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M.Sc. in Department of Mathematics

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**UNIVERSITY OF GAZİANTEP
GRADUATE SCHOOL OF
NATURAL & APPLIED SCIENCES**

POST QUANTUM CALCULUS

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IN
DEPARTMENT OF MATHEMATICS**

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UĞUR DURAN
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Post Quantum Calculus

M.Sc. Thesis
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Supervisor
Assoc. Prof. Dr. Mehmet AÇIKGÖZ

by
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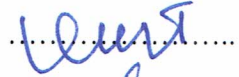
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Uğur DURAN

ABSTRACT
POST QUANTUM CALCULUS

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M.Sc. in Department of Mathematics

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This thesis consists of seventeen chapters. The first chapter is the introduction part which explains the emergence of the theory of *post quantum calculus* and its historical development.

The second chapter includes some definitions, formulas and results belonging to *quantum calculus*.

Some definitions, notations, formulas and results of the (p,q) -calculus including (p,q) -hypergeometric functions, (p,q) -power basis, (p,q) -Taylor formulas, (p,q) -binomial coefficients, (p,q) -exponential functions, (p,q) -antiderivative, (p,q) -integral, the fundamental theorem of (p,q) -calculus, the (p,q) -Gamma functions and the (p,q) -Beta functions are discussed in the third chapter to twelfth chapter. Also, some relations between (p,q) -derivative and divided differences are analyzed.

The thirteenth chapter deals with p -adic analysis with some definitions, theorems and applications. Basic information belonging to p -adic analysis, Volkenborn integral and its properties, and fermionic integral and its properties are examined under three main heading.

New generalizations of the Bernoulli, Euler and Genocchi numbers and polynomials based on (p,q) -numbers are given in the fourteenth chapter to sixteenth chapter. Then, some properties of aforementioned polynomials are investigated extensively. Also, an extension of the Haar distribution based on (p,q) -numbers is described in order to construct (p,q) -Volkenborn integral which derives Carlitz's-type (p,q) -Bernoulli polynomials and numbers.

At the final chapter of this thesis, the conclusions and finding gained in this thesis are presented.

Key Words: quantum calculus, post quantum calculus, p -adic analysis, special functions, analytic number theory.

ÖZET

POST KUANTUM ANALİZ

DURAN, Uğur

Yüksek Lisans Tezi, Matematik Bölümü

Tez Yöneticisi: Doç. Dr. Mehmet AÇIKGÖZ

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205 sayfa

Bu tez on yedi bölümden oluşmaktadır. İlk bölüm, *post kuantum analiz* teorisinin oluşumunu ve bu teorisinin tarihsel gelişimini anlatan giriş bölümüdür.

İkinci bölüm *kuantum analiz*' e ait tanımlamaları, notasyonları ve formülleri içerir.

(p,q) -Hipergeometrik fonksiyonlarını, (p,q) -kuvvet tabanını, (p,q) -Taylor formüllerini, (p,q) -binom katsayılarını, (p,q) -üstel fonksiyonlarını, (p,q) -ters türevini, (p,q) -integralini, (p,q) -analizin temel teoremini, (p,q) -Gamma fonksiyonlarını ve (p,q) -Beta fonksiyonlarını içeren (p,q) -analizin bazı tanımları, gösterimleri, formülleri ve sonuçları üçüncü bölümden on ikinci bölüme kadar tartışılmıştır. Ayrıca, bölünmüş farklar ile (p,q) -türev arasındaki bazı ilişkiler incelenmiştir.

On üçüncü bölüm bazı tanımlar, teoremler ve uygulamalarla birlikte p -adic analizi ele almıştır. p -adic analize ait temel bilgiler, Volkenborn integral ve özellikleri, fermiyonik integral ve özellikleri üç ana başlık altında incelenmiştir.

Bernoulli, Euler ve Genocchi sayılarının ve polinomlarının (p,q) -sayılarına dayanan yeni genellemeleri on dördüncü bölümden on altıncı bölüme kadar verilmiştir. Daha sonra, bahsedilen polinomların bazı özellikleri yoğun bir şekilde geliştirilmiştir. Ayrıca, Carlitz's tipinde (p,q) -Bernoulli polinomlarını ve sayılarını üreten (p,q) -Volkenborn integralini kurmak için Haar dağılımının (p,q) -sayılarına dayalı olan bir genellemesi tanımlanmıştır.

Bu tezin son bölümünde, tezde elde edilen sonuçlar ve bulgular verilmiştir.

Anahtar Kelimeler: kuantum analiz, post kuantum analiz, p -adic analiz, özel fonksiyonlar, analitik sayılar teorisi.



To My Family

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LIST OF SYMBOLS

$\mathbb{Z}$	The field of integers
$\mathbb{Z}_p$	Topological closure of $\mathbb{Z}$
$\mathbb{Q}$	The field of rational numbers
$\mathbb{Q}_p$	Topological closure of $\mathbb{Q}$
$\mathbb{C}$	The field of complex numbers
$\mathbb{C}_p$	The field of p -adic completion of an algebraic closure of $\mathbb{Q}_p$
$B_n(x)$	Bernoulli polynomial
$E_n(x)$	Euler polynomial
$G_n(x)$	Genocchi polynomial
B_n	Bernoulli number
E_n	Euler number
G_n	Genocchi number
$\mathfrak{B}_{n,q}^{(\alpha)}(x, y)$	q -Bernoulli polynomial of order α
$\mathfrak{E}_{n,q}^{(\alpha)}(x, y)$	q -Euler polynomial of order α
$\mathfrak{G}_{n,q}^{(\alpha)}(x, y)$	q -Genocchi polynomial of order α
$\mathfrak{B}_{n,q}^{(\alpha)}$	q -Bernoulli number of order α

$\mathfrak{E}_{n,q}^{(\alpha)}$	q -Euler number of order α
$\mathfrak{G}_{n,q}^{(\alpha)}$	q -Genocchi number of order α
$\mathcal{B}_n^{(\alpha)}(x, y; p, q)$	(p, q) -Bernoulli polynomial of order α
$\mathcal{E}_n^{(\alpha)}(x, y; p, q)$	(p, q) -Euler polynomial of order α
$\mathcal{B}_n^{(\alpha)}(x, y; p, q)$	(p, q) -Genocchi polynomial of order α
$\mathcal{B}_n^{(\alpha)}(p, q)$	(p, q) -Bernoulli number of order α
$\mathcal{E}_n^{(\alpha)}(p, q)$	(p, q) -Euler number of order α
$\mathcal{G}_n^{(\alpha)}(p, q)$	(p, q) -Genocchi number of order α
$[x]_q$	q -analogue of x
$[x]_{p,q}$	(p, q) -analogue of x
D_q	q -derivative operator
$D_{p,q}$	(p, q) -derivative operator
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	q -binomial coefficient
$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$	(p, q) -binomial coefficient
${}_r\phi_s$	q -hypergeometric series
${}_r\Phi_s$	(p, q) -hypergeometric series
$\Gamma_{p,q}(x)$	(p, q) -Gamma functions
$B_{p,q}(x, y)$	(p, q) -Beta functions
ord_p	p -adic value
$\mu(a + p^N \mathbb{Z}_p)$	Haar distribution

$\int_{\mathbb{Z}_p} d\mu(x)$ Volkenborn integral

$\int_{\mathbb{Z}_p} d\mu_{-1}(x)$ Fermionic integral



CHAPTER 1

INTRODUCTION

“There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.” N. I. Lobachevsky

The quantum calculus, also denoted q -calculus, which is based on the finite difference re-scaling is first considered by Euler investigating the q -binomial formula, and Euler’s identities for q -exponential functions and Gauss developing the q -binom formula. The aforementioned results lead to an intensive investigation on q -calculus in XIX century. For pq -hypergeometric series which is a generalization of the hypergeometric series, the discovery of Heine’s formula [1], relation with the Ramanujan formula, and the relation between Euler’s identities and the Jacobi triple identity, are just a few of the significant results arisen from this field. Jacobi’s formula for the number of sums of four squares, Gauss’s formula for the number of sums of two squares, explicit formulas for sums of two and of four triangular numbers and Euler’s infinite product for the classical partition function are natural outcomes of q -calculus, which are stated in [2]. The systematic development of q -calculus starts with Jackson who reconsidered the Euler-Jackson q -difference operator [3] in 1908, who gave a q -form of Taylor’s theorem [4] in 1909 and who introduced q -definite integral on a finite interval [5], which is earlier introduced Thomas [6] in 1869, in 1910. This theory has been investigated extensively since Jackson’s time and has played a crucial role in the study of almost every branch of mathematics and physics, for example, in the theory of special functions, differential equations (also fractional differential equations), combinatorics, analytic number theory, quantum theory, quantum group, special polynomials, numerical anaysis, operator theory and other related theories: namely, quantum calculus is known as one of the most active area of research in the physics and mathematics. Given the aforementioned information, q -calculus has in the three decades served as a bridge between mathematics and physics.

While one takes care of q -calculus including q -number with one base q , (p,q) -calculus including (p,q) -number with independent two variables is firstly considered by Chakrabarti and Jagannathan [7], Brodimas *et al.* [8], Wachs and White [9], and Arik *et al.* [10], roughly the same time and independently. The (p,q) -number is introduced by Chakrabarti and Jagannathan [7] in order to generalize or unify several forms of q -oscillator algebras well-known in the physics literature related to the representation theory of single parameter quantum algebras. The (p,q) -number is considered by Brodimas *et al.* [8] so that (p,q) -algebra can be derived from q -calculus by a Bargmann differential realization of the creation and annihilation operator, the Bose representation of those operators is derived, and (p,q) -Harmonic oscillator is found. The (p,q) -number is imagined by Wachs and White [9] in mathematical literature so as to obtain (p,q) -Stirling number which is the generating function the joint distribution of pairs of statistics. The (p,q) -number is considered by Arik *et al.* [10] to investigate Fibonacci oscillators. Thereby, several physical and mathematical problems lead to the necessity of (p,q) -calculus. The (p,q) -calculus have been developed, based on the papers [7-10], by many mathematicians and physicists in many different areas of research since 1991. For example, (p,q) -hypergeometric functions were introduced by Burban and Klimyk [11] and the relations between basic hypergeometric functions, q -hypergeometric functions, and (p,q) -hypergeometric functions has been studied. The (p,q) -extension of the binomial coefficients was developed by Corcino [12] and also some properties parallel to those of the ordinary and q -binomial coefficients were established. A method to embed the q -series into a (p,q) -series was given by Jagannathan and Srinivasa [13], some results corresponding (p,q) -extensions of the known q -identities were discovered and twin-basic hypergeometric series, (p,q) -hypergeometric series, based on the (p,q) -numbers was also studied. A more general (p,q) -hypergeometric series as well as defined in [11] was considered by Jagannathan [14] and some related preliminary results were derived. A (p,q) -analogue of the classical Rogers-Szegö polynomial was defined by Jagannathan and Sridhar [15] and an association between Rogers-Szegö polynomial and q -oscillator algebra was obtained. A new analogue of Bernstein operators, calling as (p,q) -Bernstein operators that are a generalization of q -Bernstein operators was introduced by Mursaleen *et al.* [16, 17] and approximation properties based on Korovkin's type approximation theorem of (p,q) -Bernstein operators were considered, and also some direct theorems were established. A Korovkin's type approximation

theorem by generalizing the q -Bernstein-Schurer operator using (p,q) -integer was acquired by Mursaleen *et al.* [18] and the convergence of the operators by using the modulus of continuity, and also some proof of direct theorems were presented. Some properties of the (p,q) -derivative and the (p,q) -integration were investigated by Sadjang [19] and two appropriate polynomials basis for the (p,q) -derivative were given. Then, various properties of these bases were derived by Sadjang. As an application, not only two (p,q) -Taylor formulas for polynomials were given but also the fundamental theorem of (p,q) -calculus was provided. The formula of (p,q) -integration by part was given. A new generalization of the Gamma and the Beta functions, calling as (p,q) -Gamma and (p,q) -Beta functions, were introduced by Sadjang [20] and some their properties which reduce to the known results as specific cases were developed. A new extension of Beta functions based on (p,q) -numbers which is different from the Sadjang's (p,q) -Beta functions was given by Milovanovic *et al.* [21]. The structure of the irreducible representations of the two-parameter quantum algebra was studied by Smirnov and Wehrhahn [22]. Motivated and inspired by [24, 25], some connections between (p,q) -derivative operator and divided differences, which are (p,q) -generalizations of the known q -results, were derived by Duran *et al.* [23] and the (p,q) -analogue of Leibnitz rule by means of the divided differences was given. Some new investigations and results for *post quantum calculus* were given by Duran *et al.* [26] and (p,q) -analogue of the chain rule for (p,q) -derivative was developed. Also, many useful results relating with (p,q) -binomial coefficients and (p,q) -antiderivative were developed. Further, (p,q) -analogue of some elementary functions including trigonometric functions and hyperbolic functions were introduced by Duran *et al.* and their properties and relations between them were researched extensively. These investigations may be a good consideration in developing the (p,q) -calculus in combinatorics, number theory and other fields of mathematics and physics.

In mathematics, the theory of special polynomials is one of the most active research areas and is studied and investigated by many mathematicians. Also, one of the most effects of the q -calculus in mathematics is occurred in the theory of special polynomials. This reason is why q -calculus is considered as one of the useful concepts to study with special numbers and polynomials. By the same motivation, the theory of special polynomials was considered and was investigated by Duran, Acikgoz and Araci in the light of (p,q) -calculus. The (p,q) -analogues of Bernoulli polynomials,

Euler polynomials, and Genocchi polynomials were described by Duran *et al.* [27], the (p,q) -analogues of known earlier formulae were obtained, and relations between the new and old polynomials were investigated by making use of the fermionic p -adic integral over the p -adic number fields. A new class of Bernoulli, Euler and Genocchi polynomials based on the theory of (p,q) -calculus were considered by Duran *et al.* [28] and their some properties including addition theorems, difference equations, derivative properties, recurrence relationships were investigated. The (p,q) -extension of Cheon's main result in [29] was acquired, and further, the (p,q) -analogue of the main results given earlier by Srivastava and Pintér in [30] was discovered. Moreover, an extension of Haar distribution based on (ρ,q) -numbers were considered by Duran *et al.* [31] as

$$\mu_{\rho,q}(a + p^N \mathbb{Z}_p) = \frac{\rho^{p^N}}{[p^N]_{\rho,q}} \left(\frac{q}{\rho} \right)^a.$$

The (ρ,q) -analogue of Volkenborn integration that is a new generalization of Kim's q -Volkenborn integration defined in [32] was derived by means of this distribution. Some properties of Volkenborn integration based on (ρ,q) -numbers were investigated. Finally, (ρ,q) -Bernoulli numbers and polynomials derived from (ρ,q) -Volkenborn integral were constructed and also some their properties were obtained. Details on the special polynomials can be found in [27-44, 46-68].

The aim of this master thesis is to study the theory of (p,q) -calculus and to give its some applications. The structure of this thesis is organized as follows.

The second chapter provides some definitions, notations and results belonging to q -calculus taken from [2, 12, 45, 69].

The (p,q) -number is defined and some properties of this number including the addition formula, the subtraction formula, multiplication rule and division rule are derived by utilizing the definition of this number in Chapter 3. The (p,q) -differentiation and then (p,q) -derivative are presented. Multiplication rule of (p,q) -derivative and quotient rule of (p,q) -derivative are investigated. Moreover, the (p,q) -analogue of Leibnitz rule is provided. As the last result of this chapter, a chain rule for (p,q) -derivative by choosing a particular form of functions is acquired.

The Chapter 4 examines the (p,q) -hypergeometric series and gives some relations between (p,q) -hypergeometric series and its correspond series in q -calculus [2, 70, 71]. A method of application of the (p,q) -hypergeometric series is provided to convert the various well-known q -identities into their (p,q) -analogues; after the

conversion the resulting (p,q) -identities offer more choices for symbolic manipulations transcending the applications of the original q -identities and, in fact, give a new look to the latter. As examples, the (p,q) -binomial theorem, the (p,q) -binomial coefficient and (p,q) -Heine transformation for ${}_2\Phi_1$ are analyzed.

In Chapter 5, (p,q) -analogue of $(x - a)^n$ or so called (p,q) -power are studied and some properties of (p,q) -power are given. Utilizing the definition of the (p,q) -derivative, some derivative properties of (p,q) -basis are obtained.

Two type Taylor formulas for polynomials under the theory of (p,q) -calculus are given in Chapter 6. Then, some of their consequences are investigated.

The Chapter 7 deals with the theory of (p,q) -binomial coefficients. Some properties and identities parallel to those of the usual and q -binomial coefficients including the triangular, vertical, and the horizontal recurrence relations, horizontal generating function, and the orthogonality and inverse relations are established. Some examples of these properties and identities are presented.

As Sofoena [30, 31] obtained some good relations between q -derivative operator D_q and divided differences under the theory of *quantum calculus*, some useful connections between (p,q) -derivative operator $D_{p,q}$ and divided differences are obtained under the theory of *post quantum calculus*, which are native generalizations of the known q -results, in Chapter 8. Also, the (p,q) -analogue of Leibnitz rule by means of the divided differences is given.

The two types (p,q) -exponential functions are examined and some relations between two functions are given in the subsequent chapter. Also, (p,q) -derivatives of these functions are developed. The (p,q) -analogues of the sine, cosine, tangent and cotangent functions are defined in the same manner with their well-known Euler expressions by means of the exponential functions. Their (p,q) -derivative properties are presented and some tables including identities and formulae belonging to the (p,q) -trigonometric functions which reduce to the both q -identities and q -formulae, and classical identities and classical formulas under the special cases are stated as ultimate results of this chapter.

The (p,q) -antiderivative of a function $f(x)$ is supplied and some simple examples of (p,q) -antiderivative are given in Chapter 10. The inverse of the (p,q) -differentiation called the (p,q) -integration is constructed using this

(p,q) -antiderivative, that is, the (p,q) -integral that gives the (p,q) -antiderivative $F(x)$ of an arbitrary function $f(x)$ is constructed. Under which conditions it ready converges to a (p,q) -antiderivative are studied. Thereafter, the definite (p,q) -integral and improper (p,q) -integral of any function $f(x)$, which turn into the counterparts in q -calculus and classical calculus as in the special cases are provided.

In Chapter 11, following the work completed in q -calculus, where the introduction of the definite integral has been motivated by an antiderivative, the relation between the (p,q) -derivative and the (p,q) -integral is more obvious. Similar to the ordinary and q -cases, fundamental theorem, or (p,q) -Newton-Leibnitz formula of (p,q) -calculus is given.

The Chapter 12 includes new generalizations of the Gamma and the Beta functions based on (p,q) -numbers. Then, some their properties which reduce to the known results belonging to the classical counterparts and counterparts in q -calculus as special cases are developed.

The definitions of p -adic numbers and the counterpart absolute value (norm), namely, the p -adic norm are given in Chapter 13. Also, a p -adic distribution is presented and then the most famous some examples of p -adic distributions including the Haar distribution μ_{Haar} , the Mazur distribution μ_{Mazur} , the Dirac distribution μ_α and the Bernoulli distribution $\mu_{B,k}$ are stated. The definition of the Volkenborn integral and its some properties are mentioned. Then Bernoulli numbers are shown by means of the Volkenborn integral. In the end of this chapter, fermionic p -adic integral defined by Kim [32] and its some properties are studied. Thereafter, Euler numbers and Genocchi numbers are denoted by means of the fermionic p -adic integral.

The Chapter 14 supplies new generating functions of the Bernoulli, the Euler and the Genocchi polynomials firstly considered by Duran *et al.* [27] under *post quantum calculus* based on the (p,q) -integer. From those generating functions, the behaviours of the defined polynomials are explored. By making use of the fermionic p -adic integral over the p -adic number fields, relations between the new and old polynomials are acquired. These turn out that some well-known formulae are obtained as special cases.

A new kind of Bernoulli, Euler and Genocchi polynomials introduced by Duran *et al.* [28] in the light of the (p,q) -calculus is mentioned and their some properties involving addition theorems, difference equations, derivative properties,

recurrence relationships are examined in Chapter 15. The (p, q) -analogues of some known formulae belonging to usual Bernoulli, Euler, and Genocchi polynomials are gained. Moreover, the (p, q) -extension of Cheon's main result in [29] is obtained. Furthermore, the (p, q) -analogue of the main results given earlier by Srivastava and Pintér in [30] is provided.

In Chapter 16, an extension of Haar distribution based on (ρ, q) -numbers firstly considered by Duran *et al.* [31] is given by

$$\mu_{\rho, q}(a + p^N \mathbb{Z}_p) = \frac{\rho^{p^N}}{[p^N]_{\rho, q}} \left(\frac{q}{\rho} \right)^a.$$

The (ρ, q) -analogue of Volkenborn integration which is a new generalization of Kim's q -Volkenborn integration defined in [32] is derived by means of this distribution. Some properties of Volkenborn integration based on (ρ, q) -numbers are discovered. Furthermore, (ρ, q) -Bernoulli numbers and polynomials derived from (ρ, q) -Volkenborn integral are constructed and some their properties are obtained.

At the end of this thesis, the conclusions and finding gained in this thesis are outlined in Chapter 17.

CHAPTER 2

QUANTUM CALCULUS

The usual quantum calculus (or recalled q -calculus) has been extensively studied for a long time by many mathematicians, physicists, and engineers. The development of q -calculus stems from the applications in many fields such as engineering, economics, mathematics, and so on. One of the most important branches of q -calculus is q -special polynomials. For example, q -Volkenborn integral is considered by Kim [32] to derive q -Bernoulli polynomials and numbers. A class of q -Bernoulli and q -Euler polynomials based on q -exponential functions are introduced by Mahmudov [58]. Moreover, Apostol q -Bernoulli polynomials, Apostol q -Euler polynomials and Apostol q -Genocchi polynomials are constructed by Srivastava *et al.* [64]. This is why q -calculus is thought as one of the most useful tools to study with special numbers and polynomials.

In this chapter, some basic definitions, notations and results belonging to q -calculus are given under five main heading.

2.1 q -Numbers

The quantum number, denoted by q -number, $[n]_q$ is defined, for any natural number n , as

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad (2.1)$$

which is a polynomial in q possessing the degree $n - 1$, see [2, 6, 69]. The deformation parameter q can be chosen as a real or complex number. In the limiting case $q \rightarrow 1$, q -numbers reduce to the ordinary numbers. Now a few examples of q -numbers are listed:

$$[0]_q = 0, [1]_q = 1, [2]_q = 1 + q, [3]_q = 1 + q + q^2.$$

The definition of the q -numbers can be extended to a fixed real number n

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

For instance, if $n = \frac{1}{2}$, then

$$\left[\frac{1}{2}\right]_q = \frac{q^{\frac{1}{2}} - 1}{q - 1} = 1 - q^{\frac{1}{2}} + q^{\frac{1}{4}} - q^{\frac{1}{8}} + q^{\frac{1}{16}} \dots.$$

Some properties of the q -numbers including q -addition formula, q -substruction formula, q -product formula and q -division formula are stated below by motivating and inspiring the book [69]:

$$[n + m]_q = \frac{1 - q^{n+m}}{1 - q} = \frac{1 - q^m + q^m - q^{n+m}}{1 - q} = [m]_q + q^m [n]_q$$

and similarly

$$[n + m]_q = \frac{1 - q^{n+m}}{1 - q} = \frac{1 - q^n + q^n - q^{n+m}}{p - q} = [n]_q + q^n [m]_q.$$

The subtraction formula can be derived by replacing $m \rightarrow -m$ as

$$[n - m]_q = [-m]_q + q^{-m} [n]_q,$$

or by using the identity

$$[-n]_q = (-q)^{-n} [n]_q,$$

the corresponding formula can be rewritten as

$$[n - m]_q = q^{-m} ([n]_q - [m]_q).$$

The other subtraction formula

$$[n-m]_q = [n]_q - q^{n-m} [m]_q$$

can be found in a like manner. The multiplication rule can be obtained by the following way

$$[nm]_q = \frac{1-q^{nm}}{1-q} = \frac{1-q^{nm}}{1-q^n} \cdot \frac{1-q^n}{1-q} = [m]_{q^n} [n]_q.$$

Also, the other multiplication formula can be acquired as

$$[nm]_q = [n]_{q^m} [m]_q.$$

The division rules of the q -number can be derived as

$$\left[\frac{n}{m} \right]_q = \frac{1-q^{\frac{n}{m}}}{1-q} = \frac{1-q^{\frac{n}{m}}}{1-q^{\frac{1}{m}}} \cdot \frac{1-q^{\frac{1}{m}}}{1-q} = \frac{[n]_{q^{\frac{1}{m}}}}{[m]_{q^{\frac{1}{m}}}}.$$

and also

$$\left[\frac{n}{m} \right]_q = \frac{1-q^{\frac{n}{m}}}{1-q} = \frac{1-q^{\frac{n}{m}}}{1-q^n} \cdot \frac{1-q^n}{1-q} = \frac{[n]_q}{[m]_{q^{\frac{n}{m}}}}.$$

The applications above are summarized as

$$[n+m]_q = [n]_q + q^n [m]_q = [m]_q + q^m [n]_q \quad (2.2)$$

$$[n-m]_q = [n]_q - q^{n-m} [m]_q = q^{-m} ([n]_q - [m]_q) \quad (2.3)$$

$$[nm]_q = [n]_{q^m} [m]_q = [m]_{q^n} [n]_q \quad (2.4)$$

$$\left[\frac{n}{m} \right]_q = \frac{[n]_{q^{\frac{1}{m}}}}{[m]_{q^{\frac{1}{m}}}} = \frac{[n]_q}{[m]_{q^{\frac{n}{m}}}}. \quad (2.5)$$

where n and m are real or complex numbers, see for details [2, 69].

2.2 q -Derivative Operator

Let $f(x)$ be an arbitrary function. The q -differential of $f(x)$

$$d_q f(x) = f(qx) - f(x)$$

and the q -derivative of $f(x)$

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}$$

hold true, see [2, 3]. Note that if $f(x)$ is differentiable function, it then turn into the standard derivative when $q \rightarrow 1$ as

$$\lim_{q \rightarrow 1} D_q f(x) = \lim_{q \rightarrow 1} \frac{f(qx) - f(x)}{(q-1)x} = \lim_{q \rightarrow 1} \frac{xf'(qx)}{x} = f'(x).$$

Applying to the q -derivative to some basic functions yields to

$$D_q(x) = \frac{d_q x}{d_q x} = 1,$$

$$D_q(\alpha x^n) = \frac{d_q(\alpha x^n)}{d_q x} = \frac{\alpha(qx)^n - \alpha x^n}{(q-1)x} = \alpha \frac{q^n - 1}{q-1} x^{n-1} = \alpha[n]_q x^{n-1},$$

$$D_q(\sin x) = \frac{d_q \sin x}{d_q x} = \frac{\sin(qx) - \sin x}{(q-1)x}.$$

The q -derivative operator has the linearity property, namely,

$$D_q(af(x) + bg(x)) = aD_q f(x) + bD_q g(x),$$

for any constants a and b , see [69]. The q -derivative operator has the following two product rule of arbitrary functions $f(x)$ and $g(x)$

$$\begin{aligned}
D_q(f(x)g(x)) &= f(x)D_qg(x) + g(qx)D_qf(x) \\
&= f(qx)D_qg(x) + g(x)D_qf(x),
\end{aligned}$$

also known as q -Leibnitz formulas. Moreover, the q -derivative operator has two division rules:

$$\begin{aligned}
D_q\left(\frac{f(x)}{g(x)}\right) &= \frac{g(qx)D_qf(x) - f(qx)D_qg(x)}{g(x)g(qx)} \\
&= \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(x)g(qx)},
\end{aligned}$$

where $g(x) \neq 0$ and $g(qx) \neq 0$, for details see [2, 69]. As a lack of q -calculus from ordinary calculus, there is no general chain rule for q -derivative operator. If only one choose a function with the following form $f(u(x))$, where $u(x) = \alpha x^\beta$ with α, β being constants, one then has the following chain rule (see Chapter 1 in [2])

$$D_q f(u(x)) = (D_{q^\beta} f)(u(x)) \cdot D_q u(x).$$

2.3 q -Taylor's Formula and Binomial Coefficients

The q -analogue of $n!$ is defined by

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$$

with the initial condition $[0]_q! = 1$. The q -analogue of $(x-a)^n$ is the polynomial [2]

$$(x \ominus a)_q^n = (x-a)(x-qa) \cdots (x-q^{n-1}a) \quad (2.6)$$

with the initial condition $(x \ominus a)_q^0 = 1$. Some properties of the aforementioned polynomial are given as

$$D_q(x \ominus a)_q^n = [n]_q (x - a)_q^{n-1} \quad (2.7)$$

$$(x \ominus a)_q^{m+n} = (x \ominus a)_q^m (x \ominus q^m a)_q^n \quad (2.8)$$

$$(x \ominus a)_q^{-n} = \frac{1}{(x \ominus q^{-n} a)_q^n} \quad (2.9)$$

$$(a \ominus x)_q^n = (-1)^n q^{n(n-1)/2} (x \ominus q^{-n+1} a)_q^n \quad (2.10)$$

$$D_q(a \ominus x)_q^n = -[n]_q (a \ominus qx)_q^{n-1} \quad (2.11)$$

$$D_q \frac{1}{(x \ominus a)_q^n} = \frac{[-n]_q}{(x \ominus q^n a)_q^{n+1}} \quad (2.12)$$

$$D_q \frac{1}{(a \ominus x)_q^n} = \frac{[n]_q}{(a \ominus x)_q^{n+1}} \quad (2.13)$$

which can be found in [2].

The following q -Taylor expansion

$$f(x) = \sum_{j=0}^N (D_q^j f)(c) \frac{(x \ominus c)_q^n}{[j]_q!},$$

holds true for any polynomial $f(x)$ of degree N and any number c , see [2]. To illustrate, consider $f(x) = x^n$ and $c = 1$, where n is a positive integer. For $j \leq n$, it becomes

$$(D_q^1 f)(x) = [n]_q x^{n-1}$$

$$(D_q^2 f)(x) = [n]_q [n-1]_q x^{n-2}$$

$$\vdots$$

$$(D_q^j f)(x) = [n]_q [n-1]_q \cdots [n-j+1]_q x^{n-j},$$

and thus,

$$(D_q^j f)(0) = [n]_q [n-1]_q \cdots [n-j+1]_q.$$

Thereby, the q -Taylor formula for x^n about $x=1$ yields to

$$x^n = \sum_{j=0}^n \frac{[n]_q [n-1]_q \cdots [n-j+1]_q}{[j]_q!} (x-1)_q^j. \quad (2.14)$$

Expanding $f(x) = (x \ominus a)_q^n$ about $x=0$ by using q -Taylor's formula it gives

$$(x \ominus a)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\frac{k(k-1)}{2}} x^{n-k} a^k,$$

where the q -binomial coefficients, similar to usual binomial coefficients, are defined for positive integers n, k by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$$

with the initial condition $\begin{bmatrix} n \\ n \end{bmatrix}_q = \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n < k$, see [2, 6, 68, 72, 73].

After defining the q -binomial coefficient, the Eq. (2.14) can be rewritten as

$$x^n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_q (x \ominus 1)_q^j.$$

If x and y are noncommutative and q -commutative, that is $yx = qxy$, noncommutative binomial formula [2] is given by

$$(x+y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}. \quad (2.15)$$

Before giving properties of the q -binomial coefficients, let usual binomial coefficients and their properties be remembered, for further details, see [12]. The usual

binomial coefficients appeared as the coefficients in the expansion of the binomial expression $(x+y)^n$, namely

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k},$$

are defined by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad (0 \leq k \leq n)$$

and $\binom{n}{k} = 0$, otherwise. The usual binomial coefficients have numerous properties and identities. To mention a few, the triangular recurrence relation, namely the Pascal rule,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \quad (2.16)$$

the identity

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}, \quad (2.17)$$

the inverse relation

$$f_n = \sum_{k=0}^n \binom{n}{k} g_k \Leftrightarrow g_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f_k, \quad (2.18)$$

Chu Shih-Chieh's identity

$$\binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}, \quad (2.19)$$

and the orthogonality relation

$$\sum_{k=i}^n (-1)^{n-k} \binom{n}{k} \binom{k}{i} = \sum_{k=i}^n (-1)^{k-i} \binom{n}{k} \binom{k}{i} = \delta_{ni} = \begin{cases} 0, & n \neq i \\ 1, & n = i \end{cases} \quad (2.20)$$

hold true for usual binomial coefficients where δ_{ni} is called the *Kronecker delta*.

Now some properties and identities of the q -binomial coefficients are analyzed below. Each q -binomial coefficient is a polynomial in q of degree $k(n-k)$, with 1 as the leading coefficient. The triangular recurrence relations, or two q -Pascal rules,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad (2.21)$$

$$= q^{n-k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q,$$

the identity

$$\sum_{k \text{ even}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{k \text{ odd}} \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad (2.22)$$

the inverse relation

$$f_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q g_k \Leftrightarrow g_n = \sum_{k=0}^n (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k, \quad (2.23)$$

the identity

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q + \dots + q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad (2.24)$$

and the orthogonality relation

$$\sum_{k=i}^n (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q = \sum_{k=i}^n (-1)^{k-i} q^{\binom{k-i}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q = \delta_{ni} \quad (2.25)$$

are valid for q -binomial coefficients, see the reference [12] for details. Note that the

Eqs. (2.21)-(2.25) are q -extensions of the Eqs. (2.16)-(2.20), respectively. Using the Eq. (2.21), simple example is given as

$$\begin{aligned}
 \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q &= \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q + q^3 \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \\
 &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q + q^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q + q^3 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q + q^6 \\
 &= 1 + q + (q^2 + q^3) \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q + q \right) q^6 \\
 &= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.
 \end{aligned}$$

Considering the q -Pascal rules, the q -analogue of the Pascal triangle are stated below:

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & & & \\
 & & & 1 & & 1 & \\
 & & 1 & & 1+q & & 1 \\
 & & & & & & \\
 & 1 & & 1+q+q^2 & & 1+q+q^2 & 1 \\
 & & & & & & \\
 & 1 & & 1+q+q^2+q^3 & & (1+q+q^2)(1+q^2) & 1+q+q^2+q^3 & 1 \\
 & & & & & & \\
 1 & & 1+q+q^2 & & (1+q+q^2)(1+q^2) & & (1+q+q^2+q^3) & & 1+q+q^2 & & 1 \\
 & & +q^3+q^4 & & +q^3(1+q+q^2+q^3) & & + (1+q+q^2)(q^2+q^4) & & +q^3+q^4 & &
 \end{array}$$

Figure 2.1. q -Pascal Triangle

Moreover, the q -analogue of Taylor expansion of the function $f(x) = 1/(1-x)^n$ is

$$\frac{1}{(1 \ominus x)_q^n} = 1 + \sum_{k=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \quad (2.26)$$

which is also called as Heine's binomial formula in [2].

2.4 Two q -Exponential Functions

The q -analogues of the classical exponential function e^x are given by (see [2])

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \quad (2.27)$$

and

$$E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}. \quad (2.28)$$

For the q -commuting variables x and y such as $yx = qxy$, note that

$$e_q(x+y) = e_q(x)e_q(y). \quad (2.29)$$

Some other properties and identities of given q -exponential functions are listed below:

$$e_q(x) = \frac{1}{(1 \ominus (1-q)x)_q^{\infty}} \text{ and } E_q(x) = (1 \oplus (1-q)x)_q^{\infty} \quad (2.30)$$

$$D_q e_q(x) = e_q(x) \text{ and } D_q E_q(x) = E_q(qx) \quad (2.31)$$

$$e_q(x)E_q(-x) = 1 \text{ and } e_{\frac{1}{q}}(x) = E_q(x). \quad (2.32)$$

The q -analogues of the sine and cosine trigonometric functions are defined in analogy with their well-known Euler expressions by means of the exponential functions (see [2, 74]):

$$\sin_q x = \frac{e_q(ix) - e_q(-ix)}{2i}, \quad \text{SIN}_q x = \frac{E_q(ix) - E_q(-ix)}{2i}, \quad (2.33)$$

$$\cos_q x = \frac{e_q(ix) + e_q(-ix)}{2}, \quad \text{COS}_q x = \frac{E_q(ix) + E_q(-ix)}{2}. \quad (2.34)$$

It follows from Eqs. (2.33) and (2.34), the following equality is true:

$$\cos_q x \text{COS}_q x + \sin_q x \text{SIN}_q x = 1, \quad (2.35)$$

which is the q -analogue of the identity $\sin^2 x + \cos^2 x = 1$. The q -trigonometric functions satisfy the following q -derivative properties:

$$D_q \sin_q x = \cos_q x, \quad D_q \text{SIN}_q x = \text{COS}_q qx, \quad (2.36)$$

$$D_q \cos_q x = -\sin_q x, \quad D_q \text{COS}_q x = -\text{SIN}_q qx, \quad (2.37)$$

for further investigations, see [2, 74].

2.5 q -Integral

As with ordinary calculus, the q -derivative is accompanied with a q -antiderivative. A function $F(x)$ is a q -antiderivative of $f(x)$ if $D_q F(x) = f(x)$. It is denoted by

$$F(x) = \int f(x) d_q x,$$

for further details, see Chapter 18 in the book [2].

Just as in ordinary calculus, any given function has multiple possible q -antiderivatives. Hence, $F(x)$ is identified as a q -antiderivative rather than the q -antiderivative.

Theorem 2.5.1 (Kac and Cheon [2]). *If $0 < q < 1$ then, up to adding a constant, any function $f(x)$ has at most one q -antiderivative continuing at $x = 0$.*

Let $u = u(x) = \alpha x^\beta$, where α and β are constants. Assuming $F(x)$ is a q -antiderivative of $f(x)$. Then, it becomes

$$F(u) = \int f(u) d_q u = \int f(u(x)) d_{\frac{1}{q^\beta}} u(x),$$

which means $f(u(x)) d_{\frac{1}{q^\beta}} u(x)$ is one of the q -antiderivatives of $f(x)$, see [2].

The q -integral of any function $f(x)$ was defined by Jackson [5] as follows:

$$\int f(x) d_q x = (1-q)x \sum_{n=0}^{\infty} f(q^n x) q^n \quad (2.38)$$

provided that the series on right hand side converges absolutely. The integral above is also called as Jackson integral. A more general formula [2] is presented by

$$\int f(x) d_q g(x) = \sum_{n=0}^{\infty} f(q^n x) (g(q^n x) - g(q^{n+1} x)).$$

The integral given in Eq. (2.38) does not always converge to a real valued function $F(x)$ even the q -antiderivative exists. Now, some of the cases under which the Jackson integral converges to a q -antiderivative are examined.

Theorem 2.5.2 (Kac and Cheon [2]). *Supposing $0 < q < 1$. If $|f(x)x^\alpha|$ is bounded on the interval $(0, A]$ for some $0 \leq \alpha < 1$, then the Jackson integral (2.38) converges to a function $F(x)$ on $(0, A]$, which is a q -antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$.*

Note that the definition of the Jackson integral (2.38) gives a q -antiderivative of $f(x)$, that is, $D_q \left((1-q)x \sum_{n=0}^{\infty} f(q^n x) q^n \right) = f(x)$. Let $f(x) = x^m$ for $m \in \mathbb{Z}^+$. It follows

$$\begin{aligned} \int x^m d_q x &= (1-q)x \sum_{n=0}^{\infty} q^{mn} x^m q^n \\ &= (1-q)x^{m+1} \sum_{n=0}^{\infty} q^{n(m+1)} \\ &= \frac{1-q}{1-q^{m+1}} x^{m+1}, \end{aligned}$$

hence

$$\int x^m d_q x = \frac{x^{m+1}}{[m+1]_q}. \quad (2.39)$$

For $0 < a < b$, the definite q -integral is defined in [2] as

$$\int_0^b f(x) d_q x := (1-q)b \sum_{n=0}^{\infty} f(q^n b) q^n \quad (2.40)$$

and

$$\int_a^b f(x) d_q x := \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \quad (2.41)$$

To see an example of the definite integral, by using the expression (2.39), the following equations satisfies

$$\begin{aligned} \int_a^b x^m d_q x &= \int_0^b x^m d_q x - \int_0^a x^m d_q x \\ &= (1-q)b \sum_{n=0}^{\infty} q^{mn} b^m q^n - (1-q)a \sum_{n=0}^{\infty} q^{mn} a^m q^n \end{aligned}$$

$$= \frac{b^{m+1} - a^{m+1}}{[m+1]_q}.$$

For example, taking $b=1$, $a=0$ and $f(x) = \ln x$ yields to

$$\begin{aligned} \int_0^1 \ln x d_q x &= (1-q) \sum_{n=0}^{\infty} q^n \ln q^n \\ &= (1-q) \frac{q \ln q}{(1-q)^2} \\ &= \frac{q \ln q}{1-q}, \end{aligned}$$

where the equality $\sum_{n=0}^{\infty} nq^n = q \frac{d}{dq} \sum_{n=0}^{\infty} q^n = q \frac{d}{dq} \left(\frac{1}{1-q} \right) = \frac{q}{(1-q)^2}$ when $|q| < 1$ has been used.

The improper Jackson integral of $f(x)$ on $[0, +\infty)$ is defined to be

$$\int_0^{\infty} f(x) d_q x = \sum_{n=-\infty}^{\infty} \int_{q^{n+1}}^{q^n} f(x) d_q x \quad (2.42)$$

if $0 < q < 1$, or

$$\int_0^{\infty} f(x) d_q x = \sum_{n=-\infty}^{\infty} \int_{q^n}^{q^{n+1}} f(x) d_q x \quad (2.43)$$

if $q > 1$.

Now, a theorem about the convergence of the improper q -integral is given as follows.

Theorem 2.5.3 (Kac and Cheon [2]). *The aforementioned improper q -integral converges if $x^\alpha f(x)$ is bounded when x is the neighbourhood of $x=0$ for some $\alpha < 1$ and for sufficiently large x for some $\alpha > 1$.*

Analogous to the ordinary case, the following *fundamental theorem*, or

Newton-Leibnitz's formula, for q -calculus is given.

Theorem 2.5.4 (Kac and Cheon [2]). *(Fundamental theorem of q -calculus) If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, the following equality holds true:*

$$\int_a^b f(x) d_q x = F(b) - F(a), \quad (2.44)$$

where $0 \leq a < b \leq \infty$.

Proposition 2.5.1 (Kac and Cheon [2]). *If $f'(x)$ exists in a neighbourhood of $x=0$ and is continuous at $x=0$, where $f'(x)$ indicates the ordinary derivative of $f(x)$, the following equation holds*

$$\int_a^b D_q f(x) d_q x = f(b) - f(a). \quad (2.45)$$

The following formula is valid:

$$\int_a^b f(x) d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) d_q f(x),$$

which is also known the formula of q -integration by parts, see [2].

CHAPTER 3

POST QUANTUM NUMBERS AND POST QUANTUM DERIVATIVE

3.1 Post Quantum Numbers

In this section, the (p,q) -numbers are defined and some properties of this numbers including the addition formula, the subtraction formula, multiplication rule and division rule are investigated. Let the following definition of the (p,q) -number be presented.

The (p,q) -numbers are defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad (3.1)$$

which is symmetric, that is $[n]_{p,q} = [n]_{q,p}$, see [7-10]. Note that when $p = 1$, the (p,q) -number $[n]_{p,q}$ reduces to the q -number $[n]_q$ (see Eq. (2.1)) and also as $p = 1$ and $q \rightarrow 1$, the (p,q) -number $[n]_{p,q}$ reduces to the ordinary number n . As it can be seen that the (p,q) -number $[n]_{p,q}$ plays the same role in (p,q) -calculus as the q -number $[n]_q$ does in q -calculus and as the integer n does in ordinary calculus. A few instances of (p,q) -numbers are stated below:

$$[0]_{p,q} = 0, \quad [1]_{p,q} = 1, \quad [2]_{p,q} = p + q, \quad [3]_{p,q} = p^2 + pq + q^2.$$

As for standard numbers and q -numbers, addition formulas for (p,q) -numbers can be determined, with similar method in [69], in the following forms

$$[n+m]_{p,q} = \frac{p^{n+m} - q^{n+m}}{p - q}$$

$$= \frac{p^{n+m} - p^n q^m + p^n q^m - q^{n+m}}{p - q}$$

$$= p^n [m]_{p,q} + q^m [n]_{p,q}$$

and similarly

$$[n+m]_{p,q} = \frac{p^{n+m} - q^{n+m}}{p - q}$$

$$= \frac{p^{n+m} - p^m q^n + p^m q^n - q^{n+m}}{p - q}$$

$$= p^m [n]_{p,q} + q^n [m]_{p,q}.$$

The subtraction formula can be derived by substituting $m \rightarrow -m$ as

$$[n-m]_{p,q} = p^n [-m]_{p,q} + q^{-m} [n]_{p,q},$$

or by using the equality

$$[-n]_{p,q} = -(pq)^{-n} [n]_{p,q},$$

the formula can be rewritten as

$$[n-m]_{p,q} = q^{-m} ([n]_{p,q} - p^{n-m} [m]_{p,q}).$$

Similarly, the other subtraction formula is obtained as

$$[n-m]_{p,q} = p^{-m} ([n]_{p,q} - q^{n-m} [m]_{p,q}).$$

The multiplication rules can be obtained by Eq. (3.1):

$$[nm]_{p,q} = \frac{p^{nm} - q^{nm}}{p - q}$$

$$\begin{aligned}
&= \frac{p^{nm} - q^{nm}}{p^n - q^n} \frac{p^n - q^n}{p - q} \\
&= [m]_{p^n, q^n} [n]_{p, q}
\end{aligned}$$

and similarly

$$[nm]_{p, q} = [n]_{p^m, q^m} [m]_{p, q}.$$

Lastly, the division rules are given as follows

$$\begin{aligned}
\left[\frac{n}{m} \right]_{p, q} &= \frac{p^{\frac{n}{m}} - q^{\frac{n}{m}}}{p - q} \\
&= \frac{p^{\frac{n}{m}} - q^{\frac{n}{m}}}{p^{\frac{1}{m}} - q^{\frac{1}{m}}} \cdot \frac{p^{\frac{1}{m}} - q^{\frac{1}{m}}}{p - q} \\
&= \frac{[n]_{p^m, q^m}^{\frac{1}{m}, \frac{1}{m}}}{[m]_{p^m, q^m}^{\frac{1}{m}, \frac{1}{m}}}
\end{aligned}$$

and also

$$\begin{aligned}
\left[\frac{n}{m} \right]_{p, q} &= \frac{p^{\frac{n}{m}} - q^{\frac{n}{m}}}{p - q} \\
&= \frac{p^{\frac{n}{m}} - q^{\frac{n}{m}}}{p^n - q^n} \cdot \frac{p^n - q^n}{p - q} \\
&= \frac{[n]_{p, q}}{[m]_{p^m, q^m}^{\frac{n}{m}, \frac{n}{m}}}.
\end{aligned}$$

As a result of this section the following conclusion is developed:

$[n + m]_{p,q} = p^n [m]_{p,q} + q^m [n]_{p,q}$ $= p^m [n]_{p,q} + q^n [m]_{p,q}$	(p,q)-addition formula
$[n - m]_{p,q} = q^{-m} ([n]_{p,q} + p^{n-m} [m]_{p,q})$ $= p^{-m} ([n]_{p,q} + p^{n-m} [m]_{p,q})$	(p,q)-subtraction formula
$[nm]_{p,q} = [m]_{p^n, q^n} [n]_{p,q}$ $= [n]_{p^m, q^m} [m]_{p,q}$	(p,q)-product rule
$\left[\frac{n}{m} \right]_{p,q} = \frac{[n]_{p^{\frac{1}{m}}, q^{\frac{1}{m}}}}{[m]_{p^{\frac{1}{m}}, q^{\frac{1}{m}}}}$ $= \frac{[n]_{p,q}}{[m]_{p^{\frac{n}{m}}, q^{\frac{n}{m}}}}$	(p,q)-division rule

(3.2)

By motivating the definition of (p,q)-number, the definition of the (p,q)-analogue of $n!$ called (p,q)-factorial of the number n is given by (see [12, 14]):

$$[n]_{p,q}! := \begin{cases} [n]_{p,q} [n-1]_{p,q} \cdots [2]_{p,q} [1]_{p,q}, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

By inspiring the definition of (p,q)-factorial, similar to usual binomial coefficients and q -binomials coefficients the (p,q)-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}, \quad (3.3)$$

which will be discussed extensively in Chapter 7, for further details, look at the references [11, 12, 14, 19].

