

**REPUBLIC OF TURKEY  
YILDIZ TECHNICAL UNIVERSITY  
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES**

**VARIOUS FINITE DIFFERENCE SOLUTIONS OF  
OPTION PRICING MODELS**



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**MSc. THESIS  
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**REPUBLIC OF TURKEY**  
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## LIST OF SYMBOLS

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$C(S, t)$	Option value for the European call options
$E$	Exercise price
$r$	Interest rate
$S(t)$	Stock price at time $t$
$T$	Expiry time
$V(S, t)$	Option value
$\sigma$	Volatility of the underlying asset
$\mu$	Drift rate

## LIST OF ABBREVIATIONS

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FDM	Finite Difference Method
FD2	Second-order Finite Difference Schemes
FD4	Fourth-order Finite Difference Schemes
FD6	Sixth-order Finite Difference Schemes
PDE	Partial Differential Equation

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**ABSTRACT**

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**VARIOUS FINITE DIFFERENCE SOLUTIONS OF  
OPTION PRICING MODELS**

Tülay YILDIRIM

Department of Mathematics

MSc. Thesis

Adviser: Assoc. Prof. Dr. Murat SARI

This thesis investigates the Black-Scholes option pricing model representing some economic processes. To achieve this, fundamental numerical algorithms based on finite difference schemes are used with the European call option pricing. These algorithms are implemented by taking into account various spatial and temporal discretizations. The obtained results revealed that the suggested difference schemes are efficient and readily applicable. The validity of the current numerical models has been verified through the produced results and the literature.

**Key words:** Black-Scholes economic model, European call option pricing, higher order difference schemes, volatility

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**OPSİYON FİYATLAMA MODELLERİNİN ÇEŞİTLİ  
SONLU FARK ÇÖZÜMLERİ**

Tülay YILDIRIM

Matematik Anabilim Dalı

Yüksek Lisans Tezi

Tez Danışmanı: Doç. Dr. Murat SARI

Bu tez, bazı ekonomik işlemleri temsil eden Black-Scholes opsiyon fiyatlama modelini inceler. Bunu gerçekleştirmek için, sonlu farklar şemasına dayalı temel sayısal algoritmalar kullanılır. Bu algoritmalar çeşitli uzaysal ve zaman ayrıklaştırmaları göz önünde bulundurularak uygulanır. Elde edilen sonuçlar, önerilen fark şemalarının daha etkili ve kolay kullanılabilirliğini göstermiştir. Mevcut sayısal modellerin geçerliliği literatür ve üretilen sonuçlar neticesinde doğrulanmıştır.

**Anahtar Kelimeler:** Black-Scholes opsiyon fiyatlama modelleri, yüksek mertebeden fark şemaları, oynaklık

## CHAPTER 1

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### INTRODUCTION

#### 1.1 Literature review

Options have been paid attention since ancient times. The first notable option buyer was known as the ancient Greek mathematician and philosopher Thales of Miletus. He was predicted that the season's olive harvest would be larger than usual, and during the off-season, he acquired the right to use a number of olive presses the following spring. When spring came and the olive harvest was larger than expected he exercised his options and then rented the presses out at a much higher price than he paid for his 'option' [1]. However, until the end of the 18th century, there was not a standard method to price options and it was really impossible to put a fair value on them. Because of the difficulty in deciding whether there was a good value for money, options were not commonly considered as a suitable financial instrument by investors and traders. Yet, when we arrived in 1973, this assumption was changed by the Black and Scholes. They found a mathematical formula to calculate a fair value for an option based on certain variables. In accordance with their study, options have only one correct price, which can be determined by the Black-Scholes formula. Also in the same year, this formula was extended by Merton [2]. Black, Scholes and Merton proved that mathematics can be applied using partial differential equations to determine a fair value for European style calls and puts. In the following years, this model became widely accepted and it contributed to options trading becoming far more popular than it might otherwise have been.

This model have been tried to apply and improve for various assumptions and terms by many researchers. Under the influence of Black and Scholes, to solve difficulties with payouts and potential bankruptcies, a paper is published on valuation of option based upon different jump and diffusion process by Cox and Ross [3] . In the following two years, a closed form of the solution for European call options with stochastic volatility is found by Heston [4]. He also explains the relation between the stochastic volatility and the asset price. Kemma and Vost [5] attempt to provide an analytical solution for the arithmetic average option before and over the final time interval, yet they could not succeed to do so. Nonetheless, they prove that average options have a lower value than an equivalent European option by using Monte Carlo simulations. Turnbull and Wakeman [6] investigate the difference between the arithmetic and geometric average option prices and concluded if the averaging period is shorter than the maturity of the option, then it would lead to similar results. Otherwise, there can be difference in prices. Ankudiova and Ehrhardt [7] solved nonlinear Black-Scholes equations numerically. They were interested in various models of the equaiton with volatility depending on several factors. They also studied on the European and American Call options analytically by transforming into convection diffusion equation with non-linear term and the free boundary problem, respectively [7]. Khan et al. [8] suggest some modifications in the Black-Scholes option pricing model on the basis of risk-free interest rate assumption without giving its implementation.

The importance of the model is that it can be possible to create a perfect hedging situation through combining options contracts and the underlying security, assuming that the contracts are priced correctly. Essentially, the aim of the theory introduces that there is only one truly correct price for an option, and that price can be calculated mathematically. On the other hand, options can not be priced correctly all the time since the price can be affected by many factors including demand and supply. Yet, by the Black-Scholes option pricing model it is now possible, theoretically, to determine whether the trading price of an option is higher or lower than its true value: which can in turn highlight potential trading opportunities.

## 1.2 Objective of Thesis

In the financial markets, the Black-Scholes model is an important tool for price options on stocks. This model gives a theoretical prediction of the price of the European style options. Black and Scholes [9] published their model in 1973. The key idea behind their model is to estimate the price of the option over time. The aim of the model is to hedge the option by buying and selling the underlying asset in just the right way and, as a result, to eliminate risk. The options can be defined as a class of derivatives such that the value of a financial asset depends on another asset which is called underlying. One of the important factor and unknown parameter in the Black-Scholes equation is volatility that is the degree of variation of a trading price series over time as measured by the standard deviation of returns. This model is extended to the mathematical insight of the options pricing model by Merton [2] and then the corresponding model began to call as Black-Scholes option pricing model.

Furthermore, to solve the model, Mellin transformation was used by various researchers [10], [11]. Eventually, they approximated the solution of the model by using quadrature approximation. Also, Company and his co-workers [12] proved the solution of the Black-Scholes option pricing model with an extensive class of payoff functions that contain both Dirac delta distributions and ordinary payoff functions with discontinuities of their derivatives. Non-linear Black-Scholes equation was analyzed by Ankudiova and Ehrhardt [13]. They also studied the European call option pricing and the American call option pricing exactly using transformation into a convection-diffusion equation with nonlinear term. In this thesis, we primarily deal with the Black-Scholes model to numerically produce results for the European call option pricing. Then we approximate the model by using explicit and implicit difference methods of various high order schemes. Finally, we see that qualitative and quantitative results are in good agreement with the literature and exact results. The thesis is organized as follows:

In the following of this Chapter, we give some background material for the option pricing in the financial market. We also discuss the model equation by considering

its derivations and discretizations. Additionally, the literature review of the model is given at the end of the chapter.

In Chapter 2, some fundamental numerical methods for the Black-Scholes PDE [9] are introduced. Also various numerical experiments for the stock derivatives are examined.

In Chapter 3, explicit and implicit finite difference methods are implemented to the model equation to solve the discretized system. This chapter discusses the implementation of the explicit and implicit finite difference methods.

In Chapter 4, quantitative and qualitative results have been presented and compared with the literature.

### **1.3 Hypothesis**

In numerical modelling of the Black-Scholes option pricing model, high order methods are more accurate than lower order methods and they are user-friendly methods in order to explain numerical behaviour of the model.

#### **1.3.1 Hedge**

In the financial market, hedge is an investment to reduce the risk of adverse price movements in an asset. In the general case, a hedge consists of taking an offsetting position in a related security, such as a futures contract [14], [15].

Hedging emerges when an individual or institution buys an asset and sells it using a futures contract. They also have access to the asset for a specified amount of time, and can then sell it in the future at a specified price according to the futures contract. In this way, reducing the risk that the future selling price will deviate unexpectedly from the markets current assessment of the future value of the asset and the individual or institution can get the benefit of holding the asset.

### **1.3.2 Derivative**

A derivative is a financial contract that derives its value from its relationship with another underlying assets. The underlying asset can be oil, stocks, bonds, interest rates, gasoline or gold, etc [14]. Financial derivatives are generally described as cash flows that are conditionally stochastically and discounted to present value and the buyer and seller agree on how much the asset price will change over a specific period. Most derivatives require that the agreement is fulfilled. That is accomplished by either by an exchange of the asset, a cash payment, or another agreement that offsets the value of the first.

### **1.3.3 Arbitrage and Risk Free Interest Rate**

Arbitrage is the process of simultaneous purchase and sale of an asset for taking advantage of a price difference between two or more markets. It is also a trade that profits by utilizing the price differences of similar financial stocks on different market places. This term can roughly be defined as the opportunity to buy an asset at a low price then immediately selling it on a different market for a higher price and to profit from these results. As a result of market inefficiencies, arbitrage has been widely used in economics and finance.

In economics and financial markets, return is known as a profit on an investment. It contains any change in value and interest or some cash flows which the investor receives from the investment. At the same time, rate of return is defined as a profit on an investment over a period of time and the time period is preferred as a year, in which case the rate of return is referred to as annual return. So, risk free interest rate is called the rate of return of an investment or hedge without risk of financial loss. This demonstrates the interest that an investor expects from an absolutely risk-free investment over a specific time. More details on this issue can be found in [16], [17].

### 1.3.4 Volatility

A measure of risk based on the standard deviation of the asset return. Volatility is described as a rate at which the price of a security increases or decreases for a given set of returns. It is measured by calculating the standard deviation of the annualized returns over a given period of time. It shows the range to which the price of a security may increase or decrease [14].

### 1.3.5 Options

Options have been paid attention nearly for a century. According to some evidence they have been around for millennia, being used as early as the sixth century B. C. in Ancient Greece. Options are known as a contract. This contract gives the right to the buyer either to sell (in the case of a put option) or buy (in the case of a call option), but not obligation, the specified underlying assets at a specified strike price on or before a certain date. The strike price is the price at which the holder of an option can buy or sell the underlying security when the option is exercised. Hence, the strike price is also known as the exercise price or the striking price [15].

There are two main classes of options: call options and put options. While call option gives the holder the right to buy an asset for a specific price, put option gives the holder the right to sell an asset for a specific price. These prices are called the exercises price,  $E$ . Moreover, the traded kinds of call and put options are generally European and American options. Thus these options are known as "vanilla" options. In the sense of American options, the holder can exercise the option at any time between the date of purchase and the expiration date. However, European options are different from the American options in that the holder may only exercise the option at the time of exercise or expiry time [14].

