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MASTER THESIS

PSEUDO-Q-ALGEBRA

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I certify that I have read this thesis and that in my opinion, it is fully adequate, inscopeand in quality, as a dissertation for the degree of master of science.

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APPROVAL PAGE

This study, titled by "PSEUDO-Q-ALGEBRAS" and presented as Master Thesis by Shwan Adnan BAJALAN has been evaluated in compliance with the provisions of Yaşar University Graduate Education and Training Regulation and Yaşar University Institute of Science Education and Training Direction. The jury members below have decided for the defence of this thesis and it has been declared by consensus/majority of the votes that the candidate has succeeded in his thesis defence examination dated 21th October, 2016.

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ABSTRACT

PSEUDO-Q-ALGEBRAS

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In this thesis we introduce the notion of pseudo-Q-algebras and investigate some of their properties. We also consider ideals, minimal elements and centre of pseudo-Q-algebras and we introduce the notion of pseudo ideal and pseudo strong ideal of pseudo-Q-algebras. We investigate the relation between any ideals and pseudo ideals and pseudo strong ideals of pseudo-Q-algebras. Additionally, we characterise these ideals and homomorphisms.

Keywords:- Q-algebras, subalgebras, ideals, pseudo-Q-algebras, minimal element, centre, pseudo ideal, pseudo strong ideal, homomorphism.

ÖZET

SÖZDE Q-CEBİRLERİ

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Bu tezde sözde Q-cebirlerin tanımı verilmış ve ilgili özellikleri elde edimiştir. Ayrıca, sözde Q-cebirlerin idealleri, minimal elemanları ve merkezi çalışılmştır. Sözde Q-cebirlerin herhangi bir ideali, sözde ideali ve sözde güçlü idealleri arasındaki ilişki çalışılmıştır. Bunun yanında bu idealler ve homomorfizmalar karakterize edilmiştir.

Anahtar Kelimeler :- Q-cebirleri, alt cebirleri, idealler, sözde Q-cebirleri, minimal eleman, merkez, sözde ideal, sözde güçü ideal, homomorfizma.

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Finally, I extend my thanks and appreciations to my dear parents who never hesitated to supports me my life and especially during my studies. My dear mother, father, brother and sister to be always being there for me for helping me always throughout my life and especially during my studies. I never forget their help. Furthermore, many thanks for all my friends , who always wishing.

TEXT OF OATH

I, Shwan Adnan BAJALAN do declare and honestly confirm that my study, titled " PSEUDO-Q-ALGEBRAS" and presented as a Master's Thesis, has been written without applying to any assistance inconsistent with scientific ethics and traditions, that all sources from which I have benefited are listed in the bibliography, and that I have benefited from these sources by means of making references.

Student Name and Signature
Shwan Adnan BAJALAN

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INTRODUCTION

BCK-algebras and BCI-algebras were introduced by Imai. and Iseki as two classes of abstract algebras in 1966 (Iseki, K. (1980)),(Iseki, K. and Tanaka, S. (1978)). It is known that BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, BCH-algebras as a wide class of abstract algebras were introduced by Hu and Li (Hu, Q. L. and Li, X. (1983)), (Q. P. and X. Li (1985)). In their study, it is given that the class of BCI-algebras are proper subclasses of BCH-algebras. In 1999, the notion of d-algebras that is another useful generalization of BCK-algebras was introduced by Neggers and Kim (Neggers, J. and Kim, H. S. (1999)). In 2001,a new notion called a Q-algebras was introduced by J.Neggers, S.S.Ahn and H.S.Kim (Joseph, N. , Sun, S. A. and Hee, S.K. (2001)). At the same time pseudo-BCK-algebras as an extension of BCK-algebras was introduced by G.Geordscu,and A.Iorgulescu (Georgescu, G. and Iorgulescu, A. (2001)). In 2008, pseudo-BCK-algebras as a natural generalization of BCI-algebras and pseudo-BCK-algebras were introduced by W.A.Dudek and Y.B.Jun (Dudek and Jun, Y. B. (2008)). These algebras have also connections with other algebras of logics such as pseudo-MV-algebras and pseudo-BL-algebras defined by G.Georgesuc and A.Iorgulescu (Georgescu, G. and Iorgulescu, A. (1999)) and (Georgescu, G. and Iorgulescu, A. (2000)), respectively. As a generalization of many algebras, pseudo of these algebras has been studied by many researchers (Dymek, G. (2012)), (Dymek, G. (2012)), (Kim, Y. H. and So, K. S. (2012)) and (Lee, K. J. and Park, Ch. H. (2009)), (Jun, Y. B. ,Kim, H. S. and Neggers, J. (2009)). In this thesis, we introduce pseudo-Q-algebras.We state some basic properties of pseudo-Q-algebras and provide some characterization of these algebras and we consider the ideals of pseudo-Q-algebras. we defined pseudo ideal and pseudo strong ideal of pseudo-Q-algebras and state properties of these ideals.

CHAPTER 1

Preliminaries

Definition 1.1 *(Joseph, N. , Sun, S. A. and Hee, S.K. (2001)) A Q-algebra* $(X; *, 0)$ *is a nonempty set X with a constant 0 and a binary operation* $*$ *satisfying the following axioms:*

(I) $x * x = 0$;

(II) $x * 0 = x$;

(III) $(x * y) * z = (x * z) * y$ *for all* $x, y, z \in X$.

For brevity, we also call X a Q-algebra. On X we can define a binary relation $\leq$ *by* $x \leq y$ *if and only if* $x * y = 0$. *Recently, Ahn and Kim (Ahn, S. S. and Kim, H. S. (1999)) introduced the notion of QS-algebras. A Q-algebra X is said to be a QS-algebra if it satisfies the additional condition:*

(IV) $(x * y) * (x * z) = z * y$, *for any* $x, y, z \in X$.

Definition 1.2 *(Joseph, N. , Sun, S. A. and Hee, S.K. (2001)) Let* $(X; *, 0)$ *be a Q-algebra and* $I(\neq \emptyset) \subseteq X$. *The set I is called an ideal of X if for any* $x, y \in X$ *the following hold:*

(1) $0 \in I$;

(2) $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.

Obviously, $\{0\}$ *and X are ideals of X. We call* $\{0\}$ *and X the zero ideal and the trivial ideal of X, respectively. An ideal I is said to be proper if* $I \neq X$.

