THE PROBLEM OF INFINITY:

A PENDULOUS SWING BETWEEN EMPIRICISM AND METAPHYSICS

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DECLARATION OF ORIGINALITY

I, Nesibe Sena Arslan, certify that

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- this thesis contains no material that has been submitted or accepted for a degree or diploma in any other educational institution;
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ABSTRACT

The Problem of Infinity:

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This thesis is an attempt to evaluate the resolutions and further discussions about Zeno's paradoxes of motion, which approach the matter from two different viewpoints: empirical and metaphysical. I examine the empirical arguments for the logical impossibility of completing an infinite series of task, and more generally, for the self-contradictoriness of the actual infinite, as well as the metaphysical arguments and analyses which attempt to provide infinity with an extensive comprehension by revising the language, or by understanding more about the language we presently use. I aim to show that the empirical arguments are not justified in claiming that the notion of infinity leads to a contradiction, and that we should strictly accept finitism. In addition, I examine the good and the rather poor examples of philosophical thought experiments about infinite processes, which helps to view infinity from new and substantial perspectives. Through such an examination, my objective is to argue that the arguments holding empirical and metaphysical concerns neither trivialize nor annihilates one another. In fact, they shed light on the notion of infinity by providing distinct understandings. In this regard, even the arguments by the empirical refusal of the actual infinite, when examined thoroughly so that their mistakes and shortcomings are revealed, are illuminating for further metaphysical inquiry.

ÖZET

Sonsuzluk Problemi:

Empirizm ve Metafizik Arasında Mekik Dokumak

Bu tez, Zeno'nun hareketin imkânsızlığını gösterdiği paradokslara getirilmiş çözüm önerileri ve bu öneriler üzerine yapılmış tartışmaların bir değerlendirmesidir. Paradoksların ele alınışında ampirik ve metafiziksel olmak üzere iki farklı yaklaşım söz konusudur. Tez, sonsuz bir serinin mantıksal olarak tamamlanmasının imkansızlığına, daha genel haliyle, sonsuzluk kavramının kendinden çelişkili olmasına dayanan ampirik argümanları inceler ve ampirik argümanların iddialarında haklı olmadığını göstermeye çalışılır. Bunun yanında, kullandığımız dile getirilen farklı yaklaşım ve dilde revizyonlar sayesinde sonsuzluğu daha kapsamlı bir bağlamda tartışmaya imkân arayan metafiziksel argümanları ele alır. Bu bağlamda, ikinci bölüm, sonsuz sayıda hareket üzerine geliştirilmiş düşünce deneylerinden örnekler ile sonsuzluğun nasıl farklı ve zengin içerikli bir bakış açısıyla değerlendirilebileceğini tartışır. Böylelikle bu tez, söz konusu iki farklı yaklaşımın birbirini geçersiz kılmadığını, bunun yerine sonsuzluğun farklı açılardan anlaşılmasına katkıda bulunduğunu savunur. Bu bağlamda, sonsuzluğu ampirik anlamda reddeden argümanların hata ve eksikliklerinin değerlendirilmesi dahi kavram üzerine metafiziksel bir soruşturma yürütülmesine yardımcı olur.

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To my parents

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CHAPTER 1

INTRODUCTION

In a sense, philosophy and mathematics started with the question of infinity. Early thinkers such as Aristotle and Plato asked whether the universe was finite or infinite and whether time had a beginning or not[1]. Scrutinizing such cosmological questions led to a vast literature as well as to new important questions. Especially, the treatment of infinity by the Ancient Greeks was quite novel; the infinite was feared because of its indefiniteness and unboundedness. This was certainly the case for their mathematics as well. However, we cannot say that Greek mathematics was not advanced enough to produce a theory of the actual infinite. On the contrary, Ancient Greek mathematics made some of the most original contributions to the theory of infinite, which "laid the foundations for a rigorous treatment of infinite processes in the nineteenth century" (Stillwell, 2010, p. 53).What was unique about it is that it was developed in accordance with a certain understanding of the notion. Expressly, "the infinitude of a process, collection, or magnitude was understood as the possibility of its indefinite continuation, and no more—certainly not the possibility of eventual completion" (Stillwell, 2010, p. 54). For instance, the natural numbers 1, 2, 3, …, when accepted as a potential infinity, can be generated by a process of aggregation of units. In other words, the natural numbers can be generated from 1 by the process of adding 1 (Stillwell, 2010, p. 54).

Perhaps, the main source of refusing the actual infinite is due to the paradoxes of Zeno, around 450 BE. In fact, Zeno was the first thinker to show that infinity was a problematic notion. Among his ten paradoxes which are known, we will only

[1] For Ancient Greek cosmological accounts which tackle these questions, see Plato's *Timaeus* and Aristotle's *Physics* and *De caelo*.

consider two of his paradoxes of motion, namely “Achilles and the Tortoise” and “The Dichotomy”. The summary of these is as follows:

1. *Achilles and the Tortoise*. Achilles is to race against a tortoise. The tortoise has a head start, so if Achilles is to overtake it, he must first run to the place where the tortoise presently is. By the time he arrives there, the tortoise moves ahead from his previous point to a new point. Now Achilles has to run the new distance that separates them, but when he runs the distance, the tortoise moves ahead to a new place, *ad infinitum*. Thus, Zeno argues that Achilles will never catch the tortoise.
2. *Dichotomy*. There are two forms of the Dichotomy. In one of the forms, a man is to walk from point A to B. Before arriving at point B, he must first walk half that distance. Then, he must walk half of the remaining distance, and so on, *ad infinitum*. Therefore, Zeno argues that he will never reach point B. The other version concludes that the motion cannot begin, since the man must walk quarter of the distance before walking half the distance, etc. *ad infinitum*.

To resolve the paradoxes, we will consider the responses from the empirical viewpoint, and the metaphysical viewpoint. Throughout our discussion, empiricism will be used in the sense of “radical empiricism.” According to radical empiricism against the backdrop of the problem of infinity, the actual infinite is objected because sense experience cannot confirm that the infinite series of tasks can be performed. Another aspect of the empirical arguments is that they mostly refer to the intuitive (or the ordinary) linguistic usage. In other words, the empirical understanding of infinity seems to be the level where we conceptualize the notion in accordance with our standard linguistic usage. The ordinary sense of infinity, for instance, implies that what is infinite is without an end, i.e. there is no definite last member. According to this definition, in an infinite process, if there is no last act to be performed, the process can never be completed. Here, what is meant by the ordinary linguistic usage is about how we understand the expressions “last act” and “completed.”

Metaphysics, on the other hand, refers to more of a “rationalistic metaphysics,” where reason plays a major role to investigate the nature of reality. What is more, the role of language is overstressed. In addition to the standard linguistic usage of the empirical viewpoint, the metaphysical approach argues that, when understanding what infinity truly is, we also need the critical examination and logical analysis of our statements about the notion. Such examinations and analyses mostly make use of mathematical conceptualizations regarding the infinite as well as other notions and expressions that we utilize when analyzing infinity. In that way, the metaphysical viewpoint suggests a more “mathematical” sense of infinity. Thus, while the ordinary sense of the infinite offers an understanding as stated above, the “mathematical” sense, for the most part, adopts the non-intuitive mathematical conceptions and applications such as the limit theory. Thus, the major difference between the empirical and the metaphysical approach stems from the difference in linguistic usage and the conceptual apparatus.

In this thesis, we argue that, among the empirical and the metaphysical viewpoints, which investigate the notion of infinity with regard to Zeno’s paradoxes of motion, one does not supersede the other. In other words, we will argue that these viewpoints have different approaches as well as specific criteria to make sense of the notion. No matter how compelling a case the empirical viewpoint supports, it does not annul the significance and the strength of the arguments from the metaphysical viewpoint, or vice versa. My position is that, between empiricism and rationalistic metaphysics, one is not a better theory about our knowledge of infinity compared to the other. Both theories, in fact, have their advantages and shortcomings, and it is not sensible that we impose one single viewpoint on the notion. Being a notion of physics and mathematics, as well as a subject-matter of philosophical inquiry, we claim that infinity both needs the empirical viewpoint and the metaphysical viewpoint.

Zeno's paradoxes do not simply dissolve by ostensibly walking from point A to B. In fact, a Moorean argument showing how motion is obviously possible cannot be able to touch the point of Zeno's arguments.[2] In this way, I will argue that the empirical account is not justified for dispelling the notion for being self-contradictory and logically impossible on empirical grounds. Even if the infinite was not contradiction-free in empirical terms, that would not mean that the notion is devoid of any substantial meaning, and it is merely a mathematical schema for appropriate uses.[3] On the contrary, it more than anything shows that the notion needs to be tackled from more than one single viewpoint, which ultimately contributes to the bigger picture.

In the second chapter, I will consider the arguments of those holding empirical concerns when investigating the nature of infinity. Examining the

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[2] G. E. Moore (1939) argues as follows:
(1) If there is one hand here, then there is an external world.
(2) Here is a hand, and here is another (ostensibly show one's own hands).
(3) Therefore, the external world exists.

[3] In terms of influencing different traditions, disciplines and application areas, one similar notion to the infinite is chaos. Like infinity, the cosmological/philosophical inquiry into the notion of chaos has its roots in Ancient Greece. According to the cosmogonical understanding of Hesiod and Pre-Socratics, chaos is described to be the unformed and infinite void, from which the universe is created. The cosmological accounts of Aristotle and Plato are substantially based on the cosmogonical understanding of chaos. Like infinity, chaos had negative connotations which are passed on to modern sciences. However, when "envisaged not as an absence or void but as a force in its own right" (Hayles, 1990, p. 3), a new mathematical theory, chaos theory emerged. Starting from the nineteenth century, chaos theory is used for predicting the behavior of inherently unpredictable systems in mathematics and physics. However, there is no universally accepted definition for chaos, only certain properties for a system to be defined as "chaotic". Like infinity, chaos is a notion that is not devoid of any problems but it still contributes greatly to our theories.

reasoning of these arguments as well as revealing their mistakes and shortcomings, I will consider the corresponding metaphysical responses on the matter, which enable us to view the notion from a new and illuminating perspective. In addition, I will show that although the empirical arguments cannot argue that completing an infinite series of acts is logically impossible, this alone does not show that it is metaphysically possible. Regarding the views which argue that exhausting infinity is metaphysically possible, in the third chapter, I will consider the metaphysical approach to the problem which searches for a proper conception of infinity by reforming language, or by understanding more about the language we presently use. By doing this, I will attempt to show that this approach enables us to touch on different and important issues raised by Zeno's paradoxes, as well as to widen our horizon in order to understand infinity.

CHAPTER 2

SUPERTASKS AND INFINITY MACHINES

Zeno's paradoxes of motion have been a matter of dispute on various levels for a long time. The paradoxes that we will be bringing into question, namely "Achilles and the Tortoise" and "Dichotomy" which, Aristotle says, are the same in principle (*Physics*, 239b), points to the inadequacy lying at the very heart of our concepts of space, time, motion, continuity, and infinity. It is exactly because of such difficulties that certain metaphysical and epistemological discussions have flourished as well as mathematical and scientific studies further developed. In this chapter, I will set forth the research on supertasks that has been carried out so far, which illuminates our understanding of infinity for bringing together infinity and logic.

Supertask is one topic of discussion and of perplexity ever since Zeno. A supertask is a (countably) infinite sequence of acts or operations that is performed in a finite interval of time. The disturbance, as Earman and Norton (1996) note, has been "profitable since it has forced us to clarify our notions of infinity, continuity and continuum" (p. 231). In a nutshell, the discussion developed in two directions: one towards the impossibility of carrying out an infinite task which is thought to bear contradictions, and the other, by showing the "defects" of a given argument against the performability of supertasks, towards the conclusion that supertasks cannot be called to be impossible to perform. However, independently of whether one argues for or against the actual infinite, the conclusion of Zeno's arguments is rejected for being absurd. In that case, either one of the premises has to be rejected. The division of opinions, in that respect, results from rejecting different premises. To clarify the upcoming discussion better, let us take a closer look at Zeno's argument, the Dichotomy:

(1) One must complete an infinite number of journeys to complete any journey. (P1)
(2) It is logically absurd that one can complete an infinite number of journeys, just as it is logically absurd that one cannot complete an infinite number of tasks. (P2)
(3) Therefore, it is logically absurd for someone to complete a journey.

In this chapter, although we will examine the different views regarding what went wrong in Zeno's paradox, our aim is not to focus on the reasons of the invalidity of the argument. This chapter will try to illuminate the shortcomings of the empirical viewpoint when tackling the paradox. The empirical account leads the discussion of infinity to a dead end for the notion is claimed to be self-contradictory on practical grounds. We will argue that infinity still needs to be clarified theoretically, for the physical impossibility of exhausting infinity does not entail the logical impossibility. This chapter also seeks a feasible approach as well as ways and means to conduct such a theoretical discussion, which can respond to the difficulties raised by the paradox more comprehensively.

2.1 Black's infinity machines

Much before the dispute that we will examine in the following, Cajori (1915) straightly describes the grounds on which the disagreement fundamentally relies:

> The question at issue is usually not so much one of logic, as it is of the postulates which the reasoner is willing to accept as reasonable and useful. An investigator who vetoes any assumption which does not appeal to his intuition or to his power of imagination can hardly find comfort in Cantor's theory of aggregates and the Cantor continuum. To him Zeno's paradoxes must necessarily remain paradoxes forever. (p. 216)

Cajori's observation is what exactly happens in Max Black's criticism. Taking both Achilles and Dichotomy into consideration, where the puzzle arises because the conclusion is absurd, Black (1951) tries to find out exactly what mistake is committed in the arguments. Denying P1, he considers a machine, “Alpha”, which counts an infinite number of marbles; moving one marble from the left-hand tray to the right in a minute and resting for a minute, and carrying out the task in this manner for each member of the sequence 1, 1/2 , 1/4,… in four minutes. (Let us call such a sequence “Z-series”, and a procedure as such “Z-procedure” from now on.) Black argues that the obvious difficulty in conceiving an infinity machine that performs such a task is this: “How are we supposed to know that there are infinitely many marbles in the left-hand tray at the outset? Or, for that matter, that there are infinitely many on the right when the machine has stopped?” (p. 97). One criticism of Black comes from Taylor (1951), who argues that, for the machine to exhaust all the marbles, “no less than an infinitely large supply would suffice; otherwise, the marbles would certainly be gone before four minutes were up” (p. 39). Therefore, under the given conditions and set-up, it is in fact logically *necessary* for the machine to perform an infinite task.

