $V = \{e_{i1} | 1 \leq i \leq n\} \subset A$.

- V is a set of vertices for A: For each $1 \leq i \leq n$ $e_{i1}e_{11} = e_{i1}$, and for each $2 \leq i \leq n$ $e_{i1}e_{i1} = 0$, $e_{11}e_{i1} = 0$.
- V is the spanning set of vertices for A: $1_A \notin < V >$, and $V \sqcup \{1_A\}$ spans A as an F-vector space.
- V is a base of vertices : $\mathcal{B} = \{1\} \cup \{e_{i1} | 1 \leq i \leq n\}$ is a basis for A.
- $G = (V, E) = (\{e_{i1} | 1 \leq i \leq n\}, \{(e_{i1}, e_{11}) | 1 \leq i \leq n\})$ is the graph induced by the base of vertices
- G induces an algebra isomorphic to A.

The algebra A is isomorphic to $A[N_{n-1} \oplus K_1]$:

- $G = (V, E)$ is isomorphic to the graph $N_{n-1} \oplus K_1$: Let the vertex set of $(N_{n-1} \oplus K_1)$ be $V' = \{v_i, w_1 | 2 \leq n\}$, where v_i is source a nilpotent vertex for all $i = 2, \ldots, n$, and w_1 is a target idempotent vertex, and E' be an edge set. Define a one-to-one correspondence $\phi : V \to V'$ is defined by for $i = 2, \ldots, n$, $\phi(e_{i1}) = v_i$, and $\phi(e_{11}) = w_1$. Thus, ϕ is an edge-preserving bijection.
- Hence, $A \cong A[N_{n-1} \oplus K_1]$.

Since G has only finitely many non-null connected components, A has a right and left indecomposable decomposition: Let $e = 1 - e_{11}$. Then $Ae_{11} = \begin{pmatrix} F & 0 & 0 & \ldots & 0 \\ F & 0 & 0 & \ldots & 0 \\ F & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ F & 0 & \ldots & \ldots & 0 \end{pmatrix}$, $Ae = \left\{ \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & a & 0 & \ldots & 0 \\ 0 & 0 & a & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & \ldots & a \end{pmatrix} | a \in F \right\}$, $e_{11}A = \begin{pmatrix} F & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}$, $eA = \sqcup_{a \in F} \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ F & a & 0 & \ldots & 0 \\ F & 0 & a & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ F & 0 & \ldots & \ldots & a \end{pmatrix}$. Therefore,

$$A = Ae \oplus Ae_{11}$$

is a left decomposition and

$$A = eA \oplus e_{11}A$$

is a right decomposition.

(4) The infinite-dimensional algebra of $\omega \times \omega$ square matrices has nonzero entries appear only finitely many entries of the first column and as a constant element (except for the a_{11} value). A typical element of A is of the form

$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{21} & a & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

A is a graph magma algebra with basis of vertices $V = \{e_{i1} | i \geq 1\}$ and edges $E = \{(e_{i1,11}) | i \geq 1\}$. Likewise, in the above example, $A = A[G] \cong A[N_\omega \oplus K_1]$. Let $e = 1 - e_{11}$.

Then $Ae_{11} = \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ F & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ F & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \end{pmatrix}$, $Ae = \left\{ \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & a & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & \dots & a \\ \dots & \dots & \dots & \dots & \end{pmatrix} | a \in F \right\}$,

$e_{11}A = \begin{pmatrix} F & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \end{pmatrix}$, $eA = \sqcup_{a \in F} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ F & a & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ F & 0 & \dots & \dots & a \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \end{pmatrix}$.

Therefore, $A = Ae \oplus Ae_{11}$ is a left decomposition and $A = eA \oplus e_{11}A$ is a right decomposition.

6. CONCLUSION

In this thesis, the ring structure of graph magma algebras is examined in details based on articles [1], [3], and [8]. Diaz-Boils and Lopez-Permouth, in [3], fully characterize graph magma algebras with finitely many non-null connected components. It is shown that such algebras are exactly semiperfect rings, and that a semiperfect graph magma algebra is semiprimary. Furthermore, it is determined under what conditions semiperfect algebras can arise as graph magma algebras. Moreover, it is proved that a graph has finitely many vertices if and only of $R = A[G]$ is right (left) Artinian, if and only if R is right (left) Noetherian. Also, it is given a characterization of commutative graph magma algebras R in which the graph has finitely many non-null connected components: R is commutative if and only if R is right duo (every right ideal is a left ideal.) if and only if every simple right ideal of R is a left ideal, if and only if R is left duo (every left ideal is a right ideal.). Under any equivalent conditions, R is of the form $R \cong B \oplus C$, where B is zero or a quotient algebra

$$B = \frac{F[x_i | i \in I]}{<x_i x_j | i, j \in I>}, \text{ and}$$

C is zero or a direct sum of copies of the field F.