Definition 1.3 *(Joseph, N. , Sun, S. A. and Hee, S.K. (2001)) An ideal I of a Q-algebra*

$(X;*,0)$ *is said to be implicative if* $(x*y)*z \in I$ *and* $y*z \in I$, *then* $x*z \in I$, *for any* $x, y, z, \in X$.

Definition 1.4 *(Walendziak, A. (2015)) An algebra* $X = (X;*,0)$ *of type* $(2;0)$ *is called a BCH-algebra if it satisfies for all* $x, y, z \in X$ *the following axioms:*

(BCH-1) $x * x = 0$ *;*

(BCH-2) $(x * y) * z = (x * z) * y$*;*

(BCH-3) $x * y = y * x = 0 \Rightarrow x = y$.

A BCH-algebra X is said to be a BCI-algebra if it satisfies the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$.

A BCK-algebra is a BCI-algebra X satisfying the law $0 * x = 0$.

Definition 1.5 *(Jun, Y. B. ,Kim, H. S. and Neggers, J. (2009)) A B-algebras is a non-empty set X with a constant 0 and binary operation * satisfiying for all* $x, y, z \in X$ *the following axioms:*

(B1) $x * x = 0$;

(B2) $x * 0 = x$;

(B3) $(x * y) * z = x * (z * (0 * y))$; *for all* $x, y, z \in X$.

Example 1.1 *(Joseph, N. , Sun, S. A. and Hee, S.K. (2001)) Let* $X = \{0, 1, 2, 3\}$ *be a set with the following table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X;*,0)$ *is a Q-algebra, which is not a BCH/BCI /BCK-algebra. Neggers and Kim (Neggers, J. and Kim, H.S. (2002)) introduced the related notion of B-algebra, that* *is, algebras* $(X; *, 0)$ *which satisfy (I)* $x * x = 0$; *(II)* $x * 0 = x$; *(V)* $(x * y) * z = x * (z * (0 * y))$, *for any* $x, y, z \in X$. *It is easy to see that B-algebras and Q-algebras are different notions. For instance, Example 1.1 illustrates a Q-algebra, but not a B-algebra, since* $(3 * 2) * 1 = 0 \neq 3 = 3 * (1 * (0 * 2))$. *Consider the following example. Let* $X = \{0, 1, 2, 3, 4, 5\}$ *be a set with the following table:*

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra but not Q-algebra since $(5*3)*4 = 3 \neq 4 = (5*4)*3$.

The following example shows that a Q-algebra may not satisfy the associative law.

Example 1.2 *(Joseph, N. , Sun, S. A. and Hee, S.K. (2001)) (a) Let* $X = \{0, 1, 2\}$ *with the table as follows:*

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then X is a Q-algebra, but associativity does not hold, since $(0 * 1) * 2 = 0 \neq 1 = 0 * (1 * 2)$.

(b)Let $\mathbb{Z}$ *be the set of all integers and let* $nZ = \{nZ|z \in \mathbb{Z}\}$ *where* $n \in \mathbb{Z}$. *Then* $(\mathbb{Z}; -, 0)$ *and* $(n\mathbb{Z}; -, 0)$ *are Q-algebras and QS - algebras, where " − " is the usual subtraction. Also,* $(\mathbb{R}; -, 0)$ *and* $(\mathbb{C}; -, 0)$ *are Q-algebras and QS-algebras where* $\mathbb{R}$ *is the set of all real numbers,* $\mathbb{C}$ *is the set of all complex numbers.*

Theorem 1.1 *(Joseph, N. , Sun, S. A. and Hee, S.K. (2001)) Every Q-algebra* $(X; *, 0)$ *satisfying the associative law is a group under the operation * .*

Proof Putting $x = y = z$ in the associative law $(x * y) * z = x * (y * z)$ and using (I) and (II) of Definition 1.1, we obtain $0 * x = x * 0 = x$. This means that 0 is the zero element of X. By (I), every element x of X has itslf as its inverse. Therefore $(X; *)$ is a group. □

CHAPTER 2

Pseudo - Q - algebras

Definition 2.1 *(Walendziak, A. (2015)) A pseudo-BCH-algebra is an algebra* $(X; *, \diamond, 0)$ *of type* $(2; 2; 0)$ *satisfying for all* $x, y, z \in X$ *the axioms:*

(pBCH-1) $x * x = x \diamond x = 0$;

(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;

(pBCH-3) $x * y = y \diamond x = 0 \Rightarrow x = y$;

(pBCH-4) $x * y = 0 \iff x \diamond y = 0$.

Definition 2.2 *(Dudek and Jun, Y. B. (2008)) A pseudo-BCI-algebra is a structure* $X = (X; \leqslant; *, \diamond, 0)$*, where* $\leqslant$ *is a binary relation on the set X,* $*$ *and* $\diamond$ *are binary operations on X and 0 is an element of X, satisfying for all* $x, y, z \in X$ *the axioms:*

(pBCI-1) $(x * y) \diamond (x * z) \leqslant z * y$, $(x \diamond y) * (x \diamond z) \leqslant z \diamond y$;

(pBCI-2) $x * (x \diamond y) \leqslant y$, $x \diamond (x * y) \leqslant y$;

(pBCI-3) $x \leqslant x$;

(pBCI-4) $x \leqslant y, y \leqslant x \Rightarrow x = y$;

(pBCI-5) $x \leqslant y \Rightarrow x * y = 0 \Rightarrow x \diamond y = 0$.

A pseudo-BCI-algebra X is called a pseudo-BCK-algebra if it satisfies the identities

(pBCK) $0 * x = 0 \diamond x = 0$.

Remark 2.1 *(Hu, Q. L. and Li, X. (1983)) Every pseudo-BCH-algebra satisfies (pBCI-2) - (pBCH-5)*

The following definition introduces the notion of pseudo-Q-algebras.

Definition 2.3 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) A pseudo-Q-algebra is a non-empty set X with a constant 0 and two binary operations $'' * ''$ and $'' \diamond ''$ satisfying for all $x, y, z \in X$ the following axioms:*

(PQ1) $x * x = x \diamond x = 0$;

(PQ2) $x * 0 = x \diamond 0 = x$;

(PQ3) $(x * y) \diamond z = (x \diamond z) * y$.