3.2 Post Quantum Derivative

As mentioned in the introduction part, some properties of post quantum derivative are investigated. Let the notion of a post quantum differential be given to start this section. Let f be a function defined on the set of the complex numbers.

Definition 3.2.1 (Chakrabarti and Jagannathan [7], Brodimas *et al.* [8]). *The (p,q) -differential of a function f is defined by*

$$d_{p,q} f(x) = f(px) - f(qx).$$

Note that, particularly, $d_{p,q} x = (p - q)x$. Similar to q -differential in [2], an interesting difference of (p,q) -differential from the ordinary ones is the lack of symmetry in the differential of the product of two functions. As

$$\begin{aligned} d_{p,q} (f(x)g(x)) &= f(px)g(px) - f(qx)g(qx) \\ &= f(px)g(px) - f(qx)g(px) + f(qx)g(px) - f(qx)g(qx) \\ &= g(px)d_{p,q} f(x) + f(qx)d_{p,q} g(x), \end{aligned}$$

and hence,

$$d_{p,q} (f(x)g(x)) = g(px)d_{p,q} f(x) + f(qx)d_{p,q} g(x). \quad (3.4)$$

Also, similarly the following equation fulfils

$$d_{p,q} (f(x)g(x)) = f(px)d_{p,q} g(x) + g(qx)d_{p,q} f(x). \quad (3.5)$$

Following to the definition of (p,q) -differential, the corresponding (p,q) -derivative is described as follows.

Definition 3.2.2 (Chakrabarti and Jagannathan [7], Brodimas *et al.* [8]). *The (p,q) -derivative of a function f is defined by*

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x} \quad (x \neq 0)$$

and $(D_{p,q} f)(0) = f'(0)$, provided that f is differentiable at 0.

Note that as $p=1$, the (p,q) -derivative reduces to the q -derivative $D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$; also as $q \rightarrow p=1$, the (p,q) -derivative reduces to the ordinary derivative, namely $\frac{df(x)}{dx}$ if $f(x)$ is differentiable, which is called as Leibnitz notation. The Leibnitz notation, a ratio of two “infinitesimals”, is fairly confusing, because the notion of the differential $df(x)$ requires an elaborate explanation. On the contrary, the notion of (p,q) -differentials are obvious and the (p,q) -derivatives are just ratios.

Note that the following notations will be used in the following chapters:

$$D_{p,q;x} f(x) := D_{p,q} f(x).$$

As with the q -derivative and the ordinary derivative, the action of applying the (p,q) -derivative of any function is a linear operator, see [26]. Since, for any constants a and b ,

$$\begin{aligned} D_{p,q} (af(x) + bg(x)) &= \frac{(af(px) + bg(px)) - (af(qx) + bg(qx))}{(p-q)x} \\ &= a \frac{f(px) - f(qx)}{(p-q)x} + b \frac{g(px) - g(qx)}{(p-q)x} \\ &= aD_{p,q} f(x) + bD_{p,q} g(x). \end{aligned}$$

Example 3.2.1 The following equations hold

$$D_{p,q} (x^n) = \frac{(px)^n - (qx)^n}{px - qx} = [n]_{p,q} x^{n-1} \quad (n \in \mathbb{Z}^+)$$

which resembles the value of the ordinary derivative of x^n , as a more specific example

$$\begin{aligned} D_{p,q}(x^2 + 3x + 1) &= D_{p,q}x^2 + D_{p,q}3x + D_{p,q}1 \\ &= [2]_{p,q}x + 3, \end{aligned}$$

and for the sine function

$$D_{p,q}(\sin x) = \frac{\sin px - \sin qx}{px - qx} = \frac{2 \cos \frac{p+q}{2}x \sin \frac{p-q}{2}x}{(p-q)x}.$$

Now the (p,q) -derivative of product and the quotient of $f(x)$ and $g(x)$ are computed below, see [19, 26] for details. By the formula (3.4), it becomes

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= \frac{d_{p,q}(f(x)g(x))}{d_{p,q}x} = \frac{g(px)d_{p,q}f(x) + f(qx)d_{p,q}g(x)}{(p-q)x} \\ &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x) \end{aligned}$$

and thus,

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x). \quad (3.6)$$

Similarly, by the formula (3.5), it turn out to be

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x). \quad (3.7)$$

By motivating and inspiring the results about multiple q -calculus in [69], the formulas (3.6) and (3.7) can be rewritten in explicitly symmetrical form:

$$D_{p,q}(f(x)g(x)) = D_{p,q}f(x) \left(\frac{g(px) + g(qx)}{2} \right) + D_{p,q}g(x) \left(\frac{f(px) + f(qx)}{2} \right). \quad (3.8)$$

More general form of the product rule of (p,q) -derivative is presented with fixed α ,

$$D_{p,q} (f(x)g(x)) = (\alpha f(qx) + (1-\alpha)f(px))D_{p,q}g(x) + (\alpha g(px) + (1-\alpha)g(qx))D_{p,q}f(x).$$

Choosing $\alpha=1$, $\alpha=0$ and $\alpha=\frac{1}{2}$ gives the formulas in Eqs. (3.6), (3.7) and (3.8), respectively.

In order to get the quotient rule of the (p,q) -derivative by using the same method for q -derivative in [2], the formula in Eq. (3.6) is applied to differentiate

$$g(x) \cdot \frac{f(x)}{g(x)} = f(x),$$

then

$$\frac{f(px)}{g(px)}D_{p,q}(g(x)) + g(qx)D_{p,q}\left(\frac{f(x)}{g(x)}\right) = D_{p,q}f(x),$$

and hence,

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. \quad (3.9)$$

Moreover, applying the (p,q) -derivative to the formula in Eq. (3.7), then the following is procured:

$$g(px)D_{p,q}\left(\frac{f(x)}{g(x)}\right) + \frac{f(qx)}{g(qx)}D_{p,q}(g(x)) = D_{p,q}f(x),$$

and thus,

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}. \quad (3.10)$$

As well as the formulas in Eqs. (3.9) and (3.10), one more representation in symmetrical form (see similar form for multiple q -derivative in [69]) can be written as

$$D_{p,q} \left(\frac{f(x)}{g(x)} \right) = \frac{1}{2} \frac{D_{p,q} f(x)(g(px) + g(qx)) - D_{p,q} g(x)(f(px) + f(qx))}{g(px)g(qx)}. \quad (3.11)$$

The formulas in Eqs. (3.9), (3.10) and (3.11) are valid, however, one of these forms may be more useful than others under special cases.

Lemma 3.2.1 (Duran et al. [23]) *Let the functions $f : D_{p,q} \rightarrow \mathbb{C}$ and $g : D_{p,q} \rightarrow \mathbb{C}$ be (p,q) -differentiable of order n . Then*

$$D_{p,q}^n (fg)(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} D_{p,q}^k (f)(xp^{n-k}) D_{p,q}^{n-k} (g)(xq^k).$$

Proof. The theorem is proved by mathematical induction method. Firstly, for $n=1$, one can with ease see that Lemma 3.2.1 is true. Assume that Lemma 3.2.1 is true for $n=m$. Then it holds also true for $n=m+1$ using the case $n=m$ and the (p,q) -Pascal rule (7.4), since

$$\begin{aligned} D_{p,q}^{m+1} (fg)(x) &= D_{p,q} (D_{p,q}^m (fg)(x)) \\ &= D_{p,q} \left(\sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} D_{p,q}^k (f)(xp^{m-k}) D_{p,q}^{m-k} (g)(xq^k) \right) \\ &= \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} \left[p^{m-k} D_{p,q}^{k+1} (f)(xp^{m-k}) D_{p,q}^{m-k} (g)(xq^{k+1}) \right. \\ &\quad \left. + D_{p,q}^k (f)(xp^{m+1-k}) D_{p,q}^{m+1-k} (g)(xq^k) q^k \right] \\ &= \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} D_{p,q}^k (f)(xp^{m+1-k}) D_{p,q}^{m+1-k} (g)(xq^k) q^k \\ &\quad + \sum_{k=1}^{m+1} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{p,q} p^{m+1-k} D_{p,q}^k (f)(xp^{m+1-k}) D_{p,q}^{m+1-k} (g)(xq^k) \\ &= f(xp^{m+1}) D_{p,q}^{m+1} (g)(x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^m \left\{ q^k \begin{bmatrix} m \\ k \end{bmatrix}_{p,q} + p^{m+1-k} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{p,q} \right\} D_{p,q}^k(f)(xp^{m+1-k}) D_{p,q}^{m+1-k}(g)(xq^k) \\
&\quad + D_{p,q}^{m+1}(f)(x)g(xq^{m+1}) \\
&= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{p,q} D_{p,q}^k(f)(xp^{m+1-k}) D_{p,q}^{m+1-k}(g)(xq^k).
\end{aligned}$$

After discovering the multiplication rule and the division rule of (p,q) -differentiation, reasonably, one can be worried about whether there is a general chain rule in (p,q) -calculus or not, like the property of q -derivative in Chapter 2 (also, see [2]). Still, there doesn't exist a general chain rule for (p,q) -derivative. That is, if the function $f(u(x))$ is considered where $u = u(x) = \lambda x^\mu$ with λ, μ being constants, then, a chain rule as special case is derived as [26]:

$$\begin{aligned}
D_{p,q}[f(u(x))] &= D_{p,q}\left[f(\lambda x^\mu)\right] = \frac{f(\lambda x^\mu p^\mu) - f(\lambda x^\mu q^\mu)}{x(p-q)} \\
&= \frac{f(\lambda x^\mu p^\mu) - f(\lambda x^\mu q^\mu)}{\lambda x^\mu p^\mu - \lambda x^\mu q^\mu} \cdot \frac{\lambda x^\mu p^\mu - \lambda x^\mu q^\mu}{x(p-q)} \\
&= \frac{f(up^\mu) - f(uq^\mu)}{up^\mu - uq^\mu} \cdot \frac{u(px) - u(qx)}{x(p-q)}
\end{aligned}$$

which yields to

$$D_{p,q} f(u(x)) = \left(D_{p^\mu, q^\mu} f \right)(u(x)) \cdot D_{p,q} u(x). \quad (3.12)$$

Conversely, if one consider the function $u(x) = \cos x$ or $u(x) = x^3 + x$, the quantity $u(px)$ and $u(qx)$ can not be derived in terms of u in a basic way, and thereby it is impossible to write a general chain rule.

CHAPTER 4

(p,q)-HYPERGEOMETRIC FUNCTIONS

The (p,q) -hypergeometric series and its some identities in conjunction with q -hypergeometric series are interested in this chapter. Before considering (p,q) -hypergeometric series, let some basic definitions be recalled.

The Pochhammer symbol is defined as

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \quad (n \geq 0),$$

and its q -analogue, namely q -Pochhammer symbol, is defined by [13, 70, 71]

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & n > 0, \end{cases} \quad (4.1)$$

which is also called q -shifted factorial. The q -hypergeometric series ${}_r\phi_s$ is defined as

$$\begin{aligned} & {}_r\phi_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z)}{(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; q, z)(q; q)_n} \left((-1)^n q^{n(n-1)/2} \right)^{1+s-r} z^n \end{aligned} \quad (4.2)$$

where

$$(a_1, a_2, \dots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n, \quad (4.3)$$

see [13, 70, 71] for further details.

Now, by motivating knowledge above, the (p,q) -analogue of q -shifted factorial (4.1) and (p,q) -analogue of the identity (4.3) are given as follows [13, 14]:

$$\left((a, b); (p, q)\right)_n = \begin{cases} 1, & n = 0, \\ (a-b)(pa-bq)(p^2a-bq^2)\cdots(p^{n-1}a-bq^{n-1}), & n > 0. \end{cases} \quad (4.4)$$

and

$$\begin{aligned} & \left((a_{1p}, a_{1q}), (a_{2p}, a_{2q}), \dots, (a_{mp}, a_{mq}); (p, q)\right)_n \\ &= \left((a_{1p}, a_{1q}); (p, q)\right)_n \left((a_{2p}, a_{2q}); (p, q)\right)_n \cdots \left((a_{mp}, a_{mq}); (p, q)\right)_n. \end{aligned} \quad (4.5)$$

Thus, the (p, q) -analogue of the q -hypergeometric series or (p, q) -hypergeometric series can be defined as [13, 14]

$$\begin{aligned} & {}_r\Phi_s \left((a_{1p}, a_{1q}), \dots, (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), \dots, (b_{sp}, b_{sq}); (p, q), z\right) \\ &= \sum_{n=0}^{\infty} \frac{\left((a_{1p}, a_{1q}), \dots, (a_{rp}, a_{rq}); (p, q)\right)_n}{\left((b_{1p}, b_{1q}), \dots, (b_{sp}, b_{sq}); (p, q)\right)_n \left((p, q); (p, q)\right)_n} z^n \left((-1)^n \left(\frac{q}{p}\right)^{n(n-1)/2}\right)^{1+s-r} \end{aligned} \quad (4.6)$$

with the condition $|q/p| < 1$. Although, $0 < q < p$ is supposed, p and q can also take other values if there is no problem with the convergence of the particular series involved in the result. Note that taking $a_{1p} = a_{2p} = \cdots = a_{rp} = b_{1p} = b_{2p} = \cdots = b_{sp} = 1$, $a_{1q} = a_1, a_{2q} = a_2, \dots, a_{rq} = a_r$, $b_{1q} = b_1, b_{2q} = b_2, \dots, b_{sq} = b_s$, and $p = 1$, ${}_r\Phi_s$ turn into the ${}_r\phi_s$. Throughout this chapter, it is assumed that $|z| < 1$. Unless one choose a specific value for any parameter, the parameter will be generic, with nonzero values. Generally, the notations (a_p, a_q) , (b_p, b_q) , etc., will be shown by different symbols according to the convenience of the situation and such notations should be clear from the context.

Define [13]

$$\Delta_{(\alpha, \beta)} f(z) = \alpha f(qz) - \beta f(pz).$$

With

$$\Delta f(z) := \Delta_{(1,1)} f(z) = f(qz) - f(pz),$$

note that

$$\hat{D} f(z) = \frac{\Delta f(z)}{\Delta z}.$$

Then, ${}_r\Phi_s$ fulfils the (p,q) -difference equation [13]

$$\left(\Delta \prod_{i=1}^s \Delta_{(b_{iq}/q, b_{ip}/p)} \right)_r \Phi_s = \left(z \prod_{i=1}^r \Delta_{(a_{iq}, a_{ip})} \right)_r \Phi_s \left(\left(\frac{q}{p} \right)^{1+s-r} z \right). \quad (4.7)$$

When $a_{1q} = a_1, a_{2q} = a_2, \dots, a_{rq} = a_r, a_{1p} = a_{2p} = \dots = a_{rp} = b_{1p} = b_{2p} = \dots = b_{sp} = 1, b_{1q} = b_1, b_{2q} = b_2, \dots, b_{sq} = b_s$, and $p = 1$, the Eq. (4.7) reduces to the q -difference equation satisfied by ${}_r\phi_s$.

Now a method to embed the usual ${}_r\phi_s$ -series (4.2) in the ${}_r\Phi_s$ -series (4.6) is provided. Note that [13]

$$((la, lb); (p, q))_n = l^n ((a, b); (p, q))_n \quad (4.8)$$

for any fixed nonzero l , and

$$(b/a; q/p)_n = a^{-n} p^{-n(n-1)/2} ((a, b); (p, q))_n. \quad (4.9)$$

Hence, it is possible to write, formally,

$${}_r\Phi_s(a_{1q}/a_{1p}, a_{2q}/a_{2p}, \dots, a_{rq}/a_{rp}; b_{1q}/b_{1p}, b_{2q}/b_{2p}, \dots, b_{sq}/b_{sp}; q/p, z) = \begin{cases} {}_r\Phi_s((a_{1p}, a_{1q}), \dots, (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), \dots, (b_{sp}, b_{sq}); (p, q), \lambda z) & \text{if } s = r-1, \\ {}_{s+1}\Phi_s((a_{1p}, a_{1q}), \dots, (a_{rp}, a_{rq}), (0, 1), \dots, (0, 1); (b_{1p}, b_{1q}), \dots, (b_{sp}, b_{sq}); (p, q), \lambda z) & \text{if } s > r-1, \\ {}_r\Phi_{r-1}((a_{1p}, a_{1q}), \dots, (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), \dots, (b_{sp}, b_{sq}), (0, 1), \dots, (0, 1); (p, q), \lambda z) & \text{if } s < r-1, \end{cases} \quad (4.10)$$

with $\lambda = \frac{b_{1p}b_{2p}\cdots b_{sp}p}{a_{1p}a_{2p}\cdots a_{rp}}$, assume that the given ${}_r\phi_s$ -series is convergent or terminating [13]. Thus, any well behaved ϕ -series can be written as a Φ -series. However, the converse is not valid, in general; in the general case, when $p \neq 1$, this can be only for an ${}_r\Phi_{r-1}$. Since

$$\begin{aligned} {}_0\Phi_0(-;-(p,q),z) &= \sum_{n=0}^{\infty} \frac{(-1)^n (q/p)^{n(n-1)/2}}{((p,q);(p,q))_n} z^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (\rho/p)^{n(n-1)/2}}{(\rho;\rho)_n} \left(\frac{z}{p}\right)^n, \text{ with } \rho = q/p, \end{aligned}$$

which shows that ${}_0\Phi_0$ becomes a ϕ -series if and only if $p = 1$. Similarly, one is readily convinced that a generic ${}_r\Phi_s$ -series can not be identified within the class of ϕ -series unless $p = 1$ or $s = r - 1$. Obviously, the (p,q) -series is a larger structure in which the q -series gets embedded. Also, in usual theory of ϕ -series there is no direct analogue for the choice $a_{ip} = 0$ or $b_{ip} = 0$, for any i , generally (surely, subject to conditions of convergence and so on), in the case of the (p,q) -series; to get a corresponding result in the case of the ϕ -series one will have to resort to the limit process of confluence, that is, substituting z by z/a_r and taking the limit $a_r \rightarrow \infty$ [13]. To see an example the following case can be considered. As it is well known, in the definition of the usual q -hypergeometric series (4.2), presence of the element $\left((-1)^n q^{n(n-1)/2}\right)^{1+s-r}$ leads to useful relation

$$\lim_{a_r \rightarrow \infty} {}_r\Phi_s(z/a_r) = {}_r\phi_s(z).$$

For the (p,q) -hypergeometric series (4.6) the corresponding property is [13]:

$$\lim_{a_{rq} \rightarrow \infty} {}_r\Phi_s(z/a_{rq}) = {}_r\Phi_s((a_{1p}, a_{1q}), \dots, (a_{(r-1)p}, a_{(r-1)q}), (0, 1); (b_{1p}, b_{1q}), \dots, (b_{sp}, b_{sq}); (p, q), z),$$

$$\lim_{a_{rp} \rightarrow \infty} {}_r\Phi_s(z/a_{rp}) = {}_r\Phi_s((a_{1p}, a_{1q}), \dots, (a_{(r-1)p}, a_{(r-1)q}); (1, 0); (b_{1p}, b_{1q}), \dots, (b_{sp}, b_{sq}); (p, q), z).$$

The converse of the Eq. (4.10) is also written down in the case $s = r - 1$ [13]:

$$\begin{aligned} & {}_r\Phi_{r-1}((a_{1p}, a_{1q}), \dots, (a_{rp}, a_{rq}); (b_{1p}, b_{1q}), \dots, (b_{(r-1)p}, b_{(r-1)q}); (p, q), z) \\ &= {}_r\phi_{r-1}(a_{1q}/a_{1p}, a_{2q}/a_{2p}, \dots, a_{rq}/a_{rp}; b_{1q}/b_{1p}, b_{2q}/b_{2p}, \dots, b_{(r-1)q}/b_{(r-1)p}; q/p, z/\lambda). \end{aligned} \quad (4.11)$$

By using the relation (4.9), some other relations are

$$\begin{aligned} \frac{(b/a; q/p)_\infty}{(d/c; q/p)_\infty} &= \lim_{N \rightarrow \infty} \frac{(b/a; q/p)_N}{(d/c; q/p)_N} \\ &= \lim_{N \rightarrow \infty} \frac{a^{-N} p^{-N(N-1)/2} ((a, b); (p, q))_N}{c^{-N} p^{-N(N-1)/2} ((c, d); (p, q))_N} \\ &= \frac{((c, bc/a); (p, q))_\infty}{((c, d); (p, q))_\infty} \\ &= \frac{((a, b); (p, q))_\infty}{((c, ad/c); (p, q))_\infty}, \end{aligned} \quad (4.12)$$

and its clear generalizations including several factors in numerator and denominator [13].

Using the relationships above, manipulations take the usual q -identities to the (p, q) -identities. The identities belonging to the (p, q) -series, of course, reduce to the identities belonging to the q -series in the case $a_{1p} = a_{2p} = \dots = a_{rp} = b_{1p} = b_{2p} = \dots = b_{sp} = 1$, and $p = 1$. Now a few examples are talked about below.

4.1 (p,q) -Binomial theorem

In q -calculus, the usual q -binomial theorem can be denoted by means of the q -hypergeometric series as follows [13]:

$${}_1\phi_0(a; -; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty}. \quad (4.13)$$

The (p,q) -analogue of Eq. (4.13), that is, (p,q) -binomial theorem is given by [13]

$${}_1\Phi_0((a,b); -; (p,q), z) = \frac{((p, bz); (p,q))_\infty}{((p, az); (p,q))_\infty}. \quad (4.14)$$

To prove the Eq. (4.14), let the Eq. (4.13) be rewritten as

$${}_1\phi_0(b/a; -; q/p, \zeta) = \frac{(b\zeta/a; q/p)_\infty}{(\zeta; q/p)_\infty}. \quad (4.15)$$

Using Eqs. (4.10) and (4.12), the following satisfy

$${}_1\Phi_0((a,b); -; (p,q), p\zeta/a) = \frac{((a, b\zeta); (p,q))_\infty}{((a, a\zeta); (p,q))_\infty}.$$

Taking $\zeta = za/p$ yields to [13]

$${}_1\Phi_0((a,b); -; (p,q), z) = \frac{((a, abz); (p,q))_\infty}{((a, a^2z); (p,q))_\infty}. \quad (4.16)$$

Given the Eqs. (4.8) and (4.12), by pulling out powers of a/p in the numerator and denominator of the right hand side of the Eq. (4.16), the (p,q) -binomial theorem follows. When $a=1$ and $p=1$, the (p,q) -binomial theorem (4.14) turns into the q -binomial theorem (4.13). The (p,q) -binomial theorem acquired in [11] is a specific circumstance of Eq. (4.14) corresponding to the special choice $(p,q) = (q^{-1/2}, p^{-1/2})$ and $(a,b) = (q^{-a/2}, p^{a/2})$, see [13]. An interesting attribute of (p,q) -binomial theorem

in Eq. (4.14) may be noted here. The product $\prod_{i=1}^n {}_1\Phi_0((a_{ip}, a_{iq}); -(p, q), z)$ is seen to be an invariant under the group of independent permutations of the p -components $(a_{1p}, a_{2p}, \dots, a_{np})$ and q -components $(a_{1q}, a_{2q}, \dots, a_{nq})$. If the n -tuple of p -components $(a_{1p}, a_{2p}, \dots, a_{np})$ is concerned with the n -tuple of q -components $(a_{1q}, a_{2q}, \dots, a_{nq})$ by a mere permutation, the aforementioned product has value 1 (see [13]).

For $n = 2$, this implies that

$${}_1\Phi_0((a, b); -(p, q), z) {}_1\Phi_0((b, a); -(p, q), z) = 1. \quad (4.17)$$

As a special case of this result, the following equality fulfils

$${}_1\Phi_0((1, 0); -(1, q), z) {}_1\Phi_0((0, 1); -(1, q), z) = 1. \quad (4.18)$$

Recognized that

$${}_1\Phi_0((1, 0); -(1, q), z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = e_q(z), \quad (4.19)$$

$${}_1\Phi_0((0, 1); -(1, q), z) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{z^n}{(q; q)_n} = E_q(-z), \quad (4.20)$$

where $e_q(z)$ and $E_q(z)$ are the canonical q -exponentials, the well-known relation [2]

$$e_q(z)E_q(-z) = 1 \quad (4.21)$$

follows from the relation in Eq. (4.18). It is observed that whereas in the usual q -theory [71] $e_q(z)$ is ${}_1\phi_0(0; -; q, z)$ and $E_q(z)$ is ${}_0\phi_0(-; -; q, -z)$, in the (p, q) -series formalism both $e_q(z)$ and $E_q(z)$ belong to the same ${}_1\Phi_0$ -series. This recommends the native definitions [13]

$$e_{p,q}(z) = {}_1\Phi_0((1,0);-;(p,q),z) = \sum_{n=0}^{\infty} p^{n(n-1)/2} \frac{z^n}{((p,q);(p,q))_n}, \quad (4.22)$$

$$E_{p,q}(z) = {}_1\Phi_0((0,1);-;(p,q),z) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{z^n}{((p,q);(p,q))_n}, \quad (4.23)$$

for the (p,q) -exponentials such that

$$e_{p,q}(z)E_{p,q}(-z) = 1 \quad (4.24)$$

which is (p,q) -extension of the relation (4.21). When $p = 1$, the (p,q) -exponentials $e_{p,q}(z)$ and $E_{p,q}(z)$ reduce to the q -exponentials $e_q(z)$ and $E_q(z)$, respectively.

For $n = 3$, the above result and the relation (4.17) implies that [13]

$${}_1\Phi_0((u,v);-;(p,q),z) {}_1\Phi_0((v,w);-;(p,q),z) = {}_1\Phi_0((u,q);-;(p,q),z).$$

Now, if $u = 1$, $v = a$, $w = ab$ are taken and $p = 1$, then the last equation becomes the well-known product formula for ${}_1\phi_0$, that is,

$${}_1\phi_0(a;-;q,z) {}_1\phi_0(b;-;q,z) = {}_1\phi_0(ab;-;q,z), \quad (4.25)$$

in view of the relation (4.11). Hence, a new consideration for the product formula (4.25) within the (p,q) -series formalism is procured.

4.2 (p,q) -Binomial coefficients

A natural generalization of the q -binomial coefficients (see [2, 6, 66])

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

is, for $k \in \{0, 1, \dots, n\}$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{((p,q);(p,q))_n}{((p,q);(p,q))_k ((p,q);(p,q))_{n-k}},$$

which are termed as (p,q) -binomial coefficient [12-14, 22, 23]. In conjunction with the (p,q) -number, (p,q) -binomial coefficients can be rewritten as (see Eq. (3.3))

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!},$$

where, as known,

$$[n]_{p,q}! = [n]_{p,q} [n-1]_{p,q} \cdots [2]_{p,q} [1]_{p,q} \text{ and } [0]_{p,q}! = 1.$$

By taking $a = p^n$ and $b = q^n$ in Eq. (4.17), the following result are deduced [13] that

$$\begin{aligned} {}_1\Phi_0\left((p^n, q^n); -; (p, q), z\right) &= \sum_{k=0}^{\infty} \begin{bmatrix} n-1+k \\ k \end{bmatrix}_{p,q} z^k \\ &= \frac{p^{n(n+1)/2}}{((p, p^n z); (p, q))_n} \\ &= \left\{ \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pq)^{k(k-1)/2} (-z)^k \right\}^{-1}, \end{aligned} \quad (4.26)$$

which is, obviously, a generalization of the result

$$\begin{aligned} {}_1\phi_0(q^n; -; q, z) &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \\ &= \frac{1}{(z; q)_n} \\ &= \left\{ \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} (-z)^k \right\}^{-1}. \end{aligned}$$

Taking $p = 0$ in the Eq. (4.26) yields to

$$\sum_{k=0}^{\infty} (q^{n-1}z)^k = \frac{1}{1-q^{n-1}z}.$$

Also taking $p \rightarrow q \neq 1$ gives a new form of the Eq. (4.26) as [13]

$$\begin{aligned} {}_1F_0(n; -; q^{n-1}z) &= \sum_{k=0}^{\infty} \binom{n-1+k}{k} (q^{n-1}z)^k \\ &= (1-q^{n-1}z)^{-n} \\ &= \left\{ \sum_{k=0}^n \binom{n}{k} (-q^{n-1}z)^k \right\}^{-1}. \end{aligned}$$

Hence, it is observed that, though an identity belonging to (p,q) -binomial coefficient may be derived an identity belonging to q -binomial coefficient, the identity of (p,q) -binomial coefficient offers more choices for manipulations. To see an example, if $(p,q) = (q^{-1}, q)$ is chosen in view of the identity (4.26), then the following

$$\begin{aligned} {}_1\Phi_0\left((q^{-n}, q^n); -; (q^{-1}, q), z\right) &= \sum_{k=0}^{\infty} \left[\begin{matrix} n-1+k \\ k \end{matrix} \right]_{q^{-1}, q} z^k \\ &= \frac{q^{-n(n+1)/2}}{\left((q^{-1}, zq^{-n}); (q^{-1}, q)\right)_n} \\ &= \left\{ \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{q^{-1}, q} (-z)^k \right\}^{-1} \end{aligned}$$

is derived with

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_{q^{-1}, q} = \frac{\left((q^{-1}, q); (q^{-1}, q)\right)_n}{\left((q^{-1}, q); (q^{-1}, q)\right)_k \left((q^{-1}, q); (q^{-1}, q)\right)_{n-k}},$$

which should be relevant in the context of quantum groups [13]. By the Eq. (4.26), the following can be taken:

$$p^{-n(n+1)/2} \left((p, p^n z); (p, q) \right)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (pq)^{k(k-1)/2} (-z)^k. \quad (4.27)$$

By using the formula (4.8) and by taking $z = \zeta_q / \zeta_p$, the Eq. (4.27) can be rewritten as

$$\left((p\zeta_p, p^n\zeta_q); (p, q) \right)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{n(n+1)/2+k(k-1)/2} q^{k(k-1)/2} (-1)^k \zeta_p^{n-k} \zeta_q^k.$$

If $p\zeta_p$ and $p^n\zeta_q$ are renamed as a and b , respectively, then it becomes [13]

$$\left((a, b); (p, q) \right)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} (-1)^k a^{n-k} b^k, \quad (4.28)$$

which proves that

$$(a \ominus b)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} (-1)^k a^{n-k} b^k.$$

From q -calculus, it is known that if x and y are noncommuting variables such that $xy = qyx$ (for further details, see Chapter 5 in [2]), then the Eq. (2.15) holds, namely

$$(x + y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} x^k. \quad (4.29)$$

A (p, q) -extension of this result is stated in [22] as

$$(ax + by)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k, \quad (4.30)$$

with $ab = p^{-1}ba$, $xy = qxy$, all other commutators among the variables $\{a, b, x, y\}$ vanish, p and q commute with $\{a, b, x, y\}$, and the associative law holds [13]. The Eq.

(4.30) can be proved by replacing q by q/p and (x, y) by (ax, by) in Eq. (4.29).

4.3 (p, q) -Heine transformation for ${}_2\Phi_1$

In quantum calculus, q -calculus, the Heine transformation of the ${}_2\phi_1$ series is denoted by [13]

$${}_2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1(c/b, z; az; q, b), \quad (4.31)$$

and its (p, q) -analogue is given by

$$\begin{aligned} {}_2\Phi_1((a, b), (c, d); (e, f); (p, q), z) &= \frac{((ce, de), (pe, bcz); (p, q))_\infty}{((ce, cf), (pe, acz); (p, q))_\infty} \\ &\times {}_2\Phi_1((de, cf), (pe, acz); (pe, bcz); (p, q), p/ce). \end{aligned} \quad (4.32)$$

In order to prove the Eq. (4.32), let the Eq. (4.31) be firstly considered as

$${}_2\Phi_1(b/a, d/c; f/e; q/p, \zeta) = \frac{(d/c, b\zeta/a; q/p)_\infty}{(f/e, \zeta; q/p)_\infty} {}_2\Phi_1(cf/de, \zeta; b\zeta/a; q/p, d/c).$$

Using the Eq. (4.10) and the following arguments of the type used in the Eq. (4.12), the Eq. (4.32) can be rewritten as

$$\begin{aligned} {}_2\Phi_1((a, b), (c, d); (e, f); (p, q), pe\zeta/ac) &= \frac{((c, d), (a, b\zeta); (p, q))_\infty}{((e, f), (ac/e, ac\zeta/e); (p, q))_\infty} \\ &\times {}_2\phi_1((de, cf), (1, \zeta); (a, b\zeta); (p, q), (pa/ce)). \end{aligned}$$

Taking $\zeta = acz/pe$ yields to

$${}_2\Phi_1((a, b), (c, d); (e, f); (p, q), pe\zeta/ac) = \frac{((c, d), (a, b\zeta); (p, q))_\infty}{((e, f), (ac/e, ac\zeta/e); (p, q))_\infty}$$

$$\times {}_2\Phi_1((de, cf), (1, \zeta); (a, b\zeta); (p, q), (pa/ce)),$$

hence, the asserted result is obtained, (p, q) -Heine transformation formula (4.32) for ${}_2\Phi_1$, see [13].

Substituting $a = 0$, $b = c = e = 1$, relabeling d as a and f as b , and taking $p = 1$ in the Eq. (4.32) gives

$${}_1\phi_1(a; b; q, z) = \frac{(a, z; q)_\infty}{(b; z)_\infty} {}_2\phi_1(0, b/a; z; q, a), \quad (4.33)$$

which can be derived directly from q -Heine transformation formula (4.31) by using the limiting process of confluence, that is, changing z by z/a and applying to the limit $a \rightarrow \infty$, and then renaming the parameters. Here, $z = b/a$ is taken in (4.33), by using the q -binomial theorem, then the summation formula is gained by [13]

$${}_1\phi_1(a; b; q, b/a) = \frac{(b/a; q)_\infty}{(b; z)_\infty}.$$

CHAPTER 5

THE (p,q) -POWER BASIS

In this chapter, firstly (p,q) -analogue of $(x-a)^n$ or so called (p,q) -power is dealt with and then some properties of (p,q) -basis are investigated extensively.

Let (p,q) -factorial, $[n]_{p,q}! = [n]_{p,q} [n-1]_{p,q} \cdots [2]_{p,q} [1]_{p,q}$ ($n \geq 1$) with $[0]_{p,q}! = 1$ be recalled (see Chapter 3).

Definition 5.1 (Jagannathan [14], Sadjang [19, 20]). *The (p,q) -analogue of $(x-a)^n$ is the polynomial*

$$(x \ominus a)_{p,q}^n := \begin{cases} (x-a)(px-aq) \cdots (p^{n-2}x-aq^{n-2})(p^{n-1}x-aq^{n-1}), & \text{if } n \neq 0, \\ 1, & \text{if } n = 0, \end{cases} \quad (5.1)$$

which is earlier mentioned in the previous chapter.

These polynomials will be used to construct some (p,q) -Taylor formulas in the next chapter.

5.1 Some Properties of the Polynomials $(x \ominus a)_{p,q}^n$

Here are some properties of the polynomial $(x \ominus a)_{p,q}^n$ as follows.

Proposition 5.1.1 (Sadjang [19], Milanovic *et al.* [21]). *For $n \geq 1$, the following expressions hold true:*

$$D_{p,q}(x \ominus a)_{p,q}^n = [n]_{p,q} (px \ominus a)_{p,q}^{n-1} \quad (5.2)$$

and $D_{p,q}(x \ominus a)_{p,q}^0 = 0$.

Proof. The above formula shall prove by mathematical induction method. When $n=1$ the formula is clearly valid. Supposing that $D_{p,q}(x \ominus a)_{p,q}^k = [k]_{p,q}(px \ominus a)_{p,q}^{k-1}$ for some integer k . By using the definition of (p,q) -power, then $(x \ominus a)_{p,q}^{k+1} = (x \ominus a)_{p,q}^k (p^k x - q^k a)$ can be written. From Eq. (3.6), it follows

$$\begin{aligned} D_{p,q}(x \ominus a)_{p,q}^{k+1} &= p^k (px \ominus a)_{p,q}^k + (p^k qx - q^k a) [k]_{p,q} (px \ominus a)_{p,q}^{k-1} \\ &= p^k (px \ominus a)_{p,q}^k + q(p^k x - q^{k-1} a) [k]_{p,q} (px \ominus a)_{p,q}^{k-1} \\ &= (p^k + q[k]_{p,q}) (px \ominus a)_{p,q}^k = [k+1]_{p,q} (px \ominus a)_{p,q}^k. \end{aligned}$$

This completes the proof of the theorem.

Proposition 5.1.2 (Sadjang [19]). *Let λ be a complex number and $n \geq 1$ be an integer, then*

$$D_{p,q}(\lambda x \ominus a)_{p,q}^n = \lambda [n]_{p,q} (\lambda px \ominus a)_{p,q}^{n-1}.$$

Proof. The above formula shall prove by mathematical induction method. When $n=1$ the formula is clearly valid. Supposing that $D_{p,q}(\lambda x \ominus a)_{p,q}^k = \lambda [k]_{p,q} (\lambda px \ominus a)_{p,q}^{k-1}$ for some integer k . By using the definition of (p,q) -power, then $(\lambda x \ominus a)_{p,q}^{k+1} = (\lambda x \ominus a)_{p,q}^k (\lambda p^k x - q^k a)$ can be written. From Eq. (3.6), this calculations

$$\begin{aligned} D_{p,q}(\lambda x \ominus a)_{p,q}^{k+1} &= \lambda p^k (\lambda px \ominus a)_{p,q}^k + \lambda (\lambda p^k qx - q^k a) [k]_{p,q} (\lambda px \ominus a)_{p,q}^{k-1} \\ &= \lambda p^k (\lambda px \ominus a)_{p,q}^k + \lambda q (\lambda p^k x - q^{k-1} a) [k]_{p,q} (\lambda px \ominus a)_{p,q}^{k-1} \\ &= \lambda (p^k + q[k]_{p,q}) (\lambda px \ominus a)_{p,q}^k = \lambda [k+1]_{p,q} (\lambda px \ominus a)_{p,q}^k \end{aligned}$$

complete the proof of the theorem.

If k times the derivative operator $D_{p,q}$ is applied to $(x \ominus a)_{p,q}^n$ successively, then it is observed that

$$\begin{aligned}
D_{p,q}(x \ominus a)_{p,q}^n &= [n]_{p,q} (px \ominus a)_{p,q}^{n-1} \\
D_{p,q}^2(x \ominus a)_{p,q}^n &= p[n]_{p,q} [n-1]_{p,q} (p^2 x \ominus a)_{p,q}^{n-2} \\
D_{p,q}^3(x \ominus a)_{p,q}^n &= pp^2 [n]_{p,q} [n-1]_{p,q} [n-2]_{p,q} (p^3 x \ominus a)_{p,q}^{n-3} \\
&\vdots \\
D_{p,q}^k(x \ominus a)_{p,q}^n &= p^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} (p^k x \ominus a)_{p,q}^{n-k}.
\end{aligned}$$

As a consequence of above applications, the following result is stated.

Proposition 5.1.3 (Sadjang [19]). *Let $n \in \mathbb{Z}$ with $n \geq 1$ and $0 \leq k \leq n$. The following holds true*

$$D_{p,q}^k(x \ominus a)_{p,q}^n = p^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} (p^k x \ominus a)_{p,q}^{n-k}. \quad (5.3)$$

Here are some other properties of the polynomial $(x - a)_{p,q}^n$.

As in q -calculus (see Eq. (3.6) in [2]), now let whether there is (p,q) -extension of the equality $(x \ominus a)_{p,q}^{n+m} = (x \ominus a)_{p,q}^n (x \ominus a)_{p,q}^m$ or not be explored. Thus, it can be seen that

$$\begin{aligned}
(x \ominus a)_{p,q}^{n+m} &= (x - a)(px - aq) \cdots (p^{m-1}x - aq^{m-1})(p^m x - aq^m) \\
&\quad \times (p^{m+1}x - aq^{m+1}) \cdots (p^{m+n-1}x - aq^{m+n-1}) \\
&= ((x - a)(px - aq) \cdots (p^{m-1}x - aq^{m-1})) \\
&\quad \times ((p^m x - aq^m)(p(p^m x) - (aq^m)q) \cdots (p^{n-1}(p^m x) - (aq^m)q^{n-1})),
\end{aligned}$$

which yields to

$$(x \ominus a)_{p,q}^{n+m} = (x \ominus a)_{p,q}^m (p^m x \ominus a q^m)_{p,q}^n. \quad (5.4)$$

If one substitutes m by n in Eq. (5.4), then our definition is extended to the following equation [19]

$$(x \ominus a)_{p,q}^{-n} = \frac{1}{(p^{-n} x \ominus q^{-n} a)_{p,q}^n}, \quad (5.5)$$

for all non-negative integer.

The following results show that this really provides a good extension.

Proposition 5.1.4 (Sadjang [19]). *The Eq. (5.4) holds true for any two integers n and m .*

Proof. The case in which $m > 0$ and $n > 0$ has earlier been proved. In addition, the case in which one of m and n is zero is clear. Now let the case $m = -m' < 0$ be firstly considered. Then, it can be observed that

$$(x \ominus a)_{p,q}^m (p^m x \ominus a q^m)_{p,q}^n = (x \ominus a)_{p,q}^{-m'} (p^{-m'} x \ominus a q^{-m'})_{p,q}^n$$

$$\text{by Eq. (5.5)} = \frac{(p^{-m'} x \ominus a q^{-m'})_{p,q}^n}{(p^{-m'} x \ominus a q^{-m'})_{p,q}^{-m'}}$$

$$\text{by Eq. (5.4)} = \begin{cases} (p^m (p^{-m'} x) \ominus (a q^{-m'}) q^m)_{p,q}^{n-m'} & \text{if } n \geq m' \\ \frac{1}{(q^n (p^{-m'} x) \ominus (a q^{-m'}) q^n)_{p,q}^{m-n'}} & \text{if } n < m' \end{cases}$$

$$\text{by Eq. (5.5)} = (x \ominus a)_{p,q}^{n-m'} = (x \ominus a)_{p,q}^{n+m}.$$

If $m \geq 0$ and $n = -n'$, then

$$(x \ominus a)_{p,q}^m (p^m x \ominus a q^m)_{p,q}^n = (x \ominus a)_{p,q}^m (p^m x \ominus a q^m)_{p,q}^{-n'}$$

$$\text{by Eq. (5.5)} = \frac{(x \ominus a)_{p,q}^m}{\left(p^{m-n'} x \ominus a q^{m-n'}\right)_{p,q}^{n'}}$$

$$\text{by Eq. (5.4)} = \begin{cases} \frac{\left(p^{m-n'} x \ominus a q^{m-n'}\right)_{p,q}^{n'} (x \ominus a)_{p,q}^{m-n'}}{\left(p^{m-n'} x \ominus a q^{m-n'}\right)_{p,q}^{n'} (x \ominus a)_{p,q}^m} & \text{if } m \geq n' \\ \frac{(x \ominus a)_{p,q}^m}{\left(p^{m-n'} x \ominus a q^{m-n'}\right)_{p,q}^{n'-m} \left(p^{n'-m} (p^{m-n'} x) \ominus (a q^{m-n'}) q^{n'-m}\right)_{p,q}^m} & \text{if } m < n' \end{cases}$$

$$= \begin{cases} (x \ominus a)_{p,q}^{m-n'} & \text{if } m \geq n' \\ \frac{1}{\left(p^{m-n'} x \ominus a q^{m-n'}\right)_{p,q}^{n'-m}} & \text{if } m < n' \end{cases}$$

$$= (x \ominus a)_{p,q}^{m-n'} = (x \ominus a)_{p,q}^{m+n}.$$

At last, if $m = -m' < 0$ and $n = -n' < 0$, then

$$\begin{aligned} (x \ominus a)_{p,q}^m (p^m x \ominus a q^m)_{p,q}^n &= (x \ominus a)_{p,q}^{-m'} (p^{-m'} x \ominus a q^{-m'})_{p,q}^{-n'} \\ &= \frac{1}{\left(p^{-m'} x \ominus a q^{-m'}\right)_{p,q}^{m'} \left(p^{-n'-m'} x \ominus a q^{-n'-m'}\right)_{p,q}^{n'}} \\ &= \frac{1}{\left(p^{-n'-m'} x \ominus a q^{-n'-m'}\right)_{p,q}^{n'+m'}} \\ &= (x \ominus a)_{p,q}^{-m'-n'} = (x \ominus a)_{p,q}^{m+n}. \end{aligned}$$

Thus, the expected results are obtained.