It should be noted, what is assumed along with infinity in the sense that Black uses it in his examples is that completing a task such as counting a heap of marbles requires exhausting each and every element or arriving at the last term of the heap. As we mentioned, Black argues that Beta would fail to achieve its task for such reasons. This, however, invokes a naïve conception of the infinite. Here, “Galileo’s paradox” which demonstrates the peculiar properties of infinite sets may provide us with a counterexample. Consider, for instance, an infinite machine that picks out all the perfect squares, i.e. the product of two equal integers, out of the positive integers. When the machine stops, the numbers that were not picked still constitute an infinite aggregate. This shows that, even if a supertask is logically impossible to achieve, it cannot be due to a matter of exhausting all the terms and coming to the last term, since it is logically consistent that, as our example illustrates, an infinity machine may as well pick out an infinite number of terms while leaving an infinite number of terms behind. Therefore, with this argument, Black fails to show that exhausting infinity is logically impossible.

To make things simpler, Black considers next a second machine, “Beta”, which performs a Z-procedure by carrying one and the same marble from the left-hand tray to the right in four minutes, where the marble is returned to its original place at each step while Beta is resting. Now, Black claims that the task of Beta is self-defeating and “shows clearly that the infinite count really is impossible” simply because “the very act of transferring the marble from left to right immediately causes it to be returned again” (p. 97). This conclusion, however, is puzzling for he now seems to suppose that, as Beta is working, it also counts the marble. If, as Black supposes, counting is involved in the task of the machine, then there is clearly no last term to count. The completion of the task is absurd. No one would deny that. However, Black’s assumption that the infinite series (qua a physical act such as “counting”) does not have an end does not imply that it is logically impossible that the machine carries the marble back and forth infinitely many times.

Against Black, I would argue that the difficulty regarding the epistemic status of a tray with infinitely many marbles is not immediately relevant to the logical possibility of completing an infinite task. That is to say, emphasizing the impossibility of performing a supertask in respect of its physical character does not altogether dispel the theoretical conflict the notion creates. Therefore, right at the beginning of the discussion, the following question needs to be asked: Are we justified in claiming that the infinity machine is impossible simply because there is no way for us to know that there are infinitely many marbles by experience (as though, the machine would stop after counting an infinite number of marbles)? However, as Black and those who hold similar concerns argue, such epistemological concerns and emphasis on the experience when rationalizing the infinite (or rather, discarding the notion of being “empirically, physically” and therefore “logically” impossible) obstruct any possibility of providing the actual infinite with a rational account. We will consider this in more detail later.

A further and equally important consideration is the use of ambiguous words in Black’s argument. Earman and Norton (1996) make note that the word “incompletable” is ambiguous:

> Black’s fallacy lies in the confusion of two senses of “incompletable” and its allure lies in the case with which we can slide between the two senses. An infinite sequence of acts is incompletable in the sense that we can nominate no last act, the act that completes it. An infinite sequence of acts may also be incompletable in the sense that we cannot carry out the totality of its acts, even though each act individually may be executable. This may become the case, for example, in the runner’s journey, if the runner is required to spend the equal time in each of the infinitely many intervals. An infinite sequence of acts cannot be completed in the first sense, but that certainly does not entail that it cannot be completed in the second sense. (p. 233)

Different versions of the definitions of infinity play a major role for this confusion. To state once again, Black’s position against supertasks and his motivation for denying P1 results from one version of the definition of infinity which states that a series which is actually infinite does not have an end. As Norton and Earman write, the series with no last term, that is the completing an infinite sequence of acts in the first sense, is indubitably impossible. However, “completion” in the first sense is not necessarily true of all the possible versions of completing an infinite sequence of acts. There is certainly no necessity imposed for such a restriction on all the statements invoking exhausting infinity.

This definition of the infinite is also in accordance with Black’s views about the infinite summation which rely on the mathematical notion of limit. The question, then, “does an infinite convergent series actually reach its limit or not?” is replied always with a “No!” in Black’s case:

> The summation of all the terms of an infinite series is not the same thing as the summation of a finite set of numbers. In one case we can get the answer by working out a finite number of additions; in the other case we must “perform a limit operation”, that is to say, prove that there is a number whose difference from the sum of the initial members of the series can be made to remain as small as we please. (p. 93)

It is thusly clear that the limit operation, according to him, necessarily includes performing subtraction of the partial sums from the sum of the infinite series, which can never be zero, but take a positive value that can be made as small as pleased. For Black, this implies that the infinity, whether it be a series of numerical (subtraction) or physical (counting) acts, cannot be exhausted for the limit value is never attained. Black’s opinion that the limit cannot be attained correlates with the definition of infinity invoking the first sense of “incompletable”, which is that an infinite series does not have an end. This assumption is also supported by the machines that are exemplified.

Black further argues, it will not help to have a machine which transfers the marbles, where each step becomes smaller in geometrical progression for he argues that the logical impossibility of performing an infinite series of operations is the result of his claim that “the logical possibility of the existence of any one of the machines depends on the logical possibility of the existence of all of them, or indeed, any machine that could count an infinite number of objects” (p. 100), as mentioned before. This conclusion, however, is found to be a hasty generalization by Benardete. Since here, we must take account of all the senses of “incompletable” to determine whether supertasks are logically impossible or not. Below, we will take a look at Benardete’s analysis of the statement about the incompletableness of supertasks, which characterizes new understandings and definitions regarding the words referred in the statement.

The questions which Benardete considers are these: How can a series without an end ever come to an end? Does an infinite convergent series actually reach its limit or not? To this, while Black answers with a strict "No!" for empirical reasons, Benardete (1964) claims that the answer "depends on the ontological conditions: in some cases, yes, in other cases, no" (p. 54). The diversity depending on the ontological conditions results from different meanings that are attached to the terms (such as "end") in question. Thus, Benardete's answer is in parallel with Norton and Earman's remark that "completing a supertask with no last act" does not logically entail that the supertask cannot be completed, if "completion" is understood in the second sense, the sense that does not refer to the end term of an infinite series:

> It is obvious that the paradox cannot be dissolved unless an equivocation is exposed in the two uses of the word 'end'. We must show that the fallacy of equivocation is being committed and that the same word is being employed in two very different senses. (p. 66)

Indeed, Benardete's idea that an equivocation is being employed in the uses of the word "end" bears a resemblance to Aristotle's distinction between two ways of "being":

> Further, being is used in several ways, so that one should not take the infinite in the way of an individual, such as a human being or a house, but rather in the way a day and games are said [to be], which have being not in the way some substance has come to be, but [have being] all the time in [a process of] coming to be or passing away, finite, but all the time different and different. (*Physics*, 206a 29-33)

The intention behind Aristotle's distinction is forced upon avoiding the absurdity of a modal scheme, according to which, being potentially *F* entails it being possible to be actually *F*. However, applying the scheme to the potential infinity, the scheme causes an absurdity, since it leads to the possibility of the actual infinity. Therefore, to escape from such an undesirable result, Aristotle makes an important distinction between the uses of the word "being" as applied to infinity. What concerns us here is that, since antiquity, revealing the equivocation in our terms is quite an important step in conceptualizing the infinite in a well-suited context. It also shows that

Benardete's argument has an Aristotelian spirit in its approach, which makes his argument, in turn, compelling next to Aristotle's account.

Turning back to Benardete, his proposal for tackling the seeming contradiction in "coming to the end of a series without an end" is to alter the definition of the infinite series which is understood and therefore defined to be without an end. What do we mean, then, when we say the series is without an end? Benardete replies, we mean that "each term in the series has an immediate successor that lies within the series" (p. 66). In this way, he redefines "coming to the end of an infinite series" as reaching certain terms that are "lying outside the series, which fall beyond each and every term occurring in the series: these post-serial terms are successors (though not immediate successors) of the terms in the series" (p. 66). With these new definitions, the solution to the contradiction is proposed as the following:

> If the Z-series is enacted in the large, then it is not possible for those post-serial terms ever to be reached. Time is of the essence. If we are asked precisely how those post-serial terms can in fact be reached, we have no other recourse but to exhibit the Z-series in the small as an object-lesson. (p. 66)

What Benardete means by the Z-series "in the small" and Z-series "in the large" is simply regarding the consideration of time. In short, the difference between these two series is, as the distance sequence converges to a limit, whether the time-sequence also converges in proportion to the decrease in the distance sequence or not. Thus, Z-series "in the small" means that the time-sequence converges as does the distance sequence, whereas in Z-series "in the large", time sequence does not converge to a limit. In light of this, what exactly does Benardete's solution show us? He simply says when exhausting an infinite supply, "time is of the essence" (p. 66). As the halves in the distance sequence become infinitely small, he claims that the series comes to an end if the same case is conditioned on the time-sequence.

Bearing this analysis in mind, can we say that Black's infinity machines are logically impossible? Benardete's argument purports that there is nothing fallacious with the claim that both Alpha and Beta in Black's example "logically" achieve their task, for we can perfectly place the machines within an ontological account. On the other hand, regarding whether Alpha and Beta can "actually" reach the end of their tasks, the question concerns part science, part philosophy.[4] With regard to science, and mainly physics, just because it cannot establish the actual infinite due to its own limitations and observational constraints (such as counting or recording each step) we cannot conclude that there is nothing physically or actually infinite. In that regard, although the Big Bang theory suggests that the universe has not existed forever, the observable universe is not all there is.

The answers to questions such as the one above ultimately depend on one's conception of the setup that is conceived to occur, whether exhausting the halves is taken to be a physical possibility or possibility "in thought". Here, it is necessary to remember the quote from Cajori once again: "The question at issue is usually not so much one of logic, as it is of the postulates which the reasoner is willing to accept as reasonable and useful". This quote then perfectly clarifies for us the bifurcation in the discussion. Nonetheless, it is important to note that the problem of infinity is not adequately identified, much less resolved, whether one holds rational (for the most part) or empirical (for the most part) concerns.

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[4] On the philosophical implications of whether the infinite can be "actually" exhausted, Benardete (1964) puts forth a new paradox, i.e. the paradox of gods: "A man decides to walk one mile from A to B. A god waits in readiness to throw up a Wall blocking the man's further advance when the man has travelled 1/2 mile. A second god (unknown to the first) waits in readiness to throw up a wall of his own blocking the man's further advance when the man has travelled 1/4 mile. A third god, etc. It is clear that this infinite sequence of mere *intentions* (assuming the contrary-to-fact conditional that each god would succeed in executing his intention if given the opportunity) logically entails the consequence that the man will be arrested at point A; he will not be able to pass beyond it, even though not a single wall will in fact be thrown down in his path (p. 259)." The paradox of gods and its variants opens up a whole new discussion. (See, for instance, [Yablo, 2000], [Priest, 1999], [Perez Laraudogoitia, 2003].) In our discussion, we will not consider the paradox of gods and the further discussions, for it is beyond the scope of the thesis.

To clarify this better, let us take a closer look at Black’s and Benardete’s accounts. In Benardete’s account, infinity is viewed in a metaphysical investigation that requires one to consult the mathematics of infinity (p. 38). Carrying ontological concerns about clarifying the actual infinite, Benardete thereby neither begins the investigation by denying P1 or P2, nor by arguing that the derivation is invalid. Benardete, for that matter, does not approach to the paradox as a physical impossibility. In fact, he finds the impossibility of supertasks and infinity machines on practical grounds to be “the least of the objections” (p. 2). For him, although infinity is almost hopeless to advocate on practical grounds, this alone should not compel anyone to consider infinity machines (as “utopian experiments”) to be theoretically impossible (p. 2).

In Zeno’s argument, traversing an infinite sequence of finite spatial intervals in a finite time is analogous to Alpha and Beta’s task, for they perform a supertask (with distinct acts every time they work, which lasts for a finite time). To this paradox, Benardete accepts Aristotle’s distinction of magnitude (time and space) being infinite with respect to divisibility and being finite, and of a certain quantity, in actuality. Benardete claims that “by sorting out these different senses, time and space are found to be strictly isomorphic: the paradox is dissolved. Hence, an infinite sequence of finite spatial intervals can be traversed in an infinite sequence of finite temporal intervals (p. 65).

Black, on the other hand, claims that the paradox is simply a matter of physical possibility. Unlike Benardete, Black has no concerns about providing infinity with an ontological account which would imply that such a theoretical notion is logically consistent even if the notion lacks empirical application and employment. The empiricism that Black’s approach hinges upon implies that (1) what is given to us is not infinite numbers of subintervals that need to be travelled in thought, but discrete and finite steps that Achilles takes. He argues that (2) we create the illusion that Achilles performs a supertask with the mathematics that we use to describe space, time, and motion; thus commit a fallacy by “confusing a series of acts with a series of numbers generated by some mathematical law” (p. 95). Black’s approach is thoroughly articulated in the following:

> The track on which he [Achilles] runs has a finite number of pebbles, grains of earth, and blades of grass, each of which in turn has a finite, though enormous number of atoms. For all of these are things that have a beginning and end in space or time. But if anybody says we must imagine that the atoms themselves occupy space and so are divisible “in thought”, he is no longer talking about spatio-temporal things. To divide a thing “in thought” is merely to halve the numerical interval which we have assigned to it.… We can of course choose to say that we shall represent a distance by a numerical interval, and that every part of that numerical interval shall also count as representing a distance; then it will be true a priori that there are infinitely many “distances”. But the class of what will then be called “distances” will be a series of pairs of numbers, not an infinite series of spatio-temporal things. (1951, p. 101)

The passage puts forth Black’s take on the paradox and his reasons for the paradoxicality in Zeno’s arguments. Can we say that Black successfully disposes the paradox without further elaborating on (1) and (2)? Yes and no! Yes, because, under the circumstances Black regards, which are mainly on practical grounds, his arguments are persuasive to show that the infinite series of tasks is physically impossible. The same, however, is not the case with the infinite series of tasks being logically impossible. Hence, no; the paradox is not resolved. It is an indubitable fact that Achilles takes a finite number of steps to catch the tortoise in reality. Even Zeno would not reject that. However, such an empirical fact is not all there is to resolve the paradox, a paradox which has been around for more than two thousand years. Infinity is not only a physical and practical concern. Our theoretical inquiry about the notion should certainly not be dismissed entirely; infinity still needs to be discussed on theoretical grounds.