Definition 2.4 *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebras and let $\emptyset \neq I \subset X$. I is called a pseudo subalgebra of X if $x*y, x\diamond y \in I$ whenever $x, y \in I$. I is called ideal of X if it satisfies:*

(I1) $0 \in I$;

(I2) $x * y$ *or* $x \diamond y \in I$ *and* $y \in I$ *imply* $x \in I$ *for all* $x, y \in X$.

We will denote by $Id(X)$ the set of all ideals of X. Obviously, $\{0\}, X \in Id(X)$.

Example 2.1 *Let $X = \{0, 1, 2, 3\}$. Define the binary operations $'' * ''$ and $'' \diamond ''$ on X by the following tables:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	0	0	0
3	3	2	3	0

$\diamond$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	1	0

Then it is easy to verify that $(X; *, 0)$ and $(X; \diamond, 0)$ are not Q-algebras $[(3 * 1) * 2 = 0 \neq 1 = (3 * 2) * 1]$,but $(X; *, \diamond, 0)$ is a pseudo-Q-algebras and not *pseudo-BCH-algebras since* $[2 * 1 = 1 \diamond 2 = 0$ *but* $2 \neq 1]$. *Let* $I = \{0, 1\}$. *Then* I *is a pseudo subalgebra of* X *,but not an ideal of* X*, since* $2 * 1 = 0$*, and* $0, 1 \in I$ *but* $2 \notin I$.

Proposition 2.1 *If* $(X; *, \diamond, 0)$ *is a pseudo-Q-algebra, then (PQ4):* $(x * (x \diamond y)) \diamond y = (x \diamond (x * y)) * y = 0$*, for any* $x, y \in X$.

Proof Let x, y be in X. Then by (PQ1) and (PQ3), we have $(x * (x \diamond y)) \diamond y = (x \diamond y) * (x \diamond y) = 0$ and $(x \diamond (x * y)) * y = (x * y) \diamond (x * y) = 0$.

□

We now investigate some relationships between pseudo-Q-algebras and pseudo-BCH-algebras. The following theorems are easily proved.

Theorem 2.1 *Every pseudo-BCH-algebra X is a pseudo-Q-algebra. Every pseudo-Q-algebra X satisfying condition for all* $x, y, z \in X$

$$(pBCH)(x * y) = (y \diamond x) = 0 \quad implies \quad x = y$$

is a pseudo-BCH-algebra.

Theorem 2.2 *Every pseudo-Q-algebra X satisfying*

$$x * (x \diamond y) = x * y$$

or

$$x \diamond (x * y) = x \diamond y \ \ for \ \ all \ \ x, y \in X$$

is a trivial algebra.

Proof Putting $x = y$ in the equation $x * (x \diamond y) = x * y$ or $x \diamond (x * y) = x \diamond y$, we obtain $x * 0 = 0$ or $x \diamond 0 = 0$. By (PQ2), we have $x = 0$. Hence X is a trivial algebra. □

Definition 2.5 *Let* $(X;*,\diamond,0)$ *be a pseudo-Q-algebras. Define the relation* $'' \leq ''$ *on* X *by*

$x \leq y$ *if and only if* $x * y = 0$ *(or equivalently,* $x \diamond y = 0$*)*

for all $x, y \in X$.

Proposition 2.2 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) In a pseudo-Q-algebra* $(X;*,\diamond,0)$ *the following properties hold for all* $x, y \in X$*:*

(1) $x \leq 0 \Rightarrow x = 0$;

(2) $x * (x \diamond y) \leq y, x \diamond (x * y) \leq y$;

(3) $0 * x = 0 \diamond x$;

(4) $x \leq y \Rightarrow 0 * x = 0 \diamond y$;

(5) $0 \diamond (0 * (0 \diamond x)) = 0 \diamond x, 0 * (0 \diamond (0 * x)) = 0 * x$;

(6) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$;

(7) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$.

Proof (1) Let $x \leq 0$. Then we get $x * 0 = 0$. By (PQ2) we have $x * 0 = x \diamond 0 = x$. So $x * 0 = 0 = x \diamond 0 = x$. Hence we get $x = 0$.

(2) By using (PQ3) and (PQ1), we have

$$(x * (x \diamond y)) \diamond y = (x \diamond y) * (x \diamond y) = 0$$

. Hence $x * (x \diamond y) \leq y$. Similarly, $x \diamond (x * y) \leq y$.

(3) By using (PQ1) and (PQ3), we get

$$0 * x = (x \diamond x) * x = (x * x) \diamond x = 0 \diamond x$$

.

(4) Let $x \leq y$. Then $x \diamond y = 0$ and therefore,

$$0 * x = (x \diamond y) * x = (x * x) \diamond y = 0 \diamond y$$

.

(5) From (2) it follows that $0 * (0 \diamond x) \leq x$ and $0 \diamond (0 * x) \leq x$. Hence, using (3) and (4) we obtain

$$0 \diamond (0 * (0 \diamond x)) = 0 \diamond x,$$

$$0 * (0 \diamond (0 * x)) = 0 * x$$

.

(6) By using (PQ1) and (PQ3), we have

$$\begin{aligned}(0 \diamond x) \diamond (0 * y) &= (((x * y) * (x * y)) \diamond x) \diamond (0 * y) \\ &= (((x \diamond x) * y) * (x * y)) \diamond (0 * y) \\ &= ((0 * y) * (x * y)) \diamond (0 * y) \\ &= ((0 * y) \diamond (0 * y)) * (x * y) \\ &= 0 * (x * y)\end{aligned}$$

.

(7) The proof is similar to the proof of (6). □

Remark 2.2 *Every pseudo-Q-algebra satisfies (pBCI-2) and (pBCI-3).*

Theorem 2.3 *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra. The following statements are equivalent:*

(i) $x * (y * z) = (x * y) * z$, for all $x, y, z \in X$;

(ii) $0 * x = x = 0 \diamond x$, for every $x \in X$;

(iii) $x * y = x \diamond y = y * x$, for all $x, y \in X$;

(iv) $x \diamond (y \diamond z) = (x \diamond y) \diamond z$, *for all* $x, y, z \in X$.

Proof (i)$\Rightarrow$ (ii) Let $x \in X$. We have $x = x * 0 = x * (x * x) = (x * x) * x = 0 * x$. By (3), we have $0 \diamond x = x$.