Even if there are many option pricing models in use, amongst the most common models is the Black-Scholes option pricing model. The importance of this model is that it can be possible to create a perfect hedging situation through combining options con-

tracts and the underlying security, assuming that the contracts are priced correctly [16]. The theory of this topic introduces that there is only one truly correct price for an option, and that price can be calculated mathematically.

### 1.3.6 The Model Problem

Financial option pricing model was first improved by Black-Scholes [9]. They found a parabolic second order partial differential equation for the estimation of an European option with no-arbitrage assumption. Therefore, this section introduces the linear Black-Scholes PDE with boundary and final conditions and also proves the derivation of the equation to compare the exact solution of the model.

### 1.3.7 Linear Black-Scholes Equations

We introduce here the well-known Black-Scholes model. This model can price the option over a specific time interval [18]. One can denote the option value as

$$V(S, R, T, \mu, \sigma, r, t) \equiv V(S, t) \quad (1.1)$$

where  $S(t)$  is the stock price at time  $t$ ;  $E$  is the exercise price;  $T$  is the expiry time and also  $r$ ,  $\sigma$ ,  $\mu$  represent interest rate, volatility of the underlying asset and drift rate, respectively. Two-dimensional linear parabolic equation representing the Black-Scholes model can be given as [9]

$$\frac{\partial V}{\partial t}(S, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, t) + rS \frac{\partial V}{\partial S}(S, t) - rV(S, t) = 0 \quad (1.2)$$

where diffusion coefficient is a function of  $S$ , advection term shows moving  $V$  in a preferred direction and  $rV$  represents a reaction term.

This formula plays an important role in the option pricing theory. The fundamental financial idea behind the equation (1.2) is to perfectly hedge the option by buying and selling the underlying asset.

### 1.3.8 Derivation of the Black-Scholes Model

At maturity, the option value is just the payoff

$$V(S(t), T) = \max\{S(t) - E, 0\}. \quad (1.3)$$

In general, to escape randomness, we use options to construct a portfolio whose value is deterministic. Let  $\Pi$  denotes the portfolio of one long position and a short position in some quantity,  $\Delta$ , of the underlying asset  $S$ :

$$\Pi = V(S, t) - \Delta S \quad (1.4)$$

differentiating  $\Pi$  and using equation (1.2), we reach

$$d\Pi = \Delta dS - dV = \left(\Delta - \frac{\partial V}{\partial S}\right) dS - \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \quad (1.5)$$

to eliminate the term  $dS$  and reduce the randomness to zero, one can choose  $\Delta = \frac{\partial V}{\partial S}$ . Yet, in the case of no-arbitrage, the risk-free interest rate  $r$  can be formed:

$$d\Pi = r\Pi dt \quad (1.6)$$

by eliminating  $d\Pi$  between (1.5) and (1.6), one can reach

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r \left(V - S \frac{\partial V}{\partial S}\right) dt \quad (1.7)$$

or

$$\frac{\partial V}{\partial t}(S, t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, t) + rS \frac{\partial V}{\partial S}(S, t) - rV(S, t) = 0. \quad (1.8)$$

Hence, a deterministic partial differential equation can be obtained by describing an option value  $V(S, t)$ . This equation can be regarded for any option with its payoff depends only on the current price of the underlying asset, or its price at maturity. Detailed discussions on this issue are given in [9].

### 1.3.9 Boundary and Final Conditions

The value of an option must be unique otherwise, the arbitrage chance could increase. So, we need to put some boundary conditions to define the behaviour of the intended

solution at some part of the solution domain. For the study, we concentrate on European call options with the value denoted by  $C(S, t)$  with exercise price  $E$  and expiry date  $T$ . So, options can only be exercised at expiry date and in the final condition at time  $t = T$ . If  $S > E$  at expiration, the option can be exercised and it can be worth  $S - E$  since the buyer of the option can buy stock for  $E$  and immediately sell it for  $S$ . If  $S < E$  at expiration, the option could not be exercised and the option expires worthless [18]. Therefore, the value of a call option at expiry is known as

$$C(S, T) = \max(S - E, 0). \quad (1.9)$$

This is the final condition for the differential equation and becomes initial condition such that

$$C(S, 0) = \max(S - E, 0). \quad (1.10)$$

Since the Black-Scholes differential equation is of the parabolic type, we impose two boundary conditions in  $S$  leading to a unique solution [18]. Thus, let

$$V(S, t) = V_{x_0}(t) \text{ for } S = x_0 \quad (1.11)$$

and

$$V(S, t) = V_{x_1}(t) \text{ for } S = x_1 \quad (1.12)$$

where  $V_{x_0}$  and  $V_{x_1}$  are prescribed functions of  $t$ . Since the equation is a kind of parabolic differential equation, there is also a final condition such as

$$V(S, t) = V_t(S), \quad t = T, \quad (1.13)$$

where  $V_t$  is a known function. Let  $p(t) = T - t$ , which transform the parabolic equation such that

$$\frac{\partial V(S, t)}{\partial t} = \frac{\partial V(S, p(t))}{\partial p(t)} p'(t) = -\frac{\partial V(S, p)}{\partial p}. \quad (1.14)$$

Substituting (1.14) into (1.2) yields

$$\frac{\partial V}{\partial t}(S, t) - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S, t) - rS \frac{\partial V}{\partial S}(S, t) + rV(S, t) = 0. \quad (1.15)$$

Furthermore, the payoff European call option is denoted by  $C(S,t)$  such that

$$\frac{\partial C}{\partial t}(S,t) - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t) - rS \frac{\partial C}{\partial S}(S,t) + rC(S,t) = 0 \quad (1.16)$$

and boundary conditions are defined as [18]

$$\begin{cases} C(0,t) = 0, \\ C(S,t)/S \rightarrow 1, \text{ as } S \rightarrow \infty. \end{cases} \quad (1.17)$$

### 1.3.10 Exact Solution of the Black-Scholes PDE

Due to their well-known advantages, numerical methods are constructed to approach the exact solution of the PDEs. The exact solution for the European call option is given by [9]

$$C(S,t) = N(d_1)S - N(d_2)Ee^{-r(T-t)} \quad (1.18)$$

where

$$d_1 = \frac{\ln(S/E) + (r + \frac{\sigma}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad (1.19)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \quad (1.20)$$

where  $N(\cdot)$  is the distribution function for the standard normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy. \quad (1.21)$$

---

### NUMERICAL METHODS FOR THE BLACK-SCHOLES MODEL

In this chapter, numerical solutions of the Black-Scholes PDE are generated by using various forms of finite difference methods. In general, the first and second order finite difference schemes are preferred in practices [10], [12], [18], [19] and [20]. However, the higher order finite difference schemes are not very common in using for intricate and relatively difficult problems even if they produce more accurate results. Therefore, we give our attention on such schemes. Discretizations of space and time are introduced with respect to asset price and year. Also various finite difference methods with the high order approximations are discussed and the finite difference schemes are discussed in terms of to their sensitivity.

#### 2.1 Spatial and Time Discretizations

The Black-Scholes model is discretized with respect to underlying asset price and time such that  $St$ -plane is divided into equal meshes and approximated with reasonable steps  $\Delta S$  and  $\Delta t$ . Direction of time and space are denoted by  $j$  and  $i$ , respectively such that

$$V_i^j \equiv V(i\Delta S, j\Delta t) \equiv V(S_i, t_j). \quad (2.1)$$

where  $t_j$  denotes the value of the derivative at time step when the underlying asset has value  $S_i$ . More precisely, the interval  $[0, T]$  is divided into  $M+1$  equally spaced grid points  $t_0, t_1, \dots, t_M$  to discretize time with  $t_{j+1} - t_j = \Delta t$  and  $\Delta t = T/M$ .

Since the value of the stock price is always greater than zero,  $S_{max}$  can be taken as  $2S_0$ . Also, the interval  $[0, S_{max}]$  is divided  $N+1$  equally spaced grid points  $S_0, S_1, \dots, S_N$  to

discretize the space with  $S_{i+1} - S_i = \Delta S$  and  $\Delta S = S_{max}/N$ .

The quantities  $V(0, j)$  and  $V(N, j)$  for  $j = 0, 1, \dots, M$  denote the boundary values,  $V(i, 0)$  for  $i = 0, 1, \dots, N$  represents the initial value and  $V(i, j)$  for  $i = 0, 1, \dots, N$  and  $j = 1, 2, \dots, M$  are referred to as interior points on a rectangular region of the  $St$ -plane.

## 2.2 Finite Difference Methods

In the case of the existence of the analytic expression of a function, it is possible to find the value of the desired points of the function, determine the derivative of the function at certain points or calculate the integral for particular ranges. However, it is usually not known value of the function for given some discrete points. Therefore, some numerical approaches have been developed to make such calculations. One of those numerical approaches is developed for the numerical approximations which is one of the most widely used methods. More details on the FDMs can be found in, for instance, references [21], [22] and [23].

## 2.3 Higher Order Approximations in Space

The finite difference methods are divided into two groups according to their sensitivity which are first and second order, and also higher order finite difference schemes. Even through first and second order finite difference methods are more preferred due to their simplicity, higher order approximations are usable for complicated and difficult problems.

The finite difference methods are applied in three different ways: forward, backward and central finite difference methods. By these methods, we introduce first and second order approximations with also higher order finite difference schemes by using Taylor series expansions. For more information, we refer readers to see [21], [22], [23], [24] and [25].

### 2.3.1 Finite Difference Approximations

Reasonable the number of points to calculate the numerical derivatives provides more accurate results for all schemes. Thus, derivation of some finite difference approximations are given.

A finite difference method is commonly preferred in the numerical calculation of the first and second order derivatives, because the amount of the error can be reduced by using more than two points by this method. However, the  $n$ -th order derivative of a given function  $V(S, t)$  must have at least  $n + 1$  points. Therefore, finite difference formulae of the second order derivatives need at least three points. Also, using more points reduces the order of the error.