In the rest of this section, we will consider Wisdom's practical concerns which draw attention to the distinction between mathematics and physics as far as the paradox is concerned. Next, we will trace such concerns back to Kant and examine his arguments in more detail. With such a historical as well as argumentative tracing, we intend to put forth that viewing infinity merely on practical grounds cannot completely resolve the problems the notion creates.

2.2 Wisdom's physical racecourse

Wisdom (1951) endorses Black's views which he finds to be a needed but an incomplete contribution. Similar to Black, Wisdom argues that there are two alternatives to resolve the paradox: to show either that the inference is invalid or that the premise is false. What matters, according to Wisdom, is not to prove how Achilles catches the tortoise but to find out where exactly the argument breaks down. For him, there is logically nothing wrong with the argument, that is to say, the inference is valid. However, even though the conclusion is validly derived, it is not proven. According to Wisdom, the argument is not sound because the premise (P1) is not true (p. 85). Similar to what we have seen in Black's opinions, Wisdom's argument is built upon denying Zeno's assumption (P1) and rejecting the mathematical representation of racecourse as the aggregate of infinitely many points. In other words, Wisdom denies that "the mathematical description is a correct description of physical distance" (p. 69).

However, Wisdom differs from Black by arguing that claiming supertasks are impossible, in fact, does contribute to Zeno's conclusion that Achilles cannot catch the tortoise or that motion is impossible. In other words, whether supertasks are self-contradictory or not does not contribute to resolving the paradox. With this remark, Wisdom can avert certain objections which Black was subjected to. In Black's argument, we saw that the logical impossibility of the infinite machines is mainly grounded on an advocacy of finitism, where the process in Zeno's argument which consists of distinct physical acts is argued to be related only to the potential infinite. Although such empiricist considerations are not completely irrelevant, they cannot prove the logical impossibility of completing a supertask.

The only way for Wisdom to tackle the absurdity is to notice that the word "distance" in the argument is used in an equivocal way, and thus to claim that one of the assumptions in the argument (P1) is false. According to Wisdom, the assumption is not explicitly expressed. He claims that if, by Achilles' distance which is represented by 1 + 1/2 + 1/4 + etc. we mean Achilles' "mathematical" distance as opposed to "physical" distance, then the premise contains no contradiction. In other words, his claim is that "Achilles' distance is equal to the geometric sum 1 + 1/2 + 1/4 + etc." and "Achilles's distance is physical" are contradictory (p. 70). Wisdom's concerns regarding the paradox, as we thusly see, are mainly empirical, which are briefly summarized as the following:

> A physical point, unlike a mathematical point, has some size, though this may be as small as we please. But, however small a physical point, since it has some size greater than zero, an infinity of them cannot be packed into a finite distance. In particular, an infinity of physical points cannot be packed to correspond to the mathematical points described by an infinite geometric series. Hence an infinite geometric series is inapplicable to a physical distance. I.e. a physical race cannot be described by repeated bisection, or Zeno's premise is false. (p. 72)

His objection against the mathematical explanation offered to the problem can further be found in the following lines:

> We wish to know what is meant by the claim that the paradoxes are resolved by modern mathematics.…The claim must depend upon the theorem that a continuous function attains its limit: that is to say, not merely does the function represented by the infinite series approach a limit, which is its "sum", but attains it. Now this theorem when properly stated is seen to hold only for "a closed interval"; it is not true for "an open interval". But an infinite series can be defined for open intervals only; consequently no infinite series can attain its limit, and no mathematical solution of the paradoxes along these lines is possible. (1941, pp. 70-71)

Similar to Black, Wisdom claims that the infinite series cannot attain its limit for more or less the same reason, which we can also find in Aristotle's account. The use of ellipsis or "ad infinitum" in Zeno's argument is not justified for the infinite is not a completed totality, but can only signify an ongoing process. Indeed, performing a limit operation which means taking the difference between the partial sums of the distances traversed at each step and the limit of the total sum (i.e. the length of the whole path) in Black's argument already assumes the potential infinite. Likewise, when Wisdom argues that the infinite series cannot attain the limit for the series in Zeno's argument is open-ended, it is implied that the end (i.e. the limit) can never be attained but approached as close as desired. Thus, Wisdom also assumes the potential infinite.

However, compared to Black's argument, there is more to Wisdom's position: He further claims that an infinite series can only be defined for open intervals. This claim, however, is not necessarily true for all infinite series. Consider, for example, a runner going from A to B. The path of the runner, then, can mathematically be represented by either one of (A,B), (A,B], [A,B), [A,B]. In all four of the intervals, it is possible to define an infinite series. The only difference would be that the endpoint B which is also the limit point of the run is not included in the intervals (A,B) and [A,B). In that case, we can have a series where the end cannot be attained for the series has no last term inside the interval. There is an important consequence of Wisdom's allowance for only open intervals. In that case, the infinite series is "incompletable" in the first sense. Hence, it cannot be completed, as we have previously seen, for the series lacks the last term. This restriction, however, is arbitrarily made. There is no reason not to consider the closed intervals.

How about (A,B] and [A,B] then? Can they offer a better context for the logical possibility of completing an infinite series of tasks? Wisdom and Benardete seem to respond to the question with two opposite answers. Let us first consider Wisdom's (1941) remarks. Making an adjustment in his argument, he assumes that "the infinite series contains an actual infinity of terms, i.e., $\aleph_0$. Then we might say that the series was a function defined for the closed interval $1 \leq n \leq \aleph_0$, and it would therefore attain its limit" (1941, p. 71). However, he claims, such an adjustment does not solve anything:

> We may, it is true, introduce $\aleph_0$ as the number of numbers, for mathematical purposes, provided no contradiction ensues -and we may assume that no mathematical contradiction does ensue- but does this apply to the physical case, where Achilles is traversing physical intervals? That is to say, can Achilles traverse $\aleph_0$ physical subintervals? Since these would become smaller without limit there would be the physical difficulty of affecting the markings, on account of the limits to the degree of accuracy with which we can make measurements. No improvement in modes of measurement can even in principle lead to measurements of absolute accuracy; and any finite improvement, however great, entails that the number of measurable subintervals is finite. This objection would seem to be fatal to the project of obtaining $\aleph_0$ subintervals corresponding to a convergent series of numbers. The concept of $\aleph_0$ therefore makes no useful addition to a mathematical solution. (Wisdom, 1941, p. 71)

Thus, Wisdom argues that infinite cannot be introduced to be really a number for the physical difficulties that may arise in our measurements. We see that, for Wisdom, the practical constraints are of utmost importance in our understanding of infinity which he views with an empiricist and finitist attitude.

From the finitist perspective, *aleph-null* is credited for not being really a number since it defies the definition of number in the primitive sense which is deeply embedded in the view "aggregate of units," viz., an attribute of *megethos*. But so does number *zero*, Benardete claims, yet we do not have a problem with it being a number. For this, he considers the example of a man which is given a pen and a piece of paper and instructed to write down any number of words, any number he pleases. He writes nothing, or 0 words (Benardete, p. 32). For Benardete, it is puzzling that zero is sometimes stands as "none" and sometimes as a number which is “something”. Thus, there seems to be an ambiguity in the meaning of zero. Benardete, at this point, deduces from this example that the language of mathematics is largely sophistical: “Under the guise of an informal proof, we are being rhetorically redirected by means of a persuasive definition to recast our concept of number into a new form” (1964, p. 36). Just as zero is not a number per se, or the straight angle is not really an angle, aleph-null can perhaps be claimed to be a cardinal number in the sense that 5 or a billion are cardinals. Equally important, consider the following passage:

> I am engaged in arguing that zero and none are strictly equivalent, and yet we commonly acknowledge the former to be a number while we deny it of the latter. How then can they be strictly equivalent? We may say that they are logically or semantically equivalent but nominally different. However, this nominal difference is by no means negligible. That very nominal difference, logically and semantically so trifling, serves both to execute and to disguise the concept-leap at once. (p. 34)

Thus, by introducing the notion of concept-leap aptly in his discussion, Benardete points out that in one sense aleph-null is not really a number (that is the finitist view which we saw previously in Wisdom’s argument), but in another sense, it certainly can be. In a way, this “other sense” is Benardete’s attempt to carry the discussion on theoretical grounds, and allows him for a “metaphysical leap”. We will hopefully clarify what Benardete means by this sense (that is, the tropological sense) toward the end of this chapter, and return to Wisdom’s argument that we initially began to investigate.

What can be said on Wisdom’s distinction of the “physical distance” and the “mathematical distance”? One important response comes from Taylor. He argues that this distinction is not present in Zeno’s argument (1952, p. 16). In fact, Zeno does not mention mathematical points or geometric sums. The paradox, in fact, can be rephrased without resorting to such abstractions and mathematical apparatuses. Regarding Wisdom’s insistence upon Achilles’s racecourse as a physical path rather than a mathematical path which contains dimensionless points and infinitely divisible continuum, Taylor argues that Zeno might have been expected to say to Wisdom the following:

> If anything is going to move from, here, over there to B, it must *first* pass through part of the distance, must it not? And before it accomplishes that, must it not traverse some part of *that* interval? And before that, a portion of the last mentioned part! And may we not say the same over and over, without ever saying anything which is not obviously true? Indeed, if you want at any time to deny this statement, that the part must be traversed before the whole, and the part of the part before the whole of the part, *ad infinitum*, you are in the extra-ordinary position of saying that something can have got from one place to another (however near) without first having gone part way. But if something could thus instantaneously traverse some minute interval, why could it not similarly traverse any distance whatever? And if it cannot do the latter, how can it do the former? (Taylor, 1952, p. 16)

It is made clear in the passage, without appealing to the different meaning of "distance", i.e. "physical" or "mathematical", Wisdom misses hitting the target at Zeno's argument for the argument still poses the same problem. Wisdom's concern regarding the intervals or distances that are too small to measure, and the question "What could be meant by an interval too small to observe?" lead to rejecting the infinite sum as the mathematical solution to the paradox for psychological difficulties that arise from observation. However, arguing for such psychological difficulties against the problem of the paradox is analogous to saying "I cannot see what is beyond the horizon, so anything beyond the horizon is physically meaningless". But does it mean that anything beyond the horizon is altogether meaningless?

That is why we are right to claim that Wisdom, and also Black, try to overcome the semantical and logical difficulty that is presented in the paradox by offering explanations which are based on empirical or physical impossibilities such as measuring distances with strict accuracy, as well as taking the meanings of the terms such as "measurement" to be empirically constructed. These explanations revolved around the distinction between the physical reality and our mathematical abstractions. Moreover, the meanings of the terms are taken to be in accordance with the spatio-temporality of each act that is performed, each step that Achilles takes, and the physical distance that cannot be divided after a certain point.

Considering what we have seen so far, compared to Benardete's position and his agenda, Black and Wisdom argue with a certain kind of empiricism which enunciates the actual infinite as self-contradictory for it cannot be experienced. In fact, such objections against the infinite for not being accessible by experience are even raised by Kant regarding the cosmological question whether the universe is finite or infinite. Below, we will examine Kant's position and Benardete's response.

2.3 Kant and the empirical concerns

In the *Critique of Pure Reason*, Kant formulates four antinomies which are supposed to affirm the limits of pure reason and to make us suspicious about the absolutes, or anything "unconditioned" in Kant's words.[5] At least, that is what Kant thinks that we should make of the antinomies. Benardete elaborates in detail on the first antinomy which says that "the world has a beginning in time and is also enclosed within bounds as regards space" (A427). For Kant, the objects of experience are not given *in themselves,* but only in experience, and have no existence outside it. Rationality is in connection with perception which is in accordance with the empirical laws. If a thing is outside of such sphere of experience, then it is *in itself*.

Before examining Benardete's criticism, let us also consider Kant's proof of the first antinomy. Kant argues that "if time has no beginning, then up to any given point in time there must have passed an infinite series of events. However, an infinite

[5] The distinction between "condition" and "conditioned" is a very general distinction in Kant, which appears throughout his first *Critique*, according to which, the causes are "conditions" of their effects which are, in this respect, "conditioned". Kant argues that human reason seeks the "conditions" in what we experience. If the conditions of a thing are not accessible through our experience, it is "unconditioned". Kant emphasizes that "the conditioned is analytically related to some condition, but not the unconditioned" [A308]. The distinction between "conditioned" and "unconditioned" resonates with his distinction between "phenomena" and "noumena".

series can never be completed through successive synthesis. Therefore, it is not possible for an infinite series of events to pass away" (A426). Benardete's attack on this proof relies on Kant's definition of infinity in the premise as "that which never ends" (similar to what we have seen in Black's argument) and also the equivocation of the word "end." For Benardete, what Kant refers to in his proof is a naïve understanding of infinity which relies on "the enumeration of all co-existing things." Such enumeration includes exhausting things one by one. And he claims that he in fact can do with an example. In the example, Benardete considers a spiral (starting from point *A* at the center) which covers an infinite timeline that starts with today. Thus, *A* is set to be where he is today, and as we move along the spiral, a different point, say *B*, is where he was yesterday, and moving a little further from *B*, another point, say *C*, is where he was the day before yesterday (Benardete, 1964, p.123). In that fashion, he claims, he can succeed in travelling throughout the entire infinite timeline of the universe. This argument, however, does not really give an answer to Kant's insistence of "exhausting the infinite series of events one by one."