(iv) $\Rightarrow$ (ii) The proof is similar to the above proof.

(ii) $\Rightarrow$ (iii) Let (ii) hold and $x, y \in X$. By using proposition 2.2(6) and (PQ3) we obtain

$$\begin{aligned} x * y &= 0 * (x * y) = (0 \diamond x) \diamond (0 * y) \\ &= x \diamond y. \\ &= (0 * x) \diamond y = (0 \diamond y) * x = y * x \end{aligned}$$

.

(iii) $\Rightarrow$ (i) Let $x, y, z \in X$. Using (iii) and (PQ3) we get

$x * (y * z) = (y \diamond z) * x = (y * x) \diamond z = (x * y) * z$.

(iii) $\Rightarrow$ (iv) has a proof similar to the proof of (iii) $\Rightarrow$ (i).

Hence all the conditions are equivalent.

□

Theorem 2.4 *Every pseudo-Q-algebra* $(X; *, \diamond, 0)$ *satisfying the associative law is a group under each operation* " $*$ " *and* " $\diamond$ ".

Proof Putting $x = y = z$ in the associative law $(x * y) * z = x * (y * z)$ and using (I) and (II), we obtain $0 * x = x * 0 = x$. This means that 0 is the zero element of X. By (I), every element x of X has as its inverse the element x itself. Therefore $(X; *)$ and $(X; \diamond)$ are a group.

□

Proposition 2.3 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) If* $(X; *, \diamond, 0)$ *is a pseudo-Q-algebra and* $a * b = a * c$, $a \diamond b = a \diamond c$, *for all* $a, b, c \in X$, *then* $0 * b = 0 * c$, $0 \diamond b = 0 \diamond c$.

Proof By (PQ3) and (PQ1), we have $(a * b) \diamond a = (a \diamond a) * b = 0 * b$ and $(a * c) \diamond a = (a \diamond a) * c = 0 * c$. Since $a * b = a * c$, we get
$0 * b = 0 * c$.

$$(a \diamond b) * a = (a * a) \diamond b = 0 \diamond b \quad \text{and} \quad (a \diamond c) * a = (a * a) \diamond c = 0 \diamond c.$$

Since $a \diamond b = a \diamond c$, we get $0 \diamond b = 0 \diamond c$. □

Definition 2.6 *An element a of a pseudo-Q-algebra X is said to be minimal if for every* $x \in X$ *the following implication*

$$x \leq a \Rightarrow x = a$$

holds.

Proposition 2.4 *Let X be a pseudo-Q-algebra and let* $a \in X$. *Then the following conditions are equivalent for every* $x \in X$:

(i) a is minimal;

(ii) $x \diamond (x * a) = a$;

(iii) $0 \diamond (0 * a) = a$;

(iv) $a * x = (0 * x) \diamond (0 * a)$;

(v) $a * x = 0 \diamond (x * a)$.

Proof

(i)⇒ (ii) By proposition 2.2(1), $x \diamond (x * a) \leq a$ for all $x \in X$. Since a is minimal, we get (ii).

(ii) $\Rightarrow$ (iii) Obvious.

(iii)$\Rightarrow$ (iv) We have $a * x = (0 \diamond (0 \diamond a)) * x = (0 * x) \diamond (0 * a)$.

(iv)$\Rightarrow$ (v) Using proposition 2.2(3) and 2.2(6) we see that $0 \diamond (x * a) = 0 * (x * a) = (0 * x) * (0 * a) = (0 * x) \diamond (0 * a) = a * x$.

(v) $\Rightarrow$ (i) Let $x \leq a$. Then $x * a = 0$ and hence $a * x = 0 \diamond (x * a) = 0$. Thus $a \leq x$. Consequently, $x = a$.

□

Replacing $'' * ''$ by $'' \diamond ''$ and $'' \diamond ''$ by $*$ in Proposition 2.4 we obtain

Proposition 2.5 *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra and let $a \in X$. Then for every $x \in X$ the following conditions are equivalent:*

(i) a is minimal;

(ii) $x * (x \diamond a) = a$;

(iii) $0 * (0 \diamond a) = a$;

(iv) $a \diamond x = (0 \diamond x) * (0 \diamond a)$;

(v) $a \diamond x = 0 * (x \diamond a)$.

Proposition 2.6 *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebras and let $a \in X$. Then a is minimal if and only if there is an element $x \in X$ such that $a = 0 * x$.*

Proof Let a be a minimal element of $(X; *, \diamond, 0)$. By Proposition 2.4, we have $a = 0 * (0 \diamond a)$. If we set $x = 0 \diamond a$, then $a = 0 * x$.

Conversely, suppose that $a = 0 * x$ for some $x \in X$. Using Proposition 2.2 (5) we get $0 * (0 \diamond a) = 0 * (0 \diamond (0 * x)) = 0 * x = a$. From Proposition 2.5 it follows that a is minimal. □

Proposition 2.7 *For $x \in X$, set*

$$\overline{x} = 0 * (0 \diamond x).$$

By Proposition 2.5, $\overline{x} = 0 * (0 * x) = 0 \diamond (0 \diamond x) = 0 * (0 \diamond x)$.

Proposition 2.8 *Let X be a pseudo-Q-algebra. For any $x, y \in X$ we have:*

(a) $\overline{x * y} = \overline{x} * \overline{y}$;

(b) $\overline{x \diamond y} = \overline{x} \diamond \overline{y}$;

(c) $\overline{\overline{x}} = \overline{x}$.

Proof

(a) Using proposition 2.2(6) and 2.2(7) we get

$$\begin{aligned}\overline{x * y} &= 0 \diamond (0 * (x * y)) = 0 \diamond [(0 \diamond x) \diamond (0 * y)] \\ &= [0 * (0 \diamond x)] * [0 \diamond (0 * y)] = \overline{x} * \overline{y}.\end{aligned}$$

(b) Has a proof similar to (a).