**Forward Finite Difference Scheme:** For the first order approximation, let  $V(S, t)$  be an analytical function, Taylor series expansion of  $V(S + \Delta S, t)$  at a neighbouring point  $S$  is [21]

$$\begin{aligned} V(S + \Delta S, t) &= V(S, t) + (\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} \\ &+ \frac{(\Delta S)^3}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots + \frac{(\Delta S)^n}{n!} \frac{\partial^n V(S, t)}{\partial S^n} + \dots \\ &= V(S, t) + \sum_{k=1}^{\infty} \frac{(\Delta S)^k}{k!} \frac{\partial^k V(S, t)}{\partial S^k}. \end{aligned} \quad (2.2)$$

Equation (2.2) leads to write

$$\frac{\partial V(S, t)}{\partial S} = \frac{V(S + \Delta S, t) - V(S, t)}{\Delta S} - \Delta S \left[ \frac{1}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} + \frac{\Delta S}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots \right] \quad (2.3)$$

by cutting next terms from the first term on the right side, following forward difference approximation is obtained:

$$\frac{\partial V(S, t)}{\partial S} = \frac{V(S + \Delta S, t) - V(S, t)}{\Delta S} + O(\Delta S) : \quad (2.4)$$

The term  $O(\Delta S)$  in equation (2.4) is called truncation error. By dropping the truncation error, we generally use the common subscript notation as follow:

$$\frac{\partial V(S_i, t_j)}{\partial S} \approx \frac{V_{i+1}^j - V_i^j}{\Delta S}. \quad (2.5)$$

To obtain the forward difference approximation for the second order derivatives, the following Taylor series approximations are generated:

$$V(S + \Delta S, t) = V(S, t) + (\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} + \frac{(\Delta S)^3}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots + \frac{(\Delta S)^n}{n!} \frac{\partial^n V(S, t)}{\partial S^n} + \dots \quad (2.6)$$

$$V(S + 2\Delta S, t) = V(S, t) + (2\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(2\Delta S)^2}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} + \frac{(2\Delta S)^3}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots + \frac{(2\Delta S)^n}{n!} \frac{\partial^n V(S, t)}{\partial S^n} + \dots \quad (2.7)$$

multiplying (2.6) by 2 and then extracting from (2.7) yields that

$$V(S + 2\Delta S, t) - 2V(S + \Delta S, t) = -V(S, t) + (\Delta S)^2 \frac{\partial^2 V(S, t)}{\partial S^2} + (\Delta S)^3 \frac{\partial^3 V(S, t)}{\partial S^3} + \dots \quad (2.8)$$

by equation (2.8), one can obtain forward difference approximation for the second order derivative with the error of the  $\Delta S$  order as follows:

$$\frac{\partial^2 V(S, t)}{\partial S^2} = \frac{V(S + 2\Delta S, t) - 2V(S + \Delta S, t) + V(S, t)}{(\Delta S)^2} + O(\Delta S), \quad (2.9)$$

or more commonly:

$$\frac{\partial^2 V(S, t)}{\partial S^2} \approx \frac{V_{i+2}^j - 2V_{i+1}^j + V_i^j}{(\Delta S)^2}. \quad (2.10)$$

**Backward Finite Difference Scheme:** For the first order derivatives, backward finite difference approximations are similar to forward finite difference approximation. Taylor series expansion of  $V(S - \Delta S, t)$  at a neighbouring point  $S$  is defined as [21]

$$\begin{aligned} V(S - \Delta S, t) &= V(S, t) - (\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} \\ &\quad - \frac{(\Delta S)^3}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots \\ &= V(S, t) + \sum_{k=1}^{\infty} \left[ (-1)^k \frac{(\Delta S)^k}{k!} \right] \frac{\partial^k V(S, t)}{\partial S^k}. \end{aligned} \quad (2.11)$$

If equation (2.11) is solved for  $\frac{\partial V(S, t)}{\partial S}$ , we have

$$\begin{aligned} \frac{\partial V(S, t)}{\partial S} &= \frac{V(S, t) - V(S - \Delta S, t)}{\Delta S} \\ &\quad + \Delta S \left[ \frac{1}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} - \frac{\Delta S}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots \right]. \end{aligned} \quad (2.12)$$

So, the first order backward difference approximation is obtained as follows :

$$\frac{\partial V(S,t)}{\partial S} = \frac{V(S,t) - V(S - \Delta S, t)}{\Delta S} + O(\Delta S), \quad (2.13)$$

or

$$\frac{\partial V(S_i, t_j)}{\partial S} \approx \frac{V_i^j - V_{i-1}^j}{\Delta S}. \quad (2.14)$$

Backward finite difference approximation for the second order derivatives:

$$\begin{aligned} V(S - \Delta S, t) = & V(S, t) - (\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} \\ & - \frac{(\Delta S)^3}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots \end{aligned} \quad (2.15)$$

$$\begin{aligned} V(S - 2\Delta S, t) = & V(S, t) - (2\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(2\Delta S)^2}{2!} \frac{\partial^2 V(S, t)}{\partial S^2} \\ & - \frac{(2\Delta S)^3}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots \end{aligned} \quad (2.16)$$

Multiplying (2.15) by 2 and then extracting from (2.16) yields that

$$\begin{aligned} V(S - 2\Delta S, t) - 2V(S - \Delta S, t) = & -V(S, t) + (\Delta S)^2 \frac{\partial^2 V(S, t)}{\partial S^2} \\ & - (\Delta S)^3 \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{7(\Delta S)^4}{12} \frac{\partial^4 V(S, t)}{\partial S^4} - \dots \end{aligned} \quad (2.17)$$

by equation (2.17), the backward difference approximation for the second order derivative with the order of the error  $\Delta S$  is obtained as follows:

$$\frac{\partial^2 V(S, t)}{\partial S^2} = \frac{V(S - 2\Delta S, t) - 2V(S - \Delta S, t) + V(S, t)}{(\Delta S)^2} + O(\Delta S). \quad (2.18)$$

The common subscript is

$$\frac{\partial^2 V(S, t)}{\partial S^2} \approx \frac{V_{i-2}^j - 2V_{i-1}^j + V_i^j}{(\Delta S)^2}. \quad (2.19)$$

**Central Finite Difference Scheme:** Extracting Taylor series expansion (2.11) from the Taylor series expansion (2.2), we have

$$\begin{aligned} V(S + \Delta S, t) - V(S - \Delta S, t) = & 2(\Delta S) \frac{\partial V(S, t)}{\partial S} \\ & + \frac{2(\Delta S)^3}{3!} \frac{\partial^3 V(S, t)}{\partial S^3} + \dots \end{aligned} \quad (2.20)$$

Dividing equation (2.20) by  $2(\Delta S)$ , one obtains:

$$\frac{\partial V(S,t)}{\partial S} = \frac{V(S+\Delta S,t) - V(S-\Delta S,t)}{2\Delta S} - (\Delta S)^2 \left[ \frac{1}{6} \frac{\partial^3 V(S,t)}{\partial S^3} + \dots \right] \quad (2.21)$$

by equation (2.21), the central difference approximation for the first order derivatives with the order of error  $(\Delta S)^2$  is produced as follows:

$$\frac{\partial V(S,t)}{\partial S} = \frac{V(S+\Delta S,t) - V(S-\Delta S,t)}{2\Delta S} + O((\Delta S)^2). \quad (2.22)$$

Then for the common subscript notation:

$$\frac{\partial V(S_i, t_j)}{\partial S} \approx \frac{V_{i+1}^j - V_{i-1}^j}{2(\Delta S)}. \quad (2.23)$$

To obtain the central finite difference approximation for the second order derivatives, adding equations (2.2) and (2.11) leads to

$$V(S-\Delta S,t) + V(S+\Delta S,t) = 2V(S,t) + (\Delta S)^2 \frac{\partial^2 V(S,t)}{\partial S^2} + \frac{(\Delta S)^4}{12} \frac{\partial^4 V(S,t)}{\partial S^4} + \dots \quad (2.24)$$

By equation (2.24), the central difference approximation for the second order derivatives with the order of error  $(\Delta S)^2$  is found to be

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{V(S+\Delta S,t) - 2V(S,t) + V(S-\Delta S,t)}{(\Delta S)^2} + O((\Delta S)^2). \quad (2.25)$$

or more commonly:

$$\frac{\partial^2 V(S,t)}{\partial S^2} \approx \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{(\Delta S)^2}. \quad (2.26)$$

### 2.3.2 Fourth Order Finite Difference Approximations

Using more than two points in the calculation of the numerical derivatives decrease the order of the error and so that the FD4 formula is expected to provide more accurate results comparison to lower order ones. These kinds of numerical approaches with the order of the error more than two are commonly called the higher order difference schemes. So, the fourth-order finite difference schemes are discussed by using five

points and the Taylor series expansions.

**Forward Finite Difference Scheme:** Firstly, we deal with fourth-order forward finite difference formulae for the first order derivatives with using five points. By taking into account the following equations [21] :

$$\begin{aligned}
V(S + \Delta S, t) &= V(S, t) + (\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
&+ \frac{(\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
&+ \frac{(\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
V(S + 2\Delta S, t) &= V(S, t) + (2\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(2\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
&+ \frac{(2\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(2\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
&+ \frac{(2\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
V(S + 3\Delta S, t) &= V(S, t) + (3\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(3\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
&+ \frac{(3\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(3\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
&+ \frac{(3\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
V(S + 4\Delta S, t) &= V(S, t) + (4\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(4\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
&+ \frac{(4\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(4\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
&+ \frac{(4\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.30}$$

By using equations (2.27) - (2.30), we get [21]:

$$\begin{aligned}
48V(S + \Delta S, t) - 36V(S + 2\Delta S, t) + 16V(S + 3\Delta S, t) \\
- 3V(S + 4\Delta S, t) = 25V(S, t) + 12(\Delta S) \frac{\partial V(S, t)}{\partial S} + O((\Delta S)^5).
\end{aligned} \tag{2.31}$$

Hence, by equation (2.31), the fourth-order forward finite difference formula for the first order derivatives can be described as follows:

$$\begin{aligned} \frac{\partial V(S,t)}{\partial S} &= \frac{-25V(S,t) + 48V(S + \Delta S,t) - 36V(S + 2\Delta S,t)}{12(\Delta S)} \\ &+ \frac{16V(S + 3\Delta S,t) - 3V(S + 4\Delta S,t)}{12(\Delta S)} + O((\Delta S)^4). \end{aligned} \quad (2.32)$$

or

$$\frac{\partial V(S,t)}{\partial S} \approx \frac{-25V_i^j + 48V_{i+1}^j - 36V_{i+2}^j + 16V_{i+3}^j - 3V_{i+4}^j}{12(\Delta S)}. \quad (2.33)$$

Now, we pay our attention to find fourth-order forward finite difference formulae for the second order derivatives by taking five points. Use of equations (2.27) - (2.30) one can reach [21]:

$$\begin{aligned} -104V(S + \Delta S,t) + 114V(S + 2\Delta S,t) - 56V(S + 3\Delta S,t) \\ + 11V(S + 4\Delta S,t) - 35V(S,t) + 12 \frac{\partial^2 V(S,t)}{(\partial S)^2} + O((\Delta S)^5). \end{aligned} \quad (2.34)$$

Hence, by the equation (2.34), the fourth-order forward finite difference formula for the second order derivatives is:

$$\begin{aligned} \frac{\partial^2 V(S,t)}{\partial S^2} &= \frac{35V(S,t) - 104V(S + \Delta S,t) + 114V(S + 2\Delta S,t)}{12(\Delta S)^2} \\ &- \frac{56V(S + 3\Delta S,t) + 11V(S + 4\Delta S,t)}{12(\Delta S)^2} + O((\Delta S)^3). \end{aligned} \quad (2.35)$$

