HEAVY MESONS IN HEAVY QUARK EFFECTIVE THEORY

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ABSTRACT

HEAVY MESONS IN HEAVY QUARK EFFECTIVE THEORY

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In this thesis heavy quark effective theory is studied. Typical interaction momentum between constituents of a meson is about $300MeV$ where QCD is in its non-perturbative region. Although there is no solution to QCD in this region information can be obtained by using effective field theories. The heaviest quarks have mass much greater than Λ_{QCD}. This property is used to construct an effective field theory in the infinite mass limit by modifying the QCD Lagrangian. The residual motion and the spin interactions of the heavy quark are shown to be suppressed by the heavy quark mass. This leads to the formation of degenerate spin doublets of B and D mesons. Also it is predicted that the change in mass when the light quark flavor is changed is the same between B and D mesons. Then HQET is used to study semileptonic $B \to D$ decays upto the first order in perturbative series. In the end V_{cb} is calculated.

Keywords: Heavy Quark Effective Theory, Non-perturbative QCD, Meson Mass, Semileptonic Decay, CKM Matrix Element

ÖZ

AĞIR QUARK ETKİN KURAMINDA AĞIR MEZONLAR

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Bu tezde ağır kuark etkin teori çalışıldı. Bir mezonun içeriğindekilerin etkileşim momentumu yaklaşık $300MeV$ düzeyindedir ki bu KRD'nin tedirgemesiz bölgesindedir. Bu bölgede KRD'nin çözümü olmasa bile etkin teori kullanılarak bilgi elde edilebilir. En ağır kuarkların kütleleri Λ_{QCD}' den çok büyüktür. Bu özellik kullanılarak sonsuz kütle limitinde KRD Lagrangian' ı modifiye edilerek etkin teori kuruldu. Ağır kuarkın hareketinin ve spin etkileşiminin kütle ile baskılandığı gösterildi. Sonuç olarak B ve D mezonlarının spin çiftleri arasında yaklaşık olarak kütle farkının olmayacağı öngörüldü. Ayrıca B ve D mezonlarının hafif serbestlik derecelerinin farklı çeşniler için kütle farkının yaklaşık olarak aynı olacağı öngörüldü. Daha sonra teori yarı leptonik $B \to D$ bozunmalarını çalışmak için kullanıldı ve tedirgenme serisi birinci seviyesine kadar hesap yapıldı. Tez V_{cb} bulunarak bitirildi.

Anahtar Kelimeler: Ağır Kuark Etkin Teori, Tedirgemesiz KRD, Mezon Kütlesi, Yarıleptonik Bozunma, CKM Matris Elemanı

To my family

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LIST OF ABBREVIATIONS

QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
HQET	Heavy Quark Effective Theory
RS	Renormalization Scheme
$\overline{\text{MS}}$	Modified Minimal Subtraction
U(n)	Unitary Group
SU(n)	Special Unitary Group

NOTATION: Latin indices ijk run from 1 to 3 and Greek indices $\mu\nu\ \lambda$ run from 1 to 4. Einstein's summation convention , i.e. $u_\mu v^\mu = \sum_{\mu=0}^{4} u_\mu v^\mu$ and units in which $\hbar = c = 1$ are used throughout the text. The spacetime is taken to be flat, which is given by the Minkowski metric; $\eta = diag(1, -1, -1, -1)$. For the gamma matrices Dirac represenation is used. The elementary charge e is taken for the charge of the proton.

CHAPTER 1

INTRODUCTION

In the sixties, the studies on hadrons suggested that they were made up of sub-hadronic constituents. This quickly led to the search for the internal dynamics. In order to probe the internal structure of a hadron with high resolution, beams of electron with high energies were used in scattering experiments. The majority of the scatterings happened in the deep inelastic region of the phase space [2]. Bjorken argued that in these deep inelastic scatterings the structure function scales with the ratio square of transferred momentum over energy transfer q^2/ν. Later this argument is confirmed with the experiment. This scaling is explained by assuming electrons scatter off almost free, point like constituents which were called partons. The scaling also implies that the interactions between partons becomes weak at short distances. Later partons are identified with quarks [3].

The behavior of interactions is explained by non-Abelian gauge field theory. Its basic property is the generators of the symmetry group do not commute. It exhibits the desired property which is called asymptotic freedom. The effective coupling constant decreases at small distances and increases at long distances. Which is different from Quantum Electrodynamics (QED) where effective coupling increases at small distances and tends to zero at large distances [4].

Meanwhile there was also a requirement for a symmetry that is associated with non-commutative algebra. $SU(3)$ color symmetry, which gives quarks a color charge, is proposed and the theory emerged by its combination with non-Abelian field theory is called Quantum Chromodynamics (QCD). Just as in QED, where the interactions between charged particles are mediated by a gauge boson called photon, in QCD interactions between color charged quarks are also mediated by a gauge boson. This boson is called gluon due to its property of binding quarks together like glue. While photons have no electric charge, gluons possess color charge and hence interacts with each other.

1.1 Properties of Quarks

Quarks are fermionic particles that interacts via electromagnetic, gravitational, weak and, strong interactions. There are currently six observed quarks that are classified under three generations [5]. Theses quarks are; up u, down d, strange s, charm c, bottom (or beauty) b and top (or truth) t quarks with generations

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix} \begin{pmatrix} b \\ t \end{pmatrix}$$

Flavors of quarks are denoted by flavor quantum number. For example the charm quark has the quantum number charmness $C = 1$ and the remaining flavor numbers, strangeness S, bottomness B and topness T are zero. Similarly for t quark the quantum numbers are $T = 1$, $S = C = B = 0$. The s quark has $S = -1$ and for b quark $B = -1$. Their anti-quark counterparts have negative of their flavor numbers. This quantum number is conserved under strong interactions [6]. The u and d quarks are associated with another quantum number called isospin which will be covered in chapter 1.4.

The quarks also possess electric charge of fractional values of elementary charge e. Up, charm and top quarks have charge of $2/3e$, while down, strange and bottom have $-1/3e$. Then hadrons obtain the total charge of their constituent quarks. Proton made of uud has charge $+e$ and neutron udd has total of zero.

The idea behind generations is clear at this point. After the first generation (ud) new observed quarks show similarities, having the same spin and interaction properties, only difference is their masses. The masses are; up quark, $m_u = 2.3^{+0.7}_{-0.5}MeV$, down quark, $m_d = 4.8^{+0.5}_{-0.3}MeV$, strange quark, $m_s = 95 \pm 5MeV$, charm quark with $m_c = 1.275 \pm 0.025GeV$, bottom quark, $m_b = 4.66 \pm 0.03GeV$ and top quark, $m_t = 173.21 \pm 0.51 \pm 0.71GeV$ [6]. Although flavor changing is not permitted with strong interactions, later generation quarks tends to decay into early generations via weak interaction.

1.2 Color Confinement

Quarks carry one of the three color charges; red r, green g, blue b. In $SU(3)$ color group quarks belong to fundamental representation $\underline{3}$. Anti-quarks are different objects, carry anti-charges; $\overline{r}$, $\overline{g}$ and $\overline{b}$ and belong to complex conjugate representation $\underline{3}^*$ [5]. Gluons on the other hand carry a charge and an anti-charge. A state with a color and an anti-color charge corresponds to one of the irreducible representations in $\underline{3} \otimes \underline{3}^*$.

$$\underline{3} \otimes \underline{3}^* = \underline{1} \oplus \underline{8} \tag{1.1}$$

Giving eight color octets; $r\overline{b}$, $r\overline{g}$, $g\overline{b}$, $g\overline{r}$, $b\overline{g}$, $b\overline{r}$, $\frac{1}{\sqrt{2}}(r\overline{r} - g\overline{g})$, $\frac{1}{\sqrt{6}}(r\overline{r} + g\overline{g} - 2b\overline{b})$ and one singlet $\frac{1}{\sqrt{3}}(r\overline{r} + g\overline{g} + b\overline{b})$ [7].

The colored objects however are not directly observed, but only colorless bound states are observed. This phenomena is hypothesized as color confinement. In other words, color carrying quarks or gluons are never observed on their own. Returning back to the gluons, the singlet state cannot interact with strong interaction, and can leave a hadron to exist on its own and even be undetectable. Hence there are only eight gluon states that are color octets.

Hadrons can only appear as singlet states. Mesons; quark q - anti-quark $\overline{q}$ bound states

$$q\overline{q} : \underline{3} \otimes \underline{3}^* = \underline{1} \oplus \underline{8} \tag{1.2}$$

and baryons with three quarks

$$qqq : \underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10} \tag{1.3}$$

has a possible singlet state required for confinement and observed experimentally. Other non-singlet states with explicit color degrees of freedom are expected not to be observed [3]. Diquarks

$$qq : \underline{3} \otimes \underline{3} = \underline{3}^* \oplus \underline{6} \tag{1.4}$$

and four quark states

$$qqqq : \underline{3} \otimes \underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{3} \oplus \underline{3} \oplus \underline{3} \oplus \underline{6}^* \oplus \underline{15} \oplus \underline{15} \oplus \underline{15} \oplus \underline{15}. \tag{1.5}$$

are examples of this situations.

1.3 Asymptotic Freedom

In order to give a meaningful description of asymptotic freedom, the concept of renormalization must be understood. Starting with any quantum field theory, while calculating an integral over all momentum space such as a Green's function, UV divergences in loop diagrams would be encountered. These divergences can be removed order by order in perturbation theory by rescaling and redefining parameters like mass, coupling constant and field normalization factors according to the energy scale at hand [8]. This process is called renormalization. There is not a unique way to renormalize a theory and each different way to do is called renormalization scheme (RS). Although they change the way of how things are calculated, the process must leave the observable values unchanged [9]. RS inevitably introduces a mass scale μ called renormalization scale. Then the intermediate parameters depend on RS and μ. The simplest and most used scheme is $\overline{\text{MS}}$ and it is perturbative.

In QED the scale μ can be fixed to the masses of charged particles since they have direct physical meaning. The perturbative series is asymptotic, meaning approximation to a quantity is improved by increasing the number of terms included up to a maximum number of $\text{N}_* = 137$ is reached. After this number terms gets too large and series loses it's meaning [8].

In QCD there is no natural choice for the scale μ as quark masses are not directly observable due to confinement. It becomes an arbitrary value and the choice depends on other things, such as convergence and simplicity of calculations. The coupling constant g_s or in a more convenient description $\alpha_s = \frac{g_s^2}{4\pi}$ is two order of magnitude larger than QED coupling and the value of N_* is of order 1 [8]. Therefore an appropriate choice of RS becomes important.

Asymptotic freedom is a property of QCD coupling scaled at high energies [5]. Introducing $\alpha_s(\mu^2)$ from its relation to unrenormalized coupling constant α_s^{ur}

$$\alpha_s(\mu) Z_\alpha \left(\alpha_s(\mu^2), \frac{\mu^2}{\Lambda_{UV}^2} \right) = \alpha_s^{ur} \tag{1.6}$$

where Z_α is the rescaling factor and Λ_{UV}^2 is the ultraviolet cut-off parameter[3]. This corresponds to the differential equation,

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s). \tag{1.7}$$

The $\beta(\alpha_s)$ function gives how fast the coupling changes while moving through the energy scales. It remains finite as $\Lambda_{UV} \to \infty$ and how to calculate it through perturbation theory is known.

$$\beta(\alpha_s) = -\alpha_s^2(\beta_0 + \beta_1\alpha_s + \beta_2\alpha_s^2 + ...) \tag{1.8}$$

At one loop, $\alpha_s(\mu^2)$ can be written in terms of an overall scale Λ_{QCD}

$$\alpha_s(\mu^2) = \frac{1}{\beta_0 ln\left(\frac{\mu^2}{\Lambda_{QCD}}\right)} \tag{1.9}$$

Λ_{QCD} is a dimensionful parameter of QCD and is measured to have the approximate value of $\approx 300MeV$ [6]. It is the characteristic scale of QCD that also gives the confinement at $R_{had} \approx 1/\Lambda_{QCD} \approx 1fm$. As the energy scale increases the relation $\alpha_s \to 0$ as $\mu \to \infty$ is obtained. Asymptotic freedom allows to use perturbation theory. However for μ values close to Λ_{QCD} coupling becomes large and perturbative QCD breaks down.

1.4 Symmetries of QCD

The QCD Lagrangian is

$$\mathcal{L} = \sum_i \overline{\psi}_i(x)(i\not{D} - m_i)\psi_i(x) - \frac{1}{4}G^{\mu\nu}(x)G_{\mu\nu}(x) \tag{1.10}$$

where $\psi_i(x)$ are fermion fields that corresponds to quarks, D_μ is the covariant derivative $D_\mu = \partial_\mu - ig_sA_\mu$ and $G^{\mu\nu}(x)$ is field strength tensor $G^{\mu\nu} = \frac{1}{g_s}[D^\mu, D^\nu]$[4]. Noether theorem states that if equations of motion is invariant under a continuous transformation of Lagrangian, then there is a conserved current corresponds to this invariance [10]. This is the notion of symmetry. This current is given by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi(x))}\delta\psi - K^\mu \tag{1.11}$$

where $\delta\psi(x)$ corresponds to difference between transformed and the original fermion field and $\partial_\mu K^\mu = \mathcal{L}(\psi'(x), \partial_\mu\psi'(x)) - \mathcal{L}(\psi(x), \partial_\mu\psi(x))$. Then the derivative of J^μ is

$$\partial_\mu J^\mu(x) = 0. \tag{1.12}$$

If this current is localized in space, then the current corresponds to a conserved charge Q.

$$Q = \int d^3x J^0, \quad \frac{dQ}{dx^0} = 0 \tag{1.13}$$

The QCD Lagrangian also shows exact and approximate symmetries.

1. CP Symmetry

   Parity is taking the symmetry of a system with respect to origin. The parity operator P acts as [11],

   $$\begin{aligned} &P : (t, x, y, z) = (t, -x, -y, -z), \\ &P^2 : (t, x, y, z) = P : (t, -x, -y, -z) = (t, x, y, z) \end{aligned} \tag{1.14}$$

   Clearly the operator satisfies $P^2 = 1$, and acting it on a vector yields

   $$P(\vec{V}) = -\vec{V} \tag{1.15}$$

   Note that a cross product of two vectors transforms as

   $$P(\vec{V} \times \vec{W}) = P(\vec{V}) \times P(\vec{W}) = (-\vec{V}) \times (-\vec{W}) = +\vec{V} \times \vec{W}. \tag{1.16}$$

   An object that acts like a vector under rotations but under parity transformation does not obtain a minus sign is obtained. This object is called an axial vector. Moreover consider a vector $\vec{V}$ and an axial vector $\vec{A}$

   $$P(\vec{V} \cdot \vec{A}) = (-\vec{V}) \cdot (\vec{A}) = -\vec{V} \cdot \vec{A} \tag{1.17}$$

   Their dot product yields a scalar object that obtains a minus sign under parity transformation. This object is called a pseudoscaler.
   For a wave function if the following argument is satisfied

   $$\Psi(t, -x, -y, -z) = n\Psi(t, x, y, z) \tag{1.18}$$

   then n is called the parity. The $n = +1$ case is called positive parity and $n = -1$ case is called negative parity. As a convention quarks are assigned positive parity and anti-quarks are assigned negative parity. Then for a general state the parity transformation gives

   $$P\left|\psi\right\rangle = (-1)^l P_{int}\left|\psi\right\rangle \tag{1.19}$$

where l is the orbital angular momentum and P_{int} is the intrinsic parity. According to the experimental results, parity is conserved in strong interactions. Another symmetry transformation is charge conjugation [11]. It transforms a particle to its anti-particle. The operator is denoted as C and acts on a state as

$$C\left|p\right\rangle = \left|\overline{p}\right\rangle. \tag{1.20}$$

Obviously it satisfies $C^2 = 1$. Most particles are not eigenstates of C operator because this requires the particles must be its own anti-particle. The eigenvalue equation gives

$$C\left|p\right\rangle = \left|\overline{p}\right\rangle = \pm\left|p\right\rangle \tag{1.21}$$

For bound fermion- anti-fermion pairs the the eigenvalue is obtained as

$$C\left|\psi\right\rangle = (-1)^{l+s}\left|\psi\right\rangle \tag{1.22}$$

The charge conjugation is also conserved in strong interactions.

2. Global $U(1)$ Symmetry
   The Lagrangian is invariant under phase transformation [12]

$$\psi(x) \rightarrow e^{i\theta}\psi(x) \tag{1.23}$$

which yields the current and a charge

$$J_B^\mu = \overline{\psi}(x)\gamma^\mu\psi(x) \quad B = \int d^3x\overline{\psi}(x)\gamma^0\psi(x) \tag{1.24}$$

B is the conserved quantity called baryon number. Quarks are associated with baryon number $1/3$ and anti-quarks are associated with $-1/3$. In every process number of quarks minus number of anti-quarks is conserved.

3. Isospin Symmetry
   The small mass difference between protons and neutrons led to the idea that these particles are different states of the same particle named nucleon. The states are given as [13]

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.25}$$

The transformation between states are given by $SU(2)$ transformations

$$\hat{T}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\hat{\tau}_3 \tag{1.26}$$

so that the nucleons are assigned eigenvalues of $T_3 = \pm 1/2$.

$$\hat{T}_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{T}_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.27}$$

and similarly other transformation matrices are defined as

$$\hat{T}_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{T}_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.28}$$

This is analogous to case of spin $1/2$ particles, there is nothing about spin when defining isospin. Proton and neutron forms doublet with total isospin $T = 1/2$ and the third component $T_3 = \pm 1/2$. If the charge difference is ignored, then there is nothing to distinguish these particles. Then the physics is expected to be invariant under rotations in the isospin space.

This idea is extended for the quarks. If the mass difference between u quark and d quark is ignored, the strong interactions becomes invariant under an isospin transformation. In other words an approximate flavor symmetry is obtained. The field transformation is generated as

$$\psi(x) \rightarrow e^{i\frac{\tau_i}{2}\theta_i}\psi(x). \tag{1.29}$$

This gives the current and the charge

$$J_i^\mu = \overline{\psi}(x)\gamma^\mu \frac{\tau_i}{2}\psi(x), \quad Q_i = \textstyle\int d^3x \psi^\dagger(x)\frac{\tau_i}{2}\psi(x). \tag{1.30}$$

4. $SU(3) Flavor Symmetry$

The idea for isospin can be expanded to include s quark. If $m_u = m_d = m_s$ is assumed, then a $SU(3)$ flavor symmetry would be obtained. The field transformations are given by $SU(3)$ group;

$$\psi(x) \rightarrow e^{i\frac{\lambda_i}{2}\theta_i}\psi(x). \tag{1.31}$$

Then the obtained current and charge are

$$J_i^\mu = \overline{\psi}(x)\gamma^\mu \frac{\lambda_i}{2}\psi(x), \quad Q_i = \int d^3x \psi^\dagger(x)\tfrac{\lambda_i}{2}\psi(x). \tag{1.32}$$

This symmetry is broken by the mass difference between s quark and lighter quarks.

A new symmetry of QCD appears when moving to low energies. It is called heavy quark symmetry. It is a symmetry of the Lagrangian in the infinite quark mass limit. The purpose of this thesis is to explain and show the applications of this symmetry. We first need to start with the question, what is an effective theory?

1.5 Effective Field Theories

When describing a physical system it is useful to concentrates on the relevant degrees of freedom. For example the motion at low velocities does not require special relativistic description. The calculations would be more complicated than Newtonian mechanics' and the solutions would approximately be the same. Although the Newtonian mechanics fails at higher velocities, for low velocities it is considered to be the correct effective theory of special relativity.

Construction of effective theories for field theories is also possible and yields great simplifications. It starts with defining energy or distance scales. A very heavy particle cannot be created in an energy scale smaller than it's mass. This is given by decoupling theorem [14]; heavy degrees of freedom decouples at low energies, meaning their contribution to correlation function is suppressed by their mass. It is then useful to construct an effective field theory by removing irrelevant degrees of freedom from the Lagrangian [15].