(c) By Proposition 2.2(5), $0 * (0 \diamond (0 * x)) = 0 * x$, that is, $0 * \overline{x} = 0 * x$. Hence $\overline{\overline{x}} = 0 \diamond (0 * \overline{x}) = 0 \diamond (0 * x) = \overline{x}$. □

Definition 2.7 *Following the terminology from BCH-algebras (Chaudhry, M. A. (1991)) the set $\{x \in X : x = \overline{x}\}$ will be called the centre of $(X; *, \diamond, 0)$. We shall denote it by CenX. By Proposition 2.4, CenX is the set of all minimal elements of X. We have*

$$CenX = \{\overline{x} : x \in X\}.$$

Proposition 2.9 *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra. Then CenX is a subalgebra of $(X; *, \diamond, 0)$.*

Proposition 2.10 *Let X be a pseudo-Q-algebra and let $x, y \in CenX$. Then for every $z \in X$ we have*

$$x \diamond (z * y) = y * (z \diamond x).$$

Proof Let $z \in X$. Using Propositions 2.5 and 2.4, we obtain

$x \diamond (z * y) = [z * (z \diamond x)] \diamond (z * y) = [z \diamond (z * y)] * (z \diamond x) = y * (z \diamond x)$. + □

Following Dymek, G. (2012) a pseudo-Q-algebra $(X; *, \diamond, 0)$ is said to be $p - semisimple$ if it satisfies for all $x \in X$,

$$0 \leq x \Rightarrow x = 0.$$

From Theorem 3.1 of Dymek, G. (2012)it follows that if $(X; \leq, *, \diamond, 0)$ is a pseudo-BCI algebra, then $(X; \leq, *, \diamond, 0)$ is $p - semisimple$ if and only if $x = \overline{x}$ for every $x \in X$ (that is,$CenX = X$).

Remark 2.3 *From Theorem 3.6 of Dymek, G. (2012) we deduce that* $(CenX; +, 0)$ *is a group, where* $x + y$ *is* $x * (0 \diamond y)$, *for all* $x, y \in CenX$.

Proposition 2.11 *Let* $(X; *, \diamond, 0)$ *be a pseudo-Q-algebra and let I Ideal. For any* $x, y \in X$, *if* $y \in I$ *and* $x \leq y$, *then* $x \in I$.

Proof Assume that $x \in I$ and $x \leq y$.

Then $y * x = 0$ and $y \diamond x = 0$ by (I1) and (I2) we have $y \in I$ □

Proposition 2.12 *Let* $(X; *, \diamond, , 0)$ *be a pseudo-Q-algebra and I be a subset of X satisfying (I1). Then I is an ideal of* $(X; *, \diamond, 0)$ *if and only if for all* $x, y \in X$,

(I2″) if $x \diamond y \in I$ *and* $y \in I$, *then* $x \in I$.

Proof Let I be an ideal of $(X; *, \diamond, 0)$. Suppose that $x \diamond y \in I$ and $y \in I$. By proposition 2.2(2), $x*(x \diamond y) \leq y$ and from Proposition 2.11 it follows that $x*(x \diamond y) \in I$. Therefore, since $x \diamond y \in I$ and I satisfies (I2), we obtain $x \in I$, that is, (I2″) holds. The proof of the implication (I2″) ⇒ (I2) is analogous. □

Definition 2.8 *An ideal I of a pseudo-Q-algebra* $(X; *, \diamond, 0)$ *is said to be closed if* $0 * x \in I$ *for every* $x \in I$.

Theorem 2.5 *An ideal I of a pseudo-Q-algebra* $(X; *, \diamond, 0)$ *is closed if and only if I is a subalgebra of* $(X; *, \diamond, 0)$.

Proof Suppose that I is a closed ideal of $(X;*,\diamond,0)$ and let $x,y\in I$. By (PQ3) and (PQ1), we have

$$\begin{aligned}((x*y)*(0*y))\diamond x &= [(x*y)\diamond x]*(0*y)\\ &= [(x\diamond x)*y]*(0*y)\\ &= (0*y)*(0*y)=0\end{aligned}$$

.

Hence $[(x*y)*(0*y)]\diamond x\in I$. Since $x, 0*y\in I$, we have $x*y\in I$. Similarly, $x\diamond y\in I$. Conversely, if I is a subalgebra of $(X;*,\diamond,0)$, then $x\in I$ and $0\in I$ imply $0*x\in I$. □

Definition 2.9 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) Let $(X;*,\diamond,0)$ be a pseudo-Q-algebra. For any nonempty subset S of X, we define*

$$G(S)=\{x\in S|0*x=x=0\diamond x\},$$

if $S=X$ then $G(x)$ is called the G-part of X.

Corollary 2.1 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) A left cancellation law holds in $G(X)$.*

Proof Let $a,b,c\in G(X)$ with $a*b=a*c$. By Lemma 2.3, $0*b=0*c$. Since $b,c\in G(X)$, we obtain $b=c$. □

Proposition 2.13 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) Let $(X;*,\diamond,0)$ be a pseudo-Q-algebra. Then $x\in G(X)$ if and only if $0*x=0\diamond x\in G(X)$.*

Proof If $x\in G(X)$, then $0*x=x=0\diamond x$ and $0*(0*x)=0*x=0\diamond x=0\diamond(0\diamond x.)$ Hence $0*x$ and $0\diamond x\in G(X)$. Conversely, if $0*x=0\diamond x\in G(x)$, then $0*(0*x)=0*x=0\diamond x=0\diamond(0\diamond x)$. By applying Corollary 2.1, we obtain $0*x=x=0\diamond x$. Therefore $x\in G(X)$.

□

Definition 2.10 *For any pseudo-Q-algebra $(X;*,\diamond,0)$, the set*

$$B(X)=\{x\in X|0*x=0=0\diamond x\}$$

is called the $p-$radical of X.

