

**FUZZY AND ROBUST OPTIMIZATION APPROACHES TO
CAPITAL RATIONING AND CAPITAL BUDGETING WITH
SEVERAL UNCERTAINTIES**

**Ph.D. Thesis by
Esra Baş, Dipl.-Ing., MBA**

Department : Industrial Engineering

Programme : Industrial Engineering

NOVEMBER 2008

**FUZZY AND ROBUST OPTIMIZATION APPROACHES TO
CAPITAL RATIONING AND CAPITAL BUDGETING WITH
SEVERAL UNCERTAINTIES**

**Ph.D. Thesis by
Esra Bař, Dipl.-Ing., MBA
(507042115)**

Date of submission : 29 July 2008

Date of defence examination : 18 November 2008

Supervisor (Chairman): Prof. Dr. Cengiz KAHRAMAN

Members of the Examining Committee Prof.Dr. Nahit SERARSLAN (İTÜ)

Assoc.Prof.Dr. Tijen ERTAY (İTÜ)

Prof.Dr. Ziya ULUKAN (GSÜ)

Asst. Prof.Dr. Tufan DEMİREL (YTÜ)

NOVEMBER 2008

**ÇEŞİTLİ BELİRSİZLİKLERİN OLDUĞU SERMAYE
PAYLAŞTIRIMI VE SERMAYE BÜTÇELEME PROBLEMLERİNE
BULANIK VE DİRENÇLİ OPTİMİZASYON YAKLAŞIMLARI**

DOKTORA TEZİ
Dipl.-Ing., MBA Esra BAŞ
(507042115)

Tezin Enstitüye Verildiği Tarih : 29 Temmuz 2008
Tezin Savunulduğu Tarih : 18 Kasım 2008

Tez Danışmanı : Prof.Dr. Cengiz KAHRAMAN

Diğer Jüri Üyeleri Prof.Dr. Nahit SERARSLAN (İ.T.Ü.)

Doç.Dr. Tijen ERTAY (İ.T.Ü.)

Prof.Dr. Ziya ULUKAN (G.S.Ü.)

Yrd.Doç.Dr. Tufan DEMİREL (Y.TÜ.)

KASIM 2008

ACKNOWLEDGMENT

Many people have contributed to this Ph.D. thesis. First, I would like to thank thesis committee members Prof. Dr. Nahit Serarslan and Prof. Dr. Ziya Ulukan for their questions and interest. I also thank Assoc.Prof. Tijen Ertay and Asst. Prof. Tufan Demirel for their attendance to the defense, and their contributions. Special thanks to my Ph.D. thesis advisor Prof.Dr. Cengiz Kahraman for listening to my ideas very carefully, and encouraging me to study with enthusiasm.

Prof. Soulaymane Kachani at Columbia University Industrial Engineering and Operations Research (IEOR) department is also gratefully acknowledged for the discussions regarding “Robust Optimization Approach to Capital Rationing and Capital Budgeting” in Chapter 4 during my stay at IEOR department as a Fulbright visiting scholar.

I definitely thank all my friends at Istanbul Technical University Industrial Engineering department, since their friendship makes workplace meaningful and fun. I also thank all my Professors at Industrial Engineering department for creating a friendly environment for all of us.

Finally, I would like to thank my husband Kemal Baş, my family, and the family of my husband for their continuous support.

November 2008

Esra BAŞ

TABLE OF CONTENTS

ABBREVIATIONS	v
LIST OF TABLES	vi
LIST OF FIGURES	vii
LIST OF SYMBOLS	ix
SUMMARY	x
ÖZET	xi
1. INTRODUCTION	1
2. FUZZY MATHEMATICAL PROGRAMMING AND ROBUST LINEAR PROGRAMMING	6
2.1. Fuzzy Mathematical Programming and Fuzzy Relations	6
2.1.1. Basics of fuzzy mathematics	6
2.1.2. A taxonomy of fuzzy ranking methods, t-norm and t-conorm fuzzy relations and their properties	11
2.1.3. Fuzzy linear programming	15
2.1.3.1. Primal model formulations	15
2.1.3.2. Dual model formulations	20
2.1.4. Fuzzy quadratic programming	23
2.1.4.1. Primal model formulation	23
2.1.4.2. Dual model formulation	24
2.2. Robust Linear Programming	25
3. FUZZY LINEAR PROGRAMMING AND FUZZY QUADRATIC PROGRAMMING APPROACHES TO CAPITAL RATIONING AND CAPITAL BUDGETING	28
3.1. Fuzzy Capital Rationing Models	28
3.1.1. Uncertainty-based approaches to capital rationing models	28
3.1.1.1. Stochastic capital rationing model	29
3.1.1.2. Robust optimization approach to capital rationing	30
3.1.2. Three fuzzy capital rationing models	31
3.1.2.1. Fuzzy capital rationing model of Chiu and Park	32
3.1.2.2. Chance-constrained programming approach to capital rationing	38
3.1.2.3. Mean-variance model for fuzzy capital budgeting	47
3.2. Fuzzy Capital Rationing Model with t-norm Fuzzy Relation	51
3.2.1. The primal model and the dual model	51
3.2.2. Sensitivity analysis	61
3.2.3. Complementary slackness analysis	69
3.2.4. Computational analysis	70
3.3. Internal Rate of Return of a Simple Project with PV-Equation Based on t-norm and t-conorm Fuzzy Relation	75

3.3.1. Preliminaries	75
3.3.2. Fuzzy approach to investment appraisal techniques	76
3.3.3. Formulation of PV-equation with fuzzy cash flows	77
3.3.4. A numerical example for a simple project	79
3.4. Fuzzy quadratic programming: A case of Bernhard's capital budgeting model	82
3.4.1. Two primal-dual model pairs for quadratic programming with fuzzy parameters	82
3.4.2. Bernhard's general model	91
3.4.3. Fuzzification of Bernhard's model with quadratic objective function	92
3.4.4. Computational analysis	101
4. ROBUST LINEAR PROGRAMMING APPROACH TO CAPITAL RATIONING AND CAPITAL BUDGETING	112
4.1. Extensions to Robust Optimization Approach to Capital Rationing	112
4.1.1. Analysis of robust counterpart of capital rationing model	119
4.1.1.1. Characterization of level of investment in projects	120
4.1.1.2. Characterization of $\hat{a}_{nj}$ and $\hat{M}_n$	122
4.1.2. Computational analysis	124
4.2. Extensions to Robust Optimization Approach to Horizon Capital Budgeting	130
4.2.1. Analysis of robust counterpart of horizon capital budgeting model	134
4.2.1.1. Characterization of level of investment in projects	135
4.2.1.2. Characterization of the budget dual variables as forward rates	136
4.2.1.3. Characterization of $\hat{a}_{nj}$ and $\hat{M}_n$	137
4.2.1.4. Characterization of v_n^* and w_n^*	138
4.2.2. Computational analysis	139
4.3. Robust Decision Rules of Investment Appraisal Techniques for Simple Projects	147
4.3.1. Preliminaries	147
4.3.2. Robust decision rules	148
4.3.2.1. NPV robust decision rules	148
4.3.2.2. IRR robust decision rules for simple projects	155
5. CONCLUSIONS	160
REFERENCES	166
APPENDIXES	171
CURRICULUM VITAE	239

ABBREVIATIONS

B/C Ratio	: Benefit-Cost Ratio
Ch	: Chance
Cr	: Credibility
FLP	: Fuzzy Linear Programming
IRR	: Internal Rate of Return
KKT	: Karush-Kuhn-Tucker
LB	: Lower Bound
LIFO	: Last in First out
LP	: Linear Programming
MARR	: Minimum Attractive Rate of Return
Nec	: Necessity
NPV	: Net Present Value
Pos	: Possibility
PV	: Present Value
t-conorm	: Triangular Conorm
t-norm	: Triangular Norm
TFN	: Triangular Fuzzy Number
TrFN	: Trapezoidal Fuzzy Number

LIST OF TABLES

	<u>Page</u>
Table 3.1	A three-project and three-period example with the nominal values 35
Table 3.2	A three-project and three-period example with the fuzzy cash flows, budget limits, and discount rates 36
Table 3.3	A three-project and three-period example with the fuzzy cash flows...43
Table 3.4	A three-project and three-period example with the fuzzy cash flows and budget limits.....49
Table 3.5	Level of investment in projects and sensitivity analysis 71
Table 3.6	A three-period project example with symmetric triangular fuzzy numbers representing the net cash flows, and MARR.....79
Table 3.7	Results for α – efficient t-norm fuzzy-relation based IRR at 5 th iteration 79
Table 3.8	Results for α – efficient t-conorm fuzzy-relation based IRR at 5 th iteration 80
Table 3.9	Results for α – efficient t-norm and t-conorm fuzzy relation based MARR 80
Table 3.10	Cash flows and budget limits for a ten-project example.....101
Table 3.11	Defuzzified interval of terminal wealth in quadratic programming model.....105
Table 3.12	Defuzzified average dividend in quadratic programming model.....110
Table 4.1	Summary of the results for robust optimization approach to pure capital rationing model 124
Table 4.2	A four-project, six-period horizon capital budgeting example with different lending and borrowing rates 140
Table 4.3	Summary of the results for robust optimization approach to horizon capital budgeting model 140
Table 4.4	NPV robust decision rules.....148
Table 4.5	Min and max NPV values 150
Table 4.6	IRR robust decision rules 155
Table 4.7	Min/max IRR values 158
Table A.1	Xpress-MP Code for fuzzy capital rationing model 172
Table A.2	Xpress-MP code for Bernhard’s model with t-norm and t-conorm fuzzy relation.....177
Table A.3	Defuzzified interval of terminal wealth in quadratic programming model.....191
Table A.4	Defuzzified average dividend in quadratic programming model.....205
Table B.1	Xpress-MP Code for robust optimization approach to capital rationing 211
Table B.2	Xpress-MP Code for robust optimization approach to horizon capital budgeting.....221

LIST OF FIGURES

	<u>Page</u>
Figure 2.1 : Triangular fuzzy number, $\tilde{A}$	8
Figure 2.2 : Trapezoidal fuzzy number, $\tilde{A}$	9
Figure 2.3 : L-R fuzzy number, $\tilde{A}$	10
Figure 2.4 : Taxonomy of fuzzy ranking methods	11
Figure 2.5 : Classification of fuzzy linear programming problems	15
Figure 3.1 : Branch-and-bound flow chart for fuzzy capital rationing model	33
Figure 3.2 : Branch-and-bound method for the numerical example of fuzzy capital rationing model proposed in [16].....	38
Figure 3.3 : Branch-and-bound method for the numerical example of chance-constrained programming approach to capital rationing	47
Figure 3.4 : Branch-and-bound method for the numerical example of fuzzy mean-variance model.....	51
Figure 3.5 : Level of investment in Project 3	74
Figure 3.6 : Triangular fuzzy numbers representing IRR and MARR.....	81
Figure 3.7 : Sample fuzzy numbers representing terminal wealth in quadratic programming model.....	102
Figure 3.8 : Sample fuzzy numbers representing terminal wealth in LP model ..	104
Figure 3.9 : Sample fuzzy numbers representing average dividend in quadratic programming model.....	107
Figure 3.10 : Sample fuzzy numbers representing average dividend in LP model	108
Figure 4.1 : Level of investment in Project 1	126
Figure 4.2 : Level of investment in Project 3	128
Figure 4.3 : Level of investment in Project 1	142
Figure 4.4 : Level of investment in Project 2	143
Figure 4.5 : Level of investment in Project 3	145
Figure 4.6 : Level of investment in Project 4	146
Figure 4.7 : Min/max NPV values of staying indifferent areas at 20% variability rate for Project 1	151
Figure 4.8 : Min/max NPV values of staying indifferent areas at 20% variability rate for Project 3	151
Figure 4.9 : Acceptability areas of a project for 5%, 10%, 15% variability rates	158
Figure A.1 : Level of investment in Project 3	173
Figure A.2 : Fuzzy numbers representing terminal wealth in quadratic programming model.....	181
Figure A.3 : Fuzzy numbers representing terminal wealth in LP model	187
Figure A.4 : Fuzzy numbers representing average dividend in quadratic programming model.....	195
Figure A.5 : Fuzzy numbers representing average dividend in LP model.....	201
Figure B.1 : Level of investment in Project 1	213
Figure B.2 : Level of investment in Project 3	217
Figure B.3 : Level of investment in Project 1	223
Figure B.4 : Level of investment in Project 2	227

Figure B.5	: Level of investment in Project 3	231
Figure B.6	: Level of investment in Project 4	235

LIST OF SYMBOLS

$\tilde{A}$	: Fuzzy set
$\mu_{\tilde{A}}$	: Membership function of a fuzzy set $\tilde{A}$
T	: T-norm
S	: T-conorm
a_{nj}	: Net cash flow for project j at period n
M_n	: Budget limit at period n
x_j	: Project selection variable for project j
r	: Discount rate or marginal cost of capital

FUZZY AND ROBUST OPTIMIZATION APPROACHES TO CAPITAL RATIONING AND CAPITAL BUDGETING WITH SEVERAL UNCERTAINTIES

SUMMARY

Capital budgeting problems are linear or non-linear programming models that address the solution of the level of investment in projects by considering basically budget limits. Additional constraints such as borrowing limit constraint or scarce resource constraint are also possible.

In this thesis, we study the uncertainty in different capital rationing and capital budgeting problems by considering fuzzy and robust optimization approaches. Fuzzy mathematical programming includes the defuzzification of a fuzzy model by using a fuzzy relation, and analysis and solution of the model by using deterministic mathematical programming techniques. On the other hand, robust optimization approach includes the best solution of the model by considering the worst realization of the uncertain parameters. Although both approaches are devoted to the solution of the models by considering uncertainty, a model solved by fuzzy mathematical programming does not necessarily have to be robust.

In this thesis, we first examine the fuzzy mathematical programming approaches to the Lorie-Savage capital rationing model in the literature, and illustrate the models with the numerical examples. In the other sections devoted to the fuzzy optimization, we propose the defuzzification of Weingartner's pure capital rationing model by triangular norm (t-norm) fuzzy relation and Bernhard's model by t-norm and triangular conorm (t-conorm) fuzzy relations and their detailed analysis. We also propose the decision rules for internal rate of return (IRR) in case of cash flow uncertainty. In the chapters devoted to robust optimization, we extend the robust optimization approach to Weingartner's pure capital rationing and horizon capital budgeting models proposed in the literature by considering additional parameters as uncertain. We also redefine the decision rules of investment appraisal techniques for simple projects analogous to robust optimization. All model propositions are accompanied by computational analysis, which proves the applicability and practicality of the proposed models.

ÇEŞİTLİ BELİRSİZLİKLERİN OLDUĞU SERMAYE PAYLAŞTIRIMI VE SERMAYE BÜTÇELEME PROBLEMLERİNE BULANIK VE DİRENÇLİ OPTİMİZASYON YAKLAŞIMLARI

ÖZET

Sermaye bütçeleme problemleri özellikle bütçe kısıtları dikkate alındığında belirli sayıdaki yatırım seçeneklerine sermaye paylaşım oranlarının incelendiği doğrusal ya da doğrusal olmayan programlama problemleridir. Bütçe kısıtları dışında borç alma ya da kısıtlı kaynak gibi kısıtlar da dikkate alınabilir.

Bu tezde, çeşitli sermaye bütçeleme problemlerindeki belirsizlik bulanık ve dirençli optimizasyon yaklaşımları dikkate alınarak incelenmiştir. Bulanık matematiksel programlama, bulanık bir modelin belirli bir bulanık bağıntı kullanılarak netleştirilmesini ve netleştirmenin ardından standart matematiksel programlama teknikleri ile incelenmesini ve çözülmesini içerir. Dirençli optimizasyon yaklaşımı ise, bir problemdeki belirsiz parametrelerin olabilecek en kötü gerçekleşme durumunda bile bu problemin eniyi çözümü verecek şekilde çözülmesini içerir. Her iki yaklaşım da belirsizliği modellemek için kullanılmasına rağmen, bulanık matematiksel programlama yaklaşımı ile çözülen bir problemin dirençli bir sonuç vermesi gerekmemektedir.

Bu tezde, başlangıçta Lorie-Savage sermaye paylaşım problemi literatürdeki bulanık matematiksel programlama yaklaşımları incelenmiş ve nümerik analizlerle inceleme ayrıntılandırılmıştır. Tezin bulanık optimizasyona ayrılan diğer bölümleri ise, Weingartner'ın saf sermaye paylaşımı doğrusal programlama modelinin t-norm bulanık bağıntısı ve Bernhard'ın ikinci dereceden programlama modelinin t-norm ve t-conorm bulanık bağıntıları dikkate alınarak bulanık yaklaşımla modellenmesine ve ayrıntılı incelenmesine ayrılmıştır. Ayrıca nakit akışlarındaki belirsizlik durumunda iç verim oranı karar kuralları oluşturulmuştur. Tezin dirençli optimizasyona ayrılan bölümünde ise, Weingartner'ın saf sermaye paylaşımı ve planlama ufku modeli için literatürde önerilen dirençli optimizasyon yaklaşımları, diğer bazı parametrelerin de belirsiz varsayılmasıyla genişletilmiştir. Ayrıca bir basit proje için yatırım değerlendirme karar kuralları da dirençli yaklaşımla yeniden tanımlanmıştır. Bütün önermelere eşlik eden nümerik analizler, önerilen modellerin uygulanabilirliğini göstermektedir.

1. INTRODUCTION

The following thesis has been devoted to two main research areas: 1. Fuzzy linear programming (FLP) and fuzzy quadratic programming approach to capital rationing and capital budgeting, including fuzzy formulations of investment appraisal techniques for simple projects. 2. Robust linear programming approach to capital rationing and capital budgeting, including robust formulations of investment appraisal techniques for simple projects.

The Lorie-Savage model is a capital rationing model for computing the level of investment in projects with fixed capital budget for each period, and with no permission for borrowing or lending, while capital budgeting problems may also include borrowing and lending permission. Several capital rationing and capital budgeting models have been proposed in the literature (for example, see [1,2, p: 279-350]). Weingartner applied linear programming (LP) to the Lorie-Savage problem as in [1, p: 16-38]. Weingartner's horizon model with borrowing and lending permission has been extended by Bernhard who also considered dividends and terminal wealth in a nonlinear objective function [2, p: 305].

Traditionally, the parameters in capital rationing and capital budgeting models have been assumed to be deterministic. But in real world applications, the parameters are unlikely to be certain. Some approaches with this concern to address uncertainty in capital rationing and capital budgeting models have been proposed in the literature.

Assuming that the cash flows and budgets follow a predetermined probability distribution has been one of the ways of modelling the uncertainty in capital rationing and capital budgeting problems. [3,4] have proposed models as an extension of Weingartner's capital rationing model by adding stochastic constraints and penalties for infeasibility. However, modelling capital rationing and capital budgeting problems with stochastic programming may generally lead to computational intractability, while it may be hard to predict the accurate probability distribution for cash flows or budgets as discussed in [5].

Robust optimization approach was applied to capital rationing and capital budgeting models by [5]. Robust optimization was also applied to portfolio selection problems (for example, see [6-8]). Robust optimization was originally proposed by [9]. [9] assumed the coefficients of the constraints to belong to a convex set, considered column-wise uncertainty and a worst-case scenario in the sense that a solution set with the worst-case realizations of each coefficient of the constraints was considered. Thereafter, several papers have appeared in the literature, especially to point out the topics of uncertainty set and over-conservatism in robust optimization methodology. [10] studied semidefinite programs with “unknown-but-bounded” uncertainty set for the coefficients. [11] proposed to partition the coefficients of each constraint into certainty set and uncertainty set to alleviate the over-conservatism in [9]. They assume the uncertainty as “unknown-but-bounded” or “random symmetric” for LP models. [12] proposed a linear robust counterpart model of a linear model where the robustness of the model can be tuned based on predefined parameters. Recently, [13] proposed adjustable robust solutions of uncertain linear programs to consider non-adjustable (here-and-now) decision variables which should be determined before the realization of the actual data and adjustable (wait-and-see) decision variables which may be determined after the realization of the actual data.

Fuzzy set theory as means of modelling imprecision and vagueness has also been applied to capital rationing and capital budgeting problems and fuzzy formulations of investment appraisal techniques. [14] studied fuzzy future value and fuzzy present value by considering cash flow, time period, and interest rate as fuzzy variables. [15] studied payback period method, IRR method, and benefit-cost (B/C) ratio method with fuzzy variables. [16] modelled the Lorie-Savage capital rationing model as a fuzzy 0-1 integer programming model by considering budget limits, periodic net cash flows, and discount rates as triangular fuzzy numbers (TFNs), and proposed the branch-and-bound method for the solution. [16] also proposed so-called “weighted method” to compare the fuzzy numbers, and discussed other fuzzy ranking methods in the literature. [17] applied chance-constrained programming to the Lorie-Savage model with fuzzy cash outflows and fuzzy net cash flows, and [18] applied chance-constrained programming to the Lorie-Savage problem with random fuzzy cash outflows and net cash flows, and proposed so-called “fuzzy simulation based genetic algorithm” and “hybrid intelligent algorithm” for the solution, respectively. [19]

proposed fuzzy mean-variance model for the Lorie-Savage model, and fuzzy simulation based genetic algorithm for the solution of the model.

Since capital rationing models and capital budgeting models can be LP models, the literature review regarding FLP is also needed. Several papers have appeared in the literature with either fully fuzzified or partially fuzzified FLP models. [20] studied a fully fuzzified linear program, discussed the notion of fuzzy inequality and maximization of an objective function, and proposed an evolutionary algorithm. Partially fuzzified models include models with fuzzy objective function coefficients or fuzzy constraints or constraint matrix with fuzzy coefficients. [21] proposed an FLP model with several objective functions. [22] proposed a general model for FLP with fuzzy constraints, constraint matrix with fuzzy coefficients, and fuzzy right-hand sides. In addition to the papers with primal models, the papers related to the dual approach in an FLP have also appeared in the literature. [23] proposed a dual approach to solve the FLP problem. [24] examined satisficing solutions and duality in interval and fuzzy linear programming, and [25] analyzed the duality in an FLP model with possibility and necessity relations. In FLP models, the solution differs according to the fuzzy ranking method selected. [26] provided a taxonomy of fuzzy ranking methods, and examined the fuzzy ranking methods in detail, while different papers discussed the advantages and drawbacks of the fuzzy ranking methods (for example, see [24]).

Quadratic programming is very important in its own right, while its significance in capital budgeting problems has also been emphasized in the literature (for example, see [27]). However, the literature in quadratic programming with fuzzy modelling for modelling uncertainty has relatively been immature. [28, p: 75-78] reviewed symmetric and non-symmetric quadratic programming models with fuzzy objective function, fuzzy constraints, and fuzzy parameters. [29] considered two models for quadratic programming with fuzzy parameters to obtain the lower and upper bound of the objective function for a given α -cut level. [30] proposed quadratic programming for portfolio optimization to obtain the lower and upper bound of the objective function for a given α -cut level, while [31] proposed a sequence of multiobjective quadratic programming models under fuzzy environment, and [32] applied the models to portfolio optimization. A fuzzy quadratic programming model,

like in an FLP model, can be transformed into various special problems by using different fuzzy ranking methods.

In this thesis, in Chapter 2, fuzzy mathematical programming and robust linear programming are reviewed. In Section 2.1, the basics of fuzzy mathematics, a taxonomy of fuzzy ranking methods provided in [26], t-norm and t-conorm fuzzy relations and their properties, and basic FLP and fuzzy quadratic programming models proposed in the literature are presented. In Section 2.2, basic robust linear programming models proposed in [9,11,12] are provided.

In this thesis, Chapter 3 is devoted to FLP and fuzzy quadratic programming approach to capital rationing and capital budgeting models, including fuzzy formulations of investment appraisal techniques for simple projects. In Section 3.1, the fuzzy capital rationing models proposed in the literature are examined in detail, the models are illustrated with the numerical examples, and the results are interpreted. In Section 3.2, Weingartner's pure capital rationing model with fuzzy net cash flows and fuzzy budget limits is provided, then a crisp primal model based on t-norm fuzzy relation for the constraints and objective function is developed. Next, the dual model based on t-conorm fuzzy relation is formulated. The α – cut interval under which the weak duality is validated for the proposed primal-dual pair is proved, then the intervals under which the current basis remains optimal are provided when a budget level or cash flow level of a non-basic variable as well as a basic variable is changed. The level of investment in projects based on duality and complementary slackness results is also analyzed. Finally, the proposed model is illustrated with computational analysis, and the results are interpreted. In Section 3.3, first, the fuzzy approach to investment appraisal techniques presented in [15] is reviewed, then the calculation of IRR for simple projects by considering cash flows and Minimum Attractive Rate of Return (MARR) as fuzzy numbers is studied, with this aim, the t-norm and t-conorm fuzzy relation based Present Value (PV) equations are formulated to compute the relevant IRRs, and the decision rules are defined. The proposed decision rules are also illustrated with a numerical example for a simple project, and the consistency of Net Present Value (NPV) and IRR method for this example is investigated. In Section 3.4, two main contributions are to be considered. One of the key contributions of Section 3.4 is to show that the primal-dual pair formulation with t-norm and corresponding dual t-conorm fuzzy relation, and the

primal-dual pair formulation with t-conorm and corresponding dual t-norm fuzzy relation suggested in [24] for FLP are not directly extended to quadratic programming. In the first model pair, it is proved that that strong duality does not hold, although weak duality holds for $\alpha \in [0.5, 1)$, while in the second model pair, the α – cut interval under which weak duality holds is unclear. Therefore, alternative dual models are proposed so that weak duality and strong duality of the primal-dual model pairs hold for each $\alpha \in [0, 1)$. Next, the α – cut interval under which one primal-dual model pair represents the lower bound (worst-case scenario, pessimistic case) and the other primal-dual model pair represents the upper bound (best-case scenario, optimistic case) is proved. The second key contribution of Section 3.4 is that the proposed primal-dual model pairs are applied to Bernhard’s capital budgeting model, an extension to Weingartner’s horizon model with a non-linear objective function, to investigate the extensions to the dual analysis in the crisp case. To take into account uncertainty in Bernhard’s model has been neglected in the literature, although variations from the predetermined crisp values in budget limits as well as in cash flows are highly likely, especially if the planning horizon is long. In this thesis, uncertainty in discount rate, lending rate, or borrowing rate is not considered; instead, cash flows and budgets are assumed to be subject to vary. To test the validity, a computational analysis is performed, and different scenarios are considered, namely quadratic programming and LP models with crisp parameters, quadratic programming and LP models with fuzzy parameters, and the differences in the performance criteria are analyzed.

In this thesis, Chapter 4 is devoted to robust linear programming approach to capital rationing and capital budgeting, including robust formulations of investment appraisal techniques for simple projects. In Section 4.1, the robust optimization approach to Weingartner’s pure capital rationing model proposed in [5] is extended by considering additional parameters to belong to a given convex set. In Section 4.2, the robust optimization approach to Weingartner’s horizon model proposed in [5] is also extended by considering additional parameters to belong to a convex set. In Section 4.3, the robust decision rules for investment appraisal techniques including NPV method, and IRR method, are developed and the results are analyzed.

Finally, in Chapter 5, the conclusions, and discussions for future research are given.

2. FUZZY MATHEMATICAL PROGRAMMING AND ROBUST LINEAR PROGRAMMING

In this chapter, we give basics of fuzzy mathematics, a taxonomy of fuzzy ranking methods and properties of t-norm and t-conorm fuzzy relations, basic models of FLP and fuzzy quadratic programming and basic models of robust linear programming.

2.1. Fuzzy Mathematical Programming and Fuzzy Relations

In Section 2.1.1, we review the basics of fuzzy mathematics, in Section 2.1.2, we provide a taxonomy of fuzzy ranking methods and properties of t-norm and t-conorm fuzzy relations, and in Section 2.1.3 and Section 2.1.4, we study basic FLP models and fuzzy quadratic programming models proposed in the literature, respectively.

2.1.1. Basics of fuzzy mathematics¹

Let a *fuzzy set* $\tilde{A}$ be defined with the membership function $\mu_{\tilde{A}} : X \rightarrow [0, 1]$.

Definition 1: $\tilde{A}$ is a *normal fuzzy set*, if there exists an element x^* such that $\mu_{\tilde{A}}(x^*) = 1$, where x^* is called the *core* of the fuzzy set $\tilde{A}$ [33, p: 88].

Definition 2: The *support* or *base* of the fuzzy set $\tilde{A}$ is defined as [28, p: 22]:

$$S(\tilde{A}) = \{x \in X \mid \mu_{\tilde{A}}(x) > 0\} \quad (2.1)$$

Definition 3: A fuzzy set $\tilde{A}$ is *convex* if for any $x_1 < x_2 < x_3$, $\mu_{\tilde{A}}(x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_3))$ holds [34,33, p: 89]². In other words, if “the membership values are strictly monotonically increasing, or strictly monotonically

¹ The models and definitions with the references [24] and [28] are reprinted or adapted from Fuzzy Sets and Systems, Vol. 135, Inuiguchi, M., Ramik, J., Tanino, T. and Vlach, M., Satisficing solutions and duality in interval and fuzzy linear programming, pp. 151-177, Copyright©2002 Elsevier Science B.V., with permission from Elsevier; Fuzzy Mathematical Programming and Fuzzy Matrix Games, Vol. 169, 2005, Bector, C.R. and Chandra, S., Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media.

² The definition is reprinted from Information and Control, Vol. 8, Zadeh, L.A., Fuzzy sets, pp: 338-353, Copyright©1965 Elsevier Inc., with permission from Elsevier.

decreasing, or strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe [33, p: 89]”, then the fuzzy set is a convex fuzzy set. In [33, p: 89], it was emphasized that the convexity in a fuzzy set is different from the convexity in mathematics, and that the definition of convexity in mathematics is not used for a fuzzy set. In [33, p: 89], it was also emphasized that an important property of two convex fuzzy sets is that their intersection is also a convex fuzzy set³.

Definition 4: α – cut and strict α – cut of a fuzzy set $\tilde{A}$ are defined as [24]:

$$[\tilde{A}]_{\alpha} = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\} \quad \alpha \in [0, 1] \quad (2.2)$$

and

$$(\tilde{A})_{\alpha} = \{x \in X \mid \mu_{\tilde{A}}(x) > \alpha\} \quad \alpha \in [0, 1] \quad (2.3)$$

respectively. Note that α – cut and strict α – cut of a fuzzy set $\tilde{A}$ are closed and bounded intervals [35, p: 8]. Also note that for $\alpha_1 \leq \alpha_2$, $[\tilde{A}]_{\alpha_2} \subseteq [\tilde{A}]_{\alpha_1}$ and $(\tilde{A})_{\alpha_2} \subseteq (\tilde{A})_{\alpha_1}$ hold [28, p: 24].

Definition 5: Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers. Then, the following equations hold for $\alpha \in [0, 1]$ [35, p: 11]⁴:

$$\text{If } \tilde{C} = \tilde{A} + \tilde{B}, \text{ then } [\tilde{C}]_{\alpha} = [\tilde{A}]_{\alpha} + [\tilde{B}]_{\alpha} \quad (2.4)$$

$$\text{If } \tilde{C} = \tilde{A} - \tilde{B}, \text{ then } [\tilde{C}]_{\alpha} = [\tilde{A}]_{\alpha} - [\tilde{B}]_{\alpha} \quad (2.5)$$

$$\text{If } \tilde{C} = \tilde{A} \cdot \tilde{B}, \text{ then } [\tilde{C}]_{\alpha} = [\tilde{A}]_{\alpha} \cdot [\tilde{B}]_{\alpha} \quad (2.6)$$

$$\text{If } \tilde{C} = \tilde{A} / \tilde{B}, \text{ then } [\tilde{C}]_{\alpha} = [\tilde{A}]_{\alpha} / [\tilde{B}]_{\alpha} \quad (2.7)$$

³ Definition 1 and Definition 3 are reprinted or paraphrased from Fuzzy Logic with Engineering Applications, Ross, T.J., International Edition, 1995, McGraw-Hill, Inc., New York, Copyright© 1995, with permission from the author.

⁴ Definiton 5, Definition 8, and Definition 10 are reprinted or paraphrased from Fuzzy Mathematics in Economics and Engineering, Vol. 91, 2002, pp. 6,11, Chapter 2 Fuzzy Sets, Buckley, J.J., Eslami, E. and Feuring, T., Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media.

Definition 6: A fuzzy set $\tilde{A}$ is a *fuzzy number* if $\tilde{A}$ is normal and convex, α – cut of $\tilde{A}$ is a closed interval for each $\alpha \in (0, 1]$, and the support of $\tilde{A}$ is bounded [28, p:42-44;36, p: 97]⁵.

Definition 7: A fuzzy number $\tilde{A}$ is called a TFN if its membership function is in the following form [28, p: 47]:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , x < x_l, x > x_r \\ \frac{x - x_l}{x_m - x_l} & , x_l \leq x \leq x_m \\ \frac{x_r - x}{x_r - x_m} & , x_m < x \leq x_r \end{cases} \quad (2.8)$$

where $\tilde{A}$ is represented by the triplet $\tilde{A} = (x_l, x_m, x_r)$ as illustrated in Figure 2.1:

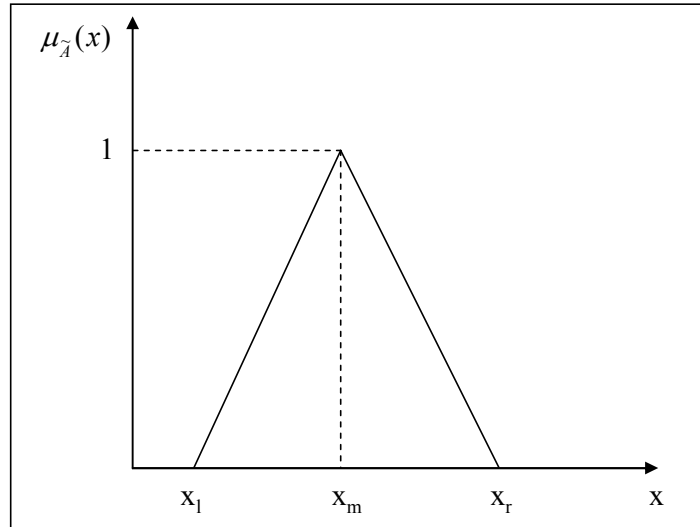


Figure 2.1 : Triangular fuzzy number, $\tilde{A}$

Definition 8: A *triangular shaped fuzzy number* is also represented by a triplet $\tilde{A} \approx (x_l, x_m, x_r)$, but its membership function is not an exact TFN, rather monotonically increasing under the interval $x_l - x_m$ and monotonically decreasing under the interval $x_m - x_r$ [35, p: 6].

⁵ This definition is also reprinted from Fuzzy Sets and Fuzzy Logic Theory and Applications, 1995, Klir, G.J. and Yuan, B., Prentice Hall.

Definition 9: A fuzzy number $\tilde{A}$ is called a *trapezoidal fuzzy number* (TrFN) if its membership function is in the following form [28, p: 51]:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & , x < x_l, x > x_r \\ \frac{x - x_l}{x_{m1} - x_l} & , x_l \leq x < x_{m1} \\ 1 & , x_{m1} \leq x \leq x_{m2} \\ \frac{x_r - x}{x_r - x_{m2}} & , x_{m2} < x \leq x_r \end{cases} \quad (2.9)$$

where $\tilde{A}$ is represented by the quadruplet $\tilde{A} = (x_l, x_{m1}, x_{m2}, x_r)$ as illustrated in Figure 2.2:

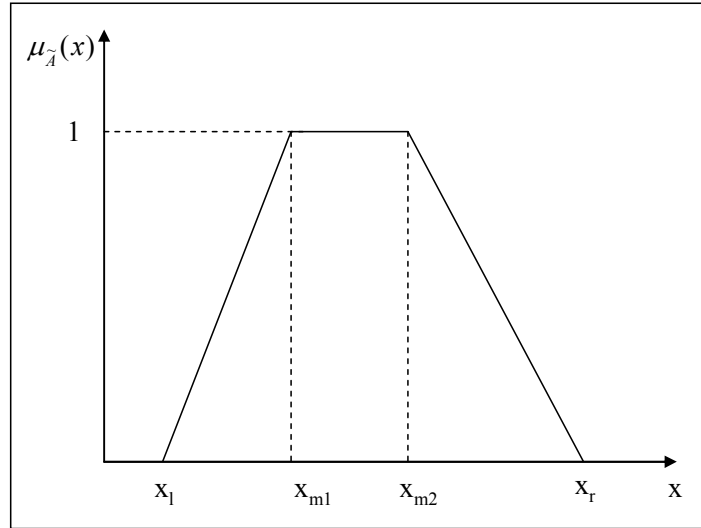


Figure 2.2 : Trapezoidal fuzzy number, $\tilde{A}$

Definition 10: A *trapezoidal shaped fuzzy number* is also represented by a quadruplet $\tilde{A} \approx (x_l, x_{m1}, x_{m2}, x_r)$, but its membership function is not an exact TrFN, rather monotonically increasing under the interval $x_l - x_{m1}$ and monotonically decreasing under the interval $x_{m2} - x_r$ [35, p: 6].

Note that the definition of a triangular shaped fuzzy number in Definition 8 and the definition of a trapezoidal shaped fuzzy number in Definition 10 are made analogously.

Definition 11: A fuzzy number $\tilde{A}$ is called an *L-R fuzzy number* if its membership function is in the following form [28, p: 52]:

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{x - x_{m1}}{\alpha}\right) & , (x_{m1} - \alpha) \leq x < x_{m1}, \quad \alpha > 0 \\ 1 & , x_{m1} \leq x \leq x_{m2} \\ R\left(\frac{x - x_{m2}}{\beta}\right) & , x_{m2} < x \leq (x_{m2} + \beta), \quad \beta > 0 \\ 0 & , \text{otherwise} \end{cases} \quad (2.10)$$

where $\tilde{A}$ is represented by $\tilde{A} = (x_{m1}, x_{m2}, \alpha, \beta)_{LR}$ as illustrated in Figure 2.3 [28, p: 52].

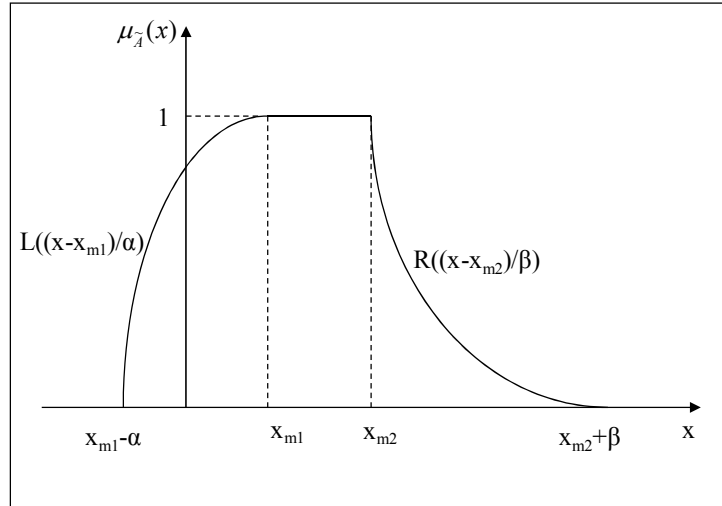


Figure 2.3 : L-R fuzzy number, $\tilde{A}$ ⁶

In Figure 2.3, x_{m1} and x_{m2} represent the starting and ending points of the flat interval, α is the left spread and β is the right spread, $L(\cdot)$ is an increasing piecewise continuous function, so-called left reference function, and $R(\cdot)$ is a decreasing piecewise continuous function, so-called right reference function, with $L(0) = R(0) = 1$ [28, p: 52].

⁶ Redrawn from Fuzzy Mathematical Programming and Fuzzy Matrix Games, Vol. 169, 2005, p. 52, Bector, C.R. and Chandra, S., Fig. 3.7, Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media.

2.1.2. A taxonomy of fuzzy ranking methods, t-norm and t-conorm fuzzy relations and their properties⁷

[26, p: 11] provided a basic classification of fuzzy ranking methods as depicted in Figure 2.4.

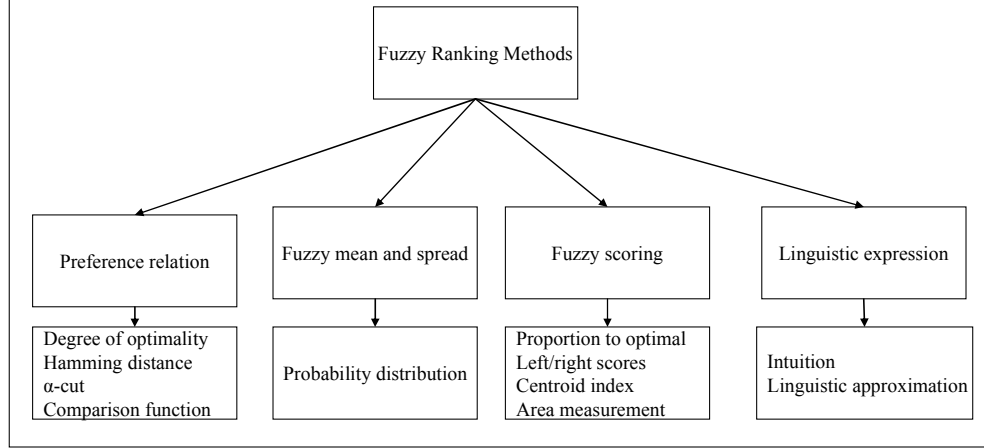


Figure 2.4 : Taxonomy of fuzzy ranking methods⁸

The details of the fuzzy ranking methods are beyond the scope of this thesis. We deal with possibility and necessity measures, which have been classified under “Comparison function” section in [26], since t-norm and t-conorm fuzzy relations, which will be used throughout the thesis, are the “extensions of possibility and necessity measures [25]”.

Definition 1: “Let X and Y be non-empty sets. A *binary relation* P between the elements of the sets X and Y is defined as a subset of the Cartesian product $X \times Y$, that is $P \subset X \times Y$ [24]”.

Definition 2: “Let X and Y be non-empty sets. A *valued relation* P defined on $X \times Y$ is a fuzzy subset of $X \times Y$ [24,37, p: 42]⁹”.

⁷ The models and definitions in this section with the reference [24] are reprinted or adapted from Fuzzy Sets and Systems, Vol. 135, Inuiguchi, M., Ramik, J., Tanino, T. and Vlach, M., Satisficing solutions and duality in interval and fuzzy linear programming, pp. 151-177, Copyright©2002 Elsevier Science B.V., with permission from Elsevier.

⁸ Adapted from Fuzzy Multiple Attribute Decision Making: Methods and Applications, 1992, p. 11, Chen, S.-J. and Hwang, C.-L., Fig. 1.3, Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media.

⁹ Adapted also from Fuzzy Preference Modelling and Multi-Criteria Decision Support, 1994, p. 42, Vol. 14, Fodor, J. and Roubens, M., Kluwer Academic Publishers, Dordrecht, Boston, London, Copyright©1994, with kind permission of Springer Science + Business Media.

Definition 3: “Let X and Y be non-empty sets, $f : X \rightarrow Y$ be a mapping. Let $F(X)$ and $F(Y)$ be the set of all fuzzy subsets of X and Y . The mapping $\tilde{f} : F(X) \rightarrow F(Y)$ defined for all $A \in F(X)$ and all $y \in Y$ by

$$\mu_{\tilde{f}(A)}(y) = \begin{cases} \sup \{ \mu_{\tilde{A}}(x) \mid x \in X, f(x) = y \} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \quad (2.11)$$

is called a *fuzzy extension* of f , where $f^{-1}(y) = \{x \in X \mid f(x) = y\}$ [24]”.

Definition 4: “Let X and Y be non-empty sets. A fuzzy subset of $F(X) \times F(Y)$ is called a *fuzzy relation* on $X \times Y$ [24]”.

Definition 5: *Possibility and necessity measures* for the fuzzy sets $\tilde{A}$ and $\tilde{B}$ are defined as follows [25,38]¹⁰:

$$Pos(\tilde{A} \preceq \tilde{B}) = \sup \{ \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \mid x \leq y, x, y \in \mathfrak{R} \} \quad (2.12)$$

$$Pos(\tilde{A} \prec \tilde{B}) = \sup \{ \inf \{ \min(\mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(y)) \mid x \leq y, y \in \mathfrak{R} \} \mid x \in \mathfrak{R} \} \quad (2.13)$$

$$Nec(\tilde{A} \preceq \tilde{B}) = \inf \{ \sup \{ \max(1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \mid x \leq y, y \in \mathfrak{R} \} \mid x \in \mathfrak{R} \} \quad (2.14)$$

$$Nec(\tilde{A} \prec \tilde{B}) = \inf \{ \max(1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(y)) \mid x > y, x, y \in \mathfrak{R} \} \quad (2.15)$$

Definition 6: “A class of functions $T : [0,1]^2 \rightarrow [0,1]$ that are commutative, associative, nondecreasing in every variable and satisfy the boundary condition $T(a,1) = a$ for all $a \in [0,1]$ are called *t-norms* [24]”.

t-norms also satisfy the following conditions [36, p: 62]¹¹:

$$b \leq d \text{ implies } T(a,b) \leq T(a,d) \text{ (monotonicity)} \quad (2.16)$$

¹⁰ Reprinted from Fuzzy Sets and Systems, Vol. 157, Ramik, J., Duality in fuzzy linear programming with possibility and necessity relations, pp. 1283-1302, Copyright©2005 Elsevier B.V., with permission from Elsevier; also adapted from Information Sciences, Vol. 30, Dubois, D. and Prade, H., Ranking fuzzy numbers in the setting of possibility theory, pp. 183-224, Copyright©1983 Elsevier Science Inc., with permission from Elsevier.

¹¹ (2.16)-(2.27) are adapted from Fuzzy Sets and Fuzzy Logic Theory and Applications, 1995, Klir, G.J. and Yuan, B., Prentice Hall, Upper Saddle River New Jersey.

$$T(a,b) = T(b,a) \quad (\text{commutativity}) \quad (2.17)$$

$$T(a,T(b,d)) = T(T(a,b),d) \quad (\text{associativity}) \quad (2.18)$$

Different examples of t-norms are available as follows [24,36, p: 63]:

$$T_M(a,b) = \min \{a,b\} \quad (2.19)$$

$$T_P(a,b) = a.b \quad (2.20)$$

$$T_L(a,b) = \max \{0, a+b-1\} \quad (2.21)$$

Definition 7: “A class of functions $S : [0,1]^2 \rightarrow [0,1]$ that are commutative, associative, nondecreasing in every variable and satisfy the boundary condition $S(a,0) = a$ for all $a \in [0,1]$ are called the *t-conorms* [24]”.

t-conorms also satisfy the following conditions [36, p: 77]:

$$b \leq d \text{ implies } S(a,b) \leq S(a,d) \quad (\text{monotonicity}) \quad (2.22)$$

$$S(a,b) = S(b,a) \quad (\text{commutativity}) \quad (2.23)$$

$$S(a,S(b,d)) = S(S(a,b),d) \quad (\text{associativity}) \quad (2.24)$$

Different examples of t-conorms are available as follows [24,36, p: 78]:

$$S_M(a,b) = \max \{a,b\} \quad (2.25)$$

$$S_P(a,b) = a + b - a.b \quad (2.26)$$

$$S_L(a,b) = \min \{1, a+b\} \quad (2.27)$$

Definition 8: “For each t-norm T , the function $T^* : [0,1]^2 \rightarrow [0,1]$ defined for all $a,b \in [0,1]$ by

$$T^*(a,b) = 1 - T(1-a, 1-b) \quad (2.28)$$

is a t-conorm [24]”, which means t-conorm T^* is a dual to t-norm T .

Definition 9: “For each t-conorm S , the function $S^* : [0, 1]^2 \rightarrow [0, 1]$ defined for all $a, b \in [0, 1]$ by

$$S^*(a, b) = 1 - S(1 - a, 1 - b) \quad (2.29)$$

is a t-norm [24]”, which means t-norm S^* is a dual to t-conorm S .

Definition 10: Let $T = \min$ be a t-norm, and $S = \max$ be a t-conorm, let $\tilde{A}$ and $\tilde{B}$ be normal and compact fuzzy sets, $\alpha \in (0, 1)$, and let $\tilde{\leq}$ be the fuzzy extension of the binary relation $\leq$. Then, the following relations are equivalent [24]:

$$\mu_{\tilde{\leq}_T}(\tilde{A}, \tilde{B}) \geq \alpha \text{ if and only if } \inf [\tilde{A}]_\alpha \leq \sup [\tilde{B}]_\alpha \quad (2.30)$$

$$\mu_{\tilde{\leq}_S}(\tilde{A}, \tilde{B}) \geq \alpha \text{ if and only if } \sup (\tilde{A})_{1-\alpha} \leq \inf (\tilde{B})_{1-\alpha} \quad (2.31)$$

$$\mu_{\tilde{\leq}_{T,S}}(\tilde{A}, \tilde{B}) \geq \alpha \text{ if and only if } \mu_{\tilde{\leq}_T}(\tilde{A}, \tilde{B}) \geq \alpha \text{ and } \sup (\tilde{A})_{1-\alpha} \leq \sup [\tilde{B}]_\alpha \quad (2.32)$$

$$\mu_{\tilde{\leq}_{S,T}}(\tilde{A}, \tilde{B}) \geq \alpha \text{ if and only if } \mu_{\tilde{\leq}_S}(\tilde{A}, \tilde{B}) \geq \alpha \text{ and } \inf [\tilde{A}]_\alpha \leq \inf (\tilde{B})_{1-\alpha} \quad (2.33)$$

Definition 11: For any normal fuzzy set $\tilde{A}$, the following definitions are also equivalent for $\alpha \in [0, 1]$ [24]:

$$\overset{-L}{A}(\alpha) = \inf \left\{ x \in R \mid x \in [\tilde{A}]_\alpha \right\} \quad (2.34)$$

$$\overset{L}{A}(\alpha) = \inf \left\{ x \in R \mid x \in (\tilde{A})_\alpha \right\} \quad (2.35)$$

$$\overset{-R}{A}(\alpha) = \sup \left\{ x \in R \mid x \in [\tilde{A}]_\alpha \right\} \quad (2.36)$$

$$\overset{R}{A}(\alpha) = \sup \left\{ x \in R \mid x \in (\tilde{A})_\alpha \right\} \quad (2.37)$$

If $\tilde{A}$ is a strictly convex and normal fuzzy set, then the following holds [24]:

$$\overset{-R}{A}(\alpha) = \overset{R}{A}(\alpha), \quad \overset{-L}{A}(\alpha) = \overset{L}{A}(\alpha) \text{ for any } \alpha \in (0, 1) \quad (2.38)$$

2.1.3. Fuzzy linear programming¹²

FLP problems have been classified into four basic groups as depicted in Figure 2.5, where the classifications in parentheses denote the terminology of Verdegay [23,28, p: 60]:

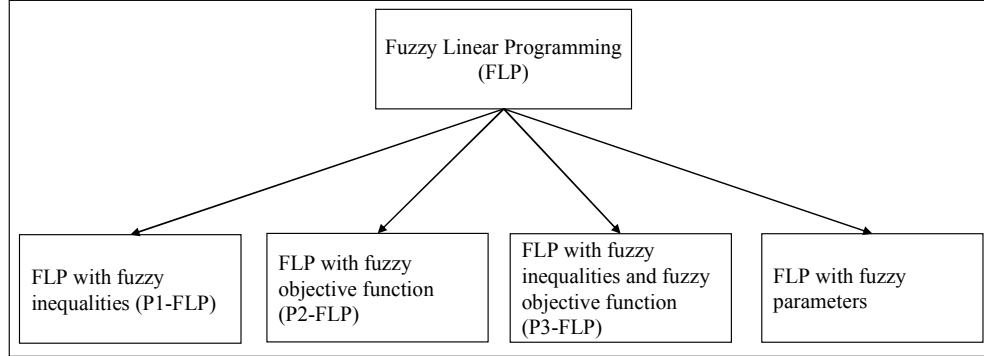


Figure 2.5 : Classification of fuzzy linear programming problems

Each type of FLP problem can be solved by considering two approaches, i.e., symmetric FLP problems or non-symmetric FLP problems, where symmetric FLP problems include the intersection of the fuzzy sets of the objective function and constraints, while non-symmetric FLP problems include a distinction between the fuzzy sets of the objective function and constraints as explained in [28, p: 60]. In this section, we provide some basic primal and dual model formulations.

2.1.3.1. Primal model formulations

FLP problems with fuzzy inequalities (P1-FLP)

An FLP problem with fuzzy inequalities and crisp objective function is formulated as follows [23,28, p: 61]:

$$\begin{aligned}
 &\max \quad c^T x \\
 &\text{s.t.} \quad A_n x \lesseqgtr b_n \quad \forall n \in \{1, 2, \dots, N\} \\
 &\quad \quad x \geq 0
 \end{aligned} \tag{2.39}$$

¹² The models in this section with the references [21],[23] and [28] are reprinted or adapted from Fuzzy Sets and Systems, Vol. 1, Zimmermann, H.-J., Fuzzy programming and linear programming with several objective functions, pp. 45-55, Copyright©1978 Elsevier Science B.V., with permission from Elsevier; Fuzzy Sets and Systems, Vol. 14, Verdegay, J.L., A dual approach to solve the fuzzy linear programming problem, pp. 131-141, Copyright©1984 Elsevier B.V., with permission from Elsevier; Fuzzy Mathematical Programming and Fuzzy Matrix Games, Vol. 169, 2005, Bector, C.R. and Chandra, S., Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media, respectively.

where the constraints are assumed to be soft constraints, $\widetilde{\leq}$ is a fuzzy extension of the classical binary relation $\leq$, and the satisfaction of each constraint is measured with the following membership function according to Verdegay's approach [23,28, p: 62]:

$$\mu_n(A_n x) = \begin{cases} 1 & , A_n x < b_n \\ 1 - \frac{A_n x - b_n}{p_n} & , b_n \leq A_n x \leq b_n + p_n \\ 0 & , A_n x > b_n + p_n \end{cases} \quad (2.40)$$

where $\mu_n(A_n x)$ is the membership function and p_n is the tolerance level for the satisfaction of the n th constraint. Verdegay proposed to solve (2.39) as a non-symmetric FLP model for each $\alpha \in [0,1]$, and obtained a standard parametric linear programming as follows [23,28, p: 62]:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & A_n x \leq b_n + (1 - \alpha)p_n \quad \forall n \in \{1, 2, \dots, N\} \\ & x \geq 0 \\ & \alpha \in [0, 1] \end{aligned} \quad (2.41)$$

where $\mu_n(A_n x) \geq \alpha$ is assumed for $\alpha \in [0, 1]$.

FLP problems with fuzzy objective function (P2-FLP)

An FLP problem with fuzzy objective function is formulated as follows [23,28, p: 64]:

$$\begin{aligned} \max \quad & \widetilde{c}^T x \\ \text{s.t.} \quad & A_n x \leq b_n \quad \forall n \in \{1, 2, \dots, N\} \\ & x \geq 0 \end{aligned} \quad (2.42)$$

Verdegay proposed to solve (2.42) analogous to (P1-FLP) problem as follows [23,28, p: 64]:

$$\begin{aligned}
& \max \quad c^T x \\
& \text{s.t.} \quad \phi(c) \geq 1 - \alpha \\
& \quad \quad A_n x \leq b_n \quad \forall n \in \{1, 2, \dots, N\} \\
& \quad \quad x \geq 0 \\
& \quad \quad \alpha \in [0, 1]
\end{aligned} \tag{2.43}$$

where $\phi(c) = \inf_j \phi_j(c_j)$ is the membership function for the satisfaction of the fuzzy objective function, where $\phi_j : \Re \rightarrow [0, 1]$ for $\forall j \in \{1, 2, \dots, J\}$ [23,28, p: 64].

FLP problems with fuzzy inequalities and fuzzy objective function (P3-FLP)

FLP problems with fuzzy inequalities and fuzzy objective function are a combination of (2.39) and (2.42) as follows [28, p: 67-68]:

$$\begin{aligned}
& \tilde{\max} \quad c^T x \\
& \text{s.t.} \quad A_n x \lesseqgtr b_n \quad \forall n \in \{1, 2, \dots, N\} \\
& \quad \quad x \geq 0
\end{aligned} \tag{2.44}$$

Zimmermann proposed to solve (2.44) as a symmetric FLP model, and proposed the following membership function for the satisfaction of the objective function, and (2.40) for the satisfaction of the n th constraint, respectively [21,28, p: 68]:

$$\mu_0(c^T x) = \begin{cases} 1 & , c^T x > Z_0 \\ 1 - \frac{Z_0 - c^T x}{p_0} & , Z_0 - p_0 \leq c^T x \leq Z_0 \\ 0 & , c^T x < Z_0 - p_0 \end{cases} \tag{2.45}$$

where $\mu_0(c^T x)$ is the membership function, Z_0 is the aspiration level, and p_0 is the tolerance level for the satisfaction of the objective function.

Finally, (2.44) is solved by considering the following model [21,28, p: 69]:

$$\begin{aligned}
& \max \quad \alpha \\
& \text{s.t.} \quad \left(1 - \frac{Z_0 - c^T x}{p_0} \right) \geq \alpha \\
& \quad \left(1 - \frac{A_n x - b_n}{p_n} \right) \geq \alpha \quad \forall n \in \{1, 2, \dots, N\} \\
& \quad \alpha \in [0, 1] \\
& \quad x \geq 0
\end{aligned} \tag{2.46}$$

FLP problems with fuzzy parameters¹³

The following model is from [22]:

$$\begin{aligned}
& \max \quad c^T x \\
& \text{s.t.} \quad \sum_{j=1}^J \tilde{a}_{nj} x_j \leq \tilde{b}_n \quad \forall n \in \{1, 2, \dots, N\} \\
& \quad x_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{2.47}$$

In FLP problems with fuzzy parameters, some or all of the parameters can be assumed to be fuzzy. In Model (2.47), the parameters of the left-hand side and right-hand side of the constraints are assumed to be fuzzy numbers. In this model, the satisfaction of $\tilde{a}_{nj}$ for $\forall n \in \{1, 2, \dots, N\}$, $\forall j \in \{1, 2, \dots, J\}$ and the satisfaction of $\tilde{b}_n$ for $\forall n \in \{1, 2, \dots, N\}$ may be defined by the membership

¹³ The explanations and models under this title are adapted from Fuzzy Sets and Systems, Vol. 109, Buckley, J.J. and Feuring, T., Evolutionary algorithm solution to fuzzy problems: Fuzzy linear programming, pp. 35-53, Copyright©2000 Elsevier Science B.V., with permission from Elsevier; Fuzzy Sets and Systems, Vol. 29, Delgado, M., Verdegay, J.L. and Vila, M.A., A general model for fuzzy linear programming, pp. 21-29, Copyright©1989 Elsevier Science B.V., with permission from Elsevier.

functions μ_{nj} and μ_n , respectively. The solution of (2.47) depends on the fuzzy ranking method chosen for ranking the fuzzy numbers in an inequality constraint with fuzzy parameters. The fuzzy ranking methods provided in Figure 2.4 can be used for comparing the left-hand side and right-hand side of the inequality constraints according to the characteristics of the problem. Note that in a fully fuzzified linear program, all decision variables as well as all parameters are assumed to be fuzzy numbers as emphasized in [20]. In this type of problem, the definition of the maximization of the objective function is also crucial. One interpretation for the maximization of an objective function in an FLP problem with fuzzy objective function coefficients has been proposed by [20] as follows: Let $\tilde{Z} = (z_1, z_2, z_3)$ be a triangular shaped fuzzy number representing the objective function of an FLP problem with fuzzy objective function parameters. Then, the maximization of the objective function can be interpreted as $(\sup z_2, \sup A_2, \inf A_1)$, where A_1 represents the area under the graph of the fuzzy number from z_1 to z_2 , and A_2 represents the area under the graph of the fuzzy number from z_2 to z_3 . In addition to the classification depicted in Figure 2.5, and to Model (2.47), [22] proposed a general model and its solution, which can be called as “FLP problems with fuzzy constraints and fuzzy coefficients”.