An example is W-boson propagator appearing in tree level of $b \to cl\overline{\nu}$. The maximum transferred momentum in this process is $q^2 = (m_b - m_c)^2$ which is small compared to m_W. Therefore the propagator can be approximated as

$$\frac{1}{p^2 - m_W^2} = -\frac{1}{m_W^2} - \frac{p^2}{m_W^4} - ... \tag{1.33}$$

Instead of working on each amplitude one by one, the weak Lagrangian is expanded in terms of local operators in powers of external momenta divided by the scale of

heavy physics. The result is called an effective Lagrangian.
A more appropriate description for constructing an effective field theory is as follows [16]. In the begining a very large scale that is above all masses, with renormalization scale μ is taken. In this region, the set of fields X describes the heaviest particles and ϕ describes the lighter particles. The full Lagrangian is in the form

$$\mathcal{L}_H(X,\phi)+\mathcal{L}(\phi) \tag{1.34}$$

Then the theory start evolving to lower energies. As long as the renormalization scale does not encounter to a particle mass, renormalization group methods describes the evolution. When μ becomes lower than a heavy particles mass, full theory is modified to an effective theory without the heavy particle.

1.6 Physical Picture of a Heavy Meson

Consider a heavy meson; a bound state of a heavy quark Q and light degrees of freedom, containing light quark q and gluon cloud [17]. Typical momentum transfer inside a hadron is approximately given by the constituent quark model which is around a third of a proton's mass $m_p/3 \approx \Lambda_{QCD} \approx 330MeV$. This is the amount how much the quarks are off-shell inside a hadron [18]. Given the masses of lightest quarks, the quarks composing proton and light quarks inside the heavy hadron are off-shell by a very large amount. However a heavy quark with the mass $m_Q \gg \Lambda_{QCD}$, being off-shell by Λ_{QCD}, is in fact nearly on-shell. This also means the hadron as a whole, is nearly on-shell. With this definition, light quarks are u, d and s while heavy quark are t, b and c [19]. However t quarks are short lived with lifetime of $\approx 0.5\times10^{-24}s$ which is less than the time scale for strong interactions $\Lambda^{-1}\approx10^{-23}s$ and hence decays via weak interaction before forming a hadron [20]. Also considering the ratio $\Lambda_{QCD}/m_c \approx 1/6$, c quark is considered a heavy quark only with some reservation. Therefore the notion of heavy quark generally refers to b quark [17].
Interaction with the light degrees of freedom changes heavy quarks momentum by Λ_{QCD} and so changes its four velocity vector by Λ_{QCD}/m_Q. Velocity change is bounded by the limit $\Lambda_{QCD}/m_Q \to 0$, which is approximately the case. Heavy quark velocity becomes constant in infinite heavy quark mass limit. This is the idea underlying HQET. A very heavy object sits in the middle of the meson. Light degrees of

freedom interacts with it as if it is an infinitely heavy object. Of course the quark is not infinitely heavy, so it is said that heavy quark symmetry is an approximate symmetry appearing in infinite mass limit. The rest of the physical picture is understood with this behavior. Interactions are not energetic enough to change heavy quarks quantum numbers. Then the light degrees of freedom perceive it only as an external color source.

Then the question is which fields are integrated out while constructing the effective theory? Removing heavy quark fields would lose the meson picture which is wanted to be described. The answer dwells on the fact that interactions cannot create a heavy quark anti-quark pair. The field mode that gives pair production is integrated out.

The author claims no originality in this work. Most of the material follows from the review of Neubert [21] and the book of Manohar and Wise [22].

CHAPTER 2

HEAVY QUARK EFFECTIVE THEORY

The effective theory is going to be constructed in the infinite mass limit for the heavy quarks. Therefore the part of QCD Lagrangian corresponding to heavy degrees of freedom needs to be restated accordingly. The Lagrangian is

$$\mathcal{L}_Q = \overline{Q}(i\not{D} - m_Q)Q \tag{2.1}$$

Consider a heavy hadron that contains a heavy quark and a light quark and moves with the four velocity v^μ. The momentum of the hadron is given by [16]

$$P_Q^\mu = M_Q v^\mu \tag{2.2}$$

Where M_Q is the mass of the bound state. Heavy quark approximately moves with the hadron's velocity v^μ. Then the heavy quark is nearly on-mass shell with momentum [23]

$$p_Q^\mu = m_Q v^\mu + k^\mu = P_Q^\mu - p_q^\mu \tag{2.3}$$

where p_q^μ is the momentum carried by light cloud, k^μ is the residual momentum and is defined through above equation.

$$k^\mu = (M_Q - m_Q)v^\mu - p_q^\mu \tag{2.4}$$

It is of order Λ_{QCD}. As expected, in the limit $m_Q \to \infty$ the velocity of the heavy quark is equal to the velocity of the hadron. This can be seen from eq.(2.3), as the k^μ term goes to zero.

$$v_Q^\mu = \frac{p_Q^\mu}{m_Q} = v^\mu + \frac{k^\mu}{m_Q} \tag{2.5}$$

Note that infinitely heavy quark cannot have well defined position and momentum at the same time but for position and velocity this is possible [24].

$$[x, v] = \lim_{m_Q \to \infty} \frac{1}{m_Q}[x, p] = \lim_{m_Q \to \infty} \frac{i\hbar}{m_Q} = 0 \tag{2.6}$$

Now suppose the bound state scatters into a new state with the same heavy quark. Momentum of the heavy quark in the new state becomes

$$p_Q'^{\mu} = m_Q v'^{\mu} + k'^{\mu} \tag{2.7}$$

so the momentum transfer is

$$p_Q^{\mu} - p_Q'^{\mu} = m_Q(v^{\mu} - v'^{\mu}) + (k^{\mu} - k'^{\mu}) \tag{2.8}$$

In order to keep the transferred momentum finite in the infinite mass limit, the velocity of the heavy quark should be considered to remain unchanged under soft interaction processes. This leaves the momentum transfer at an order of Λ_{QCD}.

The aim is to write an effective theory for a static heavy quark. Given the Lagrangian in eq.(2.1), the equation of motion reads

$$(i\not{D} - m_Q)Q = 0 \tag{2.9}$$

Plane wave solutions are natural choice for our fields. Hence, heavy quark field becomes [25, 26]

$$Q(x) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_Q}{w_p}} (\hat{b}_s(\vec{p}) u_s(\vec{p}) e^{-ip\cdot x} + \hat{d}_s^{\dagger}(\vec{p}) v_s(\vec{p}) e^{+ip\cdot x}) \tag{2.10}$$

where the index s denotes spins with $s = +, -$. $\hat{b}$ is the annihilation operator for particles and $\hat{d}$ is called the annihilation operator for antiparticles. $u(\vec{p}, s)$ spinors satisfy the equation

$$H_D u_s(\vec{p}) = (\not{p} - m_Q) u_s(\vec{p}) = 0 \tag{2.11}$$

with energy $E = \sqrt{p^2 + m_Q^2}$. Similarly $v_s(\vec{p})$ spinors are eigenfunctions of Dirac Hamiltonian H_D with negative energy $E = -\sqrt{p^2 + m_Q^2}$. Their explicit form in Dirac representation is

$$u_s(\vec{p}) = \sqrt{\frac{|E| + m_Q}{2m_Q}} \begin{pmatrix} \varphi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{|E|+m_Q}\varphi_s \end{pmatrix} \tag{2.12}$$

$$v_s(\vec{p}) = \sqrt{\frac{|E| + m_Q}{2m_Q}} \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{|E|+m_Q}\phi_s \\ \phi_s \end{pmatrix} \tag{2.13}$$

corresponding the positive energy; Q^+, and negative energy; Q^- parts of eq.(2.10). The fields $Q^\pm$ are decomposed by introducing projection operators $P_\pm = (1 \pm \not{v})/2$, satisfying $P_\pm^2 = P_\pm$ and $\not{v}^2 = 1$ [27].

$$Q^+ = \left(\frac{1+\not{v}}{2} + \frac{1-\not{v}}{2}\right) Q^+ = h_v^+ + H_v^+ \tag{2.14}$$

$$Q^- = \left(\frac{1-\not{v}}{2} + \frac{1+\not{v}}{2}\right) Q^- = h_v^- + H_v^- \tag{2.15}$$

For an infinitely heavy, stationary quark i.e. $v = (1,0,0,0)$ the values of decomposed fields have the proportionality as the following [28],

$$\begin{aligned}\lim_{m_Q\to\infty} h_v^+ &= \lim_{m_Q\to\infty} \frac{1+\not{v}}{2} Q^+ \\ &\propto \lim_{m_Q\to\infty} \sqrt{\frac{|E|+m_Q}{2m_Q}} \begin{pmatrix} \varphi_{1,2} \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_{1,2} \\ 0 \end{pmatrix}\end{aligned} \tag{2.16}$$

$$\begin{aligned}\lim_{m_Q\to\infty} h_v^- &= \lim_{m_Q\to\infty} \frac{1-\not{v}}{2} Q^- \\ &\propto \lim_{m_Q\to\infty} \sqrt{\frac{|E|+m_Q}{2m_Q}} \begin{pmatrix} 0 \\ \phi_{3,4} \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_{3,4} \end{pmatrix}\end{aligned} \tag{2.17}$$

$$\begin{aligned}\lim_{m_Q\to\infty} H_v^+ &= \lim_{m_Q\to\infty} \frac{1-\not{v}}{2} Q^+ \\ &\propto \lim_{m_Q\to\infty} \sqrt{\frac{|E|+m_Q}{2m_Q}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned} \tag{2.18}$$

$$\begin{aligned}\lim_{m_Q\to\infty} H_v^- &= \lim_{m_Q\to\infty} \frac{1+\not{v}}{2} Q^- \\ &\propto \lim_{m_Q\to\infty} \sqrt{\frac{E+m_Q}{2m_Q}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\end{aligned} \tag{2.19}$$

For non-stationary quarks the relation $|p| \ll |E| + m_Q$ gives approximate values to the above ones. Hence h_v^+ and h_v^- are called large components and H_v^+ and H_v^- are called small components. In the rest of the thesis quark fields are going to be focused on therefore superscript plus signs are going to be dropped from these fields for a simpler notation.

As a consequence of using the projection operators and $v^2 = 1$, a heavy quark, satisfies

$$\not{v} h_v = h_v, \tag{2.20}$$

$$\not{v} H_v = -H_v \tag{2.21}$$

From eq.(2.10) it is clear that in the infinite mass limit heavy quark fields oscillate with infinite frequency. A field redefinition is required to pull out mass dependence from the phase. The redefinition gives [21],

$$h_v \to e^{im_Q v\cdot x} P_+ Q^+, \quad H_v \to e^{im_Q v\cdot x} P_- Q^+, \tag{2.22}$$

$$h_v^- \to e^{-im_Q v\cdot x} P_+ Q^-, \quad H_v^- \to e^{-im_Q v\cdot x} P_- Q^- \tag{2.23}$$

so that

$$Q^+ = e^{-im_Q v\cdot x}[h_v + H_v], \tag{2.24}$$

$$Q^- = e^{im_Q v\cdot x}[h_v^- + H_v^-] \tag{2.25}$$

Then the plane wave solution for a heavy quark is rescaled to[26]

$$Q(\mathbf{x}) = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_Q}{w_p}} (\hat{b}_s(\vec{v},\vec{k}) u_s(\vec{k}) e^{-ik\cdot x} + \hat{d}_s^\dagger(\vec{v},\vec{k}) v_s(\vec{k}) e^{+ik\cdot x}) \tag{2.26}$$

Hence the phase factor added to eq.(2.24) implies that the fields h_v and H_v have residual momentum k_μ. Also now there are different fields for each velocity.

Now note that in the infinite mass limit, strong interactions cannot create a heavy quark- heavy anti-quark pair or annihilate a heavy quark or a heavy anti-quark, projected fields should not be combined to give the whole field. Instead heavy quark field and heavy anti-quark fields must be studied separately. This argument can be seen from the example action

$$\begin{aligned} S &= \int d^4x \overline{Q} Q = \int d^4x (\overline{Q}^+ + \overline{Q}^-)(Q^+ + Q^-) \\ &= \int d^4x \left[\overline{Q}_v^+ Q_v^+ + \overline{Q}_v^- Q_v^- + e^{2im_Q v\cdot x} \overline{Q}_v^+ Q_v^- + e^{-2im_Q v\cdot x} \overline{Q}_v^- Q_v^+ \right] \end{aligned} \tag{2.27}$$

where $Q_v^+ = h_v + H_v$ and $Q_v^- = h_v^- + H_v^-$. The last two terms oscillate with infinite frequency and hence does not contribute to S.

The rescaling effects the Lagrangian and the equation of motion as the covariant derivative acting on our quark field now gives

$$i\not{D} Q = e^{-im_Q v\cdot x}(m_Q \not{v} + i\not{D}) Q_v = e^{-im_Q v\cdot x}(m_Q \not{v} + i\not{D})[h_v + H_v] \tag{2.28}$$

The Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \overline{Q}_v (i\not{D} - m_Q) Q_v \\ &= \overline{h}_v (iv\cdot D) h_v - \overline{H}_v (iv\cdot D + 2m_Q) H_v + \overline{h}_v i\not{D} H_v + \overline{H}_v i\not{D} h_v \end{aligned} \tag{2.29}$$

The derivative operator can be decomposed into two parts

$$\not{D} = \not{D}_\perp + \not{D}_\parallel \tag{2.30}$$

with

$$\not{D}_\parallel = \not{v}(v \cdot D), \quad \not{D}_\perp = \not{D} - \not{v}(v \cdot D). \tag{2.31}$$

They satisfy the (anti)commutation relations

$$[\not{D}_\parallel, \not{v}] = 0, \quad \{\not{D}_\perp, \not{v}\} = 0. \tag{2.32}$$

This gives

$$h_v \not{D} H_v = h_v[\not{D}_\perp + \not{v}(v \cdot D)]H_v = h_v \not{D}_\perp H_v. \tag{2.33}$$

This equality is easy to see by writing $H_v = P_- H_v$ and anticommuting the projection operator through $\not{D}$. This would yield $\not{D}_\perp P_- = P_+ \not{D}_\perp$ and $\not{D}_\parallel P_- = P_- \not{D}_\parallel$. Then the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \overline{Q}_v(i\not{D} - m_Q)Q_v \\ &= \overline{h}_v(iv \cdot D)h_v - \overline{H}_v(iv \cdot D + 2m_Q)H_v + \overline{h}_v i\not{D}_\perp H_v + \overline{H}_v i\not{D}_\perp h_v \end{aligned} \tag{2.34}$$

It is clear that h_v describes massless degrees of freedom while H_v gives fluctuations with mass $2m_Q$. Due to being heavy, the field H_v is going to be integrated out. This is done by using its equation of motion. It gives [21]

$$H_v = \frac{1}{iv \cdot D + 2m_Q} i\not{D}_\perp h_v. \tag{2.35}$$

H_v field can be eliminated by inserting this relation in the Lagrangian.

$$\mathcal{L}_{Q_v} = \overline{h}_v iv \cdot D h_v + \overline{h}_v i\not{D}_\perp \frac{1}{iv \cdot D + 2m_Q} i\not{D}_\perp h_v. \tag{2.36}$$

The second term can be expanded in $1/m_Q$, which is calculated in Appendix A, to give

$$\mathcal{L}_{Q_v} = \overline{h}_v iv \cdot D h_v + \frac{1}{2m_Q}\overline{h}_v(iD_\perp)^2 h_v - \frac{1}{4m_Q}\overline{h}_v g_s \sigma^{\mu\nu} G_{\mu\nu} h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right) \tag{2.37}$$

where the field strength tensor is $G_{\mu\nu} = [D_\mu, D_\nu]/ig_s$. The effective Lagrangian is the leading term of the expansion,

$$\mathcal{L}_{eff} = \overline{h}_v iv \cdot D h_v \tag{2.38}$$

The mass of the heavy quark does not appear in the Lagrangian. Strong interactions are independent of the heavy quark flavor. It corresponds to $U(N_Q)$ flavor symmetry where N_Q stands for number of heavy quarks [18].

The system also has a spin symmetry since the Lagrangian does not contain any spin matrices [21]. In the rest frame an infinitesimal spin transformation takes the form

$$h_v{}' = (1 + i\vec{\theta} \cdot \vec{S}_Q)h_v \tag{2.39}$$

where

$$\vec{S}_Q = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \frac{1}{2}\gamma_5\gamma^0\vec{\gamma}. \tag{2.40}$$

The change in the Lagrangian becomes

$$\delta\mathcal{L}_{eff} = \overline{h}_v[iv \cdot D, i\vec{\theta} \cdot \vec{S}_Q]h_v = 0 \tag{2.41}$$

$SU(2)$ spin symmetry is associated with this behavior. Combining with the flavor symmetry, HQET is said to have $SU(2N_Q)$ spin-flavor symmetry [29].

The remaining two terms appearing at order $1/m_Q$ breaks these symmetries. These terms can be identified in the rest frame easily [21]. The first term;$(iD_\perp)^2$, becomes

$$O_{kin} = \frac{1}{2m_Q}\overline{h}_{v,r}(iD_\perp)^2 h_{v,r} = \frac{1}{2m_Q}\overline{h}_{v,r}(-i\vec{D})^2 h_{v,r} \tag{2.42}$$

where $h_{v,r}$ stands for at rest fields. This term is the kinetic energy arising from the residual motion of the heavy quark. It breaks the flavor symmetry. The other term; chromomagnetic operator, gives the heavy quark spin's interaction with the light cloud. In the rest frame it becomes

$$O_{mag} = \frac{1}{4m_Q}\overline{h}_{v,r}g_s\sigma^{\mu\nu}G_{\mu\nu}h_{v,r} = -\frac{1}{m_Q}\overline{h}_{v,r}g_s\vec{S}_Q \cdot \vec{B}_c h_{v,r} \tag{2.43}$$

where $B_c^i = -\frac{1}{2}\epsilon^{ijk}G_{jk}$ are the color magnetic field components. This term violates the spin symmetry in addition to flavor symmetry as the infinitesimal spin transformation eq.(2.39) now leaves $\mathcal{L}_{eff}$ changed.