If $B(X) = \{0\}$, then we say that X is a $p-$semisimple pseudo-Q-algebra. The following property is obvious:

$$G(X) \cap B(X) = \{0\}.$$

Proposition 2.14 *If $(X;*,\diamond,0)$ is a pseudo-Q-algebra and $x, y \in X$, then $y \in B(X)$ if and only if $(x * y) \diamond x = 0 = (x \diamond y) * x$.*

Proof By (PQ3) and (PQ1) we have $(x * y) \diamond x = (x \diamond x) * y = 0 * y = 0$ and $(x \diamond y) * x = (x * x) \diamond y = 0 \diamond y = 0$ if and only if $y \in B(X)$ □

Proposition 2.15 *Let $(X;*,\diamond,0)$ be a pseudo-Q-algebra. Then $B(X)$ is an ideal of X.*

Proof Since $(0 * 0) * 0 = 0$, by Proposition 2.14, we get $0 \in B(X)$. Let $x * y \in B(X)$ and $y \in B(X)$. Then by Proposition 2.14, we have $((x * y) * x) \diamond (x * y) = 0$. By (PQ3), $((x * y) \diamond (x * y)) * x = 0 * x = 0$. Hence $x \in B(X)$. Therefore $B(X)$ is an ideal of X. □

Proposition 2.16 *If S is a subalgebra of a pseudo-Q-algebra $(X;*,\diamond,0)$, then $G(X) \cap S = G(S)$.*

Proof It is obvious that $G(X) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0 * x = x$ and $x \in S \subseteq X$. Then $x \in G(X)$ and so $x \in G(X) \cap S$, which proves the proposition. □

Theorem 2.6 *Let $(X;*,\diamond,0)$ be a pseudo-Q-algebra. If $G(X) = X$,then X is p-semisimple.*

Proof Assume that $G(X) = X$.

By $G(X) \cap B(X) = \{0\}$, we have $\{0\} = G(X) \cap B(X) = X \cap B(X) = B(X)$. Hence X is p-semisimple. □

CHAPTER 3

Pseudo Ideal and Homomorphism

3.1. Pseudo Ideal and Pseudo Strong Ideal

Definition 3.1 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) Let* $(X;*,\diamond,0)$ *be a pseudo-Q-algebra and let* $\emptyset \neq I \subset X$. *I is called a pseudo ideal of X if it satisfies:*

(PI1) $0 \in I$;

(PI2) $x * y, x \diamond y \in I$ *and* $y \in I$ *imply* $x \in I$ *for all* $x, y \in X$.

Example 3.1 *Let* $X = \{0,1,2,3\}$. *Define the binary operations* $" * "$ *and* $" \diamond "$ *on X by the following tables:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	0	0	0
3	3	2	3	0

$\diamond$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	1	0

Let $I = \{0,1\}$. *Then I is pseudo ideal of X, but not ideal of X since* $2 * 1 = 0 \in I$ *but* $2 \notin I$.

Lemma 3.1 *Every ideal in a pseudo-Q-algebras is a pseudo ideal.*

Proof Let I be an ideal. Then $0 \in I$ and $x * y, y \in I$ and we have $x \in I$. Now $x \in I$ and $y \in I$ therefore $x \diamond y \in I$. Then $0 \in I$ and $x * y, x \diamond y, y \in I$ imply $x \in I$ Then I is pseudo ideal. □

Proposition 3.1 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) Let I be a pseudo ideal of a pseudo-Q-algebra X. If $x \in I$ and $y \leq x$, then $y \in I$.*

Proof Assume that $x \in I$ and $y \leq x$. Then $y * x = 0$ and $y \diamond x = 0$. By (PI1) and (PI2), we have $y \in I$.

□

Definition 3.2 *A pseudo-Q-algebra $(X; *, \diamond, 0)$ is called a pseudo-Q*-algebra if it satisfies the identities $(x * y) \diamond x = 0$ and $(x \diamond y) * x = 0$ for all $x, y \in X$.*

Proposition 3.2 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) If I is a pseudo ideal of a pseudo-Q-algebra X,then*

(i) $z * y \leq x, \forall x, y, z \in X, x, y \in I, \Rightarrow z \in I$.

(ii) $c \diamond b \leq a, \forall a, b, c \in X, a, b \in I, \Rightarrow c \in I$.

Proof Suppose that I is a pseudo ideal of X and let $x, y, z \in X$ be such that $x, y \in I$ and $z * y \leq x$. Then $(z * y) \diamond x = 0 \in I$. Since $x \in I$ and I is a pseudo ideal of X, we have $z * y \in I$. Since $y \in I$ and I is a pseudo ideal of X, we obtain $z \in I$. Thus (i) is valid.

Now let $a, b, c \in X$ be such that $a, b \in I$ and $c \diamond b \leq a$. Then $(c \diamond b) * a = 0 \in I$ and so $c \diamond b \in I$. Since $b \in I$ and I is a pseudo ideal of X, we have $c \in J$. Thus (ii) is true.

□

Definition 3.3 *A pseudo-Q-algebra $(X; *, \diamond, 0)$ is called a pseudo-Q*-algebra if it satisfies the identities $(x * y) \diamond x = 0$ and $(x \diamond y) * x = 0$ for all $x, y \in X$.*

Theorem 3.1 *Let I be a non-empty subset of a pseudo-Q*-algebra X. Then I is a pseudo ideal of X if and only if for all $x, y \in I$ and $z \in X, z \diamond x \leq y, z * x \leq y$ imply $z \in I$.*

Proof Suppose that I is a pseudo ideal of X and $z \diamond x \leq y, z * x \leq y$ for all $x, y \in I$ and $z \in X$. It follows from Proposition 3.1 that $z \diamond x \in I$ and $z * x \in I$. Using (PI2), we have $z \in I$.

Conversely, let $x \in I$, since $0 \diamond x \leq x$ and $0 * x \leq x$, we have $0 \in I$. Let $x * y, x \diamond y \in I$ and $y \in I$. Since $x \diamond y \leq x \diamond y$ and $x * y \leq x * y$, we have $x \in I$. Thus I is a pseudo ideal of X.

□

Proposition 3.3 *For any pseudo-Q*-algebra X, the set*

$$K(X) = \{x \in X | 0 \leq x\}$$

is a pseudo subalgebra of X.

Proof Let $x, y \in K(X)$. Then $0 \leq x$ and $0 \leq y$. Hence $0 = 0 * y \leq x * y$ and $0 = 0 \diamond y \leq x \diamond y$ so that $x * y, x \diamond y \in K(X)$. Thus $K(X)$ is a pseudo subalgebra of X.