Saraç and Aydoğdu, in [8], show that the simple left modules of these semiperfect rings are either injective or poor.

We determine the structure of idempotent elements and the Jacobson radical of graph magma algebras induced by graphs with infinitely many non-null connected components.

- If x is an idempotent vertex of $R = A[G]$, then x is either in the form $\sum_{i \in I} e_i + \sum_{j \in J} w_j + \sum_{j \in J} \gamma_i^{(j)} v_{ji}$ or $1 - \sum_{i \in I} e_i - \sum_{j \in J} w_j + \sum_{j \in J} \gamma_i^{(j)} v_{ji}$,

  where $\{e_1, e_2, \ldots, e_m\}$ is set of isolated idempotent vertices of $K_1^{(m)}$, $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ is set of source nilpotent vertices of component $N_{p_j} \oplus K_1$ and w_j is its target the idempotent, $\gamma_i^{(j)} \in F$, I and J are finite sets.

- Let x and y be idempotent elements in $R = A[G]$.

  (1) $x = \sum_{i\in I_1} e_i + \sum_{j\in J_1} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji}$ and $y = \sum_{i\in I_2} e_i + \sum_{j\in J_2} w_j + \sum_{j\in J_2} \theta_i^{(j)} v_{ji}$ are orthogonal idempotents if and only if $I_1 \cap I_2 = \emptyset$ and $J_1 \cap J_2 = \emptyset$.

  (2) If $x = \sum_{i\in I_1} e_i + \sum_{j\in J_1} w_j + \sum_{j\in J_1} \gamma_i^{(j)} v_{ji}$ and $y = 1 - \sum_{i\in I_2} e_i - \sum_{j\in J_2} w_j + \sum_{j\in J_2} \theta_i^{(j)} v_{ji}$ then

  (i) $xy = 0$ if and only if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$.

  (ii) $yx = 0$ if and only if $I_1 \subseteq I_2$, $J_1 \subseteq J_2$, and $\gamma_i^{(j)} + \theta_i^{(j)} = 0$ for all $1 \leq i \leq p_j$ and for all $j \in J_1$

- The Jacobson radical of $R = A[G]$ is

$$J(R) = (\bigoplus Rv_i) \oplus (\bigoplus Rv_{ji}),$$

  where $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ is set of source nilpotent vertices of component $N_{p_j} \oplus K_1$ and $\{v_1, v_2, \ldots, v_t\}$ is set of isolated nilpotent vertices. Moreover, $J(R)^2 = 0$.

Furthermore, we determine right and left socle, right and left singular ideal of a graph magma algebra induced by a graph with finitely many non-null connected components.

Let $R = A[G]$ be the graph magma algebra induced by the graph G with finitely many non-null connected components. Assume that $\{e_1, e_2, \ldots, e_m\}$ is the set of isolated idempotent vertices of $K_1^{(m)}$, $\{v_{j1}, v_{j2}, \ldots, v_{jp_j}\}$ is the set of source nilpotent vertices of component $N_{p_j} \oplus K_1$ and w_j's are corresponding target idempotents and $e = 1 - \sum_{i=1}^{m} e_i - \sum_{j=1}^{k} w_j$. Then we obtain the following ideals:

$$Soc({}_R R) = \begin{cases} Re \oplus (\bigoplus_{i=1}^{m} Re_i) \oplus (\bigoplus_{j=1}^{k} J(R)w_j), & \text{if } t = 0 \\ J(R)e \oplus (\bigoplus_{i=1}^{m} Re_i) \oplus (\bigoplus_{j=1}^{k} J(R)w_j), & \text{if } t \neq 0 \end{cases}$$

$$Soc(R_R) = \begin{cases} (\bigoplus_{j=1}^{k} \bigoplus_{i=1}^{p_j} v_{ji}R) \oplus (\bigoplus_{i=1}^{m} e_iR) \oplus (\bigoplus_{j=1}^{k} w_jR), & \text{if} \quad t = 0 \\ J(R) \oplus (\bigoplus_{i=1}^{m} e_iR) \oplus (\bigoplus_{j=1}^{k} w_jR), & \text{otherwise} \end{cases}$$

$$Z_r(R) = \begin{cases} 0, & \text{if } t = 0 \\ \bigoplus_{i=1}^{t} v_iR, & \text{if } t > 0 \end{cases}$$

$$Z_l(R) = \begin{cases} 0, & \text{if } t = 0 \\ J(R), & \text{if } t > 0 \end{cases}$$

Lastly, we explore commutative graph magma algebras with infinitely many non-null connected components, uncovering some properties of these algebras:

- Every prime ideal of $R = A[G]$ is maximal.
- R is a semiartinian ring.
- If $t = 0$ (i.e., there is no isolated nilpotent vertex), then R is a regular ring. In other words, every simple R- module is injective.
- If $t \neq 0$, then R is a semiregular ring.
- The ring R has a single non-injective simple singular module up to isomorphism.

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