2.1 Reparametrization Invariance

The particles in the effective field theory are described with velocity dependent fields with constant velocity, residual momentum and total momentum. The velocity v is arbitrarily chosen. The effective theory should be invariant under the reparametrization of the velocity and momentum [30].

$$v \to w = v + \frac{q}{m_Q} \quad k \to k' = k - q \tag{2.44}$$

where q is an arbitrary infinitesimal four vector that must satisfy

$$\left(v + \frac{q}{m_Q}\right)^2 = 1. \tag{2.45}$$

The relation $v^2 = 1$ for v must also hold for w. Recall the equality $\not{v} h_v = h_v$. Under the reparametrization this equation takes the form

$$\begin{aligned}\left(\not{v} + \frac{\not{q}}{m_Q}\right)(h_v + \delta h_v) &= (h_v + \delta h_v)\\ \not{v} h_v + \not{v}\delta h_v + \frac{\not{q}}{m_Q} h_v + \frac{\not{q}}{m_Q}\delta h_v &= h_v + \delta h_v\\ \frac{\not{q}}{m_Q} h_v &= (1 - \not{v})\delta h_v\end{aligned} \tag{2.46}$$

where eq.(2.20) is used. The term $\frac{\not{q}}{m_Q}\delta h_v$ is ignored because both factors are very small quantities. Hence their product would be much smaller. The solution to eq.(2.46) is not uniqe. A suitable choice for the δh_v field is

$$\delta h_v = \frac{\not{q}}{2m_Q} h_v. \tag{2.47}$$

which satisfies $\not{v}\delta h_v = -\delta h_v$ [22]. This is due to $v \cdot q = \mathcal{O}\left(\frac{q^2}{m}\right)$ obtained from eq.(2.45). This leads to the relation $\not{v}\not{q} = -\not{q}\not{v}$. Therefore, the large fields are reparametrized as

$$h_v \to h_w = \left(1 + \frac{\not{q}}{2m_Q}\right) h_v \tag{2.48}$$

and while reparametrizing the whole field, a phase factor must be added, which can be seen from eq.(2.24).

$$Q_v \to Q_w = e^{iq\cdot x} Q_v \tag{2.49}$$

The reparametrization of the effective Lagrangian is calculated in extend in Appendix B, and is in the form

$$\begin{aligned}\mathcal{L}_{Q_w}^{(0)} &= \overline{h}_w (iw \cdot D) h_w\\ &= \left[\overline{h}_v \left(1 + \frac{\not{q}}{2m_Q}\right) e^{-iq\cdot x}\right] \left[i\left(v + \frac{q}{m_Q}\right)\cdot D\right] \left[e^{iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right) h_v\right].\end{aligned} \tag{2.50}$$

By using the relation $v \cdot q = \mathcal{O}\left(\frac{q^2}{m}\right)$, the following can be obtained.

$$= \overline{h}_v \left(iv \cdot D + \frac{q \cdot D}{2m_Q} \right) h_v = \mathcal{L}^{(0)}_{Q_v} + \delta \mathcal{L}^{(0)}_{Q_v} \tag{2.51}$$

Therefore

$$\delta \mathcal{L}^{(0)}_{Q_v} = \overline{h}_v \frac{q \cdot D}{2m_Q} h_v. \tag{2.52}$$

The first order in the Lagrangian can also be reparametrized similarly [31]. The calculations are done in Appendix B. The first appearing term is

$$\begin{aligned} \mathcal{L}^{(1)}_{Q_w,kin} &= \overline{h}_w (iD_\perp)^2 h_w \\ &= \left[\overline{h}_v \left(1 + \frac{\not{q}}{2m_Q} \right) e^{-iq\cdot x} \right] (iD_\perp)^2 \left[e^{iq\cdot x} \left(1 + \frac{\not{q}}{2m_Q} \right) h_v \right] \quad (2.53) \\ &= \mathcal{L}^{(1)}_{Q_v,kin} + \delta \mathcal{L}^{(1)}_{Q_v,kin}. \quad (2.54) \end{aligned}$$

The term $\delta \mathcal{L}^{(1)}_{Q_v,kin}$ is calculated as

$$\frac{1}{2m_Q} \delta \mathcal{L}^{(1)}_{Q_w,kin} = -\overline{h}_v \frac{q \cdot D}{2m_Q} h_v. \tag{2.55}$$

For the chromomagnetic term variation in the first order is zero.

$$\delta \mathcal{L}^{(1)}_{Q_w,mag} = 0 \tag{2.56}$$

Hence the effective Lagrangian is reparametization invariant up to the first order.

2.2 Normalization of Effective Meson States

The construction of an effective theory in the infinite mass limit requires showing quark masses explicitly in all expressions. This principle is followed so far. However this procedure needs to be applied to meson states too. The argument goes as follows [21]. Consider the vector current composed of a light quark and a heavy quark.

$$V^\mu = \overline{q} \gamma^\mu Q \tag{2.57}$$

By using the form eq.(2.24) and implementing eq.(2.35), the heavy quark field can be stated in effective theory as

$$Q = e^{-im_Q v\cdot x} \left(1 + \frac{i\not{D}_\perp}{2m_Q} + ... \right) h_v \tag{2.58}$$

Then the vector current takes the form

$$V^{\mu} = e^{-im_Q v\cdot x}\overline{q}\gamma^{\mu}\left(1 + \frac{i\not{D}_{\perp}}{2m_Q} + ...\right) h_v \tag{2.59}$$

Matrix elements containing this current can be expressed in terms of hadronic form factors. The purpose in the effective field theory is to show the mass dependence of these form factors explicitly. Now consider the matrix element containing a general meson state $|\mathcal{M}\rangle$

$$\langle 0|V^{\mu}|\mathcal{M}(v)\rangle = \langle 0|\overline{q}\gamma^{\mu}h_v|\mathcal{M}(v)\rangle + \frac{1}{2m_Q}\langle 0|\overline{q}\gamma^{\mu}i\not{D}_{\perp}h_v|\mathcal{M}(v)\rangle + ... \tag{2.60}$$

It would be ideal if the matrix elements on the right hand side are independent of m_Q. Then the second term would act as power correction to the first one. However, from the equation of motion for h_v coming from the full Lagrangian eq.(2.37); that corresponds to QCD Lagrangian, contains m_Q dependence. Therefore all matrix elements must have m_Q dependence. A way to approach this situation is to take the effective theory Lagrangian as eq.(2.38) and treat the terms with higher dimension operators as perturbative corrections.

$$\mathcal{L}_{power} = \frac{1}{2m_Q}\mathcal{L}_1 + \frac{1}{4m_Q^2}\mathcal{L}_2 + ... \tag{2.61}$$

Then the equation of motion for h_v fields are

$$iv\cdot Dh_v = 0. \tag{2.62}$$

This also means the meson states of the effective theory are mass independent. However these effective states are now different from the full theory states. Then eq.(2.60) needs to be rewritten as

$$\begin{aligned}
& \frac{1}{\sqrt{m_{\mathcal{M}}}}\langle 0|V^{\mu}|\mathcal{M}(v)\rangle_{QCD} \\
= \;& \langle 0|\overline{q}\gamma^{\mu}h_v|\mathcal{M}(v)\rangle_{HQET} + \frac{1}{2m_Q}\langle 0|\overline{q}\gamma^{\mu}i\not{D}_{\perp}h_v|\mathcal{M}(v)\rangle_{HQET} \\
& + \frac{1}{2m_Q}\langle 0|i\int d^4xT\{\overline{q}\gamma^{\mu}h_v(0),\mathcal{L}_1(x)\}|\mathcal{M}(v)\rangle_{HQET} + ...
\end{aligned} \tag{2.63}$$

where T is the time ordering product. The contributions from O_{kin} and O_{mag} becomes different. Once these terms are included in the Lagrangian, the effective states $|\mathcal{M}(v)\rangle_{HQET}$ are no longer eigenstates of the Hamiltonian. The possibility that a state created at $t = -\infty$, then the operators O_{kin} and O_{mag} acting on this state before

decaying at $t = 0$ must be accounted. This is done with time ordering products in which these operators are inserted along the heavy quark line [32].
The matrix elements on the right hand side are now mass independent. The mass dependence of QCD states are given by the third term in eq.(2.63) in HQET.
As the effective theory states are mass independent, their normalization condition are different from the full theory states. The normalization condition for the usual relativisticly normalized heavy meson states is [19]

$$\langle \mathcal{M}(p')|\mathcal{M}(p)\rangle_{QCD} = 2p^0(2\pi)^3\delta(\vec{p} - \vec{p}'). \tag{2.64}$$

For the effective states, the relevant normalization condition is

$$\langle \mathcal{M}(v', k')|\mathcal{M}(v, k)\rangle_{HQET} = \frac{2p^0}{M_Q}\delta^v_{v'}(2\pi)^3\delta(\vec{k} - \vec{k}'). \tag{2.65}$$

One way to think is that the relativistic version of a state is related to HQET definition by [22]

$$|\mathcal{M}(p)\rangle = \sqrt{M_Q}\left[|\mathcal{M}(v)\rangle + \mathcal{O}\left(\frac{1}{m_Q}\right)\right] \tag{2.66}$$

2.3 Covariant Representation of Fields

In the heavy quark limit the heavy quark decouples from the light degrees of freedom. As a consequence, $\vec{S}_Q$ and the total angular momentum of light degrees of freedom $\vec{J}_l$ are conserved separately. Therefore it is convenient to classify mesons according to the value of $\vec{J}_l$ in doublets formed by mesons of total angular momentum $J = J_l \pm \frac{1}{2}$ and parity $P = (-1)^{l+1}$ [33]. Let us look at a ground state meson $Q\overline{q}$. Under Lorentz transformation for spinors heavy quark transforms as a particle moving with v and light quark transforms as an anti-particle moving with v.

$$Q \to D(\Lambda)Q, \quad \overline{q} \to \overline{q}D(\Lambda)^{-1} \tag{2.67}$$

The meson must then transform as a bilinear

$$Q\overline{q} \to D(\Lambda)Q\overline{q}D(\Lambda)^{-1} \tag{2.68}$$

Denoting this object with $H_v^{(Q)}$ the above equation is rewritten as [22]

$$H'^{(Q)}_{v'}(x') = D(\Lambda)H_v^{(Q)}(x)D(\Lambda)^{-1} \tag{2.69}$$

where

$$x' = \Lambda x, \quad v' = \Lambda v \tag{2.70}$$

So that

$$H'^{(Q)}_v(x) = D(\Lambda) H^{(Q)}_{\Lambda^{-1}v}(\Lambda^{-1}x) D(\Lambda)^{-1} \tag{2.71}$$

This object should contain both a pseudoscalar meson $P^{(Q)}_v$ and a vector meson $P^{*(Q)}_{v,\mu}$. A convenient way to combine two fields into a single field, a 4x4 matrix is in a linear superposition in gamma matrix base [22]

$$H^{(Q)}_v = \frac{1+\not{v}}{2}[P^{*(Q)}_{v,\mu}\gamma^\mu + iP^{(Q)}_v\gamma_5] \tag{2.72}$$

γ^μ and γ_5 convert vectors and pseudoscalars into bispinors. A more extensive illustration of how this object is obtained is shown in Appendix C. The form above satisfies the parity transformation rule.

$$H^{(Q)}_v \to \mathcal{P} H^{(Q)}_v(x) \mathcal{P} = \gamma^0 H^{(Q)}_{v_p}(x_p)\gamma^0 \tag{2.73}$$

Two constraints for $H^{(Q)}_v$ are obtained,

$$\not{v} H^{(Q)}_v = H^{(Q)}_v, \quad H^{(Q)}_v \not{v} = -H^{(Q)}_v. \tag{2.74}$$

The first one is expected for $H^{(Q)}_v$ to satisfy. But second one is a consequence of this construction. It can be seen by anti-commuting $\not{v}$ through $H^{(Q)}_v$ and using $v \cdot P^{*(Q)}_v$. The conjugate field is then defined as

$$\overline{H}^{(Q)}_v = \gamma^0 H^{(Q)\dagger}_v \gamma^0 = [P^{*(Q)\dagger}_{v,\mu}\gamma^\mu + iP^{(Q)\dagger}_v\gamma_5]\frac{1+\not{v}}{2} \tag{2.75}$$

which also must transform as a bilinear

$$\overline{H}'^{(Q)}_v(x) = D(\Lambda)\overline{H}^{(Q)}_{\Lambda^{-1}v}(\Lambda^{-1}x) D(\Lambda)^{-1} \tag{2.76}$$

The object

$$\mathrm{Tr}(\overline{H}^{(Q)}_v H^{(Q)}_v) = -2P^{(Q)\dagger}_v P^{(Q)}_v + 2P^{*(Q)\dagger}_{v,\mu} P^{*(Q)\mu}_v \tag{2.77}$$

is invariant under Lorentz transformation and heavy quark spin transformations [34]. In the rest frame; $v = v_r = (1, \vec{0})$, the field $H^{(Q)}_v$ takes the form [22]

$$H^{(Q)}_{v_r} = \begin{pmatrix} 0 & iP^{(Q)}_{v_r} - \sigma \cdot P^{*(Q)}_{v_r} \\ 0 & 0 \end{pmatrix} \tag{2.78}$$

in the Dirac basis. It transforms as a $(1/2, 1/2)$ under $S_Q \otimes S_l$

$$[S_Q^i, H_{v_r}^{(Q)}] = \frac{1}{2}\sigma_{4\times4}^i H_{v_r}^{(Q)}, \quad [S_l^i, H_{v_r}^{(Q)}] = -\tfrac{1}{2} H_{v_r}^{(Q)} \sigma_{4\times4}^i \tag{2.79}$$

where $\sigma_{4\times4}^i = i\epsilon^{ijk}[\gamma_j, \gamma_k]/4$ are Dirac rotation matrix in spinor representation. The infinitesimal heavy quark spin transformation takes the form,

$$H_{v_r}^{(Q)} \to D(R)_Q H_{v_r}^{(Q)} \tag{2.80}$$

where $D(R)_Q$ is the rotation matrix for a rotation R. The above equation means

$$\delta H_{v_r}^{(Q)} = i[\vec{\theta} \cdot \vec{S}_Q, H_{v_r}^{(Q)}] = \frac{i}{2}\vec{\theta} \cdot \vec{\sigma}_{4\times4} H_{v_r}^{(Q)} \tag{2.81}$$

Inserting the form of $H_{v_r}^{(Q)}$ in eq.(2.81), gives a result that mixes pseudoscalar and vector states.

$$\delta P_{v_r}^{(Q)} = -\frac{i}{2}\vec{\theta} \cdot \vec{P}_{v_r}^{*(Q)}, \quad \delta \vec{P}_{v_r}^{*(Q)} = \tfrac{1}{2}\vec{\theta} \times \vec{P}_{v_r}^{*(Q)} - \tfrac{1}{2}\vec{\theta} P_{v_r}^{(Q)} \tag{2.82}$$

Similarly for an infinitesimal total spin rotation

$$\delta H_{v_r}^{(Q)} = i[\vec{\theta} \cdot (\vec{S}_Q + \vec{S}_l), H_{v_r}^{(Q)}] = \frac{i}{2}[\vec{\theta} \cdot \vec{\sigma}_{4\times4}, H_{v_r}^{(Q)}] \tag{2.83}$$

which gives,

$$\delta P_{v_r}^{(Q)} = 0, \quad \delta \vec{P}_{v_r}^{*(Q)} = \vec{\theta} \times \vec{P}_{v_r}^{*(Q)}. \tag{2.84}$$

The resulting terms are spin transformation rules for pseudoscalar and vector particles.

CHAPTER 3

MESON MASSES

Consider the ground state pseudoscalar and vector mesons in the effective theory. They form degenerate doublet states in the leading order in the $1/m_Q$ expansion. The mass of the meson differs from the heavy quark mass they are constitute of, by the contribution from the part of Lagrangian that does not contain heavy quark $\bar{\Lambda}$

$$M_Q = m_Q + \bar{\Lambda} \tag{3.1}$$

which is a constant quantity and of order Λ_{QCD}. However the masses of pseudoscalar and vector mesons are not precisely degenerate. The difference in the common mass of these states can be accounted in the effective theory with the high order terms in the expansion. The physical mass of a meson takes the form of an expansion [1]

$$M_Q = m_Q + \bar{\Lambda} + \frac{\Delta M_Q^2}{2m_Q} + ... \tag{3.2}$$

where $\frac{\Delta M_Q^2}{2m_Q}$ is the correction from the first order in the effective Lagrangian. The mass of a meson is equal to the expectation value of the Hamiltonian when three momentum vanishes.

$$M_Q = \frac{\langle \mathcal{M}(0)|H|\mathcal{M}(0)\rangle}{\langle \mathcal{M}(0)|\mathcal{M}(0)\rangle} \tag{3.3}$$

In the leading order, Hamiltonian is obtained from the Lagrangian, $\mathcal{H}^{(0)} = \mathcal{L}^{(0)}$. This is due to the vanishing conjugate momentum. Adding the Hamiltonian obtained from the Lagrangian of light degrees of freedom, in the rest frame mass contribution related to this term is

$$\frac{\langle \mathcal{M}(0)|\mathcal{H}^{(0)}(x)|\mathcal{M}(0)\rangle}{\langle \mathcal{M}(0)|\mathcal{M}(0)\rangle} = \bar{\Lambda} \tag{3.4}$$

This term respects the flavor-spin symmetry. In the rest frame the $1/m_Q$ order terms, where $\mathcal{H}^{(1)} = -\mathcal{L}^{(1)}$, are

$$\Delta M_Q^2 = -\frac{\langle \mathcal{M}(0)|\mathcal{L}^{(1)}(x)|\mathcal{M}(0)\rangle}{\langle \mathcal{M}(0)|\mathcal{M}(0)\rangle} \tag{3.5}$$

ın the rest frame the normalization yields $\langle \mathcal{M}(0)|\mathcal{M}(0)\rangle = 2M_Q$ [15]. This results with two quantites in the first order.

$$\begin{aligned}\frac{\Delta M_Q^2}{2m_Q} &= \frac{\langle \mathcal{M}(0)|\overline{h}_{v_r}^{(Q)} D_\perp^2 h_{v_r}^{(Q)}|\mathcal{M}(0)\rangle}{2M_Q \times 2m_Q} \\ &\quad + \frac{\langle \mathcal{M}(0)|\overline{h}_{v_r}^{(Q)} g\sigma_{\mu\nu}G^{\mu\nu} h_{v_r}^{(Q)}|\mathcal{M}(0)\rangle}{2M_Q \times 4m_Q}\end{aligned} \tag{3.6}$$

The contributions from these terms are related to a hadronic parameter λ_i. To begin with the kinetic term

$$\frac{\langle \mathcal{M}(0)|\overline{h}_{v_r}(iD_\perp)^2 h_{v_r}|\mathcal{M}(0)\rangle}{2M_Q} = \lambda_1 \tag{3.7}$$

does not depend on m_Q since it is removed from both heavy quark fields and the heavy meson states. Thus it is independent from both flavor and spin of the hadron. The next contribution is from the chromomagnetic term. In the rest frame this term can be rewritten as $\overline{h}_{v_r}\sigma_{\mu\nu}G^{\mu\nu}h_{v_r} = \overline{h}_{v_r}\vec{\sigma}\cdot\vec{B}h_{v_r}$. The color magnetic field is created by the light degrees of freedom. Since $\vec{B}$ is a vector, it can only be proportional to the only vector quantity that light degrees of freedom possess; the total angular momentum. $l = 0$ case is the interest of this thesis, therefore the braket is proportional to $\vec{S}_Q \cdot \vec{S}_l$.

$$\frac{\langle \mathcal{M}(0)|\overline{h}_{v_r} g_s\sigma^{\mu\nu}G_{\mu\nu} h_{v_r}|\mathcal{M}(0)\rangle}{2M_Q} = 2\lambda_2 \vec{S}_Q \cdot \vec{S}_l \tag{3.8}$$

Then the form of heavy meson masses becomes

$$M_Q = m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q} + \frac{2\lambda_2 \vec{S}_Q \cdot \vec{S}_l}{m_Q} \tag{3.9}$$

Note that for $L = 0$ case the following relation holds.

$$\vec{S}_Q \cdot \vec{S}_l = (J^2 - S_Q^2 - S_l^2)/2 \tag{3.10}$$

Hence

$$\begin{aligned}
M_B &= m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2}{2m_b} \\
M_{B^*} &= m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2}{2m_b} \\
M_D &= m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} - \frac{3\lambda_2}{2m_c} \\
M_{D^*} &= m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} + \frac{\lambda_2}{2m_c}
\end{aligned} \tag{3.11}$$

The experimental results support the predicted behavior. By using the observed masses given in [6], the following arguments can be deduced [1]. First consider the mass splitting between different flavors of B and D mesons. Since the interactions are suppressed, similar difference in two systems that is equal to the difference of light degrees of freedom are expected. HQET predicts

$$\begin{aligned}
m_{B_s^0} - m_B &= \overline{\Lambda}_s - \overline{\Lambda}_d + \mathcal{O}\left(\frac{1}{m_b}\right), \\
m_{D_s^\pm} - m_{D^\pm} &= \overline{\Lambda}_s - \overline{\Lambda}_d + \mathcal{O}\left(\frac{1}{m_c}\right).
\end{aligned} \tag{3.12}$$

Experimental results corresponding to these predictions are

$$\begin{aligned}
m_{B_s^0} - m_B &= 87.35 \pm 0.24 MeV \\
m_{D_s^\pm} - m_{D^\pm} &= 98.69 \pm 0.05 MeV
\end{aligned} \tag{3.13}$$

Considering m_b as a good approximation for the heavy quark limit, the contribution from order $1/m_c$ is $\approx 11\%$. The approximate value of λ_2 can be obtained from the mass splitting between pseudoscalar and vector mesons.

$$M_{B^*}^2 - M_B^2 = (M_{B^*} - M_B)(M_{B^*} + M_B) \tag{3.14}$$

$$= \frac{4\lambda_2}{2m_b}(2m_b + 2\bar{\Lambda} + ...) \tag{3.15}$$

In the heavy quark limit it is equal to λ_2. The same argument can follow from the D mesons. The value of λ_2 is then

$$\begin{aligned}
M_{B^*}^2 - M_B^2 &= 0.49 GeV^2 \approx 4\lambda_2, \\
M_{D^*}^2 - M_D^2 &= 0.55 GeV^2 \approx 4\lambda_2
\end{aligned} \tag{3.16}$$

Mass m_b is a good approximation for the heavy quark limit, hence λ_2 is

$$\lambda_2 \approx 0.12 GeV^2 \tag{3.17}$$

For the c quark case the value is $\lambda_2 \approx 0.14 GeV^2$. The contribution from $\mathcal{O}\left(\frac{1}{m_c}\right)$ is about 14%.