What Benardete presents us with in the spiral thought experiment is similar to an example that he previously considers and calls to be "a miserable sophism." In the latter example, there is a "scoundrel" who draws a line and claims that he actually has written down all of the natural numbers (pp. 5-6). The similarity between these two arguments, namely Benardete's and the scoundrel's, is that they both falsely claim to exhaust a denumerable infinity (i.e. the natural numbers) with a higher-order infinity (i.e. continuum). From the scoundrel's sophism, Benardete points to the depth of Aristotle's account of infinity as a response to Zeno's paradoxes of motion. What Aristotle dispels when he claims that "a line is not made of points" for Benardete, is exactly this sort of sophistry. Although Aristotle accepts that a line is infinitely divisible, the infinite series of finite sub-intervals exists within the line only *potentially*, unless an *actual* division is performed. Thus, in a similar fashion, Benardete's spiral timeline cannot sufficiently fulfill Kant's "challenge" when Kant refers to an "*actual* exhaustion of the infinite series of events," a performance of enumeration which takes its *actuality* in the use of the words "one by one."

In his second criticism of Kant's proof, Benardete considers Kant's argument for the impossibility of an infinite space. Kant argues, relying on the definition that infinity cannot be completed via the enumeration of all co-existing things, that it is impossible to "think as a whole the world which fills all spaces" (A428). Here, Benardete claims that Kant has an underlying premise that "to be is to be intelligible." This premise, to a certain extent, is also what we can find in the arguments of Black and Wisdom. In theory, Benardete does not oppose this empiricist principle of conceivability, but in practice he thinks that such an inference carries difficulties for we do not know what precisely counts as the criterion of conceivability and inconceivability (Benardete, p. 128). In reply, he challenges Kant's position by establishing the intelligibility of an infinite world with the statement "The number of stars is either infinite or finite", which will be called *T* from now on. He considers the following inference:

(1) The number of stars is either infinite or finite.
(2) It is not logically necessary that the number of stars be finite.
(3) Therefore, it is logically possible that an infinite number of stars might exist. (p. 129)

Benardete proposes that we should just perform the above logical inference to conceive of an infinite world. For him, conceivability does not require a "prodigious" mental act of enumerating all of the co-existing things as Kant seems to suppose. So far, Benardete seems to offer an inspirational insight to the issue. With the insight though, can he successfully defuse finitism? I think, Benardete objects to Kant's idea of conceivability reasonably. However, the answer to this question is not in the affirmative since when tackling Brouwer's refusal of the tautology that "either the sequence '7777' occurs in the decimal expansion of π or it does not" for instance, a refusal which results from Brouwer's radical finitism, Benardete fails to defend his position adequately.

Regarding Brouwer's radical finitism and his refusal of the tautology, Benardete (1962) writes as follows:

> Of the three hallowed laws of logic-the law of identity, the law of contradiction, and the law of the excluded middle-it is only the first and second that he is willing to accept without qualification. On the strength of the law of the excluded middle nothing would seem more self-evident than that the sequence of digits 7777 either occurs somewhere in the decimal expansion of π or it does not....This is precisely what Brouwer...denies….What they [Brouwer and Wittgenstein] insist upon is that we have no *a priori* guarantee that one or the other of those proofs must even in principle be available. It might be the case that, no matter how far out we might undertake to generate the decimal expansion of π, the sequence 7777 would never be forthcoming. Moreover, we cannot assume that a mathematical proof of that fact must necessarily lie waiting for us in some Platonic heaven. This is the hard core or Brouwer's position. There is no *a priori* guarantee that every mathematical problem must be capable of a solution. (p. 59-60)

According to Benardete, Brouwer is wrong in such a refusal which relies on the impossibility of carrying out an "infinite proof." For him, there is no logical necessity of choosing finitism over the actual infinite. The reason for Benardete to hold such a view stems from, again, his handling of the proposition *T*. Benardete thinks that Brouwer's commitment to finitism is in danger because *T* can be viewed, without losing any of its force, either in terms of the actual or the potential infinite. The idea that *T* can be considered in a twofold way is in fact very exciting. To do this, he presents an example by assigning the decimal expansion of π to a cosmological model where he tries to show that Brouwer is wrong in refusing the above-mentioned tautology, which is as follows.

Assume there is a one-one correspondence between the digits (from 0 to 9) and colors (white, black, blue, purple, green, yellow, orange, scarlet, pink and brown, respectively). So, 0 corresponds to "white", 1 corresponds to "black", etc. Let it be the case that today there occurred a shower of purple sparks in the sky, tomorrow a shower of black, the next day green, the fourth day black, the fifth day yellow, etc. *ad infinitum.* In that way, let it be the case that the sequence of color showers that generates π appears in the sky (Benardete, p. 130). Then, Benardete claims, there appears either four successive showers of scarlet sparks or does not. Thus, we can decide if '7777' appears in the decimal expansion of π or not.

Benardete misses a huge point though. It is not enough to set up the example so that a shower of sparks is seen in the heavens *every other day*. This cannot suffice to be infinity in the potential sense; it stands merely as a sequence of events in time. The problem arises when he claims that we can *know* whether the sequence of four successive showers of scarlet sparks is recorded or not. If we can know that, it means that the properties of the sequence of spark-showers are already fully determinate. But then, such a sequence becomes an actual infinite. For this reason, Benardete cannot really oppose to the finitism of Brouwer.

Taking a closer look at Benardete's inference once again, we see that Benardete's attack against Kant's argument centers on the second premise, i.e. that the number of stars to be finite is not a logical necessity. This premise, I think, cannot be rejected by Kant, or Black and Wisdom, as we have previously examined. Kant's insistence for the impossibility of an infinite world relied on a physical operation, i.e. enumeration, which is empirically impossible. Similar to Kant's objection on the basis of enumeration and counting, in Black's argument, we saw that the difficulty in conceiving an infinity machine stemmed from not knowing that there are infinitely many marbles in the tray when the machine that transfers the marbles has stopped. Similarly, Wisdom argues that the inference of Zeno's argument is not logically invalid (in fact, there is nothing wrong with the inference). Nevertheless, he rejects the assumption that the mathematical representation of racecourse is the aggregate of infinitely many points, which in return causes the inference to be invalid. All three of the arguments, however different though they are, can serve as a basis for empirical evidence against the possibility of the actual infinite.

What can we, then, make of the two such seemingly-different approaches? The empirical approach does not seem to be compatible with Benardete's point of view. As we have seen with Benardete's spiral journey, the theoretical account cannot adequately tackle the problems raised by the empirical account, for both accounts hold different concerns and prioritize different aims. It seems that the two sides are trying to prove or disprove different things, and even at times, they speak different languages. Even though the notions that are referred to and the sentences that are examined are the same –in terms of what they denote– what these words and sentences seem to connote different things. After all, we are not only dealing with logical symbols and signs; much of the dispute revolves around racecourses, stars, and sequences of tasks. Far from being mere placeholders, these notions are deeply rooted in the physical world and its unique conditions.

Regarding the conflict between the two different accounts, Benardete's insight illuminates the underlying reason for the incommensurability:

> The empirical thesis that the wall is infinite logically entails the cosmological thesis that the world is infinite. If it is meaningful and intelligible to suppose that the wall is infinite, then it must also be meaningful and intelligible to suppose that the world is infinite.… I am not suggesting that there is no "logical" difference between the empirical thesis that the wall is infinite and the metaphysical thesis that the world is infinite. Let us grant that the metaphysical thesis is both unverifiable and unfalsifiable, in contradistinction to the empirical thesis which admits of falsification even as it defies verification. I insist only that if the empirical thesis is meaningful, the metaphysical thesis must also be meaningful: the one logically entails the other. (pp. 106-7)

So far Benardete is quite persuasive. That being said, there is one matter that should be pointed out. The meaningfulness of the empirical thesis and that of the metaphysical thesis, as mentioned, requires different theories of meaning. In fact, Benardete also points out this distinction. For he states:

> On the one hand we have an empirical theory of meaning and, on the other hand, we have the logical requirement of a contrast theory of meaning. These two principles are essential to any logical empiricism. According to the first principle, a concept is meaningful only if it is capable of some possible empirical application. According to the second principle, a concept is meaningful only if its logical opposite is also meaningful. These two principles prove to be incompatible in the crucial case of the finite-infinite dichotomy. (pp. 107-8)

The above remarks are quite reasonable. In that respect, I do not think that we can once and for all escape from empirical concerns about the matter for the two principles, as Benardete mentions, are tied to each other. However, this also means that the finite-infinite discussion cannot be resolved if the proponents of the two distinct approaches which clash with each other try to frame the matter purely from their own points of view, because then, an incommensurability due to different theories of meaning raises difficulties. That is why it seems that Benardete cannot directly respond to any of Black's arguments or vice versa.

Let us now finish our discussion of supertasks. In the next section, we will examine Thomson's argument and Benacerraf's objection in reply, and consider somewhat more sophisticated analyses of the logical possibility of supertasks.

2.4 Thomson's "vague" lamp

Thomson (1954) begins the discussion by claiming that we do not need to take sides in the dispute, between those which infer the falsity of the first premiss (i.e. you cannot complete an infinite number of tasks) and those which inter the falsity of the second premiss (i.e. completing an infinite number of tasks is logically absurd). For him, Zeno's argument is not valid since it commits the fallacy of equivocation of the expression "completing a supertask." Thomson tries to demonstrate that "there is an element of truth in each of the premises" (p. 1) but this is due to the different conceptions and applications of the infinite. However, the element of truth in each premiss does not affect the impossibility of supertasks.

Thomson begins by considering the second premiss. His main argument is that we cannot conceive what it would be like to perform a supertask, for we do not really know what a supertask is. If we accept that someone has performed a supertask by completing all of an infinite number of tasks, then do we know exactly what a supertask is and what it is like to perform one? The problem, he argues, is not that whether we understand the sentence "the operation so-and-so can be performed infinitely often." Certainly, we understand such a statement. "But to say that some operation can be performed infinitely often is not to say that a supertask can be performed" (p. 2). To see what he means by this more clearly, let us consider the example he gives. "Suppose (A) that every lump of chocolate can be cut in two, and (B) that the result of cutting a lump of chocolate in two is always that you get two lumps of chocolate" (p. 2). Thomson accepts that the conjunction of A and B is consistent and that a lump may be infinitely divisible. This, however, only implies that "there is an infinite number of numbers of parts into which the lump can be divided. And this is not to say that it can be divided into an infinite number of parts" (p. 2).

It is clear that such an argument entails the use of Aristotle's potential infinite, which takes infinite divisibility only in the potential sense. In other words, Aristotle's account implies that a magnitude (say, the path of a runner or a lump of chocolate) is infinitely divisible (i.e. that is being in the potential sense) but at each step of the division, it is always a finite number into which we divide the path or the lump (i.e. that is being in the actual sense). In accordance with the Aristotelian analysis, Thomson further argues that "if something is infinitely divisible, and you are to say into how many parts it shall be divided, you have $\aleph_0$ alternatives from which to choose" (p. 2). This, however, is not to say that $\aleph_0$ is one of them. In this way, unlike Benardete which we have previously mentioned, Thomson rules out the infinite numbers as one reason to justify the logical possibility of performing a supertask.

Another reason, Thomson argues, that led people to presume that supertasks are possible of performance is explained through Russell's example (p. 4). The example in question illustrates a Zenonian procedure, where it is suggested that "a man's skill in performing operations of some kind might increase so fast that he was able to perform each of an infinite sequence of operations after the first in half the time he had required for its predecessors" (p. 4). Thus, Russell argues, a supertask is only medically impossible. For Thomson, on the other hand, there is nothing inconceivable with Russell's "medically-improved superman" who performs *n* tasks in a certain amount of time for any *n*. However, Thomson once again raises the objection he raised in the example of the lump of chocolate: to say that each act in a sequence of infinity of acts is performed is not to say that all of an infinity number of tasks should have been performed. In this claim of his, we can once again see that Thomson takes the actual infinite to be conceptually vacuous. This claim is also the reason for Thomson to argue that Russell fails to show us what it would be like to perform a supertask, and does not explain the concept thoroughly.

Thomson's objection to Russell is quite compelling. Unlike Black and Wisdom, Thomson does not oppose to Russell by arguing that performing a supertask is physically impossible, nor does he find Russell's "superman" inconceivable. That being said, Thomson still claims that completing an infinite sequence of tasks is inconceivable, and that is also the reason that the word supertask is not well-defined.

2.4.1 Infinity and conceivability

By Thomson, the inconceivability of a supertask, or the actual infinite, is explained as follows:

> If we can conceive a machine doing something –e.g. calling out or writing down numbers– at a certain rate, let us call that rate *conceivable*. Then, there is obviously no upper bound to the sequence of conceivable rates. For any number *n* we can imagine a machine that calls out or writes down the first *n* numbers in just $2-\frac{1}{2^{n-1}}$ minutes. But this again is not to say that we can imagine a machine that calls out or writes down all the numerals in just two minutes. (p. 5)

Here, by introducing the method of "conceivability," Thomson tries to bridge the gap between the epistemic and modal domains. That is to say, he argues that if the task of the machine is conceivable, viz. if the task can be in any way articulated within a consistent scenario, then it is called to be possible. The possibility here is of a metaphysical kind as opposed to physical or natural. However, it should be noted that there is no consensus over the issue of what a proper conceiving truly is. Putnam (1975) shows, on the one hand, conceivability is one thing and metaphysical possibility is another. "We can perfectly well imagine having experiences that would convince us (and that would make it rational to believe) that water is not H_2O. In that sense, it is conceivable that water isn't H_2O" (p. 233). Denying that conceivability is a guide to possibility, Putnam's view ultimately yields to general skepticism about metaphysical claims. This, consequently, poses a problem about the status of our modal propositions. What do we, then, do with our modal claims? Yablo (1993) makes an important remark on this matter. He claims that, unless we are willing to accept modal skepticism, then conceivability as evidence of possibility is the only way to gain modal knowledge. What makes us hesitate is not that conceivability leads us astray, for perceptual knowledge which constitutes a great deal of our knowledge about the world may also do the same, but we do not know how to guard ourselves against modal errors. Therefore, what we must do is to develop certain strategies and rectify our understanding more and more to avoid making errors in our modal claims.