Whether the Eq. (5.2) also holds true for any number or not would like to be shown. However, before determining this, let the definition of (p,q) -number in (3.1) be extended as follows.

Definition 5.1.1 (Sadjang [19]). *For any number γ ,*

$$[\gamma]_{p,q} = \frac{p^\gamma - q^\gamma}{p - q}.$$

Now, the Eq. (5.2) is given for any integer below.

Proposition 5.1.5 (Sadjang [19]). *For any integer n ,*

$$D_{p,q}(x \ominus a)_{p,q}^n = [n]_{p,q} (px \ominus a)_{p,q}^{n-1}.$$

Proof. Note that $[0]_{p,q} = 0$, thus Eq. (5.2) is valid for $n = 0$. If $n = -n' < 0$, from the property (3.9) and Eq. (5.5) it becomes

$$\begin{aligned} D_{p,q}(x \ominus a)_{p,q}^n &= D_{p,q} \left(\frac{1}{(p^{-n'} x \ominus a q^{-n'})_{p,q}^{n'}} \right) \\ &= - \frac{D_{p,q}(p^{-n'} x \ominus a q^{-n'})_{p,q}^{n'}}{(p^{-n'} p x \ominus a q^{-n'})_{p,q}^{n'} (p^{-n'} q x \ominus a q^{-n'})_{p,q}^{n'}} \\ &= - \frac{[n']_{p,q} p^{-n'} (p^{-n'} p x \ominus a q^{-n'})_{p,q}^{n'-1}}{q^{n'} (p^{-n'} p x \ominus a q^{-n'})_{p,q}^{n'} (p^{1-n'} p x \ominus a q^{1-n'})_{p,q}^{n'}} \\ &= - \frac{p^{n'} - q^{n'}}{p - q} \frac{p^{-n'} (p^{-n'} p x \ominus a q^{-n'})_{p,q}^{n'-1}}{q^{n'} (p^{-n'} p x \ominus a q^{-n'})_{p,q}^{n'} (p^{1-n'} p x \ominus a q^{1-n'})_{p,q}^{n'}} \end{aligned}$$

$$\begin{aligned}
&= -\frac{p^{n'} - q^{n'}}{p - q} \frac{p^{-n'} q^{-n'}}{\left(p^{-1} p x - a q^{-1}\right) \left(p^{1-n'} p x \ominus a q^{1-n'}\right)_{p,q}^{n'}} \\
&= \frac{p^{-n'} - q^{-n'}}{p - q} \frac{1}{\left(p^{1-n'} p x \ominus a q^{1-n'}\right)_{p,q}^{n'+1}} \\
&= \frac{p^n - q^n}{p - q} (p x \ominus a)_{p,q}^{n-1}.
\end{aligned}$$

Therefore, the asserted result is acquired.

Similarly in q -calculus [2], another property of $(x \ominus a)_{p,q}^n$ is

$$(x \ominus a)_{p,q}^n \neq (-1)^n (a \ominus x)_{p,q}^n.$$

Alternatively,

$$\begin{aligned}
(a \ominus x)_{p,q}^n &= (a - x)(p a - x q)(p^2 a - x q^2) \cdots (p^{n-1} a - x q^{n-1}) \\
&= (a - x) p q (q^{-1} a - x p^{-1}) p^2 q^2 (q^{-2} a - x p^{-2}) \cdots p^{n-1} q^{n-1} (p^{1-n} a - x p^{1-n}) \\
&= (-1)^n (p q)^{\binom{n}{2}} (p^{1-n} x - a q^{1-n}) (p^{2-n} x - a q^{2-n}) \cdots (p^{-2} x - a q^{-2}) (p^{-1} x - a q^{-1})
\end{aligned}$$

and thus,

$$(a \ominus x)_{p,q}^n = (-1)^n (p q)^{\binom{n}{2}} (p^{1-n} x \ominus a q^{1-n})_{p,q}^n. \quad (5.6)$$

Clearly, the formula in Eq. (5.6) is true for $n = 0$. It is straightforward to confirm that it is true for $n < 0$.

5.2 (p,q) -Derivatives of the Some Known Polynomials

Firstly, the derivative of the function $\frac{1}{(x \ominus a)_{p,q}^n}$ is found by using the

Definition 5.1. For any integer n , by using Eq. (5.5) it can be computed as [19]

$$\begin{aligned}
 D_{p,q} \frac{1}{(x \ominus a)_{p,q}^n} &= D_{p,q} \left(\frac{1}{(p^{-n}(p^n x) \ominus q^{-n}(q^n a))_{p,q}^n} \right) \\
 &= D_{p,q} (p^n x \ominus q^n a)_{p,q}^{-n} \\
 &= p^n [-n]_{p,q} (p^{n+1} x - q^n a)_{p,q}^{-n-1} \\
 &= p^n \left(-(qp)^{-n} [n]_{p,q} \right) (p^{n+1} x \ominus q^n a)_{p,q}^{-n-1} \\
 &= -q[n]_{p,q} (p^{n+1} x \ominus q^{n+1} a)_{p,q}^{-n-1} \\
 &= -q[n]_{p,q} \frac{1}{(qx \ominus a)_{p,q}^{n+1}}.
 \end{aligned}$$

By the similar way, it can be deduced using Eq. (5.6), for $n \geq 0$ as [19]:

$$\begin{aligned}
 D_{p,q} (a \ominus x)_{p,q}^n &= D_{p,q} \left((-1)^n (pq) \binom{n}{2} (p^{1-n} x \ominus a q^{1-n})_{p,q}^n \right) \\
 &= (-1)^n (pq) \binom{n}{2} [n]_{p,q} p^{1-n} (p^{1-n} p x \ominus q^{1-n} a)_{p,q}^{n-1} \\
 &= -(-1)^{n-1} (pq) \binom{n-1}{2} [n]_{p,q} (p^{2-n} q x \ominus q^{2-n} a)_{p,q}^{n-1} \\
 &= -[n]_{p,q} (a \ominus qx)_{p,q}^{n-1}
 \end{aligned}$$

and

$$D_{p,q} \frac{1}{(a \ominus x)_{p,q}^n} = \frac{-D_{p,q} (a \ominus x)_{p,q}^n}{(a - px)_{p,q}^n (a \ominus qx)_{p,q}^n}$$

$$\begin{aligned}
&= \frac{[n]_{p,q} (a \ominus qx)_{p,q}^{n-1}}{(a \ominus px)_{p,q}^n (a \ominus qx)_{p,q}^n} \\
&= \frac{p[n]_{p,q}}{(a \ominus px)_{p,q}^n (p^n a \ominus q^n px)} \\
&= \frac{p[n]_{p,q}}{(a \ominus px)_{p,q}^{n+1}}.
\end{aligned}$$

As a consequence of above applications, the formulas can be stated as:

$$D_{p,q} \frac{1}{(x \ominus a)_{p,q}^n} = \frac{-q[n]_{p,q}}{(qx \ominus a)_{p,q}^{n+1}}, \quad (5.7)$$

$$D_{p,q} (a \ominus x)_{p,q}^n = -[n]_{p,q} (a \ominus qx)_{p,q}^{n-1}, \quad (5.8)$$

$$D_{p,q} \frac{1}{(a \ominus x)_{p,q}^n} = \frac{p[n]_{p,q}}{(a \ominus px)_{p,q}^{n+1}}. \quad (5.9)$$

Applying k times the derivative operator $D_{p,q}$ to $(a \ominus x)_{p,q}^n$ successively yields to

$$\begin{aligned}
D_{p,q} \frac{1}{(x \ominus a)_{p,q}^n} &= \frac{-q[n]_{p,q}}{(qx \ominus a)_{p,q}^{n+1}} \\
D_{p,q}^2 \frac{1}{(x \ominus a)_{p,q}^n} &= \frac{q^3 [n]_{p,q} [n+1]_{p,q}}{(q^2 x \ominus a)_{p,q}^{n+2}} \\
D_{p,q}^3 \frac{1}{(x \ominus a)_{p,q}^n} &= -\frac{q^6 [n]_{p,q} [n+1]_{p,q} [n+2]_{p,q}}{(q^3 x \ominus a)_{p,q}^{n+3}} \\
&\vdots \\
D_{p,q}^k \frac{1}{(x \ominus a)_{p,q}^n} &= (-1)^k q^{\binom{k}{2}} \frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!} \frac{1}{(q^k x \ominus a)_{p,q}^{n+k}}.
\end{aligned}$$

Hence, the following result is given.

Proposition 5.2.1 (Duran et al. [26]). Let $n \geq 1$ be an integer, and $0 \leq k \leq n$, the following equation satisfies

$$D_{p,q}^k \frac{1}{(x \ominus a)_{p,q}^n} = (-1)^k q^{\binom{k}{2}} \frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!} \frac{1}{(q^k x \ominus a)_{p,q}^{n+k}}. \quad (5.10)$$

In a like manner above, it can be shown that

$$\begin{aligned} D_{p,q}(a \ominus x)_{p,q}^n &= -[n]_{p,q} (a \ominus qx)_{p,q}^{n-1} \\ D_{p,q}^2(a \ominus x)_{p,q}^n &= [n]_{p,q} [n-1]_{p,q} (a \ominus q^2 x)_{p,q}^{n-2} \\ D_{p,q}^3(a \ominus x)_{p,q}^n &= -qq^2 [n]_{p,q} [n-1]_{p,q} [n-2]_{p,q} (a \ominus q^3 x)_{p,q}^{n-3} \\ &\vdots \\ D_{p,q}^k(a \ominus x)_{p,q}^n &= (-1)^k q^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} (a \ominus q^k x)_{p,q}^{n-k}, \end{aligned}$$

thus the following result is obtained.

Proposition 5.2.2 (Sadjang [19]). Let $n \geq 1$ be an integer, and $0 \leq k \leq n$, the following equality is valid:

$$D_{p,q}^k (a \ominus x)_{p,q}^n = (-1)^k q^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} (a \ominus q^k x)_{p,q}^{n-k}. \quad (5.11)$$

Further, it can be observed that

$$D_{p,q} \frac{1}{(a \ominus x)_{p,q}^n} = \frac{p[n]_{p,q}}{(a \ominus px)_{p,q}^{n+1}}$$

$$\begin{aligned}
D_{p,q}^2 \frac{1}{(a \ominus x)_{p,q}^n} &= \frac{p^3 [n]_{p,q} [n+1]_{p,q}}{(a \ominus p^2 x)_{p,q}^{n+2}} \\
D_{p,q}^3 \frac{1}{(a \ominus x)_{p,q}^n} &= \frac{p^6 [n]_{p,q} [n+1]_{p,q} [n+2]_{p,q}}{(a \ominus p^3 x)_{p,q}^{n+3}} \\
&\vdots \\
D_{p,q}^k \frac{1}{(a \ominus x)_{p,q}^n} &= p^{\binom{k}{2}} \frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!} \frac{1}{(a \ominus p^k x)_{p,q}^{n+k}}.
\end{aligned}$$

Therefore, the following result is archived.

Proposition 5.2.3 (Duran *et al.* [26]). *Let $n \geq 1$ be an integer, and $0 \leq k \leq n$, it is satisfied that*

$$D_{p,q}^k \frac{1}{(a \ominus x)_{p,q}^n} = p^{\binom{k}{2}} \frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!} \frac{1}{(a \ominus p^k x)_{p,q}^{n+k}}. \quad (5.12)$$

CHAPTER 6

(p,q)-TAYLOR FORMULAS

In this chapter, two Taylor formulas for polynomials in the light of (p,q)-calculus are given and then some of their consequences are researched extensively.

Now the first Taylor formula for polynomials is given as follows.

Theorem 6.1 (Sadjang [19]). *For any polynomial $f(x)$ of degree N , and any number a , the following Taylor expansion holds*

$$f(x) = \sum_{k=0}^N p^{-\binom{k}{2}} \frac{(D_{p,q}^k f)(ap^{-k})}{[k]_{p,q}!} (x \ominus a)_{p,q}^k. \quad (6.1)$$

Proof. Let f be a polynomial of degree N , then the following expansion is satisfied:

$$f(x) = \sum_{j=0}^N c_j (x \ominus a)_{p,q}^j. \quad (6.2)$$

Let k be an integer such that $0 \leq k \leq N$, then, applying $D_{p,q}^k$ on both sides of Eq. (6.2) and using Eq. (5.3) the following formula is obtained:

$$(D_{p,q}^k f)(x) = \sum_{j=k}^N c_j p^{\binom{k}{2}} \frac{[j]_{p,q}!}{[j-k]_{p,q}!} (p^k x \ominus a)_{p,q}^{j-k}.$$

If $x = ap^{-k}$ is chosen in above equality, then it becomes

$$(D_{p,q}^k f)(ap^{-k}) = c_k [k]_{p,q}! p^{\binom{k}{2}}.$$

Hence, it is found that

$$c_j = p^{-\binom{k}{2}} \frac{(D_{p,q}^k f)(ap^{-k})}{[k]_{p,q}!}.$$

So, this completes the proof.

Corollary 6.1 (Sadjang [19]). *The following holds*

$$x^n = \sum_{k=0}^n p^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (ap^{-k})^{n-k} (x \ominus a)_{p,q}^k.$$

Theorem 6.2 (Sadjang [19]). *For any polynomial $f(x)$ of degree N , and any number a , the following Taylor expansion holds*

$$f(x) = \sum_{k=0}^N (-1)^k q^{-\binom{k}{2}} \frac{(D_{p,q}^k f)(aq^{-k})}{[k]_{p,q}!} (a \ominus x)_{p,q}^k.$$

Proof. Let f be a polynomial of degree N , then the following expansion is fulfilled:

$$f(x) = \sum_{j=0}^N c_j (a \ominus x)_{p,q}^j. \quad (6.3)$$

Let k be an integer such that $0 \leq k \leq N$, then, applying $D_{p,q}^k$ on both sides of Eq. (6.3) and using Eq. (5.11) the following can be obtained:

$$(D_{p,q}^k f)(x) = \sum_{j=k}^N c_j q^{\binom{k}{2}} \frac{[j]_{p,q}!}{[j-k]_{p,q}!} (a \ominus q^k x)_{p,q}^{j-k}.$$

If $x = ap^{-k}$ is taken, then the following is acquired:

$$(D_{p,q}^k f)(ap^{-k}) = c_k (-1)^k [k]_{p,q}! q^{\binom{k}{2}}.$$

Hence, it becomes

$$c_j = (-1)^k q^{-\binom{k}{2}} \frac{(D_{p,q}^k f)(aq^{-k})}{[k]_{p,q}!}.$$

So, this completes the proof.

Corollary 6.2 (Sadjang [19]). *The following identity holds true*

$$x^n = \sum_{k=0}^n (-1)^k q^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (aq^{-k})^{n-k} (a \ominus x)_{p,q}^k. \quad (6.4)$$

Corollary 6.3 (Sadjang [19]). *The following relations holds true:*

$$(x \ominus b)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (a \ominus b)_{p,q}^{n-k} (x \ominus a)_{p,q}^k, \quad (6.5)$$

$$(b \ominus x)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (b \ominus a)_{p,q}^{n-k} (a \ominus x)_{p,q}^k. \quad (6.6)$$

Remark 6.1 (Sadjang [19]). *If $b = ab$ is chosen in Eqs. (6.5) and (6.6), then the following expressions are obtained:*

$$(x \ominus ab)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} (1 \ominus b)_{p,q}^{n-k} (x \ominus a)_{p,q}^k, \quad (6.7)$$

$$(ab \ominus x)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} (b \ominus 1)_{p,q}^{n-k} (a \ominus x)_{p,q}^k. \quad (6.8)$$

Note that taking $x=1$ and $p=1$ in Eqs. (6.7) and (6.8) gives the following results

$$(1 \ominus ab)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} (1 \ominus b)_q^{n-k} (1 \ominus a)_q^k, \quad (6.9)$$

$$(ab \ominus 1)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} (b \ominus 1)_{p,q}^{n-k} (a \ominus 1)_q^k. \quad (6.10)$$

The Eq. (6.9) is known as q -binomial theorem.

Notice that the Eqs. (6.7) and (6.8) are obvious generalizations of Eqs. (6.9) and (6.10), respectively.

Taking the value $a=1$ in the Eq. (5.12), (p,q) -derivative of k -th order is derived as follows:

$$D_{p,q}^k \frac{1}{(1 \ominus x)_{p,q}^n} = p^{\binom{k}{2}} \frac{[n+k-1]_{p,q}!}{[n-1]_{p,q}!} \frac{1}{(1 \ominus p^k x)_{p,q}^{n+k}}.$$

In the case $x=0$, for any $k \geq 1$, it gives

$$p^{\binom{k}{2}} [n]_{p,q} [n+1]_{p,q} \cdots [n+k-1]_{p,q}.$$

By the Eq. (6.1), it can be found that, i.e., a Taylor expansion for $\frac{1}{(1-x)_{p,q}^n}$ about $x=0$:

$$\begin{aligned} \frac{1}{(1 \ominus x)_{p,q}^n} &= 1 + \sum_{k=0}^{\infty} p^{k-\binom{k}{2}} \frac{[n]_{p,q} [n+1]_{p,q} \cdots [n+k-1]_{p,q}}{[k]_{p,q}!} x^k \\ &= 1 + \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} p^{k-\binom{k}{2}} x^k, \end{aligned} \quad (6.11)$$

which is (p,q) -analogue of Taylor's expansion of $f(x) = \frac{1}{(1-x)^n}$ in ordinary calculus (*cf.* [19]). Also, Eq. (6.11) is a (p,q) -extension of Heine's Binomial formula denoted by

$$\frac{1}{(1 \ominus x)_q^n} = 1 + \sum_{k=1}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k.$$

6.1 Gauss's Binomial Formula

Here (p,q) -binomial formula is found by inspiring the method of the Nalci and Pashaev as they have completed for multiple q -binomial formula in Chapter 9 of the book [69]. Now, this section starts with the Gauss's Binomial formula

$$(x \oplus a)_q^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} a^k x^{n-k} \quad (6.12)$$

where

$$(x \oplus a)_q^n := \begin{cases} (x+a)(x+aq)\cdots(x+aq^{n-2})(x+aq^{n-1}), & \text{if } n \neq 0 \\ 1, & \text{if } n = 0. \end{cases}$$

To get similar formula for (p,q) -polynomial, it can be written as

$$\begin{aligned} (x \oplus a)_{p,q}^n &= (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}) \\ &= (x+a)p(x+aQ)\cdots p^{n-2}(x+aQ^{n-2})p^{n-1}(x+aQ^{n-1}) \\ &= p^{\binom{n}{2}}(x \oplus a)_Q^n, \end{aligned}$$

where $Q = \frac{q}{p}$. From Eq. (6.12), it is composed by

$$(x \oplus a)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_Q Q^{\binom{k}{2}} x^{n-k} a^k, \quad (6.13)$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_Q$ are Q -binomials coefficients. Note that

$$[n]_{p,q} = p^{n-1} [n]_Q \quad (6.14)$$

and

$$[n]_{p,q}! = p^{\binom{n}{2}} [n]_Q! \quad (6.15)$$

By using Eqs. (6.13) and (6.15), it can be obtained as

$$\begin{aligned}
(x \oplus a)_{p,q}^n &= p^{\binom{n}{2}} \sum_{k=0}^n \frac{[n]_Q!}{[n-k]_Q! [k]_Q!} Q^{\binom{k}{2}} x^{n-k} a^k \\
&= p^{\binom{n}{2}} \sum_{k=0}^n \frac{p^{-\binom{n}{2}} [n]_{p,q}!}{p^{-\binom{n-k}{2}} [n-k]_{p,q}! p^{-\binom{k}{2}} [k]_{p,q}!} \left(\frac{q}{p}\right)^{\binom{k}{2}} x^{n-k} a^k \\
&= \sum_{k=0}^n \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^{n-k} a^k,
\end{aligned}$$

thus, the following (p,q) -Gauss Binomial formulas for commutative x and a ($xa = ax$) are investigated:

$$(x \oplus a)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^{n-k} a^k \quad (6.16)$$

and also

$$(x \ominus a)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^{n-k} (-a)^k. \quad (6.17)$$

As special cases of Eqs. (6.16) and (6.17), the following equalities hold:

$$(x \oplus 1)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^{n-k} \quad (6.18)$$

and

$$(x \ominus 1)_{p,q}^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^{n-k} (-1)^k. \quad (6.19)$$

CHAPTER 7

(p,q)-BINOMIAL COEFFICIENTS

Let firstly the (p,q)-binomial coefficients be remembered as (see Eq. (3.3))

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!} \quad (7.1)$$

where $[n]_{p,q}! = [n]_{p,q} [n-1]_{p,q} \cdots [2]_{p,q} [1]_{p,q}$ ($n > 0$) and $[0]_{p,q}! = 1$.

Let some properties of the (p,q)-binomials coefficients with n, k being nonnegative integers and $k \leq n$ be examined. When $p=1$ and $q \rightarrow p=1$ are chosen since again the ordinary binomial coefficients and q -binomials coefficients are found back, respectively, their (p,q)-analogues to possess similar properties are expected [12].

It may be worth noticing that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ k \end{bmatrix}_{q,p}, \quad (7.2)$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \sum_{i=0}^k \frac{p^{n-i+1} - q^{n-i+1}}{p - q} \quad (p \neq q),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \sum_{i=0}^k \frac{q^{n-i+1} - p^{n-i+1}}{q - p} \quad (p \neq q),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}}{[k]_{p,q}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}.$$

When $p = 1$, the (p,q) -binomials coefficient in Eq. (7.1) reduces to the q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$ and also when $q \rightarrow p = 1$, the (p,q) -binomials coefficient in Eq. (7.1) reduces to the usual binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Now, the properties and identities of the (p,q) -binomial coefficients are studied by motivating and by inspiring the results for q -binomials coefficients and usual binomial coefficients. Some special cases of the given a clearer description of the (p,q) -binomial coefficients are explored.

To get a rapid computation of the first amounts of (p,q) -binomial coefficients, the triangular recurrence relations which are also called by (p,q) -Pascal rules are given as follows.

Theorem 7.1 (Corcino [12]). *The (p,q) -binomial coefficients satisfy the following triangular recurrence relationships*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \quad (7.3)$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \quad (7.4)$$

with initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{p,q} = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = 0$ for $k > n$.

Proof. Let $0 < k < n$. In view of

$$\begin{aligned} [n+1]_{p,q} &= p^n + p^{n-1}q + \cdots + p^k q^{n-k} + p^{k-1} q^{n-k+1} + \cdots + pq^{n-1} + q^n \\ &= p^k (p^{n-k} + p^{n-k-1}q + \cdots + q^{n-k}) + q^{n-k+1} (p^{k-1} + p^{k-2}q + \cdots + q^{k-1}) \end{aligned}$$

$$= p^k [n-k+1]_{p,q} + q^{n-k+1} [k]_{p,q},$$

it becomes

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} &= \frac{[n+1]_{p,q}!}{[n-k+1]_{p,q}! [k]_{p,q}!} \\ &= \frac{[n+1]_{p,q} [n]_{p,q}!}{[n-k+1]_{p,q}! [k]_{p,q}!} = \frac{(p^k [n-k+1]_{p,q} + q^{n-k+1} [k]_{p,q}) [n]_{p,q}!}{[n-k+1]_{p,q}! [k]_{p,q}!} \\ &= p^k \frac{[n-k+1]_{p,q} [n]_{p,q}!}{[n-k+1]_{p,q}! [k]_{p,q}!} + q^{n-k+1} \frac{[k]_{p,q} [n]_{p,q}!}{[n-k+1]_{p,q}! [k]_{p,q}!} \\ &= p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}. \end{aligned}$$

Thus, the asserted result in Eq. (7.3) is obtained. To prove the formula (7.4), let the proved result in Eq. (7.3) be rewritten as

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{q,p} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{q,p} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{q,p},$$

by using the property in Eq. (7.2), it is found that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = q^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q},$$

as expected.

Choosing $p = 1$, Eq. (7.4) gives $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ which precisely equivalent to Eq. (2.21). Similarly, taking $p = 1$, Eq. (7.3) gives

$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$ being another form of the triangular recurrence relation for q -binomial coefficients.

Now, the first values of (p,q) -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ can be generated by using Theorem 7.1 as given in the following figure [12].

n/k	1	2	3	4	5
1	1				
2	$p+q$	1			
3	$p^2 + pq + q^2$	$p^2 + pq + q^2$	1		
4	$p^3 + p^2q + pq^2 + q^3$	$(p^2 + pq + q^2) \times (p^2 + q^2)$	$p^3 + p^2q + pq^2 + q^3$	1	
5	$p^4 + p^3q + p^2q^2 + pq^3 + q^4$	$p^2(p^2 + pq + q^2)(p^2 + q^2) + q^3(p^3 + p^2q + pq^2 + q^3)$	$p^2(p^2 + pq + q^2)(p^2 + q^2) + q^3(p^3 + p^2q + pq^2 + q^3)$	$p^4 + p^3q + p^2q^2 + pq^3 + q^4$	1

Figure 7.1 Values for $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ [12]

From Figure 7.1, one can inference with ease writing the entries of Figure 7.1 as follows.

$$\begin{array}{ccccccc}
& & & & 1 & & \\
& & & & 1 & & 1 \\
& & & 1 & p+q & 1 & \\
& & 1 & p^2+pq+q^2 & p^2+pq+q^2 & 1 & \\
& 1 & p^3+p^2q+pq^2+q^3 & (p^2+pq+q^2)(p^2+q^2) & p^3+p^2q+pq^2+q^3 & 1 & \\
1 & p^4+p^3q+p^2q^2 & p^2(p^2+pq+q^2)(p^2+q^2) & p^3(p^3+p^2q+pq^2+q^3) & p^4+p^3q+p^2q^2 & 1 & \\
& +pq^3+q^4 & +q^3(p^3+p^2q+pq^2+q^3) & +q^2(p^2+pq+q^2)(p^2+q^2) & +pq^3+q^4 & &
\end{array}$$

Figure 7.2 (p,q) -Pascal Triangle [12]

Obviously, Figure 7.2 resembles Pascal's Triangle of the familiar binomial coefficients. As it can be seen, it possesses the same symmetry as Pascal Triangle. However, the identities that there exists for the (p,q) -binomial coefficients is not possible to the same structure as those in usual binomial coefficients.

Note that applying the identities in Eq. (7.3) thrice to $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q}$, then

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} &= p^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{p,q} + q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} + p^{2(k+1)} \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}_{p,q} \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \\
&\quad + p^{2(k+1)} q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} + p^{3(k+1)} \begin{bmatrix} n-2 \\ k+1 \end{bmatrix}_{p,q}.
\end{aligned}$$

If one continues this process till $(n-k)$ -th application of Eq. (7.3), then it is observed

that

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} &= p^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{p,q} + q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\
&= q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{k+1} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \\
&\quad + p^{2(k+1)} q^{n-k-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} + p^{3(k+1)} q^{n-k-3} \begin{bmatrix} n-3 \\ k \end{bmatrix}_{p,q} + \cdots + p^{(n-k)(k+1)} \begin{bmatrix} k \\ k \end{bmatrix}_{p,q}.
\end{aligned}$$

This is a vertical recurrence relation for the (p,q) -binomial coefficients, for details see [12]. Further, rewriting the identities in Eq. (7.4) as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{-(n-k)} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} - p^{3(n-k)} q^{k+1} \begin{bmatrix} n \\ k+1 \end{bmatrix}_{p,q}$$

and applying this recurrence relation to itself $(n-k)$ -times, then a horizontal recurrence relation is determined for the (p,q) -binomial coefficients. So, these results are stated in the following theorem.

Theorem 7.2 (Corcino [12]). *The (p,q) -binomials coefficients satisfy the following vertical recurrence relation*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} = \sum_{j=k}^n p^{(n-k)(k+1)} q^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_{p,q}$$

and horizontal recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \sum_{j=0}^{n-k} (-1)^j p^{-(j+1)(n-k)+\binom{j+1}{2}} q^{jk+\binom{j+1}{2}} \begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}_{p,q}.$$

For instance, let $\begin{bmatrix} 4 \\ 3 \end{bmatrix}_{p,q}$ be calculated by using above recurrence relations as

$$\begin{aligned}
\begin{bmatrix} 4 \\ 3 \end{bmatrix}_{p,q} &= p^{(3-2)(2+1)} \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{p,q} + p^{(3-3)(2+1)} q^{3-2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} \\
&= p^3 + p^2 q + p q^2 + q^3
\end{aligned}$$

and also

$$\begin{aligned}
\begin{bmatrix} 4 \\ 3 \end{bmatrix}_{p,q} &= p^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix}_{p,q} - p^{-1} q^4 \begin{bmatrix} 5 \\ 5 \end{bmatrix}_{p,q} \\
&= p^{-1} (p^4 + p^3 q + p^2 q^2 + p q^3 + q^4) - p^{-1} q^4 \\
&= p^3 + p^2 q + p q^2 + q^3,
\end{aligned}$$

as expected from the Figure 7.1.

It is possible to consider that the vertical and horizontal recurrence relations in Theorem 7.2 are (p, q) -analogues of Hockey Stick identities, which are also known as Chu Shih-Chieh's identities in Eq. (2.19). More precisely, when $p = 1$, the results in Theorem 7.2 reduce to the following vertical and horizontal recurrence relations for the q -binomial coefficients (see [12])

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \sum_{j=k}^n q^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_q$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=0}^{n-k} (-1)^j q^{jk + \binom{j+1}{2}} \begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}_q,$$

respectively. Furthermore, If $q \rightarrow 1$, then the former reduces to the Chu Shih-Chieh's identity in Eq. (2.24) and the latter reduces to the following recurrence relation for usual binomial coefficients:

$$\binom{n}{k} = \binom{n+1}{k+1} - \binom{n+1}{k+2} + \cdots + (-1)^{n-k} \binom{n+1}{n+1}.$$

The vertical and horizontal recurrence relations in Theorem 7.2 can readily be remembered with the help of the pattern as indicated in the figure below.

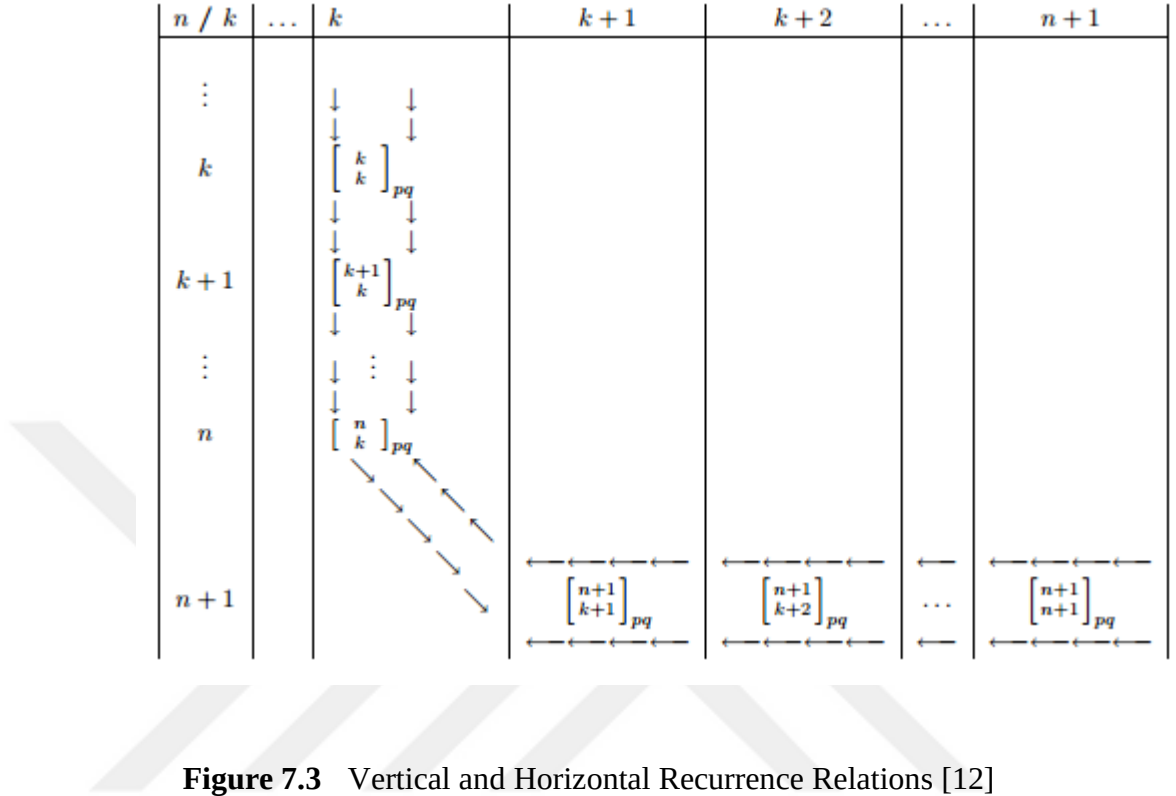


Figure 7.3 Vertical and Horizontal Recurrence Relations [12]

The entries included in the Figure 7.3 to compute the value of $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ using the vertical and horizontal recurrence relations in Theorem 7.2 apparently form a hockey stick as illustrated by the arrows in the Figure 7.3.

Now let the following observations be considered:

$$1 + x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} x$$

$$(1+x)(p+qx) = p + (p+q)x + qx^2$$

$$= p \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{p,q} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} x + q \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{p,q} x^2$$

and

$$\begin{aligned}
(1+x)(p+qx)(p^2+q^2x) &= p^3 + p(p^2+pq+q^2)x + q(p^2+pq+q^2)x^2 + q^3x^3 \\
&= p^3 \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{p,q} + p \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{p,q} x + q \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} x^2 + q^3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{p,q} x^3.
\end{aligned}$$

In view of above observation, it gives an opinion that a horizontal generating function for the (p,q) -binomial coefficients can be established. The next theorem includes the horizontal generating function aforementioned.

Theorem 7.3 (Corcino [12]). *The horizontal generating function for the (p,q) -binomial coefficients is stated as*

$$\sum_{j=0}^{n-1} (p^j + xq^j) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^k. \quad (7.5)$$

Proof. For $n = 1, 2, 3$, the generating function is already satisfied above. Assume that it is accurate for some $n \geq 1$, that is

$$\sum_{j=0}^{n-1} (p^j + xq^j) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^k.$$

Then, using the identity in Eq. (7.3), it gives

$$\begin{aligned}
\sum_{j=0}^n (p^j + xq^j) &= (p^n + xq^n) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} x^k \\
&= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}+n} q^{\binom{k}{2}} x^k + \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}+n} x^{k+1} \\
&= \sum_{k=0}^n \left\{ \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}+n} q^{\binom{k}{2}} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} p^{\binom{n-k+1}{2}} q^{\binom{k-1}{2}+n} \right\} x^k \\
&= \sum_{k=0}^n p^{\binom{n-k+1}{2}} q^{\binom{k}{2}} \left\{ p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \right\} x^k,
\end{aligned}$$

and it can be found that

$$\sum_{j=0}^n (p^j + xq^j) = \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n-k+1}{2}} q^{\binom{k}{2}} x^k.$$

As a result of Theorem 7.3, the following including an identity to those in Eqs. (2.17) and (2.22) is presented as consequence of the belowmentioned corollary.

Corollary 7.1 (Corcino [12]). *For $n \geq 1$, the following relation satisfies*

$$\sum_{k \text{ even}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} = \sum_{k \text{ odd}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}}. \quad (7.6)$$

Proof. Taking $x = -1$ in Theorem 7.3 yields to

$$\sum_{j=0}^n (p^j - q^j) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} (-1)^k.$$

The first element of the expansion of $\sum_{j=0}^{n-1} (p^j - q^j)$ equals to zero when $j = 0$, so it becomes

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}} (-1)^k = 0.$$

This exactly coincides with Eq. (7.6).

The following result is a consequence of Theorem 7.3 by taking $x = 1$.

Corollary 7.2 (cf. Corcino [12]). *For $n \geq 1$, the following equality holds true:*

$$\sum_{k=0}^n \left\{ \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}}}{\sum_{j=0}^{n-1} (p^j + q^j)} \right\} = 1. \quad (7.7)$$

Corollary 7.2 is evaluated as follows. Consider an experiment with Ω as the sample space such that

$$|\Omega| = \sum_{j=0}^{n-1} (p^j + q^j),$$

and random variable X taking the value between 0 and n determining the outcome of the experiment. Supposing a given event $E_k \subset \Omega$ is related to the value $X = k$ such that

$$|E_k| = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}}.$$

Then Eq. (7.7) shows that

$$\Pr(X = k) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} q^{\binom{k}{2}}}{\sum_{j=0}^{n-1} (p^j + q^j)}$$

is a probability distribution of X , for details, see [12].

The horizontal generating function in Eq. (7.5) may be a concept in giving a combinatorial interpretation of the (p,q) -binomial coefficients. One may try to use the method of generating functions as in the research area of combinatorics.

Now inverse relations of (p,q) -binomial coefficients which are useful tools in deriving necessary formulas of some special numbers in combinatorics are analyzed. The inverse relations for usual binomial coefficients and q -binomial coefficients given Eqs. (2.18) and (2.23) have been used many times in transforming some formulas into a fully different and interesting form. To illustrate, the second form of the horizontal recurrence relation for the generalized Stirling numbers [72] was acquired by transforming the explicit identity of generalized Stirling numbers using the inverse relation in (2.18), and the explicit formula of q -analogue for the generalized Stirling numbers [73] was transformed into a horizontal recurrence relation using Eq. (2.23). Therefore, an inverse relation for (p,q) -binomial coefficients must be established for

possible use in developing the theory of (p,q) -analogue of some special numbers.

To establish an inverse relation, the orthogonality relation is firstly presented for (p,q) -binomial coefficient by the following theorem.

Theorem 7.4 (Corcino [12]). *The orthogonality relation for the (p,q) -binomial coefficients is given by*

$$\begin{aligned} \sum_{j=i}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} q^{\binom{j-i}{2}} \\ = \sum_{j=i}^n (-1)^{j-i} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{j-i}{2}} q^{\binom{n-j}{2}} = \delta_{ni}. \end{aligned}$$

Proof. Obviously, it holds for $n = 0, 1$. Supposing that for some $n \geq 0$

$$H_n = \sum_{j=i}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} q^{\binom{j-i}{2}} = \delta_{ni}.$$

Then

$$H_{n+1} = \sum_{j=i}^{n+1} (-1)^{n+1-j} \begin{bmatrix} n+1 \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{n+1-j}{2}} q^{\binom{j-i}{2}},$$

by using Theorem 7.1, it becomes

$$\begin{aligned} H_{n+1} &= \sum_{j=i}^{n+1} (-1)^{n+1-j} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{n+1-j}{2}} q^{\binom{j-i}{2}} p^j \\ &\quad + \sum_{j=i}^{n+1} (-1)^{n+1-j} \begin{bmatrix} n \\ j-1 \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{n+1-j}{2}} q^{\binom{j-i}{2}} q^{n-j+1}. \end{aligned}$$

Using the facts that

$$\begin{bmatrix} n \\ n+1 \end{bmatrix}_{p,q} = 0, \quad p^{\binom{n+1-j}{2}+j} = p^{\binom{n-j}{2}} p^n, \quad \text{and} \quad q^{\binom{j-i}{2}+n-j+1} = q^{\binom{j-1-i}{2}} q^{n-i},$$

it is deduced that

$$H_{n+1} = -p^n \delta_{ni} + q^{n-i} \sum_{j=i-1}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j+1 \\ i \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} q^{\binom{j-i}{2}}.$$

By using Theorem 7.1, it is observed that

$$\begin{aligned} H_{n+1} &= -p^n \delta_{ni} + q^{n-i} p^i \delta_{ni} + q^{n-i} \sum_{j=i-1}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i-1 \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} q^{\binom{j-i}{2} + j-i+1} \\ &= \delta_{ni} (-p^n + q^{n-i} p^i) + q^{n-i} q \sum_{j=i-1}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i-1 \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} q^{\binom{j-i}{2}} \\ &= 0 + q^{n-i+1} \delta_{n,i-1} = \delta_{n+1,i}. \end{aligned}$$

The proof is completed for the first form of the orthogonality relation.

If one interchange p and q to get the other form, then it becomes

$$\begin{aligned} &\sum_{j=i}^n (-1)^{j-i} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{j-i}{2}} q^{\binom{n-j}{2}} \\ &= (-1)^{n-i} \sum_{j=i}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{q,p} \begin{bmatrix} j \\ i \end{bmatrix}_{q,p} q^{\binom{j-i}{2}} p^{\binom{n-j}{2}}. \end{aligned}$$

By using the symmetric property of (p,q) -binomial coefficients, that is $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ k \end{bmatrix}_{q,p}$,

and the first form of the orthogonality relation, it can be found that

$$\begin{aligned} &\sum_{j=i}^n (-1)^{j-i} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{j-i}{2}} q^{\binom{n-j}{2}} \\ &= (-1)^{n-i} \sum_{j=i}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{q,p} \begin{bmatrix} j \\ i \end{bmatrix}_{q,p} q^{\binom{j-i}{2}} p^{\binom{n-j}{2}} \\ &= (-1)^{n-i} \delta_{ni} = \delta_{ni}. \end{aligned}$$

The first form of the orthogonality relation in Theorem 7.4 can be rewritten by the following arrangement

$$\sum_{j=i}^n a_{nj} b_{ji} = \delta_{ni},$$

where

$$a_{nj} = (-1)^{n-j} p^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \text{ and } b_{ji} = q^{\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q}.$$

Let M and $\tilde{M}$ be two matrices such that $M = [a_{ij}]_{0 \leq i, j \leq m}$ and $\tilde{M} = [b_{ij}]_{0 \leq i, j \leq m}$, then $M\tilde{M} = [\delta_{ni}]_{0 \leq i, j \leq m} = I_m$, that is, $\tilde{M} = M^{-1}$ and $M = \tilde{M}^{-1}$.

Here a kind of transformation from the identity Theorem 7.3 interchanging the role of x^k and $\sum_{j=0}^{n-1} (q^j + xp^j)$ in the identity is given in the following theorem.

Theorem 7.5 (Corcino [12]). *The following equality holds true*

$$p^{\binom{n}{2}} x^n = \sum_{j=0}^n (-1)^{n-j} p^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \sum_{r=0}^{n-1} (q^r + xp^r).$$

Proof. Obviously, $p^{\binom{n}{2}} x^n = \sum_{i=0}^n \delta_{ni} p^{\binom{i}{2}} x^i$. Then, by Theorem 7.4, it is observed that

$$\begin{aligned} p^{\binom{n}{2}} x^n &= \sum_{i=0}^n \left(\sum_{j=i}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} q^{\binom{j-i}{2}} \right) p^{\binom{i}{2}} x^i \\ &= \sum_{j=0}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} \left\{ \sum_{i=0}^j \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{i}{2}} q^{\binom{j-i}{2}} x^i \right\}. \end{aligned}$$

To complete the proof, it remains to use Theorem 7.3.

It is realized that when $x = 0$, the result in Theorem 7.5 reduces precisely to

the result in Corollary 7.1. On the other hand, when $x = 1$ in Theorem 7.5, it becomes

$$\sum_{j=0}^n (-1)^{n-j} p^{\binom{n-j}{2} - \binom{n}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \sum_{r=0}^{n-1} (q^r + p^r) = 1,$$

which yields

$$\sum_{j=1}^n (-1)^{n-j} p^{\binom{n-j}{2} - \binom{n}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \sum_{r=0}^{n-1} (q^r + p^r) = 0 \quad \text{if } n \text{ is even.}$$

That means

$$\sum_{j \text{ even} (\neq 0)} p^{\binom{n-j}{2} - \binom{n}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \sum_{r=0}^{n-1} (q^r + p^r) = \sum_{j \text{ odd}} p^{\binom{n-j}{2} - \binom{n}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \sum_{r=0}^{n-1} (q^r + p^r)$$

if n is even.

Utilizing from Theorem 7.4, the following containing the inverse relation for the (p,q) -binomial coefficients is stated as follows.

Theorem 7.6 (Corcino [12]). *The inverse relation for the (p,q) -binomial coefficients is stated as*

$$f_n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} (-1)^{n-j} p^{\binom{n-j}{2}} g_j \Leftrightarrow g_n = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} q^{\binom{n-j}{2}} f_j.$$

Proof. ($\Rightarrow$) Using the given hypothesis, it is seen that

$$\begin{aligned} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} q^{\binom{n-j}{2}} f_j &= \sum_{j=0}^n \sum_{i=0}^j (-1)^{j-i} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{j-i}{2}} q^{\binom{n-j}{2}} g_i \\ &= \sum_{i=0}^n \left\{ \sum_{j=i}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} (-1)^{j-i} p^{\binom{j-i}{2}} q^{\binom{n-j}{2}} \right\} g_i. \end{aligned}$$

Applying the second form of the orthogonality relation in Theorem 7.4 yields to

$$\sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} q^{\binom{n-j}{2}} f_j = \sum_{i=0}^n \delta_{ni} g_i = g_n.$$

($\Leftarrow$) Similarly, it can be acquired that

$$\begin{aligned} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} (-1)^{n-j} p^{\binom{n-j}{2}} g_j &= \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} (-1)^{n-j} p^{\binom{n-j}{2}} \sum_{i=0}^j \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} q^{\binom{j-i}{2}} f_i \\ &= \sum_{i=0}^n \left\{ \sum_{j=i}^n (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \begin{bmatrix} j \\ i \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} q^{\binom{j-i}{2}} \right\} f_i. \end{aligned}$$

Applying the first form of orthogonality relation in Theorem 7.4 gives

$$\sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} (-1)^{n-j} p^{\binom{n-j}{2}} g_j = \sum_{j=0}^n \delta_{ni} f_j = f_n.$$

If $f_n = p^{\binom{n}{2}} x^n$ and $g_j = \sum_{r=0}^{j-1} (q^r + x p^r)$ are chosen by using Theorem 7.6, the formula in Theorem 7.5 becomes

$$\sum_{r=0}^{n-1} (q^r + x p^r) = \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} q^{\binom{n-j}{2}} p^{\binom{j}{2}} x^j$$

giving the expansion in Theorem 7.3 by substituting p into q .