Next, the value of λ_1 is obtained by introducing spin averaged meson masses. Spin dependence cancels out and λ_2 disappears. This results with

$$\begin{aligned} \overline{m}_B &= \frac{1}{4}(m_B + 3m_{B^*}) \approx 5.31 GeV, \\ \overline{m}_D &= \frac{1}{4}(m_D + 3m_{D^*}) \approx 1.97 GeV \end{aligned} \tag{3.18}$$

Then mass difference can be stated as

$$m_b - m_c = (\overline{m}_B - \overline{m}_D)\left[1 - \frac{\lambda_1}{2\overline{m}_B \overline{m}_D} + \mathcal{O}\left(\frac{1}{m_Q^3}\right)\right] \tag{3.19}$$

which gives

$$\lambda_1 = -0.3 \pm 0.2 GeV^2. \tag{3.20}$$

CHAPTER 4

SEMILEPTONIC B DECAYS

The flavor changing weak decays are governed by the electroweak Lagrangian that couples charged current J^{μ}_{cc} to W-boson fields[1].

$$\mathcal{L}_w = -\frac{g}{\sqrt{2}} J^{\mu}_{cc} W^{\dagger}_{\mu} + h.c. \tag{4.1}$$

where

$$J^{\mu}_{cc} = \begin{pmatrix} \overline{\nu}_e & \overline{\nu}_\mu & \overline{\nu}_\tau \end{pmatrix} \gamma^{\mu} \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix} + \begin{pmatrix} \overline{u}_L & \overline{c}_L & \overline{t}_L \end{pmatrix} \gamma^{\mu} V_{CKM} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \tag{4.2}$$

V_{CKM} is the Cabibbo-Kobayashi-Maskawa matrix; giving the strength of quark mixing, defined as

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{4.3}$$

Up to this point the internal dynamics of a B meson involved two scales; the heavy quark mass at $m_b \approx 4.7GeV$ and interactions at $\Lambda_{QCD} \approx 0.2GeV$. The weak decays however, takes place at a different scale at $m_W \approx 80GeV$. Effective field theory methods are proved to be helpful to tackle this multiple scale problem. The Feynman diagrams with W boson propagators represents the situation for very short distances as the weak interactions are perceived as point like at hadronic scales m_b and Λ_{QCD}[35]. Hence the physical picture of a decaying B meson can be described more appropriately with an effective Lagrangian that couples four fermions locally (see

Figure 4.1).

$$\mathcal{L}_{w,eff} = -2\sqrt{2}G_F J^{\mu}_{cc} J^{\dagger}_{cc,\mu} \tag{4.4}$$

where the effects of W-boson is accounted in Fermi constant,

$$G_F = \frac{g^2}{4\sqrt{2}m_W^2} = 1.16692 GeV^{-2}. \tag{4.5}$$

4.1 Operator Product Expansion and Inclusive Decays

The next step is to evaluate decay rate. The optical theorem states inclusive decay into a final state with c quark; X_c, is described in terms of imaginary part of transition operator T̂, defined as [1, 15]

$$\hat{T} = i\int d^4x T\{\mathcal{L}_w(x)\mathcal{L}_w^{\dagger}(0)\} \tag{4.6}$$

where T is the time ordering operator. Then the width is obtained by inserting a complete set of states inside the time ordered product

$$\begin{aligned}\Gamma(B \to X_c) &= \frac{1}{2m_B}\sum_{X_f}(2\pi)^4\delta^4(p_B - p_{X_f})|\langle X_f|\mathcal{L}_{w,eff}|B\rangle|^2 \\ &= \frac{1}{2m_B}\text{Im}\int d^4x\,\langle B|\mathcal{L}_{w,eff}(x)\mathcal{L}_{w,eff}(0)|B\rangle \\ &= \frac{1}{2m_B}\text{Im}\,\langle B|\hat{T}|B\rangle \end{aligned} \tag{4.7}$$

As a short distance approxiamation for the product of two operators that are separated by distance x, the product is expanded in terms of local operators.

$$\hat{T} = i\int d^4x T\{\mathcal{L}_w(x)\mathcal{L}_w^{\dagger}(0)\} = \sum_i C_i(\mu)O_i \tag{4.8}$$

This is known as operator product expansion (OPE). It yields [5]

$$\Gamma(B \to X_c) = G_F^2|V_{cb}|^2 m_Q^5 \sum_i \tilde{c}_i^f(\mu)\langle O_i\rangle_B \tag{4.9}$$

where O_i are local operators with the short-hand notation

$$\langle O_i\rangle_B = \frac{1}{2m_B}\langle B|O_i|B\rangle \tag{4.10}$$

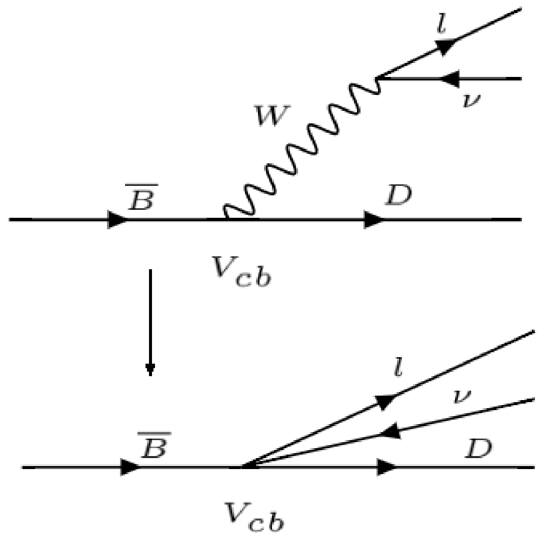

Figure 4.1: If W boson is neglected, the weak interaction is given by four-fermion local interaction. (The figure is taken from [1])

The scale μ is the normalization point, Wilson coefficients $\tilde{c}_i^f$, proportional to C_i of eq.(4.8) but has different mass dimension, depends on the quantum numbers of the final state and consists two parts $\tilde{c}_i^f = m^{3-d_i} c_i^f$; the powers of $1/m_Q$ and dimensionless coefficients c_i^f depends on the ratio of mass of final state to initial state. They can be interpreted as couplings [17]. The role of this expansion is to separate the amplitude in two parts. The short distance effects from the couplings, calculated perturbatively, and long distance effects from the calculations of $\langle O_i \rangle$ which are non-perturbative. By starting from the scale $\mu = \mathcal{O}(m_W)$ and evolving down to the hadronic scales, the physical contributions from the higher scales are implemented in c_i^f [36].
To find the amplitude first the matrix elements $\langle O_i \rangle$ need to be evaluated. Heavy quark expansion is useful in our case with B mesons. The first operator of the series, $\bar{b}b$, appears at dimension $d = 3$. The following operator at $d = 4$ is $\bar{b}i\not{D}b$. From the equations of motion this term is reduced to $m_Q \bar{b}b$ and hence there is no correction to width from order $1/m_Q$. Operators of a new form starts at $d = 5$ with $\bar{b}g_s\sigma_{\mu\nu}G^{\mu\nu}b$. The width is then in the explicit form of [1]

$$\Gamma(B \to X_c) = \frac{G_F^2 |V_{cb}|^2 m_B^5}{192\pi^3} \times \left[c_3^f(\mu) \langle \bar{b}b \rangle_B + \frac{c_5^f(\mu)}{m_B^2} \langle \bar{b}g_s\sigma_{\mu\nu}G^{\mu\nu}b \rangle_B + \mathcal{O}\left(\frac{1}{m_B^3}\right) \right] \quad (4.11)$$

These operator can be expanded upto order $1/m_Q$ in HQET by using the definition for the field H_v in eq.(2.35), the heavy quark field can be corrected as,

$$
\begin{aligned}
h_v(x) \rightarrow \quad & \left[1 + \frac{1}{iv \cdot D + 2m_Q} i\not{D}_\perp\right] h_v(x) \\
= & \left[1 + \frac{i\not{D}_\perp}{2m_Q} + ...\right] h_v(x)
\end{aligned} \tag{4.12}
$$

Expansion yields

$$
\begin{aligned}
\langle \bar{b}b\rangle_B &= \left\langle \overline{h}_v^{(b)} \left[1 - \left(\frac{i\not{D}_\perp}{2m_b}\right)^2\right] h_v^{(b)} \right\rangle_B \\
&= 1 - \frac{\mu_\pi^2(B) - \mu_G^2(B)}{2m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right), \qquad (4.13) \\
\left\langle \bar{b}\frac{g_s}{2}\sigma_{\mu\nu}G^{\mu\nu}b\right\rangle_B &= 2\mu_G^2(B) + \mathcal{O}\left(\frac{1}{m_b}\right) \qquad (4.14)
\end{aligned}
$$

for

$$
\begin{aligned}
\mu_\pi^2(B) &= \frac{1}{2m_B} \langle \overline{h}_v^{(b)} (iD)^2 h_v^{(b)} \rangle\,, \\
\mu_G^2(B) &= \frac{1}{2m_B} \langle \overline{h}_v^{(b)} \frac{g_s}{2}\sigma_{\mu\nu}G^{\mu\nu} h_v^{(b)} \rangle
\end{aligned} \tag{4.15}
$$

Then eq.(4.11)becomes

$$
\begin{aligned}
\mathit{\Gamma}(B \rightarrow X_c) &= \frac{G_F^2 |V_{cb}|^2 m_B^5}{192\pi^3} \\
\times &\left[c_3^f(\mu)\left(1 - \frac{\mu_\pi^2(B) - \mu_G^2(B)}{2m_b^2}\right) + 2c_5^f(\mu)\frac{\mu_G^2(B)}{m_b^2} + \mathcal{O}\left(\frac{1}{m_b^3}\right)\right]
\end{aligned} \tag{4.16}
$$

The terms μ_π^2 and μ_G^2 can be written in terms of λ_1 and λ_2 that were found in chapter 3 [1].

$$
\begin{aligned}
\mu_\pi^2 &= -\lambda_1 = 0.3 \pm 0.2 GeV^2, \\
\mu_G^2 &= 3\lambda_2 \approx 0.36 GeV^2.
\end{aligned} \tag{4.17}
$$

The coefficients -1 and 3 are explained in chapter 3.

4.2 Exclusive $B \rightarrow D$ Decays

One of the most important applications of HQET is obtaining the relations between form factors. Let us start with the form factors in QCD. Elastic scattering of $\overline{B}$ meson is stated as

$$\langle \overline{B}(p')|\bar{b}\gamma^\mu b|\overline{B}(p)\rangle = f_{el}(q^2)(p+p')^\mu \tag{4.18}$$

The semileptonic $B \to D$ decays induced via weak current can be expressed in terms of six form factors

$$\begin{aligned}
\langle D(p')|V^\mu|\overline{B}(p)\rangle &= f_+(q^2)(p+p')^\mu + f_-(q^2)(p-p')^\mu && (4.19)\\
\langle D^*(p',\varepsilon)|V^\mu|\overline{B}(p)\rangle &= g(q^2)\epsilon^{\mu\nu\alpha\rho}\varepsilon^*_\nu(p+p')_\alpha(p-p')_\rho && (4.20)\\
\langle D^*(p',\varepsilon)|A^\mu|\overline{B}(p)\rangle &= -f(q^2)\varepsilon^{*\mu} \\
&\quad -i\varepsilon^*\cdot p\left[a_+(q^2)(p+p')^\mu + a_-(q^2)(p-p')^\mu\right] && (4.21)
\end{aligned}$$

where $V^\mu = \bar{c}\gamma^\mu b$, $A^\mu = \bar{c}\gamma^\mu\gamma^5 b$ and $q^\mu = (p-p')^\mu$. These matrix elements can be constructed as follows[22]. The most general current matrix element must transform as Lorentz four vector. In eq.(4.19) the matrix element can only depend on the four vectors p and p'. Hence it must be in the form of their linear combination $ap^\mu + bp'^\mu$ or in a more convenient form $f_+(p+p')^\mu + f_-(p-p')^\mu$. The form factors f_+ and f_- can only have dependence on Lorentz invariants at hand. There are three independent invariants p^2, p'^2 and $p\cdot p'$. Two of them are constants; $p^2 = m^2$ and $p'^2 = m'^2$. It is convenient to choose the third one as $q^2 = (p-p')^2$

In the eq.(4.20)the left hand side as a whole is an axial vector component. The only way for the right hand side to have same sign with the left hand side after applying the equalities P $|D^*(p',\epsilon)\rangle = |D^*(\overline{p'},\overline{\epsilon})\rangle$ and P $|B(p)\rangle = -|B(\overline{p})\rangle$ is to have Levi-Civita symbol. The Lorentz vectors in this case are p,p' and ε. Contraction must leave the right hand side with one Lorentz index, therefore it takes the form in eq.(4.20).

Again the choice of the form of the last equation is determined by parity transformation. The right hand side is going to be a linear combination of three vectors. However all the terms must contain the polarization vector since the wave function of D^* is proportional to the polarization vector, hence its matrix elements should contain this vector linearly. There is only one Lorentz invariant; $\varepsilon \cdot p$.

The matrix elements are generally denoted as in [37]

$$\begin{aligned}
\langle D(p')|V^\mu|\overline{B}(p)\rangle = &\left((p+p')^\mu - \frac{m_B^2-m_D^2}{q^2}q^\mu\right)F_1(q^2)\\
&+\frac{m_B^2-m_D^2}{q^2}q^\mu F_0(q^2)
\end{aligned} \tag{4.22}$$

$$\langle D^*(p',\varepsilon)|V^\mu|\overline{B}(p)\rangle = \frac{2i}{m_B+m_D}\epsilon^{\mu\nu\alpha\rho}\varepsilon^*_\nu p_\alpha p'_\rho V(q^2) \tag{4.23}$$

$$\begin{aligned}\langle D^*(p',\varepsilon)|A^\mu|\overline{B}(p)\rangle = &\Big((m_B+m_{D^*})\varepsilon^{*\mu}A_1(q^2)\\ &-\frac{\varepsilon\cdot q}{m_B+m_{D^*}}(p+p')^\mu A_2(q^2) - 2m_{D^*}\frac{\varepsilon\cdot q}{q^2}q^\mu A_3(q^2)\Big)\\ &+2m_{D^*}\frac{\varepsilon\cdot q}{q^2}q^\mu A_0(q^2)\end{aligned} \tag{4.24}$$

Where A_3 is written as linear combination of A_1 and A_2

$$A_3(q^2) = \frac{m_B+m_{D^*}}{2m_{D^*}}A_1(q^2) - \frac{m_B-m_{D^*}}{2m_{D^*}}A_2(q^2). \tag{4.25}$$

Furthermore the poles at $q^2=0$ are canceled by imposing conditions

$$F_1(0)=F_0(0) \quad A_3(0)=A_0(0) \tag{4.26}$$

Our aim is to study these decays in HQET and compare the results with the full theory. We now move on to low energy calculations. The meson states are replaced by effective meson states as well as the currents goes to effective currents by changing fields. Heavy quarks and anti-quarks decouples and their spin transformations are generated by independent SU(2) symmetries. This property has implications while relating states with different spins. Heavy quark's interactions with light cloud cannot change its velocity or spin hence the effective meson can be written as

$$|\mathcal{M}(v)\rangle = |Q(v,s_Q)\rangle\,|light\ cloud(v,s_q)\rangle \tag{4.27}$$

Then the hadronic part of the most general decay induced by $V-A$ current would have the form [38]

$$\begin{aligned}\langle\mathcal{M}'(v')|\overline{h'}_{v'}\mathit{\Gamma}^\mu h_v|\mathcal{M}(v)\rangle &= \langle Q'(v')|\overline{h'}_{v'}\mathit{\Gamma}^\mu h_v|Q(v)\rangle\\ &\quad\times\langle light\ d.o.f(v',s_q)|light\ d.o.f(v,s_q)\rangle\end{aligned} \tag{4.28}$$

where $\mathit{\Gamma}^\mu$ takes the forms γ^μ or $i\gamma^\mu\gamma^5$. It is useful to express theses matrix elements in tems of hadron fields $H_v^{(Q)}$. Heavy quark spin transformation $D(R)_Q$ transforms h_v field as

$$h_v \to D(R)_Q h_v \tag{4.29}$$

while leaving $h'_{v'}$ field untouched. If the spin transformtion for Γ^μ is assumed as

$$\Gamma^\mu \rightarrow \Gamma^\mu D(R)_Q^{-1} \tag{4.30}$$

then the current would be invariant under heavy quark spin transformations. Then the equivalent of the matrix element should also has the same transformation propery. Hence it should be proportional to $\Gamma^\mu H_v^{(Q)}$. Also considering the rotations $D(R)_{Q'}$ and Lorentz invariance, the current must be in the form [22]

$$\overline{h'}_{v'}\Gamma^\mu h_v = \text{Tr}\left[\overline{H}_{v'}^{(Q')}\Gamma^\mu H_v^Q X\right] \tag{4.31}$$

where X is the most general bispinor,

$$X = X_0 + X_1\not{v} + X_2\not{v}' + X_3\not{v}\not{v}'. \tag{4.32}$$

Any other possible combinations can be written in terms of linear combination of given terms. The reason why γ^5 does not appear is discussed in Appendix D. In general the coefficients X_i must be a function of the only possible Lorentz invariant $v \cdot v'$ which is generally denoted as w. As a consequence of relations $\not{v}h_v = h_v$ and $\not{v}\overline{h}_v = -\overline{h}_v$ the only independent term left is X_0. The common notation is to denote X with $\xi(w)$ which is known as the Isgur-Wise function. Inserting the current eq.(4.31)in the matrix element with pseudoscalar $|M(v)\rangle$ and vector $|M(v,\varepsilon)\rangle$ meson states gives

$$\langle\mathcal{M}'(v')|\overline{h'}_{v'}\Gamma^\mu h_v|\mathcal{M}(v)\rangle = -\xi(w)\text{Tr}\left[\gamma^5\frac{1+\not{v}'}{2}\Gamma^\mu\frac{1+\not{v}}{2}(-\gamma^5)\right], \tag{4.33}$$

$$\langle\mathcal{M}'(v',\varepsilon)|\overline{h'}_{v'}\Gamma^\mu h_v|\mathcal{M}(v)\rangle = -\xi(w)\text{Tr}\left[\not{\varepsilon}^*\frac{1+\not{v}'}{2}\Gamma^\mu\frac{1+\not{v}}{2}(-\gamma^5)\right]. \tag{4.34}$$

Defining $M(v)$ and $M(v,\varepsilon)$

$$M(v) = -\frac{1+\not{v}}{2}\gamma^5, \quad \overline{M}(v) = \gamma^5\tfrac{1+\not{v}}{2} \tag{4.35}$$

$$M(v,\varepsilon) = \frac{1+\not{v}}{2}\not{\varepsilon}, \quad \overline{M}(v,\varepsilon) = \not{\varepsilon}\tfrac{1+\not{v}}{2} \tag{4.36}$$

for further usage. Then the explicit effective matrix elements corresponding to eqs.(4.19-4.21) at the leading order are [22]

$$\begin{aligned}
\langle D(v')|\overline{h}_{v'}^{(c)}\gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle &= \xi(w)\text{Tr}\left(\gamma^5\frac{1+\not{v}'}{2}\gamma^\mu\frac{1+\not{v}}{2}\gamma^5\right) \\
&= \xi(w)(v^\mu+v'^\mu), \qquad (4.37)\\
\langle D^*(v',\varepsilon)|\overline{h}_{v'}^{(c)}\gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle &= \xi(w)\text{Tr}\left(\gamma^\nu\varepsilon_\nu^*\frac{1+\not{v}'}{2}\gamma^\mu\frac{1+\not{v}}{2}\gamma^5\right) \\
&= i\xi(w)\epsilon^{\mu\nu\alpha\rho}\varepsilon_\nu^* v_\alpha v'_\rho, \qquad (4.38)\\
\langle D^*(v',\varepsilon)|\overline{h}_{v'}^{(c)}\gamma^\mu\gamma^5 h_v^{(b)}|\overline{B}(v)\rangle &= \xi(w)\text{Tr}\left(\gamma^\nu\varepsilon_\nu^*\frac{1+\not{v}'}{2}\gamma^\mu\gamma^5\frac{1+\not{v}}{2}\gamma^5\right) \\
&= \xi(w)(w+1)\varepsilon^{*\mu}-(\varepsilon^*\cdot v)v'^\mu \qquad (4.39)
\end{aligned}$$