FLP problems with fuzzy constraints and fuzzy coefficients

$$\begin{aligned}
 \max \quad & c^T x \\
 \text{s.t.} \quad & \sum_{j=1}^J \tilde{a}_{nj} x_j \lesseqgtr \tilde{b}_n \quad \forall n \in \{1, 2, \dots, N\} \\
 & x_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\}
 \end{aligned} \tag{2.48}$$

where $\tilde{a}_{nj}$ for $\forall n \in \{1, 2, \dots, N\}$, $\forall j \in \{1, 2, \dots, J\}$ and $\tilde{b}_n$ for $\forall n \in \{1, 2, \dots, N\}$ are defined by the membership functions μ_{nj} and μ_n , respectively. The satisfaction of each constraint is also defined by a membership function.

The details of an FLP problem with fuzzy constraints and fuzzy coefficients are beyond the scope of this thesis.

2.1.3.2. Dual model formulations¹⁴

Rödter and Zimmermann originally proposed the dual problem of an FLP problem in [39]. They considered the following crisp primal and dual models, respectively [39,28, p: 96]:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} \min \quad & b^T u \\ \text{s.t.} \quad & A^T u \geq c \\ & u \geq 0 \end{aligned} \tag{2.50}$$

The membership functions for the fuzzified primal problem (so-called industry problem, denoted by I as a superscript) of (2.49), and the fuzzified dual problem (so-called market problem, denoted by M as a superscript) of (2.50) have been developed as follows [39,28, p: 97]:

$$\mu^I(x) = \begin{cases} 1 & , c^T x^0 \leq c^T x \\ 1 - (c^T x^0 - c^T x) & , \text{otherwise} \end{cases} \tag{2.51}$$

where $\mu^I(x)$ is the membership function for the satisfaction of the objective function of the fuzzified model of (2.49) on $x \geq 0$, and $c^T x^0$ is the aspiration level for the satisfaction of the objective function [39,28, p: 96-97].

¹⁴ The models with the references [39] and [40] are also reprinted from Extremal Methods and System Analysis, 1980, Chapter: Duality in fuzzy linear programming, Rödter, W. and Zimmermann, H.-J., Fiacco, A.V. and Kortanek, K.O. (Eds.), Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media; Fuzzy Sets and Systems, Vol. 125, Bector, C.R. and Chandra, S., On duality in linear programming under fuzzy environment, pp. 317-325, Copyright©2002 Elsevier Science B.V., with permission from Elsevier, respectively.

$$\mu_x^I(u) = \begin{cases} 0 & , u^T(b - Ax) \leq 0 \\ u^T(b - Ax) & , \text{otherwise} \end{cases} \quad (2.52)$$

where $\mu_x^I(u)$ is the membership function on $u \geq 0$ for any given $x \geq 0$ [39,28, p: 96].

$$\mu^M(u) = \begin{cases} 1 & , b^T u \leq b^T u^0 \\ 1 - (b^T u - b^T u^0) & , \text{otherwise} \end{cases} \quad (2.53)$$

where $\mu^M(u)$ is the membership function for the satisfaction of the objective function of the fuzzified model of (2.50) on $u \geq 0$, and $b^T u^0$ is the aspiration level for the satisfaction of the objective function [39,28, p: 96-97].

$$\mu_u^M(x) = \begin{cases} 0 & , x^T(c - A^T u) \leq 0 \\ -x^T(c - A^T u) & , \text{otherwise} \end{cases} \quad (2.54)$$

where $\mu_u^M(x)$ is the membership function of the fuzzified model of (2.50) on $x \geq 0$ for any given $u \geq 0$ [28, p: 96-97].

Finally, the fuzzified models of (2.49) and (2.50) are transformed into the following models, respectively [39,28, p: 97]:

$$\begin{aligned} \max \quad & \lambda_1 \\ \text{s.t.} \quad & \lambda_1 \leq (1 - (c^T x^0 - c^T x)) \\ & \lambda_1 \leq u^T(b - Ax) \quad \text{for any given } u \geq 0 \\ & x \geq 0 \\ & \lambda_1 \in R \end{aligned} \quad (2.55)$$

and

$$\begin{aligned}
& \min \quad (-\lambda_2) \\
& \text{s.t.} \quad \lambda_2 \geq ((b^T u - b^T u^0) - 1) \\
& \quad \lambda_2 \geq x^T (c - A^T u) \quad \text{for any given } x \geq 0 \\
& \quad u \geq 0 \\
& \quad \lambda_2 \in R
\end{aligned} \tag{2.56}$$

A modified version of Rödder and Zimmermann's dual model formulation was presented in [40,28, p: 101-102]. The following primal and dual models were considered, respectively:

$$\begin{aligned}
& c^T x \lesseqgtr Z_0 \\
& Ax \lesseqgtr b
\end{aligned} \tag{2.57}$$

$$x \geq 0$$

$$b^T w \lesseqgtr W_0$$

$$A^T w \lesseqgtr c \tag{2.58}$$

$$w \geq 0$$

By considering the membership functions in (2.45) and (2.40), the primal model and dual model will be as follows, respectively [40,28, p: 102-103]:

$$\begin{aligned}
& \max \quad \lambda \\
& \text{s.t.} \quad (\lambda - 1)p_0 \leq c^T x - Z_0 \\
& \quad (\lambda - 1)p_n \leq b_n - A_n x \quad \forall n \in \{1, 2, \dots, N\}
\end{aligned} \tag{2.59}$$

$$0 \leq \lambda \leq 1$$

$$x \geq 0$$

where p_0 and p_n for $\forall n \in \{1, 2, \dots, N\}$ are the tolerance levels for the satisfaction of the objective function and the n th constraint, respectively, and Z_0 is the aspiration level for the objective function as in [40,28, p: 102], and

$$\begin{aligned}
 \min \quad & (-\eta) \\
 \text{s.t.} \quad & (\eta - 1) q_0 \leq W_0 - b^T w \\
 & (\eta - 1) q_j \leq A_j^T w - c_j \quad \forall j \in \{1, 2, \dots, J\} \\
 & 0 \leq \eta \leq 1 \\
 & w \geq 0
 \end{aligned} \tag{2.60}$$

where q_0 and q_j for $\forall j \in \{1, 2, \dots, J\}$ are the tolerance levels for the satisfaction of the objective function and the n th constraint, respectively [28, p: 102].

2.1.4. Fuzzy quadratic programming¹⁵

Although FLP has been extensively examined in the literature, the literature in fuzzy quadratic programming has been relatively scarce [28, p: 72]. In this section, we provide basic primal model and dual model formulations. Note that, we provide basic fuzzy quadratic programming primal and dual model formulations in the following sections that are analogous to FLP models in Section 2.1.3 [28, p: 72], although there are several other models proposed in the literature. Also note that, although several methods were mentioned in [28] for solving quadratic programming models, and for solving fuzzy quadratic programming models, in this section, we merely provide primal and dual models.

2.1.4.1. Primal model formulation

The following symmetric quadratic programming model with fuzzy constraints was presented in [28, p: 75]:

¹⁵ The models in this section with the references [28] and [21] are reprinted or adapted from Fuzzy Mathematical Programming and Fuzzy Matrix Games, Vol. 169, 2005, Bector, C.R. and Chandra, S., Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media; Fuzzy Sets and Systems, Vol. 1, Zimmermann, H.-J., Fuzzy programming and linear programming with several objective functions, pp. 45-55, Copyright©1978 Elsevier Science B.V., with permission from Elsevier.

$$c^T x - \frac{1}{2} x^T H x \gtrsim Z_0$$

$$A_n x \lesssim b_n \quad \forall n \in \{1, 2, \dots, N\} \quad (2.61)$$

$$x \geq 0$$

where Z_0 is the aspiration level for the satisfaction of the objective function analogous to that in FLP. The membership functions for the satisfaction of the objective function and constraints are defined analogous to (2.45) and (2.40). Finally, the model to be solved will be analogous to (2.46) as follows [21,28, p: 75]:

$$\max \quad \alpha$$

$$\text{s.t.} \quad \left(1 - \frac{Z_0 - Z}{p_0} \right) \geq \alpha$$

$$\left(1 - \frac{A_n x - b_n}{p_n} \right) \geq \alpha \quad \forall n \in \{1, 2, \dots, N\} \quad (2.62)$$

$$0 \leq \alpha \leq 1$$

$$x \geq 0$$

2.1.4.2. Dual model formulation

The following dual model of the primal problem (2.61) was considered in [28, p: 113]:

$$b^T u + \frac{1}{2} w^T H w \lesssim W_0$$

$$A^T u + H w \gtrsim c \quad (2.63)$$

$$u \geq 0$$

$$w \geq 0$$

where (2.63) was formulated analogous to (2.60) as follows [28, p: 114]:

$$\begin{aligned}
\min \quad & (-\eta) \\
\text{s.t.} \quad & (\eta - 1) q_0 \leq W_0 - F(u, w) \\
& (\eta - 1) q_j \leq H_j^T w + A_j^T u - c_j \quad \forall j \in \{1, 2, \dots, J\} \\
& 0 \leq \eta \leq 1 \\
& x, w \geq 0
\end{aligned} \tag{2.64}$$

where q_0 and q_j for $\forall j \in \{1, 2, \dots, J\}$ are the tolerance levels for the satisfaction of the objective function and constraints, respectively.

2.2. Robust Linear Programming¹⁶

[9] originally proposed the following robust formulation of an LP as presented in [12]:

$$\begin{aligned}
\max \quad & c^T x \\
\text{s.t.} \quad & \sum_j a_{nj} x_j + \sum_{j \in J_n} \hat{a}_{nj} y_j \leq b_n \quad \forall n \\
& -y_j \leq x_j \leq y_j \quad \forall j \\
& l \leq x \leq u \\
& y \geq 0
\end{aligned} \tag{2.65}$$

For any n th row, $\tilde{a}_{nj}$ for $j \in J_n$ is assumed to be a symmetric and bounded random variable such that $\tilde{a}_{nj} \in [a_{nj} - \hat{a}_{nj}, a_{nj} + \hat{a}_{nj}]$ as in [11]. [11,12] defines $\eta_{nj} = \frac{\tilde{a}_{nj} - a_{nj}}{\hat{a}_{nj}}$ as a symmetric random variable with unknown distribution such that $\eta_{nj} \in [-1, 1]$.

¹⁶ The models in this section are reprinted by permission, Bertsimas, D. and Sim, M., The price of robustness, Operations Research, 52, 1, January-February 2004. Copyright©2004 INFORMS, the Institute for Operations Research and the Management Sciences, 7240 Parkway Drive, Suite 310, Hanover, Maryland 21076 USA.

In (2.65) at optimality, $y_j = |x_j^*|$, and according to the abovementioned assumptions, the first constraint of (2.65) will be as follows:

$$\sum_j a_{nj} x_j^* + \sum_{j \in J_n} \eta_{nj} \hat{a}_{nj} x_j^* \leq \sum_j a_{nj} x_j^* + \sum_{j \in J_n} \hat{a}_{nj} |x_j^*| \leq b_n \quad \forall n \quad (2.66)$$

[11] formulated the following non-linear robust counterpart of an LP model as presented in [12]:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \sum_j a_{nj} x_j + \sum_{j \in J_n} \hat{a}_{nj} y_{nj} + \Omega_n \sqrt{\sum_{j \in J_n} \hat{a}_{nj}^2 z_{nj}^2} \leq b_n \quad \forall n \\ & -y_{nj} \leq x_j - z_{nj} \leq y_{nj} \quad \forall n, j \in J_n \\ & l \leq x \leq u \\ & y \geq 0 \end{aligned} \quad (2.67)$$

where $\Omega_n > 0$ for $\forall n$.

[12] proposed a new robust counterpart of an LP model as an alternative to (2.65) and (2.67). They introduced a parameter Γ_n , not necessarily an integer, such that $\Gamma_n \in [0, |J_n|]$ to adjust the robustness of the n th constraint, and formulated the following linear robust counterpart:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \sum_j a_{nj} x_j + \max_{\{S_n \cup \{t_n\} | S_n \subseteq J_n, |S_n| = \lfloor \Gamma_n \rfloor, t_n \in J_n \setminus S_n\}} \left\{ \sum_{j \in S_n} \hat{a}_{nj} y_j + (\Gamma_n - \lfloor \Gamma_n \rfloor) \hat{a}_{nt_n} y_{t_n} \right\} \leq b_n \quad \forall n \\ & -y_j \leq x_j \leq y_j \quad \forall j \\ & l \leq x \leq u \\ & y \geq 0 \end{aligned} \quad (2.68)$$

$$\text{where } \beta_n(x^*, \Gamma_n) = \max_{\{S_n \cup \{t_n\} | S_n \subseteq J_n, |S_n| = \lfloor \Gamma_n \rfloor, t_n \in J_n \setminus S_n\}} \left\{ \sum_{j \in S_n} \hat{a}_{nj} |x_j^*| + (\Gamma_n - \lfloor \Gamma_n \rfloor) \hat{a}_{nt_n} |x_{t_n}^*| \right\} \quad (2.69)$$

is the protection function of the n th constraint as in [12]. [12] proposed (2.70) as an equivalent problem to (2.69), and (2.71) as the dual model of (2.70), respectively:

$$\begin{aligned} \max \quad & \sum_{j \in J_n} \hat{a}_{nj} |x_j^*| z_{nj} \\ \text{s.t.} \quad & \sum_{j \in J_n} z_{nj} \leq \Gamma_n \quad [z_n] \end{aligned} \quad (2.70)$$

$$0 \leq z_{nj} \leq 1 \quad \forall j \in J_n \quad [p_{nj}]$$

$$\begin{aligned} \min \quad & \sum_{j \in J_n} p_{nj} + \Gamma_n z_n \\ \text{s.t.} \quad & z_n + p_{nj} \geq \hat{a}_{nj} |x_j^*| \quad \forall j \in J_n \\ & p_{nj} \geq 0 \quad \forall j \in J_n \\ & z_n \geq 0 \quad \forall n \end{aligned} \quad (2.71)$$

Finally, Model (2.68) will be equivalent to the following by strong duality as in [12]:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \sum_j a_{nj} x_j + \sum_{j \in J_n} p_{nj} + \Gamma_n z_n \leq b_n \quad \forall n \\ & z_n + p_{nj} \geq \hat{a}_{nj} y_j \quad \forall j \in J_n \\ & -y_j \leq x_j \leq y_j \quad \forall j \\ & p_{nj} \geq 0 \quad \forall j \in J_n \\ & z_n \geq 0 \quad \forall n \\ & l_j \leq x_j \leq u_j \quad \forall j \\ & y_j \geq 0 \quad \forall j \end{aligned} \quad (2.72)$$

3. FUZZY LINEAR PROGRAMMING AND FUZZY QUADRATIC PROGRAMMING APPROACHES TO CAPITAL RATIONING AND CAPITAL BUDGETING

This chapter is devoted to fuzzy linear programming and fuzzy quadratic programming approaches to capital rationing and capital budgeting models with t-norm and t-conorm fuzzy relations, including the formulation of decision rules for IRR from PV equations with t-norm and t-conorm fuzzy relations by considering the cash flows and MARR as fuzzy numbers. Specifically, we deal with the Lorie-Savage problem, Weingartner's pure capital rationing problem and Bernhard's capital budgeting problem with a non-linear objective function, specifically quadratic objective function in our case, and analyze the models numerically or both theoretically and numerically.

3.1. Fuzzy Capital Rationing Models¹⁷

In this section, we deal with the fuzzy capital rationing models. In Section 3.1.1, we provide some non-fuzzy uncertainty based approaches to capital rationing, stochastic and robust approaches, and in Section 3.1.2, we provide fuzzy capital rationing model proposed in [16], chance-constrained programming models proposed in [17,18], and fuzzy mean-variance model proposed in [19], and illustrate each model with a numerical analysis for the data set with the same nominal values.

3.1.1. Uncertainty-based approaches to capital rationing models

In Section 3.1.1.1, we provide stochastic capital rationing model proposed in [3], and in Section 3.1.1.2, we provide robust optimization approach to Weingartner's capital rationing model proposed in [5]. The extensions to robust optimization approach to Weingartner's capital rationing model and Weingartner's horizon capital budgeting model will be examined in Chapter 4.

¹⁷ This section is published as a book chapter, Fuzzy Engineering Economics with Applications, STUDEFUZZ Vol. 233, 2008, Chapter "Fuzzy capital rationing models", Bas, E. and Kahraman C., pp. 359-380, Kahraman, C. (Ed.), Copyright©Springer-Verlag, and is reprinted in this thesis with kind permission of Springer Science+Business Media.

3.1.1.1. Stochastic capital rationing model

[3] proposed the following model as a stochastic capital rationing model¹⁸:

$$\begin{aligned}
 & \max \sum_n \left\{ \sum_j [c_{nj}(1+r)^{-n}] x_j - \min_l \sum_l (p_n^+ y_{nl}^+ + p_n^- y_{nl}^-) P(\beta_{nl}) \right\} \\
 & \text{s.t. } \sum a_{hj} x_j = b_h \quad \forall h \in \{1, 2, \dots, m_1\} \\
 & \quad \sum t_{nj} x_j + y_{nl}^+ - y_{nl}^- = \beta_{nl} \quad \forall n \in \{1, \dots, m_2\} \\
 & \quad \quad \quad \forall l \in \{1, 2, \dots, L_n\} \quad (3.1) \\
 & \quad x_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\} \\
 & \quad y_{nl}^+ \geq 0, \quad y_{nl}^- \geq 0 \quad \forall n \in \{1, 2, \dots, m_2\} \\
 & \quad \quad \quad \forall l \in \{1, 2, \dots, L_n\}
 \end{aligned}$$

where

c_{nj} : net cash flow for project j at period n

r : discount rate

x_j : project selection variable for project j

p_n^+ : penalty associated with the shortage of funds at period n

p_n^- : penalty associated with the surplus of funds at period n

y_{nl}^+ : the shortage variable for the l th realization of the resource n

y_{nl}^- : the surplus variable for the l th realization of the resource n

β_{nl} : the l th realization of the resource n

a_{hj} : the coefficient for project j on the h th deterministic constraint

b_h : the resource constraint for the h th deterministic constraint

t_{nj} : the coefficient for project j on the n th stochastic constraint

¹⁸ Reprinted by permission from MacMillan Publishers Ltd: Kira, D.S. and Kusy, M.I, A stochastic capital rationing model, The Journal of the Operational Research Society, 41, 9, Copyright©1990, published by Palgrave Macmillan.

and the first m_1 constraints are deterministic, whereas the constraints $(m_1 + 1)$ to $(m_1 + m_2)$ are stochastic.

3.1.1.2. Robust optimization approach to capital rationing

[5] considered the following Weingartner's capital rationing model to apply robust optimization approach proposed in [12]¹⁹:

$$\begin{aligned}
 \max \quad & \sum_{j=1}^J NPV_j x_j \\
 \text{s.t.} \quad & -\sum_{j=1}^J a_{nj} x_j \leq M_n \quad \forall n \in \{0, 1, \dots, N\} \\
 & 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}
 \end{aligned} \tag{3.2}$$

where $NPV_j = \sum_{n=0}^N \frac{a_{nj}}{(1+r)^n}$ is Net Present Value of project j , a_{nj} is net cash flow for project j at period n , r is discount rate, M_n is budget limit at period n , N is project life and budget life, and x_j is project selection variable for project j , where if $x_j = 1$, the project j is fully funded, if $0 < x_j < 1$, the project j is partially funded, and if $x_j = 0$, the project j is not funded [5]. Note that the projects considered in this model are independent projects, discount rate r is assumed to be constant during the budget and project life, and (-) sign on the left-hand side of the budget limit constraint denotes the inclusion of the generated revenues to the budget limit for that period. The upper limit of 1 for project selection variable prevents the allocation of the entire budget to a project for multiple times [1, p: 17]. As a derivative model of (3.2), Weingartner's pure capital rationing model could include mutually exclusive projects and contingent projects or a combination of independent projects and mutually exclusive projects, the conditions which do not necessarily make the model theoretically or computationally more complicated as discussed in [1, p: 32-34].

¹⁹ Reprinted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

Since $\tilde{a}_{nj} \in [a_{nj} - \hat{a}_{nj}, a_{nj} + \hat{a}_{nj}]$ for $j \in J_n$ is assumed to be a symmetric and bounded random variable, and $\Gamma_n \in [0, |J_n|]$ is defined as a predetermined protection level to adjust the robustness for $\forall n \in \{0, 1, \dots, N\}$ as illustrated in Section 2.2, (3.2) is transformed into the following robust counterpart²⁰ in [5] as illustrated in Section 2.2:

$$\begin{aligned}
\max \quad & \sum_{n=0}^N \frac{x_{J+n+1}}{(1+r)^n} \\
\text{s.t.} \quad & -\sum_{j=1}^J a_{nj}x_j + z_n\Gamma_n + \sum_{j \in J_n} p_{nj} + x_{J+n+1} \leq M_n \quad \forall n \in \{0, 1, \dots, N\} \\
& z_n + p_{nj} \geq \hat{a}_{nj}x_j \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_n \\
& z_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& p_{nj} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_n \\
& x_{J+n+1} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.3}$$

where [5] transformed (3.2) into an equivalent model with objective function coefficients without a_{nj} for $\forall n, \forall j$ before formulating its robust counterpart. They also formulated the dual model of (3.3), and analyzed the complementary slackness results. They applied the proposed model to a three-project example and a two hundred-project example. The details of the robust optimization approach to capital rationing, and its extensions are left to Chapter 4.

3.1.2. Three fuzzy capital rationing models

[16-19] proposed fuzzy capital rationing models, and their solutions. Specifically, for the solutions, [16] proposed the branch-and-bound method, and [17-19] proposed fuzzy simulation based genetic algorithm or hybrid intelligent algorithm.

²⁰ Reprinted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

3.1.2.1. Fuzzy capital rationing model of Chiu and Park

[16] considered the following Weingartner's pure capital rationing model as a 0-1 integer program²¹:

$$\begin{aligned}
 \max \quad & PV_{(x_1, x_2, \dots, x_J)} \\
 \text{s.t.} \quad & \sum_{j=1}^J -P_{nj}x_j \preceq B_n \quad \forall n \in \{0, 1, \dots, N\} \\
 & x_j \in \{0, 1\} \quad \forall j \in \{1, 2, \dots, J\}
 \end{aligned} \tag{3.4}$$

where $P_{nj} = (p_{nj0}, p_{nj1}, p_{nj2})$ is a TFN representing periodic cash flow for project j at period n for $\forall n \in \{0, 1, \dots, N\}$ and $\forall j \in \{1, 2, \dots, J\}$ where N is budget life, and J is the number of the projects; B_n is the fuzzy budget limit at period n such that B_{n1} is the most possible budget, and B_{n0} is the largest possible budget for $\forall n \in \{0, 1, \dots, N\}$; x_j is project selection variable for project j , where $x_j = 1$ if project j is funded, and $x_j = 0$ if project j is not funded for $\forall j \in \{1, 2, \dots, J\}$; and $\preceq$ is a fuzzy extension of the binary relation $\leq$ [16]. $PV_{(x_1, x_2, \dots, x_J)}$ is a TFN representing PV of the combined cash flow of funded projects, and $r_n = (r_{n0}, r_{n1}, r_{n2})$ is a TFN representing discount rate at period n for $\forall n \in \{1, 2, \dots, L\}$ where $r_0 = 0$, and L is project life for all projects [16]. The calculation of present value of project j , PV_j is based on [41] as follows [16]:

$$PV_j = \left(\sum_{n=0}^L \left[\frac{\max \{p_{nj0}, 0\}}{\prod_{n'=0}^n (1+r_{n'2})} + \frac{\min \{p_{nj0}, 0\}}{\prod_{n'=0}^n (1+r_{n'0})} \right], \sum_{n=0}^L \frac{p_{nj1}}{\prod_{n'=0}^n (1+r_{n'1})}, \sum_{n=0}^L \left[\frac{\max \{p_{nj2}, 0\}}{\prod_{n'=0}^n (1+r_{n'0})} + \frac{\min \{p_{nj2}, 0\}}{\prod_{n'=0}^n (1+r_{n'2})} \right] \right) \tag{3.5}$$

²¹ Models (3.4)-(3.13) are adapted from The Engineering Economist, Vol. 43, 1998, Chiu, C.-Y. and Park, C.S., Capital budgeting decisions with fuzzy projects, pp. 125-150, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

Note that [16] considered different budget life and project life for all projects. [16] also emphasized that PV calculation proposed in [41] yields a non-linear fuzzy number, however in “reasonable discount rate ranges such as $\pm 4\%$ of these rates [16]”, PV calculation in (3.5) yields a TFN. [16] discussed that no fuzzy linear programming approach could solve (3.4), since the models available in the literature would not consider 0-1 integer program models or fuzzy parameters, thus proposed to solve (3.4) with the branch-and-bound method, and proposed the following branch-and-bound flow chart in Figure 3.1 for the solution of (3.4):

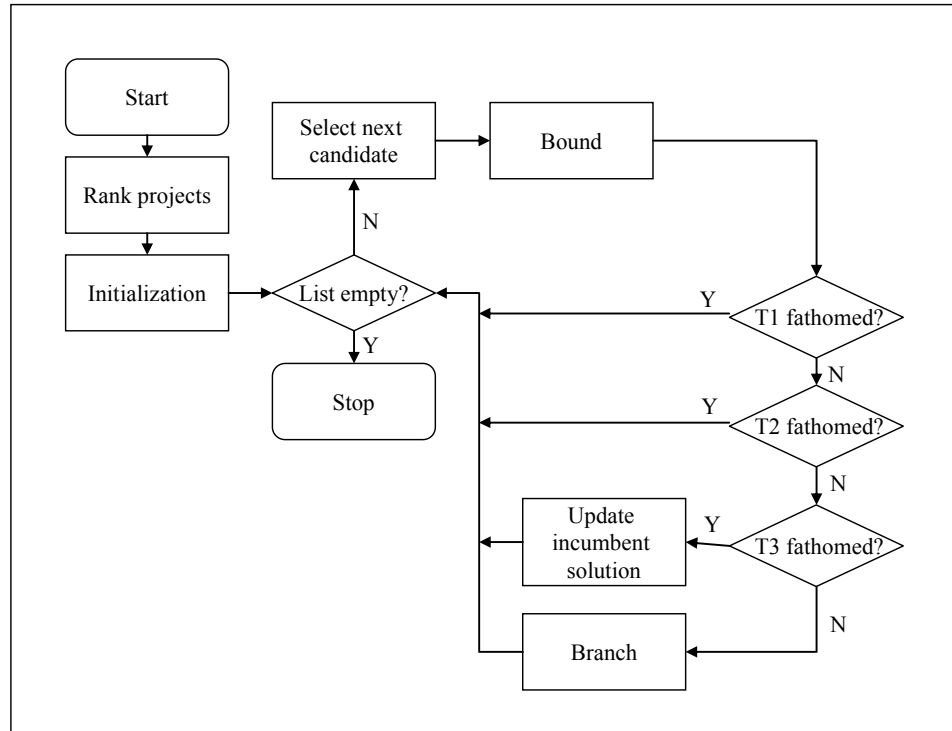


Figure 3.1 : Branch-and-bound flow chart for fuzzy capital rationing model²²

The distinctive features of the branch-and-bound method in [16] can be summarized as the so-called “weighted method” to rank the projects according to their PV, and “fathoming tests” to fathom a current candidate solution. In the following sections, we will give the details of these distinctive features.

“Weighted method” to rank the projects

Weighted method is used to rank the fuzzy numbers in [16]. If we define present value of project j as $PV_j = (pv_{j0}, pv_{j1}, pv_{j2})$, then

²² Redrawn from The Engineering Economist, Vol. 43, 1998, Chiu, C.-Y. and Park, C.S., Capital budgeting decisions with fuzzy projects, pp. 125-150, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

$$w_1 \left(\frac{pv_{j0} + pv_{j1} + pv_{j2}}{3} \right) + w_2 pv_{j1} \quad (3.6)$$

is the so-called “dominance indicator” for the comparison of PV of project j , where w_1 and w_2 are the relative weights of each criterion [16]. If we set $w_1 = 1$ as in [16], then (3.6) is equivalent to the following:

$$\left(\frac{pv_{j0} + pv_{j1} + pv_{j2}}{3} \right) + w pv_{j1} \quad (3.7)$$

where $w = 0.3$ may be recommended if pv_{j1} is fairly important, and $w = 0.1$ may be recommended if pv_{j1} is slightly important [16].

Fathoming tests to fathom a current candidate solution

Fathoming test 1 (T1):

If the current objective function is inferior to the updated lower bound (LB) of the so-called “incumbent solution [16]”, then the current candidate solution is fathomed. [16] proposed a TFN with the extreme negative numbers such as $(-9999999, -9999998, -9999997)$ as an LB for the beginning of the problem.

Fathoming test 2 (T2):

$$\sum_{j=1}^{J'} -P_{nj} x_j \succ B_n + \sum_{j=J'+1}^J \max\{P_{nj}, 0\} \quad (3.8)$$

where the left hand side represents the net expenditures of the funded projects with J' as the lastly considered project in the subset, and the right hand side represents the budget limit and revenues possibly generated by the subsequent projects for the n th period [16].

(3.8) can be transformed into the following two equations with crisp parameters [16]:

$$\sum_{j=1}^{J'} -p_{nj1} x_j > B_{n1} + \sum_{j=J'+1}^J \max\{p_{nj1}, 0\} \quad (3.9)$$

$$\sum_{j=1}^{J'} -\min\{p_{nj0}, 0\} x_j > B_{n0} + \sum_{j=1}^{J'} \max\{p_{nj2}, 0\} x_j + \sum_{j=J'+1}^J \max\{p_{nj2}, 0\} \quad (3.10)$$

where (3.9) checks the feasibility of “the most possible budget” case, and (3.10) checks the feasibility of “the largest possible budget” case. If either (3.9) or (3.10) are satisfied for any $n \in \{0, 1, \dots, N\}$, then the current candidate solution is fathomed, since the current candidate solution proves to be infeasible [16].

Fathoming test 3 (T3):

$$\sum_{j=1}^{J'} -P_{nj} x_j - P_{n,J'+1} (1 - x_{J'}) + \sum_{j=J'+2}^J -P_{nj} \leq B_n \quad (3.11)$$

where $x_{J'} = 1$ or $x_{J'} = 0$.

(3.11) can be transformed into the following two equations with crisp parameters [16]:

$$\sum_{j=1}^{J'} -p_{nj1} x_j - p_{n,J'+1,1} (1 - x_{J'}) + \sum_{j=J'+2}^J -p_{nj1} \leq B_{n1} \quad (3.12)$$

$$\begin{aligned} \sum_{j=1}^{J'} \max \{ -p_{nj0}, 0 \} x_j + \max \{ -p_{n,J'+1,0}, 0 \} (1 - x_{J'}) + \sum_{j=J'+2}^J \max \{ -p_{nj0}, 0 \} \leq B_{n0} + \\ \sum_{j=1}^{J'} \max \{ p_{nj2}, 0 \} x_j + \max \{ p_{n,J'+1,2}, 0 \} (1 - x_{J'}) + \sum_{j=J'+2}^J \max \{ p_{nj2}, 0 \} \end{aligned} \quad (3.13)$$

where (3.12) checks “the most possible budget” case, and (3.13) checks “the largest possible budget” case. If (3.12) and (3.13) are satisfied for $\forall n \in \{0, 1, \dots, N\}$, then the current candidate solution is recorded as the current incumbent solution [16].

A numerical example for fuzzy capital rationing model

The nominal values for a three-project and three-period example from [5] are given in Table 3.1:

Table 3.1 : A three-project and three-period example with the nominal values²³

r = 20%	Project 1	Project 2	Project 3	Budget
Year 0	-15	-5	-12	25
Year 1	5	5	4	25
Year 2	25	5	20	25
NPV	6.53	2.64	5.22	

²³ Redrawn from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

Table 3.1 with the nominal periodic cash flows, budget limits, and discount rate is transformed into Table 3.2 with the symmetric fuzzy periodic cash flows, fuzzy budget limits, and symmetric fuzzy discount rates. Note that we consider $\pm 10\%$ range for the symmetric triangular fuzzy cash flows, and $+10\%$ range for the fuzzy budget limits, while we consider $\pm 4\%$ range for the symmetric triangular fuzzy discount rates as suggested in [16].

Table 3.2 : A three-project and three-period example with the fuzzy cash flows, budget limits, and discount rates

$r = (19.2\%, 20\%, 20.8\%)$	Project 1	Project 2	Project 3	Budget
Year 0	$(-16.5, -15, -13.5)$	$(-5.5, -5, -4.5)$	$(-13.2, -12, -10.8)$	$(25, 27.5)$
Year 1	$(4.5, 5, 5.5)$	$(4.5, 5, 5.5)$	$(3.6, 4, 4.4)$	$(25, 27.5)$
Year 2	$(22.5, 25, 27.5)$	$(4.5, 5, 5.5)$	$(18, 20, 22)$	$(25, 27.5)$

“Weighted method” to rank the projects

According to PV calculation in (3.5), PV of each project for $\forall j \in \{1, 2, 3\}$ is calculated as follows:

$$PV_1 = \left(\left[-16.5 + \frac{4.5}{(1+0.208)^1} + \frac{22.5}{(1+0.208)^2} \right], -15 + \frac{5}{(1+0.2)^1} + \frac{25}{(1+0.2)^2}, \left[-13.5 + \frac{5.5}{(1+0.192)^1} + \frac{27.5}{(1+0.192)^2} \right] \right)$$

$$= (2.643897197, 6.5277778, 10.46851493)$$

$$PV_2 = \left(\left[-5.5 + \frac{4.5}{(1+0.208)^1} + \frac{4.5}{(1+0.208)^2} \right], -5 + \frac{5}{(1+0.2)^1} + \frac{5}{(1+0.2)^2}, \left[-4.5 + \frac{5.5}{(1+0.192)^1} + \frac{5.5}{(1+0.192)^2} \right] \right)$$

$$= (1.308912, 2.638889, 3.984978)$$

$$PV_3 = \left(\left[-13.2 + \frac{3.6}{(1+0.208)^1} + \frac{18}{(1+0.208)^2} \right], -12 + \frac{4}{(1+0.2)^1} + \frac{20}{(1+0.2)^2}, \left[-10.8 + \frac{4.4}{(1+0.192)^1} + \frac{22}{(1+0.192)^2} \right] \right)$$

$$= (2.115118, 5.222222, 8.374812)$$

If we set $w = 0.3$, then according to (3.7):

$$\left(\frac{pv_{10} + pv_{11} + pv_{12}}{3} \right) + 0.3 pv_{11} = 8.505063302$$

$$\left(\frac{pv_{20} + pv_{21} + pv_{22}}{3} \right) + 0.3 pv_{21} = 3.435926$$

$$\left(\frac{pv_{30} + pv_{31} + pv_{32}}{3} \right) + 0.3 pv_{31} = 6.804051$$

are the dominance indicators of the projects. Finally, the ranking of the projects will be $1 \succ 3 \succ 2$. According to [16], Last in First out (LIFO) method is used to decide the order for fathoming the current candidate solutions.

Fathoming tests to fathom a current candidate solution

Current candidate solution- $x_2 = 1, x_3 = 1$

(T1) is not satisfied, since the LB is determined as a TFN with the extreme negative numbers as in [16].

(T2) is not satisfied for “the most possible budget” case and for “the largest possible budget” case at any period.

(T3) is satisfied, and $(3.42403, 7.861111, 12.35979)$ is set as the LB of the current incumbent solution. According to (3.7) where $w = 0.3$ is assumed,

$$\left(\frac{3.42403 + 7.861111 + 12.35979}{3} \right) + 0.3 * 7.861111 = 10.23998 \quad \text{is the dominance}$$

indicator for the LB of the current incumbent solution.

Current candidate solution - $x_2 = 1, x_3 = 0, x_1 = 1$

(T1) is not satisfied, since the new objective function $(3.952809, 9.166667, 14.45349)$ dominates the current incumbent solution with $11.94099 > 10.23998$.

(T2) is not satisfied for “the most possible budget” case and for “the largest possible budget” case at any period.

(T3) is satisfied, and $(3.952809, 9.166667, 14.45349)$ is set as the LB of the current incumbent solution.

Current candidate solution - $x_2 = 0, x_3 = 1, x_1 = 1$

(T1) is not satisfied, since the new objective function (4.759015, 11.75, 18.84333) dominates the current incumbent solution with $15.30911 > 11.94099$.

(T2) is satisfied for “the most possible budget” case at $n = 0$.

Finally, $x_2 = 1, x_3 = 0, x_1 = 1$ is the optimal solution with a fuzzy objective function (3.952809, 9.166667, 14.45349). The branch-and-bound procedure for the numerical example of fuzzy capital rationing model proposed in [16] is summarized in Figure 3.2.

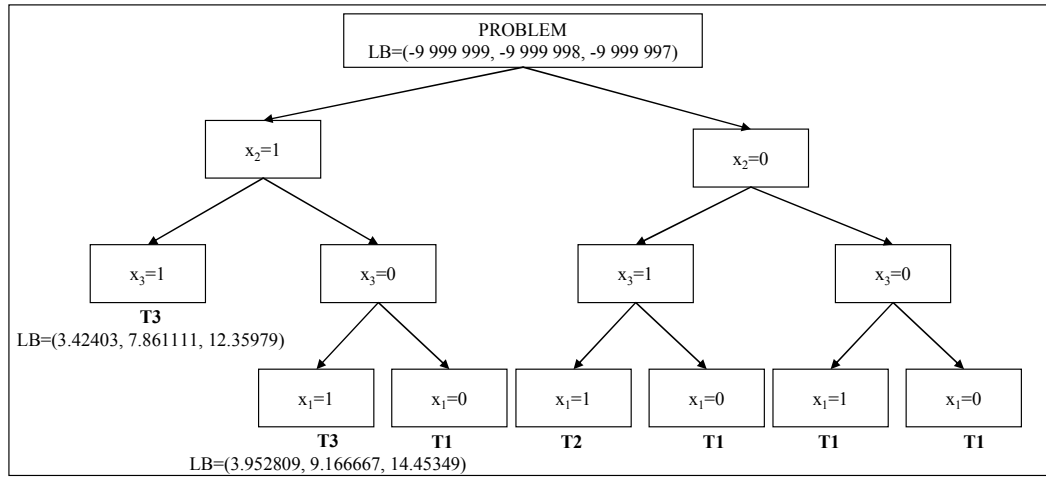


Figure 3.2 : Branch-and-bound method for the numerical example of fuzzy capital rationing model proposed in [16]

3.1.2.2. Chance-constrained programming approach to capital rationing²⁴

[17,18] considered the following Lorie-Savage problem:

$$\begin{aligned}
 \max \quad & \sum_{j=1}^J \sum_{n=1}^L \frac{d_{nj}}{(1+r)^n} x_j \\
 \text{s.t.} \quad & \sum_{j=1}^J a_{nj} x_j \leq M_n \quad \forall n \in \{1, 2, \dots, N\} \\
 & x_j \in \{0, 1\} \quad \forall j \in \{1, 2, \dots, J\}
 \end{aligned} \tag{3.14}$$

²⁴ (3.14)-(3.18) are adapted from Journal of Computational and Applied Mathematics, Vol. 198, Huang, X., Chance-constrained programming models for capital budgeting with NPV as fuzzy parameters, pp. 149-159, Copyright©2005 Elsevier B.V., with permission from Elsevier.

where d_{nj} is the net cash flow of project j at the end of the period n for $\forall n \in \{1, 2, \dots, L\}$, and $\forall j \in \{1, 2, \dots, J\}$ where L is project life for all projects, and J is the number of the projects; r is the marginal cost of capital; a_{nj} is the cash outflow of project j at the beginning of the period n for $\forall n \in \{1, 2, \dots, N\}$, $\forall j \in \{1, 2, \dots, J\}$, where N is budget life; M_n is the budget limit at the beginning of the period n for $\forall n \in \{1, 2, \dots, N\}$; and x_j is project selection variable for $\forall j \in \{1, 2, \dots, J\}$ where $x_j = 1$ if project j is funded, and $x_j = 0$ if project j is not funded. Note that the budget is assumed to be depleted at the beginning of a period, while net cash flow is assumed to occur at the end of a period [17,18]. [17,18] applied chance-constrained programming to (3.14).

Definition 1: *Credibility* index is defined as follows [17,42]:

$$\text{Cr} \{ \tilde{A} \preceq \tilde{B} \} = \frac{1}{2} (\text{Pos}(\tilde{A} \preceq \tilde{B}) + \text{Nec}(\tilde{A} \prec \tilde{B})) \quad (3.15)$$

where possibility and necessity indices were already defined in (2.12) and (2.15).

In other words, *possibility*, *necessity* of a fuzzy event $\tilde{A} \leq k$ and *credibility* of a fuzzy event $\tilde{A} \leq k$ and $\tilde{A} \geq k$ can be defined as follows, where $\tilde{A} = (x_l, x_m, x_r)$ is a TFN, and k is a real number [17,43]:

$$\text{Pos} \{ \tilde{A} \leq k \} = \begin{cases} 0 & , k \leq x_l \\ \frac{k - x_l}{x_m - x_l} & , x_l \leq k \leq x_m \\ 1 & , k \geq x_m \end{cases} \quad (3.16)$$

$$\text{Nec} \{ \tilde{A} \leq k \} = \begin{cases} 0 & , k \leq x_m \\ \frac{k - x_m}{x_r - x_m} & , x_m \leq k \leq x_r \\ 1 & , k \geq x_r \end{cases} \quad (3.17)$$

$$\text{Cr}\{\tilde{A} \leq k\} = \begin{cases} 0 & , k \leq x_l \\ \frac{k - x_l}{2(x_m - x_l)} & , x_l \leq k \leq x_m \\ \frac{x_r - 2x_m + k}{2(x_r - x_m)} & , x_m \leq k \leq x_r \\ 1 & , k \geq x_r \end{cases} \quad (3.18)$$

$$\text{Cr}\{\tilde{A} \geq k\} = \begin{cases} 1 & , k \leq x_l \\ \frac{2x_m - x_l - k}{2(x_m - x_l)} & , x_l \leq k \leq x_m \\ \frac{x_r - k}{2(x_r - x_m)} & , x_m \leq k \leq x_r \\ 0 & , k \geq x_r \end{cases} \quad (3.19)$$

Definition 2: $(X, F(X), Pos)$ is a *possibility space* if X is a non-empty set, $F(X)$ is the power set of X , $Pos\{\emptyset\}=0$, $Pos\{X\}=1$ and $Pos\{\cup_k A_k\} = \sup_k Pos\{A_k\}$ for any arbitrary collection $\{A_k\} \in F(X)$ [44, p: 148]²⁵.

Definition 3: A *random fuzzy variable* is a function from the possibility space $(X, F(X), Pos)$ to the collection of random variables $\mathfrak{R}$ [44, p: 295].

Definition 4: “Let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ be an n -dimensional random fuzzy vector, and $f: \mathfrak{R}^n \rightarrow \mathfrak{R}$ be a real-valued function over the n -dimensional Euclidean space. Then, *chance of a random fuzzy event* characterized by $f(\xi) \leq 0$ is defined as follows [18, 44, p: 299]:

²⁵ Definition 2, Definition 3, and Definition 4 are reprinted or adapted from Theory and Practice of Uncertain Programming, Vol. 102, 2002, Liu, B., Copyright©Springer-Verlag, with kind permission of Springer Science+Business Media. Definition 4 is also reprinted from International Journal of Production Economics, Vol. 106, Huang, X., Optimal project selection with random fuzzy parameters, pp. 513-522, Copyright©2006 Elsevier B.V., with permission from Elsevier. Definition 1 is also adapted from IEEE Transactions on Fuzzy Systems, Vol. 10, 2002, Liu, B. and Liu, Y.-K., Expected value of fuzzy variable and fuzzy expected value models, pp. 445-450, ©2002 IEEE, with permission from IEEE.

$$\text{Ch} \{ f(\xi) \leq 0 \}(\alpha) = \sup_{Cr\{A\} \geq \alpha} \inf_{\theta \in A} \Pr \{ f(\xi(\theta)) \leq 0 \} \quad (3.20)$$

where $A \in F(X)$, and (3.20) represents the random fuzzy event holds with probability $\text{Ch} \{ f(\xi) \leq 0 \}(\alpha)$ at credibility α [18].

[18] introduced chance-constrained programming model with random fuzzy cash outflows and random fuzzy net cash flows, chance-constrained programming model with random cash outflows and random net cash flows, and chance-constrained programming model with fuzzy cash outflows and fuzzy net cash flows, while [17] introduced chance-constrained programming model with fuzzy cash outflows and fuzzy net cash flows. In the following pages, we give three models as proposed in [17] and/or [18], and provide the solution methods as proposed in [17,18] without giving the details of the solution methods²⁶.

Chance-constrained programming model with random fuzzy cash outflows and random fuzzy net cash flows

$$\begin{aligned} \max \quad & \bar{f} \\ \text{s.t.} \quad & \text{Ch} \left\{ \sum_{j=1}^J \sum_{n=1}^L \frac{\eta_{nj}}{(1+r)^n} x_j \geq \bar{f} \right\}(\alpha) \geq \delta \\ & \text{Ch} \left\{ \sum_{j=1}^J \xi_{nj} x_j \leq M_n \right\}(\beta_n) \geq \gamma_n \quad \forall n \in \{1, 2, \dots, N\} \\ & x_j \in \{0, 1\} \quad \forall j \in \{1, 2, \dots, J\} \end{aligned} \quad (3.21)$$

where η_{nj} and ξ_{nj} are random fuzzy variables representing net cash flow of project j at the end of the period n for $\forall n \in \{1, 2, \dots, L\}$, and $\forall j \in \{1, 2, \dots, J\}$; and cash outflow of project j at the beginning of the period n for $\forall n \in \{1, 2, \dots, N\}$, and $\forall j \in \{1, 2, \dots, J\}$, respectively. α , and β_n for $\forall n \in \{1, 2, \dots, N\}$, δ , and γ_n for $\forall n \in \{1, 2, \dots, N\}$ are predetermined

²⁶ (3.21)-(3.23) are adapted from International Journal of Production Economics, Vol. 106, Huang, X., Optimal project selection with random fuzzy parameters, pp. 513-522, Copyright©2006 Elsevier B.V., with permission from Elsevier.

confidence levels, and $\max \bar{f}$ is the (α, δ) -return to the objective function

$$\sum_{j=1}^J \sum_{n=1}^L \frac{\eta_{nj}}{(1+r)^n} x_j \quad [18].$$

Chance-constrained programming model with random cash outflows and random net cash flows

$$\begin{aligned} \max \quad & \bar{f} \\ \text{s.t.} \quad & \Pr \left\{ \sum_{j=1}^J \sum_{n=1}^L \frac{\eta_{nj}}{(1+r)^n} x_j \geq \bar{f} \right\} \geq \delta \\ & \Pr \left\{ \sum_{j=1}^J \xi_{nj} x_j \leq M_n \right\} \geq \gamma_n \quad \forall n \in \{1, 2, \dots, N\} \\ & x_j \in \{0, 1\} \quad \forall j \in \{1, 2, \dots, J\} \end{aligned} \quad (3.22)$$

where η_{nj} for $\forall n \in \{1, 2, \dots, L\}$, $\forall j \in \{1, 2, \dots, J\}$ and ξ_{nj} for $\forall n \in \{1, 2, \dots, N\}$, $\forall j \in \{1, 2, \dots, J\}$ are random variables [18].

Chance-constrained programming model with fuzzy cash outflows and fuzzy net cash flows

$$\begin{aligned} \max \quad & \bar{f} \\ \text{s.t.} \quad & \text{Cr} \left\{ \sum_{j=1}^J \sum_{n=1}^L \frac{\eta_{nj}}{(1+r)^n} x_j \geq \bar{f} \right\} \geq \alpha \\ & \text{Cr} \left\{ \sum_{j=1}^J \xi_{nj} x_j \leq M_n \right\} \geq \beta_n \quad \forall n \in \{1, 2, \dots, N\} \\ & x_j \in \{0, 1\} \quad \forall j \in \{1, 2, \dots, J\} \end{aligned} \quad (3.23)$$

where η_{nj} for $\forall n \in \{1, 2, \dots, L\}$, $\forall j \in \{1, 2, \dots, J\}$ and ξ_{nj} for $\forall n \in \{1, 2, \dots, N\}$, $\forall j \in \{1, 2, \dots, J\}$ are fuzzy variables; α and β_n for $\forall n \in \{1, 2, \dots, N\}$ are predetermined credibility levels [17,18].

[17] and [18] proposed a fuzzy simulation based genetic algorithm and a hybrid intelligent algorithm for the solution of (3.23) and (3.21), respectively, while the branch-and-bound method is recommended for the solution of (3.23) with TFNs or TrFNs. In [18], fuzzy simulation and stochastic simulation are used to calculate the (α, δ) – return to the objective function $\sum_{j=1}^J \sum_{n=1}^L \frac{\eta_{nj}}{(1+r)^n} x_j$, while genetic algorithm is used to find the optimal solution.

A numerical example for chance-constrained programming approach to capital rationing

We assume the following chance-constrained programming model with fuzzy cash outflows and fuzzy net cash flows, and $\tilde{a}_{nj} = (a_{nj0}, a_{nj1}, a_{nj2})$ as a triangular, normal, and symmetric fuzzy number for $\forall n \in \{1, 2, 3\}$, $\forall j \in \{1, 2, 3\}$. We consider $(0.9 - 0.95)$ predetermined credibility levels as in [17]:

$$\begin{aligned}
 & \max \quad \bar{f} \\
 & \text{s.t.} \quad \text{Cr} \left\{ \sum_{j=1}^3 \sum_{n=1}^3 \frac{\tilde{a}_{nj}}{(1+r)^n} x_j \geq \bar{f} \right\} \geq 0.9 \\
 & \quad \text{Cr} \left\{ - \sum_{j=1}^3 \tilde{a}_{nj} x_j \leq M_n \right\} \geq 0.95 \quad \forall n \in \{1, 2, 3\} \\
 & \quad x_j \in \{0, 1\} \quad \forall j \in \{1, 2, 3\}
 \end{aligned}$$

We also consider Table 3.3 for the fuzzy cash flows, crisp budget limits, and crisp discount rate which is a slight modification of the data in Table 3.2 .

Table 3.3 : A three-project and three-period example with the fuzzy cash flows

r = 20%	Project 1	Project 2	Project 3	Budget
Year 1	(-16.5, -15, -13.5)	(-5.5, -5, -4.5)	(-13.2, -12, -10.8)	25
Year 2	(4.5, 5, 5.5)	(4.5, 5, 5.5)	(3.6, 4, 4.4)	25
Year 3	(22.5, 25, 27.5)	(4.5, 5, 5.5)	(18, 20, 22)	25

We start with the calculation of $\text{Cr} \left\{ - \sum_{j=1}^3 \tilde{a}_{nj} x_j \leq 25 \right\}$ from (3.18) as follows:

$$\text{Cr}\left\{-\sum_{j=1}^3 \tilde{a}_{nj} x_j \leq 25\right\} = \begin{cases} 0 & , 25 \leq -\sum_{j=1}^3 a_{nj2} x_j \\ \frac{25 + \sum_{j=1}^3 a_{nj2} x_j}{2\left(\sum_{j=1}^3 a_{nj2} x_j - \sum_{j=1}^3 a_{nj1} x_j\right)} & , -\sum_{j=1}^3 a_{nj2} x_j \leq 25 \leq -\sum_{j=1}^3 a_{nj1} x_j \\ \frac{-\sum_{j=1}^3 a_{nj0} x_j + 2\sum_{j=1}^3 a_{nj1} x_j + 25}{2\left(\sum_{j=1}^3 a_{nj1} x_j - \sum_{j=1}^3 a_{nj0} x_j\right)} & , -\sum_{j=1}^3 a_{nj1} x_j \leq 25 \leq -\sum_{j=1}^3 a_{nj0} x_j \\ 1 & , 25 \geq -\sum_{j=1}^3 a_{nj0} x_j \end{cases}$$

where $-\tilde{a}_{nj} = (-a_{nj2}, -a_{nj1}, -a_{nj0})$ for $\forall n \in \{1, 2, 3\}$, $\forall j \in \{1, 2, 3\}$.

Since we predetermine a 0.95 – credibility level for the constraint $-\sum_{j=1}^3 \tilde{a}_{nj} x_j \leq 25$

$\forall n \in \{1, 2, 3\}$, we consider the following equation for the rest of the calculation of

$$\text{Cr}\left\{-\sum_{j=1}^3 \tilde{a}_{nj} x_j \leq 25\right\}:$$

$$\text{Cr}\left\{-\sum_{j=1}^3 \tilde{a}_{nj} x_j \leq 25\right\} = \begin{cases} \frac{25 + \sum_{j=1}^3 a_{nj2} x_j}{2\left(\sum_{j=1}^3 a_{nj2} x_j - \sum_{j=1}^3 a_{nj1} x_j\right)} & , -\sum_{j=1}^3 a_{nj2} x_j \leq 25 \leq -\sum_{j=1}^3 a_{nj1} x_j \\ \frac{-\sum_{j=1}^3 a_{nj0} x_j + 2\sum_{j=1}^3 a_{nj1} x_j + 25}{2\left(\sum_{j=1}^3 a_{nj1} x_j - \sum_{j=1}^3 a_{nj0} x_j\right)} & , -\sum_{j=1}^3 a_{nj1} x_j \leq 25 \leq -\sum_{j=1}^3 a_{nj0} x_j \\ 1 & , 25 \geq -\sum_{j=1}^3 a_{nj0} x_j \end{cases}$$

where the right-hand sides imply mutually exclusive cases for $\text{Cr} \left\{ -\sum_{j=1}^3 \tilde{a}_{nj} x_j \leq 25 \right\}$

$\forall n \in \{1, 2, 3\}$. We next calculate $\text{Cr} \left\{ \sum_{j=1}^3 \sum_{n=1}^3 \frac{\tilde{a}_{nj}}{(1+0.2)^n} x_j \geq \bar{f} \right\}$ from (3.19) as

follows:

$$\text{Cr} \left\{ \sum_{j=1}^3 \sum_{n=1}^3 \frac{\tilde{a}_{nj}}{(1+0.2)^n} x_j \geq \bar{f} \right\} = \begin{cases} 1 & , \bar{f} \leq \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} \\ \frac{2 \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} - \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} - \bar{f}}{2 \left(\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} - \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} \right)} & , \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} \leq \bar{f} \leq \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} \\ \frac{\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2} x_j}{(1+0.2)^n} - \bar{f}}{2 \left(\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2} x_j}{(1+0.2)^n} - \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} \right)} & , \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} \leq \bar{f} \leq \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2} x_j}{(1+0.2)^n} \\ 0 & , \bar{f} \geq \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2} x_j}{(1+0.2)^n} \end{cases}$$

Since we determine a 0.90 – credibility level for the constraint

$\sum_{j=1}^3 \sum_{n=1}^3 \frac{\tilde{a}_{nj}}{(1+0.2)^n} x_j \geq \bar{f}$, we consider the following:

$$\text{Cr} \left\{ \sum_{j=1}^3 \sum_{n=1}^3 \frac{\tilde{a}_{nj}}{(1+0.2)^n} x_j \geq \bar{f} \right\} = \begin{cases} 1 & , \bar{f} \leq \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} \\ \frac{2 \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} - \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} - \bar{f}}{2 \left(\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} - \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} \right)} & , \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0} x_j}{(1+0.2)^n} \leq \bar{f} \leq \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} \\ \frac{\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2} x_j}{(1+0.2)^n} - \bar{f}}{2 \left(\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2} x_j}{(1+0.2)^n} - \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} \right)} & , \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj1} x_j}{(1+0.2)^n} \leq \bar{f} \leq \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2} x_j}{(1+0.2)^n} \end{cases}$$

where the right-hand sides imply mutually exclusive cases for

$\text{Cr} \left\{ \sum_{j=1}^3 \sum_{n=1}^3 \frac{\tilde{a}_{nj}}{(1+0.2)^n} x_j \geq \bar{f} \right\}$. As an example, we give the following problem

instance, where the credibility level is 1, and solve the model with the branch-and-bound method:

$$\begin{aligned}
& \max \quad \bar{f} \\
& \text{s.t.} \quad \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0}}{(1+0.2)^n} x_j \geq \bar{f} \\
& \quad - \sum_{j=1}^3 a_{nj0} x_j \leq 25 \quad \forall n \in \{1, 2, 3\} \\
& \quad x_j \in \{0, 1\} \quad \forall j \in \{1, 2, 3\}
\end{aligned}$$

If we set the upper bound of the first constraint as an objective function, then the following model is used for the branch-and-bound method:

$$\begin{aligned}
& \max \quad \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0}}{(1+0.2)^n} x_j \\
& \text{s.t.} \quad - \sum_{j=1}^3 a_{nj0} x_j \leq 25 \quad \forall n \in \{1, 2, 3\} \\
& \quad x_j \in \{0, 1\} \quad \forall j \in \{1, 2, 3\}
\end{aligned}$$

$$\text{Since } \sum_{n=1}^3 \frac{a_{n10}}{(1+0.2)^n} = 2.39583333, \sum_{n=1}^3 \frac{a_{n20}}{(1+0.2)^n} = 1.145833, \sum_{n=1}^3 \frac{a_{n30}}{(1+0.2)^n} = 1.916667;$$

then the ranking of the projects will be $1 \succ 3 \succ 2$. According to [16], LIFO method is used to decide the order for fathoming the current candidate solution, and the LB for the beginning of the problem is set as an extremely negative number such as $-9\,999\,999$.

Current candidate solution - $x_2 = 1, x_3 = 1$

The objective function is 3.0625. Since the feasibility condition is satisfied, 3.0625 is set as the LB of the current incumbent solution.

Current candidate solution - $x_2 = 1, x_3 = 1, x_1 = 1$

The feasibility condition is not satisfied.

Current candidate solution - $x_2 = 1, x_3 = 0, x_1 = 1$

The objective function is 3.5416667. Since the feasibility condition is satisfied, 3.5416667 is set as the LB of the current incumbent solution.

Current candidate solution - $x_2 = 0, x_3 = 1, x_1 = 1$

The feasibility condition is not satisfied.

Finally, $x_2 = 1, x_3 = 0, x_1 = 1$ is the optimal solution with an objective function 3.5416667. The branch-and-bound method for the numerical example of the chance-constrained programming approach to capital rationing is summarized in Figure 3.3.