For the common notation:

$$\frac{\partial^2 V(S,t)}{\partial S^2} \approx \frac{35V_i^j - 104V_{i+1}^j + 114V_{i+2}^j - 56V_{i+3}^j + 11V_{i+4}^j}{12(\Delta S)^2} \quad (2.36)$$

**Backward Finite Difference Scheme:** Similar to the forward difference scheme, we firstly deal with the fourth-order backward finite difference formulae for the first order derivatives by using five points [21]. So, we have

$$\begin{aligned} V(S - \Delta S,t) &= V(S,t) - (\Delta S) \frac{\partial V(S,t)}{\partial S} + \frac{(\Delta S)^2}{2} \frac{\partial^2 V(S,t)}{\partial S^2} \\ &- \frac{(\Delta S)^3}{6} \frac{\partial^3 V(S,t)}{\partial S^3} + \frac{(\Delta S)^4}{24} \frac{\partial^4 V(S,t)}{\partial S^4} \\ &- \frac{(\Delta S)^5}{120} \frac{\partial^5 V(S,t)}{\partial S^5} + \dots \end{aligned} \quad (2.37)$$

$$\begin{aligned}
V(S - 2\Delta S, t) = & V(S, t) - (2\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(2\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
& - \frac{(2\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(2\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
& - \frac{(2\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
V(S - 3\Delta S, t) = & V(S, t) - (3\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
& - \frac{(3\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(3\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
& - \frac{(3\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
V(S - 4\Delta S, t) = & V(S, t) - (4\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(4\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
& - \frac{(4\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(4\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
& - \frac{(4\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.40}$$

Use of equations (2.37) - (2.39) gives [21]:

$$\begin{aligned}
-48V(S - \Delta S, t) + 36V(S - 2\Delta S, t) - 16V(S - 3\Delta S, t) \\
+ 3V(S - 4\Delta S, t) = -25V(S, t) + 12(\Delta S) \frac{\partial^2 V(S, t)}{(\partial S)^2} + O((\Delta S)^5).
\end{aligned} \tag{2.41}$$

Hence, by equation (2.41), the fourth-order backward finite difference formula for the first order derivatives is:

$$\begin{aligned}
\frac{\partial V(S, t)}{\partial S} = & \frac{25V(S, t) - 48V(S - \Delta S, t) + 36V(S - 2\Delta S, t)}{12(\Delta S)} \\
& - \frac{16V(S - 3\Delta S, t) + 3V(S - 4\Delta S, t)}{12(\Delta S)} + O((\Delta S)^4).
\end{aligned} \tag{2.42}$$

The following notation is commonly used:

$$\frac{\partial V(S, t)}{\partial S} \approx \frac{25V_i^j - 48V_{i-1}^j + 36V_{i-2}^j - 16V_{i-3}^j + 3V_{i-4}^j}{12(\Delta S)}. \tag{2.43}$$

The fourth-order backward finite difference formula for the second order derivatives is obtained by using of equations (2.37) - (2.40). So one can obtain [21]:

$$\begin{aligned}
& -104V(S - \Delta S, t) + 114V(S - 2\Delta S, t) - 56V(S - 3\Delta S, t) \\
& + 11V(S - 4\Delta S, t) = -35V(S, t) + 12 \frac{\partial V(S, t)}{\partial S} + O((\Delta S)^5).
\end{aligned} \tag{2.44}$$

Hence, by equation (2.44), the fourth-order backward finite difference formula for the second order derivatives is:

$$\begin{aligned}
\frac{\partial^2 V(S, t)}{\partial S^2} &= \frac{35V(S, t) - 104V(S - \Delta S, t) + 114V(S - 2\Delta S, t)}{12(\Delta S)^2} \\
& - \frac{56V(S - 3\Delta S, t) + 11V(S - 4\Delta S, t)}{12(\Delta S)^2} + O((\Delta S)^3).
\end{aligned} \tag{2.45}$$

By the general notation:

$$\frac{\partial^2 V(S, t)}{\partial S^2} \approx \frac{35V_i^j - 104V_{i-1}^j + 104V_{i-2}^j + 114V_{i-2}^j - 56V_{i-3}^j + 11V_{i-4}^j}{12(\Delta S)^2}. \tag{2.46}$$

**Central Finite Difference Scheme:** By taking into consideration of the central difference method for the fourth-order finite difference scheme, we have the following Taylor series expansions [21]:

$$\begin{aligned}
V(S - \Delta S, t) &= V(S, t) - (\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
& - \frac{(\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
& - \frac{(\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
V(S + \Delta S, t) &= V(S, t) + (\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
& + \frac{(\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
& + \frac{(\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.48}$$

$$\begin{aligned}
V(S-2\Delta S, t) &= V(S, t) - (2\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(2\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
&\quad - \frac{(2\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(2\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
&\quad - \frac{(2\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.49}$$

$$\begin{aligned}
V(S+2\Delta S, t) &= V(S, t) + (2\Delta S) \frac{\partial V(S, t)}{\partial S} + \frac{(2\Delta S)^2}{2} \frac{\partial^2 V(S, t)}{\partial S^2} \\
&\quad + \frac{(2\Delta S)^3}{6} \frac{\partial^3 V(S, t)}{\partial S^3} + \frac{(2\Delta S)^4}{24} \frac{\partial^4 V(S, t)}{\partial S^4} \\
&\quad + \frac{(2\Delta S)^5}{120} \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.50}$$

Use of equations (2.47) -(2.49) yields [21]:

$$\begin{aligned}
V(S-2\Delta S, t) - 8V(S-\Delta S, t) + 8V(S+\Delta S, t) - V(S+2\Delta S, t) \\
= 12(\Delta S) \frac{\partial V(S, t)}{\partial S} - \frac{6}{15} (\Delta S)^3 \frac{\partial^5 V(S, t)}{\partial S^5} + \dots
\end{aligned} \tag{2.51}$$

Hence, by equation (2.51), the fourth-order central finite difference formula for the first order derivatives is:

$$\begin{aligned}
\frac{\partial V(S, t)}{\partial S} &= \frac{V(S-2\Delta S, t) - 8V(S-\Delta S, t)}{12(\Delta S)} \\
&\quad + \frac{8V(S+\Delta S, t) - V(S+2\Delta S, t)}{12(\Delta S)} + O((\Delta S)^4).
\end{aligned} \tag{2.52}$$

or commonly notation is:

$$\frac{\partial V(S, t)}{\partial S} \approx \frac{V_{i-2}^j - 8V_{i-1}^j + 8V_{i+1}^j - V_{i+2}^j}{12(\Delta S)}. \tag{2.53}$$

To obtain the fourth-order central finite difference formulae for the second order derivatives, firstly multiplying equations (2.47) and (2.48) by 16 and then extracting them from the summation of equations (2.49) and (2.50) gives:

$$\begin{aligned}
16V(S+\Delta S, t) + 16V(S-\Delta S, t) - V(S-2\Delta S, t) - V(S+\Delta S, t) \\
= 30V(S, t) + 12(\Delta S)^2 \frac{\partial^2 V(S, t)}{\partial S^2} - \frac{2}{15} (\Delta S)^6 \frac{\partial^6 V(S, t)}{\partial S^6} + \dots
\end{aligned} \tag{2.54}$$

Hence, by equation (2.54), the fourth-order central finite difference formula for the second order derivatives is :

$$\begin{aligned} \frac{\partial^2 V(S,t)}{\partial S^2} = & \frac{-V(S-2\Delta S,t) + 16V(S-\Delta S,t)}{12(\Delta S)^2} \\ & \frac{-30V(S,t) + 16V(S+\Delta S,t) - V(S+2\Delta S,t)}{12(\Delta S)^2} + O((\Delta S)^4). \end{aligned} \quad (2.55)$$

For the common subscript notation:

$$\frac{\partial^2 V(S,t)}{\partial S^2} \approx \frac{-V_{i-2}^j + 16V_{i-1}^j - 30V_i^j + 16V_{i+1}^j - V_{i+2}^j}{12(\Delta S)^2}. \quad (2.56)$$

### 2.3.3 Sixth Order Finite Difference Approximations

As previously mentioned, higher order difference methods are generally applied to the solution of the difficult problems which require more accuracy. Therefore, increasing the number of points provide more accurate results. In this section, higher order finite difference formulae are introduced by using seven points for the sixth order finite difference schemes such that first order approximation for the sixth order finite difference scheme is given by [27]

$$\frac{\partial V(S,t)}{\partial S} \approx \frac{-V_{i-3}^j + 9V_{i-2}^j - 45V_{i-1}^j + 45V_{i+1}^j - 9V_{i+2}^j + V_{i+3}^j}{60(\Delta S)} \quad (2.57)$$

and the sixth order finite difference scheme for the second order derivatives is [26]:

$$\frac{\partial^2 V(S,t)}{\partial S^2} \approx \frac{2V_{i-3}^j - 27V_{i-2}^j + 270V_{i-1}^j - 490V_i^j + 270V_{i+1}^j - 27V_{i+2}^j + 2V_{i+3}^j}{180(\Delta S)^2}. \quad (2.58)$$

## 2.4 First Order Approximation in Time

In this section, we first approximate first order derivative in time by using forward difference method. So, a forward difference approximation for the time derivative is given by

$$\frac{\partial V(S,t)}{\partial t} = \frac{V(S,t+\Delta t) - V(S,t)}{\Delta t} + O(\Delta t) \quad (2.59)$$

or commonly:

$$\frac{\partial V(S,t)}{\partial t} \approx \frac{V_i^{j+1} - V_i^j}{\Delta t}. \quad (2.60)$$

## 2.5 Result

In this chapter, some difference schemes are summarized by using three, five and seven points. These approaches are utilized to obtain numerical solutions of the model equation in the following sections. Also, solution of the model problem is obtained by using seven-point scheme. Since they are similar to above formulae, these approaches are not provided in this section. However, sixth order difference formulae by using seven points and fourth order finite difference formulae by using five points on the interior and boundary points are presented in Appendices-A and B, respectively [21], [26] and [27].



### NUMERICAL IMPLEMENTATION OF THE BLACK-SCHOLES MODEL

The main purpose of this chapter is to show that approximate solution of the Black-Scholes economic model (1.2) for the European call option (1.16) can be found by using various numerical methods. So, the Euler method is regarded as one of these numerical methods to understand and implement to the Black-Scholes PDE (1.2). This method includes explicit and implicit schemes related to the model problem. During this chapter, simple explicit and complicated implicit methods by using second and higher order finite difference schemes are implemented to the pricing of vanilla European call option. While the implicit method finds approximate solution of the discretized Black-Scholes equation (1.2) by solving a matrix system, explicit method evaluates the discretized equation (1.16) by the new value of each grid point independently for each time step. For more details, we refer readers to [28], [29] and [30].