Similarly the elastic scattering takes the form

$$\langle\overline{B}(v')|\overline{h}_{v'}^{(b)}\gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle = \xi(w)(v+v')^\mu \qquad (4.40)$$

By comparing eq.(4.40) to eq.(4.18), the form factor f_{el} is obtained.

$$\xi(w) = \lim_{m_B\to\infty} f_{el}(q^2) \qquad (4.41)$$

Further argument can be made about the nature of $\xi(w)$. Calculation of the following matrix element for the case $v=v'$ in the rest frame gives

$$\begin{aligned}
\langle M(v_r)|\overline{h}_{v_r}\gamma^\mu h_{v_r}|M(v_r)\rangle &= \xi(1)\text{Tr}\left(\overline{M}\gamma^\mu M\right) \\
&= 2\xi(1)v_r^\mu. \qquad (4.42)
\end{aligned}$$

The same result can be obtained without calculating any trace by pointing out that $\overline{h}_{v_r}\gamma^\mu h_{v_r}$ is a conserved current. Passing to the momentum space via Fourier transformation yields

$$\begin{aligned}
\langle M(v_r)|\overline{h}_{v_r}\gamma^\mu h_{v_r}|M(v_r)\rangle &= \langle M(v_r)|\int\frac{d^3x}{(2\pi)^3}\overline{h}_{v_r}\gamma^\mu h_{v_r}|M(v_r)\rangle \\
&= \frac{v_r^\mu}{(2\pi)^3}\langle M(v_r)|\int d^3x\overline{h}_{v_r}h_{v_r}|M(v_r)\rangle \qquad (4.43)
\end{aligned}$$

The fact that in the rest frame $\not{v}_r=\gamma^0$ turns the integral into the number operator. Since there is only one heavy quark, the charge is equal to 1. The above equation turns to

$$\begin{aligned}
\langle M(v_r)|\overline{h}_{v_r}\gamma^\mu h_{v_r}|M(v_r)\rangle &= v_r^\mu\langle M(v_r)|M(v_r)\rangle \\
&= 2v_r^\mu \qquad (4.44)
\end{aligned}$$

Comparing eqs.(4.42) and (4.44) concludes that Isgur-Wise function is normalized at $w = 1$

$$\xi(1) = \langle light\ cloud(v, s_q)|light\ cloud(v, s_q)\rangle = 1 \tag{4.45}$$

Light cloud was seeing the heavy quark as an external color source. When the heavy quark decays, if $v = v'$ light cloud would feel no difference. However if $v \neq v'$ then the light cloud will experience a sudden change in the velocity of the external source, leading to the change in the momentum of the light cloud q for [21]

$$q^2 \simeq \Lambda_{QCD}^2 (v - v')^2 \simeq \Lambda_{QCD}^2 (1 - v \cdot v') \tag{4.46}$$

The recoil energy of the released meson $\mathcal{M}'$ is

$$E = m_{\mathcal{M}'}(1 - v \cdot v'). \tag{4.47}$$

At the zero recoil point where the velocity does not change $v = v'$, the recoil energy turns out to be zero as expected.

4.3 Alternative Derivation

By using eq.(2.66), the eqs.(4.19-4.21) for low energies are modified to [39]:

$$\begin{aligned}
\langle D(v')|V^\mu|\overline{B}(v)\rangle &= \xi_+(w)(v+v')^\mu + \xi_-(w)(v-v')^\mu && (4.48)\\
\langle D^*(v',\varepsilon)|V^\mu|\overline{B}(v)\rangle &= \xi_V(w)\epsilon^{\mu\nu\alpha\rho}\varepsilon^*_\nu v_\alpha v'_\rho && (4.49)\\
\langle D^*(v',\varepsilon)|A^\mu|\overline{B}(v)\rangle &= \xi_{A_1}(w)(w+1)\varepsilon^{*\mu} \\
& \quad i\varepsilon^* \cdot v\left[\xi_{A_2}(w)v^\mu + \xi_{A_3}(w)v'^\mu\right]. && (4.50)
\end{aligned}$$

The relation between effective theory form factors ξ_i and the full theory form factors in eqs(4.22-4.24) is straight forward [39].

$$\begin{aligned}
F_1(q^2) &= \frac{m_B + m_D}{2}\left(\xi_+(w) - \frac{m_B - m_D}{m_B + m_D}\xi_-(w)\right) && (4.51)\\
F_0(q^2) &= \frac{m_B + m_D}{2}\frac{2m_B m_D}{(m_B + m_D)^2}(w+1) \\
&\quad \times\left[\xi_+ - \frac{m_B + m_D}{m_B - m_D}\frac{w-1}{w+1}\xi_-(w)\right] && (4.52)
\end{aligned}$$

$$
\begin{aligned}
V(q^2) &= \frac{m_B+m_D}{2}\xi_V(w) && (4.53)\\
A_1(q^2) &= \frac{m_B+m_{D^*}}{2}\frac{2m_Bm_{D^*}}{(m_B+m_{D^*})^2}(w+1)\xi_{A_1} && (4.54)\\
A_2(q^2) &= \frac{m_B+m_{D^*}}{2}\left[\xi_{A_3}(w)+\frac{m_{D^*}}{m_B}\xi_{A_2}(w)\right] && (4.55)\\
A_3(q^2) &= \frac{m_B+m_{D^*}}{2}\left[\frac{m_B}{(m_B+m_{D^*})^2}(w+1)\xi_{A_1}(w)\right.\\
&\quad \left.-\frac{(m_B-m_{D^*})\sqrt{m_Bm_{D^*}}}{2m_{D^*}}\left(\xi_{A_3}(w)+\frac{m_{D^*}}{m_B}\xi_{A_2}(w)\right)\right] && (4.56)\\
A_0(q^2) &= A_3(q^2)+\frac{q^2}{4\sqrt{m_Bm_{D^*}}}\sqrt{\frac{m_B}{m_{D^*}}}\left(\xi_{A_3}(w)-\frac{m_{D^*}}{m_B}\xi_{A_2}(w)\right) && (4.57)
\end{aligned}
$$

Heavy quark symmetry gives relations between these form factors. Due to eq.(2.20),

$$
(v-v')_\mu\overline{h}'_{v'}\gamma^\mu h_v = \overline{h}'_{v'}\not{v}h_v - \overline{h}'_{v'}\not{v}'h_v = \overline{h}'_{v'}h_v - \overline{h}'_{v'}h_v = 0. \qquad (4.58)
$$

Using this yields

$$
\begin{aligned}
(v-v')_\mu\langle D(v')|\overline{h}'_{v'}\gamma^\mu h_v|\overline{B}(v)\rangle &= (v-v')_\mu\left[\xi_+(w)(v+v')^\mu+\xi_-(w)(v-v')^\mu\right]\\
0 &= \xi_-(w)(2-w) && (4.59)
\end{aligned}
$$

Here w is less then 2. It follows from

$$
w = v\cdot v' = \frac{p_B^2+p_D^2-q^2}{2m_cm_b} \qquad (4.60)
$$

The maximum for value for w is obtained if the transferred momentum is zero.

$$
w_{max} = \frac{(m_B-m_D)^2}{2m_Bm_D}+1 \approx 1.6 \qquad (4.61)
$$

Hence

$$
\xi_-(w) = 0. \qquad (4.62)
$$

Transforming the spin of the heavy quark turns a pseudoscalar meson into vector. The pseudoscalar; $J^P = 0^-$, can only have the values $s = 0$ and $l = 0$ since the eigenvalues of parity and c-parity operators are $(-1)^{l+1}$ and $(-1)^{l+s}$ respectively. Then the pseudoscalar state is

$$
|P\rangle = \frac{1}{\sqrt{2}}\left(|Q\downarrow q\uparrow\rangle - |Q\uparrow q\downarrow\rangle\right) \qquad (4.63)
$$

If the third component of the heavy quark spin is acted on this state it yields [40]

$$\begin{aligned} S_Q^3\,|00\rangle = S_Q^3\,|P\rangle &= \frac{1}{\sqrt{2}} S_Q^3\left(|Q\downarrow q\uparrow\rangle - |Q\uparrow q\downarrow\rangle\right) \\ &= \left(-\frac{1}{2}\right)\frac{1}{\sqrt{2}}\left(|Q\downarrow q\uparrow\rangle + |Q\uparrow q\downarrow\rangle\right) \\ &= -\frac{1}{2}\,|V(\varepsilon_0)\rangle = -\frac{1}{2}\,|10\rangle \end{aligned} \tag{4.64}$$

where ε_0 stands for longitudal polarization. Similarly

$$S_Q^3\,|V(\varepsilon_0)\rangle = -\frac{1}{2}\,|P\rangle \tag{4.65}$$

By using these relations eq.(4.48) can be related to eq.(4.50).

$$\begin{aligned} \langle D(v')|V^\mu|\overline{B}(v)\rangle &= 4\,\langle D(v')|\left(S_Q^3\right)^2 V^\mu|\overline{B}(v)\rangle \\ &= -2\,\langle D^*(v',\varepsilon_0)|\left[S_Q^3, V^\mu\right]|\overline{B}(v)\rangle \end{aligned} \tag{4.66}$$

where the fact that S_Q^3 is defined for each field with velocity v' is used. The commutation can be calculated by using the explicit form of S_Q^3

$$S_Q^3 = \frac{1}{2}\int d^3x : h_v(x)^\dagger\gamma^5\gamma^0\gamma^3 h_v(x) : \tag{4.67}$$

The calculation of eq.(4.66) is done in Appendix E, and it yields,

$$\begin{aligned} \langle D(v')|V^3|\overline{B}(v)\rangle &= \langle D^*(v,\varepsilon_0))|A^0|\overline{B}(v)\rangle \\ \langle D(v')|V^0|\overline{B}(v)\rangle &= \langle D^*(v,\varepsilon_0))|A^3|\overline{B}(v)\rangle \\ \langle D(v')|V^1|\overline{B}(v)\rangle &= \langle D^*(v,\varepsilon_0))|iA^2|\overline{B}(v)\rangle \\ \langle D(v')|V^2|\overline{B}(v)\rangle &= \langle D^*(v,\varepsilon_0))|-iA^1|\overline{B}(v)\rangle \end{aligned} \tag{4.68}$$

Near zero recoil point; $w \approx 1$, all non-vanishing form factors can be expressed in terms of one. This calculation can easily be done choosing $v' = (1,0,0,0)$ and $\varepsilon_0 = (0,0,0,1)$. It yields

$$\begin{aligned} \xi(w) &= \xi_+(w) = \xi_{A_1}(w) = \xi_{A_3}(w) \\ \xi_{A_2} &= 0 \end{aligned} \tag{4.69}$$

This calculation also gives

$$\langle D^*(v',\varepsilon_0)|\overline{h}_{v'}^{(c)}\gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle = 0 \tag{4.70}$$

as $\langle D(v')|A^\mu|\overline{B}(v)\rangle = 0$ due to parity.

To move to eq.(4.24), the definition for $\pm$ polarization of a vector meson can be used.

$$|V(\varepsilon_+)\rangle = |\uparrow\uparrow\rangle = 2S_Q^3\,|\uparrow\uparrow\rangle \tag{4.71}$$

The following can be writted.

$$\begin{aligned}\langle D^*(v',\varepsilon_+)|V^\mu|\overline{B}(v)\rangle &= \langle D^*(v',\varepsilon_+)|2S_Q^3V^\mu|\overline{B}(v)\rangle \\ &= \langle D^*(v',\varepsilon_+)|2[S_Q^3,V^\mu]|\overline{B}(v)\rangle\end{aligned} \tag{4.72}$$

Similarly the following equalities are obtained

$$\begin{aligned}\langle D^*(v',\varepsilon_+)|V^3|\overline{B}(v)\rangle &= \langle D^*(v',\varepsilon_+)|A^0|\overline{B}(v)\rangle \\ \langle D^*(v',\varepsilon_+)|V^0|\overline{B}(v)\rangle &= \langle D^*(v',\varepsilon_+)|A^3|\overline{B}(v)\rangle \\ \langle D^*(v',\varepsilon_+)|V^1|\overline{B}(v)\rangle &= \langle D^*(v',\varepsilon_+)|iA^2|\overline{B}(v)\rangle \\ \langle D^*(v',\varepsilon_+)|V^2|\overline{B}(v)\rangle &= \langle D^*(v',\varepsilon_+)|-iA^1|\overline{B}(v)\rangle\end{aligned} \tag{4.73}$$

Using $\varepsilon_\pm = (0,1,\pm i,0)$ [41] gives

$$\xi_{A_1} = \xi_V \tag{4.74}$$

For $v = v'$ the vector current is conserved independent from the flavors of heavy quarks [42].

$$V^\mu = \overline{h'}_v\gamma^\mu h_v = \overline{h'}_v v^\mu h_v \tag{4.75}$$

where $\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}$ is used and

$$i\partial_\mu V^\mu = \overline{h'}_v iv\cdot Dh_v + \overline{h'}_v iv\cdot\overleftarrow{D}h_v = 0 \tag{4.76}$$

from the equation of motion. The charge obtained from the time component of this current generates flavor symmetry.

$$N = \frac{1}{(2\pi)^3}\int d^3xV^0 = \frac{1}{(2\pi)^3}\int d^3x\overline{h'}_v v^0 h_v \tag{4.77}$$

N is a 2×2 matrix since h_v and h'_v can be $h_v^{(b)}$ and $h_v^{(c)}$. The diagonal terms counts the number of heavy quarks and the off-diagonal terms transforms the flavor.

$$\frac{1}{(2\pi)^3}\langle D(v)|\int d^3x\overline{h'}_v v^0 h_v|\overline{B}\rangle = \langle D(v)|D(v)\rangle = 2v^0 \tag{4.78}$$

Comparing the result with eq.(4.48) gives

$$\xi_+(1) = 1 \tag{4.79}$$

4.4 $1/m_Q$ Corrections

So far the calculations had been done in the leading order. Perturbative corrections arising from heavy quark masses are calculated systematically in HQET. Using the definition for the field H_v in eq.(2.35), the heavy quark field can be corrected as,

$$\begin{aligned} h_v(x) \rightarrow \quad & \left[1 + \frac{1}{iv \cdot D + 2m_Q} i\not{D}_\perp\right] h_v(x) \\ & = \left[1 + \frac{i\not{D}_\perp}{2m_Q} + ...\right] h_v(x) \end{aligned} \tag{4.80}$$

This leads to a modification to the currents up to the second order

$$\overline{h}_{v'}^{(c)} \Gamma_\mu h_v^{(b)} \rightarrow \overline{h}_{v'}^{(c)} \Gamma_\mu h_v^{(b)} - \frac{1}{2m_c} i\overline{h}_{v'}^{(c)} \overleftarrow{\not{D}}_\perp \Gamma_\mu h_v^{(b)} - \frac{1}{2m_b} i\overline{h}_{v'}^{(c)} \Gamma_\mu \overrightarrow{\not{D}}_\perp h_v^{(b)} \tag{4.81}$$

with [22]

$$\begin{aligned} \langle \mathcal{D}(v')|\overline{h}_{v'}^{(c)} i\overleftarrow{\not{D}}_\perp \Gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle &\sim \text{Tr}(\mathcal{D}\gamma^\nu \Gamma^\mu \overline{B}\chi_\nu(v,v')), \\ \langle \mathcal{D}(v')|\overline{h}_{v'}^{(c)} \Gamma^\mu i\not{D}_\perp h_v^{(b)}|\overline{B}(v)\rangle &\sim \text{Tr}(\mathcal{D}\Gamma^\mu \gamma^\nu \overline{B}\overline{\chi}_\nu(v',v)) \end{aligned} \tag{4.82}$$

Where χ_ν is the most general form factor with relevant transformation properties. Note that χ_ν and its Dirac conjugate have their v and v' dependence switched.
Considering eq.(4.82), corrections from the b and c fields involves the same structure therefore calculating one of them at length would suffice. Let us calculate the contribution from the second term of eq.(4.82) to the matrix elements [43].

$$\begin{aligned} & \langle D(v')|\overline{h}_{v'}^{(c)} i\overleftarrow{\not{D}}_\perp \Gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle \\ = \; & \langle D(v)|\overline{h}_{v'}^{(c)} i(\overleftarrow{\not{D}} - \not{v}' v' \cdot \overleftarrow{D})\Gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle \\ = \; & \text{Tr}\left[\gamma_5 \frac{1+\not{v}'}{2}\gamma^\nu \Gamma^\mu \frac{1+\not{v}}{2}\gamma_5 (C_1 v_\nu + C_2 v'_\nu + C_3 \gamma_\nu)\right] \\ & -\text{Tr}\left[\gamma^5 \frac{1+\not{v}'}{2} v'_\rho \gamma^\rho \Gamma^\mu \frac{1+\not{v}}{2}\gamma^5 (C_1 w + C_2 + C_3 v'_\nu \gamma^\nu)\right] \\ = \; & (v_b - v_c)^\mu (C_1(w+1) - 3C_3) \end{aligned} \tag{4.83}$$

Note that instead of $\overleftarrow{\not{D}}_\perp$ there were $\overleftarrow{D}_\nu$ the trace would have the same constants C_i resulting from the derivative. This property leads to further information regarding these constants as follows [44]. Let us begin with the matrix element

$$\begin{aligned}
& \langle D(v')|\overline{h}_{v'}^{(c)} i\overleftarrow{D}^{\nu} \Gamma^{\mu} h_v^{(b)}|\overline{B}(v)\rangle \\
= & \ \mathrm{Tr}\left[\gamma_5 \frac{1+\not{v}'}{2}\Gamma^{\mu}\frac{1+\not{v}}{2}\gamma_5(C_3 v_\nu + C_2 v'_\nu + C_3\gamma_\nu)\right] \\
= & \ \{(v+v')^{\mu}(C_1 v + C_2 v')^{\nu} \\
& -C_3\left[\eta^{\mu\nu}(1-w) + v^{\nu}v'^{\mu} + v^{\mu}v'^{\nu}\right]\}
\end{aligned} \tag{4.84}$$

Details of this calculation is given in Appendix F. Multiplying both sides with v'_ν and using the equations of motion, eq.(2.9), gives

$$-m_c\,\langle D(v')|\overline{h}_{v'}^{(c)}\not{v}'\Gamma^{\mu}h_v^{(b)}|\overline{B}(v)\rangle = (C_1 w + C_2 - C_3)(v+v')^{\mu} \tag{4.85}$$

Since $\overline{h}_{v'}^{(c)}\not{v}' = \overline{h}_{v'}^{(c)}$, eq.(4.85) is simplified to

$$-m_c\,\langle D(v')|\overline{h}_{v'}^{(c)}\Gamma^{\mu}h_v^{(b)}|\overline{B}(v)\rangle = (C_1 w + C_2 - C_3)(v+v')^{\mu} \tag{4.86}$$

Comparing it with eq.(4.37) gives

$$C_1 w + C_2 - C_3 = -m_c\xi(w) \tag{4.87}$$

The calculation of the matrix element $\langle D(v')|\overline{h}_{v'}^{(c)}\Gamma^{\mu}i\vec{D}^{\nu}h_v^{(b)}|\overline{B}(v)\rangle$ is likewise and yields similar result. These two terms can be summed up in the form

$$\begin{aligned}
& \langle D(v')|i\partial^{\nu}(\overline{h}'^{(c)}\Gamma^{\mu}h_v^{(b)})|\overline{B}(v)\rangle \\
= & \ \langle D(v')|\overline{h}_{v'}^{(c)}\Gamma^{\mu}i\vec{D}^{\nu}h_v^{(b)}|\overline{B}(v)\rangle + \langle D(v')|\overline{h}_{v'}^{(c)}i\overleftarrow{D}^{\nu}\Gamma^{\mu}h_v^{(b)}|\overline{B}(v)\rangle
\end{aligned} \tag{4.88}$$