□

Example 3.2 *In Example 3.1 , K(X) = $\{0, 3\}$is a pseudo subalgebra of X, but not a pseudo ideal of X since $2 \diamond 3 = 0, 2 * 3 = 0$, and $3 \in K(X)$, but $2 \notin K(X)$.*

Proposition 3.4 *(Jun, Y. B., Kim, H. S. and Ahn, S. SH. (2016)) Let A be a pseudo ideal of a pseudo-Q-algebra X. If B is a pseudo ideal of A, then it is a pseudo ideal of X.*

Proof

Since B is a pseudo ideal of A, we have $0 \in B$. Let $y, x * y, x \diamond y \in B$ for some $x \in X$. If $x \in A$, then $x \in B$, since B is a pseudo ideal of A. If $x \in X - A$, then $y, x * y, x \diamond y \in B \subset A$ and so $x \in A$ because A is a pseudo ideal of X. Thus $x \in B$ since B is a pseudo ideal of A. This competes the proof. □

Definition 3.4 *An element w of a pseudo-Q-algebra X is called a pseudo atom if for every $x \in X, x \leq w$ implies $x = w$. Obviously, 0 is a pseudo atom of X.*

Proposition 3.5 *Let X be a pseudo-Q-algebra. If an element w of X satisfies the identity $y * (y \diamond (w * x)) = w * x$ for all $x, y \in X$, then w is a pseudo atom of X.*

Proof Let $y \in X$ be such that $y \leq w$. Then $w = w * 0 = y * (y \diamond (w * 0)) = y * (y \diamond w) = y * 0 = y$. Hence w is a pseudo atom of X. □

Lemma 3.2 *Let X be a pseudo-Q-algebra. A non-zero element $a \in X$ if is a pseudo atom of X, then $\{0, a\}$ is a pseudo ideal of X.*

Proof let X be a pseudo-Q-algebras

and a is a pseudo atom, then $\forall x \in X$; $x \leq a \Rightarrow a = x$ since $a * x = 0$ and $a \diamond x = 0$ since 0 and $a \in \{0, a\}$ then $x \in \{0, a\}$. Then $\{0, a\}$ is pseudo ideal.

□

Proposition 3.6 *If every non-zero element of a pseudo Q^*-algebra X is a pseudo atom, then any pseudo subalgebra of X is a pseudo ideal of X.*

Proof Let S be a pseudo subalgebra of X and let $x, y * x, y \diamond x \in S$. Since $y * x \leq y$ and $y \diamond x \leq y$ for all $x, y \in X$ and y is an atom of Y , we have $y * x = y, y \diamond x = y \in S$. Thus S is a pseudo ideal of X. □

From above Lemmas we obtain the following Theorem.

Theorem 3.2 *A pseudo Q^*-algebra contains only pseudo atoms if and only if its pseudo subalgebra is a pseudo ideal.*

Definition 3.5 *A non-empty subset A of a pseudo-Q-algebra X is called a pseudo strong ideal of X if it satisfies (PI1) and*

(PI3) $(x * y) \diamond z, y \in A$ imply $x * z \in A$;

(PI3′) $(x \diamond y) * z, y \in A$ imply $x \diamond z \in A$, for all $x, y, z \in X$.

Proposition 3.7 *In a pseudo-Q-algebra, any pseudo strong ideal is a pseudo ideal, but the converse is not true.*

Proof Putting $z = 0$ in (PI3) and (PI3′), we have $x * y, x \diamond y, y \in A$ imply $x \in A$. □

In Example 3.1 Let $A = \{0, 1\}$ is pseudo ideal, but not pseudo strong ideal since $(2 \diamond 0) * 1 = 0 \in A$*, but* $2 \diamond 1 = 2 \notin A$

Corollary 3.1 *Every ideal in pseudo-Q-algebras is pseudo ideal that is not pseudo strong ideal.*

Corollary 3.2 *Every pseudo strong ideal in pseudo-Q-algebras is pseudo ideal that is not ideal.*

Proposition 3.8 *In a pseudo Q^*-algebra X, any pseudo ideal is a pseudo subalgebra.*

Proof Let A be a pseudo ideal of X. Then $0 \in A$ and $(x * y) \diamond x = (x \diamond y) * x = 0$ for any $x, y \in X$. Then for any $x \in A$, we have $(x * y) \diamond x, (x \diamond y) * x \in A$, which implies $x * y, x \diamond y \in A$. □

Corollary 3.3 *Any pseudo strong ideal of pseudo Q^*-algebra is a pseudo subalgebra.*

Proposition 3.9 *Let X be a pseudo-Q-algebra. Then a pseudo subalgebra of X is a pseudo strong ideal of X if and only if for all $x, y, z \in X, x \in A, y * z, y \diamond z \in X - A$ imply $(y * x) \diamond z, (y \diamond x) * z \in X - A$.*

Proof Assume that a pseudo subalgebra A of X is a pseudo strong ideal of X and let $x, y, z \in X$ be such that $x \in A$ and $y * z, y \diamond z \in X - A$. If $(y * x) \diamond z \notin X - A$, then $(y * x) \diamond z \in A$. Since A is a pseudo strong ideal of X and $x \in A$, we have $y * z \in J$. This is a contradiction. If $(y \diamond x) * z \notin X - A$, then $(y \diamond x) * z \in A$. Since A is a pseudo strong ideal of X and $x \in A$, we have $y \diamond z \in A$. This is a contradiction.
Conversely, assume that for all $x, y, z \in X, x \in A, y * z, y \diamond z \in A$ imply $(y * x) \diamond z, (y \diamond x) * z \in X - A$. Since A is a pseudo subalgebra of X, we have $0 \in A$.

For every $x \in A$, let $(y * x) \diamond z, (y * x) \diamond z \in A$. If $y * z \notin A$ or $y \diamond z \notin A$, then $(y * x) \diamond z$ or $(y \diamond x) * z \in X - A$ by assumption. This is a contradiction. Hence $y * z \in A$ and $y \diamond z \in A$. Thus A is a pseudo strong ideal of X. □

Putting $z = 0$ in Proposition 3.9, we have the following Corollary.

Corollary 3.4 *Let A be a pseudo subalgebra of a pseudo-Q-algebra X. Then A is a pseudo ideal of X if and only if $y \in X - A, \forall x, y \in X, x \in A$ imply $y * x, y \diamond x \in X - A$.*

3.2. Homomorphism

Definition 3.6 *Let X and Y be a pseudo-Q-algebras. A mapping $f : X \to Y$ is called a* homomorphism *of pseudo-Q-algebras if*

$$f(x * y) = f(x) * f(y) \text{ and } f(x \diamond y) = f(x) \diamond f(y) \text{ for all } x, y \in X.$$

Note that if $f : X \to Y$ is a homomorphism of pseudo-Q-algebras, then $f(0_X) = 0_Y$ where 0_X and 0_Y are zero elements of X and Y, respectively.