#### 3.1 Explicit Finite Difference Formulations for the Model Equation

Considering the Black-Scholes economic model (1.2), the vanilla European call option (1.16) is discretized such as derivatives with respect to  $S(t)$  by second and high-order finite difference schemes and derivative with respect to time by forward difference scheme. Then rearranging the model equation as an explicit scheme gives one of the approximate solution of the model problem. Applying the explicit method to the model problem requires implementations.

### 3.1.1 Second Order Finite Difference Scheme

As a first attempt to solve the model equation (1.2), second order explicit method is used to obtain the approximate solution of the interested problem. So, the European Call Option (1.16) is analyzed by applying explicit scheme to the discretized derivatives such that a forward finite difference approximation (2.60) is used for the time derivative. Also, central finite difference approximation (2.23) and symmetric central finite difference approximation (2.26) are used for the first and second order derivatives,  $S$ , respectively. So, we have [18]

$$\begin{aligned} \frac{V_i^{j+1} - V_i^j}{\Delta t} - a(S, t) \left( \frac{V_{i+1}^j - 2V_i^j + V_{i-1}^j}{(\Delta S)^2} \right) \\ - b(S, t) \left( \frac{V_{i+1}^j - V_{i-1}^j}{2(\Delta S)} \right) + c(S, t)V_i^j \\ = O(\Delta t, (\Delta S)^2) \end{aligned} \quad (3.1)$$

where  $a(S, t) = \frac{1}{2}\sigma^2 S^2$ ,  $b(S, t) = rS$  and  $c(S, t) = r$ . By rearranging the difference equation (3.1) and put all  $j$  terms on the right side, we obtain

$$\begin{aligned} V_i^j = (v_1 a_i^j - \frac{1}{2} v_2 b_i^j) V_{i-1}^j + (1 - 2v_1 a_i^j + \Delta t c_i^j) V_i^j \\ + (v_1 a_i^j + \frac{1}{2} v_2 b_i^j) V_{i+1}^j + O(\Delta t^2, \Delta t \Delta S^2) \end{aligned} \quad (3.2)$$

where  $v_1 = \frac{\Delta t}{\Delta S^2}$ ,  $v_2 = \frac{\Delta t}{\Delta S}$  and  $O(\Delta t^2, \Delta t \Delta S^2)$  is called the local truncation error.

Moreover, equation (3.2) holds for  $i = 1, \dots, N-1$  since  $V_{-1}^j$  and  $V_{N+1}^j$  are not defined. Thus, there are  $N+1$  unknowns. The remaining two equations come from two boundary equations (1.17) on  $n = 0$  and  $n = N$ .

### 3.1.2 Sixth Order Finite Difference Scheme

In this section, sixth-order finite difference scheme for the approximate solution of the Black-Scholes option pricing model (1.2) is taken into consideration by using explicit method. Keeping the European call option (1.16) by using equations (2.60), (2.57) and

(2.58), one can obtain that

$$\begin{aligned}
& \frac{V_i^{j+1} - V_i^j}{\Delta t} - a(S, t) \left( \frac{2V_{i-3}^j - 27V_{i-2}^j + 270V_{i-1}^j}{180(\Delta S)^2} \right) \\
& - a(S, t) \left( \frac{-490V_i^j + 270V_{i+1}^j - 27V_{i+1}^j + 2V_{i+3}^j}{180(\Delta S)^2} \right) \\
& - b(S, t) \left( \frac{-V_{i-3}^j + 9V_{i-2}^j - 45V_{i-1}^j}{60(\Delta S)} \right) \\
& - b(S, t) \left( \frac{+45V_{i+1}^j - 9V_{i+2}^j + V_{i+3}^j}{60(\Delta S)} \right) \\
& + c(S, t)V_i^j + O(\Delta t^2, \Delta t(\Delta S)^2).
\end{aligned} \tag{3.3}$$

Rearranging difference equation (3.3), one reaches

$$\begin{aligned}
V_i^{j+1} = & \left( v_1 a_i^j \frac{2}{180} - v_2 b_i^j \frac{1}{60} \right) V_{i-3}^j + \left( -v_1 a_i^j \frac{27}{180} + v_2 b_i^j \frac{9}{60} \right) V_{i-2}^j \\
& + \left( v_1 a_i^j \frac{270}{180} - v_2 b_i^j \frac{45}{60} \right) V_{i-1}^j + \left( 1 - v_1 a_i^j \frac{490}{180} + c_i^j \Delta t \right) V_i^j \\
& + \left( v_1 a_i^j \frac{270}{180} + v_2 b_i^j \frac{45}{60} \right) V_{i+1}^j + \left( -v_1 a_i^j \frac{27}{180} - v_2 b_i^j \frac{9}{60} \right) V_{i+2}^j \\
& + \left( v_1 a_i^j \frac{2}{180} + v_2 b_i^j \frac{1}{60} \right) V_{i+3}^j + O(\Delta t^2, \Delta t(\Delta S)^2).
\end{aligned} \tag{3.4}$$

Similar to the second-order explicit method finite difference approximation, the higher-order finite difference equation (3.4) holds for  $i = 1, \dots, N-1$  and remaining equations are provided by the two boundary conditions (1.17) on  $n = 0$  and  $n = N$ .

### 3.2 Implicit Finite Difference Formulations for the Model Equation

The goal of this section is to introduce the implicit method by considering second and higher order schemes. This method is implimented to the Black-Scholes equation (1.2) to compute solution of the European call option pricing model (1.16). Thus, the equations are involved the current state of system and later one.

### 3.2.1 Second Order Finite Difference Scheme

Now, second-order implicit method is used to obtain the one of the approximate solution of the interested problem. Taking the payoff European call option (1.16), we have

$$\frac{\partial C}{\partial t}(S,t) - a(S,t) \frac{\partial^2 C}{\partial S^2}(S,t) - b(S,t) \frac{\partial C}{\partial S}(S,t) + c(S,t)C(S,t) = 0. \quad (3.5)$$

Combining equations (2.23), (2.26) and (2.60), one can obtain [18]

$$\begin{aligned} \frac{V_i^{j+1} - V_i^j}{\Delta t} - a_i^{j+1} \left( \frac{V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}}{\Delta S^2} \right) \\ - b_i^{j+1} \left( \frac{V_{i+1}^{j+1} - V_{i-1}^{j+1}}{2\Delta S} \right) + c_i^j V_i^{j+1} = 0. \end{aligned} \quad (3.6)$$

or

$$\begin{aligned} V_i^j = (-v_1 a_i^{j+1} + \frac{1}{2} v_2 b_i^{j+1}) V_{i-1}^{j+1} + (1 + 2v_1 a_i^{j+1} + \Delta t c_i^{j+1}) V_i^{j+1} \\ - (v_1 a_i^{j+1} + \frac{1}{2} v_2 b_i^{j+1}) V_{i+1}^{j+1}. \end{aligned} \quad (3.7)$$

Using the following notions:

$$l_i = \frac{1}{2} dt(-\sigma^2 i^2 + ri), \quad i = 0, \dots, N-2,$$

$$d_i = 1 + dt(\sigma^2 i^2 + r), \quad i = 1, \dots, N-1,$$

$$u_i = -\frac{1}{2} dt(\sigma^2 i^2 + ri) \quad i = 2, \dots, N,$$

equation (3.7) becomes

$$V_i^j = l_i V_{i-1}^{j+1} + d_i V_i^{j+1} + u_i V_{i+1}^{j+1} \quad (3.8)$$

for  $i = 1, \dots, N-1$  and  $j = 1, \dots, M-1$ . This approach can be formulated to obtain a linear algebraic system:

$$AV^{j+1} = b^j, \quad j = 0, \dots, M-1 \quad (3.9)$$

where  $A \in \mathbb{R}^{(N-1) \times (N-1)}$ ;  $V^{j+1}, V^j, b^j \in \mathbb{R}^{N-1}$ , such that

$$A = \begin{pmatrix} d_1 & u_2 & 0 & \cdots & 0 \\ l_1 & d_2 & u_3 & & \vdots \\ 0 & l_2 & & & \\ & & & \ddots & 0 \\ \vdots & \ddots & \ddots & & u_{N-1} \\ 0 & \cdots & l_{N-2} & d_{N-1} & \end{pmatrix}$$

and

$$V^{j+1} = \begin{pmatrix} V_1^{j+1} \\ V_2^{j+1} \\ \vdots \\ V_{N-2}^{j+1} \\ V_{N-1}^{j+1} \end{pmatrix}, b^j = \begin{pmatrix} V_1^j \\ V_2^j \\ \vdots \\ V_{N-2}^j \\ V_{N-1}^j \end{pmatrix} - \begin{pmatrix} l_0 V_0^{j+1} \\ 0 \\ \vdots \\ 0 \\ u_N V_N^{j+1} \end{pmatrix}.$$

At each time step  $j + 1$  the approximate solution can be obtained by solving the system of algebraic linear equations. Thus, at each time step, the implicit method provides a set of equations which must be solved simultaneously for all  $i = 1, 2, \dots, N - 1$ . The values  $V_i^0, V_0^j, V_N^j$  with  $i = 0, \dots, N$  and  $j = 0, \dots, M$  are known from initial and boundary conditions.

### 3.2.2 Fourth Order Finite Difference Scheme

In the previous section, to reach accurate solution of the Black-Scholes PDE, second-order implicit method is illustrated. However, fourth-order accurate solutions can be more preferable to obtain more faster approximation with fewer grid points for the same accuracy. Thus, the goal of this section is to discretize the Black-Scholes model in space and in time by combining equations (2.53), (2.56) and (2.60), one can solve the resulting linear algebraic system for each time step.