But upto $\mathcal{O}\left(\frac{1}{m_Q}\right)$ the following relation can be written.

$$\begin{aligned}
\langle D(v')|\partial^{\nu}(\overline{h}_{v'}^{(c)}\Gamma^{\mu}h_v^{(b)})|\overline{B}(v)\rangle &= (P_b - P_c)^{\nu}\,\langle D(v')|\overline{h}_{v'}^{(c)}\Gamma^{\mu}h_v^{(b)}|\overline{B}(v)\rangle \\
&= (m_b v - m_c v')^{\nu}(v+v')^{\mu}\xi(w)
\end{aligned} \tag{4.89}$$

Hence from eq.(4.88) it is obtained that

$$C_1 w + C_2 - C_3 = (m_B - m_b)\xi(w) - w m_D \xi(w) \tag{4.90}$$

Comparing this equation with eq.(4.87) gives

$$C_1(w+1) - C_3 = \overline{\Lambda}\xi(w) \tag{4.91}$$

where $\overline{\Lambda} = m_B - m_b = m_D - m_c$.
If eq.(4.91) is inserted in eq.(4.83),the coefficient C_3 can be removed.

$$\langle D(v')|\overline{h}_{v'}^{(c)} i\overleftarrow{\not{D}}_{\perp}\Gamma^{\mu}h_{v}^{(b)}|\overline{B}(v)\rangle = (v_b - v_c)^{\mu}(2C_1(w+1) - 3\overline{\Lambda}\xi(w)) \tag{4.92}$$

Second source of correction is the first order corrections to Lagrangian. In the time ordered product two terms occur at $1/m_Q$. Their appearance is explained in chapter 2.2.

$$\begin{aligned} \langle D(v')|\overline{c}\Gamma^{\mu}b|\overline{B}(v)\rangle &= ... + \frac{i}{2m_c}\langle D(v')|\int d^4xT\left\{\mathcal{L}_1(x), J^{\mu}(0)\right\}|\overline{B}(v)\rangle \\ = \quad & ... + \frac{i}{2m_c}\langle D(v')|\int d^4xT\left\{\left(\overline{h}_{v'}^{(c)}(iD_{\perp})^2h_{v'}^{(c)}\right)(x), \left(\overline{h}_{v'}^{(c)}\Gamma^{\mu}h_{v}^{(b)}\right)(0)\right\}|\overline{B}(v)\rangle \\ + \quad & \frac{i}{2m_c}\langle D(v')|\int d^4xT\left\{\left(\overline{h}_{v'}^{(c)}(g_s\sigma_{\nu\rho}G^{\nu\rho})h_{v'}^{(c)}\right)(x), \left(\overline{h}_{v'}^{(c)}\Gamma^{\mu}h_{v}^{(b)}\right)(0)\right\}|\overline{B}(v)\rangle \end{aligned} \tag{4.93}$$

This yields the following trace

$$\begin{aligned} & \frac{i}{2m_c}\langle D(v')|\int d^4xT\left\{\mathcal{L}_1(x), J^{\mu}(0)\right\}|\overline{B}(v)\rangle \\ = \quad & \frac{1}{2m_c}\left[\mathrm{Tr}\left(\gamma^5\frac{1+\not{v}'}{2}\Gamma^{\mu}\frac{1+\not{v}}{2}\gamma^5\zeta(w)\right)\right. \\ + \quad & \left.\frac{i}{2}\mathrm{Tr}\left(\gamma^5\frac{1+\not{v}'}{2}\sigma_{\nu\rho}\frac{1+\not{v}'}{2}\Gamma^{\mu}\frac{1+\not{v}}{2}\gamma^5\Xi^{\nu\rho}\right)\right]. \end{aligned} \tag{4.94}$$

This requires some explaining. ζ is a scalar function since $D_{\perp}$ does not contain any gamma matrix. For the second term, an extra $\frac{1+\not{v}'}{2}$ is obtained, due to $\overline{h}_{v'}^{(c)}$ and $h_{v'}^{(c)}$ acting on Γ^{μ} and $\sigma_{\nu\rho}$ from left and right. $\Xi^{\nu\rho}$ is a tensor, with two possible terms that does not vanish [45]

$$\Xi^{\nu\rho} = C_4v^{\nu}\gamma^{\rho} + C_5[\gamma^{\nu}, \gamma^{\rho}]. \tag{4.95}$$

Other possible terms such as $v'^{\nu}\gamma^{\rho}$ vanishes since $\frac{1+\not{v}'}{2}\gamma^{\rho}\frac{1+\not{v}'}{2} = \frac{1+\not{v}'}{2}v'^{\rho}\frac{1+\not{v}'}{2}$. Calculating the traces yields

$$\begin{aligned} & \frac{i}{2m_c}\langle D(v')|\int d^4xT\left\{\mathcal{L}_1(x), J^{\mu}(0)\right\}|\overline{B}(v)\rangle \\ = \quad & \frac{1}{2m_c}[C_4(w-1) + 3C_5](v+v')^{\mu} \end{aligned} \tag{4.96}$$

in the first order, this leads a total correction of

$$\langle D(v')|\bar{c}\Gamma^\mu b|\overline{B}(v)\rangle = ... + \frac{1}{2m_c}[(\zeta + C_4(w-1) + 3C_5)(v+v')^\mu + (2C_1(w+1) - 3\overline{\Lambda}\xi(w))(v-v')^\mu] \tag{4.97}$$

Corrections for the $\overline{B} \to D^* l\overline{\nu}$ decay calculation is similar. In fact the corrections from the current; the matrix element $\langle D^*(v',\varepsilon)|\overline{h}_{v'}^{(c)} i\overleftarrow{\not{D}}_\perp \Gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle$ has the same constants C_i as before and eq.(4.91) is still valid. Two elements are obtained

$$\langle D^*(v',\varepsilon)|\overline{h}_{v'}^{(c)} i\overleftarrow{\not{D}}_\perp \gamma^\mu h_v^{(b)}|\overline{B}(v)\rangle = \frac{1}{2m_c}\overline{\Lambda}\xi(w) i\varepsilon^*_\nu v'_\sigma v_\rho \epsilon^{\nu\sigma\rho\mu}, \tag{4.98}$$

$$\begin{aligned}&\langle D^*(v',\varepsilon)|\overline{h}_{v'}^{(c)} i\overleftarrow{\not{D}}_\perp \gamma^\mu\gamma^5 h_v^{(b)}|\overline{B}(v)\rangle \\ =&\frac{1}{2m_c}\left[\xi(w)\varepsilon^\mu(w-1)\overline{\Lambda} - v\cdot\varepsilon^* v'^\mu(\xi(w)\overline{\Lambda} - 2C_1) + 2v\cdot\varepsilon v^\mu C_1\right].\end{aligned} \tag{4.99}$$

Corrections from the time ordered product comes from the following terms

$$\begin{aligned}&\frac{i}{2m_c}\langle D^*(v',\varepsilon)|\int d^4x T\left\{\mathcal{L}_1(x), J^\mu(0)\right\}|\overline{B}(v)\rangle \\ =&\frac{i}{2m_c}\langle D^*(v',\varepsilon)|\int d^4x T\left\{\left(\overline{h}_{v'}^{(c)}(iD_\perp)^2 h_{v'}^{(c)}\right)(x), \left(\overline{h}_{v'}^{(c)}\Gamma^\mu h_v^{(b)}\right)(0)\right\}|\overline{B}(v)\rangle \\ +&\frac{i}{2m_c}\langle D^*(v',\varepsilon)|\int d^4x T\left\{\left(\overline{h}_{v'}^{(c)}(g_s\sigma_{\nu\rho}G^{\nu\rho})h_{v'}^{(c)}\right)(x), \left(\overline{h}_{v'}^{(c)}\Gamma^\mu h_v^{(b)}\right)(0)\right\}|\overline{B}(v)\rangle\end{aligned} \tag{4.100}$$

For the vector current part of J^μ it takes the form

$$\begin{aligned}&\frac{1}{2m_c}\left[\mathrm{Tr}\left(\varepsilon^*_\nu\gamma^\nu\frac{1+\not{v}'}{2}\gamma^\mu\frac{1+\not{v}}{2}\gamma^5\zeta(w)\right)\right. \\ &+ \left.\frac{i}{2}\mathrm{Tr}\left(\varepsilon^*_\nu\gamma^\nu\frac{1+\not{v}'}{2}\sigma_{\nu\rho}\frac{1+\not{v}'}{2}\gamma^\mu\frac{1+\not{v}}{2}\gamma^5\Xi^{\nu\rho}\right)\right]\end{aligned} \tag{4.101}$$

$$= \frac{1}{2m_c}(\zeta - C_5) i\varepsilon^*_\nu v'_\sigma v_\rho \epsilon^{\nu\sigma\rho\mu}, \tag{4.102}$$

and for the axial current part the correction is

$$\begin{aligned}&\frac{1}{2m_c}\left[\mathrm{Tr}\left(\varepsilon^*_\nu\gamma^\nu\frac{1+\not{v}'}{2}\gamma^\mu\gamma^5\frac{1+\not{v}}{2}\gamma^5\zeta(w)\right)\right. \\ &+ \left.\frac{i}{2}\mathrm{Tr}\left(\varepsilon^*_\nu\gamma^\nu\frac{1+\not{v}'}{2}\sigma_{\nu\rho}\frac{1+\not{v}'}{2}\gamma^\mu\gamma^5\frac{1+\not{v}}{2}\gamma^5\Xi^{\nu\rho}\right)\right]\end{aligned} \tag{4.103}$$

$$\begin{aligned}= &\frac{1}{2m_c}[(1+w)(\zeta - C_5)\varepsilon^{*\mu} - v\cdot\varepsilon^* v'^\mu(\zeta + C_4 - C_5) \\ &+ v\cdot\varepsilon^* v^\mu C_4]\end{aligned} \tag{4.104}$$

The total correction to vector current induced decays become

$$\frac{1}{2m_c}(\overline{\Lambda}\xi(w) + \zeta - C_5)i\varepsilon^*_\nu v'_\sigma v_\rho \epsilon^{\nu\sigma\rho\mu} \tag{4.105}$$

and to axial current induced decays become

$$\begin{aligned} &\frac{1}{2m_c}[\varepsilon^{*\mu}(((w-1)\overline{\Lambda}\xi(w) + (1+w)(\zeta - C_5)) \\ &-v\cdot\varepsilon^* v'^\mu(\xi(w)\overline{\Lambda} - 2C_1 + \zeta + C_4 - C_5) + v\cdot\varepsilon^* v^\mu(2C_1 + C_4)] \end{aligned} \tag{4.106}$$

Luke's theorem states that at zero recoil point all $1/m_Q$ contributions vanish [45]. The proof follows from the normalization at this point. Decays with flavor conserving currents are starting point of the argument. Due to the flavor symmetry, the structure with same unknown functions are obtained.

$$\langle D(v')|\bar{c}\gamma^\mu c|D(v)\rangle = \left[\xi + \frac{1}{m_c}(\zeta + C_4(w-1) + 3C_5)\right](v+v')^\mu, \tag{4.107}$$

$$\begin{aligned} &\langle D^*(v',\varepsilon)|\bar{c}\gamma^\mu c|D(v)\rangle = \left[\xi\left(1+\frac{2\overline{\Lambda}}{m_c}\right)\right. \\ &\left.+\frac{1}{2m_c}(2\zeta + 2C_1 - (w-1)C_4 - 2C_5)\right] i\varepsilon^*_\nu v'_\sigma v_\rho \epsilon^{\nu\sigma\rho\mu}, \end{aligned} \tag{4.108}$$

$$\begin{aligned} &\langle D^*(v',\varepsilon)|\bar{c}\gamma^\mu\gamma^5 c|D(v)\rangle = \left\{\varepsilon^{*\mu}\left[\xi\left(1+w-2(1-w)\frac{\overline{\Lambda}}{2m_c}\right)\right.\right. \\ &\left.+\frac{1}{2m_c}((1-w)(2C_1 - C_4) + 2(w+1)(C_4+\zeta)\right] \\ &\left.v\cdot\varepsilon^* v'^\mu\left[\xi(w)\left(1+\frac{2\overline{\Lambda}}{m_c}\right) + \frac{1}{2m_c}(2\zeta - 2C_1(2+w) + wC_4 + 2C_5)\right]\right\}, \end{aligned} \tag{4.109}$$

$$\begin{aligned} &\langle D^*(v',\varepsilon')|\bar{c}\gamma^\mu c|D^*(v,\varepsilon)\rangle = \left\{\left(\xi(w) + \frac{1}{m_c}(\zeta - C_5)\right)\varepsilon\cdot\varepsilon'^*(v+v')^\mu\right. \\ &+\frac{1}{2m_c}(2C_1 - C_4)v\cdot\varepsilon'^* v'\cdot\varepsilon(v+v')^\mu \\ &-\left[\xi(w) + \frac{1}{2m_c}(2\zeta + 2(1+w)C_1 + (1-w)C_4 - 2C_5)\right] v'\cdot\varepsilon\varepsilon'^{*\mu} \\ &-\left[\xi(w) + \frac{1}{2m_c}(2\zeta + 2(1+w)C_1 + (1-w)C_4 - 2C_5)\right] v\cdot\varepsilon'^*\varepsilon^\mu. \end{aligned} \tag{4.110}$$

Note that when $w = 1$, all terms except the ones containing ζ and C_5 functions are zero, due to $v\cdot\varepsilon = v\cdot\varepsilon' = 0$. The remainings are

$$\langle D(v)|\bar{c}\gamma^0 c|D(v)\rangle = 2m_D v^0\left(1 + \frac{1}{2m_c}(\zeta(1) + 3C_5(1))\right) \tag{4.111}$$

and

$$\langle D^*(v,\varepsilon')|\bar{c}\gamma^0 c|D^*(v,\varepsilon)\rangle = 2v^0\left(1+\frac{1}{2m_c}(2\zeta(1)-2C_5(1))\right) \tag{4.112}$$

Now recall that the charges correspond to number of c quarks

$$\langle D(v)|\bar{c}\gamma^0 c|D(v)\rangle = 2v^0 \quad \langle D^*(v,\varepsilon')|\bar{c}\gamma^0 c|D^*(v,\varepsilon)\rangle = 2v^0 \tag{4.113}$$

Leaving $\zeta(1)$ and $C_5(1)$ with the only possible value

$$\zeta(1) = C_5(1) = 0 \tag{4.114}$$

Thus the conclusion is that all $1/m_Q$ corrections vanish at zero recoil point.

4.5 Determination of $|V_{cb}|$

The differential semileptonic decay rates for $B \to D^* l\overline{\nu}$ is [29]

$$\begin{aligned}\frac{d\Gamma(\overline{B}\to D^* l\overline{\nu})}{dw} &= \frac{G_F^2}{48\pi^3}|V_{cb}|^2(m_B-m_{D^*})^2 m_{D^*}^3\sqrt{w^2-1}(w+1)^2 \\ &\times\left[1+\frac{4w(m_B^2-2wm_Bm_{D^*}+m_{D^*}^2)}{(w+1)(m_B-m_{D^*})^2}\right]\mathcal{F}_{D^*}^2(w)\end{aligned} \tag{4.115}$$

where $\mathcal{F}_{D^*}$ is hadronic form factors that are perturbatively corrected both in Λ_{QCD}/m_Q and matching α_s. Determination of V_{cb} is as follows. The product $|V_{cb}|\mathcal{F}_{D^*}(w)$ is measured. Then the value of V_{cb} is extracted by extrapolating the data to zero recoil point. This point allows the calculation of $\mathcal{F}_{D^*}(w)$ around $\mathcal{F}_{D^*}(1)$ with small error, this is possible since the range of w accessible in these decays is; $1 < w < 1.5$. The extrapolation is with an expansion around $w = 1$ [29].

$$\mathcal{F}_{D^*}(w) = \mathcal{F}_{D^*}(1)[1-\varrho^2(w-1)+\mathcal{O}(w-1)^2] \tag{4.116}$$

where ϱ is treated as a fit parameter. From HQET the structure of the form factor $\mathcal{F}_{D^*}(1)$ is found as

$$\begin{aligned}\mathcal{F}_{D^*}(1) &= \eta_A\left(1+0.\frac{\Lambda_{QCD}}{m_Q}+c_2\frac{\Lambda_{QCD}^2}{m_Q^2}+...\right) \\ &= \eta_A+0+\delta_{1/m^2}\end{aligned} \tag{4.117}$$

where the first term is $\xi(1)$ and there is no contribution from the second term due to Luke's theorem. η_A is the matching to the axial current [22, 46]

$$\eta_A = 1+\frac{\alpha(\mu)}{\pi}\left(\frac{m_b+m_c}{m_b-m_c}\ln\frac{m_b}{m_c}-\frac{8}{3}\right) \approx 0.960 \pm 0.007. \tag{4.118}$$

With the δ_{1/m^2} correction; calculated in [47];

$$\delta_{1/m^2} = (5.5 \pm 2.5)\% \tag{4.119}$$

the result is

$$\mathcal{F}_{D^*}(1) = 0.93 \pm 0.03. \tag{4.120}$$

Combining these results with the measurements at the $\Upsilon(4s)$ resonance from CLEO Collaboration [48];

$$\Gamma(\overline{B} \to D^* l\overline{\nu}) = [29.9 \pm 1.9(stat.) \pm 2.7(sys.) \pm 2.0(lifetime)]ns^{-1} \tag{4.121}$$

and the weighted average of data

$$\begin{aligned} |V_{cb}|\mathcal{F}_{D^*}(1) &= 0.0351 \pm 0.0019 \pm 0.0018 \pm 0.0008 \\ \hat{\varrho}^2 &= 0.084 \pm 0.12 \pm 0.08 \end{aligned} \tag{4.122}$$

CLEO Collabration determined

$$V_{cb} = 0.0362 \pm 0.0019 \pm 0.0020 \pm 0.0014 \tag{4.123}$$

where the second error is due to uncertainity in B lifetime and the third error is due to uncertainity in $\mathcal{F}(1)$.

CHAPTER 5

CONCLUSION

In this thesis heavy quark effective theory is analyzed in details. The main properties of QCD are given in the first chapter. The non-perturbative region and why there is no solution to QCD is mentioned. An effective field theory matching to the physical picture of a heavy meson is also suggested as a mean of solution for a specific case.
In the second chapter the infinite heavy quark mass limit in the QCD Lagrangian is taken to obtain the heavy quark effective theory Lagrangian. The main property that is used is in the infinite mass limit, heavy quarks velocity becomes a constant of motion. The heavy quark fields are separated into two modes via velocity projection operator. The mode with mass $2m_Q$ is integrated out, leaving the massless mode as the effective heavy quark field. However in the infinite mass limit these field were still oscillating with infinite frequency. The fields were redefined to constrain their momentum to a finite value. The resulting Lagrangian with these fields show that the kinetic energy arising from the residual motion of the heavy quark and the interaction between light degrees of freedom and heavy quarks spin is suppressed by the heavy quark's mass.
The chapter continues with noting the arbitrarines of the heavy quark velocity chosen while constructing the effective field theory. The effective theory is shown be invariant under choosing another arbitrary four velocity. Then the effective meson states are defined in a manner that all mass dependence are explicitly shown. The normalization conditions are derived for these states. Finally, instead of using the transformation properties of individual states in a spin doublet, a way to transform the doublet as a whole is found.
In the third chapter, as a first application of the theory, the form of meson masses

constructed with the tools provided by HQET. The contribution of kinetic energy of the residual motion and light degrees of freedom- heavy quark spin interaction's contribution to mass were predicted to be suppressed by heavy quark mass. For B and D mesons, the mass difference between s flavored and d flavored B and D mesons are predicted to be the same up to the first order corrections. Also HQET predicts the same mass difference between spin doublets of B and D meson up to the first order. The experimental result supports these predictions as the mass differences are found to be very small. This calculation can also be used to comment on how well HQET works for c quarks. In case of accepting b quark as a good approximation for a heavy quark, calculations with c quark yields a $11 - 14\%$ difference from the calculations with b quark.