Theorem 7.7 (Duran et al. [26]). *Each (p,q) -binomial coefficient is a polynomial including the parameters p and q of degree $k(n-k)$ whose leading coefficient is 1.*

Proof. To prove this, the following is firstly considered for any nonnegative integer n , as follows:

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n \end{bmatrix}_{p,q} = 1,$$

which is clearly a polynomial. By making use of (p,q) -Pascal rules and induction on n

for any $1 \leq k \leq n-1$, $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ is the sum of two polynomials, hence it is itself a polynomial. The (p,q) -binomial coefficient has the following explicit expression:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q} [n-1]_{p,q} \cdots [n-k+1]_{p,q}}{[k]_{p,q} [k-1]_{p,q} \cdots [1]_{p,q}} = \frac{(p^n - q^n)(p^{n-1} - q^{n-1}) \cdots (p^{n-k+1} - q^{n-k+1})}{(p^k - q^k)(p^{k-1} - q^{k-1}) \cdots (p - q)}. \quad (7.8)$$

Since both the numerator and denominator of Eq. (7.8) are polynomials based on the parameters p and q with leading coefficient 1, so is their quotient. Finally, the degree of $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ in p and q is the difference of the degrees of numerator and denominator:

that is,

$$[n + (n-1) + \cdots + (n-k+1)] - [k + (k-1) + \cdots + 1] = (n-k)k,$$

as asserted.

In the other hand, the Eq. (7.8) is considered in the following form:

$$\begin{aligned} \alpha_0 q^{(n-k)k} + \alpha_1 q^{(n-k)k-1} p + \cdots + \alpha_{k(n-k)-1} p q^{(n-k)k-1} + \alpha_{k(n-k)} p^{(n-k)k} \\ = \frac{(p^n - q^n)(p^{n-1} - q^{n-1}) \cdots (p^{n-k+1} - q^{n-k+1})}{(p^k - q^k)(p^{k-1} - q^{k-1}) \cdots (p - q)}. \end{aligned}$$

By changing p by $\frac{1}{p}$ and q by $\frac{1}{q}$, also, multiply both sides by $(pq)^{(n-k)k}$, it is observed that the right-hand side will be unchanged, whereas the left-hand side,

$$\alpha_0 p^{(n-k)k} + \alpha_1 p^{(n-k)k-1} q + \cdots + \alpha_{k(n-k)-1} p q^{(n-k)k-1} + \alpha_{k(n-k)} q^{(n-k)k},$$

has the sequence of coefficient α_m reversed in order. By comparing coefficients, it can be acquired the coefficients in the polynomial expression of $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ that are symmetric:

$$\alpha_0 = \alpha_{(n-k)k}$$

$$\alpha_1 = \alpha_{(n-k)k-1}$$

$$\alpha_2 = \alpha_{(n-k)k-2}$$

$$\vdots$$

i.e, $\alpha_m = \alpha_{(n-k)k-m}$.

Note that the (p,q) -binomial coefficients also have combinatorial interpretations like q -binomial coefficients.



CHAPTER 8

SOME RELATIONS BETWEEN (p,q) -DERIVATIVE AND DIVIDED DIFFERENCES

Let p and q be elements of complex numbers and $D = D_{p,q} \subset \mathbb{C}$ such that $x \in D$ implies $px \in D$ and $qx \in D$.

Definition 8.1 (Duran et al. [23]). Any function $f : D_{p,q} \rightarrow \mathbb{C}$ is called (p,q) -differentiable, under the restriction that if $0 \in D_{p,q}$, there is $f'(0)$ exists.

Definition 8.2 (Duran et al. [23]). A function $f : D_{p,q} \rightarrow \mathbb{C}$ is called (p,q) -differentiable of order n , iff $0 \in D_{p,q}$ implies that $f^{(n)}(0)$ exists.

The divided differences at a system of distinct points $x_0, x_1, \dots, x_n$ are denoted by $[x_0, x_1, \dots, x_n; f]$ (see [23-25]). That is

$$[x_0, x_1, \dots, x_n; f] = \sum_{k=0}^n \frac{f(x_k)}{(x_k - x_0) \cdots (x_k - x_{k-1})} \cdot \frac{1}{(x_k - x_{k+1}) \cdots (x_k - x_n)}. \quad (8.1)$$

Let the points

$$x_k = p^k q^{n-k} x \quad (k = 0, 1, \dots, n)$$

be considered as follows:

$$x_0 = q^n x, x_1 = q^{n-1} p x, \dots, x_{n-1} = q p^{n-1} x, x_n = p^n x.$$

Now the following theorem is given.

Theorem 8.1 (Duran *et al.* [23]). Let p and q be complex numbers with

$$0 < |q| < |p| \leq 1 \quad \text{and} \quad f : D_{p,q} \rightarrow \mathbb{C}.$$

Then, by taking the knots $x_k = p^k q^{n-k} x$, it can be written as

$$\begin{aligned} & \left[q^n x, q^{n-1} p x, \dots, q p^{n-1} x, p^n x; f \right] \\ &= \frac{1}{q^{\binom{n}{2}} [n]_{p,q}! x^n (p-q)^n} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{-k(2n-k-1)}{2}} q^{\binom{k}{2}} f(x p^k q^{n-k}). \end{aligned} \quad (8.2)$$

Proof. For $0 \leq l < k$, it can be observed that

$$x_k - x_l = x p^l q^{n-k} (p-q) [k-l]_{p,q}$$

and, for $k < l \leq n$, it can be found that

$$x_k - x_l = x p^k q^{n-l} (q-p) [l-k]_{p,q}.$$

Since

$$\begin{aligned} \prod_{\substack{l=0 \\ l \neq k}}^n (x_k - x_l) &= \prod_{l=0}^{k-1} (x_k - x_l) \prod_{l=k+1}^n (x_k - x_l) \\ &= x^n p^{(n-k)k} (-1)^{n-k} (p-q)^n q^{k(n-k) + \binom{n-k}{2}} [k]_{p,q}! p^{k(n-k) + \binom{k}{2}} [n-k]_{p,q}! \\ &= (-1)^{n-k} (p-q)^n x^n p^{k(2n-k-1)/2} q^{\binom{n}{2} - \binom{k}{2}} [k]_{p,q}! [n-k]_{p,q}!, \end{aligned}$$

the following consequence is obtained from Eq. (8.1):

$$[x_0, x_1, \dots, x_n; f] = \frac{q^{\binom{n}{2}}}{[n]_{p,q}! x^n (p-q)^n} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{-k(2n-k-1)/2} q^{\binom{k}{2}} f(x p^k q^{n-k}).$$

Therefore, the proof of Theorem 8.1 is completed.

By using the following expressions:

$$D_{p,q}^0 = I, \quad D_{p,q}^1 = D_{p,q} \quad \text{and} \quad D_{p,q}^k = D_{p,q} D_{p,q}^{k-1},$$

now a representation of the operator $D_{p,q}^n$ as in Theorem 8.2 is provided below.

Theorem 8.2 (Duran et al. [23]). *Let the function $f : D_{p,q} \rightarrow \mathbb{C}$ be (p,q) -differentiable of order n . Then*

$$(D_{p,q}^n f)(x) = \frac{q^{\binom{n}{2}}}{x^n (p-q)^n} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{q^{\binom{k}{2}} f(x p^k q^{n-k})}{p^{k(2n-k-1)/2}}. \quad (8.3)$$

Proof. Theorem 8.2 is proved by making use of the following results:

$$(D_{p,q} f)(x) = \frac{f(qx) - f(px)}{(q-p)x} = \frac{f(qx)}{qx - px} + \frac{f(px)}{px - qx} = [1]_{p,q}! [qx, px; f]$$

and

$$\begin{aligned} (D_{p,q}^2 f)(x) &= \frac{(D_{p,q} f)(qx) - (D_{p,q} f)(px)}{(q-p)x} \\ &= \frac{\frac{f(q^2x) - f(pqx)}{(q-p)qx} - \frac{f(pqx) - f(p^2x)}{(q-p)px}}{(p-q)x} \\ &= (p+q) \left[\frac{f(q^2x)}{(q^2-p^2)(q-p)x^2q} - \frac{f(pqx)}{(q-p)^2 x^2 pq} + \frac{f(p^2x)}{(q^2-p^2)(q-p)x^2 p} \right] \end{aligned}$$

$$= [2]_{p,q}! [q^2x, pqx, p^2x; f].$$

By mathematical induction, it is proved that

$$(D_{p,q}^n f)(x) = [n]_{p,q}! [p^n x, p^{n-1}qx, \dots, pq^{n-1}x, q^n x; f]. \quad (8.4)$$

Suppose that the above formula holds true for $n = m$. Then it holds also true for $n = m+1$, since

$$\begin{aligned} (D_{p,q}^{m+1} f)(x) &= (D_{p,q}((D_{p,q}^m f))) \\ &= \frac{(D_{p,q}^m f)(px) - (D_{p,q}^m f)(qx)}{(p-q)x} \\ &= \frac{[m]_{p,q}! [p^{m+1}x, p^m qx, \dots, p^2 q^{m-1}x, pq^m x; f]}{(p-q)x} \\ &\quad - \frac{[m]_{p,q}! [p^m qx, p^{m-1} q^2 x, \dots, pq^m x, q^{m+1} x; f]}{(p-q)x} \\ &= [m]_{p,q}! \frac{p^{m+1}x - q^{m+1}x}{(p-q)x} [p^{m+1}x, p^m qx, \dots, pq^m x, q^{m+1}x; f] \\ &= [m+1]_{p,q}! [p^{m+1}x, p^m qx, \dots, pq^m x, q^{m+1}x; f], \end{aligned}$$

in which the following formula has been used:

$$[x_0, x_1, \dots, x_n; \cdot] = \frac{[x_1, x_2, \dots, x_n; \cdot] - [x_0, x_1, \dots, x_{n-1}; \cdot]}{x_n - x_0}.$$

It follows from Theorem 8.1 that

$$(D_{p,q}^n f)(x) = q^{-\binom{n}{2}} x^{-n} (p-q)^{-n} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{-k(2n-k-1)/2} q^{\binom{k}{2}} f(xp^k q^{n-k}),$$

which completes the proof of Theorem 8.2.

In the case when

$$f(x) = x^n$$

in Theorem 8.2, the following corollary is improved.

Corollary 8.1 (Duran et al. [23]). *The following result holds true:*

$$(p-q)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k+1}{2}} q^{\binom{n-k+1}{2}} \frac{(-1)^{n-k}}{[n]_{p,q}!}.$$

Now, the (p,q) -analogue of the Leibniz rule is considered to be represented by means of the divided differences. Firstly, for the functions $f : D_{p,q} \rightarrow C$ and $g : D_{p,q} \rightarrow C$ be (p,q) -differentiable of order n , the (p,q) -analogue of the Leibniz rule is given by

$$D_{p,q}^n (fg)(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} D_{p,q}^k (f)(xp^{n-k}) D_{p,q}^{n-k} (g)(xq^k).$$

The (p,q) -Leibnitz rule is represented by means of the divided differences as follows.

Theorem 8.3 (Duran et al. [23]). *Let the functions $f : D_{p,q} \rightarrow \mathbb{C}$ and $g : D_{p,q} \rightarrow \mathbb{C}$ be (p,q) -differentiable of order n . Then $(fg)(x)$ is also (p,q) -differentiable of order n and*

$$D_{p,q}^n (fg)(x) = [n]_{p,q}! \sum_{k=0}^n \left[q^n x, q^{n-1} p x, \dots, q^{n-k+1} p^{k-1} x, q^{n-k} p^k x; f \right]$$

$$\cdot \left[q^{n-k} p^k x, q^{n-k-1} p^{k+1} x, \dots, qp^{n-1} x, p^n x; g \right].$$

Proof. From Eq. (8.4) and Lemma 3.2.1, in fact, it is observed that

$$\begin{aligned}
D_{p,q}^n (fg)(x) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} D_{p,q}^k (f)(xp^{n-k}) D_{p,q}^{n-k} (g)(xq^k) \\
&= \sum_{k=0}^n \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!} \left\{ [k]_{p,q}! [p^n x, p^{n-1} q x, \dots, p^{n-k+1} q^{k-1} x, p^{n-k} q^k x; f] \right. \\
&\quad \left. \times [n-k]_{p,q}! [p^{n-k} q^k x, p^{n-k-1} q^{k+1} x, \dots, p q^{n-1} x, q^n x; g] \right\} \\
&= \sum_{k=0}^n [n]_{p,q}! [p^n x, p^{n-1} q x, \dots, p^{n-k+1} q^{k-1} x, p^{n-k} q^k x; f] \\
&\quad \times [p^{n-k} q^k x, p^{n-k-1} q^{k+1} x, \dots, p q^{n-1} x, q^n x; g].
\end{aligned}$$

Now, a function at a point $p^n x$ by binomial expression and (p,q) -derivative of order k is given.

Theorem 8.4 (Duran et al. [23]). Let the function $f : D_{p,q} \rightarrow \mathbb{C}$ be (p,q) -differentiable of order n . Then

$$f(p^n x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k p^{\binom{k}{2}} (p-q)^k D_{p,q}^k (f(q^{n-k} x)).$$

Proof. Newton's formula is taken into consideration as follows:

$$\begin{aligned}
f(z) &= \sum_{k=0}^{n-1} (z-x_0)(z-x_1)\cdots(z-x_{k-1})[x_0, x_1, \dots, x_k; f] \\
&\quad + (z-x_0)(z-x_1)\cdots(z-x_{n-1})[x_0, x_1, \dots, x_{n-1}, z; f].
\end{aligned} \tag{8.5}$$

Upon setting

$$x_k = p^k q^{n-k} x \quad (k = 0, 1, \dots, n-1)$$

in the formula of Eq. (8.5) and $z = p^n x$, if the formula in Eq. (8.2) is used, it is found that

$$\begin{aligned}
f(p^n x) &= \sum_{k=0}^{n-1} (p^n x - q^n x)(p^n x - q^{n-1} p x) \cdots (p^n x - q^{n-k+1} p^{k-1} x) [q^n x, q^{n-1} p x, \dots, q^{n-k} p^k x; f] \\
&\quad + (p^n x - q^n x)(p^n x - q^{n-1} p x) \cdots (p^n x - q p^{n-1} x) [q^n x, q^{n-1} p x, \dots, q p^{n-1} x, p^n x; f] \\
&= \sum_{k=0}^{n-1} (p^n x - q^n x)(p^n x - q^{n-1} p x) \cdots (p^n x - q^{n-k+1} p^{k-1} x) \frac{(D_{p,q}^k f)(q^{n-k} x)}{[k]_{p,q}!} \\
&\quad + (p^n x - q^n x)(p^n x - q^{n-1} p x) \cdots (p^n x - q p^{n-1} x) \frac{(D_{p,q}^n f)(x)}{[n]_{p,q}!} \\
&= \sum_{k=0}^n (p^n x - q^n x)(p^n x - q^{n-1} p x) \cdots (p^n x - q^{n-k+1} p^{k-1} x) \frac{(D_{p,q}^k f)(x)}{[k]_{p,q}!} \\
&= \sum_{k=0}^n x^k p^{\binom{k}{2}} (p-q)^k \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!} (D_{p,q}^k f)(q^{n-k} x),
\end{aligned}$$

as asserted by Theorem 8.4.

The final result of this chapter is stated as follows.

Corollary 8.2 (Duran et al. [23]). *Let p and q be complex numbers such that*

$$0 < |q| < |p| \leq 1.$$

Also, let the function $f : D_{p,q} \rightarrow \mathbb{C}$ be (p,q) -differentiable of order n . Then

$$f(x) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} q^{k(k-n)} p^{\binom{k+1}{2}} (qx - px)^k (D_{p,q}^k f) \left(\frac{p^{n-k}}{q^n} x \right).$$

Proof. Since, for $k \in \{0, 1, \dots, n\}$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{p}, \frac{1}{q}} = \frac{[n]_{\frac{1}{p}, \frac{1}{q}}!}{[n-k]_{\frac{1}{p}, \frac{1}{q}}! [k]_{\frac{1}{p}, \frac{1}{q}}!} = \frac{(pq)^{-\binom{n}{2}}}{(pq)^{-\binom{n-k}{2}} (pq)^{-\binom{k}{2}}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q},$$

it seems

$$\left(D_{\frac{1}{p}, \frac{1}{q}} f \right)(x) = \frac{f\left(\frac{x}{q}\right) - f\left(\frac{x}{p}\right)}{(p-q)x} (pq) = pq (D_{p,q} f) \left(\frac{x}{pq} \right)$$

and

$$\begin{aligned} \left(D_{\frac{1}{p}, \frac{1}{q}}^2 f \right)(x) &= \frac{pq (D_{p,q} f) \left(\frac{x}{pq} \right) - pq (D_{p,q} f) \left(\frac{x}{pq} \right)}{\left(\frac{1}{p} - \frac{1}{q} \right) x} \\ &= \frac{(pq)^2 \left[(D_{p,q} f) \left(\frac{x}{pq^2} \right) - (D_{p,q} f) \left(\frac{x}{pq} \right) \right]}{(p-q)x} \\ &= p^2 q^2 (D_{p,q}^2 f) \left(\frac{x}{p^2 q^2} \right). \end{aligned}$$

Continuing the process, it is readily observed that

$$\left(D_{\frac{1}{p}, \frac{1}{q}}^n f \right)(x) = p^n q^n (D_{p,q}^n f) \left(\frac{x}{p^n q^n} \right).$$

From Theorem 8.4, thus it is concluded that

$$f(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{k(k-n)} p^{\binom{k+1}{2}} (qx - px)^k (D_{p,q}^k f) \left(\frac{p^{n-k}}{q^n} x \right),$$

which evidently proves Corollary 8.2.

CHAPTER 9

TWO (p,q) -EXPONENTIAL FUNCTIONS

This chapter study on the two type of the (p,q) -exponential functions, $e_{p,q}(x)$ and $E_{p,q}(x)$, and gives some correlations between these functions. Also, (p,q) -derivatives of these functions are developed. The (p,q) -analogues of the sine, cosine, tangent and cotangent functions are defined in the same manner with their well-known Euler expressions by means of the exponential functions. Their (p,q) -derivative properties are presented and some tables including identities and formulae belonging to the (p,q) -trigonometric functions which reduce to the both q -identities and q -formulae, and classical identities and classical formulas under the special cases are stated as ultimate results of this chapter.

Let the two canonical q -exponential functions, $e_q(x)$ and $E_q(x)$, be remembered as (see [2, 13])

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \quad \text{and} \quad E_q(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!},$$

which satisfy the following known relation

$$e_q(x)E_q(-x) = 1.$$

Two (p,q) -exponentials earlier defined in Chapter 4 as Eqs. (4.22) and (4.23) are given by the following definition.

Definition 9.1 (Jagannathan and Rao [23]). *The (p,q) -exponential functions are defined by*

$$e_{p,q}(x) = {}_1\Phi_0\left((1,0);-; (p,q), x\right) = \sum_{n=0}^{\infty} p^{n(n-1)/2} \frac{x^n}{((p,q);(p,q))_n}$$

and

$$E_{p,q}(x) = {}_1\Phi_0\left((0,1);-;(p,q),x\right) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{((p,q);(p,q))_n}.$$

The (p,q) -exponential functions can be shown in the following form

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \quad (9.1)$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}, \quad (9.2)$$

which are useful form of exponential functions for this thesis.

The functions $e_{p,q}(x)$ and $E_{p,q}(x)$ are natural generalizations of the functions $e_q(x)$ and $E_q(x)$, such that when $p = 1$, it becomes that

$$e_{1,q}(x) = e_q(x) \quad \text{and} \quad E_{1,q}(x) = E_q(x).$$

Note that in the limiting case $q \rightarrow p = 1$, the two exponential functions belonging to

(p,q) -calculus reduce to the classical exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Now, some properties of the functions $e_{p,q}(x)$ and $E_{p,q}(x)$ are searched.

The two (p,q) -exponential functions are closely correlated. A useful identity for the two form of (p,q) -exponential functions is stated below. Using the (p,q) -Gauss Binomial formula in Eq. (6.17), the product becomes

$$e_{p,q}(x)E_{p,q}(-x) = \left(\sum_{n=0}^{\infty} \frac{1}{[n]_{p,q}!} p^{\binom{n}{2}} x^n \right) \left(\sum_{n=0}^{\infty} \frac{1}{[n]_{p,q}!} q^{\binom{n}{2}} (-x)^n \right)$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} (-1)^{n-k} \right) \frac{x^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} (1 \ominus 1)_{p,q}^n \frac{x^n}{[n]_{p,q}!},
\end{aligned}$$

thus

$$e_{p,q}(x)E_{p,q}(-x) = 1. \quad (9.3)$$

The classical exponential function is unchanged under differentiation. Also, the functions $e_q(x)$ and $E_q(x)$ have similar behavior, look at the references [2] for details on q -exponentials. Furthermore, (p,q) -exponential functions have similar behavior. As

$$\begin{aligned}
D_{p,q}e_{p,q}(x) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{D_{p,q}x^n}{[n]_{p,q}!} \\
&= \sum_{n=1}^{\infty} p^{\binom{n}{2}} \frac{x^{n-1}}{[n-1]_{p,q}!} \\
&= \sum_{n=1}^{\infty} p^{\binom{n-1}{2}} \frac{(px)^{n-1}}{[n-1]_{p,q}!} \\
&= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{(px)^n}{[n]_{p,q}!},
\end{aligned}$$

and,

$$D_{p,q}E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{D_{p,q}x^n}{[n]_{p,q}!}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} q^{\binom{n}{2}} \frac{x^{n-1}}{[n-1]_{p,q}!} \\
&= \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} \frac{(qx)^{n-1}}{[n-1]_{p,q}!} \\
&= \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(qx)^n}{[n]_{p,q}!},
\end{aligned}$$

thus, these yields to

$$D_{p,q} e_{p,q}(x) = e_{p,q}(px) \text{ and } D_{p,q} E_{p,q}(x) = E_{p,q}(qx). \quad (9.4)$$

Note that the derivative of the (p,q) -exponentials are not precisely themselves. However, those derivatives are in the similar form of the derivative of classical exponential functions.

Similar to the relation between q -exponential functions, one has

$$e_{\frac{1}{p}, \frac{1}{q}}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{\frac{1}{p}, \frac{1}{q}}!} p^{-\binom{n}{2}} x^n = \sum_{n=0}^{\infty} \frac{1}{[n]_{p,q}!} q^{\binom{n}{2}} x^n,$$

hence, it becomes

$$e_{\frac{1}{p}, \frac{1}{q}}(x) = E_{p,q}(x). \quad (9.5)$$

In a like manner, the following relations holds true:

$$E_{\frac{1}{p}, \frac{1}{q}}(x) = e_{p,q}(x).$$

How about the additive property of the (p,q) -exponential functions? Unlike q -exponential function and usual exponential function, the answer is that the (p,q) -exponential functions have not any additive property. Indeed,

$$\begin{aligned}
e_{p,q}(x)E_{p,q}(y) &= \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{y^n}{[n]_{p,q}!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k y^{n-k} \right) \frac{1}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \frac{(x \oplus y)_{p,q}^n}{[n]_{p,q}!},
\end{aligned}$$

which does not belong to any form of (p,q) -exponential functions.

Now, (p,q) -analogues of the trigonometric and hypergeometric functions are analyzed as an applications of the (p,q) -exponential functions.

9.1 (p,q) -Trigonometric and (p,q) -Hyperbolic Functions

The q -analogues of the trigonometric functions are introduced by Kim [74] and their some properties are investigated. Now, a new extensions of the trigonometric functions, namely, (p,q) -analogues of the sine, cosine, tangent and cotangent functions are given as follows.

Definition 9.1.1 (Duran *et al.* [26]). Let $i = \sqrt{-1}$. Then two pairs of (p,q) -trigonometric functions are defined by

$\sin_{p,q} x = \frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i}$	$\text{SIN}_{p,q} x = \frac{E_{p,q}(ix) - E_{p,q}(-ix)}{2i}$
$\cos_{p,q} x = \frac{e_{p,q}(ix) + e_{p,q}(-ix)}{2}$	$\text{COS}_{p,q} x = \frac{E_{p,q}(ix) + E_{p,q}(-ix)}{2}$
$\tan_{p,q} x = \frac{\sin_{p,q} x}{\cos_{p,q} x}$	$\text{TAN}_{p,q} x = \frac{\text{SIN}_{p,q} x}{\text{COS}_{p,q} x}$
$\cot_{p,q} x = \frac{\cos_{p,q} x}{\sin_{p,q} x}$	$\text{COT}_{p,q} x = \frac{\text{COS}_{p,q} x}{\text{SIN}_{p,q} x}$

(9.6)

Using the identity in Eq. (9.3), it becomes

$$\sin_{p,q} x \text{SIN}_{p,q} x = -\frac{e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(-ix) - 2}{4}$$

and

$$\cos_{p,q} x \text{COS}_{p,q} x = \frac{e_{p,q}(ix)E_{p,q}(ix) + e_{p,q}(-ix)E_{p,q}(-ix) + 2}{4}.$$

Thus, the following formula is directly derived:

$$\sin_{p,q} x \text{SIN}_{p,q} x + \cos_{p,q} x \text{COS}_{p,q} x = 1,$$

which is the (p,q) -analogue of the formula

$$\sin^2 x + \cos^2 x = 1.$$

Now, (p,q) -derivatives of the (p,q) -trigonometric functions are discovered. By making use of the differentiation properties of the exponential functions (9.4), it is easily seen that [26]

$$\begin{aligned} D_{p,q} \sin_{p,q} x &= D_{p,q} \left(\frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i} \right) \\ &= \frac{D_{p,q} e_{p,q}(ix) - D_{p,q} e_{p,q}(-ix)}{2i} \\ &= \frac{e_{p,q}(px) + e_{p,q}(-px)}{2} = \cos_{p,q}(px), \end{aligned}$$

and also

$$D_{p,q} \text{SIN}_{p,q} x = \text{COS}_{p,q}(qx)$$

$$D_{p,q} \cos_{p,q} x = -\sin_{p,q}(px)$$

$$D_{p,q} \text{COS}_{p,q} x = -\text{SIN}_{p,q}(qx).$$

The (p,q) -exponential functions are related to the (p,q) -cosine and (p,q) -sine by

$$e_{p,q}(iz) = \cos_{p,q}(z) + i \sin_{p,q}(z)$$

$$E_{p,q}(iz) = \cos_{p,q}(z) + i \sin_{p,q}(z).$$

Definition 9.1.2 (Duran *et al.* [26]). Two pairs of (p,q) -hyperbolic functions are defined by

$\sinh_{p,q} x = \frac{e_{p,q}(x) - e_{p,q}(-x)}{2}$	$\text{SINH}_{p,q} x = \frac{E_{p,q}(x) - E_{p,q}(-x)}{2}$
$\cosh_{p,q} x = \frac{e_{p,q}(x) + e_{p,q}(-x)}{2}$	$\text{COSH}_{p,q} x = \frac{E_{p,q}(x) + E_{p,q}(-x)}{2}$
$\tanh_{p,q} x = \frac{\sinh_{p,q} x}{\cosh_{p,q} x}$	$\text{TANH}_{p,q} x = \frac{\text{SINH}_{p,q} x}{\text{COSH}_{p,q} x}$
$\coth_{p,q} x = \frac{\cosh_{p,q} x}{\sinh_{p,q} x}$	$\text{COTH}_{p,q} x = \frac{\text{COSH}_{p,q} x}{\text{SINH}_{p,q} x}$

(9.7)

By Definition 9.1.2, it is easily seen that

$$e_{p,q}(x) = \cosh_{p,q} x + \sinh_{p,q} x \quad \text{and} \quad E_{p,q}(x) = \text{COSH}_{p,q} x + \text{SINH}_{p,q} x.$$

Using the identity in Eq. (9.3), one has

$$\sinh_{p,q} x \text{SINH}_{p,q} x = \frac{e_{p,q}(x)E_{p,q}(x) + e_{p,q}(-x)E_{p,q}(-x) - 2}{4}$$

and

$$\cosh_{p,q} x \text{COSH}_{p,q} x = \frac{e_{p,q}(x)E_{p,q}(x) + e_{p,q}(-x)E_{p,q}(-x) + 2}{4}.$$

Hence, it is directly derived that

$$\cosh_{p,q} x \text{COSH}_{p,q} x - \sinh_{p,q} x \text{SINH}_{p,q} x = 1,$$

which is the (p,q) -analogue of the formula

$$\cosh^2 x - \sinh^2 x = 1.$$

Here, (p,q) -derivatives of the (p,q) -hypergeometric functions are explored. By making use of the differentiation properties of the exponential functions in Eq. (9.4), it is observed that

$$\begin{aligned} D_{p,q} \sinh_{p,q} x &= D_{p,q} \left(\frac{e_{p,q}(ix) - e_{p,q}(-ix)}{2i} \right) \\ &= \frac{D_{p,q} e_{p,q}(ix) - D_{p,q} e_{p,q}(-ix)}{2i} \\ &= \frac{e_{p,q}(px) + e_{p,q}(-px)}{2} = \cosh_{p,q}(px), \end{aligned}$$

and also

$$D_{p,q} \text{SINH}_{p,q} x = \text{COSH}_{p,q}(qx)$$

$$D_{p,q} \cosh_{p,q} x = \sinh_{p,q}(px)$$

$$D_{p,q} \text{COSH}_{p,q} x = \text{SINH}_{p,q}(qx).$$

Definition 9.1.3 (Duran *et al.* [26]). *It is defined that*

$\sec_{p,q} x := \frac{1}{\cos_{p,q} x}$	$\csc_{p,q} x := \frac{1}{\sin_{p,q} x}$
$\text{SEC}_{p,q} x := \frac{1}{\text{COS}_{p,q} x}$	$\text{CSC}_{p,q} x := \frac{1}{\text{SIN}_{p,q} x}$
$\text{sech}_{p,q} x := \frac{1}{\cosh_{p,q} x}$	$\text{csch}_{p,q} x := \frac{1}{\sinh_{p,q} x}$
$\text{SECH}_{p,q} x := \frac{1}{\text{COSH}_{p,q} x}$	$\text{CSCH}_{p,q} x := \frac{1}{\text{SINH}_{p,q} x}$

Intriguing identities for trigonometric and hyperbolic functions under the theory of (p,q) -calculus are listed as follows [26].

$\sin_{p,q} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} p^{(2n+1)n} x^{2n+1}$	$\sinh_{p,q} x = \sum_{n=0}^{\infty} p^{(2n+1)n} \frac{x^{2n+1}}{[2n+1]_{p,q}!}$
$\text{SIN}_{p,q} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} q^{(2n+1)n} x^{2n+1}$	$\text{COS}_{p,q} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} q^{(2n+1)n} x^{2n+1}$
$\cos_{p,q} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{p,q}!} p^{n(2n-1)} x^{2n}$	$\cosh_{p,q} x = \sum_{n=0}^{\infty} p^{n(2n-1)} \frac{x^{2n}}{[2n]_{p,q}!}$
$\text{SIN}_{p,q} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{p,q}!} q^{n(2n-1)} x^{2n+1}$	$\text{COSH}_{p,q} x = \sum_{n=0}^{\infty} q^{n(2n-1)} \frac{x^{2n}}{[2n]_{p,q}!}$

$\sinh_{p,q} x = -i \sin_{p,q}(ix)$ $\text{SINH}_{p,q} x = -i \text{SIN}_{p,q}(ix)$	$q \rightarrow p = 1$	$\sinh x = -i \sin(ix)$
$\cosh_{p,q} x = \cos_{p,q}(ix)$ $\text{COSH}_{p,q} x = \text{COS}_{p,q}(ix)$	$q \rightarrow p = 1$	$\cosh x = \cos(ix)$
$\tanh_{p,q} x = -i \tan_{p,q}(ix)$ $\text{TANH}_{p,q} x = -i \text{TAN}_{p,q}(ix)$	$q \rightarrow p = 1$	$\tanh x = -i \tan(ix)$
$\coth_{p,q} x = i \cot_{p,q}(ix)$ $\text{COTH}_{p,q} x = i \text{COT}_{p,q}(ix)$	$q \rightarrow p = 1$	$\coth x = i \cot(ix)$

$\sin_{p,q}(-x) = -\sin_{p,q} x$	$\text{SIN}_{p,q}(-x) = -\text{SIN}_{p,q} x$
$\cos_{p,q}(-x) = \cos_{p,q} x$	$\text{COS}_{p,q}(-x) = \text{COS}_{p,q} x$
$\tan_{p,q}(-x) = -\tan_{p,q} x$	$\text{TAN}_{p,q}(-x) = -\text{TAN}_{p,q} x$
$\cot_{p,q}(-x) = -\cot_{p,q} x$	$\text{COT}_{p,q}(-x) = -\text{COT}_{p,q} x$
$\sec_{p,q}(-x) = \sec_{p,q} x$	$\text{SEC}_{p,q}(-x) = \text{SEC}_{p,q} x$
$\csc_{p,q}(-x) = -\csc_{p,q} x$	$\text{CSC}_{p,q}(-x) = -\text{CSC}_{p,q} x$
$\sinh_{p,q}(-x) = -\sinh_{p,q} x$	$\text{SINH}_{p,q}(-x) = -\text{SINH}_{p,q} x$
$\cosh_{p,q}(-x) = \cosh_{p,q} x$	$\text{COSH}_{p,q}(-x) = \text{COSH}_{p,q} x$
$\tanh_{p,q}(-x) = -\tanh_{p,q} x$	$\text{TANH}_{p,q}(-x) = -\text{TANH}_{p,q} x$

$\coth_{p,q}(-x) = -\coth_{p,q} x$	$\text{COTH}_{p,q}(-x) = -\text{COTH}_{p,q} x$
$\text{sech}_{p,q}(-x) = \text{sech}_{p,q} x$	$\text{SECH}_{p,q}(-x) = \text{SECH}_{p,q} x$
$\text{csch}_{p,q}(-x) = -\text{csch}_{p,q} x$	$\text{CSCH}_{p,q}(-x) = -\text{CSCH}_{p,q} x$



CHAPTER 10

THE (p,q) -ANTIDERIVATIVE AND THE (p,q) -INTEGRAL

Until now, (p,q) -differentiation and its some applications solely are mentioned. Now what the integration is in (p,q) -calculus is considered. Firstly, (p,q) -antiderivative is studied in the following section.

10.1 The (p,q) -Antiderivative

Now, the inverse function of (p,q) -derivative of any function $f(x)$ is described as (p,q) -antiderivative of any function $f(x)$ as follows.

Definition 10.1.1 (Sadjang [19]). *The function $F(x)$ is a (p,q) -antiderivative of $f(x)$ if $D_{p,q}F(x) = f(x)$, and it is shown by*

$$\int f(x) d_{p,q}x.$$

As has been mentioned by a similar case in q -calculus, it is said “a” (p,q) -antiderivative rather than “the” (p,q) -antiderivative in (p,q) -calculus, since as in classical calculus, an antiderivative is not unique. In classical calculus, the uniqueness is up to a constant as the derivative of a function vanishes if and only if it is a constant. The situation in the (p,q) -calculus is more subtle. $D_{p,q}\varpi(x) = 0$ if and only if $\varpi(px) = \varpi(qx)$, which does not necessarily mean that ϖ is a constant. Adding such a function ϖ does not vary the (p,q) -derivative of a function. If ϖ is required to be a formal power series, the condition $\varpi(px) = \varpi(qx)$ implies $p^n c_n = q^n c_n$ for each n in which c_n is the coefficient of x^n . It may be only when $c_n = 0$ for any $n \geq 1$, namely, ϖ is constant.

The condition $D_{p,q}\varpi(x)=0$ if and only if $\varpi(px)=\varpi(qx)$ is called (p,q) -periodic function. By choosing $p=e^{\ln p}$ and $x=e^{\ln x}=e^y$, it can be rewritten as $\varpi(x)=\varpi(e^y)\equiv G(y)$ and

$$\varpi(px)=G(y+\ln p).$$

The condition of (p,q) -periodicity of $F(x)$ yields to

$$G(y+\ln p)=G(y+\ln q)$$

and if $y+\ln p=z$ is chosen, then it is realized that

$$G(z)=G\left(z+\ln\frac{q}{p}\right),$$

which implies that $G(z)$ is a standard periodic function

$$G(z)=G(z+t)$$

with a period $t=\ln\frac{q}{p}=\ln q-\ln p$, see [26, 69].

Example 10.1.1 If $G(z)$ is considered in the form of cosine function, then for (p,q) -periodic function it becomes

$$\varpi(x)=\cos\left(\frac{2\pi}{\ln q-\ln p}\ln x\right).$$

Applying $D_{p,q}$ derivative operator to above function gives

$$D_{p,q}\cos\left(\frac{2\pi}{\ln q-\ln p}\ln x\right)=\frac{\varpi(px)-\varpi(qx)}{px-qx}$$

$$\begin{aligned}
&= \frac{\cos\left(\frac{2\pi}{\ln q - \ln p} \ln px\right) - \cos\left(\frac{2\pi}{\ln q - \ln p} \ln qx\right)}{px - qx} \\
&= 0,
\end{aligned}$$

which proves that

$$\varpi(x) = \cos\left(\frac{2\pi}{\ln q - \ln p} \ln x\right)$$

is (p, q) -periodic function, for similar function, see Chapter 9 in [69].

Supposing

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is a formal power series, then among formal series, $f(x)$ has (p, q) -antiderivative up to constant term, which is

$$\int f(x) d_{p,q} x = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{[n+1]_{p,q}} + C.$$

Proposition 10.1.1 (Duran et al. [26]). Let $0 < \frac{q}{p} < 1$. Then, any function $f(x)$ has at most one (p, q) -antiderivative which is continuous at $x = 0$, up to adding a constant.

Proof. Let F_1 and F_2 be considered as two (p, q) -antiderivatives of f , which are both continuous at 0. Let $\varpi = F_1 - F_2$, which also must be continuous at 0. Moreover

$$D_{p,q} \varpi(x) = D_{p,q} (F_1(x) - F_2(x)) = f(x) - f(x) = 0$$

implies that $\varpi(px) = \varpi(qx)$ for any x . For some $U > 0$, let

$$s = \inf \left\{ \varpi(x) \mid \frac{q}{p}U \leq x \leq U \right\},$$

$$S = \sup \left\{ \varpi(x) \mid \frac{q}{p}U \leq x \leq U \right\},$$

which may be infinity if ϖ is unbounded above and/or below. It should be clear that because of $s \neq S$, $\varpi(0)$ can not be both s and S . It is not a problem that it is selected as s or S , so $\varpi(0) \neq s$ can be assumed. By the definition of continuous at $x = 0$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$s + \varepsilon \notin \varpi(0, \delta).$$

However, there exists for some sufficiently N such that $\left(\frac{p}{q}\right)^N U < \delta$, which implies that

$$s + \varepsilon \in (s, S) \subset \varpi \left[\frac{q}{p}U, U \right] = \varpi \left[\left(\frac{q}{p}\right)^{N+1} U, \left(\frac{q}{p}\right)^N U \right] \subset \varpi(0, \delta),$$

bringing about a contradiction. So, one has $s = S$, ϖ is a constant in that $\varpi \left[\frac{q}{p}U, U \right]$

which shows that $F_1 - F_2$ is also constant everywhere.

The proposition expresses that the uniqueness of the (p,q) -antiderivative is essentially enhanced by requiring continuity at $x = 0$. The existence problem will be discussed in the next chapter.

Let the following formula be regarded for the change of a variable $u = u(x) = \lambda x^\mu$ with λ, μ being constants. If $F(x)$ is a (p,q) -antiderivative of $f(x)$, then it is easily seen that (Duran *et al.* [26])

$$\int f(u) d_{p,q} u = F(u) = F(u(x)).$$

Using the expression in Eq. (3.12) gives the following identity for any p' and q'

$$\begin{aligned} F(u(x)) &= \int D_{p',q'} F(u(x)) d_{p',q'} x \\ &= \int \left(D_{p'^{\mu},q'^{\mu}} F \right) (u(x)) \cdot D_{p',q'} u(x) d_{p',q'} x \\ &= \int \left(D_{p'^{\mu},q'^{\mu}} F \right) (u(x)) d_{p',q'} u(x). \end{aligned}$$

If $p' = p^{1/\mu}$ and $q' = q^{1/\mu}$ are taken, then it becomes $D_{p'^{\mu},q'^{\mu}} F = D_{p,q} F = f$, and hence

$$\int f(u) d_{p,q} u = \int f(u(x)) d_{p^{1/\mu},q^{1/\mu}} u(x).$$

This formula implies that $f(u(x)) D_{p^{1/\mu},q^{1/\mu}} u(x)$ is one of the (p,q) -antiderivatives of $f(x)$, see [2] for the counterpart of q -antiderivatives.

The final results of this section are listed below by giving some simple examples of (p,q) -antiderivatives:

$$\int (1 \oplus x)_{p,q}^n d_{p,q} x = \frac{\left(\frac{1}{p} \oplus x \right)_{p,q}^{n+1}}{[n+1]_{p,q}} + C,$$

$$\int e_{p,q}(x) d_{p,q} x = e_{p,q}\left(\frac{x}{p}\right) + C$$

$$\int E_{p,q}(x) d_{p,q} x = E_{p,q}\left(\frac{x}{q}\right) + C$$

$$\int \sin_{p,q}(x) d_{p,q} x = \cos_{p,q}\left(\frac{x}{p}\right) + C$$

$$\int \cos_{p,q}(x) d_{p,q} x = -\sin_{p,q}\left(\frac{x}{p}\right) + C$$

$$\int SIN_{p,q}(x) d_{p,q}x = COS_{p,q}\left(\frac{x}{q}\right) + C$$

$$\int COS_{p,q}(x) d_{p,q}x = -SIN_{p,q}\left(\frac{x}{q}\right) + C$$

$$\int \sinh_{p,q}(x) d_{p,q}x = \cosh_{p,q}\left(\frac{x}{p}\right) + C$$

$$\int \cosh_{p,q}(x) d_{p,q}x = \sinh_{p,q}\left(\frac{x}{p}\right) + C$$

$$\int SINH_{p,q}(x) d_{p,q}x = COSH_{p,q}\left(\frac{x}{q}\right) + C$$

$$\int COSH_{p,q}(x) d_{p,q}x = SINH_{p,q}\left(\frac{x}{q}\right) + C.$$

10.2 The (p,q) -Integral

Here the inverse of the (p,q) -differentiation called the (p,q) -integration is constructed as (p,q) -integral that gives us the (p,q) -antiderivative $F(x)$ of an arbitrary function $f(x)$. Assuming that $f(x)$ be an arbitrary function and $F(x)$ be a function satisfying the equation $D_{p,q}F(x) = f(x)$, then [19]

$$\frac{F(px) - F(qx)}{px - qx} = f(x).$$

The formula above can be rewritten as $F(px) - F(qx) = (p - q)xf(x)$ where $\gamma = (p - q)$. Hence, the formula is examined by

$$F(p^1q^{-1}x) - F(p^0q^{-0}x) = (p - q)p^0q^{-1}xf(p^0q^{-1}x)$$

$$F(p^2q^{-2}x) - F(p^1q^{-1}x) = (p - q)p^1q^{-2}xf(p^1q^{-2}x)$$

$$F(p^3 q^{-3} x) - F(p^2 q^{-2} x) = (p - q) p^2 q^{-3} x f(p^2 q^{-3} x)$$

$\vdots$

$$F(p^{n+1} q^{-(n+1)} x) - F(p^n q^{-n} x) = (p - q) p^n q^{-(n+1)} x f(p^n q^{-(n+1)} x).$$

By adding these formulas terms by terms, it is deduced that

$$F(p^{n+1} q^{-(n+1)} x) - F(x) = (p - q) x \sum_{k=0}^n \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} x\right).$$

Supposing $\left|\frac{p}{q}\right| < 1$ and letting $n \rightarrow \infty$, it becomes

$$F(x) - F(0) = (q - p) x \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} x\right).$$

By the same way, for $\left|\frac{p}{q}\right| > 1$, it becomes [19]

$$F(x) - F(0) = (p - q) x \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} x\right).$$

From which, the following definition can be given.

Definition 10.2.1 (Sadjang [19]). Let f be an arbitrary function. The (p, q) -integral of any function f is defined by

$$\int f(x) d_{p,q} x = (p - q) x \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} x\right). \quad (10.1)$$

Remark 10.2.1 (Sadjang [19]). Note that this is a formal definition since the convergence of the right hand side of the expression in Eq. (10.1) is not interested.

The above-defined integral can be called as (p, q) -analogue of the Jackson

integral because when $p = 1$, the integral reduces to Jackson integral.

By the above definition, a more general formula is developed as

$$\begin{aligned} \int f(x) D_{p,q} g(x) d_{p,q} x &= (p-q) \sum_{k=0}^{\infty} \frac{q^k x}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} x\right) D_{p,q} g\left(\frac{q^k}{p^{k+1}} x\right) \\ &= (p-q) \sum_{k=0}^{\infty} \frac{q^k x}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} x\right) \frac{g\left(\frac{q^k}{p^k} x\right) - g\left(\frac{q^{k+1}}{p^{k+1}} x\right)}{(p-q) \frac{q^k}{p^{k+1}} x}, \end{aligned}$$

and hence

$$\int f(x) d_{p,q} g(x) = \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}} x\right) \left(g\left(\frac{q^k}{p^k} x\right) - g\left(\frac{q^{k+1}}{p^{k+1}} x\right) \right). \quad (10.2)$$

The following theorem gives an adequate condition for what conditions it readily converges to a (p,q) -antiderivative.