Consider the general form of the European call option (1.16), using the first and second

order derivatives with (2.53) and (2.56) respectively, one obtains [18]

$$\begin{aligned} \frac{V_i^{j+1} - V_i^j}{\Delta t} - a(S, t) \left( \frac{-V_{i-2}^{j+1} + 16V_{i-1}^{j+1} - 30V_i^{j+1} + 16V_{i+1}^{j+1} - V_{i+2}^{j+1}}{12(\Delta S)^2} \right) \\ - b(S, t) \left( \frac{V_{i-2}^{j+1} - 8V_{i-1}^{j+1} + 8V_{i+1}^{j+1} - V_{i+2}^{j+1}}{12\Delta S} \right) \\ + c(S, t)C(S, t) = 0 \end{aligned} \quad (3.10)$$

and the difference equation (3.10) can be rearranged as

$$V_i^j = \alpha_1 V_{i-2}^{j+1} + \alpha_2 V_{i-1}^{j+1} + \alpha_3 V_i^{j+1} + \alpha_4 V_{i+1}^{j+1} + \alpha_5 V_{i+2}^{j+1} \quad (3.11)$$

where

$$\alpha_1 = \Delta t \left( \frac{1}{24} \sigma^2 i^2 - \frac{1}{12} ri \right), \quad (3.12)$$

$$\alpha_2 = \Delta t \left( \frac{-16}{24} \sigma^2 i^2 + \frac{8}{12} ri \right),$$

$$\alpha_3 = 1 + \Delta t \left( \frac{30}{24} \sigma^2 i^2 + r \right),$$

$$\alpha_4 = \Delta t \left( \frac{-16}{24} \sigma^2 i^2 - \frac{8}{12} ri \right), \quad (3.13)$$

$$\alpha_5 = \Delta t \left( \frac{1}{24} \sigma^2 i^2 + \frac{1}{12} ri \right).$$

The fourth-order approximation requires special operation for the first two points at the left boundary and the last two points at the right boundary. Therefore, solutions are produced by using approximations (3.7) for  $i = 1, N + 1$  as well as approximations (3.11) for the interior points. Thus, the system of equation can be written as



sixth order finite difference formulae (2.57) and (2.58) for the space derivatives as well as difference formula (2.60) for the time derivative. So, by the general form of the European call option (1.16), one can provide that [18]

$$\begin{aligned} & \frac{V_i^{j+1} - V_i^j}{\Delta t} - a(S, t) \left( \frac{2V_{i-3}^j - 27V_{i-2}^{j+1} + 270V_{i-1}^{j+1} - 490V_i^{j+1} + 270V_{i+1}^{j+1}}{180(\Delta S)^2} \right) \\ & + a(S, t) \left( \frac{-27V_{i+2}^{j+1} + 2V_{i+3}^{j+1}}{180(\Delta S)^2} \right) - b(S, t) \left( \frac{-V_{i-3}^{j+1} + 9V_{i-2}^{j+1} - 45V_{i-1}^{j+1}}{60\Delta S} \right) \\ & + b(S, t) \left( \frac{+45V_{i+1}^{j+1} - 9V_{i+2}^{j+1} + V_{i+3}^{j+1}}{60\Delta S} \right) + c(S, t)C(S, t) = 0 \end{aligned}$$

and the difference equation can be rearranged as

$$V_i^j = \beta_1 V_{i-3}^{j+1} + \beta_2 V_{i-2}^{j+1} + \beta_3 V_{i-1}^{j+1} + \beta_4 V_i^{j+1} + \beta_5 V_{i+1}^{j+1} + \beta_6 V_{i+2}^{j+1} + \beta_7 V_{i+3}^{j+1} \quad (3.16)$$

where

$$\beta_1 = \Delta t \left( \frac{-\sigma^2 i^2}{180} + \frac{ri}{60} \right), \quad (3.17)$$

$$\beta_2 = \Delta t \left( \frac{27\sigma^2 i^2}{360} + \frac{-9ri}{60} \right),$$

$$\beta_3 = \Delta t \left( \frac{-270\sigma^2 i^2}{360} + \frac{45ri}{60} \right),$$

$$\beta_4 = 1 + \Delta t \left( \frac{490\sigma^2 i^2}{360} + r \right),$$

$$\beta_5 = \Delta t \left( \frac{-270\sigma^2 i^2}{360} + \frac{-45ri}{60} \right), \quad (3.18)$$

$$\beta_6 = \Delta t \left( \frac{27\sigma^2 i^2}{360} + \frac{9ri}{60} \right),$$

$$\beta_7 = \Delta t \left( \frac{-\sigma^2 i^2}{360} - \frac{ri}{60} \right).$$

Similar to the fourth order convergence, the sixth order approximation also requires some operations for the first three points at the left boundary and the last three points at the right boundary. So, approximate solutions are produced by using approximations (3.8) for  $i = 1, N + 1$  and (3.11) for  $i = 2, N$  as well as (3.16) approximations for the



### 3.3 Result

Various algebraic equations of the model in terms of the difference schemes are introduced by using second and higher order ones through the explicit and implicit methods. These formulations are provided by using central and forward finite differences in space derivatives and in time derivative respectively. Also, space derivatives are regarded by spatial and higher order approximations to obtain more accurate results. Since model equation (1.16) could not be produced by using central difference approximation on the boundaries, forward and backward finite difference approaches are taking into consideration on the boundary points. For each equation, derivation of the difference formulations on the boundaries are presented in Appendices A and B, respectively [21], [26] and [27].

### NUMERICAL ILLUSTRATIONS

The finite difference solutions of the European call option pricing are presented and compared with the exact solutions and with available studies in the literature [20]. Following figures indicate the model problem by using both explicit and implicit version of the FD2, FD4, and FD6 schemes.

#### 4.1 Numerical Exhibitions of the Explicit Methods

The explicit finite difference formulae for the Black-Scholes PDE are illustrated by the following figures and tables for the parameters  $r = 0.05$ ,  $\sigma = 0.2$ ,  $M = 1000$ ,  $N = 60$ ,  $E = 100$ ,  $S_{max} = 300$ . Using formula (3.2) with initial and boundary conditions of the model equation, the second order finite difference approximation to the European call option (1.16) is presented on Figure 4.1-4.2. Also by formula (3.4), the sixth order finite difference approximation to the model equation is introduced on Figure 4.3-4.4.

In continuation of this section, quantitative results are exhibited and for comparison purposes with the literature [20] and exact results, the FD2 and FD6 results have been presented as seen in Tables 4.1-4.3. As clearly seen from the tables, it is revealed that the currently produced results are in good agreement with the exact results and the literature.

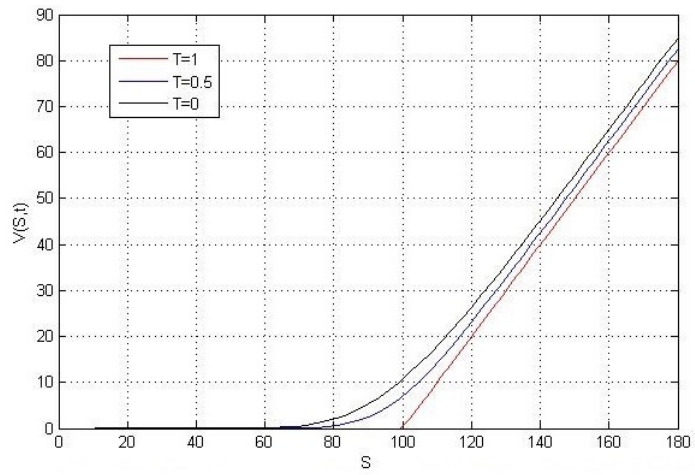


Figure 4.1 The European Call Option the Explicit Method, FD2

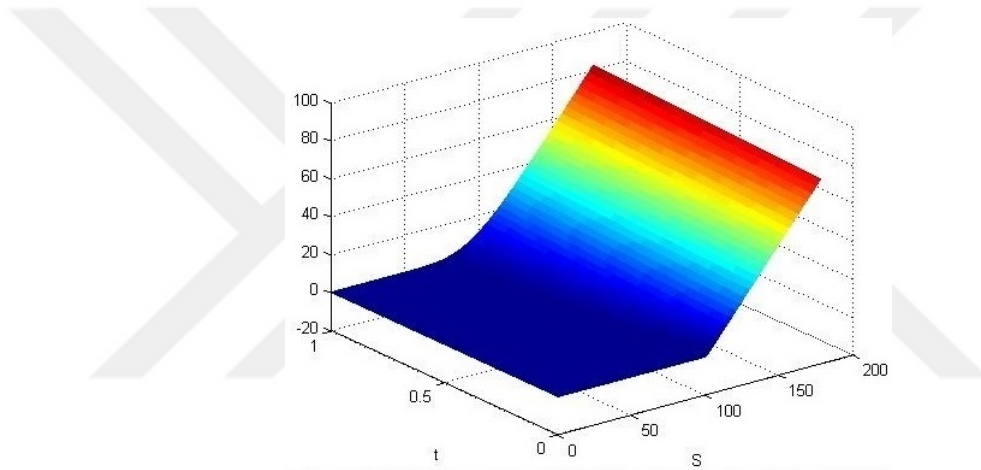


Figure 4.2 The European Call Option the Explicit Method, FD2

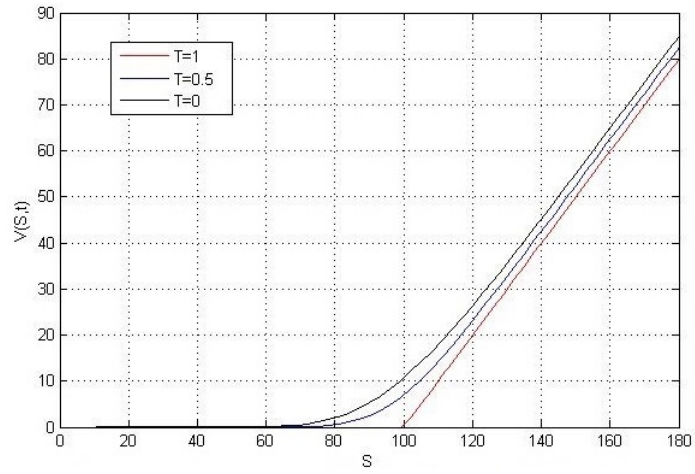


Figure 4.3 The European Call Option the Explicit Method, FD6

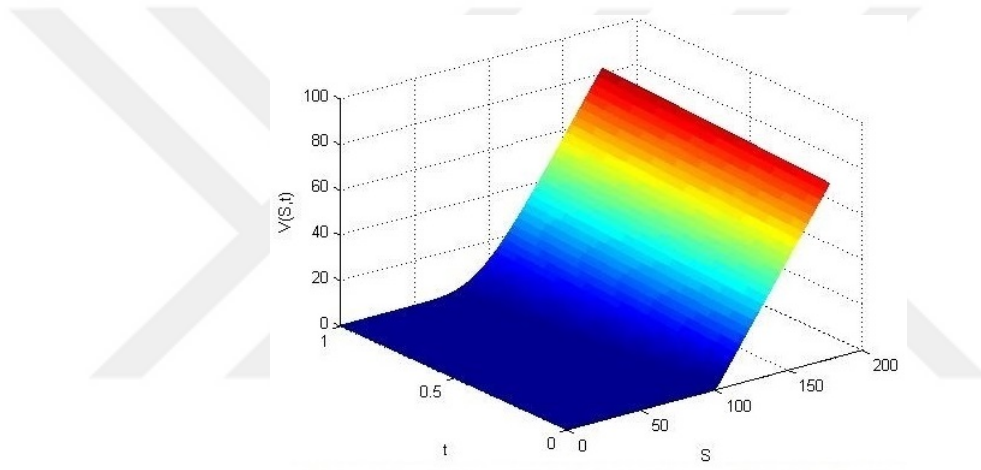


Figure 4.4 The European Call Option the Explicit Method, FD6

Table 4.1 Comparison of numerical and analytical solutions for S=120

	Time	Analytic	Daound and Oziş (2011) [19]	FD2	FD6	AE2	AE6	RE2	RE6
1	0	20	20	20	20	0	0	0	0
2	0.111	20.5586	20.554	20.5624	20.5560	0.0038	0.0027	1.82E-04	1.31E-04
3	0.222	21.1794	21.105	21.1815	21.1696	0.0021	0.0098	9.92E-05	4.62E-04
4	0.333	21.8672	21.652	21.8609	21.8511	0.0063	0.0161	2.86E-04	7.35E-04
5	0.444	22.8587	22.197	22.5721	22.5659	0.0136	0.0198	6.00E-04	8.76E-04
6	0.556	23.3198	23.744	23.3010	23.2980	0.0187	0.0218	8.03E-04	9.33E-04
7	0.667	24.0453	23.283	24.0231	24.0226	0.0222	0.0227	9.22E-04	9.44E-04
8	80.778	24.7635	23.819	24.7390	24.7404	0.0244	0.0231	9.86E-04	9.31E-04
9	0.8890	25.4718	24.352	25.4489	25.4487	0.0259	0.0231	0.0010	9.05E-04
10	1	26.1690	24.882	26.1422	26.1462	0.0269	0.0229	0.0010	8.74E-04