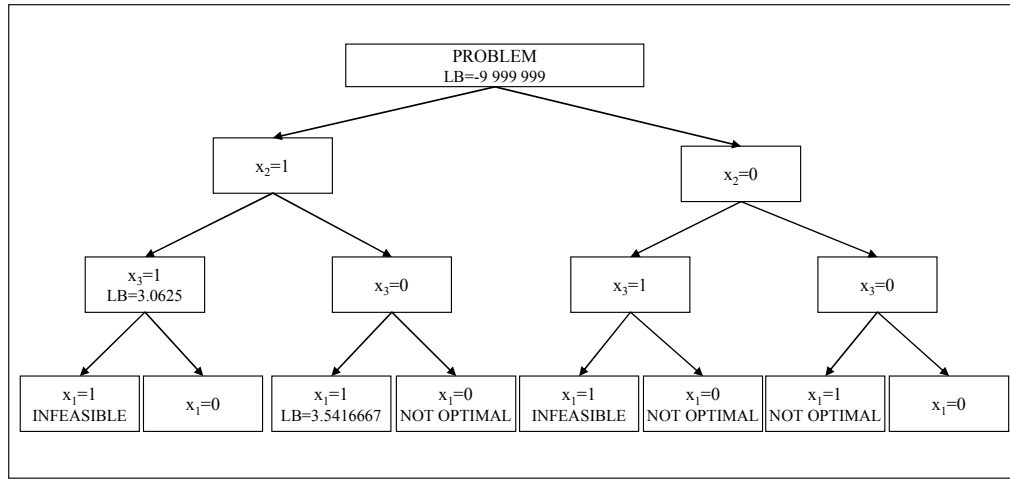


Figure 3.3 : Branch-and-bound method for the numerical example of chance-constrained programming approach to capital rationing

3.1.2.3. Mean-variance model for fuzzy capital budgeting

[19] developed the mean-variance model for fuzzy capital budgeting as shown in (3.24). In the following capital budgeting model, η_{nj} is a fuzzy variable representing the net cash flow of project j at the end of the period n for $\forall n \in \{1, 2, \dots, L_j\}$, ξ_{nj} is a fuzzy variable representing the cash outflow of project j at the beginning of the period n for $\forall n \in \{1, 2, \dots, N\}$, and ς_n is a fuzzy variable representing the budget limit at the beginning of the period n for $\forall n \in \{1, 2, \dots, N\}$, respectively. N is the budget life, and L_j is the project life of project j . E and V represent the expected value and the variance, respectively. p_1 and p_2 are the predetermined tolerable variance levels of the total fuzzy net present value, and the total fuzzy cash outflow, respectively [19].

$$\begin{aligned}
\max \quad & E \left[\sum_{j=1}^J \sum_{n=1}^{L_j} \frac{\eta_{nj}}{(1+r)^n} x_j \right] \\
\text{s.t.} \quad & V \left[\sum_{j=1}^J \sum_{n=1}^{L_j} \frac{\eta_{nj}}{(1+r)^n} x_j \right] \leq p_1 \\
& E \left[\sum_{j=1}^J \xi_{nj} x_j \right] \leq E[\varsigma_n] \quad , \quad \forall n \in \{1, 2, \dots, N\} \quad (3.24) \\
& V \left[\sum_{j=1}^J \xi_{nj} x_j \right] \leq p_2 \quad \forall n \in \{1, 2, \dots, N\} \\
& x_j \in \{0, 1\} \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned}$$

A problem instance for the generic model (3.24)

As a problem instance, let $\eta_{nj} = (\eta_{nj0}, \eta_{nj1}, \eta_{nj2})$, $\xi_{nj} = (\xi_{nj0}, \xi_{nj1}, \xi_{nj2})$ and $\varsigma_n = (\varsigma_{n0}, \varsigma_{n1}, \varsigma_{n2})$ be symmetric TFNs. Then, (3.24) is equivalent to the following model as in [19]²⁷:

$$\begin{aligned}
\max \quad & \frac{1}{4} \left(\sum_{j=1}^J \sum_{n=1}^{L_j} \frac{\eta_{nj0}}{(1+r)^n} x_j + \sum_{j=1}^J \sum_{n=1}^{L_j} \frac{2\eta_{nj1}}{(1+r)^n} x_j + \sum_{j=1}^J \sum_{n=1}^{L_j} \frac{\eta_{nj2}}{(1+r)^n} x_j \right) \\
\text{s.t.} \quad & \left(\sum_{j=1}^J \sum_{n=1}^{L_j} \frac{\eta_{nj2}}{(1+r)^n} x_j - \sum_{j=1}^J \sum_{n=1}^{L_j} \frac{\eta_{nj0}}{(1+r)^n} x_j \right)^2 \leq 24p_1 \\
& \sum_{j=1}^J \xi_{nj0} x_j + \sum_{j=1}^J 2\xi_{nj1} x_j + \sum_{j=1}^J \xi_{nj2} x_j \leq \varsigma_{n0} + 2\varsigma_{n1} + \varsigma_{n2} \quad \forall n \in \{1, 2, \dots, N\} \quad (3.25) \\
& \left(\sum_{j=1}^J \xi_{nj2} x_j - \sum_{j=1}^J \xi_{nj0} x_j \right)^2 \leq 24p_2 \quad \forall n \in \{1, 2, \dots, N\} \\
& x_j \in \{0, 1\} \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned}$$

²⁷ (3.24) and (3.25) are adapted from Computers & Industrial Engineering, Vol. 55, Huang, X., Mean-variance model for fuzzy capital budgeting, pp. 34-47, Copyright©2007 Elsevier Ltd, with permission from Elsevier.

[19] proposed the fuzzy simulation based genetic algorithm for the solution of the generic model (3.24), while the branch-and-bound algorithm has been found to be sufficient for a model with TFNs or TrFNs.

A numerical example for fuzzy mean-variance model

We assume $p_1 = p_2 = 1.5$ as the tolerable variance levels of the total fuzzy net present value, and the total fuzzy cash outflow for (3.25) as in [19]. We consider Table 3.4 which is a slight modification of the data in Table 3.2.

Table 3.4 : A three-project and three-period example with the fuzzy cash flows and budget limits

r = 20%	Project 1	Project 2	Project 3	Budget
Year 1	(-16.5, -15, -13.5)	(-5.5, -5, -4.5)	(-13.2, -12, -10.8)	(22.5, 25, 27.5)
Year 2	(4.5, 5, 5.5)	(4.5, 5, 5.5)	(3.6, 4, 4.4)	(22.5, 25, 27.5)
Year 3	(22.5, 25, 27.5)	(4.5, 5, 5.5)	(18, 20, 22)	(22.5, 25, 27.5)

Let $\tilde{a}_{nj} = (a_{nj0}, a_{nj1}, a_{nj2})$ be a normal and symmetric triangular fuzzy number for $\forall n \in \{1, 2, 3\}$, $\forall j \in \{1, 2, 3\}$, and let $\tilde{M}_n = (M_{n0}, M_{n1}, M_{n2})$ be a normal and symmetric triangular fuzzy number for $\forall n \in \{1, 2, 3\}$. Then, the following model is equivalent from (3.25):

$$\begin{aligned}
\max \quad & \frac{1}{4} \left(\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0}}{(1+r)^n} x_j + \sum_{j=1}^3 \sum_{n=1}^3 \frac{2a_{nj1}}{(1+r)^n} x_j + \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2}}{(1+r)^n} x_j \right) \\
\text{s.t.} \quad & \left(\sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj2}}{(1+r)^n} x_j - \sum_{j=1}^3 \sum_{n=1}^3 \frac{a_{nj0}}{(1+r)^n} x_j \right)^2 \leq 36 \\
& - \left(\sum_{j=1}^3 a_{nj0} x_j + \sum_{j=1}^3 2a_{nj1} x_j + \sum_{j=1}^3 a_{nj2} x_j \right) \leq M_{n0} + 2M_{n1} + M_{n2} \quad \forall n \in \{1, 2, 3\} \\
& \left(\sum_{j=1}^3 a_{nj0} x_j - \sum_{j=1}^3 a_{nj2} x_j \right)^2 \leq 36 \quad \forall n \in \{1, 2, 3\} \\
& x_j \in \{0, 1\} \quad \forall j \in \{1, 2, 3\}
\end{aligned}$$

Since all fuzzy numbers are triangular, the branch-and-bound method can be used for the solution as suggested in [19]. Since

$$\frac{1}{4} \left(\sum_{n=1}^3 \frac{a_{n10}}{(1+r)^n} + \sum_{n=1}^3 \frac{2a_{n11}}{(1+r)^n} + \sum_{n=1}^3 \frac{a_{n12}}{(1+r)^n} \right) = 5.43981481,$$

$$\frac{1}{4} \left(\sum_{n=1}^3 \frac{a_{n20}}{(1+r)^n} + \sum_{n=1}^3 \frac{2a_{n21}}{(1+r)^n} + \sum_{n=1}^3 \frac{a_{n22}}{(1+r)^n} \right) = 2.1990741,$$

$$\frac{1}{4} \left(\sum_{n=1}^3 \frac{a_{n30}}{(1+r)^n} + \sum_{n=1}^3 \frac{2a_{n31}}{(1+r)^n} + \sum_{n=1}^3 \frac{a_{n32}}{(1+r)^n} \right) = 4.351852,$$

the ranking of the projects will be $1 \succ 3 \succ 2$. According to [16], LIFO method is used to decide the order for fathoming the current candidate solutions.

Current candidate solution - $x_2 = 1, x_3 = 1$

The current objective function is 6.5509259. But, the feasibility condition is not satisfied for the first constraint, since $48.67646176 > 36$.

Current candidate solution - $x_2 = 1, x_3 = 0, x_1 = 1$

The current objective function is 7.6388889. But, the feasibility condition is not satisfied for the first constraint, since $67.14891975 > 36$.

Current candidate solution - $x_2 = 1, x_3 = 0, x_1 = 0$

The current objective function is 2.1990741, and all the feasibility conditions are satisfied. Then, 2.1990741 is set as the LB of the current incumbent solution.

Current candidate solution - $x_2 = 0, x_3 = 1, x_1 = 1$

The current objective function is 9.7916667. But, the feasibility condition is not satisfied for the first constraint, since $120.0850694 > 36$.

Current candidate solution - $x_2 = 0, x_3 = 1, x_1 = 0$

The current objective function is 4.351852, and all the feasibility conditions are satisfied. Then, 4.351852 is set as the LB of the current incumbent solution.

Current candidate solution - $x_2 = 0, x_3 = 0, x_1 = 1$

The current objective function is 5.43981481. But, the feasibility condition is not satisfied for the first constraint, since $37.06329304 > 36$.

Finally, the optimal solution is $x_2 = 0$, $x_3 = 1$, $x_1 = 0$ with an objective function 4.351852. The branch-and-bound method for the numerical example of fuzzy mean-variance model is summarized in Figure 3.4.

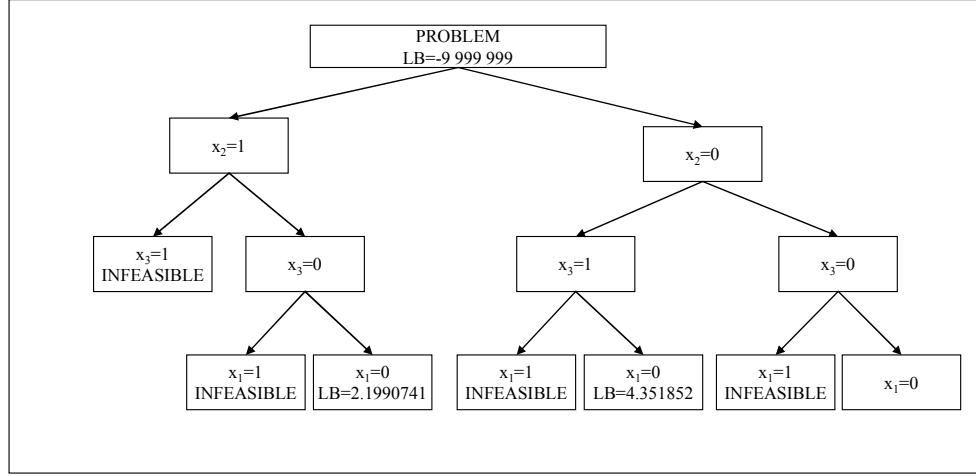


Figure 3.4 : Branch-and-bound method for the numerical example of fuzzy mean-variance model

3.2. Fuzzy Capital Rationing Model with t-norm Fuzzy Relation²⁸

In this section, we propose the defuzzification of Weingartner's pure capital rationing model with t-norm fuzzy relation for the constraints and the objective function, and propose to develop the dual model with t-conorm fuzzy relation. Then, the defuzzified capital rationing model will be examined in detail.

3.2.1. The primal model and the dual model

We consider Weingartner's pure capital rationing model in (3.2). We now develop a fuzzy pure capital rationing model based on (3.2):

$$\begin{aligned}
 \max \quad & \sum_{j=1}^J \tilde{NPV}_j x_j \\
 \text{s.t.} \quad & -\sum_{j=1}^J \tilde{a}_{nj} x_j \tilde{P}_n \tilde{M}_n \quad \forall n \in \{0, 1, \dots, N\} \\
 & 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}
 \end{aligned} \tag{3.26}$$

²⁸ This section is accepted for publication as a journal paper, Journal of Computational and Applied Mathematics, doi: 10.1016/j.cam.2008.05.044, Bas, E. and Kahraman, C., Fuzzy capital rationing model, Copyright©2008 Elsevier B.V., and is reprinted in this thesis with permission from Elsevier.

where $\tilde{NPV}_j = \sum_{n=0}^N \frac{\tilde{a}_{nj}}{(1+r)^n}$. $\tilde{a}_{nj} = (a_{nj} - \hat{a}_{nj}, a_{nj}, a_{nj} + \hat{a}_{nj})$ is a normal, symmetric, and strictly convex triangular fuzzy number for $\forall n, \forall j$. The boundaries of $\tilde{a}_{nj}$ are assumed to be either positive or negative, but not both. $\tilde{M}_n = (M_n - \hat{M}_n, M_n, M_n + \hat{M}_n)$ is a normal, symmetric, and strictly convex triangular fuzzy number for $\forall n$. $\tilde{P}_n$ for $\forall n$ is a fuzzy extension of the crisp binary relation $\leq$ for the n th constraint as in [24]. We assume all constraints as hard constraints and project selection variable x_j as crisp number for $\forall j$. Although different definitions for the maximization of the objective function are possible in an FLP model, we follow [24], and assign aspiration level for the maximization of the objective function. Finally, (3.26) will be equivalent to the following model:

$$\begin{aligned} & \sum_{j=1}^J \tilde{NPV}_j x_j \tilde{P}_{obj} \tilde{Z}_0 \\ & - \sum_{j=1}^J \tilde{a}_{nj} x_j \tilde{P}_n \tilde{M}_n \quad \forall n \in \{0, 1, \dots, N\} \\ & 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \end{aligned} \quad (3.27)$$

where $\tilde{Z}_0$ is an exogenously given fuzzy goal such that $\tilde{Z}_0 = (Z_0 - \hat{Z}_0, Z_0, Z_0 + \hat{Z}_0)$ is a normal, symmetric, and strictly convex triangular fuzzy number. $\tilde{P}_{obj}$ is a fuzzy extension of the binary relation $\geq$ for the objective function and $\tilde{P}_n$ for $\forall n$ is a fuzzy extension of the binary relation $\leq$ for the n th constraint. $\tilde{P}_{obj} = \tilde{\geq}^T$ and $\tilde{P}_n = \tilde{\leq}^T$ for $\forall n$ are assumed to be t-norm fuzzy relations.

We call an (α, α) -efficient solution set as a solution set which satisfies α -satisfaction level for the objective function and constraints. We now formulate a model with (α, α) -efficient solution set.

Proposition 1:

$$\mu_{\leq^T} \left(- \sum_{j=1}^J \tilde{a}_{nj} x_j, \tilde{M}_n \right) \geq \alpha \quad \text{if and only if} \quad - \sum_{j=1}^J \tilde{a}_{nj}(\alpha) x_j \leq \tilde{M}_n(\alpha) \quad (3.28)$$

Proof:

$$\mu_{\leq^T} \left(- \sum_{j=1}^J \tilde{a}_{nj} x_j, \tilde{M}_n \right) \geq \alpha \text{ if and only if } \inf \left[- \sum_{j=1}^J \tilde{a}_{nj} x_j \right]_{\alpha} \leq \sup \left[\tilde{M}_n \right]_{\alpha} \quad (3.29)$$

from (2.30).

$$\inf \left[- \sum_{j=1}^J \tilde{a}_{nj} x_j \right]_{\alpha} \leq \sup \left[\tilde{M}_n \right]_{\alpha} \text{ if and only if } - \sum_{j=1}^J \tilde{a}_{nj}(\alpha) x_j \leq \tilde{M}_n(\alpha) \quad (3.30)$$

since

$$\begin{aligned} \inf \left[- \sum_{j=1}^J \tilde{a}_{nj} x_j \right]_{\alpha} &= \inf \left(- \sum_{j=1}^J [\tilde{a}_{nj}]_{\alpha} x_j \right) = - \sup \left(\sum_{j=1}^J [\tilde{a}_{nj}]_{\alpha} x_j \right) \\ &= - \sum_{j=1}^J \sup [\tilde{a}_{nj}]_{\alpha} x_j = - \sum_{j=1}^J \tilde{a}_{nj}(\alpha) x_j \end{aligned} \quad (3.30a)$$

$$\text{where } \left[- \sum_{j=1}^J \tilde{a}_{nj} x_j \right]_{\alpha} = \left[\sum_{j=1}^J \tilde{a}_{nj}(-x_j) \right]_{\alpha} = \sum_{j=1}^J [\tilde{a}_{nj}]_{\alpha}(-x_j) = - \sum_{j=1}^J [\tilde{a}_{nj}]_{\alpha} x_j$$

by extended Nguyen's result by Dubois and Prade as in [24] and

$$\sup [\tilde{a}_{nj}]_{\alpha} = \tilde{a}_{nj}(\alpha) \text{ from (2.36), and}$$

$$\sup [\tilde{M}_n]_{\alpha} = \tilde{M}_n(\alpha) \quad (3.30b)$$

from (2.36). $\square$

Proposition 2:

$$\mu_{\geq^T} \left(\sum_{j=1}^J NPV_j x_j, \tilde{Z}_0 \right) \geq \alpha \text{ if and only if } \tilde{Z}_0(\alpha) \leq \sum_{j=1}^J \sum_{n=0}^N \frac{\tilde{a}_{nj}(\alpha)}{(1+r)^n} x_j \quad (3.31)$$

Proof:

$$\mu_{\geq^T} \left(\sum_{j=1}^J NPV_j x_j, \tilde{Z}_0 \right) \geq \alpha \equiv \mu_{\leq^T} \left(\tilde{Z}_0, \sum_{j=1}^J NPV_j x_j \right) \geq \alpha \equiv \mu_{\leq^T} \left(\tilde{Z}_0, \sum_{j=1}^J \sum_{n=0}^N \frac{\tilde{a}_{nj}}{(1+r)^n} x_j \right) \geq \alpha$$

$$\text{if and only if } \inf [\tilde{Z}_0]_\alpha \leq \sup \left[\sum_{j=1}^J \sum_{n=0}^N \frac{\tilde{a}_{nj}}{(1+r)^n} x_j \right]_\alpha \quad (3.32)$$

from (2.30).

$$\inf [\tilde{Z}_0]_\alpha \leq \sup \left[\sum_{j=1}^J \sum_{n=0}^N \frac{\tilde{a}_{nj}}{(1+r)^n} x_j \right]_\alpha \text{ if and only if } \tilde{Z}_0(\alpha) \leq \sum_{j=1}^J \sum_{n=0}^N \frac{a_{nj}(\alpha)}{(1+r)^n} x_j \quad (3.33)$$

since

$$\inf [\tilde{Z}_0]_\alpha = \tilde{Z}_0(\alpha) \quad (3.33a)$$

from (2.34), and

$$\begin{aligned} \sup \left[\sum_{j=1}^J \sum_{n=0}^N \frac{\tilde{a}_{nj}}{(1+r)^n} x_j \right]_\alpha &= \sup \left(\sum_{j=1}^J \sum_{n=0}^N \frac{[\tilde{a}_{nj}]_\alpha}{(1+r)^n} x_j \right) = \sum_{j=1}^J \sum_{n=0}^N \frac{\sup [\tilde{a}_{nj}]_\alpha}{(1+r)^n} x_j \\ &= \sum_{j=1}^J \sum_{n=0}^N \frac{a_{nj}(\alpha)}{(1+r)^n} x_j \end{aligned} \quad (3.33b)$$

where $\sup [\tilde{a}_{nj}]_\alpha = a_{nj}(\alpha)$ from (2.36). $\square$

By considering Proposition 1 and Proposition 2, (3.27) is equivalent to the following model as follows:

$$\begin{aligned} \sum_{j=1}^J \sum_{n=0}^N \frac{a_{nj}(\alpha)}{(1+r)^n} x_j &\geq \tilde{Z}_0(\alpha) \\ - \sum_{j=1}^J a_{nj}(\alpha) x_j &\leq \tilde{M}_n(\alpha) \quad \forall n \in \{0, 1, \dots, N\} \end{aligned} \quad (3.34)$$

$$0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}$$

Note that, when assigning aspiration level for the maximization of the objective function, the objective function is also treated as an additional constraint with fuzzy parameters in (3.27), and transformed into a crisp constraint by the same fuzzy

relation for the traditional constraints. Finally, when we include the maximization of the objective function, we obtain the following model:

$$\begin{aligned}
\max \quad & \sum_{j=1}^J \sum_{n=0}^N \frac{a_{nj}(\alpha)^{-R}}{(1+r)^n} x_j \\
\text{s.t.} \quad & -\sum_{j=1}^J a_{nj}(\alpha) x_j \leq \bar{M}_n(\alpha) \quad \forall n \in \{0, 1, \dots, N\} \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.35}$$

where (3.35) can be interpreted as an optimistic case of the generic model (3.26), since at each α -cut level, the best-case values are assigned to cash flows and budget limits. Recall that the solution of an FLP depends on the fuzzy relations, as emphasized in [24].

Since $\tilde{a}_{nj}$ for $\forall n, \forall j$ is well-defined in (3.26), (3.35) will be equivalent to the following model:

$$\begin{aligned}
\max \quad & \sum_{j=1}^J \sum_{n=0}^N \frac{a_{nj} + (1-\alpha) \hat{a}_{nj}}{(1+r)^n} x_j \\
\text{s.t.} \quad & -\sum_{j=1}^J (a_{nj} + (1-\alpha) \hat{a}_{nj}) x_j \leq M_n + (1-\alpha) \hat{M}_n \quad \forall n \in \{0, 1, \dots, N\} \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.36}$$

We next formulate the dual model of (3.26) as follows:

$$\begin{aligned}
\min \quad & \sum_{n=0}^N \tilde{M}_n \rho_n + \sum_{j=1}^J \mu_j \\
\text{s.t.} \quad & \sum_{n=0}^N -\tilde{a}_{nj} \rho_n + \mu_j \tilde{Q}_j^d \tilde{NPV}_j \quad \forall j \in \{1, 2, \dots, J\} \\
& \rho_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \mu_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.37}$$

where $\tilde{Q}_j^d = \tilde{\succeq}_s$ is a t-conorm fuzzy relation for $\forall j$ as a dual to the $\tilde{\preceq}^T$ t-norm fuzzy relation as in [24]. We assume all variables as crisp numbers. When we define $\tilde{W}_0$ as an exogenously given fuzzy goal for the objective function analogous to the primal problem, where $\tilde{W}_0 = (\tilde{W}_0 - \hat{W}_0, \tilde{W}_0, \tilde{W}_0 + \hat{W}_0)$ is a normal, symmetric, and strictly convex triangular fuzzy number, then (3.37) will be equivalent to the following model:

$$\begin{aligned} & \sum_{n=0}^N \tilde{M}_n \rho_n \tilde{+} \sum_{j=1}^J \mu_j \tilde{Q}_{obj}^d \tilde{W}_0 \\ & \sum_{n=0}^N -\tilde{a}_{nj} \rho_n \tilde{+} \mu_j \tilde{Q}_j^d \tilde{NPV}_j \quad \forall j \in \{1, 2, \dots, J\} \\ & \rho_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\ & \mu_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\} \end{aligned} \tag{3.38}$$

where $\tilde{Q}_{obj}^d = \tilde{\preceq}_s$ is also a t-conorm fuzzy relation.

Proposition 3:

$$\begin{aligned} & \mu_{\tilde{\succeq}_s} \left(\sum_{n=0}^N -\tilde{a}_{nj} \rho_n \tilde{+} \mu_j, \tilde{NPV}_j \right) \geq \alpha \\ & \text{if and only if} \quad \sum_{n=0}^N \frac{a_{nj}(1-\alpha)}{(1+r)^n} \leq -\sum_{n=0}^N a_{nj}(1-\alpha) \rho_n + \mu_j \end{aligned} \tag{3.39}$$

Proof:

$$\begin{aligned} & \mu_{\tilde{\succeq}_s} \left(\sum_{n=0}^N -\tilde{a}_{nj} \rho_n \tilde{+} \mu_j, \tilde{NPV}_j \right) \geq \alpha \equiv \mu_{\tilde{\preceq}_s} \left(\tilde{NPV}_j, \sum_{n=0}^N -\tilde{a}_{nj} \rho_n \tilde{+} \mu_j \right) \geq \alpha \\ & \text{if and only if} \quad \sup \left(\tilde{NPV}_j \right)_{1-\alpha} \leq \inf \left(\sum_{n=0}^N -\tilde{a}_{nj} \rho_n \tilde{+} \mu_j \right)_{1-\alpha} \end{aligned} \tag{3.40}$$

from (2.31).

$$\sup \left(NPV_j \right)_{1-\alpha} \leq \inf \left(\sum_{n=0}^N -\tilde{a}_{nj} \rho_n \tilde{+} \mu_j \right)_{1-\alpha}$$

$$\text{if and only if } \sum_{n=0}^N \frac{a_{nj}(1-\alpha)}{(1+r)^n} \leq -\sum_{n=0}^N a_{nj}(1-\alpha)\rho_n + \mu_j \quad (3.41)$$

since

$$\begin{aligned} \sup \left(NPV_j \right)_{1-\alpha} &= \sup \left(\sum_{n=0}^N \frac{\tilde{a}_{nj}}{(1+r)^n} \right)_{1-\alpha} = \sup \left(\sum_{n=0}^N \frac{(\tilde{a}_{nj})_{1-\alpha}}{(1+r)^n} \right) = \\ &= \sum_{n=0}^N \frac{\sup (\tilde{a}_{nj})_{1-\alpha}}{(1+r)^n} = \sum_{n=0}^N \frac{a_{nj}(1-\alpha)}{(1+r)^n} \end{aligned} \quad (3.41a)$$

where $\sup (\tilde{a}_{nj})_{1-\alpha} = a_{nj}(1-\alpha)$ from (2.37), and

$$\begin{aligned} \inf \left(\sum_{n=0}^N -\tilde{a}_{nj} \rho_n \tilde{+} \mu_j \right)_{1-\alpha} &= \inf \left(\sum_{n=0}^N -\tilde{a}_{nj} \rho_n \right)_{1-\alpha} + \mu_j \\ &= \inf \left(\sum_{n=0}^N \tilde{a}_{nj} (-\rho_n) \right)_{1-\alpha} + \mu_j = \inf \left(\sum_{n=0}^N (\tilde{a}_{nj})_{1-\alpha} (-\rho_n) \right) + \mu_j \\ &= \sum_{n=0}^N \sup (\tilde{a}_{nj})_{1-\alpha} (-\rho_n) + \mu_j = \sum_{n=0}^N a_{nj}(1-\alpha)(-\rho_n) + \mu_j \\ &= -\sum_{n=0}^N a_{nj}(1-\alpha)\rho_n + \mu_j \end{aligned} \quad (3.41b)$$

where $\sup (\tilde{a}_{nj})_{1-\alpha} = a_{nj}(1-\alpha)$ from (2.37). $\square$

Proposition 4:

$$\mu_{\leq_s} \left(\sum_{n=0}^N \tilde{M}_n \rho_n \tilde{+} \sum_{j=1}^J \mu_j, \tilde{W}_0 \right) \geq \alpha$$

$$\text{if and only if } \sum_{n=0}^N \tilde{M}_n(1-\alpha)\rho_n + \sum_{j=1}^J \mu_j \leq \tilde{W}_0(1-\alpha) \quad (3.42)$$

Proof: Follows easily from (2.31), (2.35), and (2.37). $\square$

By considering Proposition 3 and Proposition 4, (3.38) will be equivalent to (3.43). Note that the same argument holds for the objective function of the dual problem (3.43) as in (3.34) except that binary relation $\geq$ in the primal problem is changed to $\leq$ in the dual problem.

$$\begin{aligned}
& \sum_{n=0}^N \sum_{j=1}^J M_n (1-\alpha) \rho_n + \sum_{j=1}^J \mu_j \leq \bar{W}_0 (1-\alpha) \\
& - \sum_{n=0}^N \sum_{j=1}^J a_{nj} (1-\alpha) \rho_n + \mu_j \geq \sum_{n=0}^N \frac{a_{nj} (1-\alpha)}{(1+r)^n} \quad \forall j \in \{1, 2, \dots, J\} \\
& \rho_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \mu_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.43}$$

When we include the minimization of the objective function, we will obtain (3.44) and (3.45), since all fuzzy numbers are normal, symmetric, and strictly convex:

$$\begin{aligned}
& \min \quad \sum_{n=0}^N \sum_{j=1}^J M_n (1-\alpha) \rho_n + \sum_{j=1}^J \mu_j \\
& \text{s.t.} \quad - \sum_{n=0}^N \sum_{j=1}^J a_{nj} (1-\alpha) \rho_n + \mu_j \geq \sum_{n=0}^N \frac{a_{nj} (1-\alpha)}{(1+r)^n} \quad \forall j \in \{1, 2, \dots, J\} \\
& \rho_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \mu_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
& \min \quad \sum_{n=0}^N (M_n + \alpha \hat{M}_n) \rho_n + \sum_{j=1}^J \mu_j \\
& \text{s.t.} \quad - \sum_{n=0}^N (a_{nj} + \alpha \hat{a}_{nj}) \rho_n + \mu_j \geq \sum_{n=0}^N \frac{(a_{nj} + \alpha \hat{a}_{nj})}{(1+r)^n} \quad \forall j \in \{1, 2, \dots, J\} \\
& \rho_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \mu_j \geq 0 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.45}$$

Proposition 5: Weak duality

$$\sum_{j=1}^J \sum_{n=0}^N \frac{a_{nj}(\alpha)}{(1+r)^n} x_j \leq \sum_{n=0}^N M_n(\alpha) \rho_n + \sum_{j=1}^J \mu_j \quad \text{definitely holds for } \alpha \in [0.5, 1) \quad (3.46)$$

Proof: We provide the proof of weak duality based on [45, p: 147]. We define:

$$u_n = \rho_n \left(M_n(\alpha) + \sum_{j=1}^J a_{nj}(\alpha) x_j \right) \quad \text{for } \forall n \in \{0, 1, \dots, N\} \quad (3.47)$$

from (3.35).

$$w_j = \mu_j (1 - x_j) \quad \text{for } \forall j \in \{1, 2, \dots, J\} \quad (3.48)$$

from (3.35).

$$v_j = \left(- \sum_{n=0}^N a_{nj}(\alpha) \rho_n + \mu_j - \sum_{n=0}^N \frac{a_{nj}(\alpha)(1-\alpha)}{(1+r)^n} \right) x_j \quad \text{for } \forall j \in \{1, 2, \dots, J\} \quad (3.49)$$

from (3.44).

According to the definition of the dual problem:

$$\rho_n \text{ and } \left(M_n(\alpha) + \sum_{j=1}^J a_{nj}(\alpha) x_j \right) \text{ have the same sign for } \forall n \in \{0, 1, \dots, N\}.$$

$$\mu_j \text{ and } (1 - x_j) \text{ have the same sign for } \forall j \in \{1, 2, \dots, J\}.$$

$$x_j \text{ and } \left(- \sum_{n=0}^N a_{nj}(\alpha) \rho_n + \mu_j - \sum_{n=0}^N \frac{a_{nj}(\alpha)(1-\alpha)}{(1+r)^n} \right) \text{ have the same sign for } \forall j \in \{1, 2, \dots, J\}.$$

As a result, $u_n \geq 0$ for $\forall n \in \{0, 1, \dots, N\}$, $w_j \geq 0$ and $v_j \geq 0$ for $\forall j \in \{1, 2, \dots, J\}$; and

$$\sum_{n=0}^N u_n + \sum_{j=1}^J w_j + \sum_{j=1}^J v_j \geq 0 \quad (3.50)$$

$$\begin{aligned}
& \sum_{n=0}^N u_n + \sum_{j=1}^J w_j + \sum_{j=1}^J v_j = \\
& \sum_{n=0}^N \overset{-R}{M}_n(\alpha) \rho_n + \sum_{n=0}^N \sum_{j=1}^J \overset{-R}{a}_{nj}(\alpha) \rho_n x_j + \sum_{j=1}^J \mu_j - \sum_{j=1}^J \mu_j x_j - \sum_{j=1}^J \sum_{n=0}^N \overset{R}{a}_{nj}(1-\alpha) \rho_n x_j \quad (3.51) \\
& + \sum_{j=1}^J \mu_j x_j - \sum_{j=1}^J \sum_{n=0}^N \frac{\overset{R}{a}_{nj}(1-\alpha)}{(1+r)^n} x_j \geq 0
\end{aligned}$$

Since all triangular fuzzy numbers are normal and strictly convex;

$$\overset{-R}{a}_{nj}(\alpha) = \overset{R}{a}_{nj}(\alpha) \text{ for } \forall n, \forall j, \text{ and } \overset{-R}{M}_n(\alpha) = \overset{R}{M}_n(\alpha) \text{ for } \forall n \text{ from (2.38).}$$

As a result, (3.51) will be equivalent to the following:

$$\begin{aligned}
& \sum_{n=0}^N u_n + \sum_{j=1}^J w_j + \sum_{j=1}^J v_j = \\
& \sum_{n=0}^N \overset{R}{M}_n(\alpha) \rho_n + \sum_{n=0}^N \sum_{j=1}^J \overset{R}{a}_{nj}(\alpha) \rho_n x_j + \sum_{j=1}^J \mu_j - \sum_{j=1}^J \sum_{n=0}^N \overset{R}{a}_{nj}(1-\alpha) \rho_n x_j \quad (3.52) \\
& - \sum_{j=1}^J \sum_{n=0}^N \frac{\overset{R}{a}_{nj}(1-\alpha)}{(1+r)^n} x_j \geq 0
\end{aligned}$$

$$\text{If } \sum_{n=0}^N \sum_{j=1}^J \left(\overset{R}{a}_{nj}(\alpha) - \overset{R}{a}_{nj}(1-\alpha) \right) \rho_n x_j \leq 0, \text{ then } \sum_{j=1}^J \sum_{n=0}^N \frac{\overset{R}{a}_{nj}(1-\alpha)}{(1+r)^n} x_j \leq \sum_{n=0}^N \overset{R}{M}_n(\alpha) \rho_n + \sum_{j=1}^J \mu_j$$

definitely holds.

Since $\rho_n x_j \geq 0$ for $\forall n, \forall j$, then $\overset{R}{a}_{nj}(\alpha) - \overset{R}{a}_{nj}(1-\alpha) \leq 0$ for $\forall n, \forall j$

where $\overset{R}{a}_{nj}(\alpha) - \overset{R}{a}_{nj}(1-\alpha) \leq 0$ for $\forall n, \forall j$ holds for $\alpha \in [0.5, 1)$.

Since $\overset{R}{a}_{nj}(\alpha) - \overset{R}{a}_{nj}(1-\alpha) \leq 0$ for $\forall n, \forall j$ holds for $\alpha \in [0.5, 1)$, then

$$\sum_{j=1}^J \sum_{n=0}^N \frac{\overset{-R}{a}_{nj}(\alpha)}{(1+r)^n} x_j = \sum_{j=1}^J \sum_{n=0}^N \frac{\overset{R}{a}_{nj}(\alpha)}{(1+r)^n} x_j \leq \sum_{j=1}^J \sum_{n=0}^N \frac{\overset{R}{a}_{nj}(1-\alpha)}{(1+r)^n} x_j \leq \sum_{n=0}^N \overset{R}{M}_n(\alpha) \rho_n + \sum_{j=1}^J \mu_j \quad (3.53)$$

definitely holds for $\alpha \in [0.5, 1)$. $\square$

Note that the result corresponds with the result for a general FLP problem in [24].

3.2.2. Sensitivity analysis

We analyze the primal problem (3.36) for change in M_n and in a_{nj} where x_j is a basic or a non-basic variable. In all cases, we study the feasibility and optimality conditions to compute the allowable shift of the core of the triangular fuzzy numbers $\tilde{M}_n$ or $\tilde{a}_{nj}$ where x_j is a basic or a non-basic variable so that the current basis remains optimal.

Change in M_n :

We provide the sensitivity analysis for change in M_n based on [45, p: 207-208]. Let us suppose that for some n , M_n is changed to $M_n + \delta$. If we assume a fixed budget

variability rate $v_{M_n} = \frac{\hat{M}_n}{M_n} \times 100$ for each $\tilde{M}_n = (M_n - \hat{M}_n, M_n, M_n + \hat{M}_n)$

analogous to [5], then $\hat{M}_n$ is changed to $\hat{M}_n + \frac{v_{M_n} \delta}{100}$. In this analysis, we would like

to determine the range of values of δ under which the current basis remains optimal.

We assume the standard form of the primal problem. For convenience, the indices of the right-hand side vector are relabeled as $n = (0, 1, \dots, N, N+1, \dots, N+J)$. We define M as a vector with entries M_n for $\forall n \in \{0, 1, \dots, N\}$, and 1 for $\forall n \in \{N+1, N+2, \dots, N+J\}$. We define $\hat{M}$ as a vector with entries $\hat{M}_n$ for $\forall n \in \{0, 1, \dots, N\}$, and 0 for $\forall n \in \{N+1, N+2, \dots, N+J\}$. Then, the current right-hand side vector $M + (1 - \alpha)\hat{M}$ is changed to $M + \delta e_n + (1 - \alpha)\hat{M} + (1 - \alpha)e_n e_n^T \frac{v_M \delta}{100}$ where $0 \leq n \leq N$, v_M is a vector with entries v_{M_n} for $\forall n \in \{0, 1, \dots, N\}$, and 0 for $\forall n \in \{N+1, N+2, \dots, N+J\}$, and e_n is an n th unit vector. Next, we will check the feasibility and optimality conditions.

Feasibility condition:

$$B^{-1} \left(M + \delta e_n + (1 - \alpha)\hat{M} + (1 - \alpha)e_n e_n^T \frac{v_M \delta}{100} \right) \geq 0 \quad \text{for } 0 \leq n \leq N \quad (3.54)$$

where B is the current basis matrix.

For convenience, the indices of the basic variables are relabeled as $q \in B = (1, 2, \dots, N+1+J)$.

Let $g_n = (\beta_{1n}, \beta_{2n}, \dots, \beta_{(N+1+J)n})$ be the n th column of B^{-1} . Then, (3.54) will be equivalent to the following:

$$x_B + \delta g_n + \left(\frac{(1-\alpha) v_{M_n} \delta}{100} \right) g_n \geq 0 \quad (3.55)$$

where $x_B = B^{-1} (M + (1-\alpha) \hat{M})$ is the current basic feasible solution.

For the q th row, where $q \in B$ is a basic variable:

$$x_q + \delta \beta_{qn} + \left(\frac{(1-\alpha) v_{M_n} \delta}{100} \right) \beta_{qn} \geq 0 \quad (3.56)$$

As a result, the range of values of δ under which the current basis remains optimal will be as follows:

$$\max_{\{q \in B \mid \beta_{qn} > 0\}} \left(\frac{x_q}{\beta_{qn} \left(\frac{(\alpha-1) v_{M_n}}{100} - 1 \right)} \right) \leq \delta \leq \min_{\{q \in B \mid \beta_{qn} < 0\}} \left(\frac{x_q}{\beta_{qn} \left(\frac{(\alpha-1) v_{M_n}}{100} - 1 \right)} \right) \quad (3.57)$$

Optimality condition:

Since the basis matrix B does not change, the optimality condition is retained.

Change in a_{nj} :

We provide the sensitivity analysis for change in a_{nj} based on [45, p: 209] when x_j is a non-basic variable, and based on [46, p: 194-196] when x_j is a basic variable.

Two basic cases are to be distinguished as in the following:

Case 1: $a_{nj''}$ is changed to $a_{nj''} + \gamma_{NB}$, where $x_{j''}$ is a non-basic variable.

Case 1.1: $a_{nj''} > 0$

Case 1.2: $a_{nj''} < 0$

Case 2: $a_{nj'}$ is changed to $a_{nj'} + \gamma_B$, where $x_{j'}$ is a basic variable.

Case 2.1: $a_{nj'} > 0$

Case 2.2: $a_{nj'} < 0$

We start with the analysis of Case 1.

Case 1: $a_{nj''}$ is changed to $a_{nj''} + \gamma_{NB}$ where $x_{j''}$ is a non-basic variable.

Case 1.1: $a_{nj''} > 0$

Case 1.2: $a_{nj''} < 0$

Let us suppose that for any $n^* \in \{0, 1, \dots, N\}$, $a_{n^*j''}$ is changed to $a_{n^*j''} + \gamma_{NB}$. If

we assume a fixed cash flow variability rate $v_{a_{nj}} = \frac{\hat{a}_{nj}}{|a_{nj}|} \times 100$ for each

$\tilde{a}_{nj} = (a_{nj} - \hat{a}_{nj}, a_{nj}, a_{nj} + \hat{a}_{nj})$ analogous to [5], then $\hat{a}_{n^*j''}$ is changed to $\hat{a}_{n^*j''} + \frac{v_{a_{n^*j''}} \gamma_{NB}}{100}$

if $a_{n^*j''} > 0$, and $\hat{a}_{n^*j''}$ is changed to $\hat{a}_{n^*j''} - \frac{v_{a_{n^*j''}} \gamma_{NB}}{100}$ if $a_{n^*j''} < 0$, where for $\gamma_{NB} < 0$

and $a_{n^*j''} > 0$, or for $\gamma_{NB} > 0$ and $a_{n^*j''} < 0$, $|\gamma_{NB}| < |a_{n^*j''}|$ should hold. In this

analysis, we would like to determine the range of values of γ_{NB} under which the current basis remains optimal. Next, we will check the feasibility and optimality conditions.

Feasibility condition:

Since the current basis matrix B and the current right-hand side vector $M + (1 - \alpha) \hat{M}$ do not change, the feasibility condition is retained.

Optimality condition:

We should check the following optimality condition:

$$NPV_{j'', \gamma_{NB}}^\alpha - c_B^T B^{-1} \left(A_{j''} - (\gamma_{NB} \oplus (1 - \alpha) \frac{v_{a_{n^*j''}} \gamma_{NB}}{100}) e_{n^*} \right) \leq 0 \quad (3.58)$$

where

$NPV_{j'', \gamma_{NB}}^\alpha = \sum_{n=0}^N \frac{a_{n_{j''}} + (1-\alpha)\hat{a}_{nj''}}{(1+r)^n} + \frac{\gamma_{NB} \oplus (1-\alpha) \frac{v_{a_{n^* j''}} \gamma_{NB}}{100}}{(1+r)^{n^*}}$ is the modified objective function coefficient of the non-basic variable $x_{j''}$, B is the current basis matrix, $A_{j''}$ is the current column of the constraint matrix for the non-basic variable $x_{j''}$, $c_B^T B^{-1} = p^T$, and e_{n^*} is an n^* th unit vector. $\oplus = +$ for $a_{n^* j''} > 0$, and $\oplus = -$ for $a_{n^* j''} < 0$.

Then (3.58) will be equivalent to the following:

$$\overline{NPV}_{j''}^\alpha + \frac{\gamma_{NB} \oplus (1-\alpha) \frac{v_{a_{n^* j''}} \gamma_{NB}}{100}}{(1+r)^{n^*}} + \left(\left(\gamma_{NB} \oplus (1-\alpha) \frac{v_{a_{n^* j''}} \gamma_{NB}}{100} \right) p_{n^*} \right) \leq 0 \quad (3.59)$$

$$\text{where } \overline{NPV}_{j''}^\alpha = \sum_{n=0}^N \frac{a_{n_{j''}} + (1-\alpha)\hat{a}_{nj''}}{(1+r)^n} - p^T A_{j''}$$

is the current reduced cost of the non-basic variable $x_{j''}$, and p_{n^*} is the n^* th entry of p^T .

$$\overline{NPV}_{j''}^\alpha + \gamma_{NB} \left(1 \oplus (1-\alpha) \frac{v_{a_{n^* j''}}}{100} \right) \left(\frac{1}{(1+r)^{n^*}} + p_{n^*} \right) \leq 0 \quad (3.60)$$

$$\text{Let } \varphi = \left(1 \oplus (1-\alpha) \frac{v_{a_{n^* j''}}}{100} \right) \left(\frac{1}{(1+r)^{n^*}} + p_{n^*} \right) \quad (3.60a)$$

providing $1 \oplus (1-\alpha) \frac{v_{a_{n^* j''}}}{100} \neq 0$ analogous to [46, p: 195].

Finally, the range of values of γ_{NB} under which the current basis remains optimal will be provided for the different conditions of Case 1.1 and Case 1.2 when x_j is a non-basic variable as follows:

Case 1.1: $a_{n^*j''} > 0$

$$\text{i) } -a_{n^*j''} < \gamma_{NB} \leq -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} \quad \text{if } p_{n^*} > -\frac{1}{(1+r)^{n^*}} \quad (3.61)$$

$$\text{ii) } \gamma_{NB} \geq -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} \quad \text{if } p_{n^*} < -\frac{1}{(1+r)^{n^*}} \quad \text{and} \quad -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} > -a_{n^*j''} \quad (3.62)$$

$$\gamma_{NB} > -a_{n^*j''} \quad \text{if } p_{n^*} < -\frac{1}{(1+r)^{n^*}} \quad \text{and} \quad -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} \leq -a_{n^*j''} \quad (3.63)$$

since $\overline{NPV}_{j''}^\alpha \leq 0$ always holds by optimality condition and for $\gamma_{NB} < 0$,

$|\gamma_{NB}| < |a_{n^*j''}|$ should hold. $\square$

Case 1.2: $a_{n^*j''} < 0$

$$\text{i) } \gamma_{NB} \leq -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} \quad \text{if } p_{n^*} > -\frac{1}{(1+r)^{n^*}} \quad \text{and} \quad -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} < -a_{n^*j''} \quad (3.64)$$

$$\gamma_{NB} < -a_{n^*j''} \quad \text{if } p_{n^*} > -\frac{1}{(1+r)^{n^*}} \quad \text{and} \quad -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} \geq -a_{n^*j''} \quad (3.65)$$

$$\text{ii) } -\frac{\overline{NPV}_{j''}^\alpha}{\varphi} \leq \gamma_{NB} < -a_{n^*j''} \quad \text{if } p_{n^*} < -\frac{1}{(1+r)^{n^*}} \quad (3.66)$$

since $\overline{NPV}_{j''}^\alpha \leq 0$ always holds by optimality condition and for $\gamma_{NB} > 0$,

$|\gamma_{NB}| < |a_{n^*j''}|$ should hold. $\square$

Next, we analyze Case 2:

Case 2: $a_{nj'}$ is changed to $a_{nj'} + \gamma_B$, where $x_{j'}$ is a basic variable.

Case 2.1: $a_{nj'} > 0$

Case 2.2: $a_{nj'} < 0$

Let us suppose that for any $n^* \in \{0, 1, \dots, N\}$, $a_{n^*j'}$ is changed to $a_{n^*j'} + \gamma_B$. If we assume a fixed cash flow variability rate $v_{a_{nj}} = \frac{\hat{a}_{nj}}{|a_{nj}|} \times 100$ for each $\tilde{a}_{nj} = (a_{nj} - \hat{a}_{nj}, a_{nj}, a_{nj} + \hat{a}_{nj})$ analogous to [5], then $\hat{a}_{n^*j'}$ is changed to $\hat{a}_{n^*j'} + \frac{v_{a_{n^*j'}} \gamma_B}{100}$ if $a_{n^*j'} > 0$, and $\hat{a}_{n^*j'}$ is changed to $\hat{a}_{n^*j'} - \frac{v_{a_{n^*j'}} \gamma_B}{100}$ if $a_{n^*j'} < 0$, where for $\gamma_B < 0$ and $a_{n^*j'} > 0$, or for $\gamma_B > 0$ and $a_{n^*j'} < 0$, $|\gamma_B| < |a_{n^*j'}|$ should hold. In this analysis, we would like to determine the range of values of γ_B under which the current basis remains optimal.

For convenience, the indices of the basic variables are relabeled as $q = \{1, 2, \dots, N + 1 + J\}$. Since $x_{j'}$ is a basic variable, the current basis matrix B is changed to the modified basis matrix $\bar{B}$ as follows:

$$\bar{B} = B - \left(\gamma_B \oplus (1 - \alpha) \frac{v_{a_{n^*j'}} \gamma_B}{100} \right) e_{n^*} e_{j'}^T = B \left(I - \left(\gamma_B \oplus (1 - \alpha) \frac{v_{a_{n^*j'}} \gamma_B}{100} \right) B^{-1} e_{n^*} e_{j'}^T \right) \quad (3.67)$$

where e_{n^*} and $e_{j'}$ are the n^* th and j' th unit vectors, respectively. $\oplus = +$ for $a_{n^*j'} > 0$, and $\oplus = -$ for $a_{n^*j'} < 0$.

Feasibility condition:

$$\bar{B}^{-1} (M + (1 - \alpha) \hat{M}) \geq 0 \quad (3.68)$$

where

$$\bar{B}^{-1} = \left(I + \frac{\left(\gamma_B \oplus (1 - \alpha) \frac{v_{a_{n^*j'}} \gamma_B}{100} \right)}{1 - \left(\gamma_B \oplus (1 - \alpha) \frac{v_{a_{n^*j'}} \gamma_B}{100} \right) \beta_{j'n^*}} B^{-1} e_{n^*} e_{j'}^T \right) B^{-1} \quad (3.68a)$$

where $\beta_{j'n^*}$ is the $j'n^*$ th entry of B^{-1} , and $1 - \left(\gamma_B \oplus (1 - \alpha) \frac{v_{a_{n^*j'}} \gamma_B}{100} \right) \beta_{j'n^*} \neq 0$.

Proof: Follows easily based on [46, p: 195].

$$\text{Let } \phi = \frac{\left(\gamma_B \oplus (1-\alpha) \frac{v_{a_{n^*j'}} \gamma_B}{100} \right)}{1 - \left(\gamma_B \oplus (1-\alpha) \frac{v_{a_{n^*j'}} \gamma_B}{100} \right) \beta_{j'n^*}}$$

Then, (3.68a) is equivalent to the following:

$$\bar{B}^{-1} = (I + \phi B^{-1} e_{n^*} e_{j'}^T) B^{-1} \quad (3.68b)$$

Then, (3.68) is equivalent to the following:

$$(I + \phi B^{-1} e_{n^*} e_{j'}^T) B^{-1} (M + (1-\alpha) \hat{M}) \geq 0 \quad (3.69)$$

$$B^{-1} (M + (1-\alpha) \hat{M}) + (\phi B^{-1} e_{n^*} e_{j'}^T) B^{-1} (M + (1-\alpha) \hat{M}) \geq 0 \quad (3.70)$$

$$x_B + \phi g_{n^*} x_{j'} \geq 0 \quad (3.71)$$

where $x_B = B^{-1} (M + (1-\alpha) \hat{M})$ is the current basic feasible solution, and $g_{n^*} = (\beta_{1n^*}, \beta_{2n^*}, \dots, \beta_{(N+1+J)n^*})$ is the n^* th column of B^{-1} .

As a result, the range of values of ϕ under which the current basis remains feasible will be as follows:

$$\left\{ q \mid \beta_{qn^*} > 0, x_{j'} \neq 0 \right\} \left(-\frac{x_q}{\beta_{qn^*} x_{j'}} \right) \leq \phi \leq \left\{ q \mid \beta_{qn^*} < 0, x_{j'} \neq 0 \right\} \left(-\frac{x_q}{\beta_{qn^*} x_{j'}} \right) \quad (3.72)$$

Additionally, for $\gamma_B < 0$ and $a_{n^*j'} > 0$, or for $\gamma_B > 0$ and $a_{n^*j'} < 0$, $|\gamma_B| < |a_{n^*j'}|$

should hold. $\square$

Optimality condition:

We should check the following optimality condition:

$$NPV_t^\alpha - c_B^T \bar{B}^{-1} A_t \leq 0 \quad \text{for } \forall t \in NB \quad (3.73)$$

where NB is the set of non-basic variables, $NPV_t^\alpha = \sum_{n=0}^N \frac{a_{nt} + (1-\alpha)\hat{a}_{nt}}{(1+r)^n}$ is the current objective function coefficient of the non-basic variable x_t , $\bar{B}$ is the modified basis matrix, and A_t is the column of the constraint matrix for $\forall t \in NB$.

$$NPV_t^\alpha - c_B^T (I + \phi B^{-1} e_{n^*} e_{j'}^T) B^{-1} A_t \leq 0 \quad \text{for } \forall t \in NB \quad (3.74)$$

where $(I + \phi B^{-1} e_{n^*} e_{j'}^T) B^{-1} = \bar{B}^{-1}$ from (3.68b).

$$NPV_t^\alpha - c_B^T B^{-1} A_t - c_B^T \phi B^{-1} e_{n^*} e_{j'}^T B^{-1} A_t \leq 0 \quad \text{for } \forall t \in NB \quad (3.75)$$

where $c_B^T B^{-1} = p^T$.

$$NPV_t^\alpha - p^T A_t - \phi p^T e_{n^*} e_{j'}^T B^{-1} A_t \leq 0 \quad \text{for } \forall t \in NB \quad (3.76)$$

$$NPV_t^\alpha - p^T A_t - \phi p_{n^*} h_{j'} A_t \leq 0 \quad \text{for } \forall t \in NB \quad (3.77)$$

where p_{n^*} is the n^* th entry of p^T , and $h_{j'}$ is the j' th row of B^{-1} .

$$\overline{NPV}_t^\alpha - \phi p_{n^*} h_{j'} A_t \leq 0 \quad (3.78)$$

where $\overline{NPV}_t^\alpha = NPV_t^\alpha - p^T A_t$ is the current reduced cost of the non-basic variable x_t for $\forall t \in NB$.

As a result, the range of values of ϕ under which the current basis remains optimal will be as follows:

$$\max_{\{t \in NB \mid p_{n^*} h_{j'} A_t > 0\}} \left(\frac{\overline{NPV}_t^\alpha}{p_{n^*} h_{j'} A_t} \right) \leq \phi \leq \min_{\{t \in NB \mid p_{n^*} h_{j'} A_t < 0\}} \left(\frac{\overline{NPV}_t^\alpha}{p_{n^*} h_{j'} A_t} \right) \quad (3.79)$$

Additionally, for $\gamma_B < 0$ and $a_{n^* j'} > 0$, or for $\gamma_B > 0$ and $a_{n^* j'} < 0$, $|\gamma_B| < |a_{n^* j'}|$

should hold. $\square$

3.2.3. Complementary slackness analysis

We now develop the complementary slackness results as follows:

$$\rho_n^* \left(M_n(\alpha) + \sum_{j=1}^J a_{nj}(\alpha) x_j^* \right) = 0 \quad \forall n \in \{0, 1, \dots, N\} \quad (3.80)$$

from (3.35)

$$\mu_j^* (1 - x_j^*) = 0 \quad \forall j \in \{1, 2, \dots, J\} \quad (3.81)$$

from (3.35), and

$$x_j^* \left(- \sum_{n=0}^N a_{nj} (1 - \alpha) \rho_n^* + \mu_j^* - \sum_{n=0}^N \frac{a_{nj} (1 - \alpha)}{(1 + r)^n} \right) = 0 \quad \forall j \in \{1, 2, \dots, J\} \quad (3.82)$$

from (3.44).

Characterization of level of investment in projects²⁹

Based on complementary slackness results, we analyze the level of investment in projects analogous to [5]. Recall that $a_{nj}^{(R)}(\alpha) = a_{nj}^R(\alpha)$ for $\forall n, \forall j$ from (2.38).

Case 1: The project j is fully funded in the fuzzy case

$$\text{If } x_j^* = 1, \text{ then } \sum_{n=0}^N a_{nj} (1 - \alpha) \left(\rho_n^* + \frac{1}{(1 + r)^n} \right) \geq 0 \quad (3.83)$$

where x_j^* is an optimal primal variable and ρ_n^* is an optimal dual variable.

Proof: Follows easily from (3.44), (3.81), and (3.82).

The project j is fully funded in the crisp case

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_n^* \geq 0 \quad (3.84)$$

²⁹ (3.84), (3.86) and (3.88) are reprinted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

holds from [5].

Case 2: The project j is partially funded in the fuzzy case

$$\text{If } 0 < x_j^* < 1, \text{ then } \sum_{n=0}^N a_{nj} (1 - \alpha) \left(\rho_n^* + \frac{1}{(1+r)^n} \right) = 0 \quad (3.85)$$

Proof: Follows easily from (3.44), (3.81), and (3.82).

The project j is partially funded in the crisp case

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_n^* = 0 \quad (3.86)$$

holds from [5].

Case 3: The project j is not funded in the fuzzy case

$$\text{If } x_j^* = 0, \text{ then } \sum_{n=0}^N a_{nj} (1 - \alpha) \left(\rho_n^* + \frac{1}{(1+r)^n} \right) \leq 0 \quad (3.87)$$

Proof: Follows easily from (3.44), (3.81), and (3.82).

The project j is not funded in the crisp case

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_n^* \leq 0 \quad (3.88)$$

holds from [5].

Remark: In each case, a_{nj} in crisp case is replaced by $a_{nj}^R (1 - \alpha)$ in the fuzzy case.

3.2.4. Computational analysis

We consider a three-project, three-period example for the fuzzy capital rationing problem with fuzzy cash flow and fuzzy budget. We define $v_{a_{nj}}$ as the cash flow

variability rate for each $\tilde{a}_{nj}$ such that $v_{a_{nj}} = \frac{\hat{a}_{nj}}{|a_{nj}|} \times 100$ and v_{M_n} as the budget

variability rate for $\tilde{M}_n$ such that $v_{M_n} = \frac{\hat{M}_n}{M_n} \times 100$ analogous to [5]. We assume that

$v_{a_{nj}}$ is the same for $\forall n, j$ and v_{M_n} is the same for $\forall n$ such that $v_{a_{nj}} = v_a$ for $\forall n, j$ and $v_{M_n} = v_M$ for $\forall n$, and assign 5%, 10%, 15%, and 20% to v_a and v_M for each case as in [5]. α -cut levels are increased from 0.5 to 0.99 by 0.01. The data in Table 3.1 are used for the computational analysis. We use Xpress-MP software for the computation of (3.36) [47]. The Xpress-MP code which is a modification of the Xpress-MP code for the capital rationing model in [5] has been provided in Table A.1.³⁰ The summary of the results is given in Table 3.5, where the range of values of δ in M_0 and the range of values of γ_B in a_{03} under which the current basis remains optimal are also included. While some sample graphs for the level of investment in Project 3 are denoted in Figure 3.5, all the graphs for the level of investment in Project 3 are available in Figure A.1.