Returning to Thomson's "conceivability rate," we see that he makes use of a mathematical formula to show how long it takes for the machine to perform its task at each step. This, in turn, serves as a good strategy to avoid the modal errors and restricts the meaning of his use of the rather vague term "imagining." Furthermore, Thomson's conceivability rate is also in accordance with his criticism against the positions of those who argue that supertasks are possible of performance, namely Taylor (1951) and Watling (1952):

> People have, I think, confused saying (1) it is conceivable that each of an infinity of tasks be possible (practically possible) of performance, with saying (2) that is conceivable that all of an infinite number of tasks should have been performed. They have supposed that (1) entails (2). And my reason for thinking that people have supposed this is as follows. To suppose that (1) entails (2) is of course to suppose that anyone who denies thinking (2) is committed to denying (1). Now to deny (1) is to be committed to holding, what is quite absurd, (3) that for any given kind of task there is a positive integer k such that it is conceivable that k tasks of the given kind have been performed, but inconceivable, logically absurd, that k+1 of them should have been performed. But no one would hold (3) to be true unless he had confused logical possibility with physical possibility. And we do find that those who wish to assert (2) are constantly accusing their opponents of just this confusion. (p. 3)

It is clear that, Thomson's arguments regarding the actual infinite rests on this impossibility of correlation, that is, considering the rate for/the possibility of any given *n* is not the same as considering the limit case. I agree with him on the difference between the two cases when considering an infinite sequence: the case for a finite *n* and the limit case. That being said, I do not find it convincing that the limit case is inconceivable for the simple reason that these two cases are different from one another. The conceivability of the limit case, however, is only related to whether supertask is a well-defined notion or not. Below, we will examine Thomson's lamp in detail, and find out that the terms “the infinite series” and in turn “supertask” may refer to different senses of infinity in terms of convergence and divergence. In fact, in one sense of the infinite series, that is infinite divergent series, supertasks are inconceivable. That, however, does not have to be the case for the infinite convergent series, for such series can be adapted to different metaphysical models and be subject to challenging thought experiments.

2.4.2 Limit case and vagueness

It is now crucial to consider the difference between (1) the partial sums of an infinite summation and (2) its limit case. The difference is exactly what Thomson strongly emphasizes when analyzing the supertask scenarios: it seems that the rules and the conditions of (1) and (2) may remarkably differ. However, in his famous and beautifully simple scenario, this difference is what seems to be overlooked. Let us briefly state Thomson’s argument regarding the lamp:

> There are certain reading lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button, the lamp goes off. So if the lamp was originally off and you pressed the button an odd number of times, the lamp is on, and if you pressed the button an even number of times the lamp is off. Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next, and so on.…After I have completed the whole infinite sequence of jabs, i.e. at the end of two minutes, is the lamp on or off?… It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction. (p. 5)

The difference between what to do at each step (either turning the lamp on or off) and whether the lamp is on or off at the limiting case is what is missed in Thomson’s argument. Let us say from now on, for convenience, we start the task of jabbing at t_0 and finish at t_1, where the interval measures exactly two minutes. As Benacerraf (1962) argues Thomson only classifies whether the lamp is on or off only for instants before t_1. In fact, this scenario (assuming that the lamp is initially off) can be represented by a function as follows:

$$\Phi\ (n) = \begin{cases} lamp\ is\ off\ for\ j = 2n, \\ lamp\ is\ on\ for\ j = 2n+1 \end{cases} for\ n = 0, 1, 2, \ldots and\ 0 \leq t < 2,$$

where *j* counts the number of times the button is pressed, and *t* is time.

Benacerraf points out that Thomson's instructions only cover the state of the lamp at every instant between $t_0 = 0$ and $t_1 = 2$ (including t_0), and nothing has been specified for when *t* exactly measures two minutes. However, under these circumstances, we cannot argue that the notion of a supertask is contradictory. Thus, Thomson is wrong to claim that the lamp example results in a contradiction. Benacerraf claims that it is not because supertasks are contradictory that we cannot decide whether the lamp is on or off at t_1. Moreover, the given scenario is set up to present impossibility, a contradiction which results from incomplete initial conditions (p. 769). Benacerraf argues that if the infinite process ends or is completed, then there is a final state of the lamp and it is consistent that the lamp is on or off in the final state. The sense of "end" which Benacerraf appeals to is the intuitive sense of "end," where there is a final or last step of the process. However, there is no determinate final state of the lamp. Thus, when we try to view the process as actual infinity, think of it as completed or finished, then there is a final state. But this means that we invoke the intuitive sense of "end" on the lamp which does not have a determinate final state. In that regard, Benacerraf's attack against Thomson's lamp can be averted when we employ the intuitive sense of "end" which implies that there is a final state of the process.

Furthermore, Benacerraf argues that determining the status of the lamp at the final state (which is not originally specified in Thomson's example) by simply adding "let the lamp be *on* at t_1 without loss of generality" is not shown to be contradictory by Thomson's argument. The addition of the final state of the lamp as given information in contrast to being a direct result of what is initially characterized for the function Φ may perhaps be an easy way out of the argument. Though it may be refuted, if not by Thomson's argument, Benacerraf's addition is still illustrative for the following reason. It shows that the finite (viz. taking the sum of *n* terms for each *n* in an infinite series) and the infinite (viz. the total sum of the series) must be handled differently.

2.4.3 Divergence and convergence

It is not surprising that, notions signifying quite different conditions (in that sense, two different notions) require different approaches as well. What cuts the Gordian knot, in a sense, is to realize that the infinite in itself needs to be handled carefully for it may correspond to either a convergent or a divergent series, which are two distinct phenomena. The infinite divergent series, in fact, does pose peculiar problems. At this point, it is necessary that we should turn to another argument that Thomson raises, explaining why the "contradiction" arises in the case of the lamp:

> Say that the reading lamp had either of two light values, 0 (off") and 1 ("on"). To switch the lamp on is then to add 1 to its value and to switch it off is to subtract 1 from its value. Then the question whether the lamp is on or off after the infinite number of switchings have been performed is a question about the value of the lamp after an infinite number of alternating additions and subtractions of 1 to and from its value, i.e. is the question: What is the sum of the infinite divergent sequence +1, –1, +1,…? (p. 6)

The divergent series that is mentioned above is called Grandi's series, which, as a model, produced so many paradoxical examples against the infinite. In fact, Black's machine "Beta" gives out Grandi's series if we consider that, for any tray, the tray with respect to carrying the marble has two alternatives, "+1" when Beta carries it to tray, and "–1", Beta carries it away, ad infinitum. In this sense, as Thomson notices, the supertasks that are generated by the lamp and Beta can be represented by the same infinite divergent series. In 1703, the mathematical solution to the sum of Grandi's series is proposed by Guido Grandi to be 1/2. Thomson (1954) argues, though, "this answer does not help us, since we attach no sense here to saying that the lamp is half-on" (p. 6). In fact, Benardete (1964) also underlines the mathematical as well as ontological crisis that the series brings forth, adding that "almost any summation of Grandi's series would seem to be as reasonable, and as arbitrary, as any other" (p. 23). Like Benardete, Benacerraf also analyses Thomson's argument suggesting that the peculiarity of the Grandi's series is what "went wrong" in the lamp example. Both Benardete and Benacerraf propose important insights regarding the infinite divergent series and the possibility of supertask, which will be viewed in the following.

Firstly, Benardete subjects Grandi's series to an ontological model that is ready at hand, i.e. Zeno procedure. This results in an example that is nothing other than Black's machine Beta and Thomson's lamp. Regarding the arbitrariness of any solution of the summation of the series, Benardete claims, rather curiously, that "one is tempted to argue here that the question falls outside the domain of pure mathematics and the *a priori*. The question can only be answered empirically or, rather, hyper-empirically" (p. 23). With this remark, Benardete emphasizes the significance of hyper-facts when deciding the total sum of the series which is considered within the context of a Zenoesque paradox. In fact, we can recognize the same attitude when Benardete considers the question "How does a series without an end ever come to an end?" We saw that Benardete proposed to consider different ontological conditions to determine the answer of such a question when we previously discussed Black's argument in detail.

Benardete's account for the bisection paradox with regards to a procedure that generates Grandi's series, in the same manner, attempts to expose the equivocation over the word 'sum'. Consider, for instance, two summations: 1/2 + 1/2 and 1/2 + 1/4 + 1/8, &c. In the former, whereas the series possess a sum in the *literal* sense, the infinite convergent series can only possess a sum in the *tropological* sense (p. 26). The expression "total sum" means something very different in the two cases. For the infinite series, when we say that the sum is 1, we mean that the partial sums approach to 1, and nothing more than that. Thus, we can say that the series has a sum in virtue of its convergence (p. 67).

Consider now the infinite convergent series, as we have seen in the first case, and Grandi's series. In this case, we see that Grandi's series even lacks a sum in the tropological sense, for the partial sums do not approach to any fixed value. Benardete expects that, in this case, "the Z-series [infinite convergent series] must possess a sum in a literal sense, whereas Grandi's series can only possess a sum in a tropological sense" (p. 67). However, the alternating divergent series has no a terminal result, and the partial sums have no accumulation or summation. In that case, Benardete argues that:

> Merely because we are obliged to confess that the terminal result is logically indeterminate, does not entail the consequence that it is logically unintelligible. In any case, why must we allow our distress over the indeterminacy inherent in divergent series to infect and subvert our confidence in convergent series? (p. 67)

In Benardete's lines, there are some points that enable him to dispel some of the confusion and some that he does not sufficiently support. Firstly, in answer to Black's argument that the tasks of "Alpha" (i.e. the infinite machine that transfers a series of qualitatively similar but different marbles, which generates an infinite convergent series) and "Beta" (i.e. the infinite machine that transfers the same marble that is immediately returned to its original position, which generates Grandi's series) logically depend on each other (Black, 1951), Benardete's emphasis on the distinction between convergence and divergence may provide a reply. The equivocation of "sum" that appears in the cases of the convergent and divergent series, for instance, reveals the distinct ontological statuses that these series indicate. Thus, between the two machines where one generates a convergent series and the other a divergent series, Black's claim that if one succeeds in its task, so must the other is rather not a well-thought-out argument.

On the other hand, Benardete's claim regarding the consequence that the infinite divergent series is logically unintelligible is not implied by its logical indeterminacy needs to be clarified. For, it can be argued that both Black's machines and Thomson's lamp in fact successfully illustrate a case for the unintelligibility of the divergent series. However, we must be careful. We should not correlate the two notions, namely, intelligibility (which refers to what is comprehended by human mind as opposed to sense perception) and physical possibility (which highly relies on empirical facts that are collected by sense data), and thereby fall prey to a faulty reasoning. That is to say, we do not even need to view Grandi's series within a physical procedure.

When the series is considered solely as a mathematical object and examined by logical reasoning, the total sum still offers indeterminacy. Thus, even mathematics cannot offer an approach to make the series intelligible. The candidate sum 1/2 does not make things easier since the total sum may as well be "proven" to be 2/3. [6] The arbitrariness of the result of the infinite summation in the case of

[6] Holding that every infinite series must have a definite sum, Euler explicitly took (in a letter to Goldbach in 1745) the value of an infinite series to be equal to the value of the analytic function whose expansion gives rise to the series. The deceptiveness of this conviction is evidenced by the fact that there are distinct functions whose expansion produces the same series. For example,

$$\frac{1+x}{1+x+x^2} = 1 - x^2 + x^3 - x^5 + x^6 - \ldots$$

divergent series prevails. In that case, coining the term "sum" (even in the "tropological" sense of the word, whatever is meant by that) does not pave the way for the intelligibility of Grandi's series. Since, in that case, what do we mean exactly by the sum of the series? What is more, the problem is not that the expression "total sum" means two different things in the two cases 1/2 + 1/2 and 1/2 + 1/4 + 1/8, &c. It may in fact be plausible to argue for an equivocation in such a case. However, the problem is that there seems to be no fixed meaning for the expression "sum total" in the single case of the divergent series. And Benardete does not seem to offer one. Consequently, he cannot adequately argue that the logical indeterminacy of the divergent series does not entail its logical unintelligibility.

2.4.4 Benacerraf on Grandi's series

Benacerraf's rejection of Thomson's argument regarding the peculiarity of Grandi's series (which is claimed to be what "goes wrong" in the lamp example) is based on the reasons which are different from Benardete's argument. Accepting the sum of the series to be 1/2 as Grandi proposes, Thomson thinks that this shows there is no established method for deciding what is done when a supertask is performed. However, Benacerraf (1962) claims that the lamp example shows no such thing: "What reason is there to believe that its value [the value of the lamp] after all the switchings will be accurately represented by the sum of all the terms, i.e. by the limit of the partial sums?" (p. 771). He further claims that "there is no reason to expect the sum of the infinite series +1, –1, +1, –1… to represent the 'value' of the lamp after the hypothesized infinite series of switchings" (p. 772).