Table 4.2 Comparison of numerical and analytical solutions for S=180

	Time	Daound and Oziş (2011) [19]	FD2	FD6	Analytic	AE2	AE6	RE2	RE6
1	0	80	80	80	80	0	0	0	0
2	0.111	80.5535	80.5534	80.5534	80.5535	1.37E-05	1.37E-05	1.71E-07	1.71E-07
3	0.222	81.1039	81.1038	81.1038	81.1039	2.75E-05	2.74E-05	3.39E-07	3.38E-07
4	0.333	81.6512	81.6512	81.6512	81.6512	4.75E-05	4.02E-05	5.82E-07	4.92E-07
5	0.444	82.1955	82.1954	82.1955	82.1955	1.18E-04	4.63E-05	1.43E-06	5.64E-07
6	0.556	82.7418	83.7414	82.7418	82.7418	3.46E-04	3.26E-05	4.18E-06	3.94E-07
7	0.667	83.2804	83.2795	83.2804	83.2804	8.54E-04	1.28E-05	1.02E-05	1.54E-07
8	0.778	83.8164	83.8147	83.8165	83.8164	0.0017	9.56E-05	2.08E-05	1.14E-06
9	0.889	84.3502	84.3472	84.3504	84.3502	0.0031	2.15E-04	3.63E-05	2.55E-06
10	1	84.8821	84.8773	84.8825	84.8821	0.0048	3.70E-04	5.69E-05	4.36E-06

Table 4.3 Comparison of numerical and analytical solutions for S=200

	Time	Daound and Oziş (2011) [19]	FD2	FD6	Analytic	AE2	AE6	RE2	RE6
1	0	100	100	100	100	0	0	0	0
2	0.111	100.5535	100.553	100.553	100.553	1.38E-05	1.37E-05	1.37E-07	1.37E-07
3	0.222	101.1039	101.103	101.103	101.103	4.17E-05	2.55E-05	4.12E-07	2.52E-07
4	0.333	101.6512	101.650	101.651	101.651	2.82E-04	3.57E-06	2.77E-06	3.52E-08
5	0.444	102.1955	102.194	102.195	102.195	0.0012	1.23E-04	1.16E-05	1.20E-06
6	0.556	102.7417	102.738	102.742	102.741	0.0031	4.03E-04	3.00E-05	3.92E-06
7	0.667	103.28	103.274	103.280	103.280	0.0060	8.28E-04	5.77E-05	8.02E-06
8	0.778	103.8154	103.805	103.816	103.815	0.0097	0.0014	9.32E-05	1.32E-05
9	0.889	104.3479	104.333	104.349	104.347	0.0140	0.0020	1.34E-04	1.92E-05
10	1	104.8777	104.858	104.880	104.877	0.0188	0.0027	1.79E-04	2.56E-05

## 4.2 Numerical Exhibitions of the Implicit Methods

The current computer codes for the implicit methods are produced in the programming language MATLAB R2010a and then shown in this section through second and higher order schemes. Following figures and tables demonstrate results of the model equation (1.16) for the parameters  $r = 0.05$ ,  $\sigma = 0.2$ ,  $M = 1000$ ,  $N = 120$ ,  $E = 100$ ,  $S_{max} = 300$ . Use of second-order (3.7), fourth-order (3.11) and sixth-order (3.16) implicit finite difference formulae, one can obtain the following Figures 4.5-4.10.

Moreover, quantitative results are shown in Tables 4.4-4.6 for comparison of numerical and exact solutions of the European call option (1.16) with the literature [20]. Hence, it makes clear that the obtained results through FD2, FD4 and FD6 are in good agreement with the analytical results and the literature.

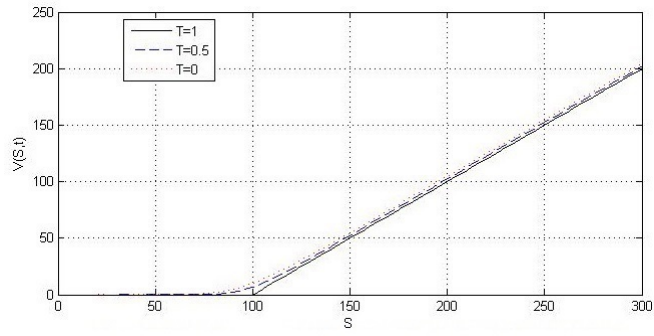


Figure 4.5 The European Call Option the Implicit Method, FD2

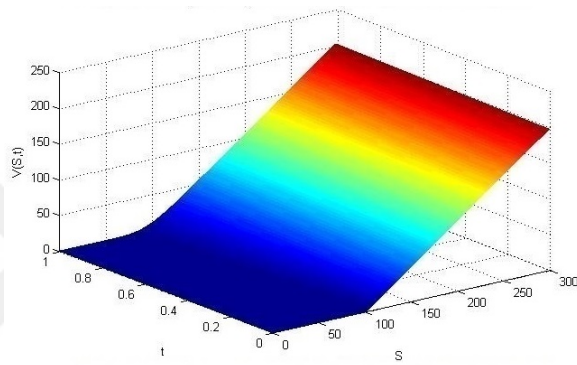


Figure 4.6 The European Call Option the Implicit Method, FD2

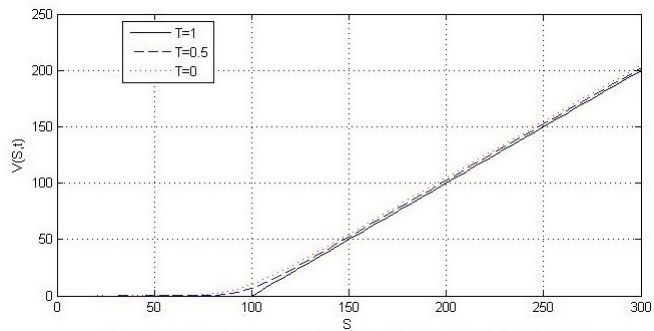


Figure 4.7 The European Call Option the Implicit Method, FD4

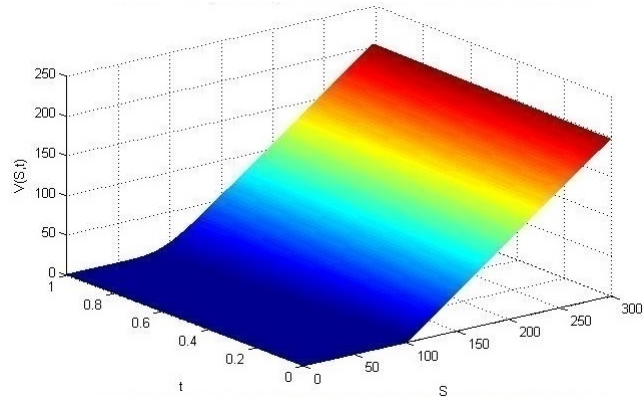


Figure 4.8 The European Call Option the Implicit Method, FD4

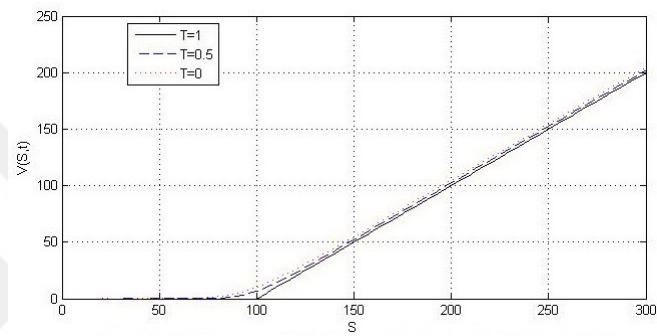


Figure 4.9 The European Call Option the Implicit Method, FD6

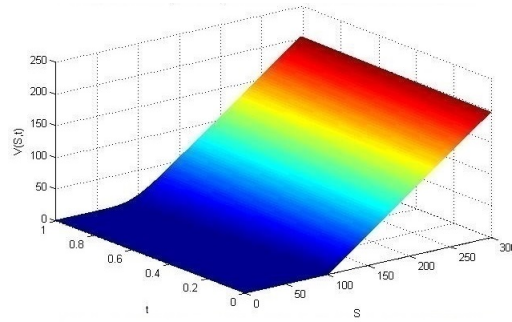


Figure 4.10 The European Call Option the Implicit Method, FD6

Table 4.4a Comparison of numerical and analytical solutions for S=120

	Time	Exact	Daound and Ozis(201 1) [19]	FD2	FD4	FD6
1	0	20	20	20	20	20
2	0.111	20.558	20.554	20.562	20.556	20.557
3	0.222	21.179	21.105	21.181	21.169	21.176
4	0.333	21.867	21.652	21.860	21.851	21.860
5	0.444	22.585	22.197	22.572	22.565	22.582
6	0.556	23.319	22.744	23.301	23.298	23.309
7	0.667	24.045	23.283	24.023	24.022	24.039
8	0.776	24.763	23.819	24.739	24.740	24.756
9	0.889	25.471	24.352	25.445	25.448	25.466
10	1	26.169	24.882	26.142	26.146	26.150