Theorem 10.2.1 (Sadjang [19]). *Let $0 < \frac{q}{p} < 1$ and let $|f(x)x^\tau|$ be bounded on the interval $(0, U]$ for some $0 \leq \tau < 1$. Then the (p,q) -integral defined in (10.1) converges to a function $F(x)$ on $(0, U]$, which is a (p,q) -antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$.*

Proof. Supposing that $|f(x)x^\tau| < M$ on $(0, U]$ and fix $0 < x \leq U$. Then for $k \geq 0$,

$$\left| f\left(\frac{q^k}{p^{k+1}} x\right) \left(\frac{q^k}{p^{k+1}} x\right)^\tau \right| < M$$

$$\left| f\left(\frac{q^k}{p^{k+1}} x\right) \left(\frac{q^k}{p^{k+1}} x\right)^\tau \right| < M$$

$$\left| f\left(\frac{q^k}{p^{k+1}}x\right) \right| < M \left(\frac{q^k}{p^{k+1}}x\right)^{-\tau}.$$

Hence, for any $0 < x \leq U$, it becomes

$$\left| \left(\frac{q^k}{p^{k+1}}\right) f\left(\frac{q^k}{p^{k+1}}x\right) \right| < M \left(\frac{q^k}{p^{k+1}}\right) \left(\frac{q^k}{p^{k+1}}x\right)^{-\tau} = Mx^{-\tau} \frac{1}{p^{1-\tau}} \left(\frac{q^{1-\tau}}{p^{1-\tau}}\right)^k. \quad (10.3)$$

If one writes in the following sum including (p,q) -integral that is majorized by a convergent geometric series, then, Eq. (10.1) converges pointwise to some functions. Namely, one can see directly that $F(0) = 0$. It is the fact that $F(x)$ is continuous at $x = 0$, i.e., $F(x)$ approaches zero as $x \rightarrow 0$ using (10.3), for $0 < x \leq U$ as

$$\begin{aligned} \left| (p-q) \sum_{k=0}^{\infty} \frac{q^k x}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right) \right| &< |p-q|x \left| \sum_{k=0}^{\infty} \frac{q^k x}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right) \right| \\ &< |p-q|x \left| \sum_{k=0}^{\infty} Mx^{-\tau} \frac{1}{p^{1-\tau}} \left(\frac{q^{1-\tau}}{p^{1-\tau}}\right)^k \right| \\ &< |p-q| \frac{1}{\left(p^{1-\tau}\right)} \frac{Mx^{1-\tau}}{1-\left(\frac{q}{p}\right)}. \end{aligned}$$

In order to determine whether $F(x)$ is a (p,q) -antiderivative of $f(x)$, it is sufficient to check that

$$\begin{aligned} D_{p,q}F(x) &= \frac{1}{(p-q)x} \left((p-q) \sum_{\tau=0}^{\infty} \frac{q^{\tau} x}{p^{\tau}} f\left(\frac{q^{\tau}}{p^{\tau}}x\right) - (p-q) \sum_{\tau=0}^{\infty} \frac{q^{\tau+1} x}{p^{\tau+1}} f\left(\frac{q^{\tau+1}}{p^{\tau+1}}x\right) \right) \\ &= \sum_{\tau=0}^{\infty} \frac{q^{\tau}}{p^{\tau}} f\left(\frac{q^{\tau}}{p^{\tau}}x\right) - \sum_{\tau=0}^{\infty} \frac{q^{\tau+1}}{p^{\tau+1}} f\left(\frac{q^{\tau+1}}{p^{\tau+1}}x\right) \\ &= f(x). \end{aligned}$$

Note that if $x \in (0, U]$ and $0 < \frac{q}{p} < 1$, then $\frac{q}{p}x \in (0, U]$, and the (p, q) -differentiation is valid.

By Proposition 10.1.1, if the hypothesis of Theorem 10.2.1 is satisfied, the (p, q) -integral gives the unique (p, q) -antiderivative being continuous at $x = 0$, up to adding a constant. On the other hand, if one know that $F(x)$ is a (p, q) -antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, $F(x)$ must be given, up to adding a constant, by the formula in Eq. (10.1), since a partial sum of the (p, q) -integral is

$$\begin{aligned} (p-q) \sum_{\tau=0}^N \frac{q^{\tau} x}{p^{\tau+1}} f\left(\frac{q^{\tau}}{p^{\tau+1}} x\right) &= (p-q) \sum_{\tau=0}^N \frac{q^{\tau} x}{p^{\tau+1}} D_{p,q} F(x) \Big|_{\frac{q^{\tau}}{p^{\tau+1}} x} \\ &= \sum_{\tau=0}^N \left[F\left(\frac{q^{\tau}}{p^{\tau}} x\right) - F\left(\frac{q^{\tau+1}}{p^{\tau+1}} x\right) \right] \\ &= F(x) - F\left(\frac{q^{N+1}}{p^{N+1}} x\right), \end{aligned}$$

approaching to $F(x) - F(0)$ as $N \rightarrow \infty$, by the continuity of $F(x)$ at the case $x = 0$.

Now, an example is given to see in which (p, q) -integral fails, like q -integral [2]. Let $f(x) = 1/x$. Since

$$D_{p,q} \ln x = \frac{\ln px - \ln qx}{(p-q)x} = \frac{\ln p - \ln q}{p-q} \frac{1}{x},$$

it becomes

$$\int \frac{1}{x} d_{p,q} x = \frac{p-q}{\ln p - \ln q} \ln x.$$

Nevertheless, the formula in Eq. (10.1) gives

$$\int \frac{1}{x} d_{p,q} x = (p-q) \sum_{k=0}^{\infty} 1 = \infty.$$

Consequently, the formula fails because $f(x)x^\tau$ is not bounded for any $0 \leq \tau < 1$.

Note that $\ln x$ is not continuous in the case $x = 0$.

By the motivation of above, the definite (p,q) -integral is defined as follows.

Definition 10.2.2 (Sadjang [19]). *Let f be an arbitrary function and a be a real number, the followings are established:*

$$\int_0^a f(x) d_{p,q} x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} a\right) \quad \text{if } \left|\frac{p}{q}\right| < 1 \quad (10.4)$$

$$\int_0^a f(x) d_{p,q} x = (p-q)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}} a\right) \quad \text{if } \left|\frac{p}{q}\right| > 1. \quad (10.5)$$

For $p = r^{1/2}$ and $q = s^{-1/2}$,

$$\left|\frac{p}{q}\right| < 1 \Leftrightarrow |rs| > 1,$$

and then, the formula in Eq. (10.4) reads

$$\int_0^a f(x) d_{p,q} x = (s^{-1/2} - r^{1/2})a \sum_{k=0}^{\infty} r^{k/2} s^{(k+1)/2} f\left(r^{k/2} s^{(k+1)/2} a\right)$$

which is the formula (11) given in [11]. Once more, for $p = r^{1/2}$ and $q = s^{-1/2}$,

$$\left|\frac{p}{q}\right| < 1 \Leftrightarrow |rs| > 1,$$

and Eq. (10.5) results in

$$\int_0^a f(x) d_{p,q} x = (r^{1/2} - s^{-1/2})a \sum_{k=0}^{\infty} s^{-k/2} r^{-(k+1)/2} f\left(s^{-k/2} r^{-(k+1)/2} a\right)$$

which is the formula (10) given in [11].

Definition 10.2.3 (Sadjang [19]). Let f be an arbitrary function and, a, b be two nonnegative numbers such that $a < b$. Then the following expression holds

$$\int_a^b f(x) d_{p,q} x = \int_0^b f(x) d_{p,q} x - \int_0^a f(x) d_{p,q} x \quad (10.6)$$

Example 10.2.1 For $\left| \frac{p}{q} \right| > 1$, it is observed that

$$\begin{aligned} \int_a^b x^n d_{p,q} x &= (p-q)b \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}b\right) - (p-q)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right) \\ &= (p-q)(b^{n+1} - a^{n+1})q^{-n-1} \sum_{k=0}^{\infty} \left(\frac{p^{(n+1)}}{q^{(n+1)}}\right)^k, \end{aligned}$$

by using the property of geometric series, it is gained that

$$\begin{aligned} (p-q)(b^{n+1} - a^{n+1})q^{-n-1} \frac{1}{1 - \frac{p^{n+1}}{q^{n+1}}} &= (b^{n+1} - a^{n+1}) \frac{p-q}{q^{n+1} - p^{n+1}} \\ &= \frac{a^{n+1} - b^{n+1}}{[n+1]_{p,q}}, \end{aligned}$$

which is the (p,q) -analogue of the usual integral $\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1}$. The (p,q) -antiderivative of x^n is $x^{n+1} / [n+1]_{p,q}$, hereby this is just

$$\left. \frac{x^{n+1}}{[n+1]_{p,q}} \right|_{x=b} - \left. \frac{x^{n+1}}{[n+1]_{p,q}} \right|_{x=a}$$

which is no coincidence.

A good definition of the improper integral can not be acquired by simply

letting $a \rightarrow \infty$ in Eq. (10.5). Instead, because

$$\begin{aligned}
\int_{q^{j+1}/p^{j+1}}^{q^j/p^j} f(x) d_{p,q}x &= \int_0^{q^j/p^j} f(x) d_{p,q}x - \int_0^{q^{j+1}/p^{j+1}} f(x) d_{p,q}x \\
&= (p-q) \left\{ \sum_{k=0}^{\infty} \frac{q^{k+j}}{p^{k+1+j}} f\left(\frac{q^{k+j}}{p^{k+1+j}}\right) - \sum_{k=0}^{\infty} \frac{q^{k+j+1}}{p^{k+2+j}} f\left(\frac{q^{k+j+1}}{p^{k+2+j}}\right) \right\} \\
&= (p-q) \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right),
\end{aligned}$$

it is natural to describe the improper (p,q) -integral as follows.

Definition 10.2.4 (Sadjang [19]). *The improper (p,q) -integral of $f(x)$ on $[0; +\infty)$ is described to be*

$$\begin{aligned}
\int_0^{\infty} f(x) d_{p,q}x &= \sum_{j=-\infty}^{\infty} \int_{q^{j+1}/p^{j+1}}^{q^j/p^j} f(x) d_{p,q}x \\
&= (p-q) \sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right)
\end{aligned} \tag{10.7}$$

if $0 < \frac{q}{p} < 1$ or

$$\int_0^{\infty} f(x) d_{p,q}x = (q-p) \sum_{j=-\infty}^{\infty} \int_{q^{j+1}/p^{j+1}}^{q^j/p^j} f(x) d_{p,q}x \tag{10.8}$$

if $1 < \frac{q}{p}$ where the formula is used.

Proposition 10.2.1 (Sadjang [19]). *Assuming that $0 < \frac{q}{p} < 1$. The improper*

(p,q) -integral converges if $x^{\tau} f(x)$ is bounded in a neighbourhood of $x=0$ with $\tau < 1$ and for sufficiently large x with some $\tau > 1$.

Proof. From the formula in Eq. (10.7), it is computed that

$$\begin{aligned}\int_0^\infty f(x) d_{p,q} x &= (p-q) \sum_{j=-\infty}^\infty \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right) \\ &= (p-q) \left\{ \sum_{j=0}^\infty \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right) + \sum_{j=1}^\infty \frac{q^{-j}}{p^{-j+1}} f\left(\frac{q^{-j}}{p^{-j+1}}\right) \right\}.\end{aligned}$$

The convergence of the former is proved by Theorem 10.2.1. For the latter, assuming for x large it is procured that $|x^\tau f(x)| < M$ where $\tau > 1$ and $M > 0$. Then, it is obtained for sufficiently large j ,

$$\begin{aligned}\left| \frac{q^{-j}}{p^{-j+1}} f\left(\frac{q^{-j}}{p^{-j+1}}\right) \right| &= p^{\tau-1} \left(\frac{q}{p}\right)^{j(\tau-1)} \left| \left(\frac{q^{-j}}{p^{-j+1}}\right)^\tau f\left(\frac{q^{-j}}{p^{-j+1}}\right) \right| \\ &< Mp^{\tau-1} \left(\frac{q}{p}\right)^{j(\tau-1)}.\end{aligned}$$

Therefore, the latter is also bounded above by a convergent geometric series, and hence converges.

Note that when $\frac{q}{p} > 1$, the similar proposition can be stated.

Definition 10.2.5 (Sadjang [19]). *Let f be an arbitrary function and a be a nonnegative real number, then it is defined that*

$$\int_a^\infty f(x) d_{p,q} x = (q-p)a \sum_{j=0}^\infty \frac{q^{-j}}{p^{-j-1}} f\left(\frac{q^{-j}}{p^{-j-1}}a\right) \text{ if } \left|\frac{p}{q}\right| < 1 \quad (10.9)$$

$$\int_a^\infty f(x) d_{p,q} x = (p-q)a \sum_{j=0}^\infty \frac{q^{-j}}{p^{-j-1}} f\left(\frac{q^{-j}}{p^{-j-1}}a\right) \text{ if } \left|\frac{p}{q}\right| > 1. \quad (10.10)$$

Remark 10.2.2 (Sadjang [19]). For $a = 1$, by comparing the formulas in Eqs. (10.4) and (10.9) with the formulas in Eqs. (10.5) and (10.10), then it is attained that

$$\int_0^\infty f(x) d_{p,q}x = (q-p)a \sum_{j=-\infty}^\infty \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}a\right) \text{ if } \left|\frac{p}{q}\right| < 1 \quad (10.11)$$

$$\int_0^\infty f(x) d_{p,q}x = (p-q)a \sum_{j=-\infty}^\infty \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}a\right) \text{ if } \left|\frac{p}{q}\right| > 1. \quad (10.12)$$



CHAPTER 11

THE FUNDAMENTAL THEOREM OF (p,q) -CALCULUS

In ordinary calculus, a derivative is defined as the limit of a ratio, and a definite integral is defined as the limit of an infinite sum. Their subtle and surprising relation is given by the Newton-Leibnitz formula, also called the fundamental theorem of calculus. Following the work completed in q -calculus in [2], where the introduction of the definite integral has been motivated by an antiderivative, the relation between the (p,q) -derivative and the (p,q) -integral is more obvious. Similar to the ordinary and q cases, the fundamental theorem of (p,q) -calculus or (p,q) -Newton-Leibnitz formula is introduced as follows.

Theorem 11.1 (Sadjang [19]). *(Fundamental theorem of (p,q) -calculus) If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, then*

$$\int_a^b f(x) d_{p,q}x = F(b) - F(a), \quad (11.1)$$

where $0 \leq a < b \leq \infty$.

Proof. For $F(x)$ is continuous at $x=0$, $F(x)$ can be given as

$$F(x) = (p-q)x \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}x\right) + F(0).$$

Since

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}a\right),$$

it can be written as

$$\int_0^a f(x) d_{p,q} x = F(a) - F(0).$$

Similarly, for a finite b , it can be written as

$$\int_0^b f(x) d_{p,q} x = F(b) - F(0).$$

Hence, it is reached that

$$\int_a^b f(x) d_{p,q} x = F(b) - F(a).$$

If $a = \frac{q^{j+1}}{p^{j+1}}$ and $b = \frac{q^j}{p^j}$ are chosen as special values and if the definition of the improper (p,q) -integral in Eq. (10.7) is taken into consideration, the Eq. (11.1) is seen to be true for $b = \infty$.

Corollary 11.1 (Sadjang [19]). *If $f'(x)$ exists in a neighbourhood of $x=0$ and is continuous at $x=0$, where $f'(x)$ means the ordinary derivative of $f(x)$, the following equation satisfies*

$$\int_a^b D_{p,q} f(x) d_{p,q} x = f(b) - f(a). \quad (11.2)$$

Proof. It is seen that

$$\begin{aligned} \lim_{x \rightarrow 0} D_{p,q} f(x) &= \lim_{x \rightarrow 0} \frac{f(px) - f(qx)}{(p-q)x} \\ &= \lim_{x \rightarrow 0} \frac{pf'(px) - qf'(qx)}{p-q} = f'(0), \end{aligned}$$

where the L'Hospital rule has been used. Thus, $D_{p,q} f(x)$ can be made continuous at

$x = 0$ if $(D_{p,q}f)(0) = f'(0)$ is introduced and Eq. (11.2) follows from the theorem.

As the q -integral, a significant difference between the (p,q) -integral and its ordinary counterpart is that even if a function on an interval like $[2;3]$ is integrated its attitude at $x = 0$ is has to be analyzed. This has to do with the condition for convergence of the (p,q) -integral and the description of the definite (p,q) -integral.

Let $f(x)$ and $g(x)$ be two functions whose ordinary derivatives exists in a neighbourhood of $x = 0$. As the property of (p,q) -derivative, it is known that

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x).$$

Because the product of differentiable functions is also differentiable in ordinary calculus, it can be applied to Corollary 11.1 so as to get

$$f(b)g(b) - f(a)g(a) = \int_a^b f(px)(D_{p,q}g(x))d_{p,q}x + \int_a^b g(qx)(D_{p,q}f(x))d_{p,q}x$$

or

$$\int_a^b f(px)(D_{p,q}g(x))d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_{p,q}f(x))d_{p,q}x, \quad (11.3)$$

which is the formula of (p,q) -integration by parts. Remark that $b = \infty$ is allowed as well.

CHAPTER 12

THE (p,q) -GAMMA AND THE (p,q) -BETA FUNCTIONS

Being related to solutions of special types of differential equations, many important functions in analysis are defined by means of the definite integral. The most important examples are the Gamma functions (represented by the symbol of capital Greek letter Γ) and the Beta functions (its symbol B is a Greek capital β rather than the similar Latin capital B) detailed below.

The Gamma functions $\Gamma(x)$ first happens in a correspondence between Euler and Goldbach in 1729 and is defined for $x > 0$ by (see [2])

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt. \quad (12.1)$$

Euler also provided equivalent representation of the Gamma function

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! n^x}{x(x+1) \cdots (x+n)}. \quad (12.2)$$

Two useful properties of the function are

$$\Gamma(x+1) = x\Gamma(x)$$

and, for any nonnegative integer n ,

$$\Gamma(n+1) = n!.$$

Closely connected with the Gamma function is the Beta function defined as (see [2])

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1}, \Re x > 0, \Re y > 0. \quad (12.3)$$

The Beta functions are interrelated with the Gamma functions as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (12.4)$$

which implies that the Beta function is symmetric, meaning that

$$B(x, y) = B(y, x). \quad (12.5)$$

Recall that the (p, q) -powers are introduced in Chapter 5 as

$$(x \ominus y)_{p,q}^n = (x - y)(px - yq) \cdots (p^{n-2}x - yq^{n-2})(p^{n-1}x - yq^{n-1}), \quad (12.6)$$

$$(x \oplus y)_{p,q}^n = (x + y)(px + yq) \cdots (p^{n-2}x + yq^{n-2})(p^{n-1}x + yq^{n-1}). \quad (12.7)$$

These formulas can be extended to (cf. [20])

$$(x \ominus y)_{p,q}^\infty = \prod_{k=0}^{\infty} (p^k x - yq^k) \quad (12.8)$$

and

$$(x \oplus y)_{p,q}^\infty = \prod_{k=0}^{\infty} (p^k x + yq^k) \quad (12.9)$$

where the convergence is required.

Here are some identities of the (p, q) -basis.

Proposition 12.1 (Sadjan [20]). *The following relations hold true*

$$(x \ominus y)_{p,q}^n = \frac{(x \ominus y)_{p,q}^\infty}{(xp^n \ominus yq^n)_{p,q}^\infty} \quad (12.10)$$

$$(x \ominus y)_{p,q}^{n+k} = (x \ominus y)_{p,q}^n (xp^n \ominus yq^n)_{p,q}^k \quad (12.11)$$

$$(xp^n \ominus yq^n)_{p,q}^k = \frac{(x \ominus y)_{p,q}^k (xp^k \ominus yq^k)_{p,q}^n}{(x \ominus y)_{p,q}^n} \quad (12.12)$$

$$(xp^k \ominus yq^k)_{p,q}^{n-k} = \frac{(x \ominus y)_{p,q}^n}{(x \ominus y)_{p,q}^k} \quad (12.13)$$

$$(xp^{2k} \ominus yq^{2k})_{p,q}^{n-k} = \frac{(x \ominus y)_{p,q}^n (xp^n \ominus yq^n)_{p,q}^k}{(x \ominus y)_{p,q}^{2k}} \quad (12.14)$$

$$(x \ominus y)_{p,q}^{2n} = (x \ominus y)_{p^2, q^2}^n (xp \ominus yq)_{p^2, q^2}^n \quad (12.15)$$

$$(x \ominus y)_{p,q}^{3n} = (x \ominus y)_{p^3, q^3}^n (xp \ominus yq)_{p^3, q^3}^n (xp^2 \ominus yq^2)_{p^3, q^3}^n \quad (12.16)$$

$$(x \ominus y)_{p,q}^{kn} = \prod_{j=0}^{k-1} (p^j x \ominus yq^j)_{p^k, q^k}^n. \quad (12.17)$$

Proof. By using the formulas in Eqs. (12.6) and (12.8), it is easily seen that, for Eq. (12.10),

$$\begin{aligned} (x \ominus y)_{p,q}^n &= (x - y)(px - yq) \cdots (p^{n-2}x - yq^{n-2})(p^{n-1}x - yq^{n-1}) \\ &= \frac{(x - y)(px - yq) \cdots (p^{n-1}x - yq^{n-1})(p^n x - yq^n)(p^{n+1}x - yq^{n+1}) \cdots}{(p^n x - yq^n)(p^{n+1}x - yq^{n+1})(p^{n+2}x - yq^{n+2}) \cdots} \\ &= \frac{(x \ominus y)_{p,q}^\infty}{(xp^n \ominus yq^n)_{p,q}^\infty}; \end{aligned}$$

for Eq. (12.11),

$$\begin{aligned} (x - y)_{p,q}^{n+k} &= (x - y)(px - yq) \cdots (p^{n-1}x - yq^{n-1})(p^n x - yq^n) \cdots (p^{n+k-1}x - yq^{n+k-1}) \\ &= [(x - y)(px - yq) \cdots (p^{n-1}x - yq^{n-1})][(p^n x - yq^n) \cdots (p^{n+k-1}x - yq^{n+k-1})] \\ &= (x - y)_{p,q}^n (xp^n \ominus yq^n)_{p,q}^k; \end{aligned}$$

for Eq. (12.12),

$$(xp^n \ominus yq^n)_{p,q}^k = (p^n x - yq^n)(p^{n+1}x - yq^{n+1}) \cdots (p^{n+k-1}x - yq^{n+k-1})$$

$$\begin{aligned}
&= \frac{(x-y)(px-yq)\cdots(p^{n-1}x-yq^{n-1})(p^n x-yq^n)(p^{n+1}x-yq^{n+1})\cdots(p^{n+k-1}x-yq^{n+k-1})}{(x-y)(px-yq)\cdots(p^{n-1}x-yq^{n-1})} \\
&= \frac{(x-y)(px-yq)\cdots(p^{k-1}x-yq^{k-1})(p^k x-yq^k)(p^{k+1}x-yq^{k+1})\cdots(p^{n+k-1}x-yq^{n+k-1})}{(x-y)(px-yq)\cdots(p^{n-1}x-yq^{n-1})} \\
&= \frac{(x\ominus y)_{p,q}^k (xp^k \ominus yq^k)_{p,q}^n}{(x\ominus y)_{p,q}^n};
\end{aligned}$$

for Eq. (12.13),

$$\begin{aligned}
(xp^k \ominus yq^k)_{p,q}^{n-k} &= (p^k x-yq^k)(p^{k+1}x-yq^{k+1})\cdots(p^{n-1}x-yq^{n-1}) \\
&= \frac{(x-y)(px-yq)\cdots(p^{k-1}x-yq^{k-1})(p^k x-yq^k)(p^{k+1}x-yq^{k+1})\cdots(p^{n-1}x-yq^{n-1})}{(x-y)(px-yq)\cdots(p^{k-1}x-yq^{k-1})} \\
&= \frac{(x\ominus y)_{p,q}^n}{(x\ominus y)_{p,q}^k};
\end{aligned}$$

for Eq. (12.14),

$$\begin{aligned}
(xp^{2k} \ominus yq^{2k})_{p,q}^{n-k} &= (p^{2k} x-yq^{2k})(p^{2k+1}x-yq^{2k+1})\cdots(p^{n+k-1}x-yq^{n+k-1}) \\
&= \frac{(x-y)(px-yq)\cdots(p^{2k-1}x-yq^{2k-1})(p^{2k} x-yq^{2k})(p^{2k+1}x-yq^{2k+1})\cdots(p^{n+k-1}x-yq^{n+k-1})}{(x-y)(px-yq)\cdots(p^{2k-1}x-yq^{2k-1})} \\
&= \frac{(x-y)(px-yq)\cdots(p^{n-1}x-yq^{n-1})(p^n x-yq^n)(p^{n+1}x-yq^{n+1})\cdots(p^{n+k-1}x-yq^{n+k-1})}{(x-y)(px-yq)\cdots(p^{2k-1}x-yq^{2k-1})} \\
&= \frac{(x\ominus y)_{p,q}^n (xp^n \ominus yq^n)_{p,q}^k}{(x\ominus y)_{p,q}^{2k}};
\end{aligned}$$

for Eq. (12.15),

$$(x\ominus y)_{p,q}^{2n} = (x-y)(px-yq)\cdots(p^{2n-1}x-yq^{2n-1})$$

$$\begin{aligned}
&= (x-y)(px-yq)(p^2x-yq^2)\cdots(p^{2n-2}x-yq^{2n-2})(p^{2n-1}x-yq^{2n-1}) \\
&= (x-y)(p^2x-yq^2)\cdots(p^{2n-2}x-yq^{2n-2})(px-yq)(p^3x-yq^3)\cdots(p^{2n-1}x-yq^{2n-1}) \\
&= (x\ominus y)_{p^2,q^2}^n (xp\ominus yq)_{p^2,q^2}^n;
\end{aligned}$$

and similarly for Eq. (12.16),

$$\begin{aligned}
(x\ominus y)_{p,q}^{3n} &= (x-y)(px-yq)\cdots(p^{3n-1}x-yq^{3n-1}) \\
&= (x-y)(px-yq)(p^2x-yq^2)\cdots(p^{3n-3}x-yq^{3n-3})(p^{3n-2}x-yq^{3n-2})(p^{3n-1}x-yq^{3n-1}) \\
&= (x-y)\cdots(p^{3n-3}x-yq^{3n-3})(px-yq)\cdots(p^{3n-2}x-yq^{3n-2})(p^2x-yq^2)\cdots(p^{3n-1}x-yq^{3n-1}) \\
&= (x\ominus y)_{p^3,q^3}^n (xp\ominus yq)_{p^3,q^3}^n (xp^2\ominus yq^2)_{p^3,q^3}^n.
\end{aligned}$$

For Eq. (12.17), if the degrees $2n$ in Eq. (12.15) and $3n$ in Eq. (12.16) up to kn is continued, then Eq. (12.17) is deduced. Therefore, the proof of the theorem is completed.

Now the (p,q) -analogues of the Gamma function and the Beta function are introduced and some their properties are examined. Firstly, the (p,q) -Gamma function is elaborated below.

12.1 The (p,q) -Gamma function

Definition 12.1.1 (Sadjang [20]). *For any complex number x , the (p,q) -Gamma function is described as*

$$\Gamma_{p,q}(x) = \frac{(p\ominus q)_{p,q}^\infty}{(p^x\ominus q^x)_{p,q}^\infty} (p-q)^{1-x}, \quad 0 < q < p. \quad (12.18)$$

Note that putting $p=1$ in the Definition 12.1.1 yields to the definition of the q -Gamma function given by

$$\Gamma_q(x) = \frac{(1 \ominus q)_q^\infty}{(1 \ominus q^x)_q^\infty} (1-q)^{1-x}, \quad 0 < q < 1.$$

The (p,q) -Gamma function satisfy the fundamental relation given below.

Proposition 12.1.1 (Sadjang [20]). *The relation*

$$\Gamma_{p,q}(x+1) = [x]_{p,q} \Gamma_{p,q}(x) \quad (12.19)$$

is true.

Proof. By using Definition 12.1.1, it is seen that

$$\begin{aligned} \Gamma_{p,q}(x+1) &= \frac{(p \ominus q)_{p,q}^\infty}{(p^{x+1} \ominus q^{x+1})_{p,q}^\infty} (p-q)^{-x} \\ &= \frac{p^x - q^x}{p-q} \frac{(p \ominus q)_{p,q}^\infty}{(p^x \ominus q^x)_{p,q}^\infty} (p-q)^{1-x} \\ &= [x]_{p,q} \Gamma_{p,q}(x) \end{aligned}$$

which ends the proof.

Corollary 12.1.1 (Sadjang [20]). *For a nonnegative integer n , the following relation holds*

$$\Gamma_{p,q}(n+1) = [n]_{p,q}! \quad (12.20)$$

Proof. It follows from Eq. (12.19) that

$$\begin{aligned} \Gamma_{p,q}(n+1) &= [n]_{p,q} \Gamma_{p,q}(n) \\ &= [n]_{p,q} [n-1]_{p,q} \Gamma_{p,q}(n-1) \\ &= [n]_{p,q} [n-1]_{p,q} [n-2]_{p,q} \Gamma_{p,q}(n-2) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & = [n]_{p,q}! \end{aligned}$$

As an immediate result of Eq. (12.20), the following identity

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)_{p,q}^n}{(p-q)^n}$$

holds true.

Proposition 12.1.2 (Sadjang [20]). *((p,q)-Legendre's multiplication formula) The following equality holds*

$$\Gamma_{p,q}(2x) \Gamma_{p^2,q^2}\left(\frac{1}{2}\right) = (p+q)^{2x-1} \Gamma_{p^2,q^2}(x) \Gamma_{p^2,q^2}\left(x + \frac{1}{2}\right).$$

Proof. By using Definition 12.1.1, it is observed that

$$\Gamma_{p,q}(2x) = \frac{(p \ominus q)_{p,q}^\infty}{(p^{2x} \ominus q^{2x})_{p,q}^\infty} (p-q)^{1-2x}$$

$$\Gamma_{p^2,q^2}\left(\frac{1}{2}\right) = \frac{(p^2 \ominus q^2)_{p^2,q^2}^\infty}{(p \ominus q)_{p^2,q^2}^\infty} (p^2 - q^2)^{\frac{1}{2}}$$

$$\Gamma_{p^2,q^2}(x) = \frac{(p^2 \ominus q^2)_{p^2,q^2}^\infty}{(p^{2x} \ominus q^{2x})_{p^2,q^2}^\infty} (p^2 - q^2)^{1-x}$$

$$\Gamma_{p^2,q^2}\left(x + \frac{1}{2}\right) = \frac{(p^2 \ominus q^2)_{p^2,q^2}^\infty}{(p^{2x+1} \ominus q^{2x+1})_{p^2,q^2}^\infty} (p^2 - q^2)^{\frac{1}{2}-x}.$$

Thus, it becomes

$$\frac{\Gamma_{p^2,q^2}(x) \Gamma_{p^2,q^2}\left(x + \frac{1}{2}\right)}{\Gamma_{p^2,q^2}\left(\frac{1}{2}\right)} = \frac{\frac{(p^2 \ominus q^2)_{p^2,q^2}^\infty}{(p^{2x} \ominus q^{2x})_{p^2,q^2}^\infty} (p^2 - q^2)^{1-x} \cdot \frac{(p^2 \ominus q^2)_{p^2,q^2}^\infty}{(p^{2x+1} \ominus q^{2x+1})_{p^2,q^2}^\infty} (p^2 - q^2)^{\frac{1}{2}-x}}{\frac{(p^2 \ominus q^2)_{p^2,q^2}^\infty}{(p \ominus q)_{p^2,q^2}^\infty} (p^2 - q^2)^{\frac{1}{2}}}$$

$$\begin{aligned}
&= \frac{(p^2 \ominus q^2)_{p,q}^\infty (p^2 - q^2)^{1-x} (p^2 \ominus q^2)_{p^2, q^2}^\infty (p^2 - q^2)^{\frac{1}{2}-x} (p \ominus q)_{p^2, q^2}^\infty}{(p^{2x} \ominus q^{2x})_{p^2, q^2}^\infty (p^2 - q^2)^{\frac{1}{2}} (p^2 \ominus q^2)_{p^2, q^2}^\infty (p^{2x+1} \ominus q^{2x+1})_{p^2, q^2}^\infty} \\
&= \frac{(p \ominus q)_{p,q}^\infty}{(p^{2x} \ominus q^{2x})_{p,q}^\infty} (p^2 - q^2)^{1-2x} \\
&= \frac{(p \ominus q)_{p,q}^\infty}{(p^{2x} \ominus q^{2x})_{p,q}^\infty} (p - q)^{1-2x} (p + q)^{1-2x} \\
&= (p + q)^{1-2x} \Gamma_{p,q}(2x),
\end{aligned}$$

which completes the proof of the theorem.

A generalization of the (p, q) -Legendre's multiplication formula is the following formula called (p, q) -Gauss' multiplication formula.

Proposition 12.1.3 (Sadjang [20]). (*(p, q) -Gauss' multiplication formula*) The following identity holds

$$\Gamma_{p,q}(nx) \prod_{k=1}^{n-1} \Gamma_{p^n, q^n}\left(\frac{k}{n}\right) = ([n]_{p,q})^{nx-1} \prod_{k=0}^{n-1} \Gamma_{p^n, q^n}\left(x + \frac{k}{n}\right).$$

Proof. By using Definition 12.1.1, it is observed that

$$\begin{aligned}
\Gamma_{p^n, q^n}\left(\frac{k}{n}\right) &= \frac{(p^n \ominus q^n)_{p^n, q^n}^\infty}{(p^k \ominus q^k)_{p^n, q^n}^\infty} (p^n - q^n)^{-\frac{k}{n}} \\
\Gamma_{p^n, q^n}\left(x + \frac{k}{n}\right) &= \frac{(p^n \ominus q^n)_{p^n, q^n}^\infty}{(p^{nx+k} \ominus q^{nx+k})_{p^n, q^n}^\infty} (p^n - q^n)^{1-\frac{k}{n}-x}.
\end{aligned}$$

Hence, it becomes

$$\prod_{k=1}^{n-1} \Gamma_{p^n, q^n}\left(\frac{k}{n}\right) = \prod_{k=1}^{n-1} \frac{(p^n \ominus q^n)_{p^n, q^n}^\infty}{(p^k \ominus q^k)_{p^n, q^n}^\infty} (p^n - q^n)^{1-\frac{k}{n}}$$

$$\begin{aligned}
&= \frac{\left[(p^n \ominus q^n)_{p^n, q^n}^\infty \right]^{n-1}}{\prod_{k=1}^{n-1} (p^k \ominus q^k)_{p^n, q^n}^\infty} (p^n - q^n)^{\sum_{k=1}^n \left(1 - \frac{k}{n}\right)} \\
&= \frac{\left[(p^n \ominus q^n)_{p^n, q^n}^\infty \right]^n}{\prod_{k=1}^{n-1} (p p^k \ominus q q^k)_{p^n, q^n}^\infty} (p^n - q^n)^{\frac{n-1}{2}} \\
&= \frac{\left[(p^n \ominus q^n)_{p^n, q^n}^\infty \right]^n}{(p \ominus q)_{p, q}^\infty} (p^n - q^n)^{\frac{n-1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\prod_{k=0}^{n-1} \Gamma_{p^n, q^n} \left(x + \frac{k}{n} \right) &= \prod_{k=0}^{n-1} \frac{(p^n \ominus q^n)_{p^n, q^n}^\infty}{(p^{nx+k} \ominus q^{nx+k})_{p^n, q^n}^\infty} (p^n - q^n)^{1 - \frac{k}{n} - x} \\
&= \frac{\left[(p^n \ominus q^n)_{p^n, q^n}^\infty \right]^n}{\prod_{k=0}^{n-1} (p^{nx+k} \ominus q^{nx+k})_{p^n, q^n}^\infty} (p^n - q^n)^{\sum_{k=1}^n \left(1 - \frac{k}{n} - x\right)} \\
&= \frac{\left[(p^n \ominus q^n)_{p^n, q^n}^\infty \right]^n}{(p^{nx} \ominus q^{nx})_{p^n, q^n}^\infty} (p^n - q^n)^{\left(\frac{n-1}{2} + 1 - nx\right)}.
\end{aligned}$$

From the applications above, it is arrived that

$$\begin{aligned}
\frac{\prod_{k=1}^{n-1} \Gamma_{p^n, q^n} \left(\frac{k}{n} \right)}{\prod_{k=1}^{n-1} \Gamma_{p^n, q^n} \left(\frac{k}{n} \right)} &= \frac{\frac{\left[(p^n \ominus q^n)_{p^n, q^n}^\infty \right]^n}{(p^{nx} \ominus q^{nx})_{p^n, q^n}^\infty} (p^n - q^n)^{\left(\frac{n-1}{2} + 1 - nx\right)}}{\frac{\left[(p^n \ominus q^n)_{p^n, q^n}^\infty \right]^n}{(p \ominus q)_{p, q}^\infty} (p^n - q^n)^{\frac{n-1}{2}}} \\
&= \frac{(p \ominus q)_{p, q}^\infty}{(p^{nx} \ominus q^{nx})_{p^n, q^n}^\infty} (p^n - q^n)^{1 - nx}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(p \ominus q)_{p,q}^{\infty}}{(p^{nx} \ominus q^{nx})_{p^n, q^n}^{\infty}} (p - q)^{1-nx} \left(\frac{p^n - q^n}{p - q} \right)^{1-nx} \\
&= [n]_{p,q}^{1-nx} \Gamma_{p,q}(nx),
\end{aligned}$$

which is completed the proof of this theorem.

12.2 The (p,q) -Beta functions

Definition 12.2.1 (Milavanovic et al. [21]). Let $m, n \in \mathbb{N}$, the (p,q) -Beta integral is described as

$$B_{p,q}(m, n) = \int_0^1 x^{m-1} (1 \ominus qx)_{p,q}^{n-1} d_{p,q} x.$$

Theorem 12.2.1 (Milavanovic et al. [21]). The (p,q) -Gamma and (p,q) -Beta functions fulfil the following fundamental relation

$$B_{p,q}(m, n) = p^{(n-1)(2m+n-2)/2} \frac{\Gamma_{p,q}(m) \Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}, \quad (12.21)$$

where $m, n \in \mathbb{N}$.

Proof. For any $m, n \in \mathbb{N}$, since

$$B_{p,q}(m, n) = \int_0^1 x^{m-1} (1 \ominus qx)_{p,q}^{n-1} d_{p,q} x,$$

using Eq. (11.3) for $f(x) = (x/p)^{m-1}$ and $g(x) = -(1 \ominus x)_{p,q}^n / [n]_{p,q}$ with the equality

$D_{p,q}(1-x)_{p,q}^n = -[n]_{p,q} (1 \ominus qx)_{p,q}^{n-1}$, it follows

$$B_{p,q}(m, n) = \int_0^1 x^{m-1} (1 \ominus qx)_{p,q}^{n-1} d_{p,q} x$$

$$\begin{aligned}
&= f(1)g(1) - f(0)g(0) + \frac{[m-1]_{p,q}}{p^{m-1}[n]_{p,q}} \int_0^1 x^{m-2} (1 \ominus qx)_{p,q}^n d_{p,q} x \\
&= \frac{[m-1]_{p,q}}{p^{m-1}[n]_{p,q}} \int_0^1 x^{m-2} (1 \ominus qx)_{p,q}^n d_{p,q} x \\
&= \frac{[m-1]_{p,q}}{p^{m-1}[n]_{p,q}} B_{p,q}(m-1, n+1). \tag{12.22}
\end{aligned}$$

Also, it can be written for positive integer n that

$$\begin{aligned}
B_{p,q}(m, n+1) &= \int_0^1 x^{m-1} (1 \ominus qx)_{p,q}^n d_{p,q} x \\
&= \int_0^1 x^{m-1} (1 \ominus qx)_{p,q}^{n-1} (p^{n-1} - q^n x) d_{p,q} x \\
&= p^{n-1} \int_0^1 x^{m-1} (1 \ominus qx)_{p,q}^{n-1} d_{p,q} x - q^n \int_0^1 x^m (1 \ominus qx)_{p,q}^{n-1} d_{p,q} x \\
&= p^{n-1} B_{p,q}(m, n) - q^n B_{p,q}(m+1, n).
\end{aligned}$$

Using Eq. (12.22), it is discovered that

$$B_{p,q}(m, n+1) = p^{n-1} B_{p,q}(m, n) - q^n \frac{[m]_{p,q}}{p^m [n]_{p,q}} B_{p,q}(m, n+1),$$

which yields to

$$B_{p,q}(m, n+1) = p^{n+m-1} \frac{p^n - q^n}{p^{n+m} - q^{n+m}} B_{p,q}(m, n).$$

Further, by the definition of (p, q) -integral, it is gained that

$$B_{p,q}(m, 1) = \int_0^1 x^{m-1} d_{p,q} x = \frac{1}{[m]_{p,q}}.$$

Immediately it becomes

$$\begin{aligned}
B_{p,q}(m,n) &= p^{n+m-2} \frac{p^{n-1} - q^{n-1}}{p^{n+m-1} - q^{n+m-1}} B_{p,q}(m, n-1) \\
&= p^{n+m-2} \frac{p^{n-1} - q^{n-1}}{p^{n+m-1} - q^{n+m-1}} p^{n+m-3} \frac{p^{n-2} - q^{n-2}}{p^{n+m-2} - q^{n+m-2}} B_{p,q}(m, n-2) \\
&= p^{n+m-2} \frac{p^{n-1} - q^{n-1}}{p^{n+m-1} - q^{n+m-1}} p^{n+m-3} \frac{p^{n-2} - q^{n-2}}{p^{n+m-2} - q^{n+m-2}} B_{p,q}(m, n-2) \dots \\
&\quad \times p^{n+m-2} \frac{p^{n-1} - q^{n-1}}{p^{n+m-1} - q^{n+m-1}} p^m \frac{p - q}{p^{m+1} - q^{m+1}} B_{p,q}(m, 1) \\
&= p^{m+(m+1)+\dots+(m+n-2)} \frac{(p \ominus q)_{p,q}^{n-1}}{(p^m \ominus q^m)_{p,q}^n} (p - q) \\
&= p^{(n-1)(2m+n-2)/2} \frac{(p \ominus q)_{p,q}^{n-1}}{(p^m \ominus q^m)_{p,q}^n} (p - q). \tag{12.23}
\end{aligned}$$

Following the formula in Eq. (5.4), $(a \ominus b)_{p,q}^{n+m} = (a \ominus b)_{p,q}^n (ap^n \ominus bq^n)_{p,q}^m$, thus Eq. (12.23) leads to

$$\begin{aligned}
B_{p,q}(m,n) &= p^{(n-1)(2m+n-2)/2} \frac{(p \ominus q)_{p,q}^{n-1}}{(p^m \ominus q^m)_{p,q}^n} (p - q) \\
&= p^{(n-1)(2m+n-2)/2} \frac{(p \ominus q)_{p,q}^{n-1}}{(p - q)^{n-1}} \cdot \frac{(p \ominus q)_{p,q}^{m-1}}{(p - q)^{m-1}} \cdot \frac{(p - q)^{m-1} (p - q)^{n-1}}{(p \ominus q)_{p,q}^{m-1} (p^m \ominus q^m)_{p,q}^n} (p - q) \\
&= p^{(n-1)(2m+n-2)/2} \frac{(p \ominus q)_{p,q}^{n-1}}{(p - q)^{n-1}} \frac{(p \ominus q)_{p,q}^{m-1}}{(p - q)^{m-1}} \frac{(p - q)^{m+n-1}}{(p \ominus q)_{p,q}^{m+n-1}} \\
&= p^{(n-1)(2m+n-2)/2} \frac{\Gamma_{p,q}(m) \Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.
\end{aligned}$$

This completes the proof.

Remark 12.2.1 (Milavanovic *et al.* [21]). *The following is valid for $m, n \in \mathbb{N}$:*

$$B_{p,q}(m, n+1) = p^{n-1} B_{p,q}(m, n) - q^n B_{p,q}(m+1, n).$$

Remark 12.2.2 (Milavanovic et al. [21]). The (p,q) -Beta integral is not commutative. In order to make commutative, the following form may be considered:

$$\tilde{B}_{p,q}(m,n) = \int_0^1 p^{m(m-1)/2} x^{m-1} (1 \ominus qx)_{p,q}^{n-1} d_{p,q} x.$$

For this form, (p,q) -Gamma and (p,q) -Beta functions fulfil the following fundamental relation

$$\tilde{B}_{p,q}(m,n) = p^{\left(2mn+m^2+n^2-3m-3n+2\right)/2} \frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)},$$

where $m, n \in \mathbb{N}$. Clearly for this form, it gives that

$$\tilde{B}_{p,q}(m,n) = \tilde{B}_{p,q}(n,m).$$

It follows from the Eq. (12.20) that:

$$B_{p,q}(x,y) = p^{(y-1)(2x+y-2)/2} \frac{[x-1]_{p,q}! [y-1]_{p,q}!}{[x+y-1]_{p,q}!}, \quad (12.24)$$

which is the (p,q) -extension of the formula (see [21])

$$B(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}.$$

Now, some identities of (p,q) -Beta function are given by using its definition, see [20, 21] for details.

Proposition 12.2.1 The following identities

$$B_{p,q}(x,y+1) = p^{x+y-1} \frac{[y]_{p,q}}{[x+y]_{p,q}} B_{p,q}(x,y)$$

$$B_{p,q}(x+1, y) = p^{y-1} \frac{[x]_{p,q}}{[x+y]_{p,q}} B_{p,q}(x, y)$$

$$B_{p,q}(x+1, y) = p^{-x} \frac{[x]_{p,q}}{[x+y]_{p,q}} B_{p,q}(x, y+1)$$

$$B_{p,q}(x+n, y) = p^{n(y-1)} \frac{(p^x \ominus q^x)_{p,q}^n}{(p^{x+y} \ominus q^{x+y})_{p,q}^n} B_{p,q}(x, y)$$

are true.