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is a series which again lead to Grandi's series for $x = 1$; we would therefore be able to 'prove' that $1 - 1 + 1 - 1 + 1 \ldots = \frac{2}{3}$, just as well as $\frac{1}{2}$. (Waismann, 1959, p. 125)

Benacerraf's objection simply rests on the preservability of the part-whole relation when infinitary operations (such as limit operation) are in question. At first blush, and on finitistic terms, the principle that the property of each part which makes a whole does carry the property to the whole is quite logical. However, in the case of infinity, considering all the examples we have seen so far, this principle is exactly what seems to be at stake. Perhaps, the whole discussion about the "logicalness" of supertasks results from such imperilments concerning our logical principles. Benacerraf, I believe, is quite accurate to voice such an important distinction of part/whole when the infinite (qua continuum) is in question. Furthermore, if we think that parts correspond to each partial sum, and the whole corresponds to the total sum, or the limit, we do not need to worry about whether the property of the each partial sum is carried to the limit in Grandi's series, since such series does not even accumulate any value. Therefore, it is not even logical to expect the principle to hold for the divergent series. For convergent series, on the other hand, such as Achilles's racecourse, Benacerraf argues that the series has "reasons" for the principle to hold, since the racecourse assumes the continuum underlying the Zenonian task.

The part-whole relation turns out to be deeply related to the hyper-facts about the infinite series in question. To make this clear, Benacerraf draws attention to another feature of the infinite, particularly the type of infinity that Thomson specifies in his example, which eventually reveals the "reasons" or the hyper-facts regarding an infinite series. Benacerraf writes the following:

> During the course of the argument a question was asked about what could be described as the result of performing a super-duper task. [If a supertask is a task sequence of order type ω, then a super-duper task is the result of tacking an extra (ωth) task at the end of a supertask.] Since the definition of the supertask specifies nothing about such an ωth task, it is no wonder that the question goes begging for an answer. (p. 772)

Benacerraf finds the super-duper tasks worth considering since this suggestion is ultimately linked to disclosing the distinct natures of different infinite series. On Achilles's racecourse, for instance, if we define the set of Z points as 0, 1/2, 3/4, 7/8… where the runner starts from 0, Thomson argues that it is impossible to run through the entire Z-series without reaching the limit of the series, i.e. 1; a point outside of Z. Furthermore, Thomson sets an analogy between these two supertasks, namely running on a racecourse and switching a lamp, by arguing that "the absurdity of having occupied all the Z-points without having occupied any point external to Z is exactly like the absurdity of having pressed the lamp-switch an infinite number of times" (p. 10). These words, at first, may seem quite unclear and unsupported. They surely seemed to me so, and apparently Benacerraf had the same problem. The reason for the lack of clarity in these lines, which results from Thomson's faulty analogy, is well spotted and refuted by Benacerraf as follows:

> In the case of the lamp we have a sequence of order type ω, the lamp switchings, and a sequence of order type ω+1, the moments at which they take place plus the first moment after we're through, which must inexorably come. The passage under discussion [the quote above] indicates that Thomson must believe that, just as we cannot go through all the Z-points without reaching a point outside of Z, the description of the lamp supertask is self-contradictory because it fails to provide an answer to his question about the state of the lamp at the ωth moment, about the outcome of an ωth act had there been one. But there need not be an ωth act of the relevant kind!…The analogy apparently fails. And the reason why is that, whereas the members of the Z-series are abstracted from a presupposed existing set of points (the line 0 to 1 inclusive), the task that constitute the supertask are, as it were, generated serially as we need them; there is not even an apparent logical necessity connected with the existence of a task of the relevant kind to fill the ωth spot in the parallel time series, although there might seem to be such a necessity concerning the points on the line. (p. 773-4)

This argument, I believe, is a successful criticism against Thomson's analogy. There are two points that must be considered. Firstly, it is in fact vital to notice the discrete nature of lamp switching and running on a racecourse with a sequence of points abstracted from a continuum. Benacerraf argues that, in the lamp example, there is no logical necessity to suppose that the sequence of tasks achieve its limit at the ωth step, thus turning the supertask into a super-duper task. This necessity, in fact, might have been sustained if the continuum is assumed in the lamp example. For Benardete, then, the continuum evidently signifies a hyper-fact that differentiates what happens to the lamp-switchings and Zenonian race in the limit case. In other words, “it does not follow that ‘the absurdity of having occupied all the Z-points without having occupied any point external to Z is exactly like the absurdity of having pressed the lamp-switch an infinite number of times,’ except possibly vacuously” (p. 778). Thus, the analogy is refuted on the grounds that the assumption of continuum does not underlie the task of Thomson’s lamp.

Secondly, Benacerraf’s criticism implicitly implies that we must be at all times cautious how to interpret the convergent or the divergent series in the ordinary language. The examples that we have seen so far, say Black’s machines or Thomson’s lamp, though being articulated within the limits of natural language, have mathematical (or more generally, logical) representations. These representations, after all, are devoid of any immediate meaning. Any meaning that may be attached to the procedures that are mathematically represented (or rather syntactically represented within a formal language) is subsequent to the given formulas. Consider, for instance, the infinite sets of order ω and of order $\omega+1$. The definitions of such sets, in logic or set theory, are not dependent on time considerations. In other words, it lacks the most critical property of a real event, which is spatio-temporality. Thus, when Cajori (1915) argues that the possibility of Achilles’ reaching the endpoint (i.e. the limit of his journey) is a consideration independent of the time sequence of Achilles’ journey, we are in the face of an assumption (viz., time-sequence and distance-sequence are independent of each other) which none of the above mentioned mathematical definitions implies. However, this assumption is found to be a serious mistake by Benardete. According to him, mathematical statements must be understood “neither literally nor tropologically, but instead neutrally, so as to be capable of either interpretation, depending on context” (p. 54). Furthermore, “this neutral mode of discourse is *in itself* unintelligible. It is intelligible only as a dummy or schema which is essentially oriented toward its diverse ontological uses” (p. 54).

Perhaps now we can understand Benardete’s position better, which we have been considering from the beginning of the chapter. To the questions “Does an infinite convergent series attain its limit?” or “Does the sum of the Z-series equal to 1?” we saw that he always proposed to consider the hyper-facts and the “auxiliary” assumptions. Whether the time sequence of the supertask in question converges or diverges, for one, changes the answer of the questions above. To be more precise, if the time sequence diverges (say, if the halves, however small they get, always takes one minute to travel, which we will call “Achilles in the large” as before) then, Benardete argues, the finitist approach is the only tenable option. In that case, Achilles never reaches the endpoint, even though the distance sequence converges to a limit. Thus, Benardete adds, we must all be finitist in the case of “Achilles in the large” which only affords us a perfect ontological model of the merely potential infinite (p. 54). However, the infinite convergent series as a mathematical object is also ambiguous, devoid of any pregiven meaning. Therefore, what we must derive from all of these is that the infinite needs a careful and diverse handling for each case and in different contexts.

With a similar precision about our interpretations of the mathematical statements, Benacerraf’s revelation about the falsity of Thomson’s analogy relies on the consideration of the hyper-facts about the two processes in question, which are found to be very distinct in nature with respect to one another. Firstly, we have the lamp that generates Grandi’s series with respect to the outcome of the ωth step;

secondly, there is the task of Achilles. In that respect, a divergent series and a convergent series is compared. Benardete's criticism, as we saw, relies on the assumption of continuum, which is found in the latter and is missing in the former. The different characteristics and features of these series assume different hyper-facts, and at this point of the discussion, this is not surprising.

To sum up the rather lengthy chapter, it is true that we have not solved the question whether supertasks are logically impossible of being performed or not. Just as Black, Wisdom and Thomson did not establish the impossibility of supertasks by destroying their opponents' views, similarly, Benardete and Benacerraf did not establish the possibility of supertasks with their insightful criticism. This, however, comes as no surprise for, as we are familiar from mathematics, it is always harder to give a direct proof, as opposed to disproving by *reductio ad absurdum*. In this regard, the notion of supertask, without an exact and unique description, is not an easy task to handle. Furthermore, the empirical arguments that we examined in this chapter showed that although supertasks are not logically impossible, this does not show us much about the metaphysical possibility. In the next chapter, to show that the notion is metaphysically possible, we will consider further arguments by Benardete and Benacerraf.

Nevertheless, we still have the merits of this incomplete discussion. It became clear that certain approaches are more proper. More particularly, we found that the language that we use to make sense of the notion must be enriched with regards to its metaphorical character. Not to mention, the terms and notions that we use must be handled carefully whether there are any cases of equivocation. Furthermore, with infinity machines and infinite lamp-switchings, it is clear that the notion entails conducting thought experiments which are extremely delicate and intricate. In the next chapter, we will further investigate how the language and thought experiments facilitate understanding infinity properly.

CHAPTER 3

INFINITY AND METAPHYSICS

In the previous chapter, we have seen that the criticisms regarding the logical possibility of completing a supertask, or exhausting an infinite series are refuted if we were to understand the terms that are used such as “incompletable” and “the end term of an infinite series” in a different way. I showed that the opponents’ arguments are rejected for being inconclusive for the very last step of a supertask, or they are weakened by employing equivocation on certain words (as mentioned above). In this regard, the language and how we understand the terms that we use are of utmost importance.

In this chapter, accordingly, we will examine the views which aim to improve and enrich the language when we talk about infinity. Firstly, I will discuss in what respects vagueness and the problem of infinity are similar to each other. Thus, I will argue that one reason for infinity to be a problematic notion is that certain words and concepts that we use are already problematic when describing and analyzing infinite processes. However, our descriptions and analyses are still not in vain, for a new proof helps to bridge the gap between the infinitistic and the finitistic statements in mathematics. This remarkable discovery shows that providing the infinite with a rational account is perhaps not a hopeless business. One way to rationalize the infinite is by means of thought experiments. Thus, in the second half of the chapter, I will present Benardete’s metaphysical rocket and Benacerraf’s shrinking genie, two thought experiments about exhausting infinity. I will argue that while the former fails to help us understand the notion better, the latter can respond to one of Thomson’s objection about the “illogicalness” of completing a supertask.

3.1 Infinity and vagueness

Among the criticisms, Thomson's objection that there is no established method for deciding what is done when a supertask is done (which in turn, he argues, makes supertasks self-contradictory) is the most interesting one, since it bears a resemblance to a different, but not at all unrelated issue, viz. vagueness. In his influential paper "Wang's Paradox" (1975), Dummett makes connections between the topic of vagueness in our observational language and finitism in the philosophy of mathematics. Below I will explain how such a connection enables us to liken the reactions to vague predicates to the empiricists' skepticism about infinity. If handled carelessly, vagueness, like infinity, creates problems and even leads to skepticism about how to use the observational predicates in our language. We find such skepticism about vague predicates similar to the skepticism about infinity when the notion is discussed strictly from the empirical viewpoint. Before explaining this parallelism, let us first state Wang's paradox:

> 0 is small;
> If *n* is small, *n + 1* is small:
> Therefore, every number is small. (Dummett, 1975, p. 303)

In its general form,

P1.	A(0)
P2.	For all *n*, A(*n*) → A(*n+1*)
C.	For all *n*, A(*n*)

where A is an observational predicate, say "is small", as Wang's Paradox interprets. It is clear that the paradox is essentially one form of Sorites paradox, which is the ancient Greek argument about the heap. If you remove one grain from a heap of sand, you still have a heap of sand; it follows, by repeated applications, that a single grain of sand makes a heap. Wang's paradox, Dummett (1975) states, is merely the contraposition of this, where "n is small" is interpreted to mean "n grains of sand are too few to make a heap" (p. 303).

Discussing what exactly went wrong in Wang's paradox and considering different approaches regarding resolving the paradox open up a vast literature before us, which is beyond the scope of this thesis. However, the paradox can perhaps provide us with a different perspective to reevaluate the finitist "retreat" from accepting the possibility of supertasks. To see this, let us first have our Zeno procedure in the form of Wang's paradox. Let us assume that we have a one-mile road, and assume that each time a walker travels only the half distance of the road that lies before her/him in a total time of 20 minutes. So, (s)he walks half a mile in 10 minutes, another quarter of a mile in 5 minutes… and so on. Let A(*n*) be that "there is ($\frac{1}{2^n}$)th of a mile left to travel." For $n = 0$, the statement vacuously says that there is the whole road left to travel. If we assume that what is described is a supertask, then, for any *n*, the *n*th step implies the existence of the (*n+1*)th step (otherwise, we would not have an infinite sequence of tasks), we get P2, i.e. for all *n*, A(*n*) → A(*n+1*). Therefore, by universal generalization, for all *n*, A(*n*). This gives us the first version of the Dichotomy which says that the journey can never end.

By this, it is clear that we can find Zeno's paradox of motion "Dichotomy" at the heart of Sorites paradox. Both paradoxes, in fact, can so be interpreted that they stir up "skepticism against the principle that insignificant differences can accumulate into a significant difference" (Sorensen, 2003, p. 54). Wang's paradox reveals that the observational predicates may cause incoherency in natural languages,[7] for the reason that such predicates often do not result in a clear-cut and complete description about an observational phenomenon. In a similar manner, Zeno's paradox makes it

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[7] Dummett argues that what is responsible for the appearance of paradox is that either the inductions step (P2) is not true or else the rules of classical logic (either the universal generalization or else modus ponens) cannot be validly applied to vague predicates for the rules that govern the vague predicates are inconsistent to begin with (p. 304). Here, we are not arguing for any of the views, nor aim to investigate in what respect vagueness leads to a paradoxical consequence. Thus, the incoherency that we mention does not refer to any single one of the views.

clear that infinity, when handled rather “thoughtlessly”, yields to absurd consequences. Can there be, then, a “thoughtful” way for the infinite, and if there can, what should it be like? Furthermore, searching for the answer for this, should we stick only to the limits of what the finite and empiricism offer as the arguments of Black and Wisdom suggest, or else should we not give up on the notion already and revise our language and way of thinking, like Benardete, so that infinity finds a happy and consistent place for itself? Before coming to that question, let us go step by step and ask this: what does the similarity of Wang’s paradox and Dichotomy show? Such a question is important since the ways to tackle the problems that vagueness raises may provide us with a better strategy in our own quest.