Table 4.4b Continuation of Table 4.4a

	AE2	AE4	AE6	RE2	RE4	RE6
1	0	0	0	0	0	0
2	0.0038	0.0027	0.0011	1.83E-04	1.21E-04	5.35E-05
3	0.0021	0.0098	0.0032	9.93E-05	4.62E-04	1.51E-04
4	0.0063	0.0161	0.0068	2.85E-04	7.35E-04	3.11E-04
5	0.0136	0.0198	0.0035	6.00E-04	8.76E-04	1.54E-04
6	0.0187	0.0218	0.0110	8.03E-04	9.33E-04	4.71E-04
7	0.0222	0.0227	0.0562	9.22E-04	9.44E-04	2.33E-04
8	0.0244	0.0231	0.0009	9.86E-04	9.31E-04	3.63E-05
9	0.0259	0.0231	0.0126	1.00E-03	9.05E-04	4.94E-04
10	0.0269	0.0229	0.0205	1.00E-03	8.74E-04	7.83E-04

Table 4.5a Comparison of numerical and analytical solutions for S=180

	Time	Exact	Daound and Ozis(201 1) [19]	FD2	FD4	FD6
1	0	80	80	80	80	80
2	0.111	80.5535	80.5553	80.5534	80.5534	80.5535
3	0.222	81.1039	81.1075	81.1038	81.1038	81.1038
4	0.333	81.6512	81.6567	81.6512	81.6512	81.6512
5	0.444	82.1955	82.2029	82.1954	82.1995	82.1955
6	0.556	82.7418	82.7511	82.7414	82.7418	82.7418
7	0.667	83.2804	83.2912	83.2795	83.2804	83.2804
8	0.776	83.8164	83.8284	83.8147	83.8165	83.8164
9	0.889	84.3502	84.3627	84.3472	84.3504	84.3502
10	1	84.8821	84.8941	84.8773	84.8825	84.882

Table 4.5b Continuation of Table 4.5a

	AE2	AE4	AE6	RE2	RE4	RE6
1	0	0	0	0	0	0
2	1.37E-05	1.37E-05	1.37E-05	1.71E-07	1.71E-07	1.71E-07
3	2.75E-05	2.74E-05	2.73E-05	3.39E-07	3.38E-07	3.30E-07
4	4.75E-05	4.02E-05	4.01E-05	5.82E-07	4.92E-07	4.92E-07
5	1.18E-04	4.63E-05	4.60E-05	1.43E-06	5.64E-07	5.40E-08
6	3.46E-04	3.26E-05	3.20E-05	4.18E-06	3.94E-07	2.01E-08
7	8.54E-04	1.28E-05	1.28E-05	1.02E-05	1.54E-07	1.5E-08
8	1.70E-03	9.56E-05	9.46E-05	2.08E-05	1.14E-06	1.10E-07
9	3.10E-03	2.15E-04	1.95E-04	3.62E-05	2.55E-06	1.63E-07
10	4.80E-03	3.70E-04	4.02E-05	5.69E-05	4.36E-06	4.3E-07

Table 4.6a Comparison of numerical and analytical solutions for S=200

	Time	Exact	Dound and Oziş(2011) [19]	FD2	FD4	FD6
1	0	100	100	100	100	100
2	0.111	100.5535	100.5537	100.534	100.5534	100.5535
3	0.222	101.1039	101.1083	101.1038	101.1038	101.1039
4	0.333	101.6512	101.658	101.6509	101.6512	101.6512
5	0.444	102.1955	102.2046	102.1944	102.1957	102.1956
6	0.556	102.7417	102.7531	102.7386	102.7421	102.7419
7	0.667	103.28	103.2937	103.2741	103.2809	103.2801
8	0.776	103.8154	103.8314	103.8057	103.8168	103.8155
9	0.889	104.3479	104.3661	104.3339	104.3499	104.3479
10	1	104.8777	104.8979	104.8589	104.8804	104.8777

Table 4.6b Continuation of Table 4.6a

	AE2	AE4	AE6	RE2	RE4	RE6
1	0	0	0	0	0	0
2	1.38E-05	1.37E-05	1.36E-05	1.37E-07	1.37E-07	1.37E-08
3	4.17E-05	2.55E-05	2.56E-06	4.12E-07	2.52E-07	2.52E-08
4	2.82E-04	3.57E-06	3.55E-06	2.77E-06	3.52E-08	3.52E-09
5	1.20E-03	1.23E-04	1.01E-05	1.16E-05	1.20E-06	1.32E-07
6	3.10E-03	4.03E-04	4.02E-05	3.00E-03	3.92E-06	3.01E-08
7	6.00E-03	8.28E-04	8.28E-05	5.77E-05	8.02E-06	8.00E-07
8	9.70E-03	1.40E-03	1.00E-04	9.32E-05	1.32E-05	1.32E-06
9	1.40E-02	2.00E-03	1.97E-04	1.34E-04	1.95E-05	1.99E-06
10	1.88E-02	2.70E-03	2.07E-04	1.79E-04	2.56E-05	2.03E-06

### RESULTS AND DISCUSSION

This study has concentrated on successfully solving the Black-Scholes economic model for the European call option. To achieve this, various forms of explicit and implicit finite difference schemes have been used. To compute the results, computer programs have been produced in MATLAB R2010a. The qualitative and quantitative results have been seen to be in good agreement with the literature and exact results.

Traditional advantages and disadvantages of the implicit and explicit methods have been observed. Lower and higher order schemes through implicit and explicit methods have been discussed in detail in discovering the numerical behaviour of the Black-Scholes model.

As a conclusion, the current study carried out a good starting and guiding for graduate and undergraduate students who are interested in the related fields of the present study. Finally, this study has been a great encourage for fresh researchers.

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SEVEN-POINT FINITE DIFFERENCE APPROXIMATIONS

For boundary point  $i = 1$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-147V_i + 360V_{i+1} - 450V_{i+2} + 400V_{i+3} - 225V_{i+4})}{60(\Delta S)} + \frac{(72V_{i+5} - 10V_{i+6})}{60(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(812V_i - 3132V_{i+1} + 5265V_{i+2} - 5080V_{i+3} + 2970V_{i+4})}{180(\Delta S)^2} + \frac{(-972V_{i+5} + 137V_{i+6})}{180(\Delta S)^2}.$$

For boundary point  $i = 2$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-10V_{i-1} - 77V_i + 150V_{i+1} - 100V_{i+2} + 50V_{i+3})}{60(\Delta S)} + \frac{(-15V_{i+4} + 2V_{i+5})}{60(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(137V_{i-1} - 1472V_i - 255V_{i+1} + 470V_{i+2} - 285V_{i+3})}{180(\Delta S)^2} + \frac{(93V_{i+4} - 13V_{i+5})}{180(\Delta S)^2}.$$

For boundary point  $i = 3$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(2V_{i-2} - 24V_{i-1} - 35V_i + 80V_{i+1} - 30V_{i+2})}{60(\Delta S)} + \frac{(8V_{i+3} - V_{i+4})}{60(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(-13V_{i-2} + 228V_{i-1} - 420V_i + 200V_{i+1} + 15V_{i+2})}{180(\Delta S)^2} + \frac{(-12V_{i+3} + 2V_{i+4})}{180(\Delta S)^2}.$$

For interior points  $i = 4, 5, \dots, N-3$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-V_{i-3} + 9V_{i-2} - 45V_{i-1} + 45V_{i+1} - 9V_{i+2} + V_{i+3})}{60(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(2V_{i-3} - 27V_{i-2} + 270V_{i-1} - 490V_i + 270V_{i+1})}{180(\Delta S)^2} + \frac{(-27V_{i+2} + 2V_{i+3})}{180(\Delta S)^2}.$$

For boundary point  $i = N-2$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(V_{i-4} - 8V_{i-3} + 30V_{i-2} - 80V_{i-1} + 35V_i)}{60(\Delta S)} + \frac{(24V_{i+1} - 2V_{i+2})}{60(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(2V_{i-4} + 12V_{i-3} + 15V_{i-2} + 200V_{i-1} - 420V_i)}{180(\Delta S)^2} + \frac{(228V_{i+1} - 13V_{i+2})}{180(\Delta S)^2}.$$

For boundary point  $i = N-1$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-2V_{i-5} + 15V_{i-4} - 50V_{i-3} + 100V_{i-2} - 150V_{i-1})}{60(\Delta S)} + \frac{(77V_i + 10V_{i+1})}{60(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(-13V_{i-5} + 93V_{i-4} - 285V_{i-3} + 470V_{i-2} - 255V_{i-1})}{180(\Delta S)^2} + \frac{(-147V_i + 137V_{i+1})}{180(\Delta S)^2}.$$

For boundary point  $i = N$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(10V_{i-6} - 72V_{i-5} + 255V_{i-4} - 400V_{i-3} + 450V_{i-2})}{60(\Delta S)} + \frac{(-360V_{i-1} + 147V_i)}{60(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(137V_{i-6} - 972V_{i-5} + 297V_{i-4} - 5080V_{i-3})}{180(\Delta S)^2} + \frac{(5265V_{i-2} - 3132V_{i-1} + 812V_i)}{180(\Delta S)^2}.$$



**FIVE-POINT FINITE DIFFERENCE APPROXIMATIONS**

For boundary point  $i = 1$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-257V_i + 48V_{i+1} - 36V_{i+2} + 16V_{i+3} - 3V_{i+4})}{12(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(35V_i - 104V_{i+1} + 114V_{i+2} - 56V_{i+3} + 11V_{i+4})}{12(\Delta S)^2}.$$

For boundary point  $i = 2$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-3V_{i-1} - 10V_i + 18V_{i+1} - 6V_{i+2} + V_{i+3})}{12(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(11V_{i-1} - 20V_i + 6V_{i+1} + 4V_{i+2} - V_{i+3})}{12(\Delta S)^2}.$$

For interior points  $i = 3, 4, \dots, N - 2$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-V_{i-2} - 8V_{i-1} + 8V_{i+1} - V_{i+2})}{12(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(-V_{i-2} + 16V_{i-1} - 30V_i + 16V_{i+1} - V_{i+2})}{12(\Delta S)^2}.$$

For boundary point  $i = N - 1$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(-V_{i-3} + 6V_{i-2} - 18V_{i-1} + 10V_i + 3V_{i+1})}{12(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(-V_{i-3} + 4V_{i-2} + 6V_{i-1} - 20V_i + 11V_{i+1})}{12(\Delta S)^2}.$$

For boundary point  $i = N$ ,

$$\frac{\partial V(S,t)}{\partial S} = \frac{(3V_{i-4} - 16V_{i-3} + 36V_{i-2} - 48V_{i-1} + 25V_i)}{12(\Delta S)},$$

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{(-11V_{i-4} - 56V_{i-3} + 114V_{i-2} - 104V_{i-1} + 35V_i)}{12(\Delta S)^2}.$$



## CURRICULUM VITAE

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1. SARI, M. and YILDIRIM, T (2016). “Numerical Methods for the Linear Black-Scholes Option Pricing Model”, Computational Sciences Research, İstanbul-Turkey
2. SARI, M. and YILDIRIM, T (2016). “Numerical Methods for Pricing European Call Options Using the Black-Scholes Economic Model”, 24th International Conference on Applied and Industrial Mathematics CAIM, Craiova-Romania.

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