Table 3.5 : Level of investment in projects and sensitivity analysis

v_a	v_M	α	Project 1	Project 2	Project 3	Obj. Func.	Change in M_0 : 6-Interval		Change in a_{03} : γ -interval	
5%	5%	0.5	1	1	0.52350	13.5143	-5.9756	5.4390	0.9179	1.0935
		0.6	1	1	0.50170	13.0654	-6.0204	5.9796	0.9268	1.0890
		0.7	1	1	0.48012	12.6241	-5.7614	6.2386	0.9316	1.0894
		0.8	1	1	0.45875	12.1904	-5.5050	6.4950	0.9360	1.0899
		0.9	1	1	0.43761	11.7641	-5.2513	6.7487	0.9459	1.0904
5%	10%	0.5	1	1	0.57692	13.8321	-6.7582	4.9560	0.9041	1.0885
		0.6	1	1	0.54422	13.3122	-6.4050	5.3642	0.9141	1.0890
		0.7	1	1	0.51184	12.8037	-6.0527	5.7726	0.9229	1.0894
		0.8	1	1	0.47980	12.3064	-5.7011	6.1812	0.9307	1.0899
		0.9	1	1	0.44807	11.8202	-5.5898	6.8854	0.9401	1.0862
5%	15%	0.5	1	1	0.63034	14.15	-7.2123	4.2296	0.8825	1.0885
		0.6	1	1	0.58674	13.559	-6.7751	4.7720	0.8992	1.0890
		0.7	1	1	0.54357	12.9832	-6.3356	5.3199	0.9132	1.0894
		0.8	1	1	0.50084	12.4224	-5.8934	5.8736	0.9250	1.0899
		0.9	1	1	0.45854	11.8764	-5.4483	6.4335	0.9351	1.0904
5%	20%	0.5	1	1	0.68376	14.4679	-7.6457	3.5361	0.8552	1.0885
		0.6	1	1	0.62925	13.8057	-7.1315	4.2018	0.8816	1.0890
		0.7	1	1	0.57530	13.1627	-6.6105	4.8801	0.9023	1.0894
		0.8	1	1	0.52189	12.5384	-6.0820	5.5719	0.9188	1.0899
		0.9	1	1	0.46901	11.9326	-5.5454	6.2782	0.9324	1.0904
10%	5%	0.5	1	1	0.58114	15.5109	-6.4634	4.6585	0.8999	1.0962
		0.6	1	1	0.54688	14.6305	-6.1765	5.1176	0.9115	1.0951
		0.7	1	1	0.51332	13.7745	-5.8867	5.5813	0.9214	1.0940
		0.8	1	1	0.48044	12.942	-5.5941	6.0495	0.9299	1.0929
		0.9	1	1	0.44823	12.1324	-5.2985	6.5224	0.9373	1.0919

³⁰ Modified from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

Table 3.5 : Level of investment in projects and sensitivity analysis (continued)

v_a	v_M	α	Project 1	Project 2	Project 3	Obj. Func.	Change in M_0 : 6-Interval	Change in a_{03} : γ -interval		
10%	10%	0.5	1	1	0.63597	15.8772	-6.9048	3.9524	0.8763	1.0962
		0.6	1	1	0.59028	14.9078	-6.5385	4.5385	0.8956	1.0951
		0.7	1	1	0.54553	13.9709	-6.1651	5.1359	0.9112	1.0940
		0.8	1	1	0.50170	13.0654	-5.7843	5.7451	0.9240	1.0929
		0.9	1	1	0.45875	12.1904	-5.3960	6.3663	0.9348	1.0919
10%	15%	0.5	1	1	0.69079	16.2435	-7.3256	3.2791	0.8462	1.0962
		0.6	1	1	0.63368	15.1851	-6.8868	3.9811	0.8767	1.0951
		0.7	1	1	0.57775	14.1673	-6.4354	4.7034	0.8997	1.0940
		0.8	1	1	0.52296	13.1888	-5.9709	5.4466	0.9177	1.0929
		0.9	1	1	0.46928	12.2484	-5.4926	6.2118	0.9321	1.0919
10%	20%	0.5	1	1	0.74561	16.6097	-7.7273	2.6364	0.8065	1.0962
		0.6	1	1	0.67708	15.4624	-7.2222	3.4444	0.8538	1.0951
		0.7	1	1	0.60997	14.3637	-6.6981	4.2830	0.8867	1.0940
		0.8	1	1	0.54422	13.3122	-6.1538	5.1538	0.9109	1.0929
		0.9	1	1	0.47980	12.3064	-5.5882	6.0588	0.9294	1.0919
15%	5%	0.5	1	1	0.64189	17.6148	-6.9512	3.8780	0.8752	1.0990
		0.6	1	1	0.59397	16.2618	-6.5686	4.4902	0.8956	1.0973
		0.7	1	1	0.54756	14.9607	-6.1823	5.1084	0.9116	1.0956
		0.8	1	1	0.50258	13.709	-5.7921	5.7327	0.9245	1.0940
		0.9	1	1	0.45897	12.5045	-5.3980	6.3632	0.9351	1.0924
15%	10%	0.5	1	1	0.69820	18.0321	-7.3810	3.1905	0.8431	1.0990
		0.6	1	1	0.63830	16.5709	-6.9231	3.9231	0.8759	1.0973
		0.7	1	1	0.58028	15.1745	-6.4563	4.6699	0.8999	1.0956
		0.8	1	1	0.52406	13.8399	-5.9804	5.4314	0.9181	1.0940
		0.9	1	1	0.46954	12.5643	-5.4950	6.2079	0.9324	1.0924
15%	15%	0.5	1	1	0.75451	18.4495	-7.7907	2.5349	0.8004	1.0990
		0.6	1	1	0.68262	16.88	-7.2641	3.3774	0.8518	1.0973
		0.7	1	1	0.61300	15.3884	-6.7225	4.2440	0.8865	1.0956
		0.8	1	1	0.54553	13.9709	-6.1651	5.1359	0.9112	1.0940
		0.9	1	1	0.48012	12.6241	-5.5911	6.0542	0.9296	1.0924
15%	20%	0.5	1	1	0.81081	18.8668	-8.1818	1.9091	0.7405	1.0990
		0.6	1	1	0.72695	17.1891	-7.5926	2.8519	0.8220	1.0973
		0.7	1	1	0.64572	15.6022	-6.9811	3.8302	0.8711	1.0956
		0.8	1	1	0.56701	14.1018	-6.3462	4.8462	0.9037	1.0940
		0.9	1	1	0.49069	12.684	-5.6863	5.9020	0.9268	1.0924
20%	5%	0.5	1	1	0.70602	19.835	-7.4390	3.0976	0.8397	1.1020
		0.6	1	1	0.64312	17.9637	-6.9608	3.8627	0.8749	1.0996
		0.7	1	1	0.58289	16.1846	-6.4778	4.6355	0.9000	1.0973
		0.8	1	1	0.52517	14.4919	-5.9901	5.4158	0.9185	1.0951
		0.9	1	1	0.46981	12.8803	-5.4975	6.2040	0.9327	1.0929
20%	10%	0.5	1	1	0.76389	20.3062	-7.8571	2.4286	0.7934	1.1020
		0.6	1	1	0.68841	18.306	-7.3077	3.3077	0.8497	1.0996
		0.7	1	1	0.61614	16.4164	-6.7476	4.2039	0.8862	1.0973
		0.8	1	1	0.54688	14.6305	-6.1765	5.1176	0.9115	1.0951
		0.9	1	1	0.48044	12.942	-5.5941	6.0495	0.9299	1.0929
20%	15%	0.5	1	1	0.82176	20.7774	-8.2558	1.7907	0.7273	1.1020
		0.6	1	1	0.73370	18.6482	-7.6415	2.7736	0.8181	1.0996
		0.7	1	1	0.64938	16.6482	-7.0096	3.7847	0.8703	1.0973
		0.8	1	1	0.56858	14.7691	-6.3592	4.8252	0.9039	1.0951
		0.9	1	1	0.49107	13.0037	-5.6897	5.8966	0.9270	1.0929

Table 3.5 : Level of investment in projects and sensitivity analysis (continued)

v_a	v_M	α	Project 1	Project 2	Project 3	Obj. Func.	Change in M_0 : 6-Interval	Change in a_{03} : γ -interval		
20%	20%	0.5	1	1	0.87963	21.2485	-8.6364	1.1818	0.6251	1.1020
		0.6	1	1	0.77899	18.9905	-7.9630	2.2593	0.7773	1.0996
		0.7	1	1	0.68262	16.88	-7.2641	3.3774	0.8518	1.0973
		0.8	1	1	0.59028	14.9078	-6.5385	4.5385	0.8956	1.0951
		0.9	1	1	0.50170	13.0654	-5.7843	5.7451	0.9240	1.0929

Remarks: As in Table 3.5, Project 1 and Project 2 are fully funded projects, while Project 3 is a partially funded project in all cases. As we increase α – cut level for a fixed $v_a - v_M$ level, the level of investment in Project 3 decreases. This is due to the reason that when we increase α – cut level, the budget level of each period and the cash flow level of each project decrease simultaneously. When we increase v_M level for a fixed $\alpha - v_a$ level or when we increase v_a level for a fixed $\alpha - v_M$ level, the level of investment in Project 3 increases, since the budget level or cash flow level increases in respective cases. As we increase α – cut level for a fixed $v_a - v_M$ level, the range of values of δ under which the current basis remains optimal shifts to the right, whereas when we increase v_M level for a fixed $\alpha - v_a$ level or when we increase v_a level for a fixed $\alpha - v_M$ level, the range of values of δ under which the current basis remains optimal shifts to the left. The range of values of γ_B under which the current basis remains optimal shifts to the right when we increase α – cut level for $v_a = 5\%$. But, the range of values of γ_B tightens when we increase α – cut level for other v_a levels, where for a fixed v_a level, the right-hand side of the interval of γ_B stays constant.

As in Figure 3.5, all the graphs for the level of investment in Project 3 have a linear characteristic with a downward slope, where the α – cut levels are on the horizontal axis, and the levels of investment in Project 3 are on the vertical axis. When we increase v_M level for a fixed v_a level for all variability rate combinations, the downward slope of the graphs increases, since for a constant cash flow variability rate, budget variability rate and directly budget level increases. The maximum level of investment attained in Project 3 is $x_3 = 0.87963$ for $v_a - v_M = 20\% - 20\%$ level and $\alpha = 0.5$ level, while the minimum level of investment attained in Project 3 is $x_3 \approx 0.42$ for $\alpha = 0.99$ in all variability rate combinations.

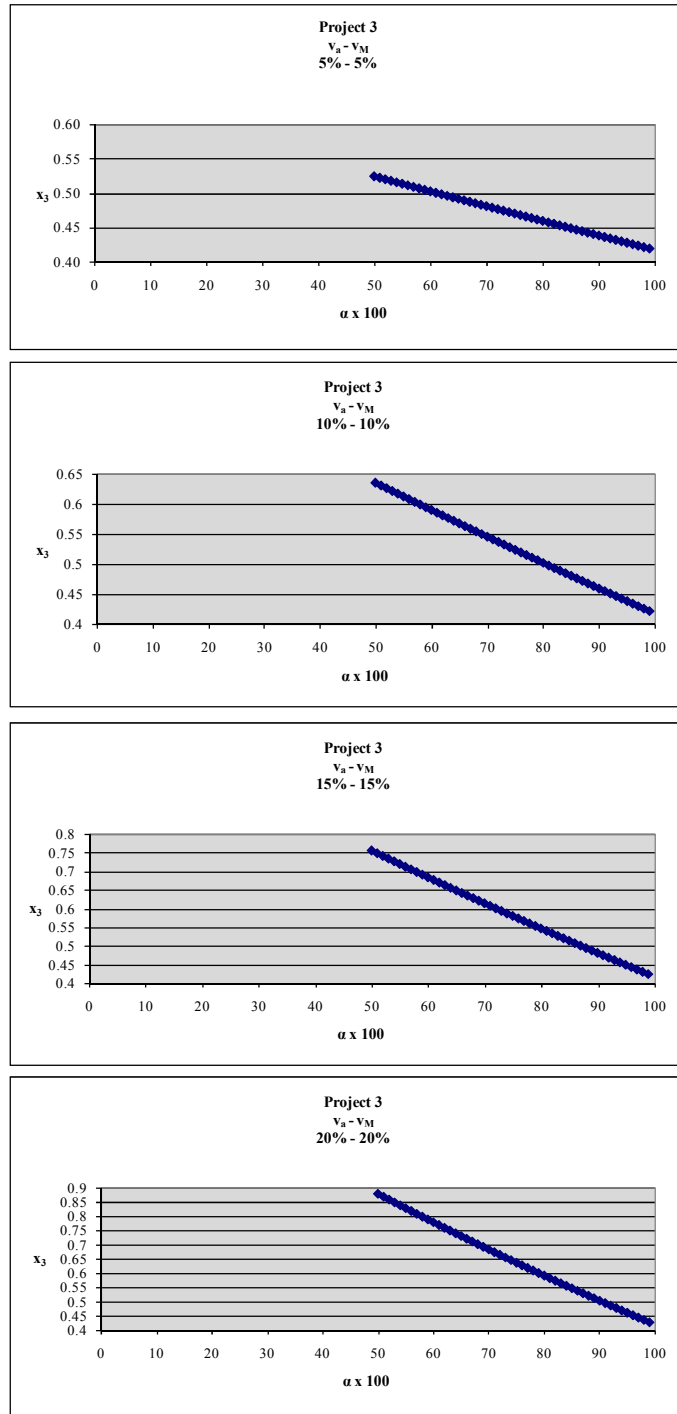


Figure 3.5 : Level of investment in Project 3

3.3. Internal Rate of Return of a Simple Project with PV-Equation Based on t-norm and t-conorm Fuzzy Relation³¹

In this section, we propose decision rules for IRR of a simple project by considering cash flows and MARR as fuzzy numbers in a PV-equation defuzzified with t-norm and t-conorm fuzzy relations. In Section 3.3.1, we provide some basics regarding IRR, and in Section 3.3.2, we review fuzzy approach to some investment appraisal techniques, and in Section 3.3.3, we formulate PV-equations with t-norm and t-conorm fuzzy relations, and define the decision rules for a simple project. Finally, in Section 3.3.4, we illustrate the proposed decision rules with a numerical example.

3.3.1. Preliminaries

Definition 1: A *simple project* is a project with only one sign change in the net cash flow during its life [2, p: 213].

Definition 2: The calculation of IRR is the calculation of r^* in the following equation [2, p: 210]:

$$PV = \sum_{n=0}^N \frac{F_n}{(1+r^*)^n} = 0 \quad (3.89)$$

where F_n represents the periodic net cash flow at period n . If $\frac{1}{(1+r^*)} = x$, then

(3.89) will be equivalent to the following:

$$PV = f(x) = \sum_{n=0}^N F_n x^n = 0 \quad (3.90)$$

Definition 3: *Newton's approximation* is an approximation technique to compute IRR as follows [2, p: 211]:

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} \quad (3.91)$$

³¹ A version of this section is published as a book chapter of conference proceedings, Bas, E., Internal rate of return of fuzzy cash flows based on pessimistic and optimistic fuzzy-relation approach, pp. 1009-1014, Proceedings of the 8th International FLINS Conference, in: Computational Intelligence in Decision and Control, Ruan, D., Montero, J., Lu, J., Martínez, L., D'hondt, P. and Kerre, E.E., Eds., World Scientific, and is reprinted in this thesis with permission from World Scientific.

where $x_j = \frac{1}{(1+r_j^*)}$ is the value at j th iteration. When $x_{j-1}^* \approx x_j^*$, we stop [2, p: 211].

Definition 4: The decision rules for IRR are as follows [2, p: 216]:

If $r^* > MARR$, accept project,

If $r^* = MARR$, stay indifferent,

If $r^* < MARR$, reject project.

3.3.2. Fuzzy approach to investment appraisal techniques

In this section, we review fuzzy approach to some of the investment appraisal techniques presented in [15]³².

Fuzzy future value analogous to fuzzy present value in (3.5) will be as follows [15]:

$$FV = \left(\sum_{n=0}^{L-1} \left[\max \{ p_{n0}, 0 \} \prod_{n'=n+1}^L (1+r_{n'0}) + \min \{ p_{n0}, 0 \} \prod_{n'=n+1}^L (1+r_{n'2}) \right] + p_{L0}, \right. \\ \left. \sum_{n=0}^{L-1} \left[\max \{ p_{n2}, 0 \} \prod_{n'=n+1}^L (1+r_{n'2}) + \min \{ p_{n2}, 0 \} \prod_{n'=n+1}^L (1+r_{n'0}) \right] + p_{L2} \right) \quad (3.92)$$

[15,48] presented the following fuzzy B/C ratio incremental formulation for two mutually exclusive projects:

$$\Delta B / \Delta C = \left(\frac{\sum_{n=0}^L (b_{n20} - b_{n12})(1+r_2)^{-n}}{\sum_{n=0}^L (c_{n22} - c_{n10})(1+r_0)^{-n}}, \frac{\sum_{n=0}^L (b_{n22} - b_{n10})(1+r_0)^{-n}}{\sum_{n=0}^L (c_{n20} - c_{n12})(1+r_2)^{-n}} \right) \quad (3.93)$$

where $\tilde{b}_{nj} = (b_{nj0}, b_{nj1}, b_{nj2})$ and $\tilde{c}_{nj} = (c_{nj0}, c_{nj1}, c_{nj2})$ are fuzzy numbers representing annual benefit and annual cost of projects for $\forall j \in \{1, 2\}$, and $\tilde{r} = (r_0, r_1, r_2)$ is a fuzzy number representing discount rate. L is project life in (3.92) and (3.93).

³² (3.92), (3.93), and (3.94) are adapted from Information Sciences, Vol. 142, Kahraman, C., Ruan D. and Tolga, E., Capital budgeting techniques using discounted fuzzy versus probabilistic cash flows, pp. 57–76, Copyright©2002 Elsevier Science Inc., with permission from Elsevier; (3.93) is also adapted from International Journal of Production Economics, Vol. 66, Kahraman, C., Tolga, E. and Ulukan, Z., Justification of manufacturing technologies using fuzzy benefit/cost ratio analysis, pp. 45–52, Copyright©2000 Elsevier Science B.V., with permission from Elsevier.

Payback-period method for a single project will also be as follows [15]:

$$\left(\sum_{n=1}^{n_p} \tilde{b}_{n0}, \sum_{n=1}^{n_p} \tilde{b}_{n1}, \sum_{n=1}^{n_p} \tilde{b}_{n2} \right) \geq (C_0, C_1, C_2) \quad (3.94)$$

where $\tilde{C} = (C_0, C_1, C_2)$ is a fuzzy number representing the initial investment of the project, and n_p is the smallest n that satisfies (3.94). [15] discussed that different fuzzy ranking methods will give different results. [15] also referred to different papers that emphasized the impossibility of the extension of the fuzzy approach to IRR calculation.

3.3.3. Formulation of PV-equation with fuzzy cash flows

We consider the following PV-equation:

$$PV = \sum_{n=0}^N \tilde{F}_n x^n = 0 \quad (3.95)$$

where $\tilde{F}_n$ is assumed to be a normal and strictly convex fuzzy number representing the periodic net cash flow for $\forall n \in \{0, 1, \dots, N\}$, and $x = \frac{1}{(1+r^*)}$ is assumed to be a crisp number.

We define an α -efficient t-norm fuzzy-relation based IRR as $r_T^*(\alpha)$ with $x_T(\alpha) = \frac{1}{1+r_T^*(\alpha)}$ and an α -efficient t-conorm fuzzy-relation based IRR as $r_S^*(\alpha)$

with $x_S(\alpha) = \frac{1}{1+r_S^*(\alpha)}$. Then, PV-equations will be as follows:

$$PV_T(\alpha) = \sum_{n=0}^{N-L} F_n(\alpha) x_T^n(\alpha) = 0 \quad \text{for } \alpha \in (0, 1) \quad (3.96)$$

from (2.30) and (2.34);

$$PV_S(\alpha) = \sum_{n=0}^{N-R} F_n(1-\alpha) x_S^n(\alpha) = 0 \quad \text{for } \alpha \in (0, 1) \quad (3.97)$$

from (2.31) and (2.37). Note that we consider 0 strictly as the right-hand side to

apply the t-norm and t-conorm fuzzy relations to the fuzzy equality (3.95).

If we assume $\tilde{F}_n$ as a symmetric, normal, and strictly convex triangular fuzzy number such that $\tilde{F}_n = (F_n - \hat{F}_n, F_n, F_n + \hat{F}_n)$ for $\forall n \in \{0, 1, \dots, N\}$, then (3.96) and (3.97) will be as follows, respectively:

$$PV_T(\alpha) = \sum_{n=0}^N (F_n + (\alpha - 1) \hat{F}_n) x_T^n(\alpha) = 0 \quad \text{for } \alpha \in (0, 1) \quad (3.98)$$

$$PV_S(\alpha) = \sum_{n=0}^N (F_n + \alpha \hat{F}_n) x_S^n(\alpha) = 0 \quad \text{for } \alpha \in (0, 1) \quad (3.99)$$

Let also $\tilde{MARR}$ be a normal and strictly convex fuzzy number. We define an α – efficient t-norm fuzzy relation based $\tilde{MARR}$ as:

$$MARR_T(\alpha) = \overset{-L}{MARR}(\alpha) \quad (3.100)$$

and an α – efficient t-conorm fuzzy relation based $\tilde{MARR}$ as:

$$MARR_S(\alpha) = \overset{R}{MARR}(1 - \alpha) \quad (3.101)$$

from (2.30), (2.34) and (2.31), (2.37), respectively. If we assume $\tilde{MARR}$ as a symmetric, normal, and strictly convex triangular fuzzy number such that $\tilde{MARR} = (MARR - \hat{MARR}, MARR, MARR + \hat{MARR})$, then (3.100) and (3.101) will be as follows, respectively:

$$\overset{-L}{MARR}(\alpha) = MARR + (\alpha - 1) \hat{MARR} \quad (3.102)$$

$$\overset{R}{MARR}(1 - \alpha) = MARR + \alpha \hat{MARR} \quad (3.103)$$

Decision Rules

We propose the following decision rules:

if $r_T^*(\alpha) > \overset{-L}{MARR}(\alpha)$ for $\forall \alpha \in (0, 1)$ and if $r_S^*(\alpha) > \overset{R}{MARR}(1 - \alpha)$ for $\forall \alpha \in (0, 1)$,

or if $r_T^*(\alpha) > \overset{-L}{MARR}(\alpha)$ for $\forall \alpha \geq \bar{\alpha}_T$ and if $r_S^*(\alpha) > \overset{R}{MARR}(1-\alpha)$ for $\forall \alpha \in (0,1)$, where $\bar{\alpha}_T$ is a predetermined α -cut level for t-norm fuzzy relation based IRR, or if $r_T^*(\alpha) > \overset{-L}{MARR}(\alpha)$ for $\forall \alpha \in (0,1)$ and if $r_S^*(\alpha) > \overset{R}{MARR}(1-\alpha)$ for $\forall \alpha \geq \bar{\alpha}_S$, where $\bar{\alpha}_S$ is a predetermined α -cut level for t-conorm fuzzy relation based IRR, accept project.

if $r_T^*(\alpha) < \overset{-L}{MARR}(\alpha)$ for $\forall \alpha \in (0,1)$ and if $r_S^*(\alpha) < \overset{R}{MARR}(1-\alpha)$ for $\forall \alpha \in (0,1)$, reject project.

otherwise, stay *indifferent*.

3.3.4. A numerical example for a simple project

A three-period project example is given in Table 3.6 (the nominal values are taken from [2, p: 211]).

Table 3.6 : A three-period project example with symmetric triangular fuzzy numbers representing the net cash flows, and MARR

A simple project	Year 0	Year 1	Year 2
MARR = (14.4%, 15%, 15.6%)	(-110, -100, -90)	(45, 50, 55)	(75.6, 84, 92.4)

Since we assume the net cash flows as symmetric, normal, and strictly convex triangular fuzzy numbers, PV-equation with an α -efficient t-norm fuzzy-relation will be as follows:

$$PV_T(\alpha) = f(x) = -100 + 10(\alpha - 1) + (50 + 5(\alpha - 1))x_T(\alpha) + (84 + 8.4(\alpha - 1))x_T^2(\alpha) = 0$$

We apply Newton's approximation for the calculation of $x_T(\alpha)$ as in (3.91), and report the results for $r_T^*(\alpha)$ in Table 3.7 as follows:

Table 3.7 : Results for α -efficient t-norm fuzzy-relation based IRR at 5th iteration

α	$r_{T5}^*(\alpha)$
0.1	0.0718
0.2	0.0853
0.3	0.0989
0.4	0.1128
0.5	0.1268
0.6	0.1411
0.7	0.1555
0.8	0.1701
0.9	0.1849

PV-equation with an α – efficient t-conorm fuzzy-relation will be as follows:

$$PV_s(\alpha) = f(x) = -100 + 10\alpha + (50 + 5\alpha) x_s(\alpha) + (84 + 8.4\alpha) x_s^2(\alpha) = 0$$

We apply Newton's approximation for the calculation of $x_s(\alpha)$, and report the results for $r_s^*(\alpha)$ in Table 3.8 as follows:

Table 3.8 : Results for α – efficient t-conorm fuzzy-relation based IRR at 5th iteration

α	$r_{ss}^*(\alpha)$
0.1	0.2153
0.2	0.2308
0.3	0.2465
0.4	0.2625
0.5	0.2787
0.6	0.2952
0.7	0.3119
0.8	0.3289
0.9	0.3463

Since we assume MARR as a symmetric, normal, and strictly convex triangular fuzzy number, then α – efficient t-norm and t-conorm fuzzy relation based MARR equations will be as follows, respectively:

$$MARR^{-L}(\alpha) = 0.15 + 0.006(\alpha - 1)$$

$$MARR^R(1 - \alpha) = 0.15 + 0.006\alpha$$

Finally, we obtain Table 3.9 with α – efficient t-norm and t-conorm fuzzy relation based MARR values as follows:

Table 3.9 : Results for α – efficient t-norm and t-conorm fuzzy relation based MARR

α	$MARR_T(\alpha)$	$MARR_S(\alpha)$
0.1	0.1446	0.1506
0.2	0.1452	0.1512
0.3	0.1458	0.1518
0.4	0.1464	0.1524
0.5	0.147	0.153
0.6	0.1476	0.1536
0.7	0.1482	0.1542
0.8	0.1488	0.1548
0.9	0.1494	0.1554

Remarks: t-norm fuzzy-relation based IRR approach yields pessimistic results, while t-conorm fuzzy-relation based IRR approach yields optimistic results. Based on the proposed decision rules, when we compare the results of α – efficient t-norm

fuzzy-relation based IRR values with α -efficient t-norm fuzzy relation based MARR values, and the results of α -efficient t-conorm fuzzy-relation based IRR values with α -efficient t-conorm fuzzy relation based MARR values, then we clearly accept the project for t-conorm fuzzy-relation based approach for each α -cut level, and we accept the project at $\alpha \geq 0.65$ levels for t-norm fuzzy-relation based approach. Note that in the crisp case, $r^* = 20\% > 15\% = MARR$, and the project is accepted. When we apply fuzzy PV-equation given in (3.5), then $PV = (-14.5001, 6.994329, 28.6794)$ is a normal triangular fuzzy number representing PV of the project. Note also that $[PV]_\alpha > 0$ if $\alpha \geq 0.674599$ which almost coincides with the results of t-norm fuzzy-relation based IRR.

Alternatively, the investment could be evaluated by the defuzzification of the normal and convex triangular fuzzy numbers representing IRR and MARR obtained from t-norm and t-conorm fuzzy relation approaches as illustrated in Figure 3.6. Note that we use “Center of Gravity” method for the defuzzification of the fuzzy numbers representing IRR and MARR. By considering this approach, it is clear that the project will be accepted like in the crisp case.

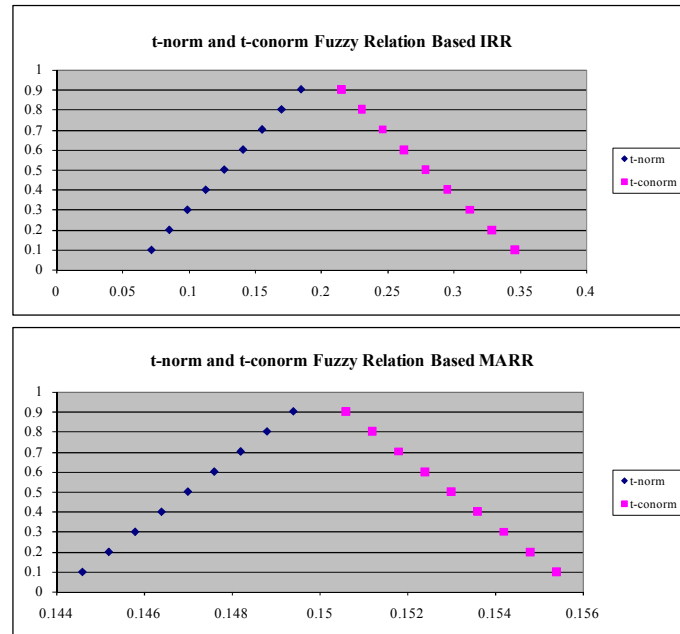


Figure 3.6 : Triangular fuzzy numbers representing IRR and MARR

As another alternative, MARR could be considered as a crisp number. However, in this case, the decision rules proposed in this section could not be used, and the defuzzification of the fuzzy numbers would be inevitable.

3.4. Fuzzy quadratic programming: A case of Bernhard's capital budgeting model³³

In this section, we deal with fuzzy quadratic programming, and the fuzzification of Bernhard's capital budgeting model with quadratic objective function. Analogous to Section 3.2, we propose to use t-norm and t-conorm fuzzy relations for the ranking of the inequality constraints, and for the maximization of the objective function.

3.4.1. Two primal-dual model pairs for quadratic programming with fuzzy parameters

We consider the following quadratic programming with fuzzy parameters:

$$\begin{aligned}
 \max \quad & \tilde{d}^T x \tilde{+} \frac{1}{2} x^T \tilde{H} x \\
 \text{s.t.} \quad & \tilde{A} x \tilde{P} \tilde{b} \\
 & x \geq 0
 \end{aligned} \tag{3.104}$$

where all fuzzy parameters are assumed to be normal and strictly convex fuzzy numbers, $\tilde{d}$ is a $J \times 1$ vector, $\tilde{A}$ is an $N \times J$ matrix, $\tilde{H}$ is an $J \times J$ negative definite symmetric matrix, all decision variables are assumed to be crisp, and $\tilde{P}$ is a fuzzy extension of the binary relation $\leq$. We propose the following two α -efficient models with $\tilde{\leq}^T$ t-norm and $\tilde{\leq}_s$ t-conorm fuzzy relations for the objective function and constraints, respectively:

$$\begin{aligned}
 \max \quad & Z_T(\alpha) = \overset{-R}{d}(\alpha)^T x_T + \frac{1}{2} x_T^T \overset{-R}{H}(\alpha) x_T \\
 \text{s.t.} \quad & \overset{-L}{A}(\alpha) x_T \leq \overset{-R}{b}(\alpha) \\
 & x_T \geq 0
 \end{aligned} \tag{3.105}$$

and

³³ This section is submitted as a journal paper, Bas, E. and Kahraman, C., Fuzzy quadratic programming: A case of Bernhard's capital budgeting model, Submitted to International Journal of Production Economics, Under Review, and is reprinted in this thesis with permission from Elsevier.

$$\begin{aligned}
\max \quad & Z_s(\alpha) = \overset{L}{d}(1-\alpha)^T x_s + \frac{1}{2} x_s^T \overset{L}{H}(1-\alpha) x_s \\
\text{s.t.} \quad & \overset{R}{A}(1-\alpha) x_s \leq \overset{L}{b}(1-\alpha)
\end{aligned} \tag{3.106}$$

$$x_s \geq 0$$

where $Z_T(\alpha)$, and $Z_s(\alpha)$ denote the objective functions, and x_T and x_s denote the solution sets of the primal models with $\widetilde{\leq}^T$ t-norm and $\widetilde{\leq}_s$ t-conorm fuzzy relations for a given α – cut level, respectively.

Proof: We show the proof of (3.105). The proof of (3.106) will be analogous. Notice that we assign an aspiration level to the objective function as the definition of maximization analogous to [24], which we also considered for the maximization of the objective function in Weingartner’s pure capital rationing model in Section 3.2.1. Then, (3.104) will be equivalent to the following for a model with $\widetilde{\leq}^T$ t-norm fuzzy relation:

$$\begin{aligned}
& \widetilde{d}^T x_T \widetilde{+} \frac{1}{2} x_T^T \widetilde{H} x_T \widetilde{\geq}^T \widetilde{Z}_{T,0} \\
& \widetilde{A} x_T \widetilde{\leq}^T \widetilde{b}
\end{aligned} \tag{3.107}$$

$$x_T \geq 0$$

where $\widetilde{Z}_{T,0}$ is the aspiration level of the objective function with $\widetilde{\leq}^T$ t-norm fuzzy relation. Consequently, (3.107) will be equivalent to the following from (2.30):

$$\begin{aligned}
& \sup \left[\widetilde{d}^T x_T \right]_{\alpha} + \sup \left[\frac{1}{2} x_T^T \widetilde{H} x_T \right]_{\alpha} \geq \inf \left[\widetilde{Z}_{T,0} \right]_{\alpha} \\
& \inf \left[\widetilde{A} x_T \right]_{\alpha} \leq \sup \left[\widetilde{b} \right]_{\alpha}
\end{aligned} \tag{3.108}$$

$$x_T \geq 0$$

and

$${}^{-R}d(\alpha)^T x_T + \frac{1}{2} x_T^T {}^{-R}H(\alpha) x_T \geq {}^{-L}Z_{T,0}(\alpha)$$

$${}^{-L}A(\alpha) x_T \leq {}^{-R}b(\alpha) \quad (3.109)$$

$$x_T \geq 0$$

follows from (2.34) and (2.36), which is equivalent to the following model:

$$\begin{aligned} \max \quad & {}^{-R}d(\alpha)^T x_T + \frac{1}{2} x_T^T {}^{-R}H(\alpha) x_T \\ \text{s.t.} \quad & {}^{-L}A(\alpha) x_T \leq {}^{-R}b(\alpha) \end{aligned} \quad (3.110)$$

$$x_T \geq 0$$

Note that $\tilde{A}x$ has been proved to be a compact and normal fuzzy number for any compact and normal fuzzy number $\tilde{A}$, and that $[\tilde{A}x]_\alpha = [\tilde{A}]_\alpha x$ in [24]. By analogous argument, $\frac{1}{2} x_T^T \tilde{H} x_T$ is a compact and normal fuzzy number for any compact and normal fuzzy number $\tilde{H} x_T$, and $\left[\frac{1}{2} x_T^T \tilde{H} x_T \right]_\alpha = \frac{1}{2} x_T^T [\tilde{H} x_T]_\alpha = \frac{1}{2} x_T^T [\tilde{H}]_\alpha x_T$. $\square$

Consequently, Karush-Kuhn-Tucker (KKT) conditions of (3.105) and (3.106) will be as follows, respectively [49, p: 200]:

$$\begin{aligned} {}^{-L}A(\alpha) x_T^* &\leq {}^{-R}b(\alpha) \\ g(x_T^*) &= {}^{-R}d(\alpha) + {}^{-R}H(\alpha) x_T^* = {}^{-L}A(\alpha)^T \lambda_T^* \end{aligned} \quad (3.111)$$

$$\lambda_T^* \geq 0$$

$$\lambda_{T,n}^* (a_n^*(\alpha)^T x_T^* - b_n^*(\alpha)) = 0 \quad \forall n \in \{1, 2, \dots, N\}$$

and

$${}^R A(1-\alpha)x_s^* \leq {}^L b(1-\alpha)$$

$$g(x_s^*) = {}^L d(1-\alpha) + {}^L H(1-\alpha)x_s^* = {}^R A(1-\alpha)^T \lambda_s^* \quad (3.112)$$

$$\lambda_s^* \geq 0$$

$$\lambda_{s,n}^* ({}^R a_n(1-\alpha)^T x_s^* - {}^L b_n(1-\alpha)) = 0 \quad \forall n \in \{1, 2, \dots, N\}$$

where λ_T^* and λ_s^* denote the optimal multiplier vectors for the constraints of the primal models with $\widetilde{\leq}^T$ t-norm and $\widetilde{\leq}_s$ t-conorm fuzzy relations, respectively. Note that we use λ for the multiplier vector instead of μ as in [49, p: 200], since in the following section, μ is used as the multiplier vector for the project selection variable constraint.

We next develop the dual model of the primal model (3.104) analogous to [2, p: 332]³⁴:

$$\begin{aligned} \min \quad & \widetilde{b}^T \lambda - \frac{1}{2} x^T \widetilde{H} x \\ \text{s.t.} \quad & \widetilde{A}^T \lambda - \widetilde{H} x \geq \widetilde{d} \end{aligned} \quad (3.113)$$

$$\lambda \geq 0$$

Then, the dual models of the primal models (3.105) and (3.106) will be as follows, respectively:

$$\begin{aligned} \min \quad & {}^R b(1-\alpha)^T \lambda_T - \frac{1}{2} x_T^T {}^L H(1-\alpha)x_T \\ \text{s.t.} \quad & {}^L A(1-\alpha)^T \lambda_T - {}^R H(1-\alpha)x_T \geq {}^R d(1-\alpha) \end{aligned} \quad (3.114)$$

$$\lambda_T \geq 0$$

³⁴ The models, explanations, and tables in Section 3.4 with the reference [2] are reprinted, adapted or paraphrased from Advanced Engineering Economics, 1990, Park, C.S. and Sharp-Bette, G.P., John Wiley & Sons, Inc., New York Chichester Brisbane, with permission from John Wiley & Sons, Inc.

$$\begin{aligned}
\min \quad & \tilde{b}^{-L}(\alpha)^T \lambda_S - \frac{1}{2} x_S^T \tilde{H}^{-R}(\alpha) x_S \\
\text{s.t.} \quad & \tilde{A}^{-R}(\alpha)^T \lambda_S - \tilde{H}^{-L}(\alpha) x_S \geq \tilde{d}^{-L}(\alpha)
\end{aligned} \tag{3.115}$$

$$\lambda_S \geq 0$$

Proof: The proof of (3.114) is analogous to the proof of (3.105). Recall that $\tilde{\succeq}_S$ t-conorm fuzzy relation is a dual to $\tilde{\preceq}^T$ t-norm fuzzy relation from (2.28), and that $\tilde{b}^T \lambda_T - \frac{1}{2} x_T^T \tilde{H} x_T \tilde{\preceq}_S \tilde{W}_{T,0}$, where $\tilde{W}_{T,0}$ is an aspiration level for the objective function analogous to [24]. The proof of (3.115) is also analogous.

Proposition 1: Weak duality

$$\tilde{d}^{-R}(\alpha)^T x_T + \frac{1}{2} x_T^T \tilde{H}^{-R}(\alpha) x_T \leq \tilde{b}^R(1-\alpha)^T \lambda_T - \frac{1}{2} x_T^T \tilde{H}^L(1-\alpha) x_T \tag{3.116}$$

holds for $\alpha \in [0.5, 1)$.

Proof: We follow the proof of weak duality in [28, p: 103-104].

$$\lambda_1^T \tilde{A}^{-L}(\alpha) x_T \leq \lambda_1^T \tilde{b}^{-R}(\alpha) \tag{3.117}$$

$$x_T^T \tilde{A}^L(1-\alpha)^T \lambda_T - x_T^T \tilde{H}^R(1-\alpha) x_T \geq x_T^T \tilde{d}^R(1-\alpha) \tag{3.118}$$

follow from models (3.105) and (3.114), respectively, since $x_T \geq 0$, $\lambda_T \geq 0$. When we combine (3.117) and (3.118), we will obtain the following:

$$\lambda_1^T \tilde{b}^{-R}(\alpha) + x_T^T \tilde{A}^L(1-\alpha)^T \lambda_T - x_T^T \tilde{H}^R(1-\alpha) x_T \geq x_T^T \tilde{d}^R(1-\alpha) + \lambda_1^T \tilde{A}^{-L}(\alpha) x_T \tag{3.119}$$

For any strictly convex and normal fuzzy number at $\alpha \in [0.5, 1)$:

$$\begin{aligned}
& \lambda_1^T \tilde{b}^{-R}(1-\alpha) + x_T^T \tilde{A}^L(1-\alpha)^T \lambda_T - \lambda_1^T \tilde{A}^{-L}(1-\alpha) x_T - \frac{1}{2} x_T^T \tilde{H}^L(1-\alpha) x_T \\
& \geq x_T^T \tilde{d}^R(\alpha) + \frac{1}{2} x_T^T \tilde{H}^R(\alpha) x_T
\end{aligned} \tag{3.120}$$

Since $x_T^T \overset{L}{A}(1-\alpha)^T \lambda_T = \lambda_T^T \overset{-L}{A}(1-\alpha)x_T$ holds from convexity;

$$\overset{-R}{d}(\alpha)^T x_T + \frac{1}{2} x_T^T \overset{-R}{H}(\alpha)x_T \leq \overset{R}{b}(1-\alpha)^T \lambda_T - \frac{1}{2} x_T^T \overset{L}{H}(1-\alpha)x_T \quad (3.121)$$

(3.120) and (3.121) follow from the fact that for any strictly convex and normal fuzzy number $\tilde{F}$ at $\alpha \in [0.5, 1)$;

$$\overset{-R}{F}(\alpha) = \overset{R}{F}(\alpha) \leq \overset{-R}{F}(1-\alpha) = \overset{R}{F}(1-\alpha) \quad (3.122)$$

$$\overset{-L}{F}(\alpha) = \overset{L}{F}(\alpha) \geq \overset{-L}{F}(1-\alpha) = \overset{L}{F}(1-\alpha) \quad (3.123)$$

hold. $\square$

Remark 1: Proposition 1 can easily be extended to FLP when $\tilde{H} = 0$, which coincides with the result in Proposition 5 of Section 3.2.1, and the result for a general FLP model in [24].

Remark 2: Since $\overset{L}{d}(1-\alpha)^T x_s + \frac{1}{2} x_s^T \overset{L}{H}(1-\alpha)x_s \leq \overset{-L}{b}(\alpha)^T \lambda_s - \frac{1}{2} x_s^T \overset{-L}{H}(\alpha)x_s$ is obtained for $\alpha \in [0.5, 1)$ similar to the proof of Proposition 1, we cannot report any general result for the primal-dual model pair with $\tilde{\leq}_s$ t-conorm and corresponding dual $\tilde{\geq}^T$ t-norm fuzzy relations, while Proposition 1 applies to the primal-dual pair (3.106) and (3.115) for FLP when $\tilde{H} = 0$.

Proposition 2: Strong duality

Strong duality does not hold for the primal-dual model pair (3.105) and (3.114) for $\alpha \in [0.5, 1)$.

Proof: We start with the first part of the proof:

$$\overset{-L}{A}(1-\alpha)^T \lambda_T^* - \overset{-R}{H}(1-\alpha)x_T^* = \overset{-R}{d}(1-\alpha) \quad (3.124)$$

always holds from KKT conditions in (3.111).

Since $\overset{-R}{d}(1-\alpha) \geq \overset{-R}{d}(\alpha)$ holds for $\alpha \in [0.5, 1)$ by convexity; then (3.124) will be equivalent to the following:

$$\overset{-L}{A}(1-\alpha)^T \lambda_T^* - \overset{-R}{H}(1-\alpha)x_T^* \geq \overset{-R}{d}(\alpha) \quad (3.125)$$

When we multiply both sides of (3.125) by x_T^{T*} , we obtain the following:

$$x_T^{T*} \overset{-L}{A}(1-\alpha)^T \lambda_T^* - x_T^{T*} \overset{-R}{H}(1-\alpha)x_T^* \geq x_T^{T*} \overset{-R}{d}(\alpha) \quad (3.126)$$

When we add both sides of (3.126) with $\frac{1}{2}x_T^{T*} \overset{-R}{H}(\alpha)x_T^*$, then we obtain the following:

$$\begin{aligned} x_T^{T*} \overset{-L}{A}(1-\alpha)^T \lambda_T^* - x_T^{T*} \overset{-R}{H}(1-\alpha)x_T^* + \frac{1}{2}x_T^{T*} \overset{-R}{H}(\alpha)x_T^* \\ \geq x_T^{T*} \overset{-R}{d}(\alpha) + \frac{1}{2}x_T^{T*} \overset{-R}{H}(\alpha)x_T^* \end{aligned} \quad (3.127)$$

Since $\overset{-R}{H}(1-\alpha) \geq \overset{-R}{H}(\alpha)$ for $\alpha \in [0.5, 1)$ holds by convexity;

$$x_T^{T*} \overset{-L}{A}(1-\alpha)^T \lambda_T^* - \frac{1}{2}x_T^{T*} \overset{-R}{H}(1-\alpha)x_T^* \geq x_T^{T*} \overset{-R}{d}(\alpha) + \frac{1}{2}x_T^{T*} \overset{-R}{H}(\alpha)x_T^* \quad (3.128)$$

The second part of the proof will be as follows:

$$\overset{-L}{A}(1-\alpha)x_T^* \leq \overset{-R}{b}(1-\alpha) \quad (3.129)$$

always holds from KKT conditions in (3.111). When we multiply both sides of (3.129) by λ_T^{T*} , then we obtain the following:

$$\lambda_T^{T*} \overset{-L}{A}(1-\alpha)x_T^* \leq \lambda_T^{T*} \overset{-R}{b}(1-\alpha) \quad (3.130)$$

and, when we subtract both sides of (3.130) from $\frac{1}{2}x_T^{T*} \overset{L}{H}(1-\alpha)x_T^*$, then (3.130) will be equivalent to the following:

$$\lambda_T^{T*} \overset{-L}{A}(1-\alpha)x_T^* - \frac{1}{2}x_T^{T*} \overset{L}{H}(1-\alpha)x_T^* \leq \lambda_T^{T*} \overset{-R}{b}(1-\alpha) - \frac{1}{2}x_T^{T*} \overset{L}{H}(1-\alpha)x_T^* \quad (3.131)$$

Since we assume $\overset{R}{H}(1-\alpha) > \overset{L}{H}(1-\alpha)$ always holds by convexity ;

$$\lambda_T^{*-L} \overset{L}{A}(1-\alpha)x_T^* - \frac{1}{2}x_T^{T*} \overset{R}{H}(1-\alpha)x_T^* < \lambda_T^{*-R} \overset{L}{b}(1-\alpha) - \frac{1}{2}x_T^{T*} \overset{L}{H}(1-\alpha)x_T^* \quad (3.132)$$

Finally, if we combine (3.128) and (3.132), then we will obtain the following:

$$\begin{aligned} \overset{-R}{d}(\alpha)^T x_T^* + \frac{1}{2}x_T^{T*} \overset{-R}{H}(\alpha)x_T^* &\leq x_T^{T*} \overset{-L}{A}(1-\alpha)^T \lambda_T^* - \frac{1}{2}x_T^{T*} \overset{-R}{H}(1-\alpha)x_T^* \\ &< \overset{-R}{b}(1-\alpha)^T \lambda_T^* - \frac{1}{2}x_T^{T*} \overset{L}{H}(1-\alpha)x_T^* \end{aligned} \quad (3.133)$$

which implies that $\overset{-R}{d}(\alpha)^T x_T^* + \frac{1}{2}x_T^{T*} \overset{-R}{H}(\alpha)x_T^* = \overset{R}{b}(1-\alpha)^T \lambda_T^* - \frac{1}{2}x_T^{T*} \overset{L}{H}(1-\alpha)x_T^*$ never

holds. $\square$

Proposition 3: Since strong duality does not hold for the primal-dual model pair (3.105) and (3.114), and α -cut level under which weak duality holds for the primal-dual model pair (3.106) and (3.115) is unclear, we propose the following alternative dual models to consider:

$$\begin{aligned} \min \quad & \overset{-R}{b}(\alpha)^T \lambda_T - \frac{1}{2}x_T^{T*} \overset{-R}{H}(\alpha)x_T \\ \text{s.t.} \quad & \overset{-L}{A}(\alpha)^T \lambda_T - \overset{-R}{H}(\alpha)x_T \geq \overset{-R}{d}(\alpha) \\ & \lambda_T \geq 0 \end{aligned} \quad (3.134)$$

and

$$\begin{aligned} \min \quad & \overset{L}{b}(1-\alpha)^T \lambda_S - \frac{1}{2}x_S^{T*} \overset{L}{H}(1-\alpha)x_S \\ \text{s.t.} \quad & \overset{R}{A}(1-\alpha)^T \lambda_S - \overset{L}{H}(1-\alpha)x_S \geq \overset{L}{d}(1-\alpha) \\ & \lambda_S \geq 0 \end{aligned} \quad (3.135)$$

Remark: Weak duality and strong duality for the primal-dual model pairs (3.105) and (3.134), and (3.106) and (3.135) hold for $\alpha \in [0, 1)$ from KKT conditions.

Proposition 4: Model (3.105) implies the upper bound (best-case scenario, optimistic scenario), while model (3.106) implies the lower bound (worst-case scenario, pessimistic scenario) for any $\alpha \in [0.5, 1)$.

Proof: For $\alpha \in [0.5, 1)$, the following conditions hold by convexity and definition:

$$\overset{-L}{A}(\alpha) < \overset{-R}{A}(\alpha) \leq \overset{-R}{A}(1-\alpha) = \overset{R}{A}(1-\alpha) \quad (3.136)$$

$$\overset{-R}{b}(\alpha) > \overset{-L}{b}(\alpha) \geq \overset{-L}{b}(1-\alpha) = \overset{L}{b}(1-\alpha) \quad (3.137)$$

$$\overset{-R}{d}(\alpha) > \overset{-L}{d}(\alpha) \geq \overset{-L}{d}(1-\alpha) = \overset{L}{d}(1-\alpha) \quad (3.138)$$

$$\overset{-R}{H}(\alpha) > \overset{-L}{H}(\alpha) \geq \overset{-L}{H}(1-\alpha) = \overset{L}{H}(1-\alpha) \quad (3.139)$$

Note that we assume for any normal and strictly convex fuzzy number $\tilde{F}$, $\overset{-L}{F}(\alpha) < \overset{-R}{F}(\alpha)$ holds for $\alpha \in [0, 1)$. Then, $\overset{-L}{A}(\alpha)$ and $\overset{R}{A}(1-\alpha)$ represent the best-case scenario and the worst-case scenario for the constraint requirements, while $\overset{-R}{b}(\alpha)$ and $\overset{L}{b}(1-\alpha)$ represent the best-case scenario and the worst-case scenario for the resources, respectively.

Finally, $\left\{x_T \in R^J \mid \overset{-L}{A}(\alpha)x_T \leq \overset{-R}{b}(\alpha)\right\} \supset \left\{x_S \in R^J \mid \overset{R}{A}(1-\alpha)x_S \leq \overset{L}{b}(1-\alpha)\right\}$. Since

$\overset{-R}{d}(\alpha)$ and $\overset{L}{d}(1-\alpha)$; $\overset{-R}{H}(\alpha)$ and $\overset{L}{H}(1-\alpha)$ represent the best-case scenario and the worst-case scenario for the objective function, respectively; then (3.105) implies the upper bound (best-case scenario, optimistic case) and (3.106) implies the lower bound (worst-case scenario, pessimistic case) for any $\alpha \in [0.5, 1)$. $\square$

Remark: Proposition 4 can easily be extended to FLP when $\tilde{H} = 0$.

3.4.2. Bernhard's general model

Bernhard's general model, an extension to Weingartner's horizon model, includes also dividends in the non-linear objective function [50,2, p: 305]. In this section, we provide the following Bernhard's general model³⁵.

$$\begin{aligned}
& \max \quad f(D_1, D_2, \dots, D_N, G) \\
& \text{s.t.} \\
& - \sum_{j=1}^J a_{nj} x_j - l_{n-1} v_{n-1} + v_n + (b_{n-1} - l_{n-1} c_{n-1}) w_{n-1} - (1 - c_n) w_n + D_n \leq M_n \quad \forall n \in \{0, 1, \dots, N\} \\
& - \sum_{j=1}^J \sum_{n=0}^{n'} a_{nj} x_j \leq 0 \\
& \sum_{j=1}^J d_j x_j \leq d \\
& - \sum_{j=1}^J \hat{a}_j x_j - v_N + (1 - c_N) w_N + G = M \\
& - G + g(D_1, D_2, \dots, D_N) \leq -K \\
& w_n \leq B_n \quad \forall n \in \{0, 1, \dots, N\} \\
& x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \\
& x_j, v_n, w_n, D_n \geq 0
\end{aligned} \tag{3.140}$$

where D_n is dividend paid at time n , G is terminal wealth at time N , a_{nj} is net cash flow for project j at time n , x_j is project selection variable where if $x_j = 1$, the project is fully funded, if $0 < x_j < 1$, the project is partially funded, and if $x_j = 0$, the project is not funded, $l_n = 1 + r_{ln}$ where r_{ln} is lending rate at time n , $b_n = 1 + r_{bn}$ where r_{bn} is borrowing rate at time n , v_n is lending amount from time n to time $n+1$, w_n is borrowing amount from time n to time $n+1$, B_n is borrowing limit at time n , c_n is compensating balance friction representing basic liquidity at time n , M_n is budget limit at time n , d_j is scarce resource consumed by project j ,

³⁵ (3.140) and the explanations are also adapted from Journal of Financial and Quantitative Analysis, Vol. 4, 1969, Bernhard, R.H., Mathematical Programming Models for Capital Budgeting –A Survey, Generalization, and Critique, pp. 111-158.

d is available amount of scarce resource, $\hat{a}_j$ is horizon time value of cash flows beyond horizon for project j , M is value at time N of post-horizon cash flows from other sources, and K is a nonnegative constant representing the minimum acceptable terminal wealth [2, p: 308].

Since the objective function is assumed to be concave, and all the constraints are convex, KKT conditions are necessary and sufficient for optimality [2, p: 307].

We consider the case of Bernhard's capital budgeting model with quadratic objective function in [2, p: 329-330] as follows:

$$\max \quad \frac{G}{c} + \sum_{n=0}^N \frac{D_n}{(1+i)^n} - c_1 \sum_{n=1}^N (D_n - D_{n-1})^2 - c_2 \left(\frac{G}{c_3 c} - D_{ave} \right)^2 \quad (3.141)$$

where $G = \sum_{j=1}^J \hat{a}_j x_j + v_N - w_N$, $c = (1+i)^N$, $D_{ave} = \frac{1}{N+1} \sum_{n=0}^N \frac{D_n}{(1+i)^n}$, i : Discount rate, and c_1, c_2, c_3 : nonnegative constant parameters. Some or all the constraints in (3.140) can be considered for further analysis. For the sake of simplicity, some of the crucial constraints will be studied as in [2, p: 330-331]. Finally, the model to be considered will be as follows:

$$\begin{aligned} \max \quad & \frac{G}{c} + \sum_{n=0}^N \frac{D_n}{(1+i)^n} - c_1 \sum_{n=1}^N (D_n - D_{n-1})^2 - c_2 \left(\frac{G}{c_3 c} - D_{ave} \right)^2 \\ \text{s.t.} \quad & -\sum_{j=1}^J a_{nj} x_j - l_{n-1} v_{n-1} + v_n + b_{n-1} w_{n-1} - w_n + D_n \leq M_n \quad \forall n \in \{0, 1, \dots, N\} \\ & D_n \geq D_{\min} \quad \forall n \in \{0, 1, \dots, N\} \\ & x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \\ & x_j, v_n, w_n, D_n \geq 0 \end{aligned} \quad (3.142)$$

3.4.3. Fuzzification of Bernhard's model with quadratic objective function

We consider the following Bernhard's capital budgeting model with quadratic objective function, fuzzy net cash flows, fuzzy horizon time value of cash flows, and fuzzy budget limits:

$$\begin{aligned}
\max \quad & \frac{\tilde{G}}{c} + \sum_{n=0}^N \frac{D_n}{(1+i)^n} - c_1 \sum_{n=1}^N (D_n - D_{n-1})^2 - c_2 \left(\frac{\tilde{G}}{c_3 c} - D_{ave} \right)^2 \\
\text{s.t.} \quad & -\sum_{j=1}^J \tilde{a}_{nj} x_j - l_{n-1} v_{n-1} + v_n + b_{n-1} w_{n-1} - w_n + D_n \leq \tilde{M}_n \quad [\rho_n] \quad \forall n \in \{0, 1, \dots, N\} \\
& D_n \geq D_{\min} \quad [\eta_n] \quad \forall n \in \{0, 1, \dots, N\} \\
& x_j \leq 1 \quad [\mu_j] \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{3.143}$$

where $\tilde{G} = \sum_{j=1}^J \tilde{a}_j x_j + v_N - w_N$, and ρ_n, η_n and μ_j are the corresponding multipliers.

Note that we borrow to use ρ_n, η_n and μ_j as the corresponding multipliers from [2, p: 330-331].

Then, the primal models with $\tilde{\leq}^T$ t-norm and $\tilde{\leq}_S$ t-conorm fuzzy relations will be as follows, respectively:

$$\begin{aligned}
\max \quad & \frac{\overset{-R}{G}_T(\alpha)}{c} + \sum_{n=0}^N \frac{D_{T,n}}{(1+i)^n} - c_1 \sum_{n=1}^N (D_{T,n} - D_{T,(n-1)})^2 - c_2 \left(\frac{\overset{-L}{G}_T(\alpha)}{c_3 c} - D_{T,ave} \right)^2 \\
\text{s.t.} \quad & -\sum_{j=1}^J \overset{-R}{a}_{nj}(\alpha) x_{T,j} - l_{n-1} v_{T,(n-1)} \\
& + v_{T,n} + b_{n-1} w_{T,(n-1)} - w_{T,n} + D_{T,n} \leq \overset{-R}{M}_n(\alpha) \quad [\rho_{T,n}] \\
& D_{T,n} \geq D_{\min} \quad [\eta_{T,n}] \\
& x_{T,j} \leq 1 \quad [\mu_{T,j}]
\end{aligned} \tag{3.144}$$

where $\overset{-R}{G}_T(\alpha) = \sum_{j=1}^J \overset{-R}{\hat{a}}_j(\alpha) x_{T,j} + v_{T,N} - w_{T,N}$, $\overset{-L}{G}_T(\alpha) = \sum_{j=1}^J \overset{-L}{\hat{a}}_j(\alpha) x_{T,j} + v_{T,N} - w_{T,N}$.

$\overset{-R}{\hat{a}}_j(\alpha), \overset{-L}{\hat{a}}_j(\alpha), \overset{-R}{a}_{nj}(\alpha), \overset{-R}{M}_n(\alpha)$ are defined as in (2.34) and (2.36), and $\rho_{T,n}, \eta_{T,n}$ and $\mu_{T,j}$ are the corresponding multipliers for the primal model with $\tilde{\leq}^T$ t-norm fuzzy relation.

$$\begin{aligned}
\max \quad & \frac{\bar{G}_S^L(1-\alpha)}{c} + \sum_{n=0}^N \frac{D_{S,n}}{(1+i)^n} - c_1 \sum_{n=1}^N (D_{S,n} - D_{S,(n-1)})^2 - c_2 \left(\frac{\bar{G}_S^R(1-\alpha)}{c_3 c} - D_{S,ave} \right)^2 \\
\text{s.t.} \quad & -\sum_{j=1}^J a_{nj}^L (1-\alpha) x_{S,j} - l_{n-1} v_{S,(n-1)} \\
& + v_{S,n} + b_{n-1} w_{S,(n-1)} - w_{S,n} + D_{S,n} \leq \bar{M}_n^L (1-\alpha) \quad [\rho_{S,n}] \\
& D_{S,n} \geq D_{\min} \quad [\eta_{S,n}] \\
& x_{S,j} \leq 1 \quad [\mu_{S,j}]
\end{aligned} \tag{3.145}$$

where

$$\begin{aligned}
\bar{G}_S^L(1-\alpha) &= \sum_{j=1}^J \hat{a}_j^L (1-\alpha) x_{S,j} + v_{S,N} - w_{S,N}, \bar{G}_S^R(1-\alpha) = \sum_{j=1}^J \hat{a}_j^R (1-\alpha) x_{S,j} + v_{S,N} - w_{S,N} \\
\hat{a}_j^L(1-\alpha), \hat{a}_j^R(1-\alpha), \hat{a}_{nj}^L(1-\alpha), \bar{M}_n^L(1-\alpha) &\text{ are defined as in (2.35) and (2.37), and} \\
\rho_{S,n}, \eta_{S,n} \text{ and } \mu_{S,j} &\text{ are the corresponding multipliers for the primal model with } \tilde{\leq}_S \text{ t-} \\
&\text{conorm fuzzy relation.}
\end{aligned}$$

Duality analysis

The following models are considered [51, p: 58; 2, p: 332]³⁶:

$$\begin{aligned}
\max \quad & d^T x + x^T H x \\
\text{s.t.} \quad & A x \leq b \\
& x \geq 0
\end{aligned} \tag{3.146}$$

and

$$\begin{aligned}
\min \quad & b^T \lambda - x^T H x \\
\text{s.t.} \quad & A^T \lambda - 2 H x \geq d \\
& \lambda \geq 0
\end{aligned} \tag{3.147}$$

³⁶ (3.146) and (3.147) are also adapted from Nonlinear Programming, A Unified Approach, 1969, Zangwill, W.I., p: 58.

where H contains entries c_{kj} such that “ c_{kj} is the coefficient of $x_k x_j$ in the objective function if $k = j$, otherwise $\frac{1}{2}$ times the coefficient of $x_k x_j$ in the objective function [2, p: 332]”.

Analysis 1:

$$\rho_{T,N}^* = \frac{2c_2}{c_3 c} \left[\frac{c_3}{2c_2} - \frac{\sum_{j=1}^J \hat{a}_j(\alpha) x_{T,j}^* + v_{T,N}^* - w_{T,N}^*}{c_3 c} + D_{T,ave}^* \right] \quad (3.148)$$

and

$$\rho_{S,N}^* = \frac{2c_2}{c_3 c} \left[\frac{c_3}{2c_2} - \frac{\sum_{j=1}^J \hat{a}_j(1-\alpha) x_{S,j}^* + v_{S,N}^* - w_{S,N}^*}{c_3 c} + D_{S,ave}^* \right] \quad (3.149)$$

holds for any $\alpha \in [0, 1)$, and nonnegative constant parameters c_1, c_2, c_3 .

Proof:

$$\rho_{T,N} + \frac{2c_2}{c_3 c} \left[\frac{\sum_{j=1}^J \hat{a}_j(\alpha) x_{T,j} + v_{T,N} - w_{T,N}}{c_3 c} - D_{T,ave} \right] \geq \frac{1}{c} \quad (3.150)$$

follows from the dual constraint of the primal variable $v_{T,N}$, and

$$-\rho_{T,N} - \frac{2c_2}{c_3 c} \left[\frac{\sum_{j=1}^J \hat{a}_j(\alpha) x_{T,j} + v_{T,N} - w_{T,N}}{c_3 c} - D_{T,ave} \right] \geq -\frac{1}{c} \quad (3.151)$$

follows from the dual constraint of the primal variable $w_{T,N}$. The combination of (3.150) and (3.151) gives (3.148). The similar argument follows also for (3.149).