Proof. By using the formulas in Eqs. (12.19) and (12.21), it is readily seen that

$$\begin{aligned} B_{p,q}(x, y+1) &= p^{(y)(2x+y-1)/2} \frac{\Gamma_{p,q}(x) \Gamma_{p,q}(y+1)}{\Gamma_{p,q}(x+y+1)} \\ &= p^{x+y-1} \frac{[y]_{p,q}}{[x+y]_{p,q}} p^{(y-1)(2x+y-2)/2} \frac{\Gamma_{p,q}(x) \Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)} \\ &= p^{x+y-1} \frac{[y]_{p,q}}{[x+y]_{p,q}} B_{p,q}(x, y), \end{aligned}$$

$$\begin{aligned} B_{p,q}(x+1, y) &= p^{(y-1)(2x+y)/2} \frac{\Gamma_{p,q}(x+1) \Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y+1)} \\ &= p^{y-1} \frac{[x]_{p,q}}{[x+y]_{p,q}} p^{(y-1)(2x+y-2)/2} \frac{\Gamma_{p,q}(x) \Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)} \\ &= p^{y-1} \frac{[x]_{p,q}}{[x+y]_{p,q}} B_{p,q}(x, y), \end{aligned}$$

and

$$\begin{aligned} B_{p,q}(x+1, y) &= p^{(y-1)(2x+y)/2} \frac{\Gamma_{p,q}(x+1) \Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y+1)} \\ &= p^{-x} \frac{[x]_{p,q}}{[y]_{p,q}} p^{y(2x+y-1)/2} \frac{\Gamma_{p,q}(x) \Gamma_{p,q}(y+1)}{\Gamma_{p,q}(x+y+1)} \end{aligned}$$

$$= p^{-x} \frac{[x]_{p,q}}{[y]_{p,q}} B_{p,q}(x, y+1).$$

Considering Eq. (12.19), then it is derived that

$$\begin{aligned}
B_{p,q}(x+n, y) &= p^{(y-1)(2x+2n+y-2)/2} \frac{\Gamma_{p,q}(x+n) \Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y+n)} \\
&= p^{y-1} \frac{[x+n-1]_{p,q}}{[x+y+n-1]_{p,q}} B_{p,q}(x+n-1, y) \\
&= p^{2(y-1)} \frac{[x+n-1]_{p,q} [x+n-2]_{p,q}}{[x+y+n-1]_{p,q} [x+y+n-2]_{p,q}} B_{p,q}(x+n-2, y) \\
&\vdots \\
&= p^{n(y-1)} \frac{[x+n-1]_{p,q} [x+n-2]_{p,q} \cdots [x]_{p,q}}{[x+y+n-1]_{p,q} [x+y+n-2]_{p,q} \cdots [x+y]_{p,q}} B_{p,q}(x, y) \\
&= p^{n(y-1)} \frac{\left(\frac{p^{x+n-1} - q^{x+n-1}}{p-q} \right) \left(\frac{p^{x+n-2} - q^{x+n-2}}{p-q} \right) \cdots \left(\frac{p^x - q^x}{p-q} \right)}{\left(\frac{p^{x+y+n-1} - q^{x+y+n-1}}{p-q} \right) \left(\frac{p^{x+y+n-2} - q^{x+y+n-2}}{p-q} \right) \cdots \left(\frac{p^{x+y} - q^{x+y}}{p-q} \right)} B_{p,q}(x, y) \\
&= p^{n(y-1)} \frac{(p^{x+n-1} - q^{x+n-1})(p^{x+n-2} - q^{x+n-2}) \cdots (p^x - q^x)}{(p^{x+y+n-1} - q^{x+y+n-1})(p^{x+y+n-2} - q^{x+y+n-2}) \cdots (p^{x+y} - q^{x+y})} B_{p,q}(x, y) \\
&= p^{n(y-1)} \frac{(p^x \ominus q^x)_{p,q}^n}{(p^{x+y} \ominus q^{x+y})_{p,q}^n} B_{p,q}(x, y).
\end{aligned}$$

Therefore, the proofs of the proposition is completed.

Here, (p, q) -binomial coefficient is derived by means of the (p, q) -Beta function.

Proposition 12.2.2 *The following equality holds true:*

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} = \frac{p^{-k(2n-k+1)/2}}{[n+1]_{p,q} B_{p,q}(n-k+1, k+1)}.$$

Proof. It follows from the definition of the (p,q) -binomial coefficient that:

$$\begin{aligned}
 \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!} \\
 &= \frac{p^{-k(2n-k+1)/2}}{[n+1]_{p,q}} \cdot p^{k(2n-k+1)/2} \frac{[n+1]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!} \\
 &= \frac{p^{-k(2n-k+1)/2}}{[n+1]_{p,q} B_{p,q}(n-k+1, k+1)}.
 \end{aligned}$$

CHAPTER 13

p -ADIC ANALYSIS

p -adic numbers were introduced in 1904 by Kurt Hensel, the German mathematician. They are used extensively in number theory. p -adic analysis was developed (principally for needs of number theory) in many directions. For example, they are used in the famous proof of Fermat's Last Theorem by Andrew Wiles [75] and also they are applied to physics by two pure mathematicians, A. Monna and F. van der Blij, in 1968. In 1972 a model of p -adic valued quantum mechanics from the positions of quantum logic is investigated by E. Beltrametti and G. Cassinelli. Since 80-th p -adic numbers are used in applications to quantum physics. p -adic strings and superstrings were the first models of p -adic quantum physics. The interests of physicists to p -adic numbers is expressed by the attempts to create new models of space-time for the description of (fantastically small) Planck distances, for further details on that what are p -adic numbers used for, see [76].

There are some evidence that the standard model based on real numbers is not adequate to Planck's domain. On the other hand, some properties of fields of p -adic numbers seem to be closely associated with Planck's domain. Especially, the field of the p -adic numbers has no ordered structure.

The pioneer developments on p -adic string theory induced investigations on p -adic quantum mechanics and field theory. This developments inspire a investigation of p -adic mathematics in many directions: theory of probability, the theory of distributions, differential and psedodifferential equations, Volkenborn integral, spectral theory of operators in a p -adic analogue of a Hilbert space, special polynomials and numbers, the theory of diophantine geometry, the theory of diophantine approximation.

13.1 Rudiments of p -adic Analysis

Some basic notations, definitions and properties belonging to the p -adic analysis are given.

Definition 13.1.1 (Koblitz [77]). Let $p \in \{2, 3, 5, 7, 11, 13, 17, \dots\}$ be a prime number. For any nonzero integer a , let $\text{ord}_p a$ be the highest power of p that divides a , i.e., the greatest m such that $a \equiv 0 \pmod{p^m}$ where the notation has been used $a \equiv b \pmod{c}$ meant a divide $a - b$.

Here are some applications of $\text{ord}_p a$:

$$\text{ord}_3 9 = 2, \text{ord}_5 21 = 0, \text{ord}_2 64 = 6, \text{ord}_7 84 = 1, \dots$$

Note that $\text{ord}_p 0 = \infty$. The following properties hold true

$$\text{ord}_p(ab) = \text{ord}_p a + \text{ord}_p b$$

and for $x = \frac{a}{b} \in \mathbb{Q}$,

$$\text{ord}_p x = \text{ord}_p a - \text{ord}_p b.$$

The p -adic absolute value (norm) of x is given by (see [76, 77])

$$|x|_p = \begin{cases} p^{-a}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

The p -adic norm provides the so-called strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

which is known as the non-Archimedean norm (see [76, 77]).

This definition of $|x|_p$ has the influence that high powers of p become

“small”, particularly $|3^n|_3 = 1/3^n$. For a $x \in \mathbb{Q} / \{0\}$, by the fundamental theorem of arithmetic, there is a unique finite set of distinct primes $p_1, \dots, p_r$ and a corresponding sequence of non-zero integers $a_1, \dots, a_r$ such that $x = p_1^{a_1} \dots p_r^{a_r}$. Thus, it becomes $|x|_{p_i} = p_i^{-a_i}$ for all $i = 1, \dots, r$, and $|x|_p = 1$ for any other prime $p \notin \{p_1, \dots, p_r\}$.

For instance, if $\frac{2200}{147} = 2^3 3^{-1} 5^2 7^{-2} 11$ is chosen, then it is observed that (see also [76])

$$\left| \frac{2200}{147} \right|_p = \begin{cases} 1/8, & \text{if } p = 2 \\ 3, & \text{if } p = 3 \\ 1/25 & \text{if } p = 5 \\ 49, & \text{if } p = 7 \\ 1/11, & \text{if } p = 11 \\ 1, & \text{if } p \geq 13. \end{cases}$$

Now the definition of p -adic integers are presented as follows.

Definition 13.1.2 (Rozikov [76]). A p -adic integer is a p -adic number of the form $\sum_{i \geq 0} a_i p^i$ where a_i are integers, and p is prime. It is sufficient to take a_i in the set $\{0, 1, \dots, p-1\}$.

The ring of all p -adic integers is denoted by $\mathbb{Z}_p$. $\mathbb{Q}_p$, the set of the all p -adic rational numbers is defined by the following definition.

Definition 13.1.3 (Koblitz [77]).

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n : 0 \leq a_i \leq p-1 \right\}.$$

For sufficiently large i in Definition 13.1.3, $a_{-n} = 0$. That is, if $x \in \mathbb{Q}_p$, it can be shown as

$$x = \cdots + a_2 p^2 + a_1 p + a^0 + \frac{a^{-1}}{p} + \cdots + \frac{a_{-k}}{p^k}.$$

In the special case $a_{-1} = a_{-2} = \cdots = a_{-k} = 0$, it becomes $x \in \mathbb{Z}_p$. $\overline{\mathbb{Q}_p}$ denotes the algebraic closure of $\mathbb{Q}_p$ and also $\mathbb{C}_p$ denotes the completion of $\overline{\mathbb{Q}_p}$.

Remark 13.1.1 (Koblitz [77]). Using the norm $|x|_p$, $\mathbb{Z}_p$ can be shown as

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$

The metric space $\mathbb{Q}_p$ has a "basis of open sets" involving in all sets of the form $a + p^n \mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x - a|_p \leq p^{-n} \right\}$ for $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$. This means that any open subset of $\mathbb{Q}_p$ is a union of open subsets of this type. An open subset of $\mathbb{Q}_p$ is compact if and only if it is a finite union of intervals. This type of open set is called as a compact-open set.

Definition 13.1.4 (Koblitz [77]). Let X and Y be two topological spaces. A map $f : X \rightarrow Y$ is called locally constant if every point $x \in X$ has a neighbourhood U such that $f(U)$ is a single element of Y .

It is obvious to see that a locally constant function is continuous.

A p -adic distribution μ on X is defined as follows.

Definition 13.1.5 (Koblitz [77]). A p -adic distribution μ on X is a $\mathbb{Q}_p$ -linear vector space homomorphism from the $\mathbb{Q}_p$ -vector space of locally constant functions on X to $\mathbb{Q}_p$. If $f : X \rightarrow \mathbb{Q}_p$ locally constant, instead of writing $\mu(f)$ for the value of μ at f , we usually write $\int f \mu$.

Theorem 13.1.1 (Koblitz [77]). Every map μ from the set of intervals contained in X to $\mathbb{Q}_p$ for which

$$\mu(a + p^n \mathbb{Z}_p) = \sum_{b=0}^{p-1} \mu(a + bp^n + p^{n+1} \mathbb{Z}_p)$$

whenever $a + p^n \mathbb{Z}_p \subset X$, extends uniquely to a p -adic distribution on X .

Now, some the most famous examples of p -adic distributions are listed below.

(1) **The Haar Distribution** μ_{Haar} : (Koblitz [77]). Define

$$\mu_{\text{Haar}}(a + p^n \mathbb{Z}_p) = \frac{1}{p^n}, a \in \mathbb{Z}_p. \quad (13.1)$$

This extends to a distribution on $\mathbb{Z}_p$ by using Theorem 3, as

$$\begin{aligned} \sum_{b=0}^{p-1} \mu_{\text{Haar}}(a + bp^n + p^{n+1} \mathbb{Z}_p) &= \sum_{b=0}^{p-1} \frac{1}{p^{n+1}} = \frac{1}{p^n} \\ &= \mu_{\text{Haar}}(a + p^n \mathbb{Z}_p). \end{aligned}$$

(2) **The Mazur Distribution** μ_{Mazur} : (Koblitz [77]). First, without loss of generality, when $a + p^n \mathbb{Z}_p$ is written it is supposed that a is a rational between 0 and $p^n - 1$. Assuming this, it is defined by

$$\mu_{\text{Mazur}}(a + p^n \mathbb{Z}_p) = \frac{a}{p^n} - \frac{1}{2}.$$

(3) **The Dirac Distribution** μ_α : (Koblitz [77]). Let $\alpha \in \mathbb{Z}_p$ and U be a compact open subset of $\mathbb{Q}_p$. Then, it is defined by

$$\mu_\alpha(U) = \begin{cases} 1, & \text{if } \alpha \in U, \\ 0, & \text{if } \alpha \notin U. \end{cases}$$

(4) **The Bernoulli Distribution** $\mu_{B,k}$: (Koblitz [77]). Fix a nonnegative integer k .

The definition of a map $\mu_{B,k}$ on intervals $a + p^n \mathbb{Z}_p$ is given by

$$\mu_{B,k}(a + p^n \mathbb{Z}_p) = p^{n(k-1)} B_k \left(\frac{a}{p^n} \right),$$

where $B_k \left(\frac{a}{p^n} \right)$ is Bernoulli polynomial with the value of $\frac{a}{p^n}$. $\mu_{B,k}$ extends to a distribution on $\mathbb{Z}_p$, which is called the k -th Bernoulli distribution. In some special cases, the Bernoulli distribution reduces to some known distribution. For example, for $k = 0$, the distribution

$$\mu_{B,0}(a + p^n \mathbb{Z}_p) = p^{n(0-1)} B_0 \left(\frac{a}{p^n} \right) = \frac{1}{p^n} = \mu_{Haar}$$

is obtained and for $k = 1$, the following distribution is derived:

$$\mu_{B,1}(a + p^n \mathbb{Z}_p) = p^{n(1-1)} B_1 \left(\frac{a}{p^n} \right) = \frac{a}{p^n} = \mu_{Mazur}.$$

Definition 13.1.6 (Koblitz [77]). A p -adic distribution μ on X is a measure if its values on compact-open $U \subset X$ are bounded by some constant $B \in \mathbb{R}$, i.e.,

$$|\mu(U)|_p \leq B \text{ for all compact-open } U \subset X.$$

The Dirac distribution μ_α for fixed $\alpha \in \mathbb{Z}_p$ is a measure, none of the Bernoulli distributions are measures. There is a standard method, called “regularization”, for turning Bernoulli distributions into measures. Let α be any rational integer not equal to 1 and not divisible by p and U be a compact-open subset of $\mathbb{Z}_p$. Then $\mu_{B,k,\alpha}$ or briefly $\mu_{k,\alpha}$ is the “regularized” Bernoulli distribution on $\mathbb{Z}_p$ defined by

$$\mu_{k,\alpha}(U) = \mu_{B,k}(U) - \alpha^{-k} \mu_{B,k}(\alpha U).$$

For $k = 1$, it turns into

$$|\mu_{1,\alpha}(U)|_p \leq 1.$$

Let μ be an unbounded p -adic distribution. If f is locally constant, $\int f \mu$ is defined by definition. However, you run into problems if you try to use limits of Riemann sums to extend integration to continuous functions f .

For instance, let $\mu = \mu_{\text{Haar}}$, and take the simple function $f: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ given by $f(x) = x$. Let the Riemann sums be formed. Given a function f , for any n , $\mathbb{Z}_p$ is divided up into

$$\bigcup_{a=0}^{p^n-1} (a + p^n \mathbb{Z}_p),$$

let $x_{a,n}$ be an arbitrary point in the a -th interval, and the n -th Riemann sum of f corresponding to $\{x_{a,n}\}$ is defined by

$$S_{n, \{x_{a,n}\}}(f) := \sum_{a=0}^{p^n-1} f(x_{a,n}) \mu(a + p^n \mathbb{Z}_p).$$

In our instance, this sum equals to

$$\sum_{a=0}^{p^n-1} x_{a,n} \frac{1}{p^n}.$$

To illustrate, choosing $x_{a,n} = a$ gives

$$p^{-n} \sum_{a=0}^{p^n-1} a = p^{-n} \frac{(p^n-1)p^n}{2} = \frac{p^n-1}{2}.$$

This sum has a limit in $\mathbb{Q}_p$ as n goes to ∞ , that is, $-1/2$. However, in place of $x_{a,n} = a \in a + p^n \mathbb{Z}_p$, if one substitute one of the $x_{a,n}$ to $a + a_0 p^n \in a + p^n \mathbb{Z}_p$ for each

n in which a_0 is some fixed p -adic integer, then it is obtained that

$$S_{n, \{x_{a,n}\}}(f) = p^{-n} \left(\sum_{a=0}^{p^n-1} a + a_0 p^n \right) = \frac{p^n - 1}{2} + a_0.$$

From which, it becomes

$$\lim_{n \rightarrow \infty} S_{n, a+a_0 p^n}(f) = a_0 - \frac{1}{2}.$$

Hence, the Riemann sums do not have a limit that is independent of the choice of points in the intervals.

Definition 13.1.7 (Koblitz [77]). Let μ be a p -adic measure on X , and let $f : X \rightarrow \mathbb{Q}_p$ be a continuous function. Then Riemann sums

$$S_{n, \{x_{a,n}\}}(f) := \sum_{\substack{0 \leq a < p^n \\ a + p^n \mathbb{Z}_p \subset X}} f(x_{a,n}) \mu(a + p^n \mathbb{Z}_p) \quad (13.2)$$

converge to a limit in $\mathbb{Q}_p$ as $n \rightarrow \infty$ which does not depend on the choice of $\{x_{a,n}\}$, where the sum in the Eq. (4.2) is taken over all a for which $a + p^n \mathbb{Z}_p \subset X$, and $x_{a,n}$ is chosen in $a + p^n \mathbb{Z}_p$.

Definition 13.1.8 (Koblitz [77]). If $f : X \rightarrow \mathbb{Q}_p$ is a continuous function and μ is a measure on X , we define $\int f \mu$ to be the limit of the Riemann sums, namely

$$\int f \mu := \lim_{n \rightarrow \infty} S_{n, \{x_{a,n}\}}(f).$$

Definition 13.1.9 (Koblitz [77]). Let X be a subset of $\mathbb{C}_p$, which does not include the discrete point. For a map $f : X \rightarrow \mathbb{C}_p$ and $x, y \in X$ with $x \neq y$, if the function $F_f(x, y)$ defined by

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

exists, the function f is called as uniformly differentiable function at a point $a \in \mathbb{Z}_p$. A map f which is uniformly differentiable function at the every point on $\mathbb{Z}_p$ is called as uniformly differentiable function. The set of all uniformly differentiable functions is shown by

$$C^1(X \rightarrow \mathbb{C}_p) = \{f \mid f : X \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

Definition 13.1.10 (Koblitz [77]). *Let*

$$f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p) = \{f \mid f : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is continuous function}\}.$$

The uncertain sum of f is denoted Sf and is defined as

$$Sf(n) = \sum_{j=0}^n f(j) \quad (n \in \mathbb{N}).$$

If $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$, then it becomes $Sf \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$.

13.2 Volkenborn Integral and Its Some Properties

Here the definition of the Volkenborn integral and its some properties are mentioned. Then Bernoulli numbers are shown by means of the Volkenborn integral.

Definition 13.2.1 (Schikhof [78]). *Let $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$ be a function from the p -adic integers taking values in the p -adic numbers. The Volkenborn integral is defined by the limit, if it exists:*

$$\int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{n \rightarrow \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x).$$

Some properties of Volkenborn integral are given in theorem below.

Theorem 13.2.1 (Schikhof [78]). Let $d\mu(x) = \mu_{\text{Haar}}(x + p^n\mathbb{Z}_p) = \frac{1}{p^n}$. The following expressions hold true

$$\int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{n \rightarrow \infty} \frac{Sf(p^n) - Sf(0)}{p^n} = (Sf)'(0).$$

$$\int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{n \rightarrow \infty} \frac{f(0) + f(1) + \cdots + f(p^n - 1)}{p^n}$$

$$\int_{\mathbb{Z}_p} (f(x+1) - f(x)) d\mu(x) = f'(0) \quad (13.3)$$

$$\int_{\mathbb{Z}_p} (f(x+n) - f(x)) d\mu(x) = \sum_{l=0}^{n-1} f'(l)$$

where

$$(Sf)'(x) = \frac{d(Sf(x))}{dx}.$$

By using Volkenborn integral, generating function of a lot of special number type can be found. For example, to get Bernoulli numbers if $f(x) = e^{xt}$ is taken in Eq. (13.3), it becomes

$$\int_{\mathbb{Z}_p} (e^{(x+1)t} - e^{xt}) d\mu(x) = t.$$

After some basic mathematical applications, it is concluded as

$$\int_{\mathbb{Z}_p} e^{xt} d\mu(x) = \frac{t}{e^t - 1}.$$

Thus, the generating function of Bernoulli numbers is given by using Volkenborn integral as (see [78])

$$\int_{\mathbb{Z}_p} e^{xt} d\mu(x) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Using the Taylor expansion of exponential function, it is observed that

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} x^n d\mu(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

which yields to, by comparing the coefficient $\frac{t^n}{n!}$ of both sides,

$$\int_{\mathbb{Z}_p} x^n d\mu(x) = B_n,$$

as expected. Details on Bernoulli numbers and polynomials can be found in [27-32, 34, 40, 41, 46, 49-60, 62-65, 68].

13.3 Fermionic Integral and Its Some Properties

Here the fermionic p -adic integral defined by Kim [48] and its some properties are mentioned. Then, Euler numbers and Genocchi numbers are denoted by means of the fermionic p -adic integral.

Definition 13.3.1 (Kim [33]). Let p be an odd prime number and $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$ be a function from the p -adic integers taking values in the p -adic numbers. The fermionic integral of f is defined by the limit, if it exists:

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \rightarrow \infty} \sum_{x=0}^{p^n-1} (-1)^x f(x).$$

Theorem 13.3.1 (Kim [33]). Let $d\mu_{-1}(x) = \mu_{-1}(x + p^n \mathbb{Z}_p) = (-1)^x$. The following identities hold true

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \rightarrow \infty} \sum_{l=0}^{p^n-1} (-1)^l f(l)$$

$$\int_{\mathbb{Z}_p} (f(x+1) + f(x)) d\mu_{-1}(x) = 2f(0) \quad (13.4)$$

$$\int_{\mathbb{Z}_p} (f(x+n) + (-1)^{n-1} f(x)) d\mu_{-1}(x) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l).$$

By using fermionic p -adic integral generating functions of not only Euler type numbers and polynomials but also Genocchi type numbers and polynomials can be derived. For instance, to obtain the generating function of Euler numbers if $f(x) = e^{xt}$ is chosen in Eq. (13.4), it gives

$$\int_{\mathbb{Z}_p} e^{(x+1)t} + e^{xt} d\mu_{-1}(x) = 2,$$

from which, it is deduced that

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1},$$

whose right side gives the generating function of the Euler numbers. Details on Euler numbers and polynomials can be found in [27-30, 33-36, 44, 46, 47, 53, 55-59, 61, 62, 64, 65]. By motivating the applications above, it is observed that

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$

which yields to

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n$$

meaningly, the Euler numbers can be shown by means of the fermionic p -adic integral (see [33]).

Note that

$$E_n(x) = \frac{G_{n+1}(x)}{n+1},$$

which implies

$$E_n = \frac{G_{n+1}}{n+1}.$$

Thus, it can be written as

$$\frac{G_{n+1}}{n+1} = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x).$$

Details on Genocchi numbers and polynomials can be found in [27, 28, 34, 39, 42, 43, 46, 55, 60, 64].



CHAPTER 14

THE (p, q) -ANALOGUES OF BERNOULLI, EULER, AND GENOCCHI NUMBERS AND POLYNOMIALS

In the recent years, many mathematicians have been studied on the Bernoulli, Euler and Genocchi polynomials and some generalizations of them. The definition of generating functions of Bernoulli, Euler and Genocchi polynomials are given in the light of (p, q) -calculus, which is firstly introduced by Duran, Acikgoz and Araci in [27].

Definition 14.1 (Duran et al. [27]). Let n be a positive integer. Define

$$\mathcal{A}(x, z : p, q) = \sum_{n=0}^{\infty} \mathcal{B}_n(x : p, q) \frac{z^n}{[n]_{p,q}!} = \frac{z}{e_{p,q}(z) - 1} e_{p,q}(xz) \quad (|z| < 2\pi),$$

$$\mathcal{D}(x, z : p, q) = \sum_{n=0}^{\infty} \mathcal{E}_n(x : p, q) \frac{z^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(z) + 1} e_{p,q}(xz) \quad (|z| < \pi),$$

$$\mathcal{M}(x, z : p, q) = \sum_{n=0}^{\infty} \mathcal{G}_n(x : p, q) \frac{z^n}{[n]_{p,q}!} = \frac{[2]_{p,q} z}{e_{p,q}(z) + 1} e_{p,q}(xz) \quad (|z| < \pi),$$

where $\mathcal{B}_n(x : p, q)$, $\mathcal{E}_n(x : p, q)$ and $\mathcal{G}_n(x : p, q)$ are called, respectively, (p, q) -Bernoulli polynomials, (p, q) -Euler polynomials and (p, q) -Genocchi polynomials. In the case when $x = 0$, it is found that

$$\mathcal{B}_n(0 : p, q) := \mathcal{B}_n(p, q), \mathcal{E}_n(0 : p, q) := \mathcal{E}_n(p, q) \text{ and } \mathcal{G}_n(0 : p, q) := \mathcal{G}_n(p, q)$$

representing (p, q) -Bernoulli numbers, (p, q) -Euler numbers and (p, q) -Genocchi numbers, respectively.

Corollary 14.1 (Duran et al. [27]). Upon setting $q \rightarrow p = 1$ in Definition 14.1, the followings hold true

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z}{e^z - 1} e^{xz} \quad (|z| < 2\pi),$$

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2}{e^z + 1} e^{xz} \quad (|z| < \pi),$$

$$\sum_{n=0}^{\infty} G_n(x) \frac{z^n}{n!} = \frac{2z}{e^z + 1} e^{xz} \quad (|z| < \pi),$$

where $B_n(x)$, $E_n(x)$ and $G_n(x)$ are called the classical Bernoulli polynomials, Euler polynomials, and Genocchi polynomials, respectively, and also in the case when $x = 0$ gives

$$B_n(0) := B_n, E_n(0) := E_n \text{ and } G_n(0) := G_n$$

representing classical Bernoulli numbers, Euler numbers and Genocchi numbers, respectively, see [29, 30].

Corollary 14.2 (Duran et al. [27]). Upon setting $p = 1$ in Definition 14.1 yields to

$$\sum_{n=0}^{\infty} B_n(x|q) \frac{z^n}{[n]_q!} = \frac{z}{e_q(z) - 1} e_q(xz) \quad (|z| < 2\pi),$$

$$\sum_{n=0}^{\infty} E_n(x|q) \frac{z^n}{[n]_q!} = \frac{[2]_q}{e_q(z) + 1} e_q(xz) \quad (|z| < \pi),$$

$$\sum_{n=0}^{\infty} G_n(x|q) \frac{z^n}{[n]_q!} = \frac{[2]_q z}{e_q(z) + 1} e_q(xz) \quad (|z| < \pi)$$

where $B_n(x|q)$, $E_n(x|q)$ and $G_n(x|q)$ are called q -Bernoulli polynomials, q -Euler polynomials and q -Genocchi polynomials, respectively. And also

$$B_n(0|q) := B_n(q), E_n(0|q) := E_n(q) \text{ and } G_n(0|q) := G_n(q)$$

represent q -Bernoulli numbers, q -Euler numbers and q -Genocchi numbers, respectively.

Assume that

$$\mathcal{A}(0, z : p, q) := \mathcal{A}(z : p, q), \quad \mathcal{D}(0, z : p, q) := \mathcal{D}(z : p, q)$$

and

$$\mathcal{M}(0, z : p, q) := \mathcal{M}(z : p, q). \quad (14.1)$$

Thus, Corollary 14.3 follows from Definition 14.1 and Eq. (14.1).

Corollary 14.3 (Duran et al. [27]). *The following functional equations*

$$\mathcal{A}(x, z : p, q) = \mathcal{A}(z : p, q) e_{p,q}(xz)$$

$$\mathcal{D}(x, z : p, q) = \mathcal{D}(z : p, q) e_{p,q}(xz)$$

$$\mathcal{M}(x, z : p, q) = \mathcal{M}(z : p, q) e_{p,q}(xz)$$

hold true.

Theorem 14.1 (Duran et al. [27]). *The following identities hold true:*

$$\mathcal{B}_n(x : p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k(p, q) p^{\binom{n-k}{2}} x^{n-k}$$

$$\mathcal{E}_n(x : p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_k(p, q) p^{\binom{n-k}{2}} x^{n-k}$$

$$\mathcal{G}_n(x : p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{G}_k(p, q) p^{\binom{n-k}{2}} x^{n-k}.$$

Proof. It is proved by using Definition 14.1 and Corollary 14.3, that is,

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x; p, q) \frac{z^n}{[n]_{p,q}!} = \left(\sum_{n=0}^{\infty} \mathcal{B}_n(p, q) \frac{z^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{z^n}{[n]_{p,q}!} \right).$$

Applying Cauchy product on the right hand side of above gives

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x; p, q) \frac{z^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k(p, q) p^{\binom{n-k}{2}} x^{n-k} \right) \frac{z^n}{[n]_{p,q}!} \quad (14.2)$$

by comparing the coefficients $\frac{z^n}{[n]_{p,q}!}$ in Eq. (14.2), it becomes

$$\mathcal{B}_n(x; p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k(p, q) p^{\binom{n-k}{2}} x^{n-k}. \quad (14.3)$$

Also, similarly, for (p, q) -Euler and (p, q) -Genocchi polynomials, the other formulas can be derived. Hence, the proof of the theorem is completed.

Theorem 14.2 (Duran *et al.* [27]). *The following identities hold true:*

$$\mathcal{B}_n(p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k(x; p, q) (-1)^{n-k} q^{\binom{n-k}{2}} x^{n-k},$$

$$\mathcal{E}_n(p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_k(x; p, q) (-1)^{n-k} q^{\binom{n-k}{2}} x^{n-k},$$

$$\mathcal{G}_n(p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{G}_k(x; p, q) (-1)^{n-k} q^{\binom{n-k}{2}} x^{n-k}.$$

Proof. Since

$$e_{p,q}(x) E_{p,q}(-x) = 1$$

and Definition 14.1, it follows

$$\sum_{n=0}^{\infty} \mathcal{B}_n(p, q) \frac{z^n}{[n]_{p,q}!} = \frac{z}{e_{p,q}(z) - 1} = \frac{z}{e_{p,q}(z) - 1} e_{p,q}(xz) E_{p,q}(-xz)$$

$$\begin{aligned}
&= \left(\sum_{n=0}^{\infty} \mathcal{B}_n(x; p, q) \frac{z^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} q^{\binom{n}{2}} (-x)^n \frac{z^n}{[n]_{p,q}!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k(x; p, q) q^{\binom{n-k}{2}} (-x)^{n-k} \right) \frac{z^n}{[n]_{p,q}!}.
\end{aligned}$$

Matching the coefficients $\frac{z^n}{[n]_{p,q}!}$ on the both sides gives the claimed result for the first equation. Similarly, it can be obtained for (p, q) -Euler and (p, q) -Genocchi numbers. Thus, our proof is completed.

Theorem 14.3 (Duran et al. [27]). *The following relations satisfy*

$$\begin{aligned}
\sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{B}_k(p, q) &= \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} \\
\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{E}_k(p, q) + \mathcal{E}_n(p, q) &= \begin{cases} [2]_{p,q}, & n = 0 \\ 0, & n \geq 1 \end{cases} \\
\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{G}_k(p, q) + \mathcal{G}_n(p, q) &= \begin{cases} [2]_{p,q}, & n = 1 \\ 0, & n > 1. \end{cases}
\end{aligned}$$

Proof. The proof of the first equation of Theorem 14.3 is given below. The others can be obtained by the similar method. Using Cauchy product rule and Definition 14.1 yields to

$$\begin{aligned}
\frac{z}{e_{p,q}(z) - 1} e_{p,q}(z) &= z + \frac{z}{e_{p,q}(z) - 1} \\
\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{B}_k(p, q) \right) \frac{z^n}{[n]_{p,q}!} &= z + \sum_{n=0}^{\infty} \mathcal{B}_n(p, q) \frac{z^n}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the both sides the coefficients $\frac{z^n}{[n]_{p,q}!}$ (for $n > 1$) gives

$$\sum_{k=0}^n p^{\binom{n-k}{2}} \frac{\mathcal{B}_k(p, q)}{[k]_{p,q}! [n-k]_{p,q}!} - \frac{\mathcal{B}_n(p, q)}{[n]_{p,q}!} = 0. \quad (14.4)$$

It follows from the Eq. (14.4) that the proof is done.

Theorem 14.4 (Duran et al. [27]). *The following identities hold true:*

$$\mathcal{E}_n(p, q) = \frac{\mathcal{G}_{n+1}(p, q)}{n+1}, \quad \mathcal{E}_n(x; p, q) = \frac{\mathcal{G}_{n+1}(x; p, q)}{n+1}$$

and

$$\mathcal{E}_n(x; p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \frac{\mathcal{G}_{k+1}(p, q)}{k+1} x^{n-k}.$$

Proof. Our assertion follows from Definition 14.1. So the proof of the theorem is omitted.

Now the k -th derivative of (p, q) -Bernoulli, (p, q) -Euler and (p, q) -Genocchi polynomials are developed by the following Theorem 14.5.

Theorem 14.5 (Duran et al. [27]). *The following identities hold true:*

$$D_{p,q;x}^k \mathcal{B}_n(x; p, q) = p^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} \mathcal{B}_{n-k}(p^k x; p, q),$$

$$D_{p,q;x}^k \mathcal{E}_n(x; p, q) = p^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} \mathcal{E}_{n-k}(p^k x; p, q),$$

$$D_{p,q;x}^k \mathcal{G}_n(x; p, q) = p^{\binom{k}{2}} \frac{[n]_{p,q}!}{[n-k]_{p,q}!} \mathcal{G}_{n-k}(p^k x; p, q).$$

Proof. Applying the (p, q) -derivative $D_{p,q;x}$ to the both sides of Definition 14.1 yields to

$$\begin{aligned}
D_{p,q;x} \left(\sum_{n=0}^{\infty} \mathcal{B}_n(x:p,q) \frac{z^n}{[n]_{p,q}!} \right) &= \sum_{n=0}^{\infty} D_{p,q} \mathcal{B}_n(x:p,q) \frac{z^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \mathcal{B}_n(px:p,q) \frac{z^{n+1}}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$ gives

$$D_{p,q;x} \mathcal{B}_n(x:p,q) = [n]_{p,q} \mathcal{B}_{n-1}(px:p,q),$$

Continuing this process, it becomes

$$\begin{aligned}
D_{p,q;x} \mathcal{B}_n(x:p,q) &= [n]_{p,q} \mathcal{B}_{n-1}(px:p,q) \\
D_{p,q;x}^2 \mathcal{B}_n(x:p,q) &= p[n]_{p,q} [n-1]_{p,q} \mathcal{B}_{n-2}(p^2x:p,q) \\
&\vdots \\
D_{p,q;x}^k \mathcal{B}_n(x:p,q) &= p^{\binom{k}{2}} [n]_{p,q} [n-1]_{p,q} \cdots [n-k+1]_{p,q} \mathcal{B}_{n-k}(p^kx:p,q)
\end{aligned}$$

which completes the proof the first equation of the theorem. Similarly, the k -th derivative of (p,q) -Euler and (p,q) -Genocchi polynomials can be derived in a like manner. Hence, the proof of the theorem is completed.

Theorem 14.6 (Duran et al. [27]). For $n \geq 0$, the following identities are true:

$$\begin{aligned}
\mathcal{B}_n(x:p,q) &= \lim_{z \rightarrow 0} D_{p,q;z}^n \frac{ze_{p,q}(xz)}{e_{p,q}(z) - 1}, \\
\mathcal{E}_n(x:p,q) &= \lim_{z \rightarrow 0} D_{p,q;z}^n \frac{[2]_{p,q}}{e_{p,q}(z) + 1} e_{p,q}(xz)
\end{aligned}$$

and

$$\mathcal{G}_n(x:p,q) = \lim_{z \rightarrow 0} D_{p,q;z}^n \frac{[2]_{p,q} z}{e_{p,q}(z) + 1} e_{p,q}(xz).$$

Proof. With the help of Definition 14.1 and Theorem 14.5, it becomes

$$D_{p,q;z}^n \frac{ze_{p,q}(xz)}{e_{p,q}(z)-1}$$

as $z \rightarrow 0$, it is derived as

$$\mathcal{B}_n(x : p, q) = \lim_{z \rightarrow 0} D_{p,q;z}^n \frac{ze_{p,q}(xz)}{e_{p,q}(z)-1}.$$

By the similar consideration, other equalities mentioned in Theorem 14.6 can be obtained similarly. Thus, the proof of the theorem is done.

Theorem 14.7 (Duran *et al.* [27]). Let $0 < |q| < |p| \leq 1$. Then the following (p, q) -integrals hold true

$$\int_0^1 \mathcal{B}_n(x : p, q) d_{p,q}x = \frac{\mathcal{B}_{n+1}\left(\frac{1}{p} : p, q\right) - \mathcal{B}_{n+1}(p, q)}{[n+1]_{p,q}},$$

$$\int_0^1 \mathcal{E}_n(x : p, q) d_{p,q}x = \frac{\mathcal{E}_{n+1}\left(\frac{1}{p} : p, q\right) - \mathcal{E}_{n+1}(p, q)}{[n+1]_{p,q}}$$

and

$$\int_0^1 \mathcal{G}_n(x : p, q) d_{p,q}x = \frac{\mathcal{G}_{n+1}\left(\frac{1}{p} : p, q\right) - \mathcal{G}_{n+1}(p, q)}{[n+1]_{p,q}}.$$

Proof. Since

$$\int_a^b D_{p,q} f(x) d_{p,q}x = f(b) - f(a) \quad (\text{see [19]})$$

and Theorem 14.5 with $0 < |q| < |p| \leq 1$, it is observed that

$$\begin{aligned}\int_0^1 \mathcal{B}_n(x; p, q) d_{p, q} x &= \frac{1}{[n+1]_{p, q}} \int_0^1 D_{p, q} \mathcal{B}_{n+1} \left(\frac{x}{p}; p, q \right) d_{p, q} x \\ &= \frac{\mathcal{B}_{n+1} \left(\frac{1}{p}; p, q \right) - \mathcal{B}_{n+1}(p, q)}{[n+1]_{p, q}}.\end{aligned}$$

Also, the (p, q) -integral representation of (p, q) -Euler polynomials and (p, q) -Genocchi polynomials can be shown in a like manner. Therefore, the proof of the theorem is completed.

Note that the results of Theorem 14.7 are (p, q) -generalization of

$$\int_0^1 B_n(x) dx = \frac{B_{n+1}(1) - B_{n+1}}{n+1}, \quad \int_0^1 E_n(x) dx = \frac{E_{n+1}(1) - E_{n+1}}{n+1}$$

and

$$\int_0^1 G_n(x) dx = \frac{G_{n+1}(1) - G_{n+1}}{n+1}.$$

Now the fermionic p -adic integral representation of aforementioned polynomials is given. First, the definition of fermionic p -adic integral on $\mathbb{Z}_p$ originally constructed by Kim [33] is needed.

Let f be a continuous function at a point $a \in \mathbb{Z}_p$, which is denoted by $f \in C(\mathbb{Z}_p)$. Then the fermionic p -adic integral on $\mathbb{Z}_p$ of a function $f \in C(\mathbb{Z}_p)$ is defined as follows:

$$I_{-1}(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x.$$

Theorem 14.8 (Duran et al. [27]). *The followings hold true*

$$I_{-1}(\mathcal{B}_n(x; p, q)) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p, q} p^{\binom{n-k}{2}} \mathcal{B}_k(p, q) E_{n-k},$$

$$I_{-1}(\mathcal{E}_n(x: p, q)) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{E}_k(p, q) E_{n-k},$$

$$I_{-1}(\mathcal{G}_n(x: p, q)) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{G}_k(p, q) E_{n-k}.$$

Proof. Since,

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x)$$

earlier given in previous chapter and Theorem 14.1, it becomes

$$\begin{aligned} \int_{\mathbb{Z}_p} \mathcal{B}_n(x: p, q) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{B}_k(p, q) x^{n-k} \right) d\mu_{-1}(x) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{B}_k(p, q) \int_{\mathbb{Z}_p} x^{n-k} d\mu_{-1}(x) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{B}_k(p, q) E_{n-k} \end{aligned}$$

which is a linear combination of the product $\mathcal{B}_k(p, q) E_{n-k}$. Moreover, the fermionic p -adic integral representation of other equalities in Theorem 14.8 can be acquired in a like manner. Thus, the proof of the theorem is completed.

In summary, the new generating functions of the Bernoulli, the Euler and the Genocchi polynomials based on *post quantum numbers* have been given, which lead to q -analogues of Bernoulli, Euler and Genocchi polynomials known in the literature when $p = 1$. From those generating functions, the aforementioned polynomials are analyzed. In the final part of this chapter, by making use of the fermionic (p, q) -adic integral over the (p, q) -adic number fields, some relations between our new polynomials and earlier defined polynomials are derived.

CHAPTER 15

A CLASS OF (p,q) -BERNOULLI, EULER AND GENOCCHI POLYNOMIALS

The following definition plays an important role in deriving the main results of this chapter. Now, new extensions of usual polynomials including Bernoulli, Euler and Genocchi polynomials which is firstly considered by Duran *et al.* [28] are stated as follows.

Definition 15.1 (Duran *et al.* [28]). Let $p, q \in \mathbb{C}$, $\alpha \in \mathbb{N}$ and $0 < |q| < |p| \leq 1$. The (p,q) -Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x, y; p, q)$ of order α , the (p,q) -Euler polynomials $\mathcal{E}_n^{(\alpha)}(x, y; p, q)$ of order α and the (p,q) -Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x, y; p, q)$ of order α are defined by means of the following Taylor series expansion about $z = 0$:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, y; p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz), \quad |z| < 2\pi,$$

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x, y; p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{[2]_{p,q}}{e_{p,q}(z) + 1} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz), \quad |z| < \pi,$$

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x, y; p, q) \frac{z^n}{[n]_{p,q}!} = \left(\frac{[2]_{p,q} z}{e_{p,q}(z) + 1} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz), \quad |z| < \pi.$$

Letting $x = 0$ and $y = 0$ in Definition 15.1 yields to $\mathcal{B}_n^{(\alpha)}(0, 0; p, q) := \mathcal{B}_n^{(\alpha)}(p, q)$, $\mathcal{E}_n^{(\alpha)}(0, 0; p, q) := \mathcal{E}_n^{(\alpha)}(p, q)$ and $\mathcal{G}_n^{(\alpha)}(0, 0; p, q) := \mathcal{G}_n^{(\alpha)}(p, q)$ which are called, respectively, n -th (p,q) -Bernoulli number of order α , n -th (p,q) -Euler number of order α and n -th (p,q) -Genocchi number of order α . In the case $\alpha = 1$ gives

$$\mathcal{B}_n^{(1)}(x, y: p, q) := \mathcal{B}_n(x, y: p, q), \quad \mathcal{E}_n^{(1)}(x, y: p, q) := \mathcal{E}_n(x, y: p, q)$$

and

$$\mathcal{G}_n^{(1)}(x, y: p, q) := \mathcal{G}_n(x, y: p, q)$$

termed as n -th (p, q) -Bernoulli polynomial, n -th (p, q) -Euler polynomial and n -th (p, q) -Genocchi polynomial. Note that if $\alpha = 1$ and $y = 0$ are chosen as special values, then the three polynomials in Definition 15.1 reduce to the polynomials defined in Chapter 14.

Now, some special cases of Definition 15.1 are given by Corollary 15.1 and Corollary 15.2 as follows.

Corollary 15.1 (Mahmudov [58, 60], Duran *et al.* [28]). *If $p = 1$ is taken in Definition 15.1, then it transforms*

$$\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(\alpha)}(x, y) \frac{z^n}{[n]_q!} = \left(\frac{z}{e_q(z) - 1} \right)^{\alpha} e_q(xz) E_q(yz), \quad (|z| < 2\pi),$$

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(\alpha)}(x, y) \frac{z^n}{[n]_q!} = \left(\frac{[2]_q}{e_q(z) + 1} \right)^{\alpha} e_q(xz) E_q(yz), \quad (|z| < \pi),$$

$$\sum_{n=0}^{\infty} \mathfrak{G}_{n,q}^{(\alpha)}(x, y) \frac{z^n}{[n]_q!} = \left(\frac{[2]_q z}{e_q(z) + 1} \right)^{\alpha} e_q(xz) E_q(yz), \quad (|z| < \pi)$$

where $\mathfrak{B}_{n,q}^{(\alpha)}(x, y)$, $\mathfrak{E}_{n,q}^{(\alpha)}(x, y)$ and $\mathfrak{G}_{n,q}^{(\alpha)}(x, y)$ are called n -th q -Bernoulli polynomial of order α , n -th q -Euler polynomial of order α and n -th q -Genocchi polynomial of order α , respectively. Substituting $x = y = 0$ in the Corollary 15.1 gives $\mathfrak{B}_{n,q}^{(\alpha)}(0, 0) := \mathfrak{B}_{n,q}^{(\alpha)}$, $\mathfrak{E}_{n,q}^{(\alpha)}(0, 0) := \mathfrak{E}_{n,q}^{(\alpha)}$ and $\mathfrak{G}_{n,q}^{(\alpha)}(0, 0) := \mathfrak{B}_{n,q}^{(\alpha)}$ that are called n -th q -Bernoulli number of order α , n -th q -Euler number of order α and n -th q -Genocchi number of order α , respectively (see [55-60]).

Corollary 15.2 (Cheon [29], Duran et al. [28]). Letting p and q to 1 yields to for $y = 0$ and $\alpha = 1$ in Definition 15.1 as follows

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z}{e^z - 1} e^{xz}, \quad (|z| < 2\pi),$$

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2}{e^z + 1} e^{xz}, \quad (|z| < \pi),$$

$$\sum_{n=0}^{\infty} G_n(x) \frac{z^n}{n!} = \frac{2z}{e^z + 1} e^{xz}, \quad (|z| < \pi)$$

where $B_n(x)$, $E_n(x)$, and $G_n(x)$ are called n -th Bernoulli polynomial, n -th Euler polynomial and n -th Genocchi polynomial, respectively. Substituting $x = 0$ in the Corollary 15.2 gives $B_n(0) := B_n$, $E_n(0) := E_n$ and $G_n(0) := G_n$ that are called n -th Bernoulli number, n -th Euler number and n -th Genocchi number, respectively (see [29, 30]).

Now, some results of the aforementioned polynomials are analyzed as follows.