Probing into the nature of vague notions, Sainsbury (1990) argues that the essence of vagueness “is to be found in the idea that vague concepts are concepts without boundaries” (p. 251), adding that:

> Some concepts classify by setting boundaries but some do not. In the philosophical tradition, the former have received all the attention, and have lent a distinctive character to attempts to study classificatory concepts and their linguistic correlates. Within what I shall call the “classical picture”, a picture which dominates most thinking about thought and language, there is no room for the thesis I wish to put forward: that concepts can classify without setting boundaries. According to this classical picture, the job of classificatory concepts is to sort or segregate things into *classes* by providing a system of pigeonholes, by placing a grid over reality, by demarcating areas of logical space. Boundaries are what count, for a concept must use a boundary to segregate the things which fall under it from the things which do not. (p. 251-252)

Between vagueness and infinity, while perceptual vague notions (such as “red”, “bald”, etc.) even lack any kind of formal definition, the notion of infinity gained at least a mathematical definition with Cantor’s transfinite set theory. After centuries, infinity was finally defined to be only in terms of the most elementary binary relations such as “less than or equal to” as well as the use of unbounded quantifiers, thus the notion was employed to represent a variable that is considered to be greater than any number assigned. In that way, the infinite mathematically gained its rigor, while still implying unboundedness and indefiniteness

Nonetheless, although infinity had a proper mathematical definition, this did not altogether solve the problems that the notion creates in mathematics. Indeed, the indefiniteness aspect of the notion in a way prevailed. However, being bigger than any assignable number does not actually specify the number that is assigned, and for that reason exactly, infinity, in some philosophies of mathematics, was either restricted to the potential infinite (constructivism) or completely banned (strict finitism), depending on how radical the philosophy of mathematics one assumes regarding the actual infinite.[8] Traditional constructivism, for one, claims that mathematics can only allow the constructions which we are capable of affecting or which we can in practice carry out. That is why the expressions "to be capable of affecting" and "in practice" are interpreted to imply only the potential infinite. Strict finitism, on the other hand, which is a more radical interpretation of constructivism, assigns meanings to the expressions in question with "stricter" empiricism. Thus, it becomes clear that these expressions too are in fact expressions without clear boundaries.

We started by the vagueness of our terms and notions when describing infinite processes, and ended up with that many terms and expressions in our language and definitions are already "contaminated" with vagueness. What do all these say to us? A solution is nowhere to be found. Are we to sink into despair even

[8] Dummett (1975) explains the degrees of finitism, varying from moderate to radical, in the philosophy of mathematics as follows: "Constructivist philosophies of mathematics insist that the meanings of all terms, including logical constants, appearing in mathematical statements must be given in relation to constructions which we are capable of effecting, and of our capacity to recognise such constructions as providing proofs of those statements; and, further, that the principles of reasoning which, in assessing the cogency of such proofs, we acknowledge as valid must be justifiable in terms of the meanings of the logical constants and of other expressions as so given.…Strict finitism rejects this concession to traditional views, and insists, rather, that the meanings of our terms must be given by reference to constructions which we can in practice carry out, and to criteria of correct proof on which we are in practice prepared to rely: and the strict finitist employs against the old-fashioned constructivist arguments of exactly the same form as the constructivist has been accustomed to use against the platonist; for, after all, it is, and must necessarily be, by reference only to constructions which we can in practice carry out that we learn the use of mathematical expressions." (p. 301-302)

more? Does it perhaps mean that empiricism about the actual infinite is not to be blamed for since empiricism at least functions as a “boundary-setter”, which is, under these circumstances, what is needed the most? However, whether we accept empirical facts to initially decide the extent and content of our notions or not, we cannot escape that these very notions are still contingent on the language. That is to say, the choice between the different views about how we give meaning to our terms and notions is arbitrary; one thinker can perfectly have strong empirical concerns while the other takes the nature of experience to be something different than what the empiricists understand. I believe that both cases contribute to our understanding of a certain concept or a phenomenon in different ways, since different epistemological theories come with different schemes determining the limits of what/how to perceive, as well as distinct understandings of perception and experience. Indeed, there is no way to decide which view encapsulates the gist of reality, and I do not think it is likely that one single view could achieve that. Thus, we cannot say that the strict empiricist approach (for instance, that of Wisdom) considers the issue at hand with such a thorough examination, for it only shows one façade of the building that stands before us. Because of this, I believe that Benardete’s analysis which approaches the issue from various angles is worth considering.

In the next section, we will further look into Benardete’s position and reveal its advantages and disadvantages. Firstly, we will consider Benardete’s definitions of finite/infinite and how he uses them to the advantage of the infinite, whereby he argues that the infinite should as well assume a rational account as the finite. Then, I will explain that Benardete’s argument, dating back to 1964, becomes more tenable thanks to the current work of two mathematicians, Yokoyama and Patey (2016). According to their recent paper, a new proof helps to bridge the gap between “finitistic” and “infinitistic” mathematical statements. Consequently, I will argue that their remarkable discovery is significant in respect of Benardete’s analysis.

3.2 Definitions of the infinite

Boundaryless concepts, Sainsbury (1990) asserts, “tend to come in systems of contraries: opposed pairs like child/adult, hot/cold, weak/strong, true/false” (p. 258), where we tend to grasp one together with its opposite. Following a similar spirit, Benardete’s evaluation of the pair “finite/infinite” elucidates the resistance against the actual infinite which is perhaps undeserved. He argues as follows:

> When we entertain the possibility that there may be an infinite number of stars in the heavens, we may mean by “infinite” any one of the following: (1) the sequence of stars does come to an end –there is no star, (2) for every star there exists another star that lies beyond it, (3) for any natural number *n*, there exists *n* stars, (4) there exists a relation of one-one correspondence between all stars and a proper subset of all stars. These four definitions are not all on a par: the first alone is couched in negative terms; the second, third, and fourth are couched in positive terms.…Nominally, the word ‘infinite’ is the negate of the word ‘finite’, but logically it is not difficult to reverse the relationship, so as to view the infinite in a positive light. When we say that there may be only a finite number of stars in the heavens, we may mean by ‘finite’ any one of the following: (1) the sequence of stars does come to an end –there is a last star, (2) there exists a star such that no star lies beyond it, (3) there exists a natural number *n* such that it is not the case that there are *n* stars, (4) there does not exist a relation of one-one correspondence between all stars and any proper subset of all stars. Here it is the finite which proves to be the negative idea, in three out of the four of the definitions. Although we are tempted to regard the actual infinite as being, at best, a highly problematic concept, it would seem to be as readily susceptible of a *logos*, a rational account, as its correlative, the finite: the one is as intelligible as the other. Equally intelligible, they are not, however, equally empirical. Indeed, it is only the concept of the finite which would seem to lend itself to any empirical application. (p. 109-110)

The actual infinite’s susceptibility of a *logos*, which Benardete mentions, might have easily come as a naïve optimism. However, a recent discovery provides this “optimism” with a substantial support. Before coming to Yokoyama and Patey’s finding, let us consider the historical background of the subject so that we can appreciate the significance of their discovery.

3.2.1 Hilbert's program

During the early 20th century, the foundation of mathematics was going through a serious crisis with paradoxes and inconsistencies. The validity of the infinitistic reasoning was challenged by the great mathematicians such as Kronecker, Poincaré, and Brouwer. Against such threats, Hilbert still defended the Cantorian paradise for the freedom and creativity it provides to mathematics. The notion of axioms as "self-evident truths" was increasingly replaced with logical concepts such as consistency and completeness. From 1922 onwards, Hilbert's approach, which is what he called the finitary standpoint, shifted to this modern axiomatic method where axioms were not taken to be self-evident truths. Hilbert characterized the domain of finitary reasoning as follows:

> As a further precondition for using logical deduction and carrying out logical operations, something must be given in conception, viz., certain extralogical concrete objects which are intuited as directly experienced prior to all thinking. For logical deduction to be certain, we must be able to see every aspect of these objects, and their properties, differences, sequences, and contiguities must be given, together with the objects themselves, as something which cannot be reduced to something else and which requires no reduction. This is the basic philosophy which I find necessary, not just for mathematics, but for all scientific thinking, understanding, and communicating. (1925/1983, p. 192)

Thus, the finitary reasoning restricted the mathematical thought to those objects which are "intuited as directly experienced prior to all thinking," and to the operations and methods of reasoning which do not appeal to the actual infinity. Although it is not clear what exactly he means by finitism, in terms of the mathematical objects that he advocates, Hilbert often refers to Kantian intuition. Hilbert explicates these objects, for instance, by considering the domain of finitary number theory. He argues that "in number theory, we have the numerical symbols 1, 11, 111, 1111,… where each numerical symbol is intuitively recognizable by the fact that it contains only 1's" (1925/1983, p. 192).

Hilbert's finitism in this "atomic" sense, which is underlies the foundation of mathematics just like a giant wall is composed of a pile of bricks, was for tackling the problem of the infinite which he thought what was above all necessary to clarify and justify. To achieve this, he argues that infinitistic mathematics can be validated by means of a three-step program, which includes (1) isolating the finitistic part of mathematics, (2) reconstituting, or axiomatizing infinite reasoning, and finally (3) giving a finitistically correct consistency proof (i.e. a finitistically correct mathematical proof which demonstrates that the theory is consistent) of this axiomatic system. The program, however, was doomed to failure due to Gödel's incompleteness theorems which showed that proof theories (such as that of Hilbert's) cannot prove their own consistency. Yet, toward the end of the 20th century, Hilbert's program was reconsidered for a partial realization, which aimed to show that substantial portion of infinitistic mathematics is finitistically reducible (Simpson, 1988, p, 251).[9] Yokoyama and Patey's study is a part of this "new program."

3.2.2 Partial success of Hilbert's program

Because it is highly technical, we cannot go through the proof that Yokoyama and Patey give, which is also beyond the scope of this thesis. Instead, we will only consider the significance of their proof. Yokoyama and Patey (2016) take Ramsey's

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[9] Feferman (1988) explains the pattern to achieve Hilbert's program as follows: "A part of mathematics M is represented in a formal system T_1 which is justified by a foundational or conceptual framework F_1. T_1 is reduced proof-theoretically to a system T_2 which is justified by another, more elementary such framework F_2" (p. 364). Here, "M refers to an informal part of mathematics (such as number theory, analysis, algebra, etc.); T refers to a formal axiomatic system in L (e.g. the system of first-order Peano Arithmetic PA in the language of elementary number theory) and F refers to a general foundational framework (e.g. finitary, constructive, set-theoretical, etc.)" (Feferman, 1993/1998, pp. 187-8) Under these definitions, the basic idea of a *proof-theoretic reduction* of T_1 to T2 is that "we have an effective method of transforming each proof in T_1 into a proof in T2; moreover we should be able to establish that transformation provably within T2" (Feferman, 1993/1998, p. 193). With Gödel's incompleteness theorems, the program has failed to be achieved for PA with second-order arithmetic as the system capturing infinitistic reasoning. However, instead of PA, a weaker system than PA, namely the primitive recursive arithmetic (PRA), with the suitable subsystems of second-order arithmetic over PRA is proposed for a partial realization of Hilbert's program.

theory for pairs which is “a branch of mathematics studying the conditions under which some structure appears among sufficiently large collections of objects” (p. 3). In layman’s terms, Ramsey’s theorem for pairs takes an infinite set of objects, such as the set of all natural numbers and pairs each number with all the other numbers in the set. Then, each pair is colored (say, red or blue) according to some rule which is defined for the numbers in pairs. With this set-up, the general conditions for the existence of substructures with regular properties (more specifically, the subsets of pairs of the same color) are sought. Ramsey’s theorem for pairs states that, when the coloring is done, there exists an infinite monochromatic subset (i.e. a subset of infinitely many numbers such that all the pairs which are formed by any two numbers chosen from this subset are the same color.) Yokoyama and Patey’s proof shows that these infinite sets in Ramsey’s theorem for pairs are finitistically reducible.

Thus, with this proof, the infinitistic statements (which are proved with the assumption of the actual infinite) and the finitistic ones (which can be proved without invoking the actual infinite) are found to be not completely separated from one another. In other words, the theorem which is a statement invoking the actual infinite (for it assumes the set of all natural numbers) is proved to be finitistically reducible, which means that it can be proof-theoretically reduced to a system of logic that does not invoke infinity. In this way, the infinitary framework is shown to be reducible to a more elementary, or finitistic framework. Such a bridge between the infinite and the finite is in a way anticipated by Benardete’s definitions of these opposite notions, and his claim that the infinite is susceptible of a logos just like the finite.

The significance of the discovery further purports that the empirical standpoint does not adequately provide infinity with a proper and sufficient understanding. The empirical standpoint, as we have seen in Black's and Wisdom's arguments, is not justified for discarding the possibility of providing infinity with a rational account. In other words, it is not justified that such a possibility is disregarded since infinity cannot be placed within the physical reality in accordance with what the empirical facts suggests. In contrast to the empirical standpoint, the metaphysical viewpoint suggests that we may, in fact, make sense of infinity by providing a rational and logical account and accordingly, investigates the ways to offer such an account.

Benardete's arguments, as we have seen in the previous chapter, suggest that analyzing the language used and viewing the terms and notions in different ways shed light on the notion and improve our understanding of infinity in metaphysical terms. However, such an analysis must be conducted carefully. In the next section, we will consider Benardete's thought experiment, namely the metaphysical rocket, which illustrates a rather poor example of a metaphysical account for the actual infinite. The experiment shows that the presenting a metaphysical account for infinity is indeed a delicate task to handle.

3.3 The metaphysical rocket

In the example, space is assumed to consist of two Euclidean subspaces, S_1 and S_2, and we simply send a rocket from S_1 to S_2. Mimicking the procedure in Zeno's paradoxes of motion, Benardete (1964) says that if the rocket takes one thousand miles in 1/2 minute, another one thousand miles in 1/4 minute… and so on, then "at the end of one minute the rocket travels an infinite distance" (p. 149). Furthermore, travelling in a spiral, the rocket might visit an infinite number of stars, thus "enumeration of all the co-existent things" is also achieved within a minute.