Note that $\rho_N^* = \frac{2c_2}{c_3c} \left[\frac{c_3}{2c_2} - \frac{G}{c_3c} + D_{ave}^* \right]$ holds in the crisp case, where

$$G = \sum_{j=1}^J \hat{a}_j x_j^* + v_N^* - w_N^* \quad [2, \text{p: 332}].$$

Remark 1: If we assume $c_1 = c_2 = 0$, then (3.144) and (3.145) reduce to an LP model, and $\rho_{T,N}^* = \rho_{S,N}^* = \frac{1}{c}$.

Remark 2: Remark 1 holds also for $c_2 = 0$.

Analysis 2:

$$-\sum_{n=0}^{N-R} a_{nj}(\alpha) \rho_{T,n} + \mu_{T,j} + \frac{2c_2}{c_3c} \frac{\sum_{j=1}^J \hat{a}_j(\alpha) x_{T,j} + v_{T,N} - w_{T,N}}{c_3c} - D_{T,ave} \geq \frac{\hat{a}_j(\alpha)}{c}$$

$$\text{for } \forall j \in \{1, 2, \dots, J\} \quad (3.152)$$

follows from the dual constraint of the primal variable $x_{T,j}$, and

$$-\sum_{n=0}^{N-L} a_{nj}(1-\alpha) \rho_{S,n} + \mu_{S,j} + \frac{2c_2}{c_3c} \frac{\sum_{j=1}^J \hat{a}_j(1-\alpha) x_{S,j} + v_{S,N} - w_{S,N}}{c_3c} - D_{S,ave} \geq \frac{\hat{a}_j(1-\alpha)}{c}$$

$$\text{for } \forall j \in \{1, 2, \dots, J\} \quad (3.153)$$

follows from the dual constraint of the primal variable $x_{S,j}$.

Note that

$$-\sum_{n=0}^N a_{nj} \rho_n + \mu_j + \frac{2c_2 \hat{a}_j}{c_3c} \left(\frac{G}{c_3c} - D_{ave} \right) \geq \frac{\hat{a}_j}{c} \quad \forall j \in \{1, 2, \dots, J\} \quad (3.154)$$

holds in the crisp case, where $G = \sum_{j=1}^J \hat{a}_j x_j + v_N - w_N$ [2, p: 331].

Remark 1: If we assume $c_1 = c_2 = 0$, then (3.144) and (3.145) reduce to an LP model, and consequently (3.152) reduces to

$$-\sum_{n=0}^N a_{nj}^{-R}(\alpha) \rho_{T,n} + \mu_{T,j} \geq \frac{\hat{a}_j^{-R}(\alpha)}{c} \quad (3.155)$$

and (3.153) reduces to

$$-\sum_{n=0}^N a_{nj}^L(1-\alpha) \rho_{S,n} + \mu_{S,j} \geq \frac{\hat{a}_j^L(1-\alpha)}{c} \quad (3.156)$$

Remark 2: Remark 1 holds also for $c_2 = 0$.

Characterization of level of investment in projects

We analyze the characterization of level of investment in projects analogous to [5]. Note that we also analyzed the characterization of level of investment in projects for Weingartner's pure capital rationing model in Section 3.2.3 analogous to [5]. As in [5] and in Section 3.2.3, three distinguishing cases are to be considered: Fully funded project, partially funded project, and unfunded project. We analyze these three cases for quadratic programming model and extend the analysis to LP model for $c_1 = c_2 = 0$ in quadratic programming model.

Case 1: The project j is fully funded

Since $x_{T,j}^* = 1$,

$$\frac{\hat{a}_j^{-R}(\alpha)}{c} + \sum_{n=0}^N a_{nj}^{-R}(\alpha) \rho_{T,n}^* - \frac{2c_2 \hat{a}_j^{-L}(\alpha)}{c_3 c} \left(\frac{\sum_{j=1}^J \hat{a}_j^{-L}(\alpha) x_{T,j}^* + v_{T,N}^* - w_{T,N}^*}{c_3 c} - D_{T,ave}^* \right) \geq 0 \quad (3.157)$$

Since $x_{S,j}^* = 1$,

$$\frac{\hat{a}_j^L(1-\alpha)}{c} + \sum_{n=0}^N a_{nj}^L(1-\alpha) \rho_{S,n}^* - \frac{2c_2 \hat{a}_j^R(1-\alpha)}{c_3 c} \left(\frac{\sum_{j=1}^J \hat{a}_j^R(1-\alpha) x_{S,j}^* + v_{S,N}^* - w_{S,N}^*}{c_3 c} - D_{S,ave}^* \right) \geq 0 \quad (3.158)$$

Proof: Follows from (3.152) and (3.153), respectively, and corresponding complementary slackness results.

Remark 1: If we assume $c_1 = c_2 = 0$ or $c_2 = 0$; then (3.157) and (3.158) reduce to the following, respectively:

$$\frac{\hat{a}_j^{-R}(\alpha)}{c} + \sum_{n=0}^N a_{nj}^{-R}(\alpha) \rho_{T,n}^* \geq 0 \quad (3.159)$$

$$\frac{\hat{a}_j^L(1-\alpha)}{c} + \sum_{n=0}^N a_{nj}^L(1-\alpha) \rho_{S,n}^* \geq 0 \quad (3.160)$$

Case 2: The project j is partially funded

Since $0 < x_{T,j}^* < 1$,

$$\frac{\hat{a}_j^{-R}(\alpha)}{c} + \sum_{n=0}^N a_{nj}^{-R}(\alpha) \rho_{T,n}^* - \frac{2c_2 \hat{a}_j^{-L}(\alpha)}{c_3 c} \left(\frac{\sum_{j=1}^J \hat{a}_j^{-L}(\alpha) x_{T,j}^* + v_{T,N}^* - w_{T,N}^*}{c_3 c} - D_{T,ave}^* \right) = 0 \quad (3.161)$$

Since $0 < x_{S,j}^* < 1$,

$$\frac{\hat{a}_j^L(1-\alpha)}{c} + \sum_{n=0}^N a_{nj}^L(1-\alpha) \rho_{S,n}^* - \frac{2c_2 \hat{a}_j^R(1-\alpha)}{c_3 c} \left(\frac{\sum_{j=1}^J \hat{a}_j^R(1-\alpha) x_{S,j}^* + v_{S,N}^* - w_{S,N}^*}{c_3 c} - D_{S,ave}^* \right) = 0 \quad (3.162)$$

Remark 2: If we assume $c_1 = c_2 = 0$ or $c_2 = 0$; then (3.161) and (3.162) reduce to the following, respectively:

$$\frac{\hat{a}_j^{-R}(\alpha)}{c} + \sum_{n=0}^N a_{nj}^{-R}(\alpha) \rho_{T,n}^* = 0 \quad (3.163)$$

$$\frac{\hat{a}_j^L(1-\alpha)}{c} + \sum_{n=0}^N a_{nj}^L(1-\alpha) \rho_{S,n}^* = 0 \quad (3.164)$$

Case 3: The project j is not funded

Since $x_{T,j}^* = 0$,

$$\frac{\hat{a}_j^{(-R)}(\alpha)}{c} + \sum_{n=0}^N \hat{a}_{nj}^{(-R)}(\alpha) \rho_{T,n}^* - \frac{2c_2 \hat{a}_j^{(-L)}(\alpha)}{c_3 c} \left(\frac{\sum_{j=1}^J \hat{a}_j^{(-L)}(\alpha) x_{T,j}^* + v_{T,N}^* - w_{T,N}^*}{c_3 c} - D_{T,ave}^* \right) \leq 0 \quad (3.165)$$

Since $x_{S,j}^* = 0$,

$$\frac{\hat{a}_j^{(L)}(1-\alpha)}{c} + \sum_{n=0}^N \hat{a}_{nj}^{(L)}(1-\alpha) \rho_{S,n}^* - \frac{2c_2 \hat{a}_j^{(R)}(1-\alpha)}{c_3 c} \left(\frac{\sum_{j=1}^J \hat{a}_j^{(R)}(1-\alpha) x_{S,j}^* + v_{S,N}^* - w_{S,N}^*}{c_3 c} - D_{S,ave}^* \right) \leq 0 \quad (3.166)$$

Remark 3: If we assume $c_1 = c_2 = 0$ or $c_2 = 0$; then (3.165) and (3.166) reduce to the following, respectively:

$$\frac{\hat{a}_j^{(-R)}(\alpha)}{c} + \sum_{n=0}^N \hat{a}_{nj}^{(-R)}(\alpha) \rho_{T,n}^* \leq 0 \quad (3.167)$$

$$\frac{\hat{a}_j^{(L)}(1-\alpha)}{c} + \sum_{n=0}^N \hat{a}_{nj}^{(L)}(1-\alpha) \rho_{S,n}^* \leq 0 \quad (3.168)$$

Analysis 3:

$$\rho_{T,0}^* - \frac{2c_2}{N+1} \left(\frac{\hat{G}_T^{(-L)}(\alpha)}{c_3 c} - D_{T,ave}^* \right) + 2c_1(D_{T,0}^* + D_{T,1}^*) = 1 \quad D_{T,0}^* > D_{\min} \quad (3.169)$$

$$\rho_{T,n}^* - \frac{2c_2}{(N+1)(1+i)^n} \left(\frac{\hat{G}_T^{(-L)}(\alpha)}{c_3 c} - D_{T,ave}^* \right) + 2c_1(D_{T,(n-1)}^* + 2D_{T,n}^* + D_{T,(n+1)}^*) = \frac{1}{(1+i)^n}$$

$$\text{for } \forall n \in \{1, 2, \dots, N-1\} \quad D_{T,n}^* > D_{\min} \quad (3.170)$$

$$\rho_{T,N}^* \left(1 + \frac{c_3}{N+1} \right) + 2c_1(D_{T,(N-1)}^* + D_{T,N}^*) = \frac{1+c_3/(N+1)}{c} \quad D_{T,N}^* > D_{\min} \quad (3.171)$$

$$\rho_{S,0}^* - \frac{2c_2}{N+1} \left(\frac{\hat{G}_S^{(R)}(1-\alpha)}{c_3 c} - D_{S,ave}^* \right) + 2c_1(D_{S,0}^* + D_{S,1}^*) = 1 \quad D_{S,0}^* > D_{\min} \quad (3.172)$$

$$\rho_{S,n}^* - \frac{2c_2}{(N+1)(1+i)^n} \left(\frac{G_S^R(1-\alpha)}{c_3 c} - D_{S,ave}^* \right) + 2c_1(D_{S,(n-1)}^* + 2D_{S,n}^* + D_{S,(n+1)}^*) = \frac{1}{(1+i)^n}$$

$$\text{for } \forall n \in \{1, 2, \dots, N-1\} \quad D_{S,n}^* > D_{\min} \quad (3.173)$$

$$\rho_{S,N}^* \left(1 + \frac{c_3}{N+1} \right) + 2c_1(D_{S,(N-1)}^* + D_{S,N}^*) = \frac{1+c_3/(N+1)}{c} \quad D_{S,N}^* > D_{\min} \quad (3.174)$$

Note that the equations (3.169)-(3.174) follow from the dual constraints of $D_{T,n}$ and $D_{S,n}$. “Since the objective of the company is to have the difference between the terminal wealth and dividends as low as possible [2, p: 337]”, the effect of terminal wealth on the range of parameter c_1 vanishes, and range of parameter c_1 in the fuzzy case equals to that of the crisp case in [2, p: 337], i.e.,

$$c_1 \leq \frac{1}{2(D_{T,0}^* + D_{T,1}^*)}, \quad D_{T,0}^* > D_{\min} \quad (3.175)$$

$$c_1 \leq \frac{1}{2(D_{T,(n-1)}^* + 2D_{T,n}^* + D_{T,(n+1)}^*)(1+i)^n}, \quad \forall n \in \{1, 2, \dots, N-1\}, \quad D_{T,n}^* > D_{\min} \quad (3.176)$$

$$c_1 \leq \frac{1+c_3/(N+1)}{2c(D_{T,(N-1)}^* + D_{T,N}^*)} \quad D_{T,N}^* > D_{\min} \quad (3.177)$$

where the similar argument follows also for the model with $\widetilde{\leq}_s$ t-conorm fuzzy relation. Also note that (3.175)-(3.177) follow from (3.169)-(3.171) since $\rho_{T,n}^* \geq 0$ for $\forall n \in \{0, 1, \dots, N\}$. The following inequalities are used to determine the range of c_2 :

$$\frac{c_3}{c_2} \geq 2 \left(\frac{\sum_{j=1}^J \tilde{a}_j(\alpha) x_{T,j}^* + v_{T,N}^* - w_{T,N}^*}{c_3 c} - D_{T,ave}^* \right) \quad (3.178)$$

$$\frac{c_3}{c_2} \geq 2 \left(\frac{\sum_{j=1}^J \hat{a}_j (1 - \alpha) x_{S,j}^* + v_{S,N}^* - w_{S,N}^*}{c_3 c} - D_{S,ave}^* \right) \quad (3.179)$$

where (3.178) and (3.179) follow from (3.148) and (3.149) as in [2, p: 338], respectively, since $\rho_{T,N}^* \geq 0$, $\rho_{S,N}^* \geq 0$. $c_3 \approx \frac{1}{i}$, since c_3 serves for the “equalizing dividends within the planning period with those past the horizon [2, p: 338]”. Subsequently, c_2 can be determined from (3.178) for the primal model with t-norm $\tilde{\leq}^T$ fuzzy relation and from (3.179) for the primal model with t-conorm $\tilde{\leq}_s$ fuzzy relation; and c_1 can be determined from (3.171) or (3.169) or (3.170) for the primal model with t-norm $\tilde{\leq}^T$ fuzzy relation, and from (3.174) or (3.172) or (3.173) for the primal model with t-conorm $\tilde{\leq}_s$ fuzzy relation analogous to [2, p: 337-338].

3.4.4. Computational analysis

We use the following data set in Table 3.10 from [2, p: 339].

Table 3.10 : Cash flows and budget limits for a ten-project example

	Project 1	Project 2	Project 3	Project 4	Project 5	Project 6	Project 7	Project 8	Project 9	Project 10	Budget limit
Year 0	-10	-60		-30	-15	-50	0		-30	-10	70
Year 1	-10	7.5	-20	17.5	-15	0	0		33	-10	15
Year 2	-10	17.5	-10	17.05	-15	0	-40			35	15
Year 3	15	5.831	5		-15	20		-60		-13.09	15
Year 4	15	48.438	-5		20	25	48.4	-120			15
Year 5	9.405		10		25	28.825		60			15
Year 6			20		40.957			80			
Year 7			16.77					80			

We assume a horizon of 5 years, an external discount rate, a lending rate, and a borrowing rate of 9.5%, 9%, and 10.1%, respectively, and $D_{\min} = 5$ as in [2, p: 338]. We also assume a variability rate for the symmetric triangular fuzzy numbers of cash flows, horizon time value of cash flows, and budget limits analogous to [5], assign values 5%, 10%, 15%, and 20%, and obtain total 16 variability rate combinations such as 5% – 10%, where the first percentage represents the cash flow variability rate, and the second percentage represents the budget variability rate. We conduct 27 runs for quadratic programming by assigning three different nonnegative values for

each c_1, c_2, c_3 constant parameter as in [2, p: 338], for each $\alpha \in [0, 1)$ with 0.1 increments, and for each variability rate combination. We assume the same nonnegative values for c_1, c_2, c_3 as in [2, p: 338]. We solve the problem with the crisp parameters for quadratic programming model and for LP model, and with the fuzzy parameters for quadratic programming model and for LP model where $c_1 = c_2 = 0$. We use Xpress-MP software [47] for the computations that uses Newton's Barrier algorithm for the solution of the quadratic programming model. The Xpress-MP code is available in Table A.2. The performance criteria have been determined as terminal wealth and average dividend as in [2, p: 338-339]. In this computational analysis, we report the fuzzy numbers representing terminal wealth and average dividend for quadratic programming model, and for LP model, defuzzified interval of terminal wealth and defuzzified average dividend for quadratic programming model. Figure 3.7 denotes some sample fuzzy numbers representing terminal wealth for quadratic programming model and for constant parameters $c_1 = 10^{-7}$, $c_2 = 0.001$, and $c_3 = 10$, while other sample graphs are available in Figure A.2.

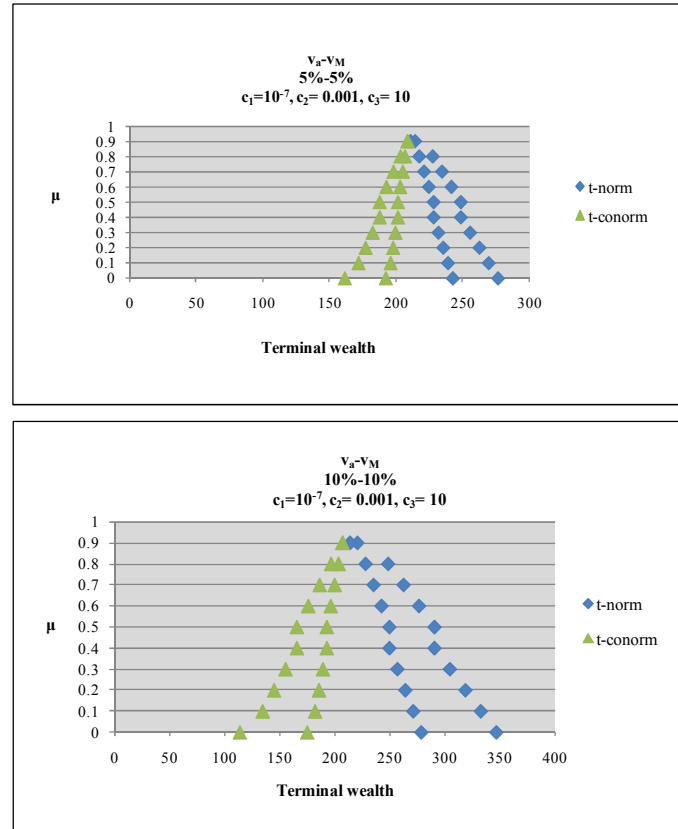


Figure 3.7 : Sample fuzzy numbers representing terminal wealth in quadratic programming model

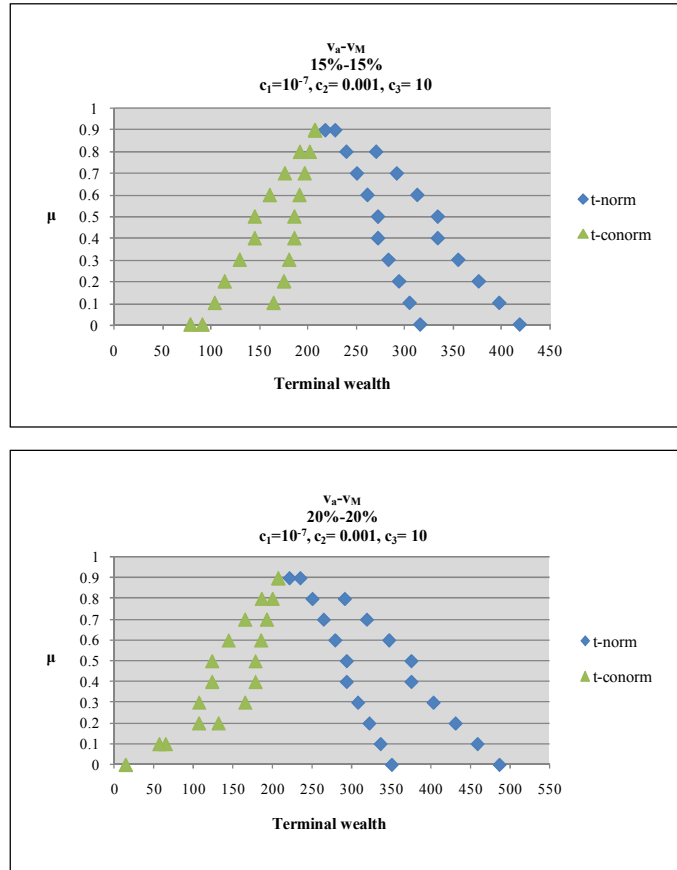


Figure 3.7 : Sample fuzzy numbers representing terminal wealth in quadratic programming model (continued)

Remarks: As anticipated, fuzzy numbers representing terminal wealth are normal, and can be interpreted as convex triangular shaped fuzzy numbers. It is not also amazing to observe that the cash flow variability has more impact than the budget variability rate on the variability of terminal wealth in both models. As the cash variability rate increases, in the model with t-conorm fuzzy relation, the triangular shape of terminal wealth deteriorates, especially for 20% variability rate, and terminal wealth approaches to zero for very low membership values.

When we solve the problem as an LP model, we observe that the fuzzy set representing terminal wealth in LP covers the fuzzy set representing terminal wealth in quadratic programming at some variability rate and parameter combinations, for example at 5% and 10% cash flow variability rates. Figure 3.8 denotes some sample fuzzy numbers representing terminal wealth with different variability rate combinations for LP, while all the graphs are available in Figure A.3. Note that the outputs of the two models yield together a triangular shaped fuzzy number for terminal wealth, and that terminal wealth equals to 294.061 in the crisp case.

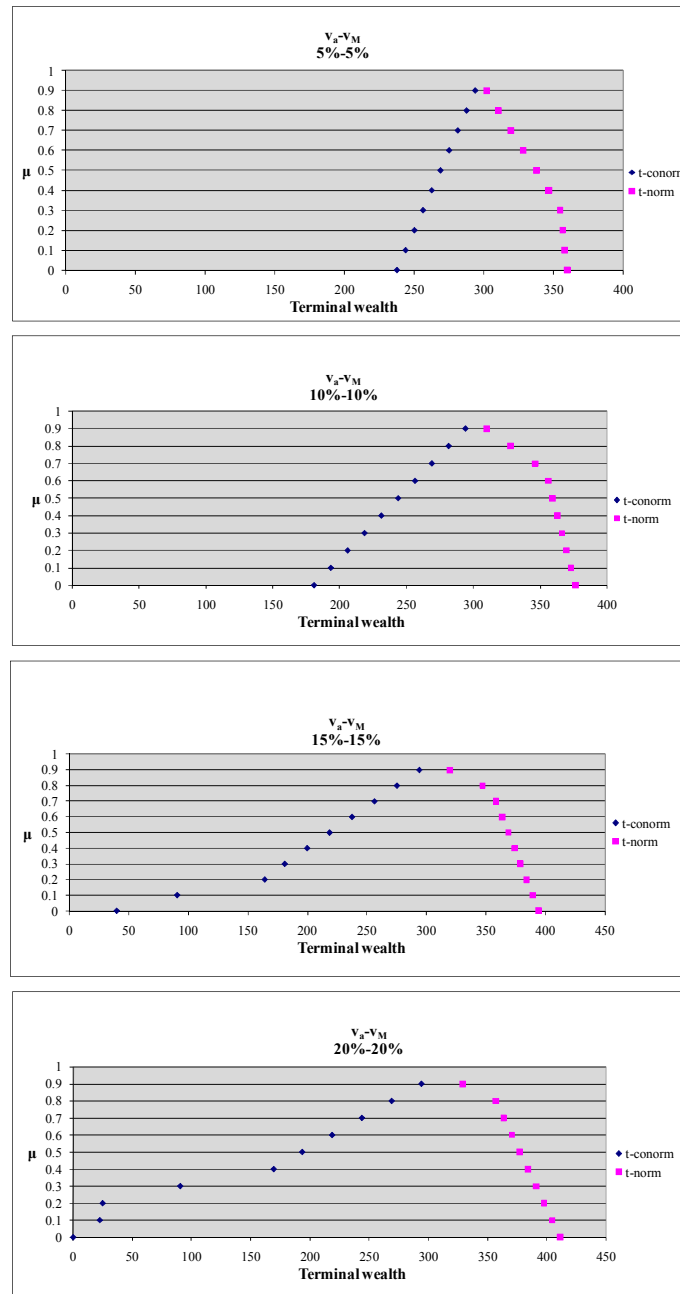


Figure 3.8 : Sample fuzzy numbers representing terminal wealth in LP model

Remarks: Like in quadratic programming model, the fuzzy numbers representing terminal wealth are normal, and can be interpreted as convex triangular shaped fuzzy numbers. The cash flow variability has more impact on the variability of the fuzzy number representing terminal wealth, and as the cash flow variability rate increases, the triangular shape of the fuzzy number representing terminal wealth deteriorates. The deterioration of the triangular shape starts at 20% cash flow variability rate in quadratic programming model, while it starts at 15% cash flow variability rate in LP model.

By Proposition 4, the model with t-norm $\widetilde{\leq}^T$ fuzzy relation represents the best-case scenario, and the model with t-conorm $\widetilde{\leq}_s$ fuzzy relation represents the worst-case scenario for $\alpha \in [0.5, 1)$. Therefore, we perform defuzzification to the triangular shaped fuzzy numbers representing terminal wealth obtained from quadratic programming model for $\alpha \in [0.5, 1)$, and obtain a defuzzified interval for each parameter and variability rate combination to see the variation of the terminal wealth between the most pessimistic value and the most optimistic value. Note that we use “Center of Gravity” method for the defuzzification. Table 3.11 denotes some sample defuzzified intervals of terminal wealth for different parameter and variability rate combinations as well as terminal wealth for each parameter combination in the crisp case, while all the tables are available in Table A.3.

Table 3.11 : Defuzzified interval of terminal wealth in quadratic programming model

		Crisp model	c_3	10	13	16		
		c_1	c_2	Terminal wealth				
10^{-7}			0.001	207.897	227.542	241.814		
			0.003	207.697	227.357	241.650		
			0.006	207.647	227.310	241.608		
10^{-6}			0.001	210.509	229.942	243.933		
			0.003	208.590	228.181	242.380		
			0.006	208.096	227.726	241.977		
10^{-5}			0.001	229.942	246.943	258.397		
			0.003	216.594	235.427	248.699		
			0.006	212.343	231.611	245.395		

5%-5%	Worst-case		Best-case	Worst-case		Best-case	Worst-case		Best-case
	c_3	10		13			16		
c_1	c_2								
10^{-7}	0.001	193.869	230.935	212.835	252.040	226.614	267.371		
	0.003	193.678	230.715	212.660	251.833	226.459	267.186		
	0.006	193.631	230.660	212.616	251.781	226.420	267.140		
10^{-6}	0.001	196.350	233.827	215.105	254.719	228.607	269.761		
	0.003	194.527	231.703	213.439	252.753	227.146	268.010		
	0.006	194.057	231.157	213.009	252.245	226.767	267.555		
10^{-5}	0.001	214.818	255.336	231.185	273.700	242.210	286.075		
	0.003	202.133	240.344	220.293	260.843	233.090	275.139		
	0.006	198.094	235.632	216.683	256.581	229.982	271.411		

Table 3.11 : Defuzzified interval of terminal wealth in quadratic programming model (continued)

10%-10%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	179.840	254.680	198.128	277.308	211.414	293.748
	0.003	179.660	254.436	197.963	277.079	211.269	293.541
	0.006	179.615	254.375	197.921	277.021	211.232	293.489
10^{-6}	0.001	182.192	257.862	200.268	280.280	213.280	296.422
	0.003	180.464	255.524	198.697	278.099	211.912	294.462
	0.006	180.019	254.922	198.292	277.536	211.557	293.954
10^{-5}	0.001	199.694	280.965	215.427	301.330	226.022	314.671
	0.003	187.673	264.244	205.159	287.048	217.480	302.437
	0.006	183.843	259.430	201.756	282.217	214.568	298.266

15%-15%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	153.066	278.385	163.980	302.532	171.909	320.075
	0.003	159.256	278.135	173.618	302.301	183.584	319.869
	0.006	162.586	278.073	177.501	302.242	188.589	319.817
10^{-6}	0.001	154.517	281.638	165.313	305.533	173.086	322.737
	0.003	159.586	279.248	174.075	303.331	184.374	320.786
	0.006	162.731	278.632	177.887	302.762	188.931	320.280
10^{-5}	0.001	165.313	305.251	174.779	326.795	181.354	340.910
	0.003	164.485	288.144	178.583	312.386	188.507	328.728
	0.006	165.183	283.229	180.244	307.541	191.032	324.574

Remarks: The decision maker can determine her position according to her risk behavior. Note that as mentioned before, in case of 20% cash flow variability rates, the triangular shape of terminal wealth deteriorates for the model with t-conorm fuzzy relation, and that “Center of Gravity” method is unable to yield a feasible defuzzified value. Each 5% increase in budget variability rate causes merely an approximate change of 3 units in the worst-case and best-case values, while each 5% increase in cash flow variability rate causes a considerable change in the worst-case and best-case values as in Table A.3.

Next, we deal with the fuzzy numbers representing average dividend for quadratic programming model and LP model. Figure 3.9 denotes some sample fuzzy numbers representing average dividend with different variability rate combinations, and constant parameters $c_1 = 10^{-7}$, $c_2 = 0.001$, $c_3 = 10$ for quadratic programming model,

while other sample graphs for quadratic programming model are available in Figure A.4.

As in Figure 3.9 and in Figure A.4, the fuzzy numbers representing average dividend in quadratic programming model are triangular fuzzy numbers until 15% cash flow variability rate. The deterioration starts at 15% cash flow variability rate because of the t-conorm fuzzy relation based results, and the increase in budget cash flow variability rate has little impact on the deterioration.

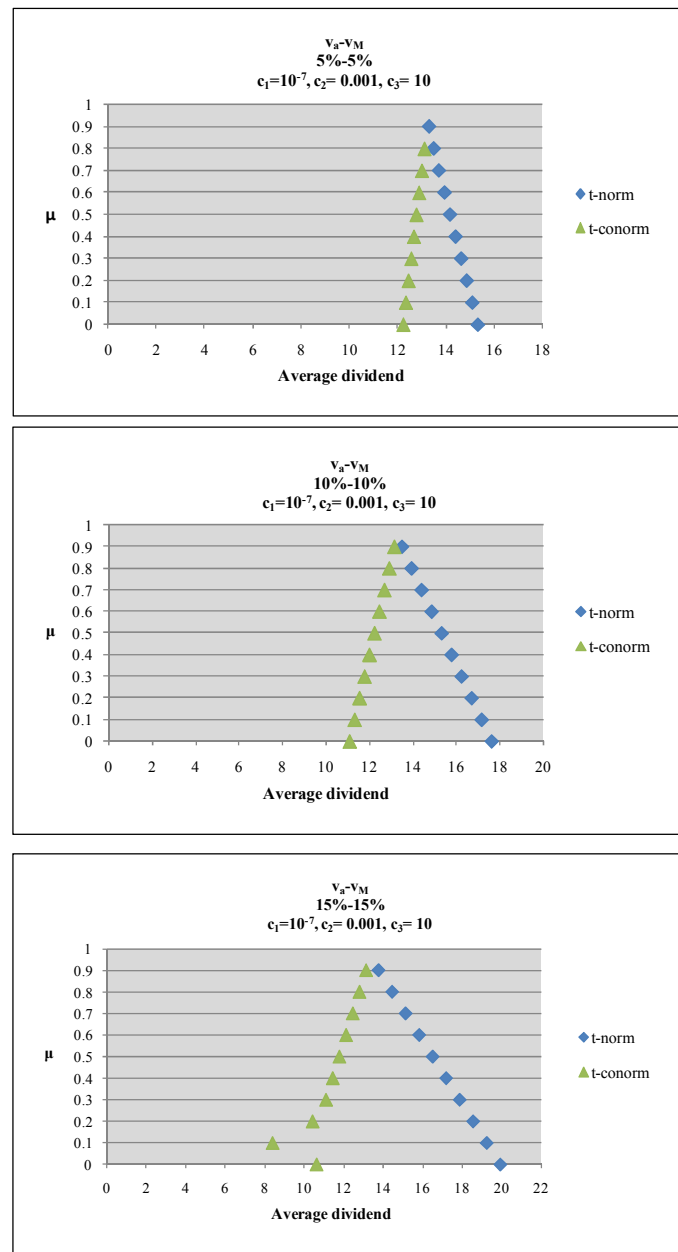


Figure 3.9 : Sample fuzzy numbers representing average dividend in quadratic programming model

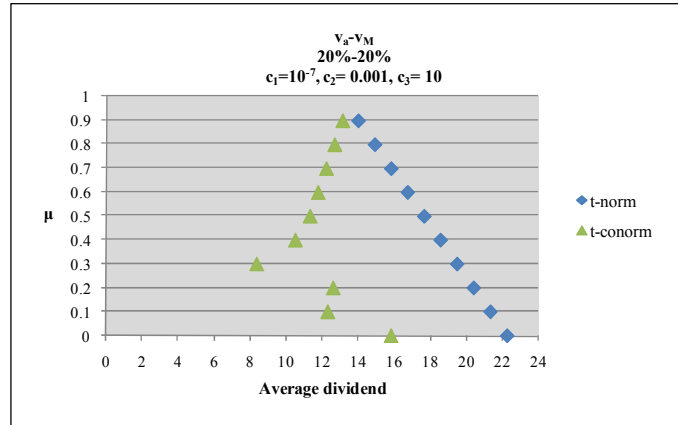


Figure 3.9 : Sample fuzzy numbers representing average dividend in quadratic programming model (continued)

Figure 3.10 denotes some sample fuzzy numbers representing average dividend for LP model, while all the graphs for LP model are available in Figure A.5, respectively.

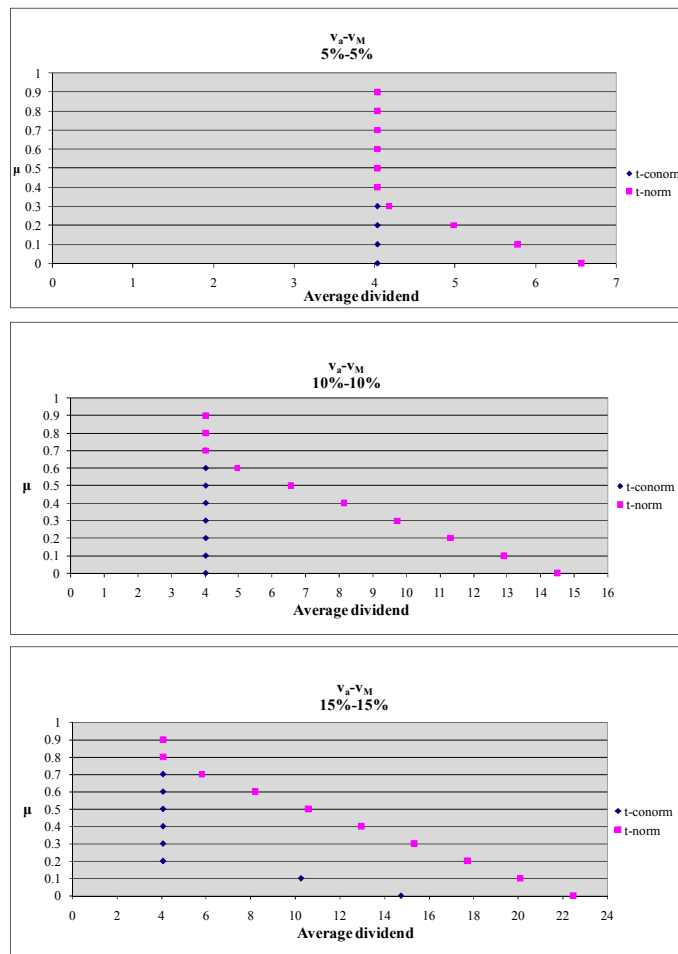


Figure 3.10 : Sample fuzzy numbers representing average dividend in LP model

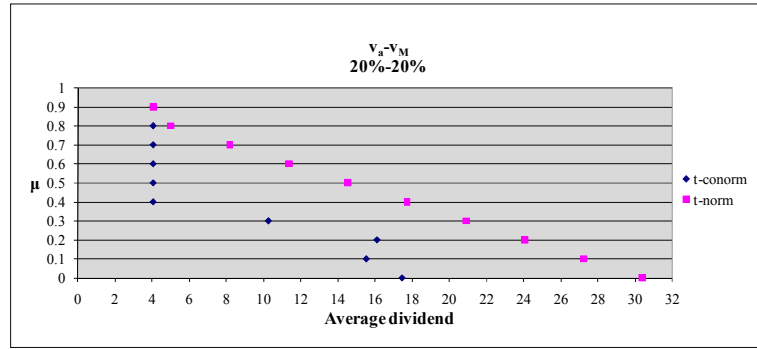


Figure 3.10 : Sample fuzzy numbers representing average dividend in LP model (continued)

Remarks: Although the fuzzy numbers representing average dividend in quadratic programming model and in LP model share some similar patterns, additional insights are also to be observed. The fuzzy numbers representing average dividend in LP model are non-convex, and the results from LP model are much lower than those from quadratic programming model for any given α – cut level. In LP model, for the model with t-norm fuzzy relation, the variability of the average dividend is affected by each 5% increase of cash flow variability rate or budget variability rate, while for the model with t-conorm fuzzy relation, it is unaffected at 5% and 10% cash flow variability rates. For 5% and 10% cash flow variability rates, the average dividend is 4 for the model with t-conorm fuzzy relation. The triangular shape of average dividend deteriorates starting from 15% cash flow variability rate in quadratic programming model and LP model.

By following the same argument for terminal wealth, we perform defuzzification to the fuzzy numbers representing average dividend obtained from quadratic programming model for $\alpha \in [0.5, 1)$, and obtain the defuzzified average dividend values for each parameter and variability rate combination. Table 3.12 denotes some sample defuzzified average dividend values for different parameter and variability rate combinations as well as the average dividend value for each parameter combination in the crisp case, while all the tables are available in Table A.4.

As in Table 3.12, 20% cash flow variability rate has the most impact on the variability of average dividend, while other variability rates have less impact. Although, the deterioration of the fuzzy numbers representing average dividend starts at 15% cash flow variability rate, the “Center of Gravity” method managed to give defuzzified values for 15% and 20% cash flow variability rates. However, with a

close look, it is possible to observe that the defuzzified values may get inaccurate as the cash flow variability rate increases, especially for 20% cash flow variability rate.

Table 3.12 : Defuzzified average dividend in quadratic programming model

Crisp model				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	13.155	11.076	9.565
	0.003	13.177	11.095	9.582
	0.006	13.182	11.100	9.586
10^{-6}	0.001	12.879	10.821	9.340
	0.003	13.082	11.008	9.505
	0.006	13.134	11.056	9.547
10^{-5}	0.001	10.821	9.021	7.809
	0.003	12.235	10.241	8.836
	0.006	12.685	10.645	9.185
5%-5%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.088	11.860	10.242
	0.003	14.111	11.882	10.261
	0.006	14.117	11.887	10.266
10^{-6}	0.001	13.167	11.062	9.547
	0.003	13.376	11.255	9.718
	0.006	13.430	11.305	9.762
10^{-5}	0.001	11.042	9.198	7.956
	0.003	12.519	10.461	9.023
	0.006	12.984	10.879	9.386
10%-10%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.722	12.416	11.281
	0.003	14.747	12.440	11.305
	0.006	14.753	12.445	9.641
10^{-6}	0.001	12.682	10.749	10.536
	0.003	12.891	10.940	10.759
	0.006	12.945	10.989	6.704
10^{-5}	0.001	10.641	8.927	8.570
	0.003	12.105	10.160	9.863
	0.006	12.535	10.579	6.432

Table 3.12 : Defuzzified average dividend in quadratic programming model
(continued)

15%-15%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	15.656	13.149	11.333
	0.003	15.467	13.123	11.433
	0.006	15.351	12.986	10.227
10^{-6}	0.001	14.164	12.611	10.930
	0.003	13.333	11.582	10.778
	0.006	12.922	11.083	7.084
10^{-5}	0.001	12.631	10.828	9.362
	0.003	12.669	10.883	10.172
	0.006	12.564	10.719	6.810
20%-20%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	18.900	15.248	13.174
	0.003	15.837	15.526	13.173
	0.006	16.478	13.760	12.725
10^{-6}	0.001	16.562	13.776	11.687
	0.003	16.325	14.088	11.985
	0.006	15.311	13.533	12.085
10^{-5}	0.001	15.022	12.616	11.015
	0.003	15.921	13.639	11.707
	0.006	15.099	13.355	11.953

General Remarks for Computational Analysis: As the variability rates increase, especially cash flow variability rates, the decision maker has to consider a wider interval. Although the fuzzy numbers seem to be convex for low variability rates, as the cash flow variability rate increases, the triangular shape deteriorates, and the fuzzy numbers turn out to be non-convex, which makes the defuzzification impossible or inaccurate. Starting from definite cash flow variability rates, the defuzzified fuzzy number values representing terminal wealth approach to zero for very low membership values, which implies the most pessimistic case. The decision maker may consider either pessimistic or optimistic models, or determine her risk behaviour by taking the values between the pessimistic and optimistic results into account, although the latter may give a very wide range. The discrepancy between the results of quadratic programming and LP models is considerably high. In LP model, the variability of the fuzzy numbers representing terminal wealth is affected by the cash flow variability rates more than quadratic programming model.

4. ROBUST LINEAR PROGRAMMING APPROACH TO CAPITAL RATIONING AND CAPITAL BUDGETING

In this chapter, we extend the robust optimization approach to capital rationing and capital budgeting proposed by [5]. We also propose a robust approach to some investment appraisal techniques for simple projects, i.e., to NPV and IRR, and illustrate the proposed robust decision rules with the numerical examples.

4.1. Extensions to Robust Optimization Approach to Capital Rationing

We extend the robust optimization approach to capital rationing proposed by [5]. Unlike in [5], we consider the following generic LP problem:

$$\begin{aligned}
 \max \quad & \sum_{j \in J_c} \tilde{c}_j x_j + \sum_{j \notin J_c} c_j x_j \\
 \text{s.t.} \quad & \sum_{j \in J_{A_n}} \tilde{a}_{nj} x_j + \sum_{j \notin J_{A_n}} a_{nj} x_j \leq \tilde{b}_n \quad \forall n \in B_b \\
 & \sum_{j \in J_{A_n}} \tilde{a}_{nj} x_j + \sum_{j \notin J_{A_n}} a_{nj} x_j \leq b_n \quad \forall n \notin B_b \\
 & 0 \leq x \leq u
 \end{aligned} \tag{4.1}$$

where $\tilde{c}_j$ for J_c is defined as a random variable such that $\tilde{c}_j \in [c_j, c_j + \hat{c}_j]$, c_j is the nominal (mean) value, $\hat{c}_j > 0$, and J_c is the set of the objective function coefficients that are definitely subject to uncertainty analogous to [52]. Although $\tilde{c}_j \in [c_j, c_j + \hat{c}_j]$ is defined for J_c , in order to tune the robustness, we also define a protection function $\beta_c(x^*, \Gamma_c)$ with a protection level of Γ_c analogous to [12,52].

$\tilde{a}_{nj}$ for $j \in J_{A_n}$ is defined as a random variable with symmetric interval uncertainty such that $\tilde{a}_{nj} \in [a_{nj} - \hat{a}_{nj}, a_{nj} + \hat{a}_{nj}]$, where a_{nj} is the nominal (mean) value, $\hat{a}_{nj} > 0$,

and J_{A_n} is the set of the entries of the n th constraint that are definitely subject to uncertainty analogous to [12]. We define a protection function $\beta_{A_n}(x^*, \Gamma_{A_n})$ for $\forall n \in \{1, 2, \dots, N\}$ with a protection level of Γ_{A_n} to tune the robustness such that $\Gamma_{A_n} = \lfloor 0, |J_{A_n}| \rfloor$, where Γ_{A_n} is not necessarily an integer, analogous to [12]. We define the worst-case scenario for the n th constraint as $\Gamma_{A_n} = \lfloor |J_{A_n}| \rfloor$.

$\tilde{b}_n$ for $\forall n \in B_b$ is defined as a random variable with symmetric interval uncertainty such that $\tilde{b}_n \in [b_n - \hat{b}_n, b_n + \hat{b}_n]$, where b_n is the nominal (mean) value, $\hat{b}_n > 0$, and B_b is the set of the constraints with the right-hand-sides that are definitely subject to uncertainty. We define a protection function $\beta_{b_n}(\Gamma_{b_n})$ for $\forall n \in B_b$ with a protection level of Γ_{b_n} to tune the robustness analogous to [12].

Finally, u is a J -dimensional vector with the entries u_j such that $u_j = 1$ for $\forall j$ as in [5].

By considering the protection functions, (4.1) will be equivalent to the following:

$$\begin{aligned}
& \max \quad c^T x - \beta_c(x^*, \Gamma_c) \\
& \text{s.t.} \quad \sum_{j=1}^J a_{nj} x_j + \beta_{A_n}(x^*, \Gamma_{A_n}) - b_n + \beta_{b_n}(\Gamma_{b_n}) \leq 0 \quad \forall n \in B_b \\
& \quad \sum_{j=1}^J a_{nj} x_j + \beta_{A_n}(x^*, \Gamma_{A_n}) \leq b_n \quad \forall n \notin B_b \\
& \quad 0 \leq x \leq 1
\end{aligned} \tag{4.2}$$

where all protection functions are defined analogous to [12] and as in Section 2.2 as follows:

$$\beta_c(x^*, \Gamma_c) = \max \left\{ \sum_{j \in S_c} \hat{c}_j x_j^* + (\Gamma_c - \lfloor \Gamma_c \rfloor) \hat{c}_t x_t^* : \begin{aligned} & S_c \cup \{t\}, S_c \subseteq J_c, |S_c| = \lfloor \Gamma_c \rfloor, t \in J_c \setminus S_c \end{aligned} \right\} \tag{4.2a}$$

We obtain the following problem by introducing the variable z_j such that:

$$\beta_c(x^*, \Gamma_c) = \max \left\{ \sum_{j \in J_c} \hat{c}_j x_j^* z_j, \sum_{j \in J_c} z_j \leq \Gamma_c, \quad 0 \leq z_j \leq 1, \quad \forall j \in J_c \right\} \quad (4.2b)$$

The equivalence of two problems like in (4.2a) and (4.2b) was proved in [12].

We take the dual of $\beta_c(x^*, \Gamma_c)$ as in [12]:

$$\beta_c(x^*, \Gamma_c) = \min \left\{ \sum_{j \in J_c} p_{cj} + \Gamma_c z_c : z_c + p_{cj} \geq \hat{c}_j x_j^*, \quad z_c \geq 0, \quad p_{cj} \geq 0, \quad \forall j \in J_c \right\} \quad (4.2c)$$

$$\text{Note that by strong duality } \sum_{j \in J_c} \hat{c}_j x_j^* z_j^* = \sum_{j \in J_c} p_{cj}^* + \Gamma_c z_c^* \quad (4.2d)$$

□

$$\beta_{A_n}(x^*, \Gamma_{A_n}) = \max \left\{ \sum_{j \in S_{A_n}} \hat{a}_{nj} |x_j^*| + (\Gamma_{A_n} - \lfloor \Gamma_{A_n} \rfloor) \hat{a}_{nt_n} |x_{t_n}^*| : \right. \\ \left. S_{A_n} \cup \{t_n\}, \quad S_{A_n} \subseteq J_{A_n}, \quad |S_{A_n}| = \lfloor \Gamma_{A_n} \rfloor, \quad t_n \in J_{A_n} \setminus S_{A_n} \right\} \quad (4.2e)$$

We obtain the following problem by introducing the variable z_{nj} such that:

$$\beta_{A_n}(x^*, \Gamma_{A_n}) = \max \left\{ \sum_{j \in J_{A_n}} \hat{a}_{nj} |x_j^*| z_{nj}, \sum_{j \in J_{A_n}} z_{nj} \leq \Gamma_{A_n}, \quad 0 \leq z_{nj} \leq 1, \quad \forall j \in J_{A_n} \right\} \\ \forall n \in \{1, 2, \dots, N\} \quad (4.2f)$$

We take the dual of $\beta_{A_n}(x^*, \Gamma_{A_n})$ as in [12]:

$$\min \left\{ \sum_{j \in J_{A_n}} p_{anj} + \Gamma_{A_n} z_{A_n} : z_{A_n} + p_{anj} \geq \hat{a}_{nj} |x_j^*|, \quad z_{A_n} \geq 0, \quad p_{anj} \geq 0, \quad \forall j \in J_{A_n} \right\} \\ \forall n \in \{1, 2, \dots, N\} \quad (4.2g)$$

$$\text{Note that by strong duality } \sum_{j \in J_{A_n}} \hat{a}_{nj} |x_j^*| z_{nj}^* = \sum_{j \in J_{A_n}} p_{anj}^* + \Gamma_{A_n} z_{A_n}^* \quad (4.2h)$$

□

$$\beta_{b_n}(\Gamma_{b_n}) = \max \left\{ \hat{b}_n \Gamma_{b_n} : 0 < \Gamma_{b_n} \leq 1 \right\} \quad \forall n \in B_b \quad (4.2i)$$

We obtain the following problem by introducing the variable t_{b_n} such that:

$$\beta_{b_n}(\Gamma_{b_n}) = \max \left\{ \hat{b}_n t_{b_n} : t_{b_n} \leq \Gamma_{b_n}, 0 \leq t_{b_n} \leq 1, \forall n \in B_b \right\} \quad (4.2j)$$

We take the dual of $\beta_{b_n}(\Gamma_{b_n})$ as in [12]:

$$\min \left\{ p_{b_n} + \Gamma_{b_n} z_{b_n} : z_{b_n} + p_{b_n} \geq \hat{b}_n, z_{b_n} \geq 0, p_{b_n} \geq 0, \forall n \in B_b \right\} \quad (4.2k)$$

$$\text{Note that by strong duality } \hat{b}_n t_{b_n}^* = p_{b_n}^* + \Gamma_{b_n} z_{b_n}^*. \quad (4.2l)$$

□

Finally, by considering (4.2a)-(4.2l), (4.2) will be equivalent to the following model:

$$\begin{aligned} \max \quad & c^T x - \sum_{j \in J_c} p_{cj} - \Gamma_c z_c \\ \text{s.t.} \quad & \sum_{j=1}^J a_{nj} x_j + \sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n} + p_{b_n} + \Gamma_{b_n} z_{b_n} \leq b_n \quad \forall n \in B_b \\ & \sum_{j=1}^J a_{nj} x_j + \sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n} \leq b_n \quad \forall n \notin B_b \\ & z_c + p_{cj} \geq \hat{c}_j x_j \quad \forall j \in J_c \\ & z_{A_n} + p_{a_{nj}} \geq \hat{a}_{nj} |x_j| \quad \forall n \in \{1, 2, \dots, N\} \quad \forall j \in J_{A_n} \\ & z_{b_n} + p_{b_n} \geq \hat{b}_n \quad \forall n \in B_b \\ & 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \\ & -y_j \leq x_j \leq y_j \quad \forall j \in \{1, 2, \dots, J\} \\ & p_{cj} \geq 0 \quad \forall j \in J_c \\ & z_c \geq 0 \\ & p_{a_{nj}} \geq 0 \quad \forall n \in \{1, 2, \dots, N\} \quad \forall j \in J_{A_n} \end{aligned} \quad (4.3)$$

$$z_{A_n} \geq 0 \quad \forall n \in \{1, 2, \dots, N\}$$

$$p_{b_n} \geq 0 \quad \forall n \in B_b$$

$$z_{b_n} \geq 0 \quad \forall n \in B_b$$

When we apply the idea of the generic model (4.1) and its robust counterpart (4.3) to the pure capital rationing model (3.2), then we obtain the following pure capital rationing model, and its robust counterpart, respectively:

$$\begin{aligned}
\max \quad & \sum_{n=0}^N \frac{1}{(1+r)^n} \left(\sum_{j \in J_{A_n}} \tilde{a}_{nj} x_j + \sum_{j \notin J_{A_n}} a_{nj} x_j \right) \\
\text{s.t.} \quad & - \sum_{j \in J_{A_n}} \tilde{a}_{nj} x_j - \sum_{j \notin J_{A_n}} a_{nj} x_j \leq \tilde{M}_n \quad \forall n \in B_M \\
& - \sum_{j \in J_{A_n}} \tilde{a}_{nj} x_j - \sum_{j \notin J_{A_n}} a_{nj} x_j \leq M_n \quad \forall n \notin B_M \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{4.4}$$

where $B_M \cup \bar{B}_M = \{0, 1, \dots, N\}$, $\tilde{a}_{nj} \in [a_{nj} - \hat{a}_{nj}, a_{nj} + \hat{a}_{nj}]$ for $j \in J_{A_n}$ is assumed to be a symmetric and bounded random variable, and $\tilde{M}_n \in [M_n - \hat{M}_n, M_n + \hat{M}_n]$ for $n \in B_M$ is assumed to be a symmetric and bounded random variable.

$$\begin{aligned}
\max \quad & \sum_{j=1}^J NPV_j x_j - \sum_{n=0}^N \frac{\beta_{A_n}(x^*, \Gamma_{A_n})}{(1+r)^n} \\
\text{s.t.} \quad & - \sum_{j=1}^J a_{nj} x_j + \beta_{A_n}(x^*, \Gamma_{A_n}) + \beta_{M_n}(\Gamma_{M_n}) \leq M_n \quad \forall n \in B_M \\
& - \sum_{j=1}^J a_{nj} x_j + \beta_{A_n}(x^*, \Gamma_{A_n}) \leq M_n \quad \forall n \notin B_M \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{4.5}$$

where $\beta_{A_n}(x^*, \Gamma_{A_n})$ and $\beta_{M_n}(\Gamma_{M_n})$ are defined as in (4.2e) and (4.2i), respectively.

Finally, (4.5) will be equivalent to the following:

$$\begin{aligned}
& \max \quad \sum_{j=1}^J NPV_j x_j - \sum_{n=0}^N \frac{\sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n}}{(1+r)^n} \\
& \text{s.t.} \quad - \sum_{j=1}^J a_{nj} x_j + \sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n} + p_{M_n} + \Gamma_{M_n} z_{M_n} \leq M_n \quad \forall n \in B_M \quad [\rho_{A_n, M_n}] \\
& \quad - \sum_{j=1}^J a_{nj} x_j + \sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n} \leq M_n \quad \forall n \notin B_M \quad [\rho_{A_n, M_n}] \\
& \quad z_{A_n} + p_{a_{nj}} \geq \hat{a}_{nj} x_j \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \quad [\theta_{a_{nj}, M_n}] \\
& \quad z_{M_n} + p_{M_n} \geq \hat{M}_n \quad \forall n \in B_M \quad [\theta_{A_n, M_n}] \\
& \quad 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \quad [\mu_{j, A, M}] \\
& \quad p_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \\
& \quad z_{A_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \quad p_{M_n} \geq 0 \quad \forall n \in B_M \\
& \quad z_{M_n} \geq 0 \quad \forall n \in B_M
\end{aligned} \tag{4.6}$$

where ρ_{A_n, M_n} , θ_{a_{nj}, M_n} for $\forall j \in J_{A_n}$, θ_{A_n, M_n} for $\forall n \in B_M$, and $\mu_{j, A, M}$ are the respective dual variables in case of cash flow and budget uncertainty. Note that (4.6) is an extension of the case of cash flow uncertainty as in (3.3) with an exception that (3.3) does not involve net cash flows in the objective function, but an introduced variable x_{J+n+1} for $\forall n \in \{0, 1, \dots, N\}$ to avoid the protection in the objective function [5]. (3.3) has been obtained from (3.2) by introducing a slack variable x_{J+n+1} for $\forall n \in \{0, 1, \dots, N\}$ to transform the inequality constraints of (3.2) into equality constraints. Then, as a result of a series of mathematical operations, the variables in the objective function were also transformed into x_{J+n+1} for $\forall n \in \{0, 1, \dots, N\}$. Finally, it was shown that the model with x_{J+n+1} for $\forall n \in \{0, 1, \dots, N\}$ in the objective function and inequality constraints involving x_{J+n+1} for $\forall n \in \{0, 1, \dots, N\}$ is equivalent to the model with x_{J+n+1} for $\forall n \in \{0, 1, \dots, N\}$ in the objective function and equality constraints involving x_{J+n+1} for $\forall n \in \{0, 1, \dots, N\}$ as a result of the equivalence of the dual models [5].

The dual models of (4.6) and (3.3) will be as follows, respectively³⁷:

$$\begin{aligned}
& \min \sum_{n=0}^N M_n \rho_{A_n, M_n} + \sum_{n \in B_M} \hat{M}_n \theta_{A_n, M_n} + \sum_{j=1}^J \mu_{j, A, M} \\
& \text{s.t. } \sum_{n=0}^N (-a_{nj} \rho_{A_n, M_n} + \hat{a}_{nj} \theta_{A_n, M_n}) + \mu_{j, A, M} \geq NPV_j \quad \forall j \in \{1, 2, \dots, J\} \quad (x_j) \\
& \Gamma_{A_n} \rho_{A_n, M_n} - \sum_{j \in J_{A_n}} \theta_{A_n, M_n} \geq -\frac{\Gamma_{A_n}}{(1+r)^n} \quad \forall n \in \{0, 1, \dots, N\} \quad (z_{A_n}) \\
& \rho_{A_n, M_n} - \theta_{A_n, M_n} \geq -\frac{1}{(1+r)^n} \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} (p_{A_n}) \\
& \Gamma_{M_n} \rho_{A_n, M_n} + \theta_{A_n, M_n} \geq 0 \quad \forall n \in B_M \quad (z_{M_n}) \\
& \rho_{A_n, M_n} + \theta_{A_n, M_n} \geq 0 \quad \forall n \in B_M \quad (p_{M_n}) \\
& \rho_{A_n, M_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \theta_{A_n, M_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \\
& \theta_{A_n, M_n} \leq 0 \quad \forall n \in B_M \\
& \mu_{j, A, M} \geq 0 \quad \forall j \in \{1, 2, \dots, J\}
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
& \min \sum_{n=0}^N M_n \rho_{A_n} + \sum_{j=1}^J \mu_{j, A} \\
& \text{s.t. } \sum_{n=0}^N (-a_{nj} \rho_{A_n} + \hat{a}_{nj} \theta_{A_n}) + \mu_{j, A} \geq 0 \quad \forall j \in \{1, 2, \dots, J\} \quad (x_j) \\
& \Gamma_{A_n} \rho_{A_n} - \sum_{j \in J_{A_n}} \theta_{A_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad (z_n) \\
& \rho_{A_n} \geq \frac{1}{(1+r)^n} \quad \forall n \in \{0, 1, \dots, N\} \quad (x_{J+n+1})
\end{aligned} \tag{4.8}$$

³⁷ Model (4.8) is adapted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

$$\rho_{A_n} - \theta_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n}(p_{a_{nj}})$$

$$\mu_{j,A} \geq 0 \quad \forall j \in \{1, 2, \dots, J\}$$

$$\theta_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n}$$

Finally, the complementary slackness results for the primal-dual pair (4.6) and (4.7) will be as follows:

$$\begin{aligned} \rho_{A_n, M_n}^* \left(M_n + \sum_{j=1}^J a_{nj} x_j^* - \sum_{j \in J_{A_n}} p_{a_{nj}}^* - \Gamma_{A_n} z_{A_n}^* - p_{M_n}^* - \Gamma_{M_n} z_{M_n}^* \right) &= 0 & \forall n \in B_M \\ \rho_{A_n, M_n}^* \left(M_n + \sum_{j=1}^J a_{nj} x_j^* - \sum_{j \in J_{A_n}} p_{a_{nj}}^* - \Gamma_{A_n} z_{A_n}^* \right) &= 0 & \forall n \notin B_M \\ \theta_{a_{nj}, M_n}^* (-\hat{a}_{nj} x_j^* + z_{A_n}^* + p_{a_{nj}}^*) &= 0 & \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \\ \theta_{A_n, M_n}^* (-\hat{M}_n + z_{M_n}^* + p_{M_n}^*) &= 0 & \forall n \in B_M \\ \mu_{j, A, M}^* (1 - x_j^*) &= 0 & \forall j \in \{1, 2, \dots, J\} \\ x_j^* \left(-NPV_j + \sum_{n=0}^N (-a_{nj} \rho_{A_n, M_n}^* + \hat{a}_{nj} \theta_{a_{nj}, M_n}^*) + \mu_{j, A, M}^* \right) &= 0 & \forall j \in \{1, 2, \dots, J\} \\ z_{A_n}^* \left(\Gamma_{A_n} \rho_{A_n, M_n}^* - \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* + \frac{\Gamma_{A_n}}{(1+r)^n} \right) &= 0 & \forall n \in \{0, 1, \dots, N\} \\ p_{a_{nj}}^* \left(\rho_{A_n, M_n}^* - \theta_{a_{nj}, M_n}^* + \frac{1}{(1+r)^n} \right) &= 0 & \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \\ z_{M_n}^* (\Gamma_{M_n} \rho_{A_n, M_n}^* + \theta_{A_n, M_n}^*) &= 0 & \forall n \in B_M \\ p_{M_n}^* (\rho_{A_n, M_n}^* + \theta_{A_n, M_n}^*) &= 0 & \forall n \in B_M \end{aligned} \tag{4.9}$$

In the following sections, we analyze the robust counterpart of pure capital rationing model, and extend the computational analysis in [5].