In the fourth chapter, semileptonic B decays are studied. First the weak interaction is described with four fermion coupling. The decay rate of inclusive $B \to D$ decay is written. The non-local interaction is expanded in terms of local operators. The way to handle the local operators are important. The local operators are described in terms of tools defined by HQET. It is useful because their expectation values were predicted by the effective theory. In this way the form of decay rate for the inclusive decays in HQET is obtained.

The exclusive $B \to D$ decays are also examined in the fourth chapter. All form factors appearing in different matrix elements are shown to be equal in the infinite mass limit which are equal to Isgur-Wise function. The normalization condition for this function is shown to be at zero recoil limit. This derivation is done in two different ways. In the first way, matrix elements are written in terms of effective states. The same function ξ appeared in each trace. In the second derivation all form factors are related to each other in the infinite mass limit. It turned out they were either equal to each other or equal to zero.

The $1/m_Q$ correction to decay matrix elements are calculated. They received corrections from the fact that both effective fields and the Lagrangian received first order corrections. In the zero recoil limit the corrections are shown to be equal to zero due to Luke's theorem.

At the end of the chapter the form factor $\mathcal{F}_{D^*}(w)$ is expanded around $w = 1$. Luke's theorem showed to be useful in this process. Then V_{cb} is extracted by from the data of $|V_{cb}|\mathcal{F}_{D^*}$.

REFERENCES

[1] Matthias Neubert. Introduction to B physics. In *Proceedings, Summer School in Particle Physics*, pages 244–295, 2000.

[2] Peskin Me and DV Schroeder. *An introduction to quantum field theory*. Addison-Wesley, New York, 1995.

[3] Taizō Muta. *Foundations of quantum chromodynamics: an introduction to perturbative methods in gauge theories*, volume 78. World Scientific, 2010.

[4] Boris L Ioffe, Viktor Sergeevich Fadin, and Lev Nikolaevich Lipatov. *Quantum chromodynamics: Perturbative and nonperturbative aspects*, volume 30. Cambridge University Press, 2010.

[5] Stephan Narison. *QCD as a theory of hadrons: from partons to confinement*, volume 17. Cambridge University Press, 2004.

[6] K. A. Olive et al. Review of Particle Physics. *Chin. Phys.*, C38:090001, 2014.

[7] David J Griffiths. *Introduction to elementary particles*. Wiley, 2008.

[8] G. M. Prosperi, M. Raciti, and C. Simolo. On the running coupling constant in QCD. *Prog. Part. Nucl. Phys.*, 58:387–438, 2007.

[9] Matthew D Schwartz. *Quantum field theory and the standard model*. Cambridge University Press, 2014.

[10] Ashok Das. *Lectures on quantum field theory*, volume 4. World Scientific, 2008.

[11] Robert Mann. *An introduction to particle physics and the standard model*. CRC Press, 2011.

[12] Edward V Shuryak. *The QCD vacuum, hadrons and superdense matter*, volume 71. World Scientific, 2004.

[13] Walter Greiner, Stefan Schramm, and Eckart Stein. *Quantum chromodynamics*. Springer Science & Business Media, 2007.

[14] Thomas Appelquist and J. Carazzone. Infrared Singularities and Massive Fields. *Phys. Rev.*, D11:2856, 1975.

[15] Thomas Mannel. *Effective field theories in flavour physics*, volume 203. Springer, 2004.

[16] Howard Georgi. Effective field theory. *Annual review of nuclear and particle science*, 43(1):209–252, 1993.

[17] Ikaros I. Y. Bigi, Mikhail A. Shifman, and N. Uraltsev. Aspects of heavy quark theory. *Ann. Rev. Nucl. Part. Sci.*, 47:591–661, 1997.

[18] Benjamin Grinstein. Light quark, heavy quark systems. *Ann. Rev. Nucl. Part. Sci.*, 42:101–145, 1992.

[19] Matthias Neubert. Heavy quark effective theory. In *Nonperturbative particle theory and experimental tests. Proceedings, 20th Johns Hopkins Workshop on Current Problems in Particle Theory, Heidelberg, Germany, June 27-29, 1996*, 1996.

[20] Ikaros I. Y. Bigi, Yuri L. Dokshitzer, Valery A. Khoze, Johann H. Kuhn, and Peter M. Zerwas. Production and Decay Properties of Ultraheavy Quarks. *Phys. Lett.*, B181:157–163, 1986.

[21] Matthias Neubert. Heavy-quark symmetry. *Physics Reports*, 245(5):259–395, 1994.

[22] Aneesh V Manohar and Mark B Wise. *Heavy quark physics*, volume 10. Cambridge university press, 2007.

[23] Thomas Mannel. Review of heavy quark effective theory. In *Heavy quarks at fixed target. Proceedings, Workshop, St. Goar, Germany, October 3-6, 1996*, pages 107–129, 1996.

[24] D Espriu, A Pich, and Quark Mixing. *Advanced School on Electroweak Theory*. World Scientific, 1998.

[25] Walter Greiner and Joachim Reinhardt. *Field quantization*. Springer Science & Business Media, 2013.

[26] Miguel A Sanchis-Lozano. Heavy quark effective fields as operator-valued distributions. *arXiv preprint hep-ph/9710408*, 1997.

[27] Thomas Mannel, Winston Roberts, and Zbigniew Ryzak. A Derivation of the heavy quark effective Lagrangian from QCD. *Nucl. Phys.*, B368:204–217, 1992.

[28] Yue-Liang Wu. Large component QCD and theoretical framework of heavy quark effective field theory. *Int. J. Mod. Phys.*, A21:5743–5792, 2006.

[29] Fernando Cornet and Maria Jose Herrero. *Advanced School on Effective Theories: Almunecar, Granada, Spain: 26 June-1 July 1995*. World scientific, 1997.

[30] Michael E. Luke and Aneesh V. Manohar. Reparametrization invariance constraints on heavy particle effective field theories. *Phys. Lett.*, B286:348–354, 1992.

[31] Raman Sundrum. Reparameterization invariance to all orders in heavy quark effective theory. *Phys. Rev.*, D57:331–336, 1998.

[32] Adam F. Falk. The Heavy quark expansion of QCD. In *The Strong interaction, from hadrons to partons: Proceedings, 24th SLAC Summer Institute on Particle Physics (SSI 96), Stanford, Calif., 19-30 Aug 1996*, pages 43–66, 1996.

[33] F De Fazio. Two lectures on heavy quark effective theory and spectroscopy of heavy-light mesons. `http://www.fe.infn.it/cabeo_school/2012/proceedings/defazio.pdf`, 2012. [Online; accessed 2016-08-20].

[34] Mark B. Wise. Heavy quark physics: Course. In *Probing the standard model of particle interactions. Proceedings, Summer School in Theoretical Physics, NATO Advanced Study Institute, 68th session, Les Houches, France, July 28-September 5, 1997. Pt. 1, 2*, pages 1051–1089, 1997.

[35] Thomas Blake, Gaia Lanfranchi, and David M Straub. Rare b decays as tests of the standard model. *arXiv preprint arXiv:1606.00916*, 2016.

[36] Andrzej J. Buras. Weak Hamiltonian, CP violation and rare decays. In *Probing the standard model of particle interactions. Proceedings, Summer School in Theoretical Physics, NATO Advanced Study Institute, 68th session, Les Houches, France, July 28-September 5, 1997. Pt. 1, 2*, pages 281–539, 1998.

[37] M. Wirbel, B. Stech, and Manfred Bauer. Exclusive Semileptonic Decays of Heavy Mesons. *Z. Phys.*, C29:637, 1985.

[38] Adam F. Falk. Hadrons of arbitrary spin in the heavy quark effective theory. *Nucl. Phys.*, B378:79–94, 1992.

[39] Matthias Neubert and Volker Rieckert. New approach to the universal form-factors in decays of heavy mesons. *Nucl. Phys.*, B382:97–119, 1992.

[40] Nathan Isgur and Mark B. Wise. Weak transition form-factors between heavy mesons. *Phys. Lett.*, B237:527–530, 1990.

[41] Benjamin Grinstein. An Introduction to heavy mesons. In *Particles and fields. Proceedings, 6th Mexican School, Villahermosa, Tabasco, Mexico, October 3-7, 1994*, pages 122–184, 1995.

[42] Nathan Isgur and Mark B. Wise. Weak Decays of Heavy Mesons in the Static Quark Approximation. *Phys. Lett.*, B232:113–117, 1989.

[43] Michael E. Luke. Effects of subleading operators in the heavy quark effective theory. *Phys. Lett.*, B252:447–455, 1990.

[44] Faheem Hussain and George Thompson. An Introduction to the heavy quark effective theory. In *Proceedings, Summer School in High-energy physics and cosmology: Trieste, Italy, June 13-July 29, 1994*, pages 0045–115, 1994.

[45] Michael E Luke. Effects of subleading operators in the heavy quark effective theory. *Physics Letters B*, 252(3):447–455, 1990.

[46] Andrzej Czarnecki and Kirill Melnikov. Two loop QCD corrections to b —> c transitions at zero recoil: Analytical results. *Nucl. Phys.*, B505:65–83, 1997.

[47] Matthias Neubert. Theoretical update on the model independent determination of |V(cb)| using heavy quark symmetry. *Phys. Lett.*, B338:84–91, 1994.

[48] Barry Barish et al. Measurement of the anti-B —> D* lepton anti-neutrino branching fractions and |V(cb)|. *Phys. Rev.*, D51:1014–1033, 1995.

APPENDIX A

EFFECTIVE LAGRANGIAN

The second term in eq.(2.36) can be expanded as follows.

$$\overline{h}_v i\not{D}_\perp \frac{1}{iv\cdot D + 2m_Q} i\not{D}_\perp h_v = \frac{1}{2m_Q}\overline{h}_v i\not{D}_\perp \frac{1}{1+\frac{iv\cdot D}{2m_Q}} i\not{D}_\perp h_v \tag{A.1}$$

$1/m_Q$ expansion yields

$$\begin{aligned} &\frac{1}{2m_Q}\overline{h}_v i\not{D}_\perp \left(1+\frac{iv\cdot D}{2m_Q}+...\right) i\not{D}_\perp h_v \\ &= \frac{1}{2m_Q}\overline{h}_v (i\not{D}_\perp)^2 h_v + \mathcal{O}\left(\frac{1}{4m_Q^2}\right). \end{aligned} \tag{A.2}$$

The $1/m_Q$ order takes the form

$$-\frac{1}{2m_Q}\overline{h}_v (D_{\perp\mu}D_{\perp\nu}\gamma^\mu\gamma^\nu) h_v. \tag{A.3}$$

Since $D_\perp$ resides in the color space and the gamma matrices reside in spacetime, they commute. Now using the identity

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\sigma^{\mu\nu} \tag{A.4}$$

gives

$$-\frac{1}{2m_Q}\overline{h}_v (D_{\perp\mu}D_{\perp\nu}\eta^{\mu\nu} - iD_{\perp\mu}D_{\perp\nu}\sigma^{\mu\nu}) h_v. \tag{A.5}$$

For the second term the multiplication of covarian derivatives can be seperated into symmetric and anti-symmetric parts.

$$D_{\perp\mu}D_{\perp\nu} = \frac{1}{2}(\{D_{\perp\mu}, D_{\perp\nu}\} + [D_{\perp\mu}, D_{\perp\nu}]) \tag{A.6}$$

Hence the Lagrangian becomes

$$\mathcal{L}_Q = \overline{h}_v iv\cdot D h_v + \frac{1}{2m_Q}\overline{h}_v (iD_\perp)^2 h_v - \frac{1}{4m_Q}\overline{h}_v g_s \sigma^{\mu\nu} G_{\mu\nu} h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right) \tag{A.7}$$

APPENDIX B

REPARAMETRIZATION INVARIANCE

B.1 Reparametrization of the Effective Lagrangian

Keeping in mind that q is infinitesimal and

$$v \cdot q = \mathcal{O}\left(\frac{q^2}{m}\right), \tag{B.1}$$

The leading order in Lagrangian

$$\begin{aligned}
\mathcal{L}^{(0)}_{Q_w} &= \left[\overline{h}_v \left(1 + \frac{\not{q}}{2m_Q}\right) e^{-iq\cdot x}\right] \left[i(v + \frac{q}{m_Q}) \cdot D\right] \left[e^{iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right) h_v\right] \\
&= \left[\overline{h}_v \left(1 + \frac{\not{q}}{2m_Q}\right) e^{-iq\cdot x}\right] ie^{iq\cdot x}\left(v + \frac{q}{m_Q}\right) \cdot \left[iq\left(1 + \frac{\not{q}}{2m_Q}\right)\right. \\
&\quad \left. + \left(1 + \frac{\not{q}}{2m_Q}\right) D\right] h_v \\
&= \left[\overline{h}_v \left(1 + \frac{\not{q}}{2m_Q}\right)\right] i \left[iv \cdot q\left(1 + \frac{\not{q}}{2m_Q}\right) + \left(1 + \frac{\not{q}}{2m_Q}\right) v \cdot D\right. \\
&\quad \left. + \frac{iq^2}{m_Q}\left(1 + \frac{\not{q}}{2m_Q}\right) + \left(1 + \frac{\not{q}}{2m_Q}\right)\frac{q \cdot D}{m_Q}\right] h_v \\
&= \left[\overline{h}_v \left(1 + \frac{\not{q}}{2m_Q}\right)\right] \left[iv \cdot D + i\frac{q \cdot D}{m_Q}\right] h_v \\
&= \overline{h}_v iv \cdot D h_v + \overline{h}_v i \frac{q \cdot D}{m_Q} h_v
\end{aligned} \tag{B.2}$$

B.2 The First Order Terms

The kinetic term in the first order Lagrangian

$$\mathcal{L}^{(1)}_{Q_w,kin} = i^2 \left[\overline{h}_v \left(1 + \frac{\not{q}}{2m_Q}\right) e^{iq\cdot x}\right] (D_\perp)^2 \left[e^{-iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right) h_v\right] \tag{B.3}$$

Let us expand the covariant derivative term

$$
\begin{aligned}
(D_\perp)^2 &= \left(D - \left(v + \frac{q}{m_Q}\right)\left(\left(v + \frac{q}{m_Q}\right)\cdot D\right)\right)^2 \\
&= \left[D^\mu - \left(v^\mu + \frac{q^\mu}{m_Q}\right)\left(\left(v + \frac{q}{m_Q}\right)\cdot D\right)\right] \\
&\quad \cdot \left[D_\mu - \left(v_\mu + \frac{q_\mu}{m_Q}\right)\left(\left(v + \frac{q}{m_Q}\right)\cdot D\right)\right] \\
&= D^2 - 2(v\cdot D) - 2\frac{q\cdot D}{m_Q}(v\cdot D) + v^2(v\cdot D)^2 + 2\frac{q\cdot v}{m_Q}(v\cdot D)^2 \\
&\quad + \frac{q^2}{m_Q^2}(v\cdot D)^2 \\
&= D^2 - (v\cdot D)^2
\end{aligned} \tag{B.4}
$$

In the previous part, the following is already calculated.

$$
(v\cdot D)\left[e^{iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right)h_v\right] = e^{iq\cdot x}(v\cdot D)h_v \tag{B.5}
$$

and

$$
D_\mu\left[e^{iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right)h_v\right] = e^{iq\cdot x}\left[\left(1 + \frac{\not{q}}{2m_Q}\right)D_\mu + iq_\mu\left(1 + \frac{\not{q}}{2m_Q}\right)\right]h_v \tag{B.6}
$$

Thus

$$
\begin{aligned}
(v\cdot D)^2\left[e^{iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right)h_v\right] &= e^{iq\cdot x}\left[-iv\cdot q(v\cdot D) + (v\cdot D)^2\right]h_v \\
&= e^{iq\cdot x}(v\cdot D)^2 h_v
\end{aligned} \tag{B.7}
$$

and

$$
\begin{aligned}
D^2\left[e^{iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right)h_v\right] &= e^{iq\cdot x}\left[-q^2\left(1 + \frac{\not{q}}{2m_Q}\right)\right. \\
&\quad \left. + 2i\left(1 + \frac{\not{q}}{2m_Q}\right)q\cdot D + \left(1 + \frac{\not{q}}{2m_Q}\right)D^2\right]h_v \\
&= e^{iq\cdot x}\left[2iq\cdot D + D^2 + \frac{\not{q}}{2m_Q}D^2\right]h_v
\end{aligned} \tag{B.8}
$$

Hence

$$
\begin{aligned}
& i^2\left[\overline{h}_v\left(1 + \frac{\not{q}}{2m_Q}\right)e^{-iq\cdot x}\right](D_\perp)^2\left[e^{iq\cdot x}\left(1 + \frac{\not{q}}{2m_Q}\right)h_v\right] \\
&= i^2\left[\overline{h}_v\left(1 + \frac{\not{q}}{2m_Q}\right)\right]\left[D^2 - (v\cdot D)^2 + 2iq\cdot D + \frac{\not{q}}{2m_Q}D^2\right]h_v \\
&= i^2\overline{h}_v\left[D^2 - (v\cdot D)^2 - 2iq\cdot D\right]h_v + i^2\overline{h}_v\left[2\frac{\not{q}}{2m_Q}D^2\right. \\
&\quad \left. + \frac{\not{q}}{2m_Q}(v\cdot D)^2 - 2i\frac{\not{q}}{2m_Q}q\cdot D + \left(\frac{\not{q}}{2m_Q}\right)^2 D^2\right]h_v
\end{aligned} \tag{B.9}
$$

All of the second term is of high order. Thus leaving

$$i^2\overline{h}_v(D_\perp)^2 h_v - \overline{h}_v 2iq \cdot D h_v \tag{B.10}$$

The chromomagnetic term in the first order Lagrangian

$$\begin{aligned}
& \overline{h}_w g_s \sigma^{\mu\nu} G_{\mu\nu} h_w = -i\overline{h}_w \sigma^{\mu\nu}[D_\mu, D_\nu] h_w \\
= & -i\left[\overline{h}_v\left(1+\frac{\not{q}}{2m_Q}\right)e^{-iq\cdot x}\right]\sigma^{\mu\nu}D_\mu D_\nu\left[e^{iq\cdot x}\left(1+\frac{\not{q}}{2m_Q}\right)h_v\right] + \mu \leftrightarrow \nu \\
= & -i\left[\overline{h}_v\left(1+\frac{\not{q}}{2m_Q}\right)e^{-iq\cdot x}\right]D_\mu e^{iq\cdot x} \\
& \times\left[iq_\nu\left(1+\frac{\not{q}}{2m_Q}\right)+\left(1+\frac{\not{q}}{2m_Q}\right)D_\nu\right]h_v + \mu \leftrightarrow \nu \\
= & -i\left[\overline{h}_v\left(1+\frac{\not{q}}{2m_Q}\right)\right]\left[-q_\mu q_\nu + i\left(1+\frac{\not{q}}{2m_Q}\right)q_\mu D_\nu + iq_\nu D_\mu\right. \\
& \left.+\left(1+\frac{\not{q}}{2m_Q}\right)D_\mu D_\nu\right]h_v + \mu \leftrightarrow \nu \\
= & -i\overline{h}_v\left[iq_\mu D_\nu + iq_\nu D_\mu + \left(1+\frac{\not{q}}{2m_Q}\right)D_\mu D_\nu\right]h_v + \mu \leftrightarrow \nu \\
= & -i\overline{h}_v \sigma^{\mu\nu}[D_\mu, D_\nu] h_v
\end{aligned} \tag{B.11}$$

APPENDIX C

COVARIANT REPRESENTATION OF STATES

The most general 4×4 matrix can be written is

$$H_v^{(Q)} = S\mathbb{1} + V_\mu \gamma^\mu + T_{\mu\nu}\sigma^{\mu\nu} + A_\mu \gamma^\mu \gamma^5 + P\gamma^5 \tag{C.1}$$