Example 3.3 *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebras, then the function $f : X \to X$ such that $f(x) = 0 \diamond x$ for any $x \in X$ is a homomorphism of pseudo-Q-algebras. Certainly,*

$$\begin{aligned} f(x) * f(y) &= (0 \diamond x) * (0 \diamond y) \\ &= (0 \diamond x) * (0 * y) \\ &= (0 * (0 * y)) \diamond x \\ &= (0 \diamond (0 * y)) \diamond x \\ &= (0 \diamond x) \diamond (0 * y) \\ &= 0 * (x * y) \\ &= 0 \diamond (x * y) = f(x * y), \forall x, y \in X. \end{aligned}$$

$$\begin{aligned} f(x) \diamond f(y) &= (0 \diamond x) \diamond (0 \diamond y) \\ &= (0 * x) \diamond (0 \diamond y) \\ &= (0 \diamond (0 \diamond y)) * x \\ &= (0 * (0 \diamond y)) * x \\ &= (0 * x) * (0 \diamond y) \\ &= 0 \diamond (x \diamond y) = f(x \diamond y), \forall x, y \in X. \end{aligned}$$

Example 3.4 *Define $\Phi : X \to CenX$ by $\Phi(x) = \overline{x}$ for all $x \in X$. By Proposition 2.8, Φ is a homomorphism from X onto $CenX$.*

Theorem 3.3 *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras. If B is a pseudo strong ideal of Y, then $f^{-1}(B)$ is a pseudo strong ideal of X.*

Proof Assume that B is a pseudo strong ideal of Y. Obviously, $0_x \in f^{-1}(B)$. Let $x, y, z \in X$ be such that $(x * y) \diamond z, (x \diamond y) * z, y \in f^{-1}(B)$. Then $(f(x) * f(y)) \diamond f(z) = f((x * y) \diamond z), f(y) \in B$. Since B is a pseudo strong ideal of Y, it follows from (PI3) and (PI3') that $f(x*z) = f(x)*f(z), f(x\diamond z) = f(x)\diamond f(z) \in B$ so that $x*z, x\diamond z \in f^{-1}(B)$. Hence $f^{-1}(B)$ is a pseudo strong ideal of X. □

Theorem 3.4 *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras.*
(i) If B is a pseudo ideal of Y, then $f^{-1}(B)$ is a pseudo ideal of X.
(ii) If f is surjective and I is a pseudo ideal of X, then $f(I)$ is a pseudo ideal of Y.

Proof (i) Straightforward.

(ii) Assume that f is surjective and let I be a pseudo ideal of X. Obviously, $0_Y \in f(I)$. For every $y \in f(I)$, let $a, b \in Y$ be such that $a * y \in f(I), b \diamond y \in f(I)$. Then there exist $x_*, x_\diamond \in I$ such that $f(x_*) = a * y$ and $f(x_\diamond) = b \diamond y$. Since $y \in f(I)$, there exists $x_y \in I$ such that $f(x_y) = y$. Also f is surjective, there exist $x_a, x_b \in X$ such that $f(x_a) = a$ and $f(x_b) = b$. Hence $f(x_a * x_y) = f(x_a) * f(x_y) = a * y \in f(I)$ and $f(x_b \diamond x_y) = f(x_b) \diamond f(x_y) = b \diamond y \in f(I)$, which imply that $x_a * x_y \in I$ and $x_b * x_y \in I$. Since I is a pseudo ideal of X, we get $x_a, x_b \in I$ and thus $a = f(x_a), b = f(x_b) \in f(I)$. Therefore $f(I)$ is a pseudo ideal of X □

Corollary 3.5 *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras. Then Ker$f$ $= \{x \in X | f(x) = 0\}$ is a pseudo strong ideal(ideal) of X.*

Proposition 3.10 *Let $f : (X; *_1, \diamond_1, 0) \to (Y; *_2, \diamond_2, 0)$ be a homomorphism of pseudo-Q-algebras. Then $x *_1 y, y \diamond_1 x \in Kerf$ if $f(x) = f(y), \forall x \in X$.*

Proof Assume that $f(x) = f(y), \forall x \in X$. Then $f(x) *_2 f(y) = f(x *_1 y) = 0$ and $f(x) \diamond_2 f(y) = f(x \diamond_1 y) = 0$. Hence $x *_1 y, y \diamond_1 x \in$ Ker$f$. □

Proposition 3.11 *Let $f : (X; *_1, \diamond_1, 0) \to (Y; *_2, \diamond_2, 0)$ be a homomorphism of pseudo-Q-algebras. If $y \in Kerf$, then $x *_1 (x *_1 y), (x *_1 y) *_1 x, x \diamond_1 (x *_1 y), (x *_1 y) \diamond_1 x, x *_1 (x \diamond_1 y), (x *_1 y) *_1 x, x \diamond_1 (x \diamond_1 y), (x \diamond_1 y) \diamond_1 x \in Kerf$.*

Lemma 3.3 *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras. Then f is a monomorphism if and only if $Kerf = \{0\}$.*

Theorem 3.5 *Let X, Y and Z be pseudo-Q-algebras, and $h : X \to Y$ be an onto homomorphism of pseudo-Q-algebras and $g : X \to Z$ be a homomorphism of pseudo-Q-algebras. If $Kerh \subset Kerg$, then there exists a unique homomorphism of pseudo-Q-algebras $f : Y \to Z$ satisfying $f \circ h = g$.*

Theorem 3.6 *Let X, Y and Z be pseudo-Q-algebras, and $g : X \to Z$ be a homomorphism of pseudo-Q-algebras and $h : Y \to Z$ be an one-to-one homomorphism of pseudo-Q-algebras. If $Img \subset Imh$, then there exists a unique homomorphism of pseudo-Q-algebras $f : X \to Y$ satisfying $h \circ f = g$.*

CONCLUSION

In this thesis, we have studied pseudo-Q-algebras and we derived some properties about these algebras. We also characterized the ideals, pseudo ideals and pseudo strong ideals of pseudo-Q-algebras. Additionally, we studied homomorphisms of pseudo-Q-algebras and gave examples for them; and also defined kernel and centre of the homomorphisms. We will study on characterizing these algebras.

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