4.1.1. Analysis of robust counterpart of capital rationing model

We next analyze the level of investment in projects, and the characterization of $\hat{a}_{nj}$ and $\hat{M}_n$ by considering (4.6), (4.7), and (4.9).

4.1.1.1. Characterization of level of investment in projects³⁸

The characterization of level of investment in projects is an extension of the analysis in [5].

Case 1: The project j is fully funded in case of budget and cash flow uncertainty

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_{A_n, M_n}^* \geq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}, M_n}^* \quad (4.10)$$

follows from (4.7) and (4.9).

The project j is fully funded in case of cash flow uncertainty

$$\sum_{n=0}^N a_{nj} \rho_{A_n}^* \geq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}}^* \quad (4.11)$$

follows from [5], where $\rho_{A_n}^*$ and $\theta_{a_{nj}}^*$ are the dual variables in case of cash flow uncertainty.

The project j is fully funded in the deterministic case

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_n^* \geq 0 \quad (4.12)$$

follows from [5], where ρ_n^* is the dual variable in the deterministic case.

Case 2: The project j is partially funded in case of budget and cash flow uncertainty

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_{A_n, M_n}^* = \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}, M_n}^* \quad (4.13)$$

follows from (4.7) and (4.9).

The project j is partially funded in case of cash flow uncertainty

$$\sum_{n=0}^N a_{nj} \rho_{A_n}^* = \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}}^* \quad (4.14)$$

³⁸ (4.11), (4.12), (4.14), (4.15), (4.17), and (4.18) are reprinted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

follows from [5].

The project j is partially funded in the deterministic case

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_n^* = 0 \quad (4.15)$$

follows from [5].

Case 3: The project j is not funded in case of budget and cash flow uncertainty

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_{A_n, M_n}^* \leq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}, M_n}^* \quad (4.16)$$

follows from (4.7) and (4.9).

The project j is not funded in case of cash flow uncertainty

$$\sum_{n=0}^N a_{nj} \rho_{A_n}^* \leq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}}^* \quad (4.17)$$

follows from [5].

The project j is not funded in the deterministic case

$$NPV_j + \sum_{n=0}^N a_{nj} \rho_n^* \leq 0 \quad (4.18)$$

follows from [5].

Remarks: In robust counterpart of pure capital rationing model with budget and cash flow uncertainty, the level of investment in projects is characterized by the input data $(NPV_j, a_{nj}, \hat{a}_{nj})$, and the optimal budget dual variables $(\rho_{A_n, M_n}^*, \theta_{a_{nj}, M_n}^*)$, whereas in robust counterpart of pure capital rationing model with cash flow uncertainty, it is characterized by the input data $(a_{nj}, \hat{a}_{nj})$, and the optimal budget dual variables $(\rho_{A_n}^*, \theta_{a_{nj}}^*)$ as emphasized in [5]. In other words, in pure capital rationing model with additional budget uncertainty, the discount rate r is also instrumental in the characterization of level of investment in projects as in the deterministic case.

4.1.1.2. Characterization of $\hat{a}_{nj}$ and $\hat{M}_n$

The characterization of $\hat{a}_{nj}$ and $\hat{M}_n$ is also analogous to [5].

Case 1: Characterization of $\hat{a}_{nj}$

Since $\hat{a}_{nk} > 0$ for $k \in J_{A_n}$, then $z_{A_n}^* + p_{a_{nk}}^* > 0$, and $z_{A_n}^*$ and/or $p_{a_{nk}}^*$ are strictly positive from (4.6) analogous to [5], if project k is not unfunded.

Case 1.1: $z_{A_n}^* > 0$, $p_{a_{nk}}^* > 0$ for $k \in J_{A_n}$

$$\text{If } z_{A_n}^* > 0, \text{ then } \Gamma_{A_n} \rho_{A_n, M_n}^* - \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* + \frac{\Gamma_{A_n}}{(1+r)^n} = 0 \quad (4.19)$$

from (4.9).

$$\text{If } p_{a_{nk}}^* > 0 \text{ for } k \in J_{A_n}, \text{ then } \rho_{A_n, M_n}^* - \theta_{a_{nk}, M_n}^* + \frac{1}{(1+r)^n} = 0 \quad (4.20)$$

from (4.9).

Remark 1.1.1: If $z_{A_n}^* > 0$ and $p_{a_{nk}}^* > 0$ for $k \in J_{A_n}$, then

$$\Gamma_{A_n} \theta_{a_{nk}, M_n}^* = \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* \quad (4.21)$$

Proof: Follows easily from (4.19) and (4.20).

Remark 1.1.2: Since we define the worst-case scenario as $\Gamma_{A_n} = |J_{A_n}|$, then

$$\sum_{j \in J_{A_n}} (\theta_{a_{nj}, M_n}^* - \theta_{a_{nk}, M_n}^*) = 0 \text{ for } k \in J_{A_n} \quad (4.22)$$

holds.

Proof: If $\Gamma_{A_n} = |J_{A_n}|$, then $|J_{A_n}| \theta_{a_{nk}, M_n}^* = \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^*$ from (4.21) which implies

$$\sum_{j \in J_{A_n}} \theta_{a_{nk}, M_n}^* = \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* \rightarrow \sum_{j \in J_{A_n}} (\theta_{a_{nj}, M_n}^* - \theta_{a_{nk}, M_n}^*) = 0 \quad (4.23)$$

By the same argument in Case 1.1, Case 1.2 and Case 1.3 will follow:

Case 1.2: $z_{A_n}^* = 0$, $p_{a_{nk}}^* > 0$ for $k \in J_{A_n}$

Remark 1.2.1:

$$\Gamma_{A_n} \theta_{a_{nk}, M_n}^* \geq \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* \text{ for } k \in J_{A_n} \quad (4.24)$$

Remark 1.2.2: Since we define the worst-case scenario as $\Gamma_{A_n} = |J_{A_n}|$, then

$$\sum_{j \in J_{A_n}} (\theta_{a_{nj}, M_n}^* - \theta_{a_{nk}, M_n}^*) \leq 0 \text{ for } k \in J_{A_n} \quad (4.25)$$

Case 1.3: $p_{a_{nk}}^* = 0$ for $k \in J_{A_n}$, $z_{A_n}^* > 0$

Remark 1.3.1:

$$\Gamma_{A_n} \theta_{a_{nk}, M_n}^* \leq \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* \text{ for } k \in J_{A_n} \quad (4.26)$$

Remark 1.3.2: Since we define the worst-case scenario as $\Gamma_{A_n} = |J_{A_n}|$, then

$$\sum_{j \in J_{A_n}} (\theta_{a_{nj}, M_n}^* - \theta_{a_{nk}, M_n}^*) \geq 0 \text{ for } k \in J_{A_n} \quad (4.27)$$

Case 2: Characterization of $\hat{M}_n$

The characterization of $\hat{M}_n$ is analogous to the characterization of $\hat{a}_{nj}$. Since $\hat{M}_n > 0$ for $n \in B_M$, then $z_{M_n}^* + p_{M_n}^* > 0$, and $z_{M_n}^*$ and/or $p_{M_n}^*$ are strictly positive from (4.6).

Case 2.1: $z_{M_n}^* > 0$, $p_{M_n}^* > 0$ for $n \in B_M$

Case 2.2: $z_{M_n}^* = 0$, $p_{M_n}^* > 0$ for $n \in B_M$

Case 2.3: $p_{M_n}^* = 0$, $z_{M_n}^* > 0$ for $n \in B_M$

Remark 2.1: If $p_{M_n}^* > 0$ and $0 < \Gamma_{M_n} < 1$, then $\rho_{A_n, M_n}^* = 0$ for $n \in B_M$. (4.28)

Proof: Since $\theta_{A_n, M_n}^* = -\rho_{A_n, M_n}^*$ from (4.9), $\Gamma_{M_n} \rho_{A_n, M_n}^* - \rho_{A_n, M_n}^* = \rho_{A_n, M_n}^* (\Gamma_{M_n} - 1) \geq 0$

from (4.7). Since $\Gamma_{M_n} - 1 < 0$ and $\rho_{A_n, M_n}^* \geq 0$, then $\rho_{A_n, M_n}^* (\Gamma_{M_n} - 1) \geq 0$ if and only if $\rho_{A_n, M_n}^* = 0$.

Remark 2.2: If $p_{M_n}^* > 0$ or $z_{M_n}^* > 0$ and $\Gamma_{M_n} = 1$, then $\rho_{A_n, M_n}^* + \theta_{A_n, M_n}^* = 0$ (4.29)

for $n \in B_M$.

4.1.2. Computational analysis

In this section, we extend the computational analysis in [5]. We consider the data in Table 3.1. The definitions for budget and cash flow variability rates in Section 3.2.4 hold. We assume Γ_{A_n} and Γ_{M_n} as $\Gamma_{A_n} = \Gamma_A$ and $\Gamma_{M_n} = \Gamma_M$ for $\forall n$ as in [5]. Γ_A values are increased by 0.1 from 0 to 3 as in [5], whereas Γ_M values are increased by 0.1 from 0.1 to 1. For the sake of simplicity, the first B constraints starting from 0 are assumed to belong to the set B_M . Xpress-MP software is used for the computations [47]. The Xpress-MP code, an extension of the Xpress-MP code in [5], is provided in Table B.1³⁹. The summary of the results is provided in Table 4.1.

Table 4.1 : Summary of the results for robust optimization approach to pure capital rationing model

v_a	v_M	Γ_A	Γ_M	Project 1	Project 2	Project 3	Objective function	Robust objective function % of deterministic
5%	5%	0	1	1	1	0.3125	10.8013	95.21%
		1	1	0.6098	1	0.7622	9.4055	82.90%
		2	1	0.5952	1	0.7440	8.1523	71.86%
		3	1	1	1	0.2183	7.5322	66.39%
5%	10%	0	1	1	1	0.2083	10.2575	90.41%
		1	1	0.5691	1	0.7114	8.9405	78.81%
		2	1	0.5556	1	0.6944	7.7709	68.50%
		3	1	1	1	0.1190	7.1592	63.10%
5%	15%	0	1	1	1	0.1042	9.7138	85.62%
		1	1	0.5285	1	0.6606	8.4756	74.71%
		2	1	0.5159	1	0.6448	7.3896	65.13%
		3	1	1	1	0.0198	6.7863	59.82%

³⁹ Table B.1 is extended from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

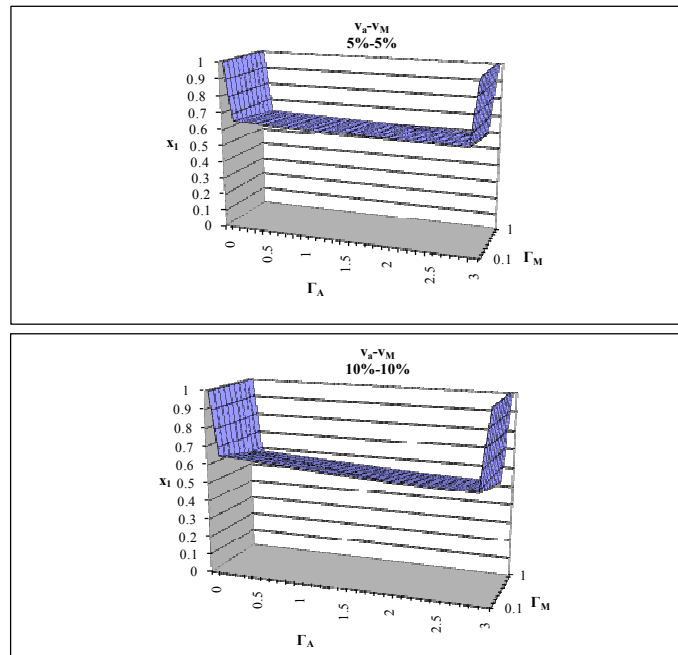
Table 4.1 : Summary of the results for pure capital rationing model (continued)

v_a	v_M	Γ_A	Γ_M	Project 1	Project 2	Project 3	Objective function	Robust objective function % of deterministic
5%	20%	0	1	1	1	0.0000	9.1700	80.83%
		1	1	0.4878	1	0.6098	8.0107	70.61%
		2	1	0.4762	1	0.5952	7.0082	61.77%
		3	1	0.9365	1	0.0000	6.4131	56.53%
10%	5%	0	1	1	1	0.3125	10.8013	95.21%
		1	1	0.5952	1	0.7440	8.0680	71.11%
		2	1	0.5682	1	0.7102	5.7270	50.48%
		3	1	1	1	0.1326	4.5581	40.18%
10%	10%	0	1	1	1	0.2083	10.2575	90.41%
		1	1	0.5556	1	0.6944	7.6784	67.68%
		2	1	0.5303	1	0.6629	5.4934	48.42%
		3	1	1	1	0.0379	4.3405	38.26%
10%	15%	0	1	1	1	0.1042	9.7138	85.62%
		1	1	0.5159	1	0.6448	7.2887	64.25%
		2	1	0.4924	1	0.6155	5.2598	46.36%
		3	1	0.9545	1	0.0000	4.1227	36.34%
10%	20%	0	1	1	1	0.0000	9.1700	80.83%
		1	1	0.4762	1	0.5952	6.8991	60.81%
		2	1	0.4545	1	0.5682	5.0262	44.30%
		3	1	0.8788	1	0.0000	3.9047	34.42%
15%	5%	0	1	1	1	0.3125	10.8013	95.21%
		1	1	0.5814	1	0.7267	6.7831	59.79%
		2	1	0.5435	1	0.6793	3.4944	30.80%
		3	1	1	1	0.0543	1.8407	16.23%
15%	10%	0	1	1	1	0.2083	10.2575	90.41%
		1	1	0.5426	1	0.6783	6.4652	56.99%
		2	1	0.5072	1	0.6341	3.3958	29.93%
		3	1	0.9710	1	0.0000	1.7648	15.56%
15%	15%	0	1	1	1	0.1042	9.7138	85.62%
		1	1	0.5039	1	0.6298	6.1473	54.19%
		2	1	0.4710	1	0.5888	3.2971	29.06%
		3	1	0.8986	1	0.0000	1.6886	14.88%
15%	20%	0	1	1	1	0.0000	9.1700	80.83%
		1	1	0.4651	1	0.5814	5.8294	51.38%
		2	1	0.4348	1	0.5435	3.1985	28.19%
		3	1	0.8261	1	0.0000	1.6125	14.21%
20%	5%	0	1	1	1	0.3125	10.8013	95.21%
		1	1	0.5682	1	0.7102	5.5470	48.89%
		2	1	0.3333	1	0.4167	1.5659	13.80%
		3	1	0	1	0.0000	0.1123	0.99%
20%	10%	0	1	1	1	0.2083	10.2575	90.41%
		1	1	0.5303	1	0.6629	5.2977	46.70%
		2	1	0.3333	1	0.4167	1.5659	13.80%
		3	1	0	1	0.0000	0.1123	0.99%
20%	15%	0	1	1	1	0.1042	9.7138	85.62%
		1	1	0.4924	1	0.6155	5.0483	44.50%
		2	1	0.3333	1	0.4167	1.5659	13.80%
		3	1	0	1	0.0000	0.1123	0.99%

Table 4.1 : Summary of the results for pure capital rationing model (continued)

20%	20%	0	1	1	1	0.0000	9.1700	80.83%
		1	1	0.4545	1	0.5682	4.7990	42.30%
		2	1	0.3333	1	0.4167	1.5659	13.80%
		3	1	0	1	0.0000	0.1123	0.99%

Project 2 is a fully funded project for all combinations, and Project 1 and Project 3 are in competition as in [5]. Unlike in case of cash flow uncertainty as in [5], no robust objective function is 100% of the deterministic NPV value, since in our formulation of budget and cash flow uncertainty, at least $\Gamma_M \neq 0$. It is not amazing to observe that as we increase the cash flow variability rate for a constant budget variability rate or vice versa for a given Γ_A and Γ_M combination, the level of investment in Project 1 and Project 3, and the objective function decrease. For any variability rate combination at $\Gamma_A = 0$ and $\Gamma_M = 1$, Project 1 is also a fully funded project, while at $v_M = 20\%$ when $\Gamma_A = 0$ or $\Gamma_A = 3$ and $\Gamma_M = 1$, Project 3 is not funded. The best objective function values are attained for $v_M = 5\%$ when $\Gamma_A = 0$ and $\Gamma_M = 1$ with 95.21% of the deterministic value, while the worst objective function values are attained for $v_a = 20\%$ when $\Gamma_A = 3$ and $\Gamma_M = 1$ with 0.99% of the deterministic value. Some sample graphs for the level of investment in Project 1 and Project 3 are provided in Figure 4.1 and Figure 4.2, respectively, while all the graphs are available in Figure B.1 and Figure B.2, respectively.

**Figure 4.1** : Level of investment in Project 1

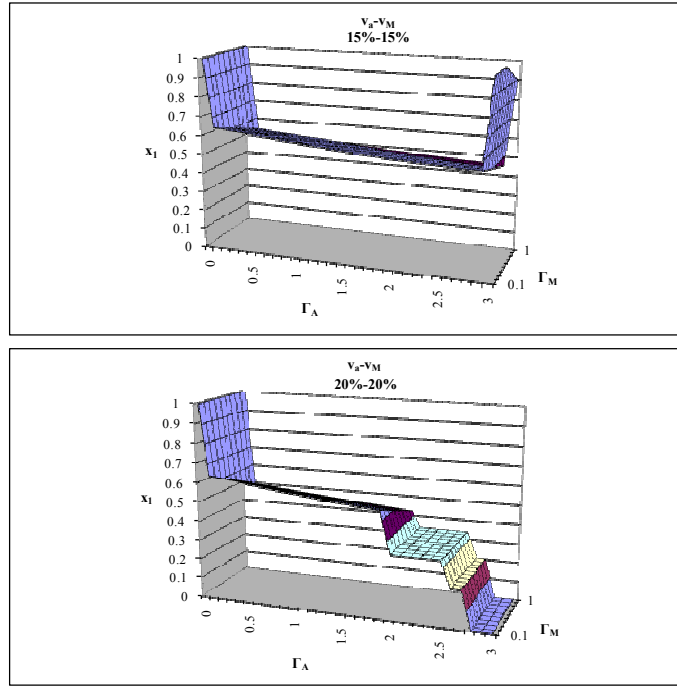


Figure 4.1 : Level of investment in Project 1 (continued)

Remarks: All the graphs except for $v_a = 20\%$ have a similar pattern. In case of cash flow uncertainty as in [5], the characteristics of the graphs for $v_a = 5\%$, 15% , 20% were variable such that the graphs had a stepwise behavior after an initial monotonically increasing level of investment phase. The graph for $v_a = 10\%$ had a monotonically increasing level of investment phase and consequently monotonically decreasing level of investment phase. The main difference in case of budget and cash flow uncertainty and in case of cash flow uncertainty is that Project 1 has an initial decreasing level of investment phase in case of cash flow and budget uncertainty for a constant Γ_M value, while it has an initial increasing level of investment phase in case of cash flow uncertainty. For $v_a = 5\%$, 10% , 15% , as we increase Γ_A in $\Gamma_A = [0, 0.1]$ for a constant Γ_M value, the level of investment in Project 1 decreases monotonically. As we increase Γ_A in $\Gamma_A = [0.1, 2.8]$ for a constant Γ_M value, the level of investment in Project 1 decreases, while as the budget variability rate increases for a constant cash flow variability rate, the level of investment in Project 1 decreases with a larger slope. As we increase Γ_A in $\Gamma_A = [2.8, 3]$ for a constant Γ_M value, the level of investment in Project 1 increases monotonically. The main

difference in the pattern of $v_a = 20\%$ is that for $\Gamma_A = [1.8, 3]$, the level of investment in Project 1 decreases stepwise with different slopes.

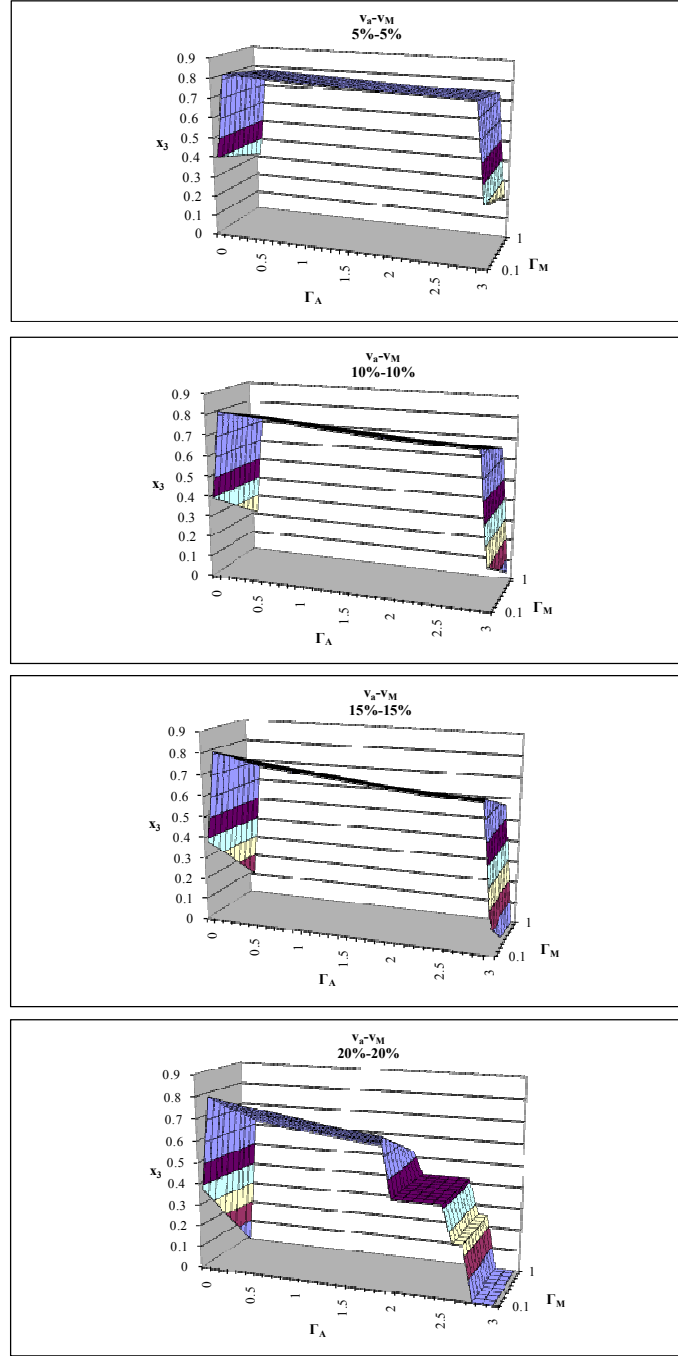


Figure 4.2 : Level of investment in Project 3

Remarks: All the graphs except for $v_a = 20\%$ have a similar pattern. In case of cash flow uncertainty as in [5], the characteristics of the graphs for $v_a = 5\%$, 10% , 20% were variable such that the graphs had a stepwise behavior after an initial monotonically decreasing level of investment phase. The graph for $v_a = 15\%$ had a

stepwise monotonically decreasing level of investment phase. The main difference in case of budget and cash flow uncertainty and in case of cash flow uncertainty is that Project 3 has an initial increasing level of investment phase in case of cash flow and budget uncertainty for a constant Γ_M , while it has an initial decreasing level of investment phase in case of cash flow uncertainty. For $v_a = 5\%, 10\%, 15\%$, as we increase Γ_A in $\Gamma_A = [0, 0.1]$ for a constant Γ_M value, the level of investment in Project 3 increases monotonically, while as we increase budget variability rate for a constant cash flow variability rate, the level of investment in Project 3 decreases for $\Gamma_A = 0$ and $\Gamma_M = 1$. As we increase Γ_A in $\Gamma_A = [0.1, 2.8]$ for a constant Γ_M value, the level of investment in Project 3 decreases, while as the budget variability rate increases for a constant cash flow variability rate, the level of investment in Project 3 decreases with a larger slope. As we increase Γ_A in $\Gamma_A = [2.8, 3]$ for a constant Γ_M value, the level of investment in Project 3 decreases monotonically. The main difference in the pattern of $v_a = 20\%$ is that for $\Gamma_A = [1.8, 3]$, the level of investment in Project 3 decreases with different slopes.

General Remarks for Project 1 and Project 3: For $v_a = 5\%, 10\%, 15\%$ when $\Gamma_A = [0, 0.2]$ and $\Gamma_A = [2.8, 3]$, Project 1 and Project 3 have an opposite behavior in case of budget and cash flow uncertainty such that for a constant Γ_M , the level of investment changes in an opposite direction when we increase Γ_A . This opposite behavior was especially considerable when $\Gamma_A = [0, 0.2]$ in case of cash flow uncertainty as in [5]. For $v_a = 20\%$ when $\Gamma_A = [0, 0.1]$, Project 1 and Project 3 have an opposite behavior, while they share a similar characteristic for $\Gamma_A = [0.1, 3]$. We note that for $v_a = 20\%$ when $\Gamma_A = [0.1, 3]$ and Γ_M is constant, the graphs in case of budget and cash flow uncertainty as depicted in Figure 4.1 and Figure 4.2, and in case of cash flow uncertainty as in [5], share a similar characteristic, especially share the similar stepwise behavior for $\Gamma_A = [1.8, 3]$.

4.2. Extensions to Robust Optimization Approach to Horizon Capital Budgeting

We extend the robust optimization approach to Weingartner's horizon model proposed by [5]. The model that we consider will be as follows:

$$\begin{aligned}
\max \quad & \sum_{j=1}^J h_j x_j + v_N - w_N \\
\text{s.t.} \quad & - \sum_{j \in J_{A_0}} \tilde{a}_{0j} x_j - \sum_{j \notin J_{A_0}} a_{0j} x_j + v_0 - w_0 \leq M_0 \\
& - \sum_{j \in J_{A_n}} \tilde{a}_{nj} x_j - \sum_{j \notin J_{A_n}} a_{nj} x_j \\
& \quad + v_n - (1+r_l)v_{n-1} + (1+r_b)w_{n-1} - w_n \leq \tilde{M}_n \quad \forall n \in B_M \\
& - \sum_{j \in J_{A_n}} \tilde{a}_{nj} x_j - \sum_{j \notin J_{A_n}} a_{nj} x_j \\
& \quad + v_n - (1+r_l)v_{n-1} + (1+r_b)w_{n-1} - w_n \leq M_n \quad \forall n \notin B_M, n \neq 0 \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \\
& v_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& w_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\}
\end{aligned} \tag{4.30}$$

where h_j is post-horizon cash flow for project j . By reconsidering all assumptions in Section 4.1, (4.30) will be equivalent to the following:

$$\begin{aligned}
\max \quad & \sum_{j=1}^J h_j x_j + v_N - w_N \\
\text{s.t.} \quad & - \sum_{j=1}^J a_{0j} x_j + \beta_{A_0}(x^*, \Gamma_{A_0}) + v_0 - w_0 \leq M_0 \\
& - \sum_{j=1}^J a_{nj} x_j + \beta_{A_n}(x^*, \Gamma_{A_n}) - M_n + \beta_{M_n}(\Gamma_{A_n}) \\
& \quad + v_n - (1+r_l)v_{n-1} + (1+r_b)w_{n-1} - w_n \leq 0 \quad \forall n \in B_M \\
& - \sum_{j=1}^J a_{nj} x_j + \beta_{A_n}(x^*, \Gamma_{A_n}) + v_n - (1+r_l)v_{n-1} + (1+r_b)w_{n-1} - w_n \leq M_n \quad \forall n \notin B_M, n \neq 0 \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \\
& v_n \geq 0, w_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\}
\end{aligned} \tag{4.31}$$

Finally, the linear robust counterpart of (4.30) will be as follows by reconsidering (4.2e)-(4.2g) and (4.2i)-(4.2k):

$$\begin{aligned}
& \max \sum_{j=1}^J h_j x_j + v_N - w_N \\
& \text{s.t.} - \sum_{j=1}^J a_{0j} x_j + \sum_{j \in J_{A_0}} p_{a_{0j}} + \Gamma_{A_0} z_{A_0} + v_0 - w_0 \leq M_0 \quad [\rho_{A_0, M_0}] \\
& - \sum_{j=1}^J a_{nj} x_j + \sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n} + p_{M_n} + \Gamma_{M_n} z_{M_n} \\
& \quad + v_n - (1 + r_l) v_{n-1} + (1 + r_b) w_{n-1} - w_n \leq M_n \quad \forall n \in B_M \quad [\rho_{A_n, M_n}] \\
& - \sum_{j=1}^J a_{nj} x_j + \sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n} + v_n - (1 + r_l) v_{n-1} + (1 + r_b) w_{n-1} - w_n \leq M_n \quad \forall n \notin B_M \quad [\rho_{A_n, M_n}] \\
& z_{A_n} + p_{a_{nj}} \geq \hat{a}_{nj} x_j \quad \forall n \in \{0, 1, \dots, N\} \forall j \in J_{A_n} \quad [\theta_{a_{nj}, M_n}] \\
& z_{M_n} + p_{M_n} \geq \hat{M}_n \quad \forall n \in B_M \quad [\theta_{A_n, M_n}] \quad (4.32) \\
& 0 \leq x_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \quad [\mu_{j, A, M}] \\
& z_{A_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& p_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \forall j \in J_{A_n} \\
& z_{M_n} \geq 0 \quad \forall n \in B_M \\
& p_{M_n} \geq 0 \quad \forall n \in B_M \\
& v_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& w_n \geq 0 \quad \forall n \in \{0, 1, \dots, N\}
\end{aligned}$$

Note that (4.32) is an extension of the case of cash flow uncertainty in [5]⁴⁰ with an exception that in [5], a $J + 2N + 2$ -dimensional vector y with all variables of the model and a $(N + 1) \times (J + 2N + 2)$ -dimensional matrix B with all parameters of the left-hand side of the model are introduced before formulating the linear robust counterpart. Finally, the linear robust counterpart in case of cash flow uncertainty will be as follows [5]:

⁴⁰ (4.33) is adapted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

$$\begin{aligned}
& \max \sum_{j=1}^{J+2N+2} h_j y_j \\
& \text{s.t.} - \sum_{j=1}^{J+2N+2} b_{nj} y_j + \sum_{j \in J_{A_n}} p_{a_{nj}} + \Gamma_{A_n} z_{A_n} \leq M_n \quad [\rho_{A_n}] \\
& \quad z_{A_n} + p_{a_{nj}} \geq \hat{a}_{nj} y_j \quad \forall n \in \{0, 1, \dots, N\} \forall j \in J_{A_n} \quad [\theta_{a_{nj}}] \\
& \quad 0 \leq y_j \leq 1 \quad \forall j \in \{1, 2, \dots, J\} \quad [\mu_{j,A}] \\
& \quad y_j \geq 0 \quad \forall j \in \{J+1, J+2, \dots, J+2N+2\} \\
& \quad z_{A_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \quad p_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \forall j \in J_{A_n}
\end{aligned} \tag{4.33}$$

The dual model of (4.32) will be as follows:

$$\begin{aligned}
& \min \sum_{n=0}^N M_n \rho_{A_n, M_n} + \sum_{n \in B_M} \hat{M}_n \theta_{A_n, M_n} + \sum_{j=1}^J \mu_{j,A,M} \\
& \text{s.t.} \left(\sum_{n=0}^N -a_{nj} \rho_{A_n, M_n} + \hat{a}_{nj} \theta_{a_{nj}, M_n} \right) + \mu_{j,A,M} \geq h_j \quad \forall j \in \{1, 2, \dots, J\} \quad (x_j) \\
& \quad \Gamma_{A_n} \rho_{A_n, M_n} - \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad (z_{A_n}) \\
& \quad \rho_{A_n, M_n} - \theta_{a_{nj}, M_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \quad (p_{a_{nj}}) \\
& \quad \Gamma_{M_n} \rho_{A_n, M_n} + \theta_{A_n, M_n} \geq 0 \quad \forall n \in B_M \quad (z_{M_n}) \\
& \quad \rho_{A_n, M_n} + \theta_{A_n, M_n} \geq 0 \quad \forall n \in B_M \quad (p_{M_n}) \\
& \quad \rho_{A_n, M_n} - (1+r_l) \rho_{A_{n+1}, M_{n+1}} \geq 0 \quad \forall n \in \{0, 1, \dots, N-1\} \quad (v_n) \\
& \quad -\rho_{A_n, M_n} + (1+r_b) \rho_{A_{n+1}, M_{n+1}} \geq 0 \quad \forall n \in \{0, 1, \dots, N-1\} \quad (w_n) \\
& \quad \rho_{A_N, M_N} \geq 1 \quad (v_N) \\
& \quad -\rho_{A_N, M_N} \geq -1 \quad (w_N) \\
& \quad \rho_{A_n, M_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\
& \quad \theta_{a_{nj}, M_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n}
\end{aligned} \tag{4.34}$$

$$\theta_{A_n, M_n} \leq 0 \quad \forall n \in \{0, 1, \dots, N\}$$

$$\mu_{j, A, M} \geq 0 \quad \forall j \in \{1, 2, \dots, J\}$$

The dual model of (4.33) will be as follows from [5]⁴¹:

$$\begin{aligned} \min \quad & \sum_{n=0}^N M_n \rho_{A_n} + \sum_{j=1}^J \mu_{j, A} \\ \text{s.t.} \quad & \sum_{n=0}^N (-a_{nj} \rho_{A_n} + \hat{a}_{nj} \theta_{a_{nj}}) + \mu_{j, A} \geq h_j \quad \forall j \in \{1, 2, \dots, J\} \quad (x_j) \\ & \rho_{A_n} - (1 + r_l) \rho_{A_{n+1}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad (v_n) \\ & -\rho_{A_n} + (1 + r_b) \rho_{A_{n+1}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad (w_n) \\ & \rho_N \geq 1 \\ & -\rho_N \geq -1 \quad (4.35) \\ & \Gamma_{A_n} \rho_{A_n} - \sum_{j \in J_{A_n}} \theta_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad (z_n) \\ & \rho_{A_n} - \theta_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \quad (p_{a_{nj}}) \\ & \rho_{A_n} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \\ & \mu_{j, A} \geq 0 \quad \forall j \in \{1, 2, \dots, J\} \\ & \theta_{a_{nj}} \geq 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \end{aligned}$$

Note that the dual model in case of budget and cash flow uncertainty in (4.34) is also an extension of the case of cash flow uncertainty in (4.3). Finally, the complementary slackness results of the primal-dual pair (4.32) and (4.34) will also be an extension of the case of cash flow uncertainty. Here, we omit to reprint the complementary slackness results in case of cash flow uncertainty in [5], and provide merely the complementary slackness results in case of budget and cash flow uncertainty as follows:

⁴¹ (4.35) is adapted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

$$\begin{aligned}
\rho_{A_0, M_0}^* (M_0 + \sum_{j=1}^J a_{0j} x_j^* - \sum_{j \in J_{A_0}} p_{a_{0j}}^* - \Gamma_{A_0} z_{A_0}^* - v_0^* + w_0^*) &= 0 \\
\rho_{A_n, M_n}^* (M_n + \sum_{j=1}^J a_{nj} x_j^* - \sum_{j \in J_{A_n}} p_{a_{nj}}^* - \Gamma_{A_n} z_{A_n}^* - p_{M_n}^* - \Gamma_{M_n} z_{M_n}^* - v_n^* + (1+r_l) v_{n-1}^* - (1+r_b) w_{n-1}^* + w_n^*) &= 0 \\
&\forall n \in B_M \\
\rho_{A_n, M_n}^* (M_n + \sum_{j=1}^J a_{nj} x_j^* - \sum_{j \in J_{A_n}} p_{a_{nj}}^* - \Gamma_{A_n} z_{A_n}^* - v_n^* + (1+r_l) v_{n-1}^* - (1+r_b) w_{n-1}^* + w_n^*) &= 0 \quad \forall n \notin B_M \\
\theta_{a_{nj}, M_n}^* (-\hat{a}_{nj} x_j^* + z_{A_n}^* + p_{a_{nj}}^*) &= 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \\
\theta_{A_n, M_n}^* (-\hat{M}_n + z_{M_n}^* + p_{M_n}^*) &= 0 \quad \forall n \in B_M \\
\mu_{j, A, M}^* (1 - x_j^*) &= 0 \quad \forall j \in \{1, 2, \dots, J\} \\
x_j^* (-h_j + \sum_{n=0}^N -a_{nj} \rho_{A_n, M_n}^* + \hat{a}_{nj} \theta_{a_{nj}, M_n}^*) + \mu_{j, A, M} &= 0 \quad \forall j \in \{1, 2, \dots, J\} \\
z_{A_n}^* (\Gamma_{A_n} \rho_{A_n, M_n}^* - \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^*) &= 0 \quad \forall n \in \{0, 1, \dots, N\} \\
p_{a_{nj}}^* (\rho_{A_n, M_n}^* - \theta_{a_{nj}, M_n}^*) &= 0 \quad \forall n \in \{0, 1, \dots, N\} \quad \forall j \in J_{A_n} \\
z_{M_n}^* (\Gamma_{M_n} \rho_{A_n, M_n}^* + \theta_{A_n, M_n}^*) &= 0 \quad \forall n \in B_M \\
p_{M_n}^* (\rho_{A_n, M_n}^* + \theta_{A_n, M_n}^*) &= 0 \quad \forall n \in B_M \\
v_n^* (\rho_{A_n, M_n}^* - (1+r_l) \rho_{A_{n+1}, M_{n+1}}^*) &= 0 \quad \forall n \in \{0, 1, \dots, N-1\} \\
w_n^* (-\rho_{A_n, M_n}^* + (1+r_b) \rho_{A_{n+1}, M_{n+1}}^*) &= 0 \quad \forall n \in \{0, 1, \dots, N-1\} \\
v_N^* (\rho_{A_N, M_N}^* - 1) &= 0 \\
w_N^* (1 - \rho_{A_N, M_N}^*) &= 0
\end{aligned} \tag{4.36}$$

4.2.1. Analysis of robust counterpart of horizon capital budgeting model

We next analyze the level of investment in projects, the characterization of the budget dual variables as forward rates, the characterization of $\hat{a}_{nj}$, $\hat{M}_n$ and v_n^* , w_n^* by considering (4.32), (4.34), (4.36) analogous to [5]. Note that the characterization of the level of investment in projects and the characterization of $\hat{a}_{nj}$ and $\hat{M}_n$ are also analogous to the analysis in Section 4.1.1.

4.2.1.1. Characterization of level of investment in projects⁴²

The analysis of the characterization of level of investment in projects is an extension of the analysis in [5].

Case 1: The project j is fully funded in case of budget and cash flow uncertainty

$$h_j + \sum_{n=0}^N a_{nj} \rho_{A_n, M_n}^* \geq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}, M_n}^* \quad (4.37)$$

follows from (4.34) and (4.36).

The project j is fully funded in case of cash flow uncertainty

$$h_j + \sum_{n=0}^N a_{nj} \rho_{A_n}^* \geq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}}^* \quad (4.38)$$

follows from [5], where $\rho_{A_n}^*$ and $\theta_{a_{nj}}^*$ are the dual variables in case of cash flow uncertainty.

The project j is fully funded in the deterministic case

$$h_j + \sum_{n=0}^N a_{nj} \rho_n^* \geq 0 \quad (4.39)$$

follows from [5], where ρ_n^* is the dual variable in the deterministic case.

Case 2: The project j is partially funded in case of budget and cash flow uncertainty

$$h_j + \sum_{n=0}^N a_{nj} \rho_{A_n, M_n}^* = \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}, M_n}^* \quad (4.40)$$

follows from (4.34) and (4.36).

The project j is partially funded in case of cash flow uncertainty

$$h_j + \sum_{n=0}^N a_{nj} \rho_{A_n}^* = \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}}^* \quad (4.41)$$

⁴² (4.38), (4.39), (4.41), (4.42), (4.44), (4.45) are reprinted from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

follows from [5].

The project j is partially funded in the deterministic case

$$h_j + \sum_{n=0}^N a_{nj} \rho_n^* = 0 \quad (4.42)$$

follows from [5].

Case 3: The project j is not funded in case of budget and cash flow uncertainty

$$h_j + \sum_{n=0}^N a_{nj} \rho_{A_n, M_n}^* \leq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}, M_n}^* \quad (4.43)$$

follows from (4.34) and (4.36).

The project j is not funded in case of cash flow uncertainty

$$h_j + \sum_{n=0}^N a_{nj} \rho_{A_n}^* \leq \sum_{n=0}^N \hat{a}_{nj} \theta_{a_{nj}}^* \quad (4.44)$$

follows from [5].

The project j is not funded in the deterministic case

$$h_j + \sum_{n=0}^N a_{nj} \rho_n^* \leq 0 \quad (4.45)$$

follows from [5].

Remark: The parameters, variables and their relationship are exactly the same as in case of cash flow uncertainty without borrowing limits.

4.2.1.2. Characterization of the budget dual variables as forward rates

The characterization of the budget dual variables as forward rates is analogous to [5].

$$(1 + r_l) \rho_{A_{n+1}, M_{n+1}}^* \leq \rho_{A_n, M_n}^* \leq (1 + r_b) \rho_{A_{n+1}, M_{n+1}}^* \quad \forall n \in \{0, 1, \dots, N-1\} \quad (4.46)$$

Then, (4.46) will be equivalent to the following:

$$(1 + r_l) \leq \frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} \leq (1 + r_b), \quad \rho_{A_{n+1}, M_{n+1}}^* \neq 0 \text{ and } \rho_{A_N, M_N}^* = 1 \quad (4.47)$$

Remark: The characterization of the budget dual variables as forwards rates is the same as in case of cash flow uncertainty and in the deterministic case as shown in [5].

4.2.1.3. Characterization of $\hat{a}_{nj}$ and $\hat{M}_n$

The characterization of $\hat{a}_{nj}$ and $\hat{M}_n$ is analogous to Section 4.1.1.2.

Case 1: Characterization of $\hat{a}_{nj}$

If $\hat{a}_{nk} > 0$ for $k \in J_{A_n}$, then $z_{A_n}^* + p_{a_{nk}}^* > 0$, and $z_{A_n}^*$ and/or $p_{a_{nk}}^*$ are strictly positive from (4.32) analogous to [5].

Case 1.1: $z_{A_n}^* > 0$, $p_{a_{nk}}^* > 0$ for $k \in J_{A_n}$

$$\text{If } z_{A_n}^* > 0, \text{ then } \Gamma_{A_n} \rho_{A_n, M_n}^* - \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* = 0 \quad (4.48)$$

from (4.36).

$$\text{If } p_{a_{nk}}^* > 0 \text{ for } k \in J_{A_n}, \text{ then } \rho_{A_n, M_n}^* - \theta_{a_{nk}, M_n}^* = 0 \quad (4.49)$$

from (4.36).

Remark 1.1.1: If $z_{A_n}^* > 0$ and $p_{a_{nk}}^* > 0$ for $k \in J_{A_n}$, then

$$\Gamma_{A_n} \theta_{a_{nk}, M_n}^* = \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* \quad (4.50)$$

Proof: Follows easily from (4.48) and (4.49).

Remark 1.1.2: Since we define the worst-case scenario as $\Gamma_{A_n} = |J_{A_n}|$, then

$$\sum_{j \in J_{A_n}} (\theta_{a_{nj}, M_n}^* - \theta_{a_{nk}, M_n}^*) = 0 \text{ for } k \in J_{A_n} \quad (4.51)$$

Proof: Since $\Gamma_{A_n} = |J_{A_n}|$, then $|J_{A_n}| \theta_{a_{nk}, M_n}^* = \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^*$ from (4.50) which implies

$$\sum_{j \in J_{A_n}} \theta_{a_{nk}, M_n}^* = \sum_{j \in J_{A_n}} \theta_{a_{nj}, M_n}^* \rightarrow \sum_{j \in J_{A_n}} (\theta_{a_{nj}, M_n}^* - \theta_{a_{nk}, M_n}^*) = 0 \quad (4.52)$$

Note that Remark 1.1.1 and Remark 1.1.2 in Section 4.2.1.3 give the same results as Remark 1.1.1 and Remark 1.1.2 in Section 4.1.1.2.

Case 1.2: $z_{A_n}^* = 0$, $p_{a_{nk}}^* > 0$ for $k \in J_{A_n}$

Case 1.3: $p_{a_{nk}}^* = 0$ for $k \in J_{A_n}$, $z_{A_n}^* > 0$

The results of Case 1.2 and Case 1.3 will also be the same as in Section 4.1.1.2.

Case 2: Characterization of $\hat{M}_n$

If $\hat{M}_n > 0$ for $n \in B_M$, then $z_{M_n}^* + p_{M_n}^* > 0$, and $z_{M_n}^*$ and/or $p_{M_n}^*$ are strictly positive from (4.32) analogous to [5].

Case 2.1: $z_{M_n}^* > 0$, $p_{M_n}^* > 0$ for $n \in B_M$

Case 2.2: $z_{M_n}^* = 0$, $p_{M_n}^* > 0$ for $n \in B_M$

Case 2.3: $p_{M_n}^* = 0$, $z_{M_n}^* > 0$ for $n \in B_M$

The results of Case 2 in Section 4.2.1.3 will be the same as in Section 4.1.1.2.

4.2.1.4. Characterization of v_n^* and w_n^*

The characterization of v_n^* and w_n^* is analogous to that in [5].

Characterization of v_n^* , w_n^* for $n \in \{0, 1, \dots, N-1\}$

Case 1: Equal lending and borrowing rates

We assume $r_l = r_b = r$.

Then, the following condition holds:

$$\frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} = (1 + r_l) = (1 + r_b) = (1 + r) \quad (4.53)$$

Proof: Follows from (4.34).

Case 2: Different lending and borrowing rates

We assume $r_l \neq r_b$.

Then, the following conditions hold:

If $v_n^* = 0$, then $\frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} = (1 + r_l)$ and $w_n^* = 0$

$$\text{or } \frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} > (1 + r_l) \text{ and } w_n^* \geq 0 \quad (4.54)$$

If $w_n^* = 0$, then $\frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} = (1 + r_b)$ and $v_n^* = 0$

$$\text{or } \frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} < (1 + r_b) \text{ and } v_n^* \geq 0 \quad (4.55)$$

$$\text{If } v_n^* > 0, \text{ then } \frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} = (1 + r_l) \text{ and } w_n^* = 0 \quad (4.56)$$

$$\text{If } w_n^* > 0, \text{ then } \frac{\rho_{A_n, M_n}^*}{\rho_{A_{n+1}, M_{n+1}}^*} = (1 + r_b) \text{ and } v_n^* = 0 \quad (4.57)$$

Proof: Follows from (4.34) and (4.36).

Characterization of v_N^* , w_N^*

$$v_N^* \geq 0, w_N^* \geq 0 \text{ since } \rho_{A_N, M_N}^* = 1$$

follows from (4.34) and (4.36).

4.2.2. Computational analysis

We consider a four-project, six-period example from [5] as in Table 4.2. All the assumptions and explanations in Section 4.1.2 hold. We also assume a lending rate $r_l = 20\%$ and a borrowing rate $r_b = 30\%$ as in [5]. We use Xpress-MP software for the computations [47], and provide the summary of the results in Table 4.3. The Xpress-MP code, an extension of the Xpress-MP code for robust optimization approach to horizon capital budgeting in [5], is available in Table B.2⁴³.

⁴³ Table 4.2 is reprinted and Table B.2 is extended from The Engineering Economist, Vol. 50, 2005, Kachani, S. and Langella, J., A robust optimization approach to capital rationing and capital budgeting, pp. 195-229, by permission of Taylor & Francis Ltd, <http://www.tandf.co.uk/journals>.

Table 4.2 : A four-project, six-period horizon capital budgeting example with different lending and borrowing rates

$r_l=20\%, r_b=30\%$	Project 1	Project 2	Project 3	Project 4	Budget
Year 0	-600	-1200	-900	-1500	270
Year 1	360	480	360	420	150
Year 2	330	360	330	480	30
Year 3	60	0	300	510	0
Year 4	-150	660	270	540	-60
Year 5	330	510	240	540	0
h_i	150	300	150	330	

Table 4.3 : Summary of the results for robust optimization approach to horizon capital budgeting model

v_a	v_M	Γ_A	Γ_M	Project 1	Project 2	Project 3	Project 4
5%	5%	0	1	1	0	0.0907	0
		1	1	0.1475	0.0738	0.0984	0
		2	1	0.1096	0.0549	0.0732	0.0439
		3	1	0.4286	0	0	0
5%	10%	0	1	1	0	0.0815	0
		1	1	0.1475	0.0738	0.0984	0
		2	1	0.1098	0.0549	0.0732	0.0439
		3	1	0.4286	0	0	0
5%	15%	0	1	1	0	0.0722	0
		1	1	0.1475	0.0738	0.0984	0
		2	1	0.1098	0.0549	0.0732	0.0439
		3	1	0.4286	0	0	0
5%	20%	0	1	1	0	0.0629	0
		1	1	0.1475	0.0738	0.0984	0
		2	1	0.1098	0.0549	0.0732	0.0439
		3	1	0.4286	0	0	0
10%	5%	0	1	1	0	0.0907	0
		1	1	0.1098	0.0549	0.0732	0.0439
		2	1	0	0	0	0
		3	1	0	0	0	0
10%	10%	0	1	1	0	0.0815	0
		1	1	0.1098	0.0549	0.0732	0.0439
		2	1	0	0	0	0
		3	1	0	0	0	0
10%	15%	0	1	1	0	0.0722	0
		1	1	0.1098	0.0549	0.0732	0.0439
		2	1	0	0	0	0
		3	1	0	0	0	0
10%	20%	0	1	1	0	0.0629	0
		1	1	0.1098	0.0549	0.0732	0.0439
		2	1	0	0	0	0
		3	1	0	0	0	0
15%	5%	0	1	1	0	0.0907	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0

Table 4.3 : Summary of the results for robust optimization approach to horizon capital budgeting model (continued)

v_a	v_M	Γ_A	Γ_M	Project 1	Project 2	Project 3	Project 4
15%	10%	0	1	1	0	0.0815	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0
15%	15%	0	1	1	0	0.0722	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0
15%	20%	0	1	1	0	0.0629	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0
20%	5%	0	1	1	0	0.0907	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0
20%	10%	0	1	1	0	0.0815	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0
20%	15%	0	1	1	0	0.0722	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0
20%	20%	0	1	1	0	0.0629	0
		1	1	0	0	0	0
		2	1	0	0	0	0
		3	1	0	0	0	0

Based on Table 4.3, no project is a fully funded project for any combinations. For any variability rate combination when $\Gamma_A = 0$ and $\Gamma_M = 1$, Project 1 is fully funded, Project 2 and Project 4 are not funded, and Project 3 is partially funded but with very small proportion. For any variability rate combination except for $v_a = 5\%$, when $\Gamma_A = 3$ and $\Gamma_M = 1$, no project is funded. Note that some sample graphs are indicated in Figure 4.3, Figure 4.4, Figure 4.5, and Figure 4.6, while all the graphs are available in Figure B.3, Figure B.4, Figure B.5, and Figure B.6. For Project 1, all the graphs have a common characteristic except for $v_a = 5\%$ as in Figure 4.3. Varying v_M value for a specific v_a value does not cause any considerable change in the graphs. This may be since the budget variability rate has a minor impact on the outcomes. Increasing v_a value for a constant v_M value tightens the width of Γ_A values for which Project 1 is fully funded. A monotonically sharp decreasing level of

investment phase after a fully-funding phase is the common case for all graphs. A monotonically increasing level of investment phase when $\Gamma_A = [2.5, 3]$ is the only case for $v_a = 5\%$. Project 1 is not funded for $v_a = 10\%$ when $\Gamma_A = [1.5, 3]$, for $v_a = 15\%$ when $\Gamma_A = [0.9, 3]$, and for $v_a = 20\%$ when $\Gamma_A = [0.7, 3]$. The graphs of Project 1 in case of cash flow uncertainty as in [5] share a similar characteristic with the graphs of Project 1 in case of budget and cash flow uncertainty.

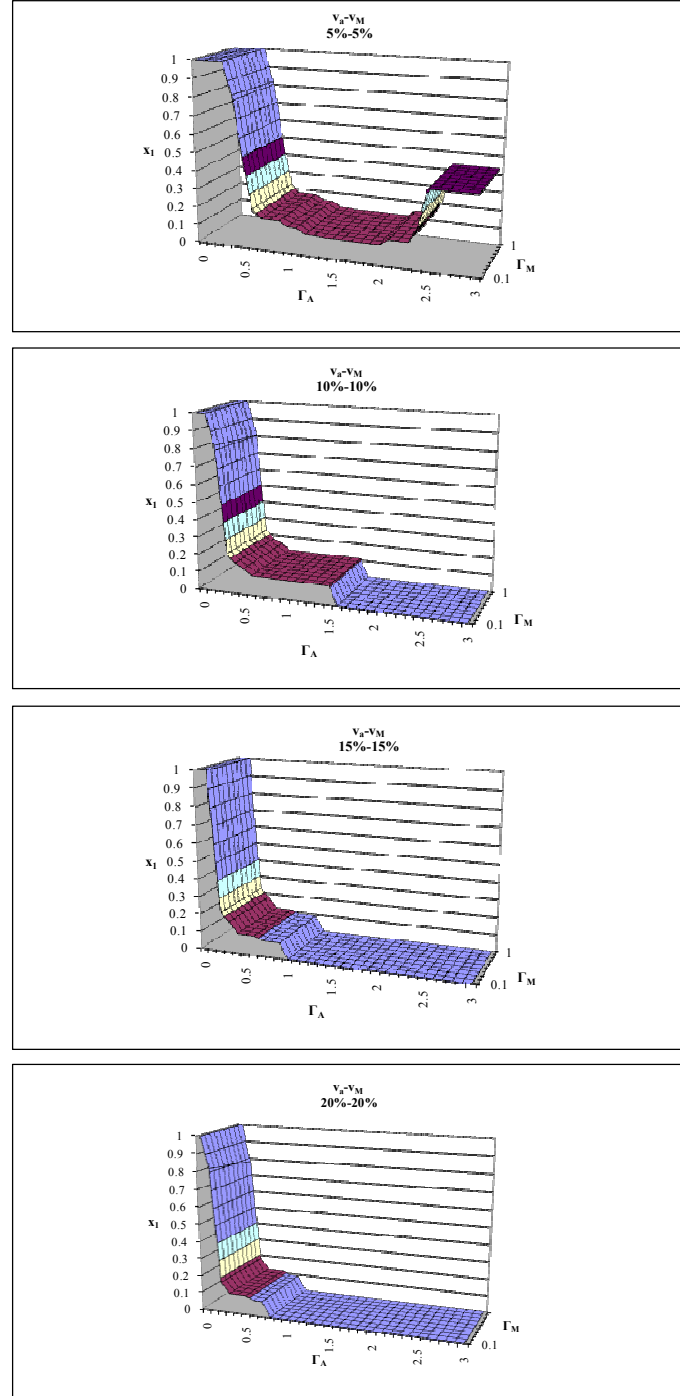


Figure 4.3 : Level of investment in Project 1

For Project 2, all the graphs have a similar characteristic with an initially no-funding phase, a monotonically increasing level of investment phase, a monotonically decreasing level of investment phase, an approximately constant level of investment phase, a monotonically decreasing level of investment phase, and a finally no-funding phase as indicated in Figure 4.4. The phases between monotonically increasing level of investment phase and monotonically decreasing level of investment phase differ according to v_a value. Varying v_M value for a specific v_a value does not cause any considerable change in the graphs. The graphs for $v_a = 15\%$ and $v_a = 20\%$ have almost the same characteristic. The width of Γ_A values for which Project 2 is not funded after the monotonically decreasing level of investment phase enlarges as v_a value increases. The maximum level of investment in Project 2 is much lower than that in Project 1 for any specific v_a, v_M combination. The graphs of Project 2 in case of cash flow uncertainty as in [5] share a similar characteristic with the graphs of Project 2 in case of budget and cash flow uncertainty.

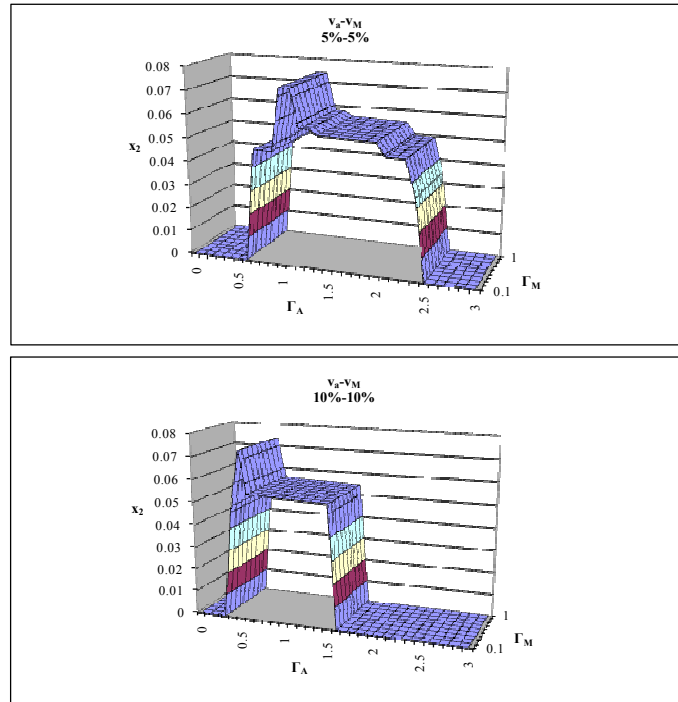


Figure 4.4 : Level of investment in Project 2

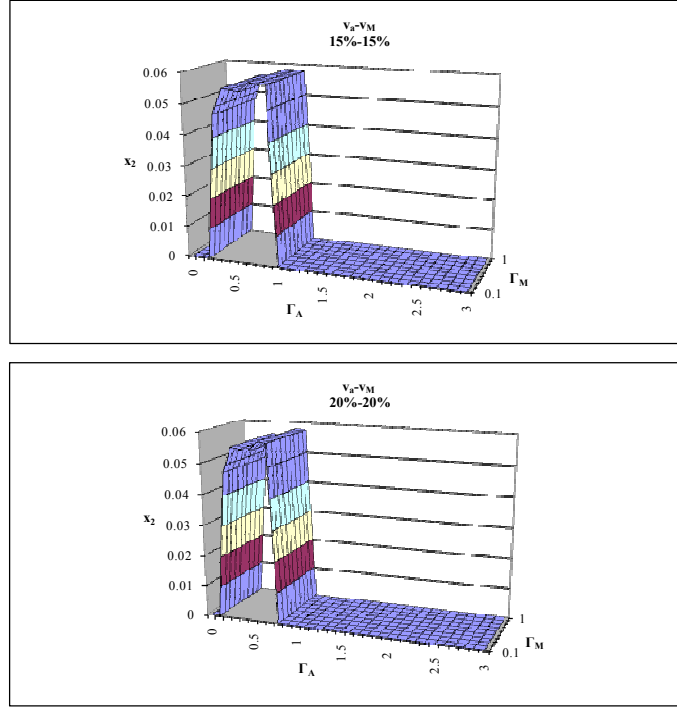


Figure 4.4 : Level of investment in Project 2 (continued)

Project 3 is characterized by a monotonically increasing level of investment phase, an immediate monotonically decreasing level of investment phase, an approximately constant level of investment phase, a monotonically decreasing level of investment phase, and a no-funding phase for all graphs as indicated in Figure 4.5. Varying v_M value for a specific v_a value does not cause any considerable change in the graphs. The graphs for $v_a = 10\%$, 15% , 20% have a similar characteristic, but the width of Γ_A values for the approximately constant level of investment phase and the no-funding phase varies according to v_a level. The maximum level of investment in Project 3 is much lower than that in Project 1, but higher than that in Project 2 for any specific v_a, v_M combination. The graphs of Project 3 in case of cash flow uncertainty as in [5] share a similar characteristic with the graphs of Project 3 in case of budget and cash flow uncertainty.