The condition $\not{v} H_v^{(Q)} = H_v^{(Q)}$ has to be implemented on this object. Trace of $H_v^{(Q)}$ is

$$\text{Tr}\left(H_v^{(Q)}\right) = 4S \tag{C.2}$$

This is equal to Tr $\left(\not{v} H_v^{(Q)}\right)$,

$$\text{Tr}\left(\not{v} H_v^{(Q)}\right) = v_\alpha \text{Tr}\left(\gamma^\alpha H_v^{(Q)}\right) = 4v_\mu V^\mu \tag{C.3}$$

This gives the first relation between two terms

$$S = v_\mu V^\mu \tag{C.4}$$

Next,

$$\text{Tr}\left(\gamma^\alpha H_v^{(Q)}\right) = 4V^\alpha \tag{C.5}$$

and

$$\begin{aligned} \text{Tr}\left(\gamma^\alpha H_v^{(Q)}\right) = \text{Tr}\left(\gamma^\alpha \not{v} H_v^{(Q)}\right) &= v_\beta T_{\mu\nu} \text{Tr}\left(\gamma^\alpha \gamma^\beta \sigma^{\mu\nu}\right) + S v_\beta \text{Tr}\left(\gamma^\alpha \gamma^\beta\right) \\ &= 16 v_\beta T^{\alpha\beta} + 4Sv^\alpha \end{aligned} \tag{C.6}$$

which gives

$$V^\alpha = 4v_\beta T^{\alpha\beta} + Sv^\alpha \tag{C.7}$$

Next,

$$\text{Tr}\left(\gamma^\alpha \gamma^5 H_v^{(Q)}\right) = -4A^\alpha \tag{C.8}$$

and

$$\begin{aligned}
& \text{Tr}\left(\gamma^{\alpha}\gamma^{5}H_{v}^{(Q)}\right) = \text{Tr}\left(\gamma^{\alpha}\gamma^{5}\not{v}H_{v}^{(Q)}\right) \\
= \; & v_{\beta}T_{\mu\nu}\text{Tr}\left(\gamma^{\alpha}\gamma^{5}\gamma^{\beta}\sigma^{\mu\nu}\right) + v_{\beta}P\text{Tr}\left(\gamma^{\alpha}\gamma^{5}\gamma^{\beta}\gamma^{5}\right) \\
= \; & 8iv_{\beta}T_{\mu\nu}\epsilon^{\alpha\beta\nu\mu} - 4v^{\alpha}P
\end{aligned} \tag{C.9}$$

gives the third relation

$$-A^{\alpha} = 2iv_{\beta}T_{\mu\nu}\epsilon^{\alpha\beta\nu\mu} - v^{\alpha}P \tag{C.10}$$

Next

$$\text{Tr}\left(\gamma^{5}H_{v}^{(Q)}\right) = 4P \tag{C.11}$$

then,

$$\text{Tr}\left(\gamma^{5}H_{v}^{(Q)}\right) = \text{Tr}\left(\gamma^{5}\not{v}H_{v}^{(Q)}\right) = v_{\alpha}A_{\mu}\text{Tr}\left(\gamma^{5}\gamma^{\alpha}\gamma^{\mu}\gamma^{5}\right) = 4v_{\mu}A^{\mu} \tag{C.12}$$

gives the relation

$$P = v_{\mu}A^{\mu} \tag{C.13}$$

Finally

$$\text{Tr}\left(\sigma^{\alpha\beta}H_{v}^{(Q)}\right) = T_{\mu\nu}\text{Tr}\left(\left[\gamma^{\alpha},\gamma^{\beta}\right]\left[\gamma^{\mu},\gamma^{\nu}\right]\right) = 16T^{\beta\alpha} \tag{C.14}$$

Here the fact that $T^{\mu\nu}$ is anti-symmetric is used. Similarly this expression is equal to

$$\begin{aligned}
\text{Tr}\left(\sigma^{\alpha\beta}\not{v}H_{v}^{(Q)}\right) &= v_{\delta}V_{\mu}\text{Tr}\left(\left[\gamma^{\alpha},\gamma^{\beta}\right]\gamma^{\delta}\gamma^{\mu}\right) + v_{\delta}A_{\mu}\text{Tr}\left(\left[\gamma^{\alpha},\gamma^{\beta}\right]\gamma^{\delta}\gamma^{\mu}\gamma^{5}\right) \\
&= 8\left[v^{\beta}V^{\alpha} - v^{\alpha}V^{\beta}\right] - 8iv_{\delta}A_{\mu}\epsilon^{\alpha\beta\delta\mu}
\end{aligned} \tag{C.15}$$

Hence the last relation is

$$T^{\beta\alpha} = \frac{1}{2}[(v^{\beta}V^{\alpha} - v^{\alpha}V^{\beta}) - iv_{\delta}A_{\mu}\epsilon^{\alpha\beta\delta\mu}] \tag{C.16}$$

All of the relations then can be summed up then.

$$S = v_{\mu}V^{\mu} \tag{C.17}$$

$$V^{\alpha} = 4v_{\beta}T^{\alpha\beta} + Sv^{\alpha} \tag{C.18}$$

$$-A^{\alpha} = 2iv_{\beta}T_{\mu\nu}\epsilon^{\alpha\beta\nu\mu} - v^{\alpha}P \tag{C.19}$$

$$P = v_{\mu}A^{\mu} \tag{C.20}$$

$$T^{\beta\alpha} = \frac{1}{2}[(v^{\beta}V^{\alpha} - v^{\alpha}V^{\beta}) - iv_{\delta}A_{\mu}\epsilon^{\alpha\beta\delta\mu}] \tag{C.21}$$

Now separating V_μ and A_μ into parallel and orthogonal parts gives

$$V_\mu = V_\mu^\perp + v_\mu S, \tag{C.22}$$

$$A_\mu = A_\mu^\perp + v_\mu P \tag{C.23}$$

with

$$v^\mu V_\mu^\perp = 0, \tag{C.24}$$

$$v^\mu A_\mu^\perp = 0. \tag{C.25}$$

The eq.(C.21) can be rewritten as

$$T^{\beta\alpha} = v^\beta V^{\alpha\perp} - \frac{1}{2} i v_\delta A_\mu^\perp \epsilon^{\alpha\beta\delta\mu} \tag{C.26}$$

Hence eq.(C.1) takes the form

$$\begin{aligned} H_v^{(Q)} &= S + (V_\mu^\perp + v_\mu S)\gamma^\mu + (A_\mu^\perp + v_\mu P)\gamma^\mu\gamma^5 + P\gamma^5 \\ &\quad + (2v_\mu V_\nu^\perp - i v^\alpha A^{\beta\perp}\epsilon_{\mu\nu\alpha\beta})\sigma^{\mu\nu} \\ &= (1+\not{v})S + (1+\not{v})P\gamma^5 + A_\mu^\perp\gamma^\mu\gamma^5 + V_\mu^\perp\gamma^\mu \\ &\quad + v_\mu V_\nu^\perp - \frac{1}{2} i v^\alpha A^{\beta\perp}\epsilon_{\mu\nu\alpha\beta}\sigma^{\mu\nu} \end{aligned} \tag{C.27}$$

Now using $\gamma^\nu\gamma^\mu = g^{\nu\mu} - \gamma^\mu\gamma^\nu$ to obtain

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu) = i(\gamma^\mu\gamma^\nu - g^{\nu\mu}) \tag{C.28}$$

This gives

$$\begin{aligned} H_v^{(Q)} &= (1+\not{v})S + (1+\not{v})P\gamma^5 + A_\mu^\perp\gamma^\mu\gamma^5 + V_\mu^\perp\gamma^\mu \\ &\quad + i\not{v}V_\mu^\perp\gamma^\mu + \not{v}A_\mu^\perp\gamma^\mu\gamma^5. \end{aligned} \tag{C.29}$$

Hence

$$H_v^{(Q)} = (1+\not{v})[(S + A_\mu^\perp\gamma^\mu\gamma^5) + (P\gamma^5 + V_\mu^\perp\gamma^\mu)] \tag{C.30}$$

The $(S + A_\mu^\perp\gamma^\mu\gamma^5)$ terms were not in eq.(2.72). However just like $P\gamma^5 + V_\mu^\perp\gamma^\mu$ they form a doublet of total angular momentum $J = 0, 1$ hence their appearance is expected. This can be seen from the following argument. The total angular momentum is given by $J = J_l \pm \frac{1}{2}$ and the angular momentum of the light degrees of freedom is given by $\vec{J_l} = \vec{l} + \vec{S_q}$. For $l = 0$ one obtains

$$l = 0, \quad J_l^p = S_l^p = \tfrac{1}{2}^- \tag{C.31}$$

Thus

$$J^p = J_l^p \pm \frac{1}{2} = 0^-, 1^- \tag{C.32}$$

They are the pseudoscalar and vector meson states. For $l = 1$,

$$l = 1, \quad J_l^p = 1 + S_l^p = \tfrac{1}{2}^+, \tfrac{3}{2}^+ \tag{C.33}$$

For $J_l^p = \frac{1}{2}^+$ case one finds

$$J^p = \frac{1}{2}^+ \pm \frac{1}{2} = 0^+, 1^+ \tag{C.34}$$

The claim was doublets are formed according to the value of J_l. Hence the ground state doublet is given by

$$H_v^{(Q)} = \frac{1 + \not{v}}{2}[P_{v,\mu}^{*(Q)}\gamma^\mu + iP_v^{(Q)}\gamma_5] \tag{C.35}$$

These fields are required to turn into each other under heavy quark spin transformation. The formal derivation is given in chapter 2.3.

APPENDIX D

OTHER LORENTZ INVARIANT TERMS IN FUNCTION X

There is one more term that that contains a different Lorentz invariant. That is $X_4 i\gamma^5$. However X_4 is found to be zero due to parity. Placing $P^2 = 1$ inside the corresponding matrix element yields

$$\begin{aligned}
& \text{Tr}\left(\overline{\mathcal{M}}(v')(\gamma^\mu - i\gamma^\mu\gamma^5)\mathcal{M}(v)X_4\left(i\gamma^5\right)\right) \\
= \; & \text{Tr}\left(\mathrm{P}\overline{\mathcal{M}}(v')(\gamma^\mu - i\gamma^\mu\gamma^5)\mathcal{M}(v)X_4\left(i\gamma^5\right)\mathrm{P}\right) \\
= \; & \text{Tr}\left(\overline{\mathcal{M}}'(v'_p)(\mathrm{P}\gamma^\mu\mathrm{P} - \mathrm{P}i\gamma^\mu\gamma^5\mathrm{P})\mathcal{M}'(v_p)X_4\left(\mathrm{P}i\gamma^5\mathrm{P}\right)\right) \\
= \; & \text{Tr}\left(\overline{\mathcal{M}}'(v'_p)(-\gamma^{\overline{\mu}} - i\gamma^{\overline{\mu}}\gamma^5)\mathcal{M}'(v_p)X_4\left(-i\gamma^5\right)\right) \\
= \; & \text{Tr}\left(\overline{\mathcal{M}}'(v'_p)(\gamma^{\overline{\mu}} + i\gamma^{\overline{\mu}}\gamma^5)\mathcal{M}'(v_p)X_4\left(i\gamma^5\right)\right)
\end{aligned} \tag{D.1}$$

Where $\mathcal{M}(v)$ stands for $M(v)$ or $M(v,\varepsilon)$. Anti-commuting P through $M(v)$ would yield a minus sign and while anti-commutation through $M(v,\varepsilon)$ would not. This effect is given by the convention $\mathcal{M}'$. The only way to have this equality is $X_4 = 0$. The argument goes same for products of γ^5 with velocities.

APPENDIX E

TURNING A PSEUDOSCALAR MESON INTO A VECTOR MESON

The commutation relation in eq.(4.66) can be calculated as follows.

$$\begin{aligned}
&\left[S_Q^3, V^\mu(0) - A^\mu(0)\right] \\
&= \left[\frac{1}{2}\int d^3x : h_{v'}^{(c)}(x)^\dagger \gamma^5\gamma^0\gamma^3 h_{v'}^{(c)}(x) :, \overline{h}_{v'}^{(c)}(0)\gamma^\mu h_v^{(b)}(0) - \overline{h}_{v'}^{(c)}(0)\gamma^\mu\gamma^5 h_v^{(b)}(0)\right]
\end{aligned} \tag{E.1}$$

First of all the current can get inside integral without a problem since it has no x dependence. Although there is a normal ordering in the definition in our case it has no implication as the positive and negative energy fields decouples. From eq.(2.10) the commution relation for fermion fields is given as

$$[\Psi_\alpha(x_0), \Psi_\beta(y_0)] = [\Psi_\alpha^\dagger(x_0), \Psi_\beta^\dagger(y_0)] = 0 \tag{E.2}$$

$$\left[\Psi_\alpha(x_0), \Psi_\beta^\dagger(y_0)\right] = \delta_\beta^\alpha \delta^3(x_0 - y_0) \tag{E.3}$$

Hence the commutation becomes

$$\left[S_Q^3, V^\mu(0) - A^\mu(0)\right] = \frac{1}{2}\int d^3x h_{v'}^{(c)}(x)^\dagger [\gamma^5\gamma^0\gamma^3, \gamma^0\gamma^\mu - \gamma^0\gamma^\mu\gamma^5] h_{v'}^{(c)}(0)\delta^3(x-0) \tag{E.4}$$

The result is; commutation turns the components of vector current in to axial vector components and vice versa. For different values of μ the result is

$$\begin{aligned}
\left[S_Q^3, V^3(0) - A^3(0)\right] &= A^0(0) - V^0(0) \\
\left[S_Q^3, V^0(0) - A^0(0)\right] &= A^3(0) - V^3(0) \\
\left[S_Q^3, V^1(0) - A^1(0)\right] &= i(A^2(0) - V^2(0)) \\
\left[S_Q^3, V^2(0) - A^2(0)\right] &= -i(A^1(0) - V^1(0))
\end{aligned} \tag{E.5}$$

APPENDIX F

1/M CORRECTIONS

The calculation of eq.(4.84) is as follows.

$$
\begin{aligned}
& \langle D(v)|\overline{h}'^{(c)} i(\overleftarrow{\not{D}} - \not{v}' v' \cdot \overleftarrow{D}) \Gamma^\mu h_v^{(b)} |\overline{B}(v)\rangle \\
= & \sqrt{m_B m_D} \left\{ \mathrm{Tr}\left[\gamma_5 \frac{1+\not{v}'}{2} \gamma^\nu \Gamma_\mu \frac{1+\not{v}}{2} \gamma_5 (C_1 v^\nu + C_2 v'^\nu + C_3 \gamma^\nu) \right] \right. \\
& \left. - \mathrm{Tr}\left[\gamma^5 \frac{1+\not{v}'}{2} v'_\rho \gamma^\rho \Gamma^\mu \frac{1+\not{v}}{2} \gamma^5 (C_1 w + C_2 + C_3 v'_\nu \gamma^\nu) \right] \right\} \\
= & \frac{1}{4} \left\{ [\mathrm{Tr}(\gamma^\nu \gamma^\mu a_\nu) + v'_\lambda \mathrm{Tr}(\gamma^\lambda \gamma^\nu \gamma^\mu a_\nu) - v_\rho \mathrm{Tr}(\gamma^\nu \gamma^\mu \gamma^\rho a_\nu) \right. \\
& + v'_\lambda v_\rho \mathrm{Tr}(\gamma^\lambda \gamma^\nu \gamma^\mu \gamma^\rho a_\nu) - \mathrm{Tr}(\gamma^5 \gamma^\nu \gamma^\mu a_\nu) + v'_\lambda \mathrm{Tr}(\gamma^5 \gamma^\lambda \gamma^\nu \gamma^\mu a_\nu) \\
& + v_\rho \mathrm{Tr}(\gamma^5 \gamma^\nu \gamma^\mu \gamma^\rho a_\nu) + v'_\lambda v_\rho \mathrm{Tr}(\gamma^5 \gamma^\lambda \gamma^\nu \gamma^\mu \gamma^\rho a_\nu)] \\
& - v'_\sigma [\mathrm{Tr}(\gamma^\sigma \gamma^\mu b) - v'_\lambda \mathrm{Tr}(\gamma^\lambda \gamma^\sigma \gamma^\mu b) - v_\rho \mathrm{Tr}(\gamma^\sigma \gamma^\mu \gamma^\rho b) \\
& + v'_\lambda v_\rho \mathrm{Tr}(\gamma^\lambda \gamma^\sigma \gamma^\mu \gamma^\rho b) - \mathrm{Tr}(\gamma^5 \gamma^\sigma \gamma^\mu b) - v'_\lambda \mathrm{Tr}(\gamma^5 \gamma^\lambda \gamma^\sigma \gamma^\mu n) \\
& \left. + v'_\rho \mathrm{Tr}(\gamma^5 \gamma^\sigma \gamma^\mu \gamma^\rho b) + v'_\lambda v_\rho \mathrm{Tr}(\gamma^5 \gamma^\lambda \gamma^\sigma \gamma^\mu \gamma^\rho)] \right\} \qquad \text{(F.1)}
\end{aligned}
$$

where a^ν and b are the short notation for $(C_1 v^\nu + C_2 v'^\nu + C_3 \gamma^\nu)$ and $(C_1 w + C_2 + C_3 v'_\nu \gamma^\nu)$. Note that the terms arising from axial vector current, traces with γ^5, does not contribute. Trace of γ_5 multiplied with two or odd numbers of gamma matrices is zero and or the trace gives Levi-Civita symbol, an antisymmetric object, multiplied with symmetric velocity multiplication. Remaining traces yields

$$
\begin{aligned}
& \sqrt{m_B m_D} \{ [C_1 (v^\mu + 2 w v^\mu - v'^\mu) + C_2 (v + v')^\mu + C_3 (4 v'^\mu - 2 v^\mu)] \\
& - [(C_1 w + C_2 - C_3)(v - v')^\mu] \} \\
= & \sqrt{m_B m_D} (v_b - v_c)_\mu (C_1 (w+1) - 3 C_3) \qquad \text{(F.2)}
\end{aligned}
$$

$$
\begin{aligned}
& \langle D(v')|\overline{h}'^{(c)} i\overleftarrow{D}^{\nu} \Gamma^{\mu} h_v^{(b)}|\overline{B}(v)\rangle \\
= \ & \sqrt{m_B m_D}\mathrm{Tr}\left[\gamma_5 \frac{1+\not{v}'}{2}\Gamma^{\mu}\frac{1+\not{v}}{2}\gamma_5(C_1 v^{\nu} + C_2 v'^{\nu} + C_3\gamma^{\nu})\right] \\
= \ & \frac{1}{4}[-\mathrm{Tr}(\gamma^{\mu}a^{\nu}) - v'_{\lambda}\mathrm{Tr}(\gamma^{\lambda}\gamma^{\mu}a^{\nu}) - v_{\rho}\mathrm{Tr}(\gamma^{\mu}\gamma^{\rho}a^{\nu}) \\
& -v_{\rho}v'_{\lambda}\mathrm{Tr}(\gamma^{\lambda}\gamma^{\mu}\gamma^{\rho}a^{\nu}) - \mathrm{Tr}(\gamma^{5}\gamma^{\mu}a^{\nu}) - v'_{\lambda}\mathrm{Tr}(\gamma^{5}\gamma^{\lambda}\gamma^{\mu}a^{\nu}) \\
& +v_{\rho}\mathrm{Tr}(\gamma^{5}\gamma^{\mu}\gamma^{\rho}a^{\nu}) + v_{\rho}v'_{\lambda}\mathrm{Tr}(\gamma^{5}\gamma^{\lambda}\gamma^{\mu}\gamma^{\rho}a^{\nu})]
\end{aligned} \tag{F.3}
$$

Again a^{ν} stands for $(C_1 v^{\nu} + C_2 v'^{\nu} + C_3\gamma^{\nu})$. Similarly there is no contribution from the matrix element induced by the axial vector current. Calculating the traces yields

$$
\sqrt{m_B m_D}\left\{(v+v')^{\mu}(C_1 v + C_2 v')^{\nu} - C_3\left[\eta^{\mu\nu}(1-w) + v^{\nu}v'^{\mu} + v^{\mu}v'^{\nu}\right]\right\} \tag{F.4}
$$