Theorem 15.1 (Duran et al. [28]). The following identities holds:

$$\mathcal{B}_n^{(\alpha)}(x, y; p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{B}_k^{(\alpha)}(x, 0; p, q) y^{n-k},$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{B}_k^{(\alpha)}(0, y; p, q) x^{n-k},$$

$$\mathcal{E}_n^{(\alpha)}(x, y; p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{E}_k^{(\alpha)}(x, 0; p, q) y^{n-k},$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{E}_k^{(\alpha)}(0, y; p, q) x^{n-k},$$

$$\mathcal{G}_n^{(\alpha)}(x, y; p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{G}_k^{(\alpha)}(x, 0; p, q) y^{n-k},$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{G}_k^{(\alpha)}(0, y: p, q) x^{n-k}.$$

Proof. It is proved by making use of Definition 15.1 and Cauchy product:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} &= \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} e_{p,q}(xz) E_{p,q}(yz) \\ &= \left(\sum_{k=0}^n \mathcal{B}_k^{(\alpha)}(x, 0: p, q) \frac{z^k}{[k]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} q^{n(n-1)/2} y^n \frac{z^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha)}(x, 0: p, q) q^{(n-k)(n-k-1)/2} y^{n-k} \right) \frac{z^n}{[n]_{p,q}!}. \end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$, the expected result in the first equation is proved.

The other equations of this theorem can be proved in a like manner.

Corollary 15.3 (Duran *et al.* [28]). Taking $x = 0$ (or $y = 0$) in Theorem 15.1 gives the following formulas

$$\mathcal{B}_n^{(\alpha)}(x, 0: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{B}_k^{(\alpha)}(p, q) x^{n-k},$$

$$\mathcal{B}_n^{(\alpha)}(0, y: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{B}_k^{(\alpha)}(p, q) y^{n-k},$$

$$\mathcal{E}_n^{(\alpha)}(x, 0: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{E}_k^{(\alpha)}(p, q) x^{n-k},$$

$$\mathcal{E}_n^{(\alpha)}(0, y: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{E}_k^{(\alpha)}(p, q) y^{n-k},$$

$$\mathcal{G}_n^{(\alpha)}(x, 0: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{G}_k^{(\alpha)}(p, q) x^{n-k},$$

$$\mathcal{G}_n^{(\alpha)}(0, y: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{G}_k^{(\alpha)}(p, q) y^{n-k}.$$

Corollary 15.4 (Duran *et al.* [28]). *Setting $x=1$ (or $y=1$) in Theorem 15.1 yields to*

$$\mathcal{B}_n^{(\alpha)}(x, 1: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{B}_k^{(\alpha)}(x, 0: p, q), \quad (15.1)$$

$$\mathcal{B}_n^{(\alpha)}(1, y: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{B}_k^{(\alpha)}(0, y: p, q), \quad (15.2)$$

$$\mathcal{E}_n^{(\alpha)}(x, 1: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{E}_k^{(\alpha)}(x, 0: p, q), \quad (15.3)$$

$$\mathcal{E}_n^{(\alpha)}(1, y: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{E}_k^{(\alpha)}(0, y: p, q), \quad (15.4)$$

$$\mathcal{G}_n^{(\alpha)}(x, 1: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} q^{(n-k)(n-k-1)/2} \mathcal{G}_k^{(\alpha)}(x, 0: p, q), \quad (15.5)$$

$$\mathcal{G}_n^{(\alpha)}(1, y: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{(n-k)(n-k-1)/2} \mathcal{G}_k^{(\alpha)}(0, y: p, q). \quad (15.6)$$

Obviously, Eqs. (15.1)-(15.6) are (p, q) -analogues of

$$B_n^{(\alpha)}(x+1) = \sum_{k=0}^n \binom{n}{k} B_k^{(\alpha)}(x),$$

$$E_n^{(\alpha)}(x+1) = \sum_{k=0}^n \binom{n}{k} E_k^{(\alpha)}(x),$$

$$G_n^{(\alpha)}(x+1) = \sum_{k=0}^n \binom{n}{k} G_k^{(\alpha)}(x).$$

Now, the addition properties are stated as follows.

Theorem 15.2 (Duran et al. [28]). (Addition Properties) Let $n \in \mathbb{N}$. Then the following formulas fulfil

$$\mathcal{B}_n^{(\alpha+\beta)}(x, y: p, q) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} \mathcal{B}_{n-k}^{(\alpha)}(x, 0: p, q) \mathcal{B}_k^{(\beta)}(0, y: p, q),$$

$$\mathcal{E}_n^{(\alpha+\beta)}(x, y: p, q) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} \mathcal{E}_{n-k}^{(\alpha)}(x, 0: p, q) \mathcal{E}_k^{(\beta)}(0, y: p, q),$$

$$\mathcal{G}_n^{(\alpha+\beta)}(x, y: p, q) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} \mathcal{G}_{n-k}^{(\alpha)}(x, 0: p, q) \mathcal{G}_k^{(\beta)}(0, y: p, q).$$

Proof. From Definition 15.1 and by using Cauchy product, it becomes

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha+\beta)}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} &= \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha+\beta} e_{p,q}(xz) E_{p,q}(yz) \\ &= \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(\beta)}(0, y: p, q) \frac{z^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} \mathcal{B}_k^{(\alpha)}(x, 0: p, q) \mathcal{B}_{n-k}^{(\beta)}(0, y: p, q) \right) \frac{z^n}{[n]_{p,q}!}. \end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$, then the desired result is obtained. By using the same proof technique, the proofs of remained equalities can be shown directly.

Corollary 15.5 (Duran et al. [28]). Taking $\alpha = 1$ and $\beta = \alpha - 1$ gives

$$\mathcal{B}_n^{(\alpha)}(x, y: p, q) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} \mathcal{B}_{n-k}(x, 0: p, q) \mathcal{B}_k^{(\alpha-1)}(0, y: p, q),$$

$$\mathcal{E}_n^{(\alpha)}(x, y: p, q) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{p,q} \mathcal{E}_{n-k}(x, 0: p, q) \mathcal{E}_k^{(\alpha-1)}(0, y: p, q),$$

$$\mathcal{G}_n^{(\alpha)}(x, y: p, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{G}_{n-k}(x, 0: p, q) \mathcal{G}_k^{(\alpha-1)}(0, y: p, q).$$

Here are (p, q) -derivatives of the polynomials $\mathcal{B}_n^{(\alpha)}(x, y: p, q)$, $\mathcal{E}_n^{(\alpha)}(x, y: p, q)$ and $\mathcal{G}_n^{(\alpha)}(x, y: p, q)$, with respect to x and y , as follows.

Theorem 15.3 (Duran et al. [28]). (*Derivative Property*) The following properties hold true

$$D_{p,q;x} \mathcal{B}_n^{(\alpha)}(x, y: p, q) = [n]_{p,q} \mathcal{B}_{n-1}^{(\alpha)}(px, y: p, q),$$

$$D_{p,q;y} \mathcal{B}_n^{(\alpha)}(x, y: p, q) = [n]_{p,q} \mathcal{B}_{n-1}^{(\alpha)}(x, qy: p, q),$$

$$D_{p,q;x} \mathcal{E}_n^{(\alpha)}(x, y: p, q) = [n]_{p,q} \mathcal{E}_{n-1}^{(\alpha)}(px, y: p, q),$$

$$D_{p,q;y} \mathcal{E}_n^{(\alpha)}(x, y: p, q) = [n]_{p,q} \mathcal{E}_{n-1}^{(\alpha)}(x, qy: p, q),$$

$$D_{p,q;x} \mathcal{G}_n^{(\alpha)}(x, y: p, q) = [n]_{p,q} \mathcal{G}_{n-1}^{(\alpha)}(px, y: p, q),$$

$$D_{p,q;y} \mathcal{G}_n^{(\alpha)}(x, y: p, q) = [n]_{p,q} \mathcal{G}_{n-1}^{(\alpha)}(x, qy: p, q).$$

Proof. From Definition 15.1 and definition of the (p, q) -derivative operator, it becomes

$$\begin{aligned} \sum_{n=0}^{\infty} D_{p,q;x} \mathcal{B}_n^{(\alpha)}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} &= \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} D_{p,q;x} e_{p,q}(xz) E_{p,q}(yz) \\ &= z \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} e_{p,q}(pxz) E_{p,q}(yz) \\ &= \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(px, y: p, q) \frac{z^{n+1}}{[n]_{p,q}!} \end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$, then the expected result for (p,q) -Bernoulli polynomials of order α is acquired. Similarly, the others be proved.

Now, the new addition formulas are stated below.

Theorem 15.4 (Duran et al. [28]). For $n \in \mathbb{N}_0$ and $x, y, a \in \mathbb{C}$, the following relationships hold true:

$$\mathcal{B}_n^{(\alpha)}(x+a, y: p, q) = \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u}^{(\alpha)}(0, y: p, q) p^{\binom{u}{2}} \sum_{s=0}^u \binom{u}{s} x^s a^{u-s},$$

$$\mathcal{B}_n^{(\alpha)}(x, y+a: p, q) = \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u}^{(\alpha)}(x, 0: p, q) q^{\binom{u}{2}} \sum_{s=0}^u \binom{u}{s} y^s a^{u-s},$$

$$\mathcal{E}_n^{(\alpha)}(x+a, y: p, q) = \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{E}_{n-u}^{(\alpha)}(0, y: p, q) p^{\binom{u}{2}} \sum_{s=0}^u \binom{u}{s} x^s a^{u-s},$$

$$\mathcal{E}_n^{(\alpha)}(x, y+a: p, q) = \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{E}_{n-u}^{(\alpha)}(x, 0: p, q) q^{\binom{u}{2}} \sum_{s=0}^u \binom{u}{s} y^s a^{u-s},$$

$$\mathcal{G}_n^{(\alpha)}(x+a, y: p, q) = \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{G}_{n-u}^{(\alpha)}(0, y: p, q) p^{\binom{u}{2}} \sum_{s=0}^u \binom{u}{s} x^s a^{u-s},$$

$$\mathcal{G}_n^{(\alpha)}(x, y+a: p, q) = \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{G}_{n-u}^{(\alpha)}(x, 0: p, q) q^{\binom{u}{2}} \sum_{s=0}^u \binom{u}{s} y^s a^{u-s}.$$

Proof. From Definition 15.1 and by using Cauchy product, it becomes

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x+a, y: p, q) \frac{z^n}{[n]_{p,q}!} &= \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} E_{p,q}(yz) e_{p,q}((x+a)z) \\ &= \left(\sum_{n=0}^{\infty} \mathcal{B}_n(0, y: p, q) \frac{z^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} p^{\binom{u}{2}} (x+a)^n \frac{z^n}{[n]_{p,q}!} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u}(0, y: p, q) p^{\binom{u}{2}} (x+a)^u \right) \frac{z^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{B}_{n-u}(0, y: p, q) p^{\binom{u}{2}} \sum_{s=0}^u \binom{u}{s} x^s a^{u-s} \right) \frac{z^n}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$ gives the desired result for (p, q) -Bernoulli polynomials. The others in this theorem can be proved similar way.

The following theorem includes some useful relationships.

Theorem 15.5 (Duran et al. [28]). For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the following relationships hold true:

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha)}(x, 0: p, q) \mathcal{B}_{n-k}^{(-\alpha)}(p, q) = p^{\binom{n}{2}} x^n,$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha)}(0, y: p, q) \mathcal{B}_{n-k}^{(-\alpha)}(p, q) = q^{\binom{n}{2}} y^n,$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_k^{(\alpha)}(x, 0: p, q) \mathcal{E}_{n-k}^{(-\alpha)}(p, q) = p^{\binom{n}{2}} x^n,$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_k^{(\alpha)}(0, y: p, q) \mathcal{E}_{n-k}^{(-\alpha)}(p, q) = q^{\binom{n}{2}} y^n,$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{G}_k^{(\alpha)}(x, 0: p, q) \mathcal{G}_{n-k}^{(-\alpha)}(p, q) = p^{\binom{n}{2}} x^n,$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{G}_k^{(\alpha)}(0, y: p, q) \mathcal{G}_{n-k}^{(-\alpha)}(p, q) = q^{\binom{n}{2}} y^n.$$

Proof. From Definition 15.1 and by using Cauchy product, it becomes

$$\begin{aligned}
\sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{z^n}{[n]_{p,q}!} &= e_{p,q}(xz) \\
&= \left(\frac{z}{e_{p,q}(z)-1} \right)^{\alpha} e_{p,q}(xz) \left(\frac{z}{e_{p,q}(z)-1} \right)^{-\alpha} \\
&= \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, 0; p, q) \frac{z^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(-\alpha)}(p, q) \frac{z^n}{[n]_{p,q}!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha)}(x, 0; p, q) \mathcal{B}_{n-k}^{(-\alpha)}(p, q) \right) \frac{z^n}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$ yields to the expected result for first equation.

The remain formulas of this theorem can be proved with a similar procedure.

Theorem 15.6 (Duran et al. [28]). *(Difference equations) The following relations hold true:*

$$\mathcal{B}_n^{(\alpha)}(1, y; p, q) - \mathcal{B}_n^{(\alpha)}(0, y; p, q) = [n]_{p,q} \mathcal{B}_{n-1}^{(\alpha-1)}(0, y; p, q), \quad (15.7)$$

$$\mathcal{B}_n^{(\alpha)}(x, 0; p, q) - \mathcal{B}_n^{(\alpha)}(x, -1; p, q) = [n]_{p,q} \mathcal{B}_{n-1}^{(\alpha-1)}(x, -1; p, q),$$

$$\mathcal{E}_n^{(\alpha)}(1, y; p, q) + \mathcal{E}_n^{(\alpha)}(0, y; p, q) = [2]_{p,q} \mathcal{E}_n^{(\alpha-1)}(0, y; p, q), \quad (15.8)$$

$$\mathcal{E}_n^{(\alpha)}(x, 0; p, q) + \mathcal{E}_n^{(\alpha)}(x, -1; p, q) = [2]_{p,q} \mathcal{E}_n^{(\alpha-1)}(x, -1; p, q),$$

$$\mathcal{G}_n^{(\alpha)}(1, y; p, q) + \mathcal{G}_n^{(\alpha)}(0, y; p, q) = [2]_{p,q} [n]_{p,q} \mathcal{G}_{n-1}^{(\alpha-1)}(0, y; p, q), \quad (15.9)$$

$$\mathcal{G}_n^{(\alpha)}(x, 0; p, q) + \mathcal{G}_n^{(\alpha)}(x, -1; p, q) = [2]_{p,q} [n]_{p,q} \mathcal{G}_{n-1}^{(\alpha-1)}(x, -1; p, q).$$

Proof. From Definition 15.1 and by using Cauchy product, it becomes

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha-1)}(0, y: p, q) \frac{z^n}{[n]_{p,q}!} &= \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha-1} E_{p,q}(yz) \\
&= \frac{1}{z} \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} (e_{p,q}(z) - 1) E_{p,q}(yz) \\
&= \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(1, y: p, q) \frac{z^{n-1}}{[n]_{p,q}!} - \mathcal{B}_n^{(\alpha)}(0, y: p, q) \frac{z^{n-1}}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \left[\mathcal{B}_n^{(\alpha)}(1, y: p, q) - \mathcal{B}_n^{(\alpha)}(0, y: p, q) \right] \frac{z^{n-1}}{[n]_{p,q}!}
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$ yields to the required result for (p, q) -Bernoulli polynomials. Similarly, the others can be proved.

By the Eqs. (15.1) and (15.7), Eqs. (15.3) and (15.8), Eqs. (15.5) and (15.9), the following formulas are acquired.

Corollary 15.6 (Duran *et al.* [28]). *The following relations hold true*

$$\mathcal{B}_n^{(\alpha-1)}(0, y: p, q) = \frac{1}{[n+1]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} \mathcal{B}_k^{(\alpha)}(0, y: p, q), \quad (15.10)$$

$$\mathcal{E}_n^{(\alpha-1)}(0, y: p, q) = \frac{1}{[2]_{p,q}} \left[\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{E}_k^{(\alpha)}(0, y: p, q) + \mathcal{E}_n^{(\alpha)}(0, y: p, q) \right], \quad (15.11)$$

$$\mathcal{G}_n^{(\alpha-1)}(0, y: p, q) = \frac{1}{[2]_{p,q} [n+1]_{p,q}} \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} \mathcal{G}_k^{(\alpha)}(0, y: p, q) + \mathcal{G}_{n+1}^{(\alpha)}(0, y: p, q). \quad (15.12)$$

Note that

$$\mathcal{B}_n^{(0)}(0, y : p, q) = \mathcal{E}_n^{(0)}(0, y : p, q) = \mathcal{G}_n^{(0)}(0, y : p, q) = q^{n(n-1)/2} y^n,$$

$$\mathcal{B}_n^{(0)}(x, 0 : p, q) = \mathcal{E}_n^{(0)}(x, 0 : p, q) = \mathcal{G}_n^{(0)}(x, 0 : p, q) = p^{n(n-1)/2} x^n.$$

From Theorem 15.5, choosing $\alpha = 1$ in Corollary 15.6 gives the following expressions:

$$\begin{aligned} y^n &= \frac{1}{[n+1]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} \mathcal{B}_k(0, y : p, q), \\ &= q^{-\binom{n}{2}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha)}(0, y : p, q) \mathcal{B}_{n-k}^{(-\alpha)}(p, q), \\ y^n &= \frac{1}{[2]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{E}_k(0, y : p, q) + \mathcal{E}_n(0, y : p, q), \\ &= q^{-\binom{n}{2}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_k^{(\alpha)}(0, y : p, q) \mathcal{E}_{n-k}^{(-\alpha)}(p, q), \\ y^n &= \frac{1}{[2]_{p,q} [n+1]_{p,q}} \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} \mathcal{G}_k(0, y : p, q) + \mathcal{G}_{n+1}(0, y : p, q), \\ &= q^{-\binom{n}{2}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{G}_k^{(\alpha)}(0, y : p, q) \mathcal{G}_{n-k}^{(-\alpha)}(p, q). \end{aligned}$$

Notice that the first equations of the last binary expressions of the above are generalizations of the formulas

$$y^n = \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} B_k(y)$$

$$y^n = \frac{1}{2} \sum_{k=0}^n \binom{n}{k} E_k(y) + E_n(y)$$

$$y^n = \frac{1}{2(n+1)} \sum_{k=0}^{n+1} \binom{n+1}{k} G_k(y) + G_{n+1}(y)$$

respectively.

The recurrence relationships are given as follows.

Theorem 15.7 (Duran *et al.* [28]). (Recurrence relationships) The following expressions hold true

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{B}_k^{(\alpha)}(x, 0: p, q) - \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{B}_k^{(\alpha)}(x, -1: p, q) \quad (15.13)$$

$$= [n]_{p,q} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\binom{n-1-k}{2}} m^{k+1} \mathcal{B}_k^{(\alpha-1)}(x, -1: p, q),$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{E}_k^{(\alpha)}(x, 0: p, q) + \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{E}_k^{(\alpha)}(x, -1: p, q) \quad (15.14)$$

$$= [2]_{p,q} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{E}_k^{(\alpha-1)}(x, -1: p, q),$$

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{G}_k^{(\alpha)}(x, 0: p, q) + \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} m^k \mathcal{G}_k^{(\alpha)}(x, -1: p, q) \quad (15.15)$$

$$= [2]_{p,q} [n]_{p,q} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\binom{n-1-k}{2}} m^{k+1} \mathcal{G}_k^{(\alpha-1)}(x, -1: p, q).$$

Proof. From Definition 15.1 and by using Cauchy product, then it becomes

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, 0: p, q) \frac{(mz)^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!} \right) \\ & - \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, -1: p, q) \frac{(mz)^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!} \right) \\ & = \left(\frac{mz}{e_{p,q}(mz) - 1} \right)^{\alpha} e_{p,q}(x mz) e_{p,q}(z) - \left(\frac{mz}{e_{p,q}(mz) - 1} \right)^{\alpha} E_{p,q}(-mz) e_{p,q}(x mz) e_{p,q}(z) \end{aligned}$$

$$\begin{aligned}
&= mz \left(\frac{mz}{e_{p,q}(mz) - 1} \right)^{\alpha-1} e_{p,q}(x mz) e_{p,q}(z)^{\frac{1-E_{p,q}(-mz)}{e_{p,q}(mz)-1}} \\
&= mz \left(\frac{mz}{e_{p,q}(mz) - 1} \right)^{\alpha-1} e_{p,q}(x mz) E_{p,q}(-mz) e_{p,q}(z) \\
&= mz \left(\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha-1)}(x, -1; p, q) \frac{(mz)^n}{[n]_{p,q}!} \right) \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha-1)}(x, -1; p, q) m^{k+1} p^{\binom{n-k}{2}} \right) \frac{z^{n+1}}{[n]_{p,q}!}.
\end{aligned}$$

The required result for (p, q) -Bernoulli polynomials is obtained by comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$. With the same procedure, the remain formulas can be proved.

Some new connections among the polynomials $\mathcal{B}_n^{(\alpha)}(x, y; p, q)$, $\mathcal{E}_n^{(\alpha)}(x, y; p, q)$ and $\mathcal{G}_n^{(\alpha)}(x, y; p, q)$ are investigated as follows.

Theorem 15.8 (Duran *et al.* [28]). For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the following relationships hold true:

$$\begin{aligned}
\mathcal{B}_n^{(\alpha)}(x, y; p, q) &= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{B}_{n-l}^{(\alpha)}(0, y; p, q) \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathcal{E}_k^{(\alpha)}(x, 0; p, q) \mathcal{E}_{l-k}^{(-\alpha)}(p, q) \\
&= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{B}_{n-l}^{(\alpha)}(0, y; p, q) \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathcal{G}_k^{(\alpha)}(x, 0; p, q) \mathcal{G}_{l-k}^{(-\alpha)}(p, q) \\
\mathcal{E}_n^{(\alpha)}(x, y; p, q) &= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{E}_{n-l}^{(\alpha)}(0, y; p, q) \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha)}(x, 0; p, q) \mathcal{B}_{l-k}^{(-\alpha)}(p, q) \\
&= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{E}_{n-l}^{(\alpha)}(0, y; p, q) \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathcal{G}_k^{(\alpha)}(x, 0; p, q) \mathcal{G}_{l-k}^{(-\alpha)}(p, q)
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_n^{(\alpha)}(x, y: p, q) &= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{G}_{n-l}^{(\alpha)}(0, y: p, q) \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathcal{B}_k^{(\alpha)}(x, 0: p, q) \mathcal{B}_{l-k}^{(-\alpha)}(p, q) \\
&= \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{G}_{n-l}^{(\alpha)}(0, y: p, q) \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathcal{E}_k^{(\alpha)}(x, 0: p, q) \mathcal{E}_{l-k}^{(-\alpha)}(p, q).
\end{aligned}$$

Proof. From Definition 15.1 and by using Cauchy product, it becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} \\
&= \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} E_{p,q}(yz) \left(\frac{[2]_{p,q}}{e_{p,q}(z) + 1} \right)^{\alpha} e_{p,q}(xz) \left(\frac{[2]_{p,q}}{e_{p,q}(z) + 1} \right)^{-\alpha} \\
&= \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(0, y: p, q) \frac{z^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} \mathcal{E}_n^{(-\alpha)}(p, q) \frac{z^n}{[n]_{p,q}!} \\
&= \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(0, y: p, q) \frac{z^n}{[n]_{p,q}!} \left[\sum_{n=0}^{\infty} \left(\sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{E}_l^{(\alpha)}(x, 0: p, q) \mathcal{E}_{n-l}^{(-\alpha)}(p, q) \right) \frac{z^n}{[n]_{p,q}!} \right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \mathcal{B}_{n-l}^{(\alpha)}(0, y: p, q) \sum_{k=0}^l \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} \mathcal{E}_k^{(\alpha)}(x, 0: p, q) \mathcal{E}_{l-k}^{(-\alpha)}(p, q) \right] \frac{z^{n-1}}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$ yields to the expected result in the first equation.

The others can be proved in a like manner.

Now, some recurrence relationships for the (p, q) -Bernoulli polynomials, the (p, q) -Euler polynomials and the (p, q) -Genocchi polynomials are researched below.

Theorem 15.9 (Duran et al. [28]). For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the (p, q) -Bernoulli polynomials, (p, q) -Euler polynomials and (p, q) -Genocchi polynomials satisfy the following recurrence relations:

$$\mathcal{B}_n(x, y: p, q) = \frac{q^{n-1}}{[n]_{p,q}} \mathcal{B}_n\left(\frac{p}{q}x, \frac{p}{q}y: p, q\right)$$

$$\begin{aligned}
& +xq^{n-1}\mathcal{B}_{n-1}\left(\frac{p}{q}x, \frac{p}{q}y:p, q\right) + yq^{n-1}\mathcal{B}_{n-1}(x, y:p, q) \\
& -\frac{1}{[n]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{B}_{n-k}\left(\frac{p}{q}, 0:p, q\right) q^{n-1-k} p^k \mathcal{B}_k(x, y:p, q), \\
\mathcal{E}_n(x, y:p, q) & = xq^{n-1}\mathcal{E}_{n-1}\left(\frac{p}{q}x, \frac{p}{q}y:p, q\right) + yq^{n-1}\mathcal{E}_{n-1}(x, y:p, q) \\
& -\frac{1}{[2]_{p,q}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \mathcal{E}_{n-1-k}\left(\frac{p}{q}, 0:p, q\right) q^{n-1-k} p^k \mathcal{E}_k(x, y:p, q), \\
\mathcal{G}_n(x, y:p, q) & = \frac{q^{n-1}}{[n]_{p,q}} \mathcal{G}_n\left(\frac{p}{q}x, \frac{p}{q}y:p, q\right) \\
& + xq^{n-1}\mathcal{G}_{n-1}\left(\frac{p}{q}x, \frac{p}{q}y:p, q\right) + yq^{n-1}\mathcal{G}_{n-1}(x, y:p, q) \\
& -\frac{1}{[2]_{p,q} [n]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{G}_{n-k}\left(\frac{p}{q}, 0:p, q\right) q^{n-1-k} q^k \mathcal{G}_k(x, y:p, q).
\end{aligned}$$

Proof. For $\alpha=1$ in Definition 15.1, taking the (p,q) -derivative of the (p,q) -Euler polynomials $\mathcal{E}_n^{(\alpha)}(x, y:p, q)$ with respect to z yields to

$$\begin{aligned}
D_{p,q;z} \left\{ \sum_{n=0}^{\infty} \mathcal{E}_n(x, y:p, q) \frac{z^n}{[n]_{p,q}!} \right\} & = D_{p,q;z} \left\{ \frac{[2]_{p,q}}{e_{p,q}(z)+1} e_{p,q}(xz) E_{p,q}(yz) \right\} \\
& = [2]_{p,q} x \frac{e_{p,q}(xz) E_{p,q}(pyz)}{e_{p,q}(qz)+1} + [2]_{p,q} y \frac{e_{p,q}(qxz) E_{p,q}(pqyz)}{e_{p,q}(qz)+1} \\
& \quad - \frac{1}{[2]_{p,q}} \frac{[2]_{p,q} e_{p,q}(pxz) E_{p,q}(pyz)}{e_{p,q}(qz)+1} \frac{[2]_{p,q} e_{p,q}(pz)}{e_{p,q}(pz)+1}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{E}_{n+1}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} &= x \sum_{n=0}^{\infty} \mathcal{E}_n\left(\frac{p}{q}x, \frac{p}{q}y: p, q\right) q^n \frac{z^n}{[n]_{p,q}!} \\
&+ y \sum_{n=0}^{\infty} \mathcal{E}_n(x, y: p, q) q^n \frac{z^n}{[n]_{p,q}!} \\
&- \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{E}_{n-k}\left(\frac{p}{q}, 0: p, q\right) q^{n-k} p^k \mathcal{E}_k(x, y: p, q) \right\} \frac{z^n}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}!}$ gives the desired result for (p,q) -Euler polynomials. The others in this theorem can be proved in a like manner.

The following theorem includes some useful relations for the (p,q) -Bernoulli polynomials, the (p,q) -Euler polynomials and the (p,q) -Genocchi polynomials.

Theorem 15.10 (Duran *et al.* [28]). *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the following relations*

$$\begin{aligned}
\mathcal{B}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{(\alpha)}(x, 0: p, q) m^{s-u} - \mathcal{B}_u^{(\alpha)}(x, 0: p, q) \right) \mathcal{B}_{n-u+1}(0, my: p, q) m^{u-n},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{(\alpha)}(0, y: p, q) m^{s-u} p^{\binom{u-s}{2}} - \mathcal{B}_u^{(\alpha)}(0, y: p, q) \right) \mathcal{B}_{n-u+1}(mx, 0: p, q) m^{u-n},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[2]_{p,q}} \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{E}_s^{(\alpha)}(x, 0: p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{E}_u^{(\alpha)}(x, 0: p, q) \right) \mathcal{E}_{n-u}(0, my: p, q) m^{u-n},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[2]_{p,q}} \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{E}_s^{(\alpha)}(0, y: p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{E}_u^{(\alpha)}(0, y: p, q) \right) \mathcal{E}_{n-u}(mx, 0: p, q) m^{u-n}, \\
\mathcal{G}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[2]_{p,q} [n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{G}_s^{(\alpha)}(x, 0: p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{G}_u^{(\alpha)}(x, 0: p, q) \right) \mathcal{G}_{n-u+1}(0, my: p, q) m^{u-n}, \\
\mathcal{G}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[2]_{p,q} [n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{G}_s^{(\alpha)}(0, y: p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{G}_u^{(\alpha)}(0, y: p, q) \right) \mathcal{G}_{n-u+1}(mx, 0: p, q) m^{u-n}
\end{aligned}$$

are valid.

Proof. Indeed

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} \\
&= \left(\frac{[2]_{p,q} z}{e_{p,q}(z) + 1} \right)^{\alpha} e_{p,q}(xz) \frac{e_{p,q}\left(\frac{z}{m}\right) + 1}{\frac{z}{m}} \frac{\frac{z}{m} [2]_{p,q}}{e_{p,q}\left(\frac{z}{m}\right) + 1} E_{p,q}\left(my \frac{z}{m}\right) \frac{1}{[2]_{p,q}} \\
&= \frac{m}{[2]_{p,q} z} \left[\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{1}{m^n} \frac{z^n}{[n]_{p,q}!} \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} \mathcal{G}_n(0, my: p, q) \frac{1}{m^n} \frac{z^n}{[n]_{p,q}!} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{m}{[2]_{p,q}} \sum_{n=0}^{\infty} \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \left\{ \sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{G}_s^{(\alpha)}(x, 0 : p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{G}_u^{(\alpha)}(x, 0 : p, q) \right\} \\
&\quad \times \mathcal{G}_{n-u}(0, my : p, q) m^{u-n} \frac{z^{n-1}}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{[n]_{p,q}}$ yields to the expected result for the fifth equation. The others in this theorem can be proved in a like manner.

Here new correlations including the polynomials $\mathcal{B}_n^{(\alpha)}(x, y : p, q)$, $\mathcal{E}_n^{(\alpha)}(x, y : p, q)$ and $\mathcal{G}_n^{(\alpha)}(x, y : p, q)$ are presented by the following theorem.

Theorem 15.11 (Duran et al. [28]). For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{C}$, the following correlations hold true:

$$\begin{aligned}
\mathcal{B}_n^{(\alpha)}(x, y : p, q) &= \frac{1}{[2]_{p,q}} \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \\
&\quad \times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{(\alpha)}(x, 0 : p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{B}_u^{(\alpha)}(x, 0 : p, q) \right) \mathcal{E}_{n-u}(0, my : p, q) m^{u-n}, \\
\mathcal{B}_n^{(\alpha)}(x, y : p, q) &= \frac{1}{[2]_{p,q} [n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\quad \times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{(\alpha)}(x, 0 : p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{B}_u^{(\alpha)}(x, 0 : p, q) \right) \mathcal{G}_{n-u+1}(0, my : p, q) m^{u-n}, \\
\mathcal{E}_n^{(\alpha)}(x, y : p, q) &= \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\quad \times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{E}_s^{(\alpha)}(x, 0 : p, q) m^{s-u} p^{\binom{u-s}{2}} - \mathcal{E}_u^{(\alpha)}(x, 0 : p, q) \right) \mathcal{B}_{n-u+1}(0, my : p, q) m^{u-n},
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[2]_{p,q} [n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \mathcal{E}_s^{(\alpha)}(x, 0: p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{E}_u^{(\alpha)}(x, 0: p, q) \right) \mathcal{G}_{n-u+1}(0, my: p, q) m^{u-n}, \\
\mathcal{G}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[n+1]_{p,q}} \sum_{u=0}^{n+1} \begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{G}_s^{(\alpha)}(x, 0: p, q) m^{s-u} p^{\binom{u-s}{2}} - \mathcal{G}_u^{(\alpha)}(x, 0: p, q) \right) \mathcal{B}_{n-u+1}(0, my: p, q) m^{u-n}, \\
\mathcal{G}_n^{(\alpha)}(x, y: p, q) &= \frac{1}{[2]_{p,q}} \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \\
&\times \left(\sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{G}_s^{(\alpha)}(x, 0: p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{G}_u^{(\alpha)}(x, 0: p, q) \right) \mathcal{E}_{n-u}(0, my: p, q) m^{u-n}.
\end{aligned}$$

Proof. It is proved by using the generating function of the (p, q) -Bernoulli polynomials of order α as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} \\
&= \left(\frac{z}{e_{p,q}(z) - 1} \right)^{\alpha} e_{p,q}(xz) \frac{e_{p,q}\left(\frac{z}{m}\right) + 1}{\frac{z}{m}} \frac{\frac{z}{m} [2]_{p,q}}{e_{p,q}\left(\frac{z}{m}\right) + 1} E_{p,q}\left(my \frac{z}{m}\right) \frac{1}{[2]_{p,q}} \\
&= \frac{m}{[2]_{p,q} z} \left[\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} m^{-n} \frac{z^n}{[n]_{p,q}!} \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} \right] \sum_{n=0}^{\infty} \mathcal{G}_n(0, my: p, q) \frac{1}{m^n} \frac{z^n}{[n]_{p,q}!}
\end{aligned}$$

$$\begin{aligned}
&= \frac{m}{[2]_{p,q}} \sum_{n=0}^{\infty} \sum_{u=0}^n \begin{bmatrix} n \\ u \end{bmatrix}_{p,q} \left\{ \sum_{s=0}^u \begin{bmatrix} u \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{(\alpha)}(x, 0: p, q) m^{s-u} p^{\binom{u-s}{2}} + \mathcal{B}_u^{(\alpha)}(x, 0: p, q) \right\} \\
&\quad \times \mathcal{G}_{n-u}(0, my: p, q) m^{u-n} \frac{z^{n-1}}{[n]_{p,q}!}.
\end{aligned}$$

Equating the coefficients of $\frac{z^n}{[n]_{p,q}}$ gives the desired result for second equation. The others in this theorem can be proved in a like manner.

The new results for the (p, q) -Bernoulli polynomials, the (p, q) -Euler polynomials and the (p, q) -Genocchi polynomials are developed as follows.

Theorem 15.12 (Duran *et al.* [28]). *For $n \in \mathbb{N}_0$, the following relationships are valid:*

$$\begin{aligned}
&\mathcal{B}_n^{(\alpha)}(x, y: p, q) \\
&= \frac{1}{[2]_{p,q}} \frac{1}{m^n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left[m^k \mathcal{B}_k^{(\alpha)}(x, 0: p, q) + \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^s \mathcal{B}_s^{(\alpha)}(x, -1: p, q) \right. \\
&\quad \left. + [k]_{p,q} \sum_{s=0}^{k-1} \begin{bmatrix} k-1 \\ s \end{bmatrix}_{p,q} p^{\binom{k-1-s}{2}} m^{s+1} \mathcal{B}_s^{(\alpha-1)}(x, -1: p, q) \right] \mathcal{E}_{n-k}(0, my: p, q),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}_n^{(\alpha)}(x, y: p, q) = \frac{1}{[2]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \\
&\quad \times \left[m^{k+1} \mathcal{B}_{k+1}^{(\alpha)}(x, 0: p, q) + \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{B}_s^{(\alpha)}(x, -1: p, q) \right. \\
&\quad \left. + [k+1]_{p,q} \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^{s+1} \mathcal{B}_s^{(\alpha-1)}(x, -1: p, q) \right] \mathcal{G}_{n-k}(0, my: p, q),
\end{aligned}$$

$$\mathcal{E}_n^{(\alpha)}(x, y : p, q)$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \left[[2]_{p,q} \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{E}_s^{(\alpha-1)}(x, -1 : p, q) \right. \\ \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{E}_s^{(\alpha)}(x, -1 : p, q) - m^{k+1} \mathcal{E}_{k+1}^{(\alpha)}(x, 0 : p, q) \right] \mathcal{B}_{n-k}(0, my : p, q),$$

$$\mathcal{E}_n^{(\alpha)}(x, y : p, q)$$

$$= \frac{1}{[2]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \left[[2]_{p,q} \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{E}_s^{(\alpha-1)}(x, -1 : p, q) \right. \\ \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{E}_s^{(\alpha)}(x, -1 : p, q) + m^{k+1} \mathcal{E}_{k+1}^{(\alpha)}(x, 0 : p, q) \right] \mathcal{G}_{n-k}(0, my : p, q),$$

$$\mathcal{G}_n^{(\alpha)}(x, y : p, q)$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \left[[2]_{p,q} [k+1]_{p,q} \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^{s+1} \mathcal{G}_s^{(\alpha-1)}(x, -1 : p, q) \right. \\ \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{G}_s^{(\alpha)}(x, -1 : p, q) - m^{k+1} \mathcal{G}_{k+1}^{(\alpha)}(x, 0 : p, q) \right] \mathcal{B}_{n-k}(0, my : p, q),$$

$$\mathcal{G}_n^{(\alpha)}(x, y : p, q) = \frac{1}{[2]_{p,q} m^n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$$

$$\times \left[m^k \mathcal{G}_k^{(\alpha)}(x, 0 : p, q) + [2]_{p,q} [k]_{p,q} \sum_{s=0}^{k-1} \begin{bmatrix} k-1 \\ s \end{bmatrix}_{p,q} p^{\binom{k-1-s}{2}} m^{s+1} \mathcal{G}_s^{(\alpha-1)}(x, -1 : p, q) \right. \\ \left. - \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^s \mathcal{G}_s^{(\alpha)}(x, -1 : p, q) \right] \mathcal{E}_{n-k}(0, my : p, q).$$

Proof. Since

$$\begin{aligned} & \left(\frac{z}{e_{p,q}(z)-1} \right)^\alpha e_{p,q}(xz) E_{p,q}(yz) \\ &= \frac{[2]_{p,q}}{e_{p,q}\left(\frac{z}{m}\right)+1} E_{p,q}\left(\frac{z}{m}my\right) \frac{e_{p,q}\left(\frac{z}{m}\right)+1}{[2]_{p,q}} \left(\frac{z}{e_{p,q}(z)-1} \right)^\alpha e_{p,q}(zx), \end{aligned}$$

it becomes

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, y: p, q) \frac{z^n}{[n]_{p,q}!} \\ &= \frac{1}{[2]_{p,q}} \left[\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{m^n [n]_{p,q}!} \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} + \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x, 0: p, q) \frac{z^n}{[n]_{p,q}!} \right] \\ & \quad \times \sum_{n=0}^{\infty} \mathcal{E}_n(0, my: p, q) \frac{z^n}{m^n [n]_{p,q}!} \\ &= \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left(\sum_{s=0}^n \begin{bmatrix} n \\ s \end{bmatrix}_{p,q} \mathcal{B}_s^{(\alpha)}(x, 0: p, q) p^{\binom{n-s}{2}} m^{s-n} + \mathcal{B}_n^{(\alpha)}(x, 0: p, q) \right) \frac{z^n}{[n]_{p,q}!} \\ & \quad \times \sum_{n=0}^{\infty} \mathcal{E}_n(0, my: p, q) \frac{z^n}{m^n [n]_{p,q}!} \\ &= \frac{1}{[2]_{p,q}} \left(\sum_{n=0}^{\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left[\sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} m^s p^{\binom{k-s}{2}} \mathcal{B}_s^{(\alpha)}(x, 0: p, q) + m^k \mathcal{B}_k^{(\alpha)}(x, 0: p, q) \right] \right. \\ & \quad \left. \times \mathcal{E}_{n-k}(0, my: p, q) m^{k-n} \right) \frac{z^n}{[n]_{p,q}!}. \end{aligned}$$

Using Theorem 15.7 yields to claimed result for first equation. The others in this theorem can be proved in a like manner.

Now, some special cases of Theorem 15.12 are considered as follows.

Corollary 15.7 (Duran *et al.* [28]). For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, the relations

$$\begin{aligned}
\mathfrak{B}_{n,q}^{(\alpha)}(x, y) &= \frac{1}{[2]_q m^n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \left[m^k \mathfrak{B}_{k,q}^{(\alpha)}(x, 0) + \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_q m^s \mathfrak{B}_{s,q}^{(\alpha)}(x, -1) \right. \\
&\quad \left. + [k]_q \sum_{s=0}^{k-1} \begin{bmatrix} k-1 \\ s \end{bmatrix}_q m^{s+1} \mathfrak{B}_{s,q}^{(\alpha-1)}(x, -1) \right] \mathfrak{E}_{n-k,q}(0, my), \\
\mathfrak{B}_{n,q}^{(\alpha)}(x, y) &= \frac{1}{[2]_q} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \left[m^{k+1} \mathfrak{B}_{k+1,q}^{(\alpha)}(x, 0) + \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_q m^s \mathfrak{B}_{s,q}^{(\alpha)}(x, -1) \right. \\
&\quad \left. + [k+1]_q \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_q m^{s+1} \mathfrak{B}_{s,q}^{(\alpha-1)}(x, -1) \right] \mathfrak{E}_{n-k,q}(0, my), \\
\mathfrak{E}_{n,q}^{(\alpha)}(x, y) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \left[[2]_q \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_q m^s \mathfrak{E}_{s,q}^{(\alpha-1)}(x, -1) \right. \\
&\quad \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_q m^s \mathfrak{E}_{s,q}^{(\alpha)}(x, -1) - m^{k+1} \mathfrak{E}_{k+1,q}^{(\alpha)}(x, 0) \right] \mathfrak{B}_{n-k,q}(0, my), \\
\mathfrak{E}_{n,q}^{(\alpha)}(x, y) &= \frac{1}{[2]_q} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \left[[2]_q \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_q m^s \mathfrak{E}_{s,q}^{(\alpha-1)}(x, -1) \right. \\
&\quad \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_q m^s \mathfrak{E}_{s,q}^{(\alpha)}(x, -1) + m^{k+1} \mathfrak{E}_{k+1,q}^{(\alpha)}(x, 0) \right] \mathfrak{B}_{n-k,q}(0, my), \\
\mathfrak{G}_{n,q}^{(\alpha)}(x, y) &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{m^n [k+1]_q} \left[[2]_q [k+1]_q \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_q m^{s+1} \mathfrak{G}_{s,q}^{(\alpha-1)}(x, -1) \right. \\
&\quad \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_q m^s \mathfrak{G}_{s,q}^{(\alpha)}(x, -1) - m^{k+1} \mathfrak{G}_{k+1,q}^{(\alpha)}(x, 0) \right] \mathfrak{B}_{n-k,q}(0, my),
\end{aligned}$$

hold true among the q -Bernoulli polynomials, q -Euler polynomials and q -Genocchi polynomials of order α , see [55-60].

Corollary 15.8 (Duran *et al.* [28]). For $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the following relationships hold true:

$$\begin{aligned} & \mathcal{B}_n(x, y: p, q) \\ &= \frac{1}{[2]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left[m^k \mathcal{B}_k(x, 0: p, q) + \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^s \mathcal{B}_s(x, -1: p, q) \right. \\ & \quad \left. + [k]_{p,q} \sum_{s=0}^{k-1} \begin{bmatrix} k-1 \\ s \end{bmatrix}_{p,q} p^{\binom{k-1-s}{2}} m^{s+1} (x \ominus 1)_{p,q}^s \right] \mathcal{E}_{n-k}(0, my: p, q), \end{aligned}$$

$$\begin{aligned} \mathcal{B}_n(x, y: p, q) &= \frac{1}{[2]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \\ & \times \left[m^{k+1} \mathcal{B}_{k+1}(x, 0: p, q) + \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{B}_s(x, -1: p, q) \right. \\ & \quad \left. + [k+1]_{p,q} \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^{s+1} (x \ominus 1)_{p,q}^s \right] \mathcal{G}_{n-k}(0, my: p, q), \end{aligned}$$

$$\begin{aligned} & \mathcal{E}_n(x, y: p, q) \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \left[[2]_{p,q} \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s (x \ominus 1)_{p,q}^s \right. \\ & \quad \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{E}_s(x, -1: p, q) - m^{k+1} \mathcal{E}_{k+1}(x, 0: p, q) \right] \mathcal{B}_{n-k}(0, my: p, q), \end{aligned}$$

$$\begin{aligned} & \mathcal{E}_n(x, y: p, q) \\ &= \frac{1}{[2]_{p,q}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \left[[2]_{p,q} \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s (x \ominus 1)_{p,q}^s \right. \end{aligned}$$

$$-\sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{E}_s(x, -1: p, q) + m^{k+1} \mathcal{E}_{k+1}(x, 0: p, q) \Big] \mathcal{G}_{n-k}(0, my: p, q),$$

$$\mathcal{G}_n(x, y: p, q)$$

$$= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{1}{m^n [k+1]_{p,q}} \left[[2]_{p,q} [k+1]_{p,q} \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^{s+1} (x \ominus 1)_{p,q}^s \right. \\ \left. - \sum_{s=0}^{k+1} \begin{bmatrix} k+1 \\ s \end{bmatrix}_{p,q} p^{\binom{k+1-s}{2}} m^s \mathcal{G}_s(x, -1: p, q) - m^{k+1} \mathcal{G}_{k+1}(x, 0: p, q) \right] \mathcal{B}_{n-k}(0, my: p, q),$$

$$\mathcal{G}_n(x, y: p, q) = \frac{1}{[2]_{p,q} m^n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \\ \times \left[m^k \mathcal{G}_k(x, 0: p, q) + [2]_{p,q} [k]_{p,q} \sum_{s=0}^{k-1} \begin{bmatrix} k-1 \\ s \end{bmatrix}_{p,q} p^{\binom{k-1-s}{2}} m^{s+1} (x \ominus 1)_{p,q}^s \right. \\ \left. - \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_{p,q} p^{\binom{k-s}{2}} m^s \mathcal{G}_s(x, -1: p, q) \right] \mathcal{E}_{n-k}(0, my: p, q).$$

Corollary 15.9 (Duran et al. [28]). For $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the following identities hold true:

$$\begin{aligned} B_n(x+y) &= \sum_{k=0}^n \binom{n}{k} \frac{E_{n-k}(my)}{2m^n} \left[m^k B_k(x) + m^k B_k\left(x-1+\frac{1}{m}\right) + km(1+m(x-1))^{k-1} \right] \\ &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \frac{G_{n-k}(my)}{m^n(k+1)} \left[m^{k+1} B_{k+1}(x) + m^{k+1} B_{k+1}\left(x-1+\frac{1}{m}\right) + (k+1)m(1+m(x-1))^k \right], \\ E_n(x+y) &= \sum_{k=0}^n \binom{n}{k} \frac{B_{n-k}(my)}{m^n(k+1)} \left[2(1+m(x-1))^{k+1} - m^{k+1} E_{k+1}\left(x-1+\frac{1}{m}\right) - m^{k+1} E_{k+1}(x) \right] \\ &= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \frac{G_{n-k}(my)}{m^n(k+1)} \left[2(1+m(x-1))^{k+1} - m^{k+1} E_{k+1}\left(x-1+\frac{1}{m}\right) - m^{k+1} E_{k+1}(x) \right], \end{aligned}$$

$$G_n(x+y)$$

$$\begin{aligned} &= \sum_{k=0}^n \binom{n}{k} \frac{B_{n-k}(my)}{m^n(k+1)} \left[2(k+1)m(1+m(x-1))^k - m^{k+1}G_{k+1}\left(x-1+\frac{1}{m}\right) - m^{k+1}G_{k+1}(x) \right] \\ &= \sum_{k=0}^n \binom{n}{k} \frac{E_{n-k}(my)}{2m^n} \left[m^k G_k(x) + 2km(1+m(x-1))^{k-1} - m^k G_k\left(x-1+\frac{1}{m}\right) \right]. \end{aligned}$$

Corollary 15.10 (cf. Duran et al. [28]). For $n \in \mathbb{N}_0$, the following identities hold true:

$$\mathcal{B}_n(0, y: p, q) = \frac{2}{[2]_{p,q}} \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{B}_k(p, q) \mathcal{E}_{n-k}(0, y: p, q) \quad (15.16)$$

$$\mathcal{B}_n(0, y: p, q) = \frac{2}{[2]_{p,q}} \sum_{k=0}^n \binom{n}{k}_{p,q} \frac{1}{[k+1]_{p,q}} \mathcal{B}_{k+1}(p, q) \mathcal{G}_{n-k}(0, y: p, q) + \mathcal{G}_n(0, y: p, q),$$

$$\mathcal{E}_n(0, y: p, q) = -2 \sum_{k=0}^n \binom{n}{k}_{p,q} \frac{\mathcal{E}_{k+1}(p, q) \mathcal{B}_{n-k}(0, y: p, q)}{[k+1]_{p,q}}, \quad (15.17)$$

$$\mathcal{G}_n(0, y: p, q) = - \sum_{k=0}^n \binom{n}{k}_{p,q} \frac{1}{[k+1]_{p,q}} \mathcal{G}_{k+1}(p, q) \mathcal{B}_{n-k}(0, y: p, q) + [2]_{p,q} \mathcal{B}_n(0, y: p, q).$$

The formula in Eq. (15.16) is (p, q) -extension of Cheon's main result in [29].