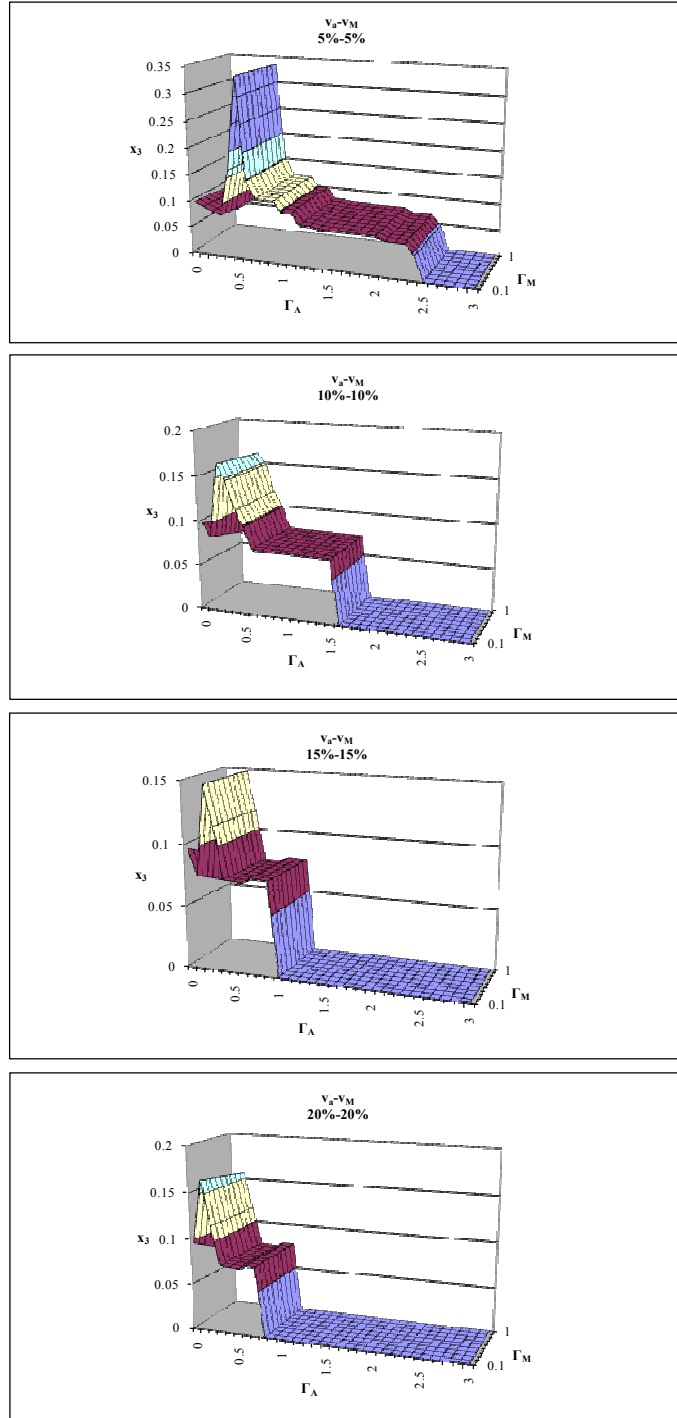


Figure 4.5 : Level of investment in Project 3

Project 4 is characterized by a no-funding phase, a monotonically increasing level of investment phase, an approximately constant level of investment phase, a monotonically decreasing level of investment phase, and a no-funding phase as indicated in Figure 4.6. Varying v_M value for a specific v_a value does not cause any considerable change in the graphs. All the graphs have a similar characteristic except

for the width of Γ_A values for the approximately constant level of investment phase and the final no-funding phase. The width of Γ_A values for the approximately constant level of investment phase tightens as v_a value increases. The graphs of Project 4 in case of cash flow uncertainty as in [5] share a similar characteristic with the graphs of Project 4 in case of budget and cash flow uncertainty.

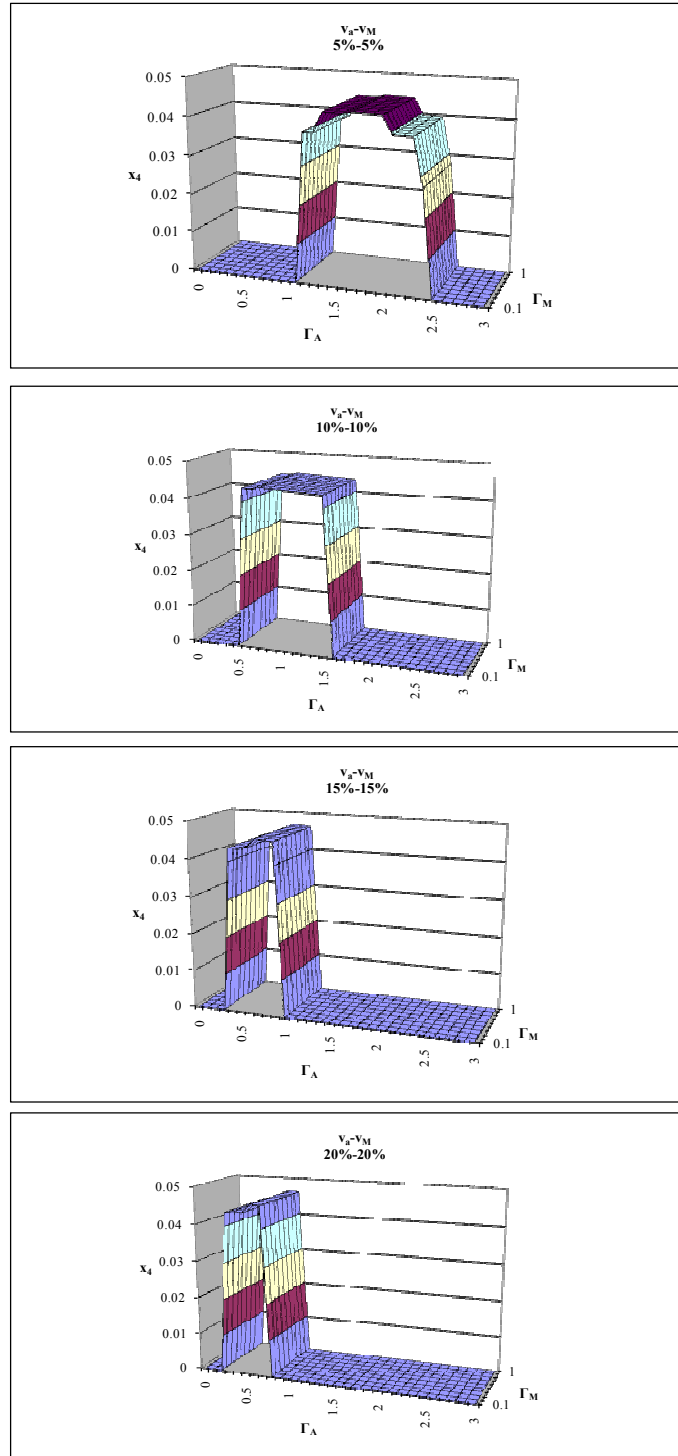


Figure 4.6 : Level of investment in Project 4

4.3. Robust Decision Rules of Investment Appraisal Techniques for Simple Projects

In this section, we define so-called robust decision rules of NPV and IRR for simple projects by assuming the uncertain cash flows to belong to a given convex set, unknown-but-bounded uncertainty set in this thesis, by borrowing from robust optimization methodology presented in [11,12]. In these decision rules, we propose different levels of acceptability degrees for a project. Then, we compute the probability of over-conservatism and over-optimism for NPV decision rules analogous to [12]. Note that we define over-conservatism as the decision of rejecting a project or staying indifferent to a project, although it could have been accepted, and over-optimism as the decision of accepting a project, although it could have been not accepted if the actual realization of the cash flows were known. As an illustration, we perform computational analysis for NPV and IRR for simple projects.

4.3.1. Preliminaries

An NPV equation is as follows:

$$NPV = \sum_{n=0}^N \frac{F_n}{(1+r)^n} \quad (4.58)$$

where $F_n = b_n - c_n$ is the net cash flow at period n , b_n is the cash inflow at period n , c_n is the cash outflow at period n , and r is the discount rate. If $NPV > 0$, we accept the project; if $NPV < 0$, we reject the project; and if $NPV = 0$, we stay indifferent [2, p: 202-204].

We consider the definition of the netted B/C Ratio in [2, p: 228]:

$$\text{Netted B/C Ratio} = \frac{B - C'}{I} \quad (4.59)$$

where B is the annual benefits, $I = \sum_{n=0}^m c_n (1+r)^{-n}$ is the initial investment, and

$C' = \sum_{n=m+1}^N c_n (1+r)^{-n}$ is the annual costs. If $\frac{B - C'}{I} > 1$, we accept the project [2, p:

226-228]. The definition and the decision rules of IRR are already available in Section 3.3.

4.3.2. Robust decision rules

We redefine the decision rules for NPV and IRR by borrowing from the robust optimization methodology presented in Section 2.2.

4.3.2.1. NPV robust decision rules

We outline the NPV robust decision rules in Table 4.4:

Table 4.4 : NPV robust decision rules

calculate	$\min \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n}$ and $\max \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n}$
if	$\min \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n} > 0$, then <i>clearly accept (CA)</i> the project, and stop.
if	$\max \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n} < 0$, then <i>clearly reject (CR)</i> the project, and stop.
if	$\min \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n} < 0$ and $\max \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n} > 0$, then for ε as a predetermined small positive number, and M as a predetermined big positive number,
if	$\frac{\left \min \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n} \right }{\max \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n}} \leq \varepsilon$, then <i>accept (A)</i> the project for ε – level, and stop.
if	$\frac{\left \min \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n} \right }{\max \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n}} \geq M$, then <i>reject (R)</i> the project for M – level, and stop.
if	$\varepsilon < \frac{\left \min \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n} \right }{\max \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n}} < M$, then
calculate	$\sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C)$ and $\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C)$
if	$\sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) > 0$, then <i>clearly accept (CA)</i> the project for (Γ_B, Γ_C) – level, and stop.
if	$\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C) < 0$, then <i>clearly reject (CR)</i> the project for (Γ_B, Γ_C) – level, and stop.

Table 4.4 : NPV robust decision rules (continued)

if $\sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) < 0$ and $\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C) > 0$, then for ε_Γ as a predetermined small positive number, and M_Γ as a predetermined big positive number,	
if	$\frac{\left \sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) \right }{\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C)} \leq \varepsilon_\Gamma$, then <i>accept (A)</i> the project for $(\Gamma_B, \Gamma_C, \varepsilon_\Gamma)$ – level, and stop.
if	$\frac{\left \sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) \right }{\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C)} \geq M_\Gamma$, then <i>reject (R)</i> the project for $(\Gamma_B, \Gamma_C, M_\Gamma)$ – level, and stop.
otherwise, stay indifferent.	

Note that by borrowing from the robust optimization methodology in [12] as presented in Section 2.2, we define $\beta(\Gamma_B)$ and $\beta(\Gamma_C)$ as the functions to adjust the uncertainty level of cash inflows, and cash outflows, respectively:

$$\beta(\Gamma_B) = \left\{ \begin{array}{l} \max_{n \in S_B \cup S_{BC}} \sum \frac{\hat{b}_n}{(1+r)^n} + (\Gamma_B - \lfloor \Gamma_B \rfloor) \frac{\hat{b}_{n_{b_n}}}{(1+r)^n} \mid (S_B \cup S_{BC}) \subseteq (J_B \cup J_{BC}), \\ \Gamma_B = [0, |J_B \cup J_{BC}|], \mid S_B \cup S_{BC} \mid = \lfloor \Gamma_B \rfloor, \\ n_{b_n} \in (J_B \cup J_{BC}) \setminus (S_B \cup S_{BC}), J_B = \{n \mid \tilde{b}_n \in [b_n - \hat{b}_n, b_n + \hat{b}_n]\}, \\ J_{BC} = \{n \mid \tilde{b}_n \in [b_n - \hat{b}_n, b_n + \hat{b}_n], \tilde{c}_n \in [c_n - \hat{c}_n, c_n + \hat{c}_n]\} \end{array} \right\} \quad (4.60)$$

$$\beta(\Gamma_C) = \left\{ \begin{array}{l} \max_{n \in S_C \cup S_{BC}} \sum \frac{\hat{c}_n}{(1+r)^n} + (\Gamma_C - \lfloor \Gamma_C \rfloor) \frac{\hat{c}_{n_{c_n}}}{(1+r)^n} \mid (S_C \cup S_{BC}) \subseteq (J_C \cup J_{BC}) \\ \Gamma_C = [0, |J_C \cup J_{BC}|], \mid S_C \cup S_{BC} \mid = \lfloor \Gamma_C \rfloor, \\ n_{c_n} \in (J_C \cup J_{BC}) \setminus (S_C \cup S_{BC}), J_C = \{n \mid \tilde{c}_n \in [c_n - \hat{c}_n, c_n + \hat{c}_n]\}, \\ J_{BC} = \{n \mid \tilde{b}_n \in [b_n - \hat{b}_n, b_n + \hat{b}_n], \tilde{c}_n \in [c_n - \hat{c}_n, c_n + \hat{c}_n]\} \end{array} \right\} \quad (4.61)$$

Remark 1: We assume that $\Gamma_B \neq 0$, $\Gamma_C \neq 0$.

Remark 2: We define the worst-case scenario as $\Gamma_B = |J_B \cup J_{BC}|$ and $\Gamma_C = |J_C \cup J_{BC}|$.

Remark 3: The decision rules for the netted B/C Ratio method follow easily, since it is easy to verify that the decision rules for the netted B/C Ratio are equivalent to the decision rules for NPV [2, p: 228-229].

Numerical Analysis:

We set $\varepsilon = \varepsilon_\Gamma = 0.01$ and $M = M_\Gamma = 100$, consider the data set in Table 3.1, and

obtain $\min \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n}$ and $\max \sum_{n=0}^N \frac{\tilde{F}_n}{(1+r)^n}$ values for different cash flow variability rates as in Table 4.5:

Table 4.5 : Min and max NPV values

NPV(min) r = 20%	Project 1	Project 2	Project 3
5%	4.7014	2.0069	3.7611
10%	2.875	1.375	2.3
15%	1.0486	0.7431	0.8389
20%	-0.7778	0.1111	-0.6222

NPV(max) r = 20%	Project 1	Project 2	Project 3
5%	8.3542	3.2708	6.6833
10%	10.1806	3.9028	8.1444
15%	12.0069	4.5347	9.6056
20%	13.8333	5.1667	11.0667

Remarks: Project 2 is *clearly accepted (CA)* for all variability rates. Project 1 and Project 3 are *clearly accepted (CA)* for all variability rates except for 20% variability rate. The graphs of Project 1 and Project 3 with staying indifferent areas for (Γ_B, Γ_C) -levels at 20% variability rate are provided in Figure 4.7 and Figure 4.8, respectively. For the sake of clarity of the graphs, *CA* and *A* areas for ε -level are not shown in the graphs. *A* areas of Project 1 are for $(\Gamma_B, \Gamma_C) = (1.5, 0.9)$, $(\Gamma_B, \Gamma_C) = (1.1, 1)$, $(\Gamma_B, \Gamma_C) = (1.2, 1)$, and *A* areas of Project 3 are for $(\Gamma_B, \Gamma_C) = (1.5, 0.9)$, $(\Gamma_B, \Gamma_C) = (1.1, 1)$ for the given ε -level. The remaining areas are *CA* areas for both projects. Project 1 and Project 3 have a similar characteristic such that they are *clearly accepted (CA)* or *accepted (A)* except for $(\Gamma_B, \Gamma_C) = ([1.6, 2], 0.9)$ and $(\Gamma_B, \Gamma_C) = ([1.3, 2], 1)$. For both projects, min/max NPV

ratios increase with a constant slope as we increase Γ_B – levels for a constant Γ_C – level, which can be interpreted as the acceptability degree of a project decreases linearly as the uncertainty level of the cash inflows increases.

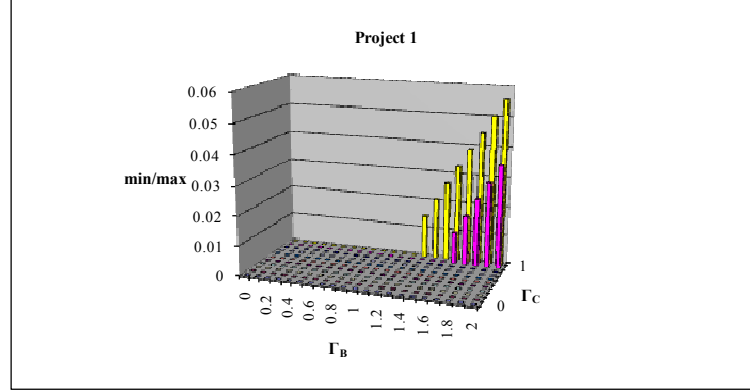


Figure 4.7 : Min/max NPV values of staying indifferent areas at 20% variability rate for Project 1

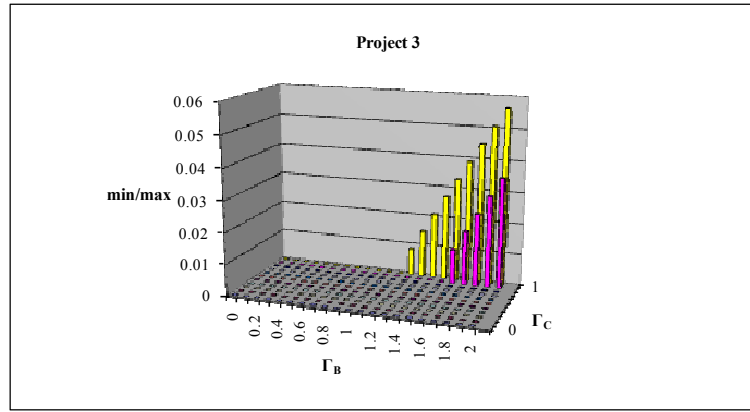


Figure 4.8 : Min/max NPV values of staying indifferent areas at 20% variability rate for Project 3

We next calculate the probability of over-conservatism, and over-optimism by borrowing the idea from [12]. Recall that we define over-conservatism as the decision of rejecting a project or staying indifferent to a project, although it could have been accepted, and over-optimism as the decision of accepting a project, although it could have been rejected or staying indifferent would be the decision, if the actual realization of the cash flows were known. First, we calculate the probability of over-conservatism based on the proposed decision rules in Table 4.4, equations (4.60) and (4.61), and also by introducing the symmetrically distributed random variables for the cash inflows and the cash outflows as in [12]. Afterwards, the calculation of the probability of over-optimism will be analogous.

Probability of over-conservatism

1st part of proof:

$$\Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\tilde{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\tilde{c}_n}{(1+r)^n} + \sum_{n \in J_C} \frac{b_n}{(1+r)^n} - \sum_{n \in J_B} \frac{c_n}{(1+r)^n} + \sum_{n \notin J} \frac{F_n}{(1+r)^n} \geq 0 \right\} \quad (4.62)$$

where $J = J_B \cup J_C \cup J_{BC}$.

$$= \Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{b_n + \eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{c_n + \eta_{c_n} \hat{c}_n}{(1+r)^n} + \sum_{n \in J_C} \frac{b_n}{(1+r)^n} - \sum_{n \in J_B} \frac{c_n}{(1+r)^n} + \sum_{n \notin J} \frac{F_n}{(1+r)^n} \geq 0 \right\} \quad (4.63)$$

$$= \Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} + \sum_{n=0}^N \frac{F_n}{(1+r)^n} \geq 0 \right\} \quad (4.64)$$

$$= \Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} \geq - \sum_{n=0}^N \frac{F_n}{(1+r)^n} \right\} \quad (4.65)$$

Note that (4.63) follows, since we define $\eta_{b_n} = \frac{\tilde{b}_n - b_n}{\hat{b}_n}$ and $\eta_{c_n} = \frac{\tilde{c}_n - c_n}{\hat{c}_n}$ as

symmetrically distributed random variables with unknown distribution such that $\eta_{b_n} \in [-1, 1]$ and $\eta_{c_n} \in [-1, 1]$ as in Section 2.2.

2nd part of proof:

The following inequality is a given condition from Table 4.4, since we assume that we are in an area of staying indifferent to a project or rejecting it:

$$\frac{\left| \sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) \right|}{\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C)} > \varepsilon_\Gamma \quad (4.66)$$

Since $\sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) < 0$ from Table 4.4, (4.66) will be equivalent to

the following:

$$\varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C) \right) - \beta(\Gamma_B) - \beta(\Gamma_C) < - \sum_{n=0}^N \frac{F_n}{(1+r)^n} \quad (4.67)$$

From (4.65) and (4.67), we obtain the following:

$$\Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} \geq - \sum_{n=0}^N \frac{F_n}{(1+r)^n} \right\} \quad (4.68)$$

$$\leq \Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} > \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C) \right) - \beta(\Gamma_B) - \beta(\Gamma_C) \right\}$$

$$= \Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} > \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1)\beta(\Gamma_B) + (\varepsilon_\Gamma - 1)\beta(\Gamma_C) \right\} \quad (4.69)$$

where $\beta(\Gamma_B)$ and $\beta(\Gamma_C)$ were already defined in (4.60) and (4.61). Then, (4.69) will be equivalent to the following:

$$\Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} > \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1) \max \left(\sum_{n \in S_B \cup S_{BC}} \frac{\hat{b}_n}{(1+r)^n} + (\Gamma_B - \lfloor \Gamma_B \rfloor) \frac{\hat{b}_{n_{b_n}}}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1) \max \left(\sum_{n \in S_C \cup S_{BC}} \frac{\hat{c}_n}{(1+r)^n} + (\Gamma_C - \lfloor \Gamma_C \rfloor) \frac{\hat{c}_{n_{c_n}}}{(1+r)^n} \right) \right\} \quad (4.70)$$

$$= \Pr \left\{ \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1) \left(\sum_{n \in S_B \cup S_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} + \sum_{n \in (J_B \cup J_{BC}) \setminus (S_B \cup S_{BC})} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in S_C \cup S_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} - \sum_{n \in (J_C \cup J_{BC}) \setminus (S_C \cup S_{BC})} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} \right) > \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1) \left(\sum_{n \in S_B \cup S_{BC}} \frac{\hat{b}_n}{(1+r)^n} + (\Gamma_B - \lfloor \Gamma_B \rfloor) \frac{\hat{b}_{n_{b_n}}}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1) \left(\sum_{n \in S_C \cup S_{BC}} \frac{\hat{c}_n}{(1+r)^n} + (\Gamma_C - \lfloor \Gamma_C \rfloor) \frac{\hat{c}_{n_{c_n}}}{(1+r)^n} \right) \right\} \quad (4.71)$$

$$= \Pr \left\{ \sum_{n \in S_B \cup S_{BC}} \frac{(\eta_{b_n} - (\varepsilon_\Gamma - 1))\hat{b}_n}{(1+r)^n} + \sum_{n \in (J_B \cup J_{BC}) \setminus (S_B \cup S_{BC})} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - (\varepsilon_\Gamma - 1)(\Gamma_B - \lfloor \Gamma_B \rfloor) \frac{\hat{b}_{n_{b_n}}}{(1+r)^n} + \sum_{n \in S_C \cup S_{BC}} \frac{-(\eta_{c_n} + (\varepsilon_\Gamma - 1))\hat{c}_n}{(1+r)^n} - \sum_{n \in (J_C \cup J_{BC}) \setminus (S_C \cup S_{BC})} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} - (\varepsilon_\Gamma - 1)(\Gamma_C - \lfloor \Gamma_C \rfloor) \frac{\hat{c}_{n_{c_n}}}{(1+r)^n} > \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) \right\} \quad (4.72)$$

$$= \Pr \left\{ \begin{aligned} & \sum_{n \in J_B \cup J_{BC}} \frac{\mu_{b_n} \hat{b}_n}{(1+r)^n} - (\varepsilon_\Gamma - 1)(\Gamma_B - \lfloor \Gamma_B \rfloor) \frac{\hat{b}_{n_{b_n}}}{(1+r)^n} + \sum_{n \in J_C \cup J_{BC}} \frac{\mu_{c_n} \hat{c}_n}{(1+r)^n} - (\varepsilon_\Gamma - 1)(\Gamma_C - \lfloor \Gamma_C \rfloor) \frac{\hat{c}_{n_{c_n}}}{(1+r)^n} \\ & > \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) \end{aligned} \right\} \quad (4.73)$$

$$= \Pr \left\{ \begin{aligned} & \sum_{n \in J_B \cup J_{BC}} \frac{\mu_{b_n} \hat{b}_n}{(1+r)^n} + \sum_{n \in J_C \cup J_{BC}} \frac{\mu_{c_n} \hat{c}_n}{(1+r)^n} > \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1)(\Gamma_B - \lfloor \Gamma_B \rfloor) \frac{\hat{b}_{n_{b_n}}}{(1+r)^n} \\ & + (\varepsilon_\Gamma - 1)(\Gamma_C - \lfloor \Gamma_C \rfloor) \frac{\hat{c}_{n_{c_n}}}{(1+r)^n} \end{aligned} \right\} \quad (4.74)$$

(4.73) follows, since we define the following random variables by borrowing the idea from [12]:

$$\mu_{b_n} = \begin{cases} (\eta_{b_n} - (\varepsilon_\Gamma - 1)) & \text{if } n \in S_B \cup S_{BC} \\ \eta_{b_n} & \text{if } n \in (J_B \cup J_{BC}) \setminus (S_B \cup S_{BC}) \end{cases} \quad (4.75)$$

$$\mu_{c_n} = \begin{cases} -(\eta_{c_n} + (\varepsilon_\Gamma - 1)) & \text{if } n \in S_C \cup S_{BC} \\ -\eta_{c_n} & \text{if } n \in (J_C \cup J_{BC}) \setminus (S_C \cup S_{BC}) \end{cases} \quad (4.76)$$

□

Probability of over-optimism

The probability of over-optimism will be calculated analogous to the calculation of probability of over-conservatism.

1st part of proof:

$$\Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\tilde{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\tilde{c}_n}{(1+r)^n} + \sum_{n \in J_C} \frac{b_n}{(1+r)^n} - \sum_{n \in J_B} \frac{c_n}{(1+r)^n} + \sum_{n \notin J} \frac{F_n}{(1+r)^n} \leq 0 \right\} \quad (4.77)$$

$$= \Pr \left\{ \sum_{n \in J_B \cup J_{BC}} \frac{\eta_{b_n} \hat{b}_n}{(1+r)^n} - \sum_{n \in J_C \cup J_{BC}} \frac{\eta_{c_n} \hat{c}_n}{(1+r)^n} \leq - \sum_{n=0}^N \frac{F_n}{(1+r)^n} \right\} \quad (4.78)$$

Note that equations (4.62) through (4.64) are repeated for computing the probability of over-optimism with a slight difference that $\geq$ is replaced by $\leq$.

2nd part of proof:

The following inequality from Table 4.4 is a given condition, since we assume that we are in an area of accepting a project:

$$\frac{\left| \sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) \right|}{\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C)} \leq \varepsilon_\Gamma \quad (4.79)$$

When we repeat the equations (4.67) through (4.76) with a slight difference that $\geq$ is replaced by $\leq$, then we obtain the probability of over-optimism as follows:

$$\leq \Pr \left\{ \begin{aligned} & \sum_{n \in J_B \cup J_{BC}} \frac{\mu_{b_n} \hat{b}_n}{(1+r)^n} + \sum_{n \in J_C \cup J_{BC}} \frac{\mu_{c_n} \hat{c}_n}{(1+r)^n} \leq \varepsilon_\Gamma \left(\sum_{n=0}^N \frac{F_n}{(1+r)^n} \right) + (\varepsilon_\Gamma - 1)(\Gamma_B - \lfloor \Gamma_B \rfloor) \frac{\hat{b}_{n_{b_n}}}{(1+r)^n} \\ & + (\varepsilon_\Gamma - 1)(\Gamma_C - \lfloor \Gamma_C \rfloor) \frac{\hat{c}_{n_{c_n}}}{(1+r)^n} \end{aligned} \right\} \quad (4.80)$$

□

4.3.2.2. IRR robust decision rules for simple projects

We outline the IRR robust decision rules for simple projects in Table 4.6:

Table 4.6 : IRR robust decision rules

calculate $r_{\min}^*$ for $\sum_{n=0}^N \frac{\min(\tilde{F}_n)}{(1+r)^n} = 0$ and $r_{\max}^*$ for $\sum_{n=0}^N \frac{\max(\tilde{F}_n)}{(1+r)^n} = 0$ if $r_{\min}^* > MARR$, then <i>clearly accept (CA)</i> the project, and stop. if $r_{\max}^* < MARR$, then <i>clearly reject (CR)</i> the project, and stop. if $r_{\min}^* < MARR$ and $r_{\max}^* > MARR$, then for ε as a predetermined small positive number, and M as a predetermined big positive number, if $\frac{ r_{\min}^* - MARR }{r_{\max}^* - MARR} \leq \varepsilon$, then <i>accept (A)</i> the project for ε-level, and stop. if $\frac{ r_{\min}^* - MARR }{r_{\max}^* - MARR} \geq M$, then <i>reject (R)</i> the project for M-level, and stop. if $\varepsilon < \frac{ r_{\min}^* - MARR }{r_{\max}^* - MARR} < M$, then calculate $r_{\max, \Gamma_B, \Gamma_C}^*$ for $\sum_{n=0}^N \frac{F_n}{(1+r)^n} - \beta(\Gamma_B) - \beta(\Gamma_C) = 0$, and $r_{\min, \Gamma_B, \Gamma_C}^*$ for $\sum_{n=0}^N \frac{F_n}{(1+r)^n} + \beta(\Gamma_B) + \beta(\Gamma_C) = 0$,

Table 4.6 : IRR robust decision rules (continued)

if $r_{\min, \Gamma_B, \Gamma_C}^* > MARR$, then *clearly accept (CA)* the project for (Γ_B, Γ_C) – level, and stop.
if $r_{\max, \Gamma_B, \Gamma_C}^* < MARR$, then *clearly reject (CR)* the project for (Γ_B, Γ_C) – level, and stop.
if $r_{\min, \Gamma_B, \Gamma_C}^* < MARR$ and $r_{\max, \Gamma_B, \Gamma_C}^* > MARR$, then for ε_Γ as a predetermined small positive number, and M_Γ as a predetermined big positive number,
if $\frac{|r_{\min, \Gamma_B, \Gamma_C}^* - MARR|}{r_{\max, \Gamma_B, \Gamma_C}^* - MARR} \leq \varepsilon_\Gamma$, then *accept (A)* the project for $(\Gamma_B, \Gamma_C, \varepsilon_\Gamma)$ – level, and stop.
if $\frac{|r_{\min, \Gamma_B, \Gamma_C}^* - MARR|}{r_{\max, \Gamma_B, \Gamma_C}^* - MARR} \geq M_\Gamma$, then *reject (R)* the project for $(\Gamma_B, \Gamma_C, M_\Gamma)$ – level, and stop.
otherwise, stay indifferent.

Note that $\beta(\Gamma_B)$ and $\beta(\Gamma_C)$ are defined as in NPV method, however, in IRR method, we propose to apply parametric programming to obtain $\beta(\Gamma_B)$ and $\beta(\Gamma_C)$ as a function of r . We now consider the following auxiliary problems as the equivalent problems of $\beta(\Gamma_B)$ and $\beta(\Gamma_C)$ analogous to (2.70):

$$\begin{aligned}
 \max \quad & \sum_{n \in J_B \cup J_{BC}} \frac{\hat{b}_n}{(1+r)^n} z_{b_n} \\
 \text{s.t.} \quad & \sum_{n \in J_B \cup J_{BC}} z_{b_n} \leq \Gamma_B \quad [z_B] \\
 & 0 \leq z_{b_n} \leq 1 \quad \forall n \in J_B \cup J_{BC} \quad [p_{b_n}]
 \end{aligned} \tag{4.81}$$

$$\begin{aligned}
 \max \quad & \sum_{n \in J_C \cup J_{BC}} \frac{\hat{c}_n}{(1+r)^n} z_{c_n} \\
 \text{s.t.} \quad & \sum_{n \in J_C \cup J_{BC}} z_{c_n} \leq \Gamma_C \quad [z_C] \\
 & 0 \leq z_{c_n} \leq 1 \quad \forall n \in J_C \cup J_{BC} \quad [p_{c_n}]
 \end{aligned} \tag{4.82}$$

where z_B and p_{b_n} are the respective dual variables for (4.81), and z_C and p_{c_n} are the respective dual variables for (4.82).

By strong duality, (4.81) and (4.82) are equivalent to the following models as in (2.71):

$$\begin{aligned}
\min \quad & \Gamma_B z_B + \sum_{n \in J_B \cup J_{BC}} p_{b_n} \\
\text{s.t.} \quad & z_B + p_{b_n} \geq \frac{\hat{b}_n}{(1+r)^n} \quad \forall n \in J_B \cup J_{BC} \\
& z_B \geq 0 \\
& p_{b_n} \geq 0 \quad \forall n \in J_B \cup J_{BC}
\end{aligned} \tag{4.83}$$

and

$$\begin{aligned}
\min \quad & \Gamma_C z_C + \sum_{n \in J_C \cup J_{BC}} p_{c_n} \\
\text{s.t.} \quad & z_C + p_{c_n} \geq \frac{\hat{c}_n}{(1+r)^n} \quad \forall n \in J_C \cup J_{BC} \\
& z_C \geq 0 \\
& p_{c_n} \geq 0 \quad \forall n \in J_C \cup J_{BC}
\end{aligned} \tag{4.84}$$

By parametric programming methodology, we initially solve (4.83) and (4.84) for some $r = r_0$ analogous to [53, p: 107-108]. We propose to solve initially for $r = 0$.

Let B_{Γ_B} and B_{Γ_C} be the optimal bases, and z_{Γ_B} and z_{Γ_C} be the optimal costs for the problems (4.83) and (4.84) when $r = 0$. Then, (4.83) and (4.84) remain always feasible, since $B_{\Gamma_B}^{-1}(\hat{b}_r) \geq 0$ and $B_{\Gamma_C}^{-1}(\hat{c}_r) \geq 0$ always hold for the vector $\hat{b}_r$ with the

entries $\hat{b}_m = \frac{\hat{b}_n}{(1+r)^n}$ for $\forall n \in J_B \cup J_{BC}$, and for the vector $\hat{c}_r$ with the entries

$\hat{c}_m = \frac{\hat{c}_n}{(1+r)^n}$ for $\forall n \in J_C \cup J_{BC}$, where $r > -1$, since $B_{\Gamma_B}^{-1}\hat{b} \geq 0$ and $B_{\Gamma_C}^{-1}\hat{c} \geq 0$ hold

for the problem with $r = 0$, where $\hat{b}$ is a vector with the entries $\hat{b}_n$ for $\forall n \in J_B \cup J_{BC}$, and $\hat{c}$ is a vector with the entries $\hat{c}_n$ for $\forall n \in J_C \cup J_{BC}$. Finally, the optimal solutions for (4.83) and (4.84) will be $z_{\Gamma_B}(r) = c_{B_{\Gamma_B}} B_{\Gamma_B}^{-1}(\hat{b}_r)$ and $z_{\Gamma_C}(r) = c_{B_{\Gamma_C}} B_{\Gamma_C}^{-1}(\hat{c}_r)$, where $c_{B_{\Gamma_B}}$ and $c_{B_{\Gamma_C}}$ are the optimal cost vectors for the problems (4.83) and (4.84) when $r = 0$.

Numerical Analysis:

We consider the example in [2, p: 211] as in Section 3.3.4. We assume $\varepsilon = \varepsilon_{\Gamma} = 0.1$ and $M = M_{\Gamma} = 10$. Min/max IRR values are provided in Table 4.7.

Table 4.7 : Min/max IRR values

A simple project MARR = 15%	min	max
5%	12.5%	28%
10%	6%	36%
15%	0%	45%
20%	0%	57%

Based on the proposed decision rules in Table 4.6, the project is not clearly accepted or clearly rejected. The project is also not accepted for ε – level or not rejected for M – level. However, the project is *clearly accepted* for some (Γ_B, Γ_C) – levels, and *accepted* for some $(\Gamma_B, \Gamma_C, \varepsilon_{\Gamma})$ – levels. We show the graphs for 5%, 10%, 15% cash flow variability rates in Figure 4.9. The long bars represent the *clearly accept* areas for the relevant (Γ_B, Γ_C) – levels, the middle bars represent the *accept* areas for the relevant $(\Gamma_B, \Gamma_C, \varepsilon_{\Gamma})$ – levels, and the short bars represent *stay indifferent* areas.

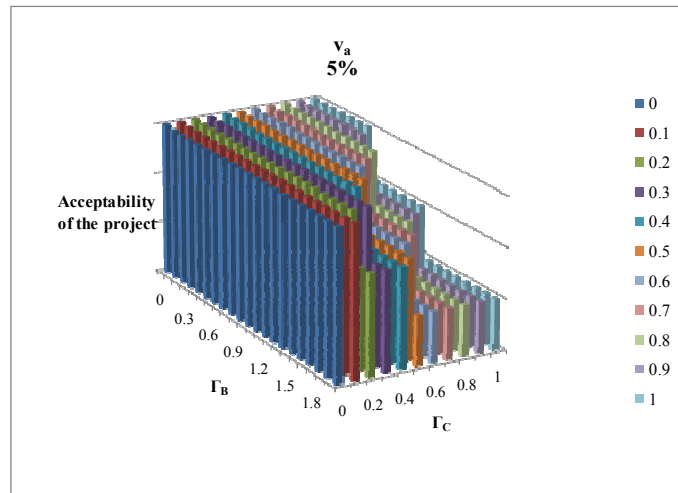


Figure 4.9 : Acceptability areas of a project for 5%, 10%, 15% variability rates

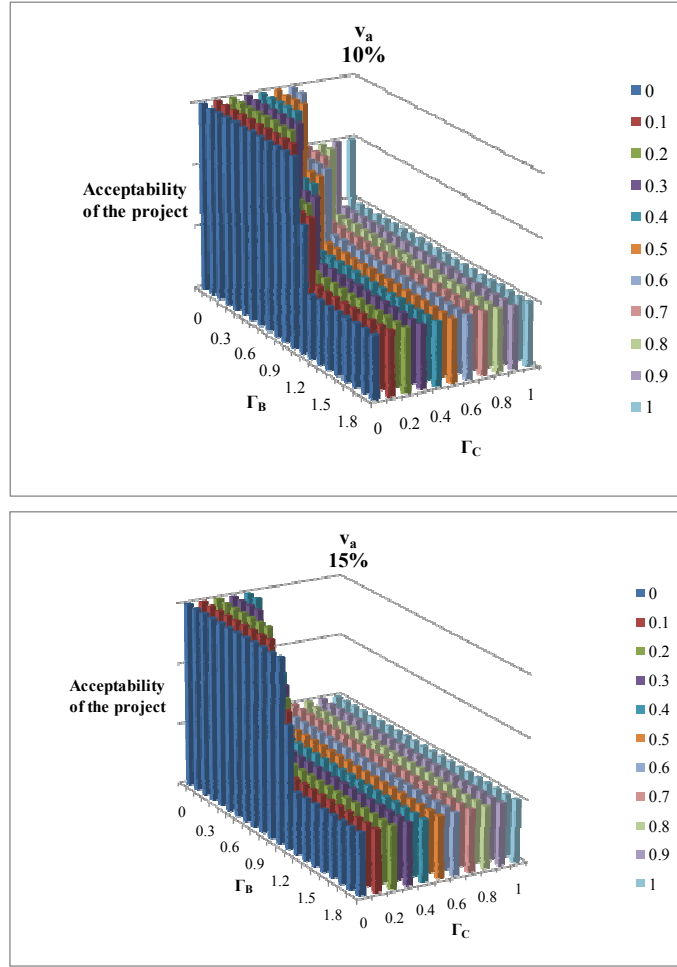


Figure 4.9 : Acceptability areas of a project for 5%, 10%, 15% variability rates (continued)

Remarks: As the variability rate increases, it is clear that the number of *stay indifferent* areas increases. For $(\Gamma_B, \Gamma_C) = ([0, 0.6], 0)$, the project is *clearly accepted* for all variability rates. At $v_a = 5\%$ variability rate, the project is *clearly accepted* when $\Gamma_B = [0, 0.6]$ regardless of the uncertainty level in the cash outflows, while it is *clearly accepted* when $\Gamma_C = 0$ regardless of the uncertainty level in the cash inflows. At $v_a = 15\%$ variability rate, the project is mainly characterized with *clearly accept* areas and *stay indifferent* areas for the given (Γ_B, Γ_C) – levels.

5. CONCLUSIONS

Capital rationing and capital budgeting are the problems to address the problem of allocating funds to a limited number of projects with consideration of some constraints such as budget constraints or borrowing limit constraints. The fuzzy set theory is an effective and powerful approach to address uncertainty in cases where the parameters or variables can be identified by the membership functions. Robust optimization approach is an emerging approach to address the worst-case in optimization problems in which the unknown parameters may belong to a given convex set.

In this thesis, Chapter 3 is devoted to fuzzy linear programming and fuzzy quadratic programming approach to capital rationing and capital budgeting problems. Chapter 4 is devoted to robust optimization approach to capital rationing and capital budgeting problems.

In Section 3.1, several fuzzy capital rationing models proposed in the literature have been provided, and illustrated with the numerical examples. While the dominance method to compare the present value of each project determines the results in [16], the predetermined credibility levels, confidence levels, and tolerable variance levels in the models in [17], [18], and [19] determine the results, respectively. The branch-and-bound method has been suggested to be sufficient for the models with triangular or trapezoidal fuzzy numbers in [17-19], while the evolutionary methods such as genetic algorithm have been suggested to be inevitable in case of non-triangular or non-trapezoidal fuzzy numbers. We used the same data set with the nominal values to illustrate each model. According to the results, the fuzzy capital rationing model proposed in [16] and the chance constrained programming model with fuzzy net cash flows proposed in [17,18] give the same solution set, i.e., in both models, Project 1 and Project 2 are fully funded, and Project 3 is not funded. However, the objective function in the latter model is smaller than the fuzzy objective function of the fuzzy capital rationing model proposed in [16]. Note also that we use one problem instance for the latter, and that the other problem instances may give different results. On the

other hand, in the fuzzy mean-variance model proposed in [19], Project 3 is funded, and Project 1 and Project 2 are not funded. In summary, given the data set, the results of the numerical examples for the models proposed in [16-18] contradict with the results of the numerical example for the model proposed in [19]. As a future research for this section, the numerical examples with large scale problems could also be recommended as well as changing the dominance method, predetermined credibility levels, confidence levels, and tolerable variance levels could be considered to see the effects on the results.

In Section 3.2, $\widetilde{\leq}^T$ t-norm fuzzy relation is used to defuzzify Weingartner's pure capital rationing model. Then, the dual model is formulated based on $\widetilde{\leq}_s$ t-conorm fuzzy relation. We show that weak duality holds for $\alpha \in [0.5, 1)$, which coincides with the result for a general FLP problem as in [24], although we follow the proof of [45], which is different from that of [24]. The detailed sensitivity analysis to determine the range of the shift of the core of triangular fuzzy numbers under which the current basis remains optimal gives the decision maker the opportunity to consider the conditions for the confidence of the solution set. Thereafter, we extend the analysis of the characterization of the level of investment projects in [5], and discover that the net cash flows in the crisp case are replaced by the best values of the fuzzy numbers representing net cash flows when we use $(1 - \alpha)$ – cut level. In the numerical analysis, the proposed primal model based on $\widetilde{\leq}^T$ t-norm fuzzy relation gives very optimistic results for the defined fuzzy numbers, where for the given data set, Project 1 and Project 2 are fully funded, and Project 3 is partially funded with a very optimistic proportion even in the worst-case. As a next research for this section, Weingartner's pure capital rationing model could be defuzzified by t-conorm fuzzy relation, and its dual model could be formulated by t-norm fuzzy relation, and the theoretical and numerical analysis performed in Section 3.2 could be extended.

Since we use the data set with the same nominal values, and the fuzzy numbers with the same assumptions, the numerical analysis in Section 3.1 and Section 3.2 gives us the opportunity to compare the results. The results of the models in [16], [17], and [18] can be interpreted to coincide with the results of the proposed model in Section 3.2, whereas the results of the model in [19] contradict with the results of all considered models.

In Section 3.3, we propose a t-norm and a t-conorm fuzzy relation based calculation of IRR from fuzzy cash flows, where we also define MARR as a fuzzy number and propose a t-norm and a t-conorm fuzzy relation based formulation of MARR. We define the decision rules for simple projects, and illustrate the decision rules with a numerical example for a simple project, and inspect the PV-consistency for this example. Given the data for a simple project, the project is accepted for t-conorm fuzzy relation based PV-equation at each α – cut level, but it is accepted for t-norm fuzzy relation based PV-equation only after a threshold α – cut level. It is evident that based on the proposed decision rules; the decision for a project depends on the predetermined threshold α – cut level. If the decision maker has a rather high predetermined threshold α – cut level for either case, then she may prefer to stay indifferent. We should also emphasize that the proposed two formulations could also be interpreted as the computation of the interval under which IRR can vary for a given α – cut level. However, this will lead to the well-known problem of not obtaining a unique IRR, even for a simple project considered in this thesis, which could be alleviated by the defuzzification of the convex fuzzy numbers obtained from two formulations. As a future research for this section, the research could also be extended to the case of non-simple projects for which multiple rates of return may arise.

In the first part of Section 3.4, we propose two primal-dual model pairs for quadratic programming model with fuzzy parameters. We formulate the primal model with $\lesseqgtr^T$ t-norm, and $\lesseqgtr_s$ t-conorm fuzzy relation, but with the corresponding dual $\lesseqgtr_s$ t-conorm and $\lesseqgtr^T$ t-norm fuzzy relations as suggested in [24] for FLP, we cannot obtain dual models which satisfy strong duality. We then propose alternative dual models, and show that weak duality and strong duality always hold. The key point of the two primal-dual model pairs is that one primal-dual model pair gives the best-case scenario (the optimistic case), and the other primal-dual model pair gives the worst-case scenario (the pessimistic case) under some α – cut level, which makes the models practically appealing. In the second part of Section 3.4, we apply the proposed primal-dual model pairs to Bernhard’s capital budgeting model with a non-linear objective function, with quadratic objective function in the special case, to see the extensions to the dual analysis. In duality analysis, not amazingly, we observe that the distinguishing part in the fuzzy case is horizon time value of cash flows for

the budget dual variable at terminal time as compared to the crisp case, and both net cash flows and horizon time value of cash flows for the rest of the duality analysis, and for the characterization of level of investment in projects. In addition to the duality for quadratic programming model, we analyze the duality for LP model. We perform the computational analysis for a ten-project example, and observe that for terminal wealth in quadratic programming model and LP model, we obtain convex triangular shaped fuzzy numbers, however, as the cash flow variability increases, the shape deteriorates, starting at 20% variability rate for quadratic programming model, and at 15% for LP model. Not amazingly, we observe that the budget variability rates have less impact on the outcomes as compared to the cash flow variability rates. The defuzzification of the fuzzy numbers representing terminal wealth obtained from t-norm and t-conorm fuzzy relations confirms the observation that the cash flow variability rates have more impact on the outcomes as compared to the budget variability rates. Since, the deterioration of the triangular shaped fuzzy numbers obtained from t-conorm fuzzy relation starts at 20% cash flow variability rate, the defuzzification cannot be managed at this cash flow variability rate. For average dividend, we obtain convex triangular fuzzy numbers in quadratic programming model, but non-convex triangular shaped fuzzy numbers in LP model. In both cases, the deterioration of the shape of the fuzzy numbers starts at 15% variability rate. We note that the values for any α – cut level are much lower in LP model than those in quadratic programming model.

In this thesis, Chapter 4 is devoted to the robust optimization approach to capital rationing and capital budgeting. In Section 4.1, robust optimization approach to Weingartner's pure capital rationing model in case of cash flow uncertainty as in [5] is extended to the case of budget and cash flow uncertainty. However, unlike in case of cash flow uncertainty as in [5], we also assume the objective function coefficients as uncertain as in [52]. We develop the robust linear counterpart of pure capital rationing model, and extend the computational analysis as well as the theoretical analysis in [5]. In the theoretical analysis part, we observe that the theoretical analysis part is characterized as either a kind of combination of the deterministic case and the case of cash flow uncertainty as in the characterization of level of investment in projects, or the separate analysis of the cash flow uncertainty and the budget uncertainty is similar as in the characterization of $\hat{a}_{nj}$ and $\hat{M}_n$. The main results of

the computational analysis section in case of budget and cash flow uncertainty coincide with the case of cash flow uncertainty, i.e., one of the projects is fully funded, and the other two projects are in competition. The graphs of the partially funded projects in case of budget and cash flow uncertainty share a similar characteristic with those in case of cash flow uncertainty, but they have an opposite behavior for very low Γ_A values. The stepwise behavior is common in case of budget and cash flow uncertainty and in case of cash flow uncertainty. Finally, not amazingly, the budget variability rates have less impact on the characteristics of the graphs as compared to the cash flow variability rates.

In Section 4.2, robust optimization approach to Weingartner's horizon capital budgeting model in case of cash flow uncertainty as in [5] is extended to the case of budget and cash flow uncertainty. We develop the robust linear counterpart of Weingartner's horizon capital budgeting model, and extend the computational analysis as well as the theoretical analysis in [5]. In the theoretical analysis part, we observe that the analysis is analogous to that in Section 4.1, namely, the characterization of level of investment in projects is a kind of combination of the deterministic case and the case of cash flow uncertainty, and the characterization of $\hat{a}_{nj}$ and $\hat{M}_n$ gives the similar results as in Section 4.1. In the computational analysis part, all the graphs of the projects share similar characteristics with those in case of cash flow uncertainty. Finally, not amazingly, as in Section 4.1, the budget variability rates have less impact on the characteristics of the graphs as compared to the cash flow variability rates.

In Section 4.3, we develop robust decision rules for NPV and IRR by borrowing from robust optimization methodology as in [12]. We define different acceptability degrees according to the cash flow variability rate. We also compute the probability of over-conservatism and over-optimism for the defined decision rules, and clarify the proposed decision rules with numerical analysis. The numerical analysis gives different acceptability degrees for different (Γ_B, Γ_C) -levels, and the decision maker has the flexibility of reaching a decision based on her uncertainty perception.

As a next research, we may suggest a search for a fuzzy relation which addresses the problems more realistically. It is clear that a primal model with t-norm fuzzy relation gives the best-case scenario results, while a primal model with t-conorm fuzzy

relation gives the worst-case scenario results. However, most problems are characterized as neither a best-case scenario nor a worst-case scenario. Moreover, the decision maker may decide to tune the level of uncertainty rather than accepting the results of a best-case or a worst-case scenario model. With this concern, a flexible model which enables the decision maker to adjust the uncertainty level in a fuzzy model would be more appropriate. To study other capital budgeting models may also be suggested as a next research for both fuzzy and robust optimization approaches.

REFERENCES

- [1] **Weingartner, H.M.**, 1967. Mathematical Programming and the Analysis of Capital Budgeting Problems, Markham Publishing Company, Chicago.
- [2] **Park, C.S. and Sharp-Bette, G.P.**, 1990. Advanced Engineering Economics, John Wiley & Sons, Inc., New York Chicester Brisbane.
- [3] **Kira, D.S. and Kusy, M.I.**, 1990. A stochastic capital rationing model, *The Journal of the Operational Research Society*, **41**(9), 853-863.
<http://www.palgrave-journals.com/jors>
- [4] **Kira, D., Kusy, M. and Rakita, I.**, 2000. The effect of project risk on capital rationing under uncertainty, *The Engineering Economist*, **45**(1), 37-55.
- [5] **Kachani, S. and Langella, J.**, 2005. A robust optimization approach to capital rationing and capital budgeting, *The Engineering Economist*, **50**(3), 195-229.
- [6] **Goldfarb, D. and Iyengar, G.**, 2003. Robust portfolio selection problems, *Mathematics of Operations Research*, **28**(1), 1-38.
- [7] **Bertsimas, D. and Pachamanova, D.**, 2008. Robust multiperiod portfolio management in the presence of transaction costs, *Computers & Operations Research*, **35**(1), 3-17.
- [8] **Lutgens, F., Sturm, J. And Kolen, A.**, 2006. Robust one-period option hedging, *Operations Research*, **54**(6), 1051-1062.
- [9] **Soyster, A.L.**, 1973. Convex programming with set-inclusive constraints and applications to inexact linear programming, *Operations Research*, **21**(5), 1154-1157.
- [10] **El Ghaoui, L., Oustry, F. and Lebret, H.**, 1998. Robust solutions to uncertain semidefinite programs, *SIAM Journal on Optimization*, **9**(1), 33-52.
- [11] **Ben-Tal, A. and Nemirovski, A.**, 2000. Robust solutions of linear programming problems contaminated with uncertain data, *Mathematical Programming*, **88**(3), 411-424.

- [12] **Bertsimas, D. and Sim, M.**, 2004. The price of robustness, *Operations Research*, **52**(1), 35-53.
- [13] **Ben-Tal, A., Goryashko, A., Guslitzer, E. and Nemirovski, A.**, 2004. Adjustable robust solutions of uncertain linear programs, *Mathematical Programming*, **99**(2), 351-376.
- [14] **Buckley, J.J.**, 1987. The fuzzy mathematics of finance, *Fuzzy Sets and Systems*, **21**(3), 257-273.
- [15] **Kahraman, C., Ruan D. and Tolga, E.**, 2002. Capital budgeting techniques using discounted fuzzy versus probabilistic cash flows, *Information Sciences*, **142**(1-4), 57–76.
- [16] **Chiu, C.-Y. and Park, C.S.**, 1998. Capital budgeting decisions with fuzzy projects, *The Engineering Economist*, **43**(2), 125-150.
- [17] **Huang, X.**, 2007. Chance-constrained programming models for capital budgeting with NPV as fuzzy parameters, *Journal of Computational and Applied Mathematics*, **198**(1), 149-159.
- [18] **Huang, X.**, 2007. Optimal project selection with random fuzzy parameters, *International Journal of Production Economics*, **106**(2), 513-522.
- [19] **Huang, X.**, 2008. Mean-variance model for fuzzy capital budgeting, *Computers & Industrial Engineering*, **55**(1), 34-47.
- [20] **Buckley, J.J. and Feuring, T.**, 2000. Evolutionary algorithm solution to fuzzy problems: Fuzzy linear programming, *Fuzzy Sets and Systems*, **109**(1), 35-53.
- [21] **Zimmermann, H.-J.**, 1978. Fuzzy programming and linear programming with several objective functions, *Fuzzy Sets and Systems*, **1**(1), 45-55.
- [22] **Delgado, M., Verdegay, J.L. and Vila, M.A.**, 1989. A general model for fuzzy linear programming, *Fuzzy Sets and Systems*, **29**(1), 21-29.
- [23] **Verdegay, J.L.**, 1984. A dual approach to solve the fuzzy linear programming problem, *Fuzzy Sets and Systems*, **14**(2), 131-141.
- [24] **Inuiguchi, M., Ramik, J., Tanino, T. and Vlach, M.**, 2003. Satisficing solutions and duality in interval and fuzzy linear programming, *Fuzzy Sets and Systems*, **135**(1), 151-177.
- [25] **Ramik, J.**, 2006. Duality in fuzzy linear programming with possibility and necessity relations, *Fuzzy Sets and Systems*, **157**(10), 1283-1302.

- [26] **Chen, S.-J. and Hwang, C.-L.**, 1992. Fuzzy Multiple Attribute Decision Making: Methods and Applications, Springer, Berlin Heidelberg New York.
- [27] **Laughunn, D.J.**, 1970. Quadratic binary programming with application to capital budgeting problems, *Operations Research*, **18**(3), 454-461.
- [28] **Bector, C.R. and Chandra, S.**, 2005. Fuzzy Mathematical Programming and Fuzzy Matrix Games, Springer, Berlin Heidelberg New York.
- [29] **Liu, S.-T.**, 2007. Quadratic programming with fuzzy parameters: A membership function approach, *Chaos, Solitons and Fractals*, doi: 10.1016/j.chaos.2007.07.054.
- [30] **Ammar, E. and Khalifa, H.A.**, 2003. Fuzzy portfolio optimization a quadratic programming approach, *Chaos, Solitons and Fractals*, **18**(5), 1045-1054.
- [31] **Ammar, E.E.**, 2007. On fuzzy random multiobjective quadratic programming, *European Journal of Operational Research*, doi:10.1016/j.ejor.2007.11.031.
- [32] **Ammar, E.E.**, 2008. On solutions of fuzzy random multiobjective quadratic programming with applications in portfolio problem, *Information Sciences*, **178**(2), 468-484.
- [33] **Ross, T.J.**, 1995. Fuzzy Logic with Engineering Applications, International Edition, McGraw-Hill, Inc., New York St. Louis San Francisco.
- [34] **Zadeh, L.A.**, 1965. Fuzzy sets, *Information and Control*, **8**(3), 338-353.
- [35] **Buckley, J.J., Eslami, E. and Feuring, T.**, 2002. Fuzzy Mathematics in Economics and Engineering, Physica-Verlag, A Springer-Verlag Company, Heidelberg New York.
- [36] **Klir, G.J. and Yuan, B.**, 1995. Fuzzy Sets and Fuzzy Logic Theory and Applications, Prentice Hall, Upper Saddle River New Jersey.
- [37] **Fodor, J. and Roubens, M.**, 1994. Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Dordrecht, Boston, London.
- [38] **Dubois, D. and Prade, H.**, 1983. Ranking fuzzy numbers in the setting of possibility theory, *Information Sciences*, **30**(3), 183-224.

- [39] **Rödder, W. and Zimmermann, H.-J.**, 1980. Duality in fuzzy linear programming, in: *Extremal Methods and System Analysis*, Fiacco, A.V. and Kortanek, K.O. (Eds.), Springer-Verlag, Heidelberg.
- [40] **Bector, C.R. and Chandra, S.**, 2002. On duality in linear programming under fuzzy environment, *Fuzzy Sets and Systems*, **125**(3), 317-325.
- [41] **Chiu, C.-Y. and Park, C.S.**, 1994. Fuzzy cash flow analysis using present worth criterion, *The Engineering Economist*, **39**(2), 113-138.
- [42] **Liu, B. and Liu, Y.-K.**, 2002. Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, **10**(4), 445-450.
- [43] **Zheng, Y. and Liu, B.**, 2006. Fuzzy vehicle routing model with credibility measure and its hybrid intelligent algorithm, *Applied Mathematics and Computation*, **176**(2), 673–683.
- [44] **Liu, B.**, 2002. *Theory and Practice of Uncertain Programming*, Physica-Verlag, A Springer-Verlag Company, Studies in Fuzziness and Soft Computing, Heidelberg New York.
- [45] **Bertsimas, D. and Tsitsiklis, J.N.**, 1997. *Introduction to Linear Optimization*, Athena Scientific, Belmont Massachusetts.
- [46] **Dantzig, G.B. and Thapa, M.N.**, 1997. *Linear Programming 1: Introduction*, Springer Series in Operations Research, New York Berlin Heidelberg.
- [47] **Xpress-MP**, Fair Isaac, Leamington Spa, UK.
- [48] **Kahraman, C., Tolga, E. and Ulukan, Z.**, 2000. Justification of manufacturing technologies using fuzzy benefit/cost ratio analysis, *International Journal of Production Economics*, **66**(1), 45-52.
- [49] **Gill, P.E., Murray, W., Saunders, M.A. and Wright, M.H.**, 1989. Constrained Nonlinear Programming, in: *Optimization Handbooks in Operations Research and Management Science Volume 1*, pp. 171-210, Eds. Nemhauser, G.L., Rinnooy Kan, A.H.G. & Todd, M.J., North-Holland, Amsterdam New York Oxford.
- [50] **Bernhard, R.H.**, 1969. Mathematical programming models for capital budgeting- A survey, generalization, and critique, *Journal of Financial and Quantitative Analysis*, **4**(2), 111-158.