Travelling an infinite distance, the metaphysical rocket is now outside of S_1 and in S_2. After exhausting S_2 in a similar fashion like S_1, he even considers S_3 and then S_4… and so on, but for our discussion, travelling from S_1 to S_2 is simply enough.

How does such a thought experiment contribute to providing infinity with a rational account? What does Benardete's metaphysical rocket reveal to us about the notion? To answer these questions, we will first and foremost investigate the function of thought experiments. Regarding these questions, Kuhn (1964/1977) argues as follows:

> Granting that every successful thought experiment embodies in its design some prior information about the world, that information is not itself at issue in the experiment. On the contrary, if we have to do with a real thought experiment, the empirical data upon which it rests must have been both well-known and generally accepted before the experiment was even conceived. (p. 241)

How, then, can a thought experiment yields new knowledge or a new understanding of the world if it exclusively relies on preexisting knowledge? In other words, if a thought experiment does not lead to new knowledge about the world, what can it reveal? Furthermore, does that mean that thought experiments can only teach us about our conceptual and methodological apparatus? If this is the case, then invoking a thought experiment does not help us to learn about the nature of infinity. However, Kuhn suggests that, as a result of an effective thought experiment, we can learn something new about our concepts as well as the world (p. 253). I think that such an understanding of the role of thought experiments is quite compelling. What is more, when the history of scientific development is concerned, from Einstein's train to Maxwell's demon, the role of a thought experiment is indeed in accordance with what Kuhn characterizes it to be.

Before coming to what we learn from thought experiments about the infinite, let us first find out in what ways Benardete's metaphysical rocket fails to be an effective thought experiment. The most important problem with the metaphysical rocket is that Benardete misses considering the metrical and topological properties of the space the rocket travels in. Let us consider that the rocket at point p_1 in S_1 starts travelling at $t_1 = 0$ to the point p_2 in S_2, finishing the journey in one minute at $t_2 = 1$. The path function then is $f: [0,1] \rightarrow S_1 \cup S_2$. Benardete claims that the motion and the path are continuous so that the assumption that the enumeration of all the co-existing things which leads to infinity as a completed (actual) object is preserved (p. 149).[10] In this way, Benardete ultimately wants to place the continuum in the physical reality which enables one to divide space and time infinitely many times, so that he is not stuck only with the "smallest" infinite which is the set of natural numbers. But if *f*, or the path is continuous, then its range is either completely in S_1 or completely in S_2. That is because *f*, being a continuous function, preserves certain topological properties between the sets it is defined on, and openness or closedness of the set is one of them.[11] Thus, if *f* is continuous, it must map a closed set to a closed set, but in this case, $S_1 \cup S_2$ is not closed. Therefore the path cannot simultaneously be continuous and lying both in S_1 and S_2. Furthermore, Benardete does not only violate certain mathematical truths but also physical principles. Take, for example, the instant when the head of the rocket is just at the boundary of S_2 and its tail still is in S_1. In that case, what happens to its mass and volume? Does the rocket have "infinite" volume so that its head touches S_2 while its tail stays at S_1? Perhaps, this is the reason for it being a "metaphysical" rocket, rather simply "a rocket".

[10] Benardete proposes his thought experiment against Kant's antinomies of infinity. Kant's position holds that an infinite world as a whole necessitates a 'reputed addition of unit to unit', where infinity is implicitly defined to be enumeration of all co-existing things (Benardete, 1964, p. 149). For details see relevant section in Chapter 2.

[11] In topology, a set *S* is open if every point in *S* has a neighborhood lying in the set. A closed set is defined to be the complement of an open set. In our case, we have the closed interval [0,1] and both open and closed $S_1 \cup S_2$.

If we return to Kuhn's analysis regarding the function of thought experiments, we can see that the above objection against Benardete's rocket in fact finds support, for Kuhn argues that thought experiments "must rest entirely on information already at hand" (p. 261). Thus, the familiar apparatus which the scientist use in thought experiments both enables her/him to disclose the confusion and contradictions in her/his mode of thought, and to revise the theory anew. That is why, Kuhn claims, "thought experiments give the scientist access to information which is simultaneously at hand and yet somehow inaccessible to him" (p. 261). While the information at hand allows the scientist to efficiently disclose the confusion and contradictions in the theory, the "inaccessible" part of the information at hand becomes accessible as the theory undergoes revision and necessary corrections. In our case, though, such a transition from the accessible apparatus to the inaccessible part of our knowledge about the infinite is not possible, since Benardete fails even obeying the most basic mathematical truths and well-supported physical knowledge at hand. Moving further from a rather poor example of metaphysical inquiry, let us now consider Benacerraf's "shrinking genie" to see how illustrative and revealing thought experiments can in fact be.

3.4 The shrinking genie

Benacerraf's thought experiment is an example against Thomson's objection that a supertask necessarily assumes a super-duper task, more specifically, his objection regarding the difference between sequences of order type ω and type $\omega+1$, which we have seen in the previous chapter. Thomson puts forth a strong argument and claims that the runner cannot cover all the Z-points (that is, each and every element of the Z-series 0, 1/2, 3/4,…) and stay in the Z-set (Benacerraf, 1962, p. 774). In other words, if we consider 0 the starting point and 1 the end point, Thomson argues that it is impossible to run through all the Z-points without reaching a point outside of Z, i.e. 1. Here, we are not interested in Thomson's argument and whether it is sound. What we should pay attention to is how Benacerraf puts forward a thought experiment to show that running through all the Z-points without reaching a point outside of Z-set can as well be logically possible. So, let us start:

> Let t_0 be the time at which the genie started from 0, and where applicable, for each *i*, let t_i be the time at which he is at *i*. The question then becomes: Does this imply that at t_1 he occupies 1?…If the genie has carried out my instructions, at t_1 he cannot be at 1, because at t_1 he is no more. To be sure, he vanishes *at* a point: 1. But what does this mean? In particular, does this mean that 1 is *the last point he occupied*? Of course not. There need not be any last point he occupied –any more than there need be a *first* point he *didn't* occupy (although there *must* be one or the other). To disappear at a point is neutral with respect to the question of "having occupied" that point. There is no necessity either way. "He disappeared at 1" could mean either that 1 is the last point he occupied *or* that 1 is the first point he didn't occupy, just as to have disappeared at t_1 could involve *either* that t_1 was his last moment on earth or that t_1 was earth's first moment without him. *Which* we say is a function of how we choose to regard trajectories and time intervals. (Benacerraf, 1962, p. 775)

At first blush, the claim that "to disappear at a point is neutral with respect to the question of 'having occupied' that point" is an argument exploiting the ambiguity between the expressions "disappearing at a point and occupying a point". However, Benacerraf further illustrates a remarkable example to tackle this ambiguity, which offers a clarification on the confusion. Regarding the question "Given that the genie disappeared at 1, did he occupy 1?" Benacerraf suggests considering the distance-line and timeline of the genie. These lines are a direct duplicate of the trajectories and time intervals of Zeno's runner in *Dichotomy*: starting from 0, travelling half the distance at each step with suitably decreasing durations, and finishing the journey at 1. Thus, Benacerraf suggests:

> We may view each line in two different ways, corresponding to the ways in which each point may be seen as dividing its line into two disjoint and jointly exhaustive sets of points: any point may be seen as dividing its line either into (a) the set of point to the right of and including it, and the set of points to the left of it; or into (b) the set of points to the right of it and the set of points to the left of and including it. That is, we may assimilate each point to its right-hand segment (a) or to its left-hand segment (b). (p. 775-776)

Benacerraf further argues that whichever we choose between (a) and (b) is arbitrary, however, it is exactly this choice which determines whether the genie occupies 1 or disappears at 1. To obtain that the genie covers all the points to the left of 1, but disappears at 1, which corresponds to exhausting the infinite (of order type ω), we should regard the distance-line and timeline according to (b). There is one problem, though, that Benacerraf points at. He states that "normally it makes no difference, but in *this case how* we view it makes the *only* difference. […] This holds only if method b of viewing the line is *mandatory*" (p. 776). However, the choice between (a) and (b) in terms of viewing the lines is, as Benacerraf admits, arbitrary. To support this possibility of viewing the lines according to (b), then, a thought experiment is proposed, which is no less extraordinary than Benardete's metaphysical rocket. In his new and improved example, the genie shrinks in proportion to the ratio of the distance he covers and what is left of him is "always equal to the ratio of the unrun portion of the course to the whole course" (p. 776). Thus, the genie is full grown at 0, half-shrunk at 1/2; only 1/8 of him is left at 7/8, and eventually disappears at 1. So, even if he vanished at 1, he need not have occupied 1. In other words, he could occupy every Z-point without occupying any point external to Z (p. 776).

Can Benacerraf's shrinking genie successfully object to those (such as Black and Thomson) who argue that supertasks are impossible of performance and self-contradictory since there is no "end term" in an infinite sequence of tasks? I think so, for it shows, though with a bizarre scenario, that there is logically nothing self-contradictory about exhausting infinity of order type ω. Compared to Benardete's rocket, furthermore, the shrinking genie fulfills the functions of a thought experiment in the Kuhnian sense, which emphasizes the role of rectifying our conceptual apparatus at hand. What is more, by carrying out an analysis of language (such as differentiating the methods (a) and (b) to clarify the meaning of our statements involving “disappear at” and “occupy” as well as revealing the equivocation in our terms and notions), Benacerraf perfectly accesses a new understanding regarding our statements which were already present but in certain respects “inaccessible”.

CHAPTER 4

CONCLUSION

In this thesis, I have argued that the metaphysical investigation of infinity is as important and expositional as the empirical and scientific considerations. To view the importance of metaphysical inquiry about infinity from a new level of perception, the following paragraph on vagueness is quite illuminating:

> The classical picture [about the use of the vague concepts] has a totalitarian aspect: there is no difference between its being not mandatory to apply a concept and its being mandatory not to apply it. If the very nature of the concept *prime*, together with the nature of some number, say eight, does not require you to apply the concept to it, then the very nature of the concept, together with the nature of the number, requires you not to apply the concept to it. For a rational and fully informed thinker, there is no freedom.
>
> By contrast, vagueness offers freedom. It can be permissible to draw a line even where it is not mandatory to do so. No one can criticize an art materials shop for organizing its tubes of paints on various shelves, including one labelled “red” and another “yellow”, even though there is a barely detectable, or perhaps even in normal circumstances undetectable, difference between the reddest paint on the shelf marked "yellow" and the yellowest paint on the shelf marked "red". Hence one can consistently combine the following: *red* draws no boundaries, that is, there is no adjacent pair in the series of tubes of paint such that the nature of the concept, together with the colour in the tube, requires one to apply *red* to one member of the pair but withhold it from the other; yet one can draw a boundary to the reds, that is, one may behave consistently with the nature of the concept in drawing a line between adjacent pairs.
>
> The envisaged attack on boundarylessness can be set out as the following argument, which makes plain how the recent observation addresses it. A boundaryless concept is one which, for closely similar pairs, never makes it mandatory to apply the concept to one member of the pair, and withhold it from the other; hence, the argument runs, a boundaryless concept is one which, for closely similar pairs, makes it mandatory never to apply the concept to one member of the pair, and withhold it from the other. (Sainsbury, 1996, pp. 259-260)

This remark also puts forth the reason behind the vast and rich literature on infinity and supertasks. Regarding these notions, we have seen that the metaphysical standpoint offers various examples and analysis in an attempt to provide a well-defined description and account for them so that we can apply these notions better. However, the empirical viewpoint has left us with accepting finitism –if not strict finitism– simply because the actual infinite is claimed to be self-contradictory and logically impossible. Such a claim, nonetheless, cannot be acceptable since whether infinity is logically possible or not is not relevant to the fact that the actual infinite is not instantiated in the physical world. Thus, the empirical constraints, in a way, hinder a possible metaphysical and rational understanding of the notion. Just like the Ancient Greeks feared and dispelled *apeiron* for being indefinite and unbounded, the empirical viewpoint rules out the actual infinite for not being observable and measurable.

But perhaps, just like the liberty of the "boundaryless" vague notions, the liberty that the metaphysical standpoint and investigation provides for infinity with its distinct takes on the meaning of the terms that are used as well as compelling thought experiments, is what we need and must protect to a certain extent, not something that we should dispel. Benardete also points to the "liberty" that such an approach provides in mathematics: "The essence of mathematics lies in its freedom. The classical approach [the rigorous reformulation of calculus in the nineteenth century] is very much the opposite: the essence of mathematics lies in its necessity" (p. 27). In a way, Benardete's views can be taken to illustrate the different approaches and concerns the empirical account and the metaphysical account hold. However, we should note that our contention is not with the empirical viewpoint and its arguments. Quite the contrary, we admit that the empirical viewpoint is indeed important in certain regards. However, it should not be considered to be the only authority. Similarly, the approaches other than the classical approach should not be seen to be any less significant. As Benardete states: "The finite and the infinite are polar opposites, and though only one of the two may be empirically accessible, neither is intelligible apart from the other" (p. 285). And the metaphysical inquiry is significant for contributing to the intelligibility of the infinite in a stimulating way.

Another reason one account does not trivialize the other is that the empirical and the metaphysical viewpoints contribute to our understanding of infinity in different terms, although they both address the same problems. Moreover, the problems of the arguments presented to resolve the paradox (be it empirical or metaphysical) and the questions further brought forth build a body of cumulative work which illuminates the notion of infinity in different aspects. In light of this, therefore, we conclude that investigating the nature of infinity from a broader perspective is something we can truly benefit from. A many-perspectival investigation combining the empirical and the metaphysical considerations embraces the complexity of Zeno's paradoxes which have tremendous impacts on various fields.

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