Eq. (15.17) is a (p, q) -extension of the formula of Srivastava-Pinter in [30].

Corollary 15.11 (Duran et al. [28]). For $n \in \mathbb{N}_0$, the following equalities hold true:

$$\mathcal{B}_n(p, q) = \frac{2}{[2]_{p,q}} \sum_{k=0}^n \binom{n}{k}_{p,q} \mathcal{B}_k(p, q) \mathcal{E}_{n-k}(p, q),$$

$$\mathcal{B}_n(p, q) = \frac{2}{[2]_{p,q}} \sum_{k=0}^n \binom{n}{k}_{p,q} \frac{1}{[k+1]_{p,q}} \mathcal{B}_{k+1}(p, q) \mathcal{G}_{n-k}(p, q) + \mathcal{G}_n(p, q),$$

$$\mathcal{E}_n(p, q) = -\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p, q} \frac{2}{[k+1]_{p, q}} \mathcal{E}_{k+1}(p, q) \mathcal{B}_{n-k}(p, q),$$

$$\mathcal{G}_n(p, q) = -\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p, q} \frac{1}{[k+1]_{p, q}} \mathcal{G}_{k+1}(p, q) \mathcal{B}_{n-k}(p, q) + [2]_{p, q} \mathcal{B}_n(p, q).$$

Corollary 15.12 (Duran et al. [28]). For $n \in \mathbb{N}_0$, the following expressions

$$B_n(x) = \frac{1}{2} \sum_{k=0}^n \binom{n}{k} E_{n-k} [2B_k(x) + kx^{k-1}]$$

$$= \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \frac{G_{n-k}}{(k+1)} [2B_{k+1}(x) + (k+1)x^k],$$

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{B_{n-k}}{k+1} [2x^{k+1} - 2E_{k+1}(x)],$$

$$G_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{B_{n-k}}{k+1} [2(k+1)x^k - 2G_{k+1}(x)]$$

hold true for classical Bernoulli polynomials, classical Euler polynomials, and classical Genocchi polynomials.

Corollary 15.13 (Duran et al. [28]). For $n \in \mathbb{N}_0$, the following relationships

$$B_n = \sum_{k=0}^n \binom{n}{k} E_{n-k} B_k + \frac{1}{2} E_{n-1}$$

$$= \sum_{k=0}^n \binom{n}{k} \frac{G_{n-k}}{(k+1)} B_{k+1}(x) + \frac{1}{2} G_n,$$

$$E_n = -\sum_{k=0}^n \binom{n}{k} \frac{2}{k+1} B_{n-k} E_{k+1},$$

$$G_n = -\sum_{k=0}^n \binom{n}{k} \frac{2}{k+1} B_{n-k} G_{k+1} + 2B_n$$

are true for classical Bernoulli numbers, classical Euler numbers and classical Genocchi numbers.

In summary, a new type generalizations of Bernoulli polynomials, Euler polynomials and Genocchi polynomials which are called (p,q) -Bernoulli polynomials, (p,q) -Euler polynomials and (p,q) -Genocchi polynomials are defined. Their some properties including additions theorems, difference equations, differential relations, recurrence relationships are investigated and analyzed. Also, the (p,q) -extension of Cheon's main result in [29] and the (p,q) -extension of the formula of Srivastava-Pintér in [30] are given. The results obtained here reduce to known properties of q -polynomials mentioned in this chapter when approached $p=1$. The results obtained here reduce to known properties of q -polynomials mentioned in this chapter when $q \rightarrow p = 1$.

CHAPTER 16

\$(\rho, q)\$-VOLKENBORN INTEGRATION

Before going to ${}^1(\rho, q)$ -Volkenborn integration, some basic definitions and properties belonging to familiar p -adic distribution and p -adic q -distribution are given firstly given. The q -analogue of the Haar distribution firstly introduced by Kim [32] is described as

$$\mu_q(a + p^N \mathbb{Z}_p) = \frac{1}{[p^N]_q} q^a \quad (16.1)$$

to construct q -Volkenborn integral of a function $f \in UD(\mathbb{Z}_p)$ as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x.$$

For $n \in \mathbb{N}^*$, Bernoulli polynomials $B_n(x)$ and Bernoulli numbers B_n are defined by the following Volkenborn integral as follows [78]:

$$B_n(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) \quad (16.2)$$

$$B_n = \int_{\mathbb{Z}_p} y^n d\mu_1(y) \quad (16.3)$$

where $\mu_1(y) = \mu(y + p^N \mathbb{Z}_p) = 1/p^N$.

¹ **Note:** p and q (or (p, q)) have been used as parameters of (p, q) -calculus in other chapters of this thesis, however in this chapter (ρ, q) is used in order to avoid confusions with the notations of p -adic q -integral.

16.1 (ρ, q) -Volkenborn integration

In this section, the notations ρ and q can be variously considered as indeterminates, complex numbers ρ and $q \in \mathbb{C}$ with $0 < |q| < |\rho| < 1$, or p -adic numbers ρ and $q \in \mathbb{C}_p$ with $|\rho - 1|_p < p^{-\frac{1}{p-1}}$ and $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $\rho^x = \exp(x \log \rho)$ and $q^x = \exp(x \log q)$ for $|x|_p \leq 1$, where $|\cdot|_p$ denotes the p -adic absolute value on $\mathbb{C}_p$ normalized by $|p|_p = 1/p$, see [33].

Now, the (p, q) -analogue of the Haar distribution is stated as follows.

Definition 16.1.1 (Duran et al. [31]). Let N be a fixed positive integer. p -adic distribution $\mu_{\rho, q}$ on X is defined by

$$\mu_{\rho, q}(a + p^N \mathbb{Z}_p) = \frac{\rho^{p^N}}{[p^N]_{\rho, q}} \left(\frac{\rho}{q} \right)^a. \quad (16.4)$$

By verifying Theorem 13.1.1, the p -adic distribution $\mu_{\rho, q}$ can be extended to a p -adic distribution on X .

For a fixed positive integer N , $\mu_{\rho, q}$ from the set of intervals contained in X to $\mathbb{Q}_p$ for which

$$\sum_{b=0}^{p-1} \mu_{\rho, q}(a + bp^N + p^{N+1} \mathbb{Z}_p) = \mu_{\rho, q}(a + p^N \mathbb{Z}_p)$$

whenever $a + dp^N \mathbb{Z}_p \subset \mathbb{Z}_p$, extends to a p -adic distribution on X . Indeed,

$$\begin{aligned} \sum_{b=0}^{p-1} \mu_{\rho, q}(a + bp^N + p^{N+1} \mathbb{Z}_p) &= \sum_{b=0}^{p-1} \frac{\rho^{p^{N+1}}}{[p^{N+1}]_{\rho, q}} \left(\frac{\rho}{q} \right)^{a+bp^N} \\ &= \frac{\rho - q}{\rho^{p^{N+1}} - q^{p^{N+1}}} \left(\frac{\rho}{q} \right)^a (\rho)^{p^{N+1}} \sum_{b=0}^{p-1} \left(\frac{\rho}{q} \right)^{bp^N} \end{aligned}$$

$$\begin{aligned}
&= \frac{\rho - q}{\rho^{p^{N+1}} - q^{p^{N+1}}} \left(\frac{\rho}{q} \right)^a \rho^{p^{N+1}} \frac{1 - \left(\frac{\rho}{q} \right)^{p^{N+1}}}{1 - \left(\frac{\rho}{q} \right)^{p^N}} \\
&= \frac{\rho - q}{\rho^{p^{N+1}} - q^{p^{N+1}}} \left(\frac{\rho}{q} \right)^a \rho^{p^{N+1}} \frac{\rho^{p^{N+1}} - q^{p^{N+1}}}{\rho^{p^N} - q^{p^N}} \frac{\rho^{p^N}}{\rho^{p^{N+1}}} \\
&= \frac{1}{\left[p^N \right]_{\rho, q}} \left(\frac{\rho}{q} \right)^a \rho^{p^N} \\
&= \mu_{\rho, q} \left(a + p^N \mathbb{Z}_p \right).
\end{aligned}$$

Remark 16.1.1 (Duran *et al.* [31]). In the case $\rho = 1$ in Eq. (16.4) gives the p -adic q -distribution μ_q in Eq. (16.1) and also letting $q \rightarrow \rho = 1$ in Eq. (16.4) reduces to p -adic Haar distribution μ_0 in Eq. (13.1).

Define

$$\sum_{0 \leq a < p^N} f(a) \mu_{\rho, q} \left(a + p^N \mathbb{Z}_p \right) = \frac{\rho^{p^N}}{\left[p^N \right]_{\rho, q}} \sum_{0 \leq a < p^N} \left(\frac{\rho}{q} \right)^a f(a)$$

which represents (ρ, q) -analogue of Riemann sums for f . The integral of f on $\mathbb{Z}_p$ is defined as the limit of those sums (as $N \rightarrow \infty$), when it exists. For

$$f \in UD(\mathbb{Z}_p) = \left\{ f \mid f : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function} \right\},$$

the (ρ, q) -Volkenborn integral on $\mathbb{Z}_p$ is defined by Duran *et al.* [31] as

$$I_{\rho, q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{\rho, q}(x) = \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{\left[p^N \right]_{\rho, q}} \sum_{x=0}^{p^N-1} f(x) \left(\frac{q}{\rho} \right)^x \quad (16.5)$$

which has a sense as the limit is convergent.

Note that if $f_n \rightarrow f$ in $UD(\mathbb{Z}_p)$, then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_{\rho,q}(x) \rightarrow \int_{\mathbb{Z}_p} f(x) d\mu_{\rho,q}(x).$$

Here some properties of (ρ, q) -Volkenborn integral in Eq. (16.5) are investigated by the following Theorems 16.1.1 and 16.1.2.

Theorem 16.1.1 (Duran et al. [31]). For $f \in UD(\mathbb{Z}_p)$, the following identity holds true

$$qI_{\rho,q}(f_1) - \rho I_{\rho,q}(f) = \rho(q - \rho) \left(\frac{f'(0)}{\ln q - \ln \rho} + f(0) \right) \quad (16.6)$$

where $f_1(x) = f(x+1)$.

Proof. From the integral in Eq. (16.5), it becomes

$$\begin{aligned} \frac{q}{\rho} I_{\rho,q}(f_1) &= \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{[p^N]_{\rho,q}} \sum_{x=0}^{p^N-1} f(x+1) \left(\frac{q}{\rho} \right)^{x+1} \\ &= \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{[p^N]_{\rho,q}} \left\{ f(1) \left(\frac{q}{\rho} \right)^1 + f(2) \left(\frac{q}{\rho} \right)^2 + \dots \right. \\ &\quad \left. + f(p^N - 1) \left(\frac{q}{\rho} \right)^{p^N-1} + f(p^N) \left(\frac{q}{\rho} \right)^{p^N} \right\} \\ &= I_{\rho,q}(f) + \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{[p^N]_{\rho,q}} \left\{ f(p^N) \left(\frac{q}{\rho} \right)^{p^N} - f(0) \right\} \\ &= I_{\rho,q}(f) + \lim_{N \rightarrow \infty} \frac{\rho - q}{1 - \left(\frac{q}{\rho} \right)^{p^N}} \left\{ f(p^N) \left(\frac{q}{\rho} \right)^{p^N} - f(0) \right\} \end{aligned}$$

$$\begin{aligned}
&= I_{\rho,q}(f) - (\rho - q) \lim_{N \rightarrow \infty} \frac{f'(p^N) + f(p^N) \ln\left(\frac{q}{\rho}\right)}{\ln\left(\frac{q}{\rho}\right)} \\
&= I_{\rho,q}(f) - (\rho - q) \left(\frac{f'(0)}{\ln q - \ln \rho} + f(0) \right).
\end{aligned}$$

Therefore, the expected result in Eq. (16.6) is archived.

Theorem 16.1.2 (Duran et al. [31]). For $f \in UD(\mathbb{Z}_p)$, the following formula holds true

$$q^n I_{\rho,q}(f_n) - \rho^n I_{\rho,q}(f) = \rho^n (q - \rho) \sum_{a=0}^{n-1} \left(\frac{q}{\rho}\right)^a \left[\frac{f'(a)}{\ln q - \ln \rho} + f(a) \right] \quad (16.7)$$

where $f_n(x) = f(x+n)$.

Proof. From Eq. (16.5), it can be written as

$$\begin{aligned}
\left(\frac{q}{\rho}\right)^n I_{\rho,q}(f_n) &= \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{[p^N]_{\rho,q}} \sum_{x=0}^{p^N-1} f(x+n) \left(\frac{q}{\rho}\right)^{x+n} \\
&= \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{[p^N]_{\rho,q}} \left\{ f(n) \left(\frac{q}{\rho}\right)^n + f(n+1) \left(\frac{q}{\rho}\right)^{n+1} + \dots \right. \\
&\quad \left. + f(p^N + n - 2) \left(\frac{q}{\rho}\right)^{p^N + n - 2} + f(p^N + n - 1) \left(\frac{q}{\rho}\right)^{p^N + n - 1} \right\} \\
&= I_{\rho,q}(f) + \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{[p^N]_{\rho,q}} \sum_{a=0}^{n-1} \left\{ f(p^N + a) \left(\frac{q}{\rho}\right)^{p^N + a} - f(a) \left(\frac{q}{\rho}\right)^a \right\}
\end{aligned}$$

$$\begin{aligned}
&= I_{\rho,q}(f) + (\rho - q) \sum_{a=0}^{n-1} \left(\frac{q}{\rho} \right)^a \frac{f'(a) + f(a) \ln \frac{q}{\rho}}{\ln \frac{q}{\rho}} \\
&= I_{\rho,q}(f) - (\rho - q) \sum_{a=0}^{n-1} \left(\frac{q}{\rho} \right)^a \left\{ \frac{f'(a)}{\ln q - \ln \rho} + f(a) \right\}.
\end{aligned}$$

Therefore, the claimed result in Eq. (16.7) is obtained.

16.2 On Carlitz's-type (ρ, q) -Bernoulli polynomials and numbers

This section defines Carlitz's-type (ρ, q) -Bernoulli numbers and Carlitz's-type (ρ, q) -Bernoulli polynomials and researchs their some properties.

For $a \in \mathbb{Q}$, Carlitz's-type (ρ, q) -Bernoulli polynomials are defined as follows.

Definition 16.2.1 (Duran et al. [31]). (Witt Type Formula) For $a \in \mathbb{Q}$, Carlitz's-type (ρ, q) -Bernoulli polynomials $\beta_{n,a}(x; \rho, q)$ are defined by the following (ρ, q) -Volkenborn integral

$$\beta_{n,a}(x; \rho, q) = \int_{\mathbb{Z}_p} \rho^{ay} [x + y]_{\rho,q}^n d\mu_{\rho,q}(y). \quad (16.8)$$

Note that

$$\beta_{n,a}(0; \rho, q) := \beta_{n,a}(\rho, q)$$

are called Carlitz's-type (ρ, q) -Bernoulli numbers.

Now, special cases of Definition 16.2.1 are presented as Corollary 16.2.1 and Corollary 16.2.2.

Corollary 16.2.1 (Duran et al. [31]). Choosing $\rho=1$ in Definition 16.2.1 gives Carlitz's q -Bernoulli polynomials, as follows:

$$B_n(x; q) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y)$$

which are given by Kim in [32].

Corollary 16.2.2 (Duran et al. [31]). Choosing $q \rightarrow \rho = 1$ in Definition 16.2.1 yields to well-known Bernoulli polynomials in Eq. (16.2).

Note that

$$[x+y]_{\rho,q} = \rho^y [x]_{\rho,q} + q^x [y]_{\rho,q}. \quad (16.9)$$

Carlitz's-type (ρ, q) -Bernoulli polynomials can be written as a linear combination in terms of Carlitz's-type (ρ, q) -Bernoulli numbers by the following theorem.

Theorem 16.2.1 (cf. Duran et al. [31]). For $n \in \mathbb{N}^*$ and $a \in \mathbb{Q}$, the following recurrence relation is valid:

$$\begin{aligned} \beta_{n,a}(x; \rho, q) &= \sum_{k=0}^n \binom{n}{k} q^{kx} \beta_{k;a+n-k}(\rho, q) [x]_{\rho,q}^{n-k} \\ &= \left(q^x \beta_a(\rho, q) + [x]_{\rho,q} \right)^n \end{aligned} \quad (16.10)$$

with the usual convention of replacing $\beta_a^k(x; \rho, q)$ by $\beta_{k;a+n-k}(x; \rho, q)$ where $k \in \{0, 1, \dots, n\}$.

Proof. By using Eq. (16.9), it can be observed that

$$\begin{aligned} \beta_{n,a}(x; \rho, q) &= \int_{\mathbb{Z}_p} \rho^{ay} [x+y]_{\rho,q}^n d\mu_{\rho,q}(y) \\ &= \int_{\mathbb{Z}_p} \rho^{ay} \left(\rho^y [x]_{\rho,q} + q^x [y]_{\rho,q} \right)^n d\mu_{\rho,q}(y) \\ &= \sum_{k=0}^n \binom{n}{k} [x]_{\rho,q}^{n-k} q^{kx} \int_{\mathbb{Z}_p} \rho^{(a+n-k)y} [y]_{\rho,q}^k d\mu_{\rho,q}(y) \\ &= \sum_{k=0}^n \binom{n}{k} [x]_{\rho,q}^{n-k} q^{kx} \beta_{k;a+n-k}(\rho, q) \end{aligned}$$

$$= \left(q^x \beta_a(\rho, q) + [x]_{\rho, q} \right)^n.$$

This ends the proof of the theorem.

Corollary 16.2.3 (Duran et al. [31]). *Choosing $\rho = 1$ in Eq. (16.10) gives the following recurrence relation*

$$B_n(x; q) = \left(q^x B(q) + [x]_q \right)^n$$

with the usual convention of replacing $B^n(q)$ by $B_n(q)$.

Corollary 16.2.4 (Duran et al. [31]). *Choosing $q \rightarrow \rho = 1$ in Eq. (16.10) yields to*

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} = (B + x)^n$$

with the usual convention of replacing B^n by B_n .

The following explicit formula is valid for Carlitz's-type (ρ, q) -Bernoulli numbers.

Theorem 16.2.2 (Duran et al. [31]). *For $n \in \mathbb{N}^*$ and $a \in \mathbb{Q}$, the following equality holds true*

$$\beta_{n;a}(\rho, q) = \frac{1}{(\rho - q)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{l+1 + (n+a) \log_{\frac{q}{\rho}} \rho}{1 - \rho^{n+a-l-1} q^{l+1}}. \quad (16.11)$$

Proof. From Definition 16.2.1, it becomes

$$\beta_{n;a}(\rho, q) = \int_{\mathbb{Z}_p} \rho^{ay} [y]_{\rho, q}^n d\mu_{\rho, q}(y)$$

$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \frac{\rho^{p^N}}{[p^N]_{\rho, q}} \sum_{k=0}^{p^N-1} \rho^{ak} [k]_{\rho, q}^n \left(\frac{q}{\rho} \right)^k \\
&= \lim_{N \rightarrow \infty} \sum_{k=0}^{p^N-1} \frac{\left(\frac{q}{\rho} \right)^k}{[p^N]_{\rho, q}} \sum_{l=0}^n \binom{n}{l} \frac{\rho^{p^N}}{(\rho - q)^n} \rho^{k(n+a-l)} q^{kl} (-1)^l \\
&= \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(\rho - q)^{n-1}} \lim_{N \rightarrow \infty} \sum_{k=0}^{p^N-1} \left(\rho^{n+a-l-1} q^{l+1} \right)^k \frac{\rho^{p^N}}{\rho^{p^N} - q^{p^N}} \\
&= \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(\rho - q)^{n-1}} \lim_{N \rightarrow \infty} \frac{1}{\left(\frac{q}{\rho} \right)^{p^N}} \frac{1 - \left(\rho^{n+a-l-1} q^{l+1} \right)^{p^N}}{1 - \rho^{n+a-l-1} q^{l+1}} \\
&= \frac{1}{(\rho - q)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{l+1 + (n+a) \log_{\frac{q}{\rho}} \rho}{1 - q^{l+1} \rho^{n+a-l-1}}.
\end{aligned}$$

Therefore, the proof is completed.

From Theorems 16.2.1 and 16.2.2, the following result is obtained.

Corollary 16.2.5 (Duran et al. [31]). For $n \in \mathbb{N}^*$ and $a \in \mathbb{Q}$, the following result holds true

$$\beta_{n;a}(x; \rho, q) = \frac{1}{(\rho - q)^{k-1}} \sum_{k=0}^n \binom{n}{k} [x]_{\rho, q}^{n-k} q^{kx} \sum_{l=0}^k \binom{k}{l} (-1)^l \frac{l+1 + (k+a) \log_{\frac{q}{\rho}} \rho}{1 - q^{l+1} \rho^{k+a-l-1}}.$$

Lemma 16.2.1 (Duran et al. [31]). For $n \in \mathbb{N}^*$ and $a \in \mathbb{Q}$, the following integral

$$\int_X \rho^{ax} [x]_{\rho, q}^n d\mu_{\rho, q}(x) = \int_{\mathbb{Z}_p} \rho^{ax} [x]_{\rho, q}^n d\mu_{\rho, q}(x)$$

holds true.

Proof. For $d \in \mathbb{N}^*$, it can be seen that

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{\rho^{dp^N}}{[dp^N]_{\rho,q}} \sum_{y=0}^{dp^N-1} \rho^{ay} [y]_{\rho,q}^n \left(\frac{q}{\rho} \right)^y \\
&= \lim_{N \rightarrow \infty} \frac{\rho - q}{1 - \left(\frac{q}{\rho} \right)^{dp^N}} \sum_{y=0}^{dp^N-1} \rho^{ay} [y]_{\rho,q}^n \left(\frac{q}{\rho} \right)^y \\
&= \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(\rho - q)^{n-1}} \lim_{N \rightarrow \infty} \frac{1}{1 - \left(\frac{q}{\rho} \right)^{dp^N}} \frac{1 - (\rho^{n+a-l-1} q^{l+1})^{dp^N}}{1 - \rho^{n+a-l-1} q^{l+1}} \\
&= \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(\rho - q)^{n-1}} \lim_{N \rightarrow \infty} \frac{1}{1 - \left(\frac{q}{\rho} \right)^{p^N}} \frac{1 - (\rho^{n+a-l-1} q)^{p^N}}{1 - \rho^{n+a-l-1} q^{l+1}} \\
&= \frac{1}{(\rho - q)^{n-1}} \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{l+1 + (n+a) \log_{\frac{q}{\rho}} \rho}{1 - \rho^{n+a-l-1} q^{l+1}}.
\end{aligned}$$

Thereby, the proof is completed.

By Definition 16.2.1 and using Theorems 16.1.1 and 16.1.2, the following is satisfied.

Proposition 16.2.1 (Duran *et al.* [31]). *The following relation holds true:*

$$q \int_{\mathbb{Z}_p} \rho^{a(x+1)} [x+1]_{\rho,q}^n d\mu_{\rho,q}(x) - \rho \int_{\mathbb{Z}_p} \rho^{ax} [x]_{\rho,q}^n d\mu_{\rho,q}(x) = \begin{cases} \rho(q - \rho) \left(\frac{a \ln \rho}{\ln q + \ln \rho} + 1 \right), & \text{if } n = 0, \\ \rho, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

By Definition 16.2.1 and Proposition 16.2.1, it is observed that

$$\beta_{0;a}(\rho, q) = \rho \left(\frac{a \ln \rho}{\ln q + \ln \rho} + 1 \right)$$

and

$$q\beta_{n;a}(1; \rho, q) - \rho\beta_{n;a}(\rho, q) = \begin{cases} \rho, & \text{if } n=1, \\ 0, & \text{if } n>1. \end{cases}$$

Now, the following result including Raabe type formula holds true for Carlitz's-type (ρ, q) -Bernoulli polynomials.

Theorem 16.2.3 (Duran et al. [31]). (Raabe Type Formula) For $n, d \in \mathbb{N}^*$ and $a \in \mathbb{Q}$, the following expression is valid:

$$\beta_{n;a}(x; \rho, q) = [d]_{\rho, q}^{n-1} \sum_{s=0}^{d-1} \left(\frac{\rho}{q} \right)^s \rho^{as} \beta_{n;a} \left(\frac{x+s}{d}; \rho^d, q^d \right). \quad (16.12)$$

Proof. From Definition 16.2.1, it becomes

$$\begin{aligned} \beta_{n;a}(x; \rho, q) &= \int_{\mathbb{Z}_p} \rho^{ay} [x+y]_{\rho, q}^n d\mu_{\rho, q}(y) \\ &= \lim_{N \rightarrow \infty} \frac{\rho^{dp^N}}{[dp^N]_{\rho, q}} \sum_{l=0}^{dp^N-1} \rho^{al} [x+l]_{\rho, q}^n \left(\frac{q}{\rho} \right)^l \\ &= \lim_{N \rightarrow \infty} \frac{\rho^{dp^N}}{[d]_{\rho, q} [p^N]_{\rho^d, q^d}} \sum_{s=0}^{d-1} \sum_{l=0}^{p^N-1} \rho^{a(s+dl)} [x+s+dl]_{\rho, q}^n \left(\frac{q}{\rho} \right)^{s+dl} \\ &= \lim_{N \rightarrow \infty} \frac{(\rho^d)^{p^N}}{[d]_{\rho, q} [p^N]_{\rho^d, q^d}} \sum_{s=0}^{d-1} \left(\frac{q}{\rho} \right)^s \rho^{as} \sum_{l=0}^{p^N-1} (\rho^d)^{al} \left[\frac{x+s}{d} + l \right]_{\rho^d, q^d}^n [d]_{\rho, q}^n \left(\frac{q^d}{\rho^d} \right)^l \\ &= \sum_{s=0}^{d-1} \left(\frac{q}{\rho} \right)^s \rho^{as} [d]_{\rho, q}^{n-1} \lim_{N \rightarrow \infty} \frac{(\rho^d)^{p^N}}{[p^N]_{\rho^d, q^d}} \sum_{l=0}^{p^N-1} (\rho^d)^{al} \left[\frac{x+s}{d} + l \right]_{\rho^d, q^d}^n \left(\frac{q^d}{\rho^d} \right)^l \\ &= \sum_{s=0}^{d-1} \left(\frac{q}{\rho} \right)^s \rho^{as} [d]_{\rho, q}^{n-1} \beta_{n;a} \left(\frac{x+s}{d}; \rho^d, q^d \right). \end{aligned}$$

So the proof of the theorem is completed.

Corollary 16.2.6 (Duran *et al.* [31]). *In the case $x = 0$ in Eq. (16.12), the following definition holds true:*

$$[d]_{\rho,q} \beta_{n;a}(\rho, q) = \sum_{k=0}^n \binom{n}{k} \beta_{k;a+n-k}(\rho, q) [d]_{\rho,q}^k \sum_{s=0}^{d-1} \left(\frac{\rho}{q}\right)^s \rho^{as} q^{ks/d} [s]_{\rho,q}^{n-k}.$$

As summary, it has been mentioned that (ρ, q) -Haar distribution is given and by means of this distribution, (ρ, q) -Volkenborn integral which is a generalization of known q -Volkenborn integral earlier originally introduced by Kim [32] is constructed. As applications, the properties of this integral are derived. Eventually, by using (ρ, q) -Volkenborn integral, a new generalizations of Carlitz's q -Bernoulli numbers and polynomials are introduced and some their properties are developed.

CHAPTER 17

CONCLUSIONS

In this thesis, the theory of *post quantum calculus*, denoted by (p,q) -calculus which is a native generalization of *quantum calculus*, denoted by q -calculus has been studied extensively. Some notations, definitions, theorems, and results belonging to (p,q) -calculus including (p,q) -number and its some properties, (p,q) -derivative and its some properties, (p,q) -hypergeometric functions and their some properties and results, (p,q) -power basis its some properties, two (p,q) -Taylor formulas, (p,q) -binomial theorem and (p,q) -binomial coefficient and its some identities and properties, two exponential functions and their some properties are mentioned. By relating (p,q) -derivative and divided difference, some their results have been given. By defining (p,q) -antiderivative, the (p,q) -integral has been defined and it has been examined extensively. After defining (p,q) -integral, introducing improper (p,q) -integral and definite (p,q) -integral has become possible. Afterthat, the fundamental theorem of (p,q) -calculus has been developed.

In mathematics, the theory of special polynomials is one of the most active research areas and is studied and investigated by many mathematicians. Also, one of the most effects of the (p,q) -calculus in mathematics is occurred in the theory of special polynomials. The theory of special polynomials has been considered and has been investigated extensively by Duran, Acikgoz and Araci in the light of (p,q) -calculus, see [27, 28, 31]. To mention a few, (p,q) -analogues of Bernoulli polynomials, Euler polynomials, and Genocchi polynomials have been introduced and the (p,q) -analogues of known earlier formulae have been obtained and some relations between the new and old polynomials have been derived by making use of the fermionic p -adic integral over the p -adic number fields. A new class of Bernoulli, Euler and Genocchi polynomials associated with (p,q) -calculus has been described and some their properties including addition theorems, difference equations, derivative properties, recurrence relationships are developed, (p,q) -extension of Cheon's main result in [29] and further, (p,q) -analogue of the main results given

earlier by Srivastava and Pintér in [30] have been discovered. An analogue of Haar distribution based on (ρ, q) -number has been described as

$$\mu_{\rho, q}(a + p^N \mathbb{Z}_p) = \frac{\rho^{p^N}}{[p^N]_{\rho, q}} \left(\frac{q}{\rho} \right)^a$$

and by means of this distribution, (ρ, q) -analogue of

Volkenborn integration which yields to (ρ, q) -Bernoulli numbers and polynomials. Some properties of these numbers and polynomials have been examined.



LIST OF REFERENCES

- [1] Heine, E. (1846). Über die reihe. *Journal für die reine und angewandte Mathematik*. **32**, 210-212.
- [2] Kac, V., Cheung, P. (2002). Quantum Calculus, New York: Springer.
- [3] Jackson, F. H. (1908). On q -functions and a certain difference operator. *Transactions of the Royal Society of Edinburgh Earth Sciences*. **46**, 253-281.
- [4] Jackson, F. H. (1909). A q -form of Taylor's theorem. *Messenger of Mathematics*. **38**, 62-64.
- [5] Jackson, F. H. (1910). On q -definite integrals. *Pure and Applied Mathematics Quarterly*. **41**, 193-203.
- [6] Ernst, T. (2000). The history of q -calculus and a new method. Report 2000: 16. U.U.D.M. Department of Mathematics, Uppsala University, Uppsala.
- [7] Chakrabarti, R., Jagannathan, R. (1991). A (p,q) -oscillator realization of two-parameter quantum algebras. *Journal of Physics A Mathematical and General*. **24**, L711.
- [8] Brodimas, G., Jannussis, A., Mignani, R. (1991). Two-parameter quantum groups. *Universita di Roma Preprint*. **820**, 1-16.
- [9] Wachs, M., White, D. (1991). p, q -Stirling numbers and set partition statistics, *Journal of Combinatorial Theory, Series A*. **56**, 27-46.
- [10] Arik, M., Demircan, E., Turgut, T., Ekinici, L., Mungan, M. (1991). Fibonacci oscillators. *Zeitschrift für Physik C Particles and Fields*. **55**, 89-95.
- [11] Burban, M., Klimyk, A. U. (1994). P, Q -differentiation, P, Q -integration and P, Q -hypergeometric functions related to quantum group. *Integral Transforms and Special Functions*. **2**, 15-36.
- [12] Corcino, R. B. (2008). On P, Q -Binomial coefficients. *Electronic Journal of Combinatorial Number Theory*. **8**, #A29.
- [13] Jagannathan, R., Rao, K. S. (2005). Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series. in *Proceeding*

of the International Conference on Number Theory and Mathematical Physics, Srinivasa Ramanujan Centre, Kumbakonam, India.

- [14] Jagannathan, R. (1997). (P, Q) -Special functions, *Special Functions and Differential Equations, Proceedings of a Workshop held at The Institute of Mathematical Sciences, Matras, India, January*. 13-24.
- [15] Jagannathan, R., Sridhar, R. (2010). (p, q) -Rogers-Szegö polynomial and the (p, q) -oscillator. *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*. 491-501.
- [16] Mursaleen, M., Ansari, K. J., Khan, A. (2015). On (p, q) -analogue of Bernstein operators. *Applied Mathematics and Computation*. **266**, 874-882.
- [17] Mursaleen, M., Ansari, K. J., Khan, A. (2016). Erratum to "On (p, q) -analogue of Bernstein Operators" [Appl. Math. Comput. 266 (2015) 874–882]. *Applied Mathematics and Computation*. **278**, 70-71.
- [18] Mursaleen, M., Nasiruzzaman, M., Nurgali, A. (2015). Some approximation results on Bernstein-Schurer operators defined by (p, q) -integer. *Journal of Inequalities and Applications*. **249**, 1-12.
- [19] Sadjang, P. N. (2013). On the fundamental theorem of (p, q) -calculus and some (p, q) -Taylor formulas. arXiv:1309.3934 [math.QA].
- [20] Sadjang, P. N. (2015). On the (p, q) -Gamma and the (p, q) -Beta functions. arXiv 1506.07394v1 [math.QA].
- [21] Milovanovic, G. V., Gupta, V., Malik, N. (2016). (p, q) -Beta functions and applications in approximation. arXiv:1602.06300v2 [math.CA].
- [22] Smirnov, Y., Wehrhahn, R. F. (1992). The Clebsch-Gordan coefficients for the two-parameter quantum algebra $SU_{p,q}(2)$ in the Lowdin-Shapiro approach. *Journal of Physics A: Mathematical and General*. **25**, 5563-5576.
- [23] Duran, U., Acikgoz, M., Araci, S., Srivastava, H. M. (2016). A certain (p, q) -derivative operator and associated divided differences. Accepted in *Journal of Computational and Theoretical Nanoscience*.
- [24] Sofonea, D. F. (2008). Some properties in q -calculus, *General Mathematics*. **16**, 47–54.
- [25] Sofonea, D. F. (2003). Numerical Analysis and q -Calculus (I). **11**, 151-156.
- [26] Duran, U., Acikgoz, M., Araci, S. (2016). A Study on Some New Results Arising from (p, q) -Calculus. Submitted to *Ukranian Mathematical Journal*.

- [27] Duran, U., Acikgoz, M., Araci, S. (2016). On some polynomials derived from (p,q) -calculus. Accepted in *Journal of Computational and Theoretical Nanoscience*.
- [28] Duran, U., Acikgoz, M., Araci, S. (2016). On (p,q) -Bernoulli, (p,q) -Euler and (p,q) -Genocchi polynomials. Accepted in *Journal of Computational and Theoretical Nanoscience*.
- [29] Cheon, G. S. (2003). A note on the Bernoulli and Euler polynomials. *Applied Mathematics Letters*. **16**, 365-368.
- [30] Srivastava, H. M., Pinter, A. (2004). Remarks on Some Relationships Between the Bernoulli and Euler Polynomials. *Applied Mathematics Letters*. **17**, 375-380.
- [31] Duran, U., Acikgoz, M., Araci, S. (2016). (p,q) -Volkenborn Integration. Submitted to *Journal of Number Theory*.
- [32] Kim, T. (2002). q -Volkenborn integration. *Russian Journal of Mathematical Physics*. **9**, 288-299.
- [33] Kim, T. (2006). A note on p -adic invariant integral in the rings of p -adic integers. arXiv:math/0606097 [math.NT].
- [34] Acikgoz, M., Araci, S., Duran, U. (2015). New extensions of some known special polynomials under the theory of multiple q -calculus. *Turkish Journal of Analysis and Number Theory*. **3**, 128-139.
- [35] Araci, S., Duran, U., Acikgoz, M. (2016). Theorems on q -Frobenius-Euler polynomials under Sym (4). Accepted in *Utilitas Mathematica*. (To be published in Nov. 2016).
- [36] Araci, S., Duran, U., Acikgoz, M. (2015). Symmetric identities involving q -Frobenius-Euler polynomials under Sym (5). *Turkish Journal of Analysis and Number Theory*. **3**, 90-93.
- [37] Araci, S., Acikgoz, M., Sen, E. (2013). On the extended Kim's p -adic q -deformed fermionic integrals in the p -adic integer ring. *Journal of Number Theory*. **133**, 3348-3361.
- [38] Araci, S., Acikgoz, M., Park, K. H., Jolany, H. (2014). On the unification of two families of multiple twisted type polynomials by using p -adic q -integral at $q = -1$. *Bulletin of the Malaysian Mathematical Sciences Society*. **37**, 17-41.
- [39] Araci, S., Ağyuz, E., Acikgoz, M. (2015). On a q -analog of some numbers and polynomials. *Journal of Inequalities and Applications*. **19**, 1-9.
- [40] Bagdasaryan, A., Araci, S., Acikgoz, M., He, Y. (2016). Some new identities on the Apostol-Bernoulli polynomials of higher order derived from Bernoulli basis. *The*

Journal of Nonlinear Science and Applications. **9**, 2697-2704.

- [41] Carlitz, L. (1948). q -Bernoulli numbers and polynomials. *Duke Mathematical Journal.* **15**, 987–1000.
- [42] Duran, U., Acikgoz, M., Esi, A., Araci, S. (2015). Some new symmetric identities involving q -Genocchi polynomials under S_4 . *Journal of Mathematical Analysis.* **4**, 22-31.
- [43] Duran, U., Acikgoz, M., Araci, S. (2015). Symmetric identities involving weighted q -Genocchi polynomials under S_4 . *Proceedings of the Jangjeon Mathematical Society.* **18**, 455-465.
- [44] Duran, U., Acikgoz, M. (2016). New identities for Carlitz's twisted (h,q) -Euler polynomials under symmetric group of degree n . Accepted in *Journal of Analysis & Number Theory* (To be published in July,1 2016).
- [45] Duran, U., Acikgoz, M., Araci, S. (2015). Research on some new results arising from multiple q -calculus. Accepted in *Filomat*.
- [46] Duran, U., Acikgoz, M., Araci, S. (2015). On a class of multiple q -Bernoulli, multiple q -Euler and multiple q -Genocchi polynomials of order α . Submitted to *Advances and Applications in Mathematical Sciences*.
- [47] Duran, U., Acikgoz, M. (2016). A note on Carlitz's twisted (h,q) -Euler polynomials under S_5 . *General Mathematics Notes.* **33**, 1-8.
- [48] Duran, U., Acikgoz, M. (2016). Symmetric identities involving Carlitz's-type twisted (h,q) -Tangent polynomials under S_5 . *Journal of New Theory.* **12**, 51-59.
- [49] Jang, L. C. (2008). A new q -analogue of Bernoulli polynomials associated with p -adic q -integrals. *Abstract and Applied Analysis.* ID: 295307, 1-6.
- [50] Kim, T. (1999). On a q -analogue of the p -adic log Gamma functions and related integrals. *Journal of Number Theory.* **76**, 320-329.
- [51] Kim, T. (2008). An invariant p -adic q -integral on $\mathbb{Z}_p$. *Applied Mathematics Letters.* **21**, 105–108.
- [52] Kim, H. S., Lim, P. S., Kim, T. (1996). A remark on p -adic q -Bernoulli measure. *Bulletin of the Korean Mathematical Society.* **33**, 39-44.
- [53] Kim, T. (2008). Symmetry p -adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials. *Journal of Differential Equations.* **14**, 1267-1277.

- [54] Kim, T. (2007). Carlitz q -Bernoulli numbers and q -Stirling numbers. arXiv:0708.3306, [math.NT].
- [55] Kurt, V. (2014). New identities and relations derived from the generalized Bernoulli polynomials, Euler polynomials and Genocchi polynomials. *Advances in Difference Equations*. **5**, 1-9.
- [56] Luo, Q. M. (2010). Some results for the q -Bernoulli and q -Euler polynomials. *Journal of Mathematical Analysis and Applications*. **363**, 7-10.
- [57] Luo, Q. M., Srivastava, H. M. (2011). q -extension of some relationships between the Bernoulli and Euler polynomials. *Taiwanese Journal of Mathematics*. **15**, 241-257.
- [58] Mahmudov, N. I. (2013). On a class of q -Bernoulli and q -Euler polynomials. *Advance in Difference Equations*. **108**, 1-11.
- [59] Mahmudov, N. I., Keleshteri, M. E. (2013). On a class of generalized q -Bernoulli and q -Euler polynomials. *Advance in Difference Equations*. **115**, 1-10.
- [60] Mahmudov, N. I. (2012). q -Analogues of the Bernoulli and Genocchi polynomials and the Srivastava-Pintér addition theorems. *Discrete Dynamics in Nature and Society*. ID: 169348, 1-8.
- [61] Ozden, H., Simsek, Y., Cangul, I. N. (2007). Euler polynomials associated with p -adic q -Euler measure. *General Mathematics*. **15**, 24-37.
- [62] Ozden, H. (2007). p -Adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials. *Applied Mathematics and Computation*. **218**, 970-973.
- [63] Ryoo, C. S. (2007). A note on q -Bernoulli numbers and polynomials. *Applied Mathematics Letters*. **20**, 524-531.
- [64] Srivastava, H. M. (2011). Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials. *Applied Mathematics & Information Sciences*. **5**, 390-444.
- [65] Choi, J., Anderson, P. J., Srivastava, H. M. (2009). Carlitz's q -Bernoulli and q -Euler numbers and polynomials and a class of generalized q -Hurwitz zeta functions. *Applied Mathematics and Computation*. **215**, 1185–1208.
- [66] Srivastava, H. M., Choi, J. (2012). Zeta and q -Zeta functions and associated series and integrals. Amsterdam, London and New York: Elsevier Science Publishers.
- [67] Apostol, T. M. (1976). Introduction to Analytic Number Theory. New York: Springer.
- [68] Koblitz, N. (1982). On Carlitz's q -Bernoulli numbers. *Journal of Number*

Theory. **14**, 332-339.

- [69] Nalci, S., Pashaev, O. K. (2014). Exactly solvable q -extended nonlinear classical and quantum models. Lambert Academic Publishing.
- [70] Exton, H. (1983). q -Hypergeometric Functions and Applications, New York: Halstead Press, Chichester: Ellis Horwood.
- [71] Gasper, G., Rahman, M. (1935). Basic Hypergeometric Series, Cambridge: Cambridge University Press.
- [72] Corcino, R. B. (2001). Some theorems on generalized Stirling numbers. *Ars Combinatoria*. **60**, 273-286.
- [73] Corcino, R. B., Hsu, L. C., Tan, E. L. (2006). A q -analogue of generalized Stirling numbers. *Fibonacci Quarterly*. **44**, 154-165.
- [74] Kim, T. (2007). q -extension of the Euler formula and trigonometric functions. *Russian Journal of Mathematical Physics*. **14**, 275–278.
- [75] Manin, Y. I., Panchishkin, A. A. 2007. Introduction to Modern Number Theory. Berlin: Springer.
- [76] Rozikov, U. A. (2013). What are p -adic numbers? What are they used for?. Asia Pasific Mathematics Newsletter. **3**, 1-6.
- [77] Koblitz, N. (1977). p -adic numbers p -adic analysis and Zeta functions. New York: Springer-Verlag.
- [78] Schikhof, W., H. (1984). Ultrametric Calculus: An Introduction to p -Adic Analysis. Cambridge University Press.