- [51] **Zangwill, W.I.**, 1969. Nonlinear Programming: A Unified Approach, Prentice-Hall, Englewood Cliffs, New Jersey.
- [52] **Bertsimas, D. and Sim, M.**, 2003. Robust discrete optimization and network flows, *Mathematical Programming*, **98**(1-3), 49-71.
- [53] **Goldfarb, D. and Todd, M.J.**, 1989. Linear Programming, in: *Optimization Handbooks in Operations Research and Management Science Volume 1*, pp. 73-170, Eds. Nemhauser, G.L., Rinnooy Kan, A.H.G. and Todd, M.J., North-Holland, Amsterdam New York Oxford.

APPENDIXES

APPENDIX A - FUZZY LINEAR PROGRAMMING AND FUZZY QUADRATIC PROGRAMMING APPROACHES TO CAPITAL RATIONING AND CAPITAL BUDGETING

APPENDIX B -ROBUST LINEAR PROGRAMMING APPROACH TO CAPITAL RATIONING AND CAPITAL BUDGETING

Table A.1 : Xpress-MP Code for fuzzy capital rationing model

```
model "Fuzzy Capital Rationing Problem"
uses "mmxprs"
declarations
J: integer !number of projects
N: integer !number of time periods
end-declarations
initializations from 'rationing_fuzzy.txt'
J N
end-initializations
declarations
PROJECT = 1..J
TIME = 0..N
ALPHA=50..99
M: array (TIME) of real
A: array (TIME,PROJECT) of real
AUNCERTAINTY: array (TIME,PROJECT) of real
MUNCERTAINTY: array (TIME) of real
DISCOUNT: array (TIME) of real
NPV: array (PROJECT) of real
CHOICE: array (PROJECT) of mpvar
end-declarations
initializations from 'rationing_fuzzy.txt'
M A AUNCERTAINTY MUNCERTAINTY DISCOUNT NPV
end-initializations
forall (a in ALPHA) do
!Define Objective function
Objective:= sum (j in PROJECT)NPV(j)*CHOICE(j)+
sum (n in TIME) (sum (j in PROJECT)(1-0.01*a)*
AUNCERTAINTY(n,j)*DISCOUNT (n)* CHOICE(j))
!Constraints
forall (n in TIME)
sum (j in PROJECT) -A(n,j)*CHOICE(j)+ sum (j in PROJECT) -(1-0.01*a)*
AUNCERTAINTY(n,j)* CHOICE(j) <= M(n)+(1-0.01*a)*MUNCERTAINTY(n)
forall (j in PROJECT) do
CHOICE(j)<=1
CHOICE(j)>=0
end-do
maximize(Objective)
! Solution printing
forall(j in PROJECT) writeln(getsol(CHOICE(j)))
writeln(getobjval)
end-do
end-model
```

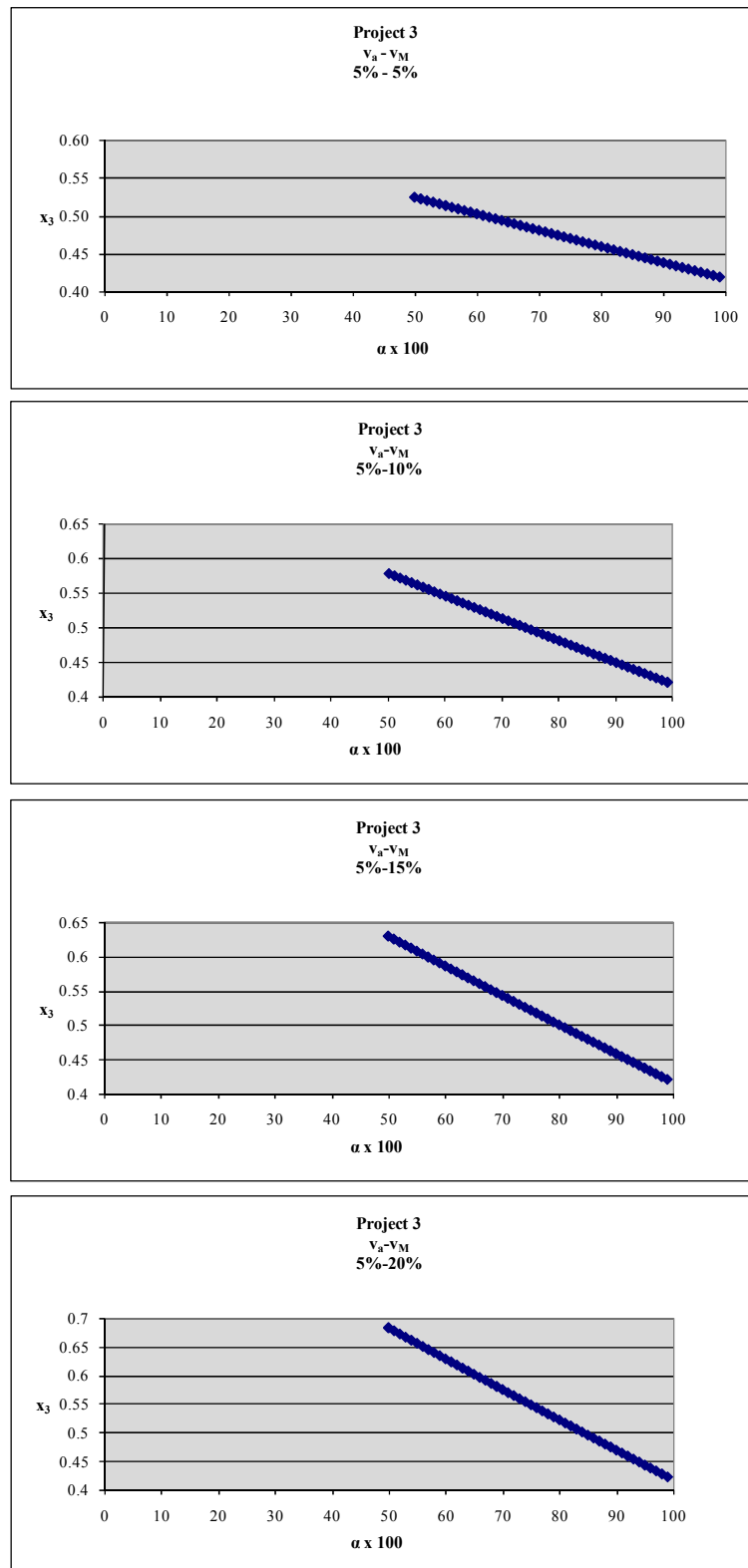


Figure A.1 : Level of investment in Project 3

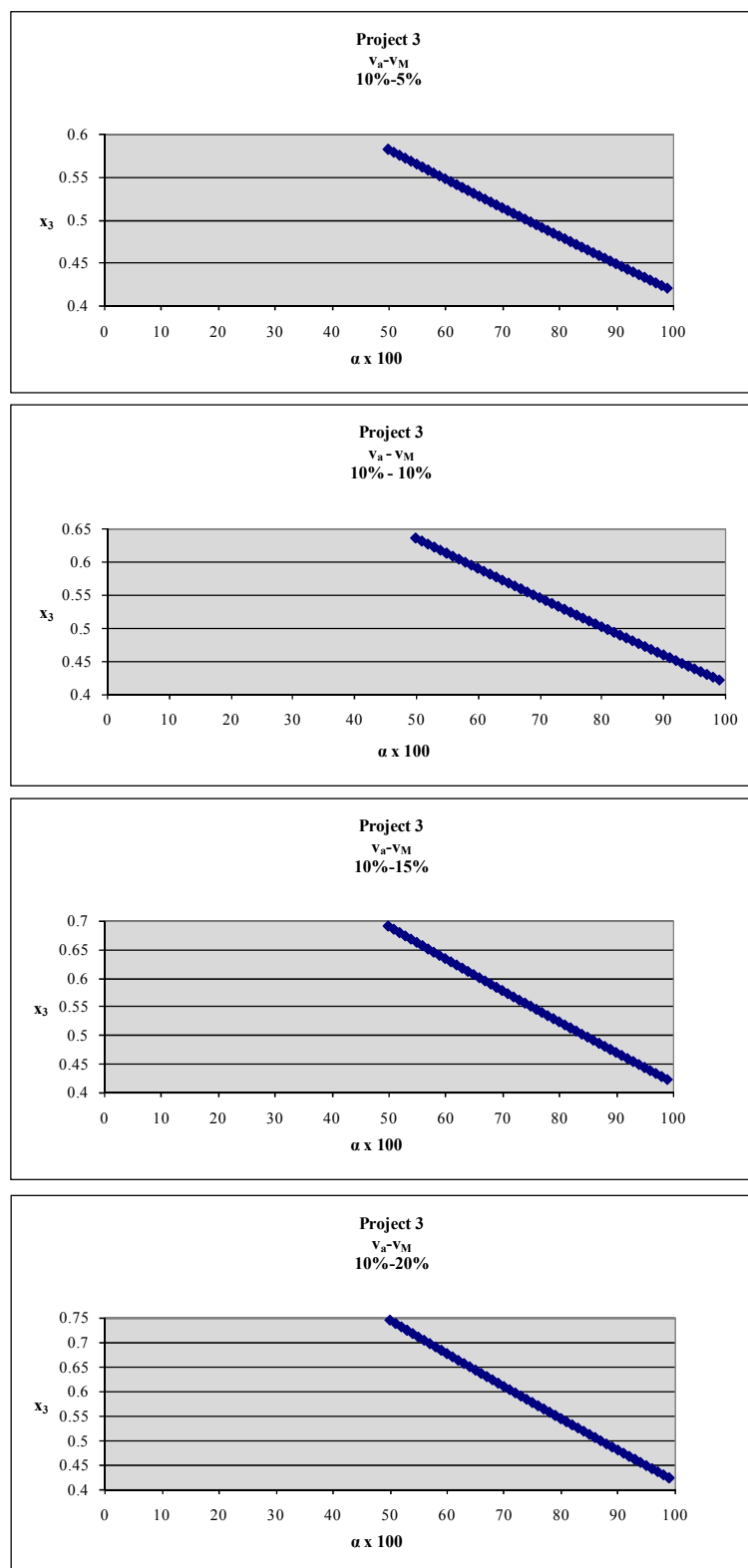


Figure A.1 : Level of investment in Project 3 (continued)

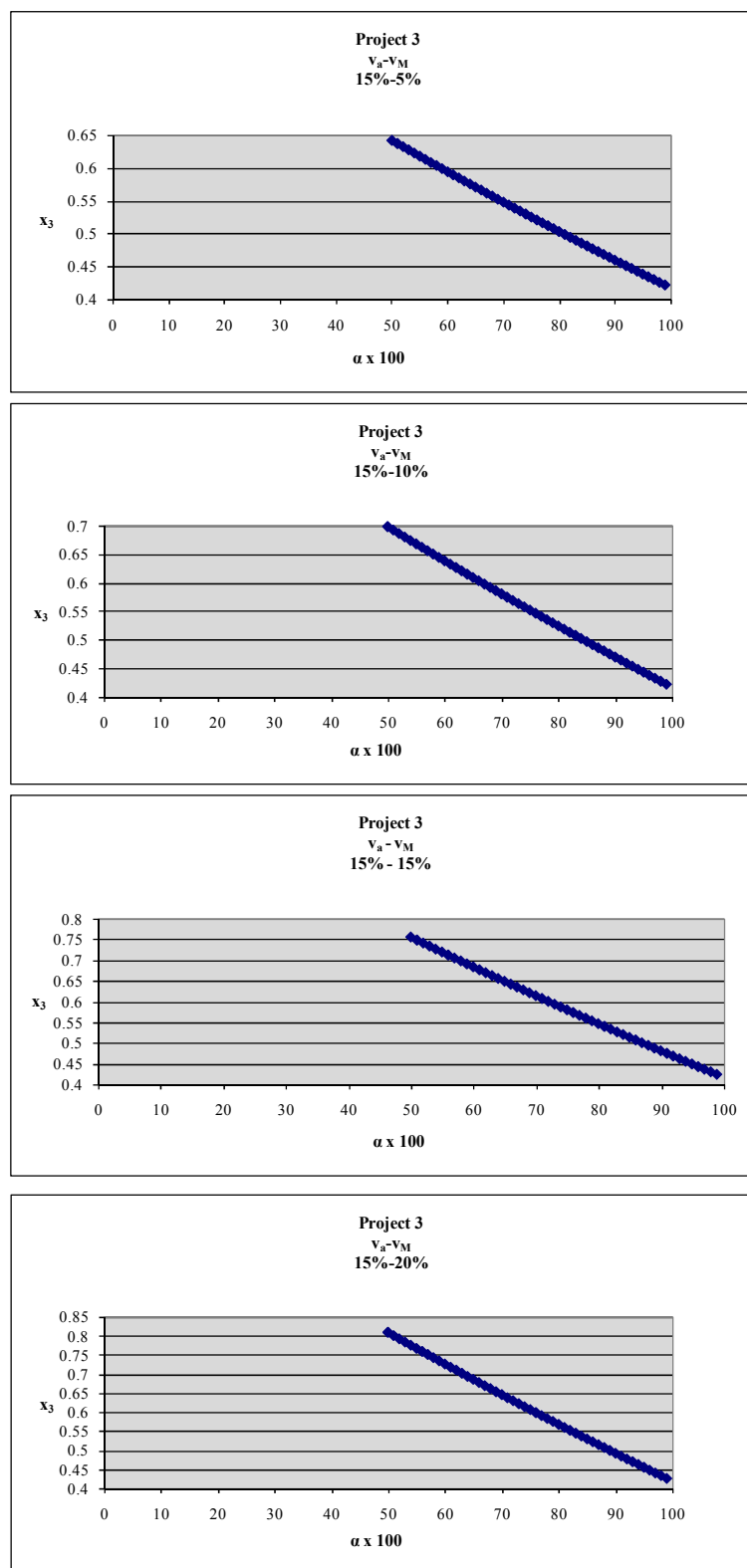


Figure A.1 : Level of investment in Project 3 (continued)

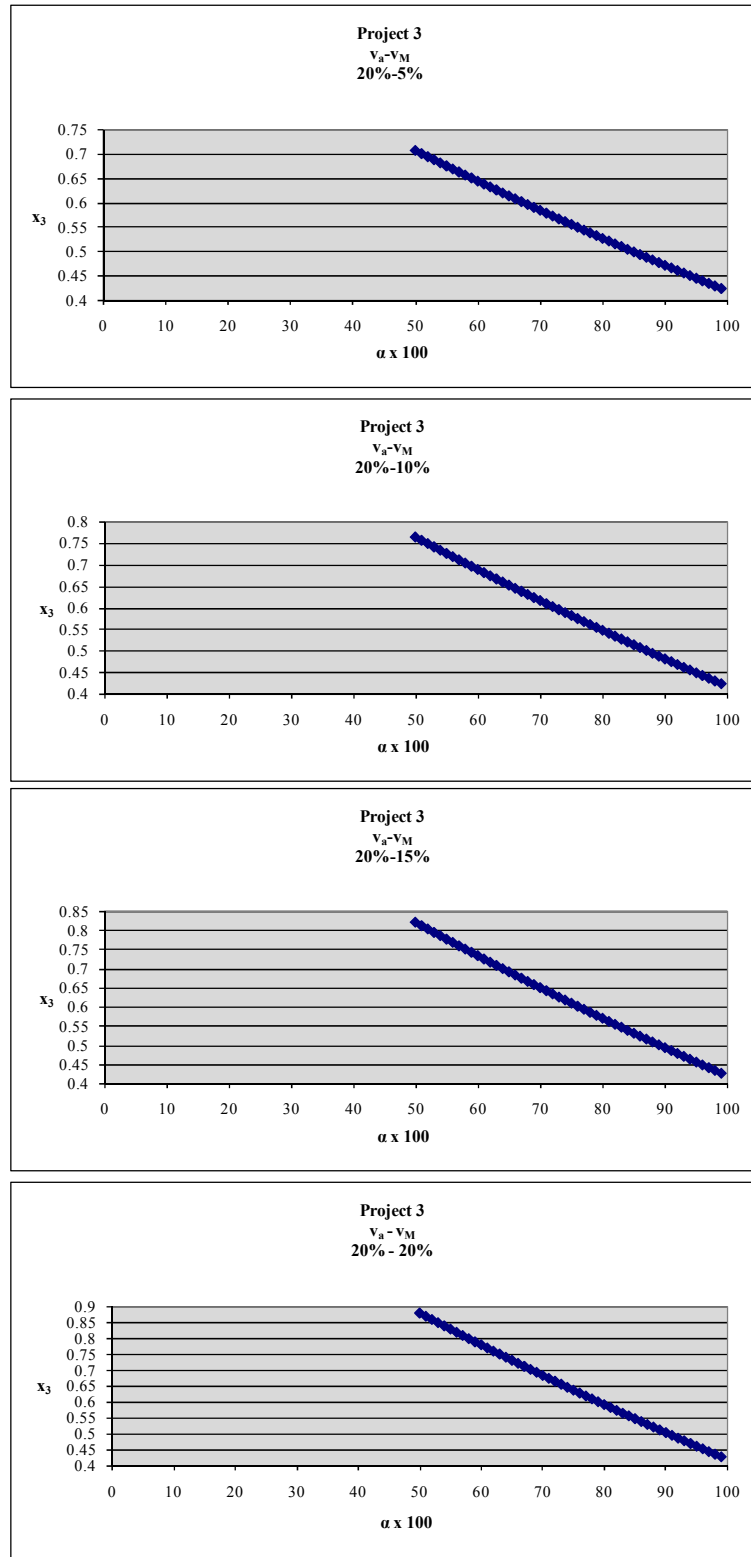


Figure A.1 : Level of investment in Project 3 (continued)

Table A.2 : Xpress-MP code for Bernhard's model with t-norm and t-conorm fuzzy relation

```

model "Bernard model with t-norm fuzzy relation"
uses "mmxprs", "mmquad"

declarations
J: integer !number of projects
N: integer !number of time periods
end-declarations

initializations from 'Bernard_fuzzy.txt'
J N
end-initializations

declarations
PROJECT = 1..J
TIME = 0..N
TIME_1 = 1..N
ALPHA=0..9
M: array (TIME) of real
A: array (TIME,PROJECT) of real
APOST:array (PROJECT) of real
AUNCERTAINTY: array (TIME,PROJECT) of real
MUNCERTAINTY: array (TIME) of real
APOSTUNCERTAINTY:array (PROJECT) of real
LENDINGRATE: array (TIME) of real
BORROWINGRATE:array (TIME) of real
DISCOUNT:array (TIME) of real
CHOICE: array (PROJECT) of mpvar
DIVIDEND: array (TIME) of mpvar
LENDING: array (TIME) of mpvar
BORROWING: array (TIME) of mpvar

end-declarations

initializations from 'Bernard_fuzzy.txt'
M A APOST AUNCERTAINTY APOSTUNCERTAINTY MUNCERTAINTY
LENDINGRATE BORROWINGRATE DISCOUNT
end-initializations

!Define Objective function
forall (a in ALPHA) do
GTR:=sum (j in PROJECT)APOST(j)*CHOICE(j)+ sum (j in PROJECT)(1-
0.1*a)*APOSTUNCERTAINTY(j)*CHOICE(j)+LENDING(N)-BORROWING(N)
GTL:=sum (j in PROJECT)APOST(j)*CHOICE(j)+ sum (j in PROJECT)(0.1*a-
1)*APOSTUNCERTAINTY(j)*CHOICE(j)+LENDING(N)-BORROWING(N)

```

Table A.2 : Xpress-MP code for Bernhard's model with t-norm and t-conorm fuzzy relation (continued)

```

D:=sum (n in TIME)DIVIDEND(n)*DISCOUNT(n)
DVARIANCE:=sum(n in TIME_1)((DIVIDEND(n)-DIVIDEND(n-1))^2)
Objective:= GTR*DISCOUNT(N)+D-0.0000001*DVARIANCE-
0.001*(GTL*0.1*DISCOUNT(N)-D/6)^2

!Constraints
sum (j in PROJECT) -A(0,j)*CHOICE(j)+ sum (j in PROJECT) -(1-0.1*a)*
AUNCERTAINTY(0,j)* CHOICE(j)+LENDING(0)-
BORROWING(0)+DIVIDEND(0)<= M(0)+(1-0.1*a)*MUNCERTAINTY(0)
    forall (n in TIME_1)
sum (j in PROJECT) -A(n,j)*CHOICE(j)+ sum (j in PROJECT) -(1-0.1*a)*
AUNCERTAINTY(n,j)* CHOICE(j)-LENDINGRATE(n-1)*LENDING(n-
1)+LENDING(n)+BORROWINGRATE(n-1)*BORROWING(n-1)-
BORROWING(n)+DIVIDEND(n) <= M(n)+(1-0.1*a)*MUNCERTAINTY(n)
    forall (n in TIME)do
DIVIDEND(n)>=5
LENDING(n)>=0
BORROWING(n)>=0
DIVIDEND(n)>=0
forall (j in PROJECT) do
    CHOICE(j)<=1
    CHOICE(j)>=0
end-do
end-do
maximize(Objective)
! Solution printing
forall(j in PROJECT) writeln(getsol(CHOICE(j)))
forall(n in TIME) writeln(getsol(LENDING(n)))
forall(n in TIME) writeln(getsol(BORROWING(n)))
forall(n in TIME) writeln(getsol(DIVIDEND(n)))
writeln(getsol (GTR))
writeln(getsol (GTL))
writeln(getsol(D/6))
writeln(getsol(DVARIANCE))
writeln (getsol((GTL*0.1*DISCOUNT(N)-D/6)^2))
writeln(getobjval)
end-do
end-model

```

Table A.2 : Xpress-MP code for Bernhard's model with t-norm and t-conorm fuzzy relation (continued)

```

model "Bernard model with t-conorm fuzzy relation"
uses "mmxprs", "mmquad"

declarations
J: integer !number of projects
N: integer !number of time periods
end-declarations

initializations from 'Bernard_fuzzy.txt'
J N
end-initializations

declarations
PROJECT = 1..J
TIME = 0..N
TIME_1= 1..N
ALPHA=0..9
M: array (TIME) of real
A: array (TIME,PROJECT) of real
APOST:array (PROJECT) of real
AUNCERTAINTY: array (TIME,PROJECT) of real
MUNCERTAINTY: array (TIME) of real
APOSTUNCERTAINTY:array (PROJECT) of real
LENDINGRATE: array (TIME) of real
BORROWINGRATE:array (TIME) of real
DISCOUNT:array (TIME) of real
CHOICE: array (PROJECT) of mpvar
DIVIDEND: array (TIME) of mpvar
LENDING: array (TIME) of mpvar
BORROWING: array (TIME) of mpvar

end-declarations

initializations from 'Bernard_fuzzy.txt'
M A APOST AUNCERTAINTY APOSTUNCERTAINTY MUNCERTAINTY
LENDINGRATE BORROWINGRATE DISCOUNT
end-initializations

!Define Objective function
forall (a in ALPHA) do
GTL:=sum (j in PROJECT)APOST(j)*CHOICE(j)+sum (j in PROJECT)-
(0.1*a)*APOSTUNCERTAINTY(j)*CHOICE(j)+LENDING(N)-
BORROWING(N)
D:=sum (n in TIME)DIVIDEND(n)*DISCOUNT(n)
Objective:= GTL*DISCOUNT(N)+D

```

Table A.2 : Xpress-MP code for Bernhard's model with t-norm and t-conorm fuzzy relation (continued)

```

!Constraints
sum (j in PROJECT) -A(0,j)*CHOICE(j)+ sum (j in PROJECT) (0.1*a)*
AUNCERTAINTY(0,j)* CHOICE(j)+LENDING(0)-
BORROWING(0)+DIVIDEND(0)<= M(0)-(0.1*a)*MUNCERTAINTY(0)
    forall (n in TIME_1)
sum (j in PROJECT) -A(n,j)*CHOICE(j)+ sum (j in PROJECT) (0.1*a)*
AUNCERTAINTY(n,j)* CHOICE(j)-LENDINGRATE(n-1)*LENDING(n-
1)+LENDING(n)+BORROWINGRATE(n-1)*BORROWING(n-1)-
BORROWING(n)+DIVIDEND(n) <= M(n)-(0.1*a)*MUNCERTAINTY(n)
    forall (n in TIME)do
DIVIDEND(n)>=5
LENDING(n)>=0
BORROWING(n)>=0
DIVIDEND(n)>=0
forall (j in PROJECT) do
    CHOICE(j)<=1
    CHOICE(j)>=0
end-do
end-do
maximize(Objective)
! Solution printing
forall(j in PROJECT) writeln(getsol(CHOICE(j)))
forall(n in TIME) writeln(getsol(LENDING(n)))
forall(n in TIME) writeln(getsol(BORROWING(n)))
forall(n in TIME) writeln(getsol(DIVIDEND(n)))
writeln(getsol (GTL))
writeln(getsol(D/6))
writeln(getobjval)
end-do
end-model

```

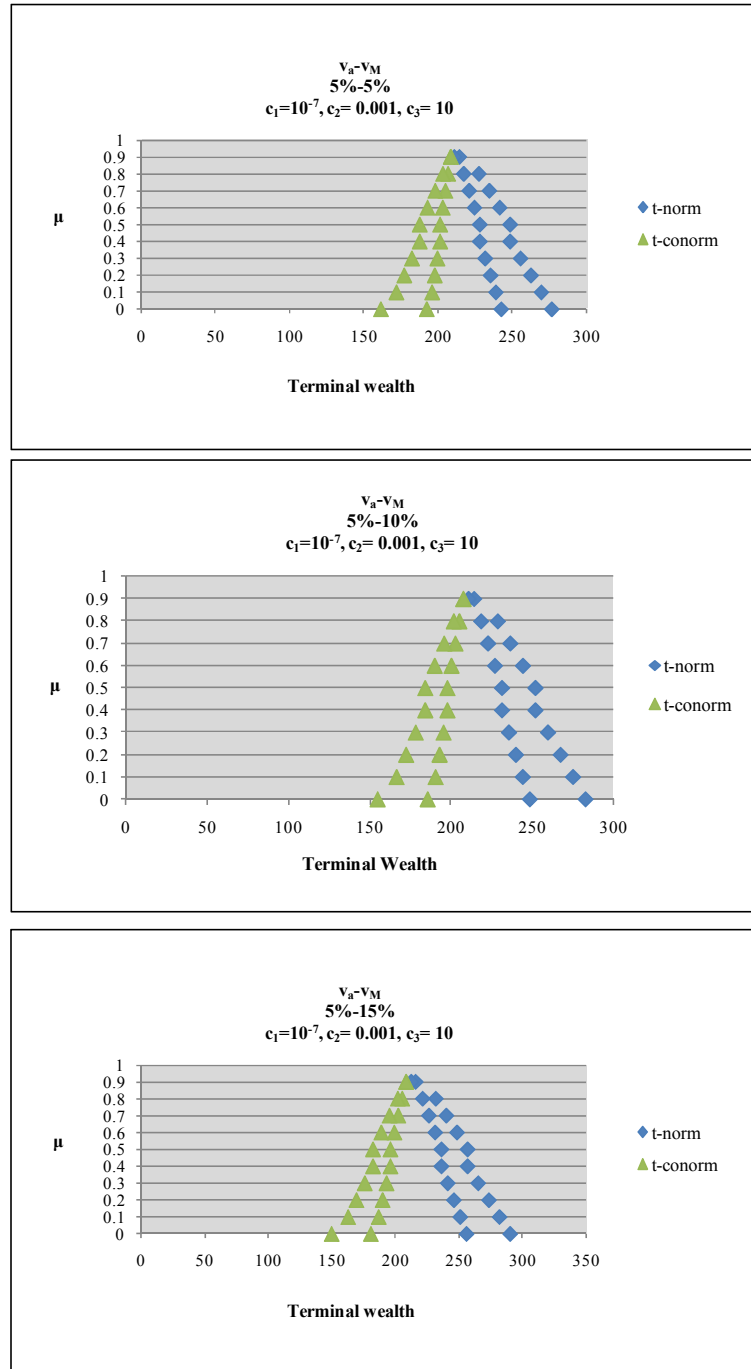


Figure A.2 : Fuzzy numbers representing terminal wealth in quadratic programming model

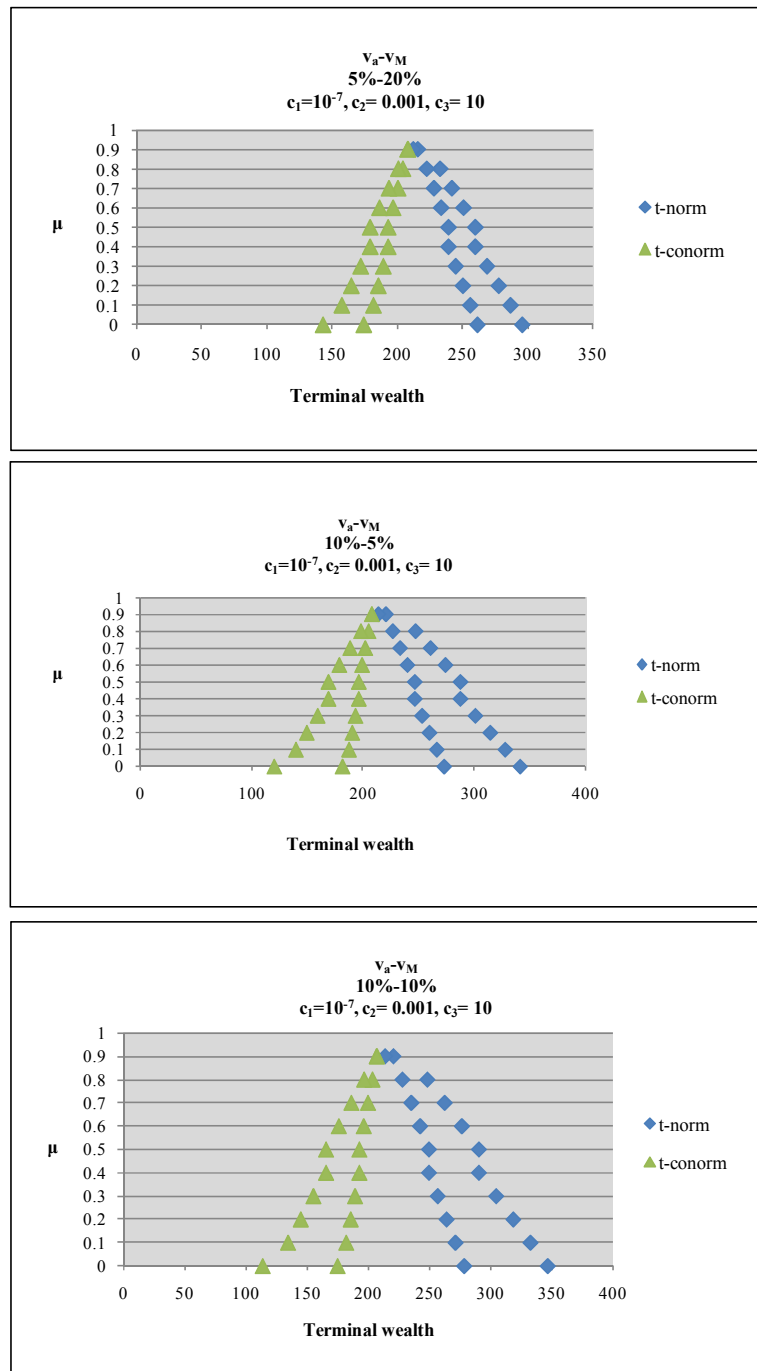


Figure A.2 : Fuzzy numbers representing terminal wealth in quadratic programming model (continued)

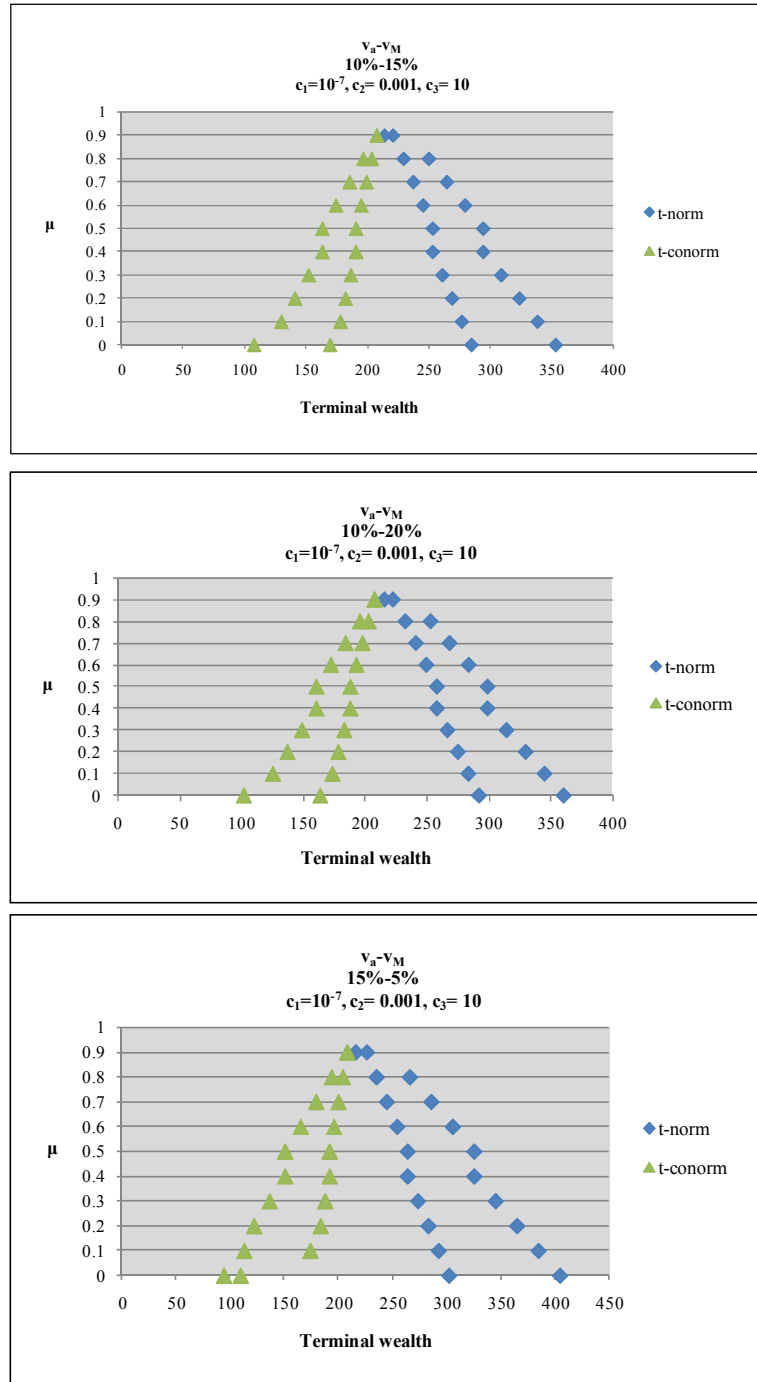


Figure A.2 : Fuzzy numbers representing terminal wealth in quadratic programming model (continued)

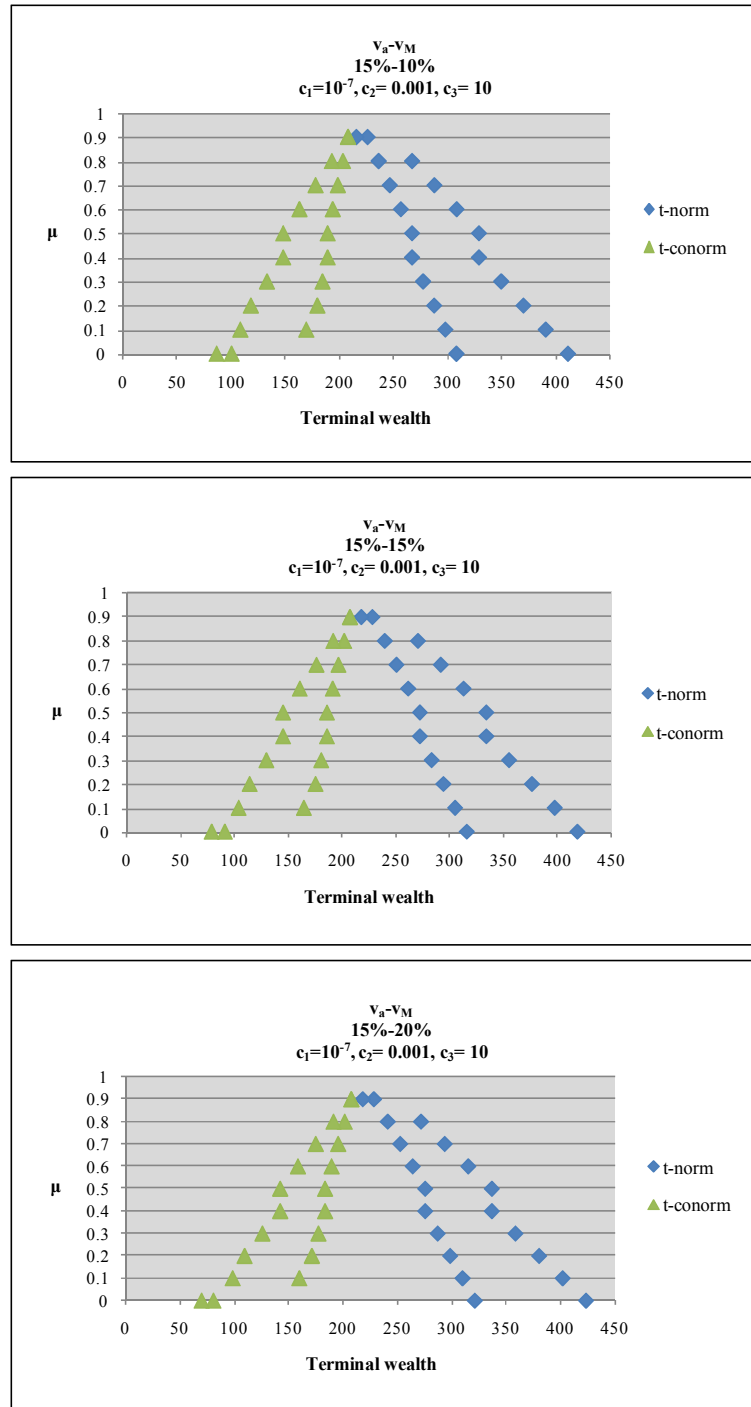


Figure A.2 : Fuzzy numbers representing terminal wealth in quadratic programming model (continued)

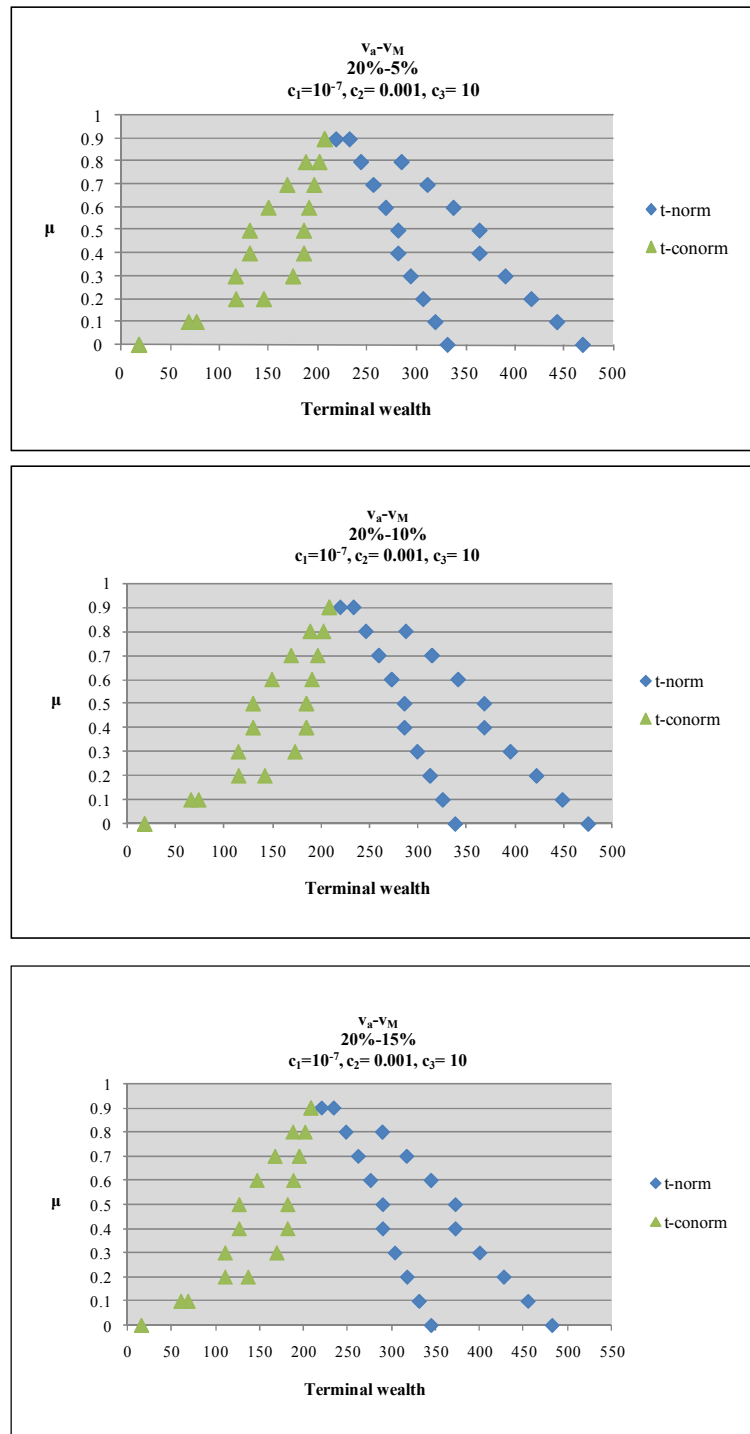


Figure A.2 : Fuzzy numbers representing terminal wealth in quadratic programming model (continued)

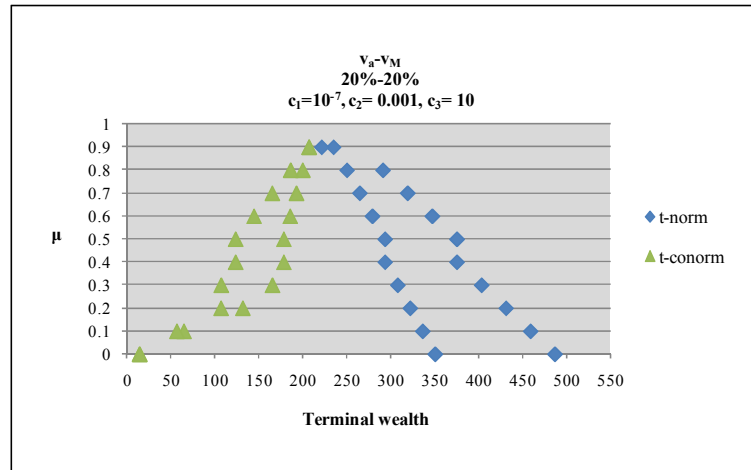


Figure A.2 : Fuzzy numbers representing terminal wealth in quadratic programming model (continued)

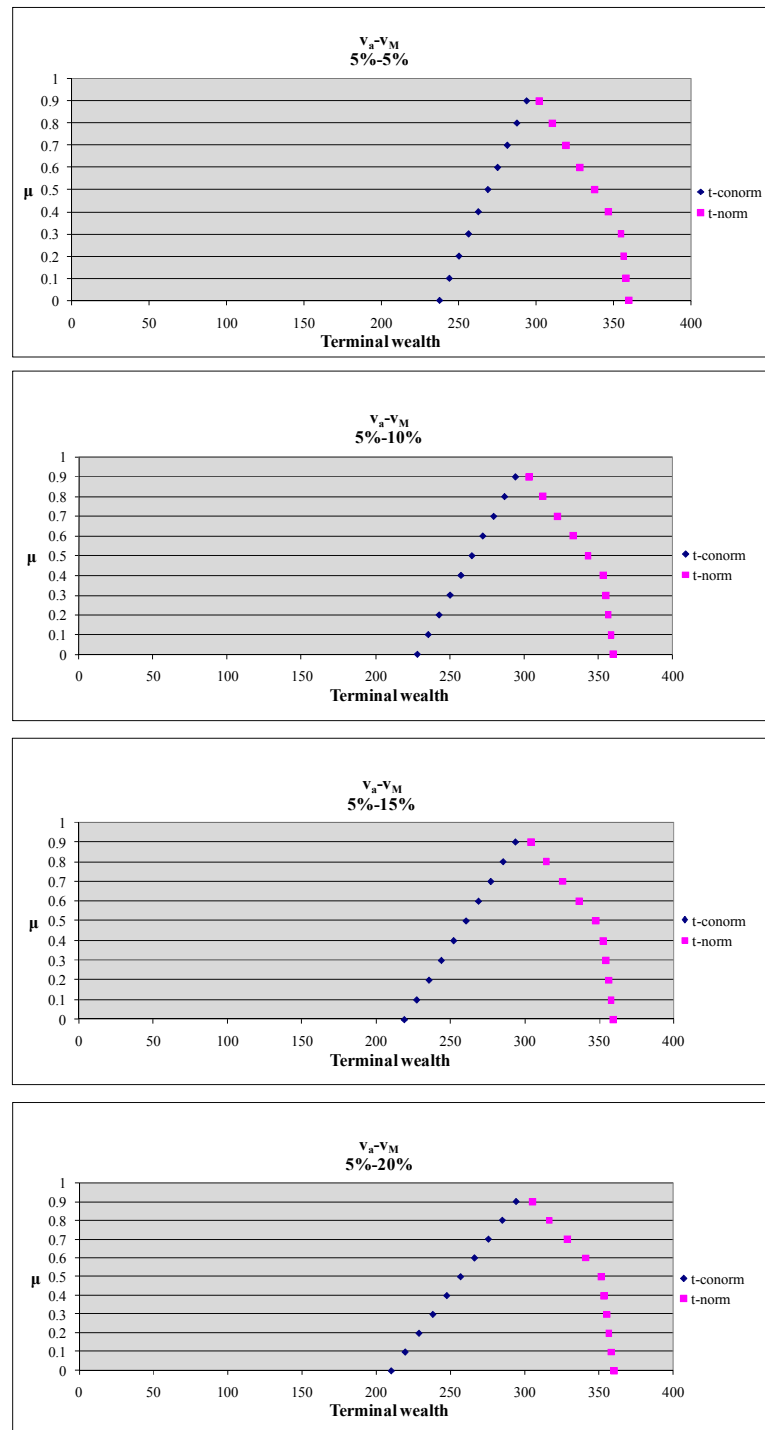


Figure A.3 : Fuzzy numbers representing terminal wealth in LP model

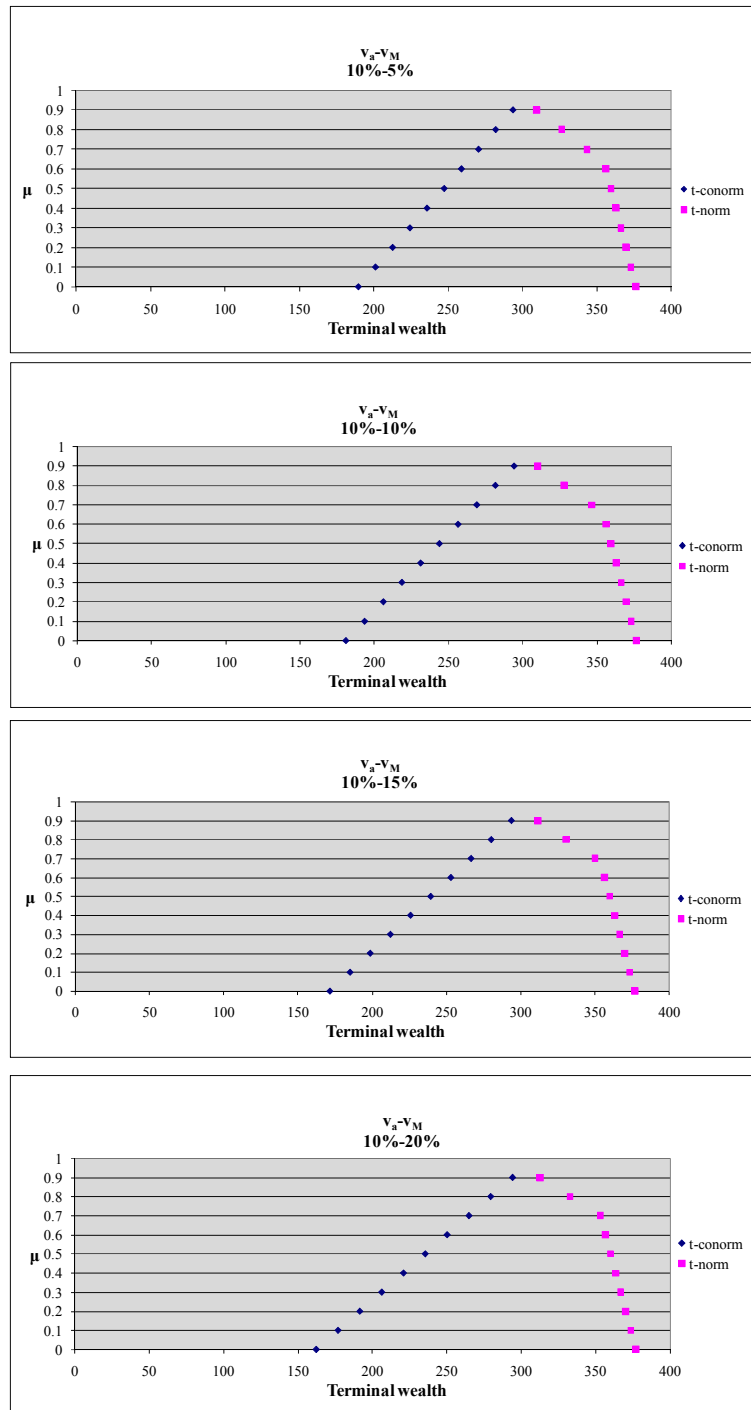


Figure A.3 : Fuzzy numbers representing terminal wealth in LP model (continued)

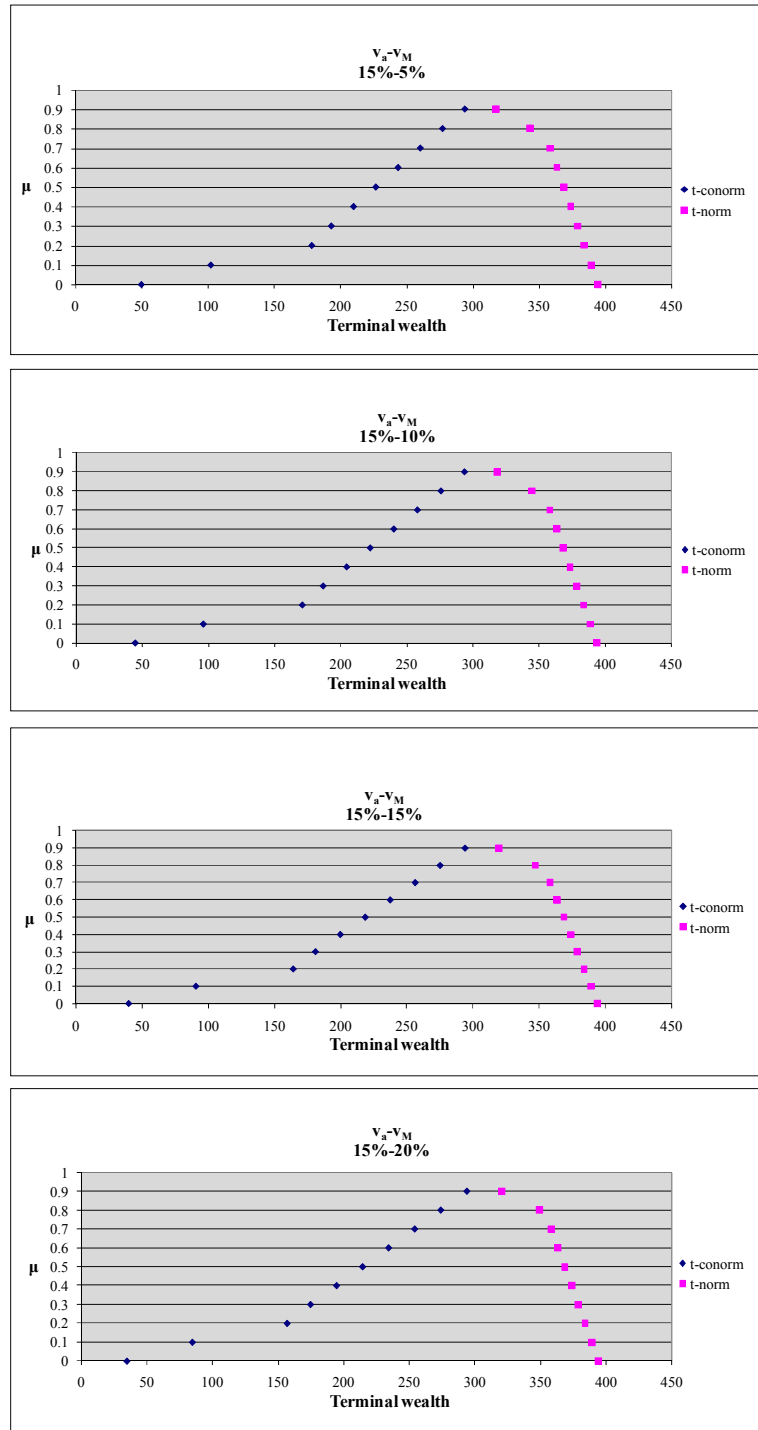


Figure A.3 : Fuzzy numbers representing terminal wealth in LP model (continued)

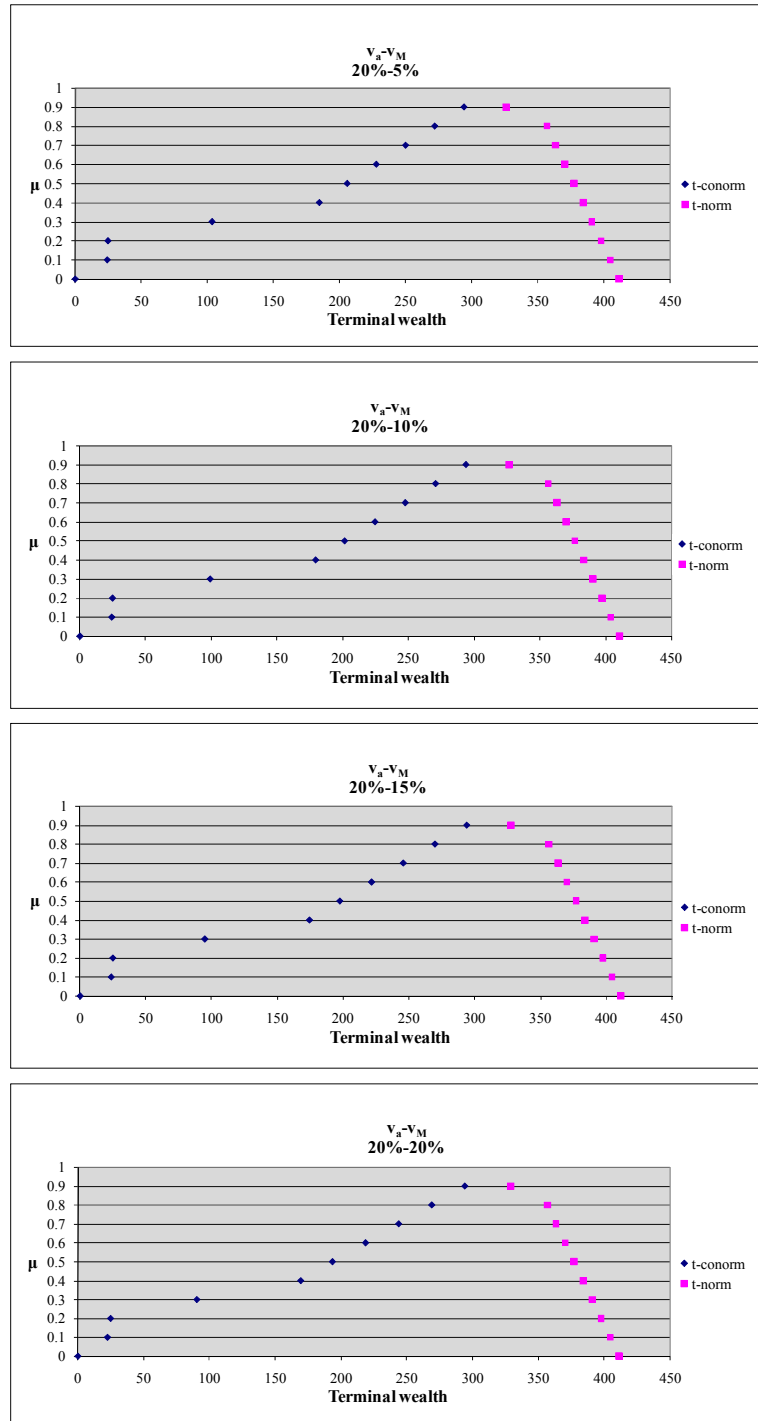


Figure A.3 : Fuzzy numbers representing terminal wealth in LP model (continued)

Table A.3 : Defuzzified interval of terminal wealth in quadratic programming model

5%-5%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	193.869	230.935	212.835	252.040	226.614	267.371
	0.003	193.678	230.715	212.660	251.833	226.459	267.186
	0.006	193.631	230.660	212.616	251.781	226.420	267.140
10^{-6}	0.001	196.350	233.827	215.105	254.719	228.607	269.761
	0.003	194.527	231.703	213.439	252.753	227.146	268.010
	0.006	194.057	231.157	213.009	252.245	226.767	267.555
10^{-5}	0.001	214.818	255.336	231.185	273.700	242.210	286.075
	0.003	202.133	240.344	220.293	260.843	233.090	275.139
	0.006	198.094	235.632	216.683	256.581	229.982	271.411

5%-10%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	191.301	233.789	210.024	255.163	223.626	270.690
	0.003	191.114	233.564	209.853	254.952	223.476	270.502
	0.006	191.067	233.508	209.809	254.899	223.437	270.454
10^{-6}	0.001	193.736	236.732	212.248	257.894	225.574	273.131
	0.003	191.946	234.570	210.616	255.890	224.147	271.343
	0.006	191.486	234.014	210.194	255.372	223.777	270.878
10^{-5}	0.001	211.858	258.624	227.998	277.241	238.869	289.788
	0.003	199.411	243.221	217.329	264.135	229.956	278.621
	0.006	195.446	238.494	213.794	259.793	226.918	274.814

5%-15%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	188.733	236.643	207.213	258.286	220.639	274.009
	0.003	188.549	236.414	207.045	258.071	220.492	273.817
	0.006	188.503	236.356	207.003	258.017	220.455	273.768
10^{-6}	0.001	191.122	239.637	209.390	261.068	222.542	276.500
	0.003	189.366	237.437	207.793	259.027	221.148	274.675
	0.006	188.915	236.871	207.380	258.499	220.786	274.201
10^{-5}	0.001	208.898	261.905	224.811	280.781	235.529	293.499
	0.003	196.688	246.103	214.366	267.425	226.822	282.103
	0.006	192.799	241.357	210.904	263.004	223.855	278.218

**Table A.3 : Defuzzified interval of terminal wealth in quadratic programming model
(continued)**

5%-20%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
	c_3	10		13		16	
c_1	c_2						
10^{-7}	0.001	186.165	239.496	204.402	261.409	217.652	277.329
	0.003	185.985	239.263	204.238	261.191	217.508	277.132
	0.006	185.940	239.205	204.197	261.135	217.472	277.082
10^{-6}	0.001	188.507	242.541	206.533	264.244	219.510	279.869
	0.003	186.786	240.304	204.970	262.164	218.148	278.008
	0.006	186.343	239.728	204.566	261.626	217.795	277.524
10^{-5}	0.001	205.938	265.200	221.625	284.322	232.188	297.211
	0.003	193.966	248.987	211.402	270.721	223.688	285.586
	0.006	190.152	244.221	208.015	266.211	220.791	281.622

10%-5%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
	c_3	10		13		16	
c_1	c_2						
10^{-7}	0.001	182.408	251.827	200.938	274.185	214.401	290.429
	0.003	182.224	251.587	200.770	273.960	214.253	290.226
	0.006	182.178	251.527	200.727	273.903	214.215	290.174
10^{-6}	0.001	184.806	254.957	203.125	277.105	216.313	293.052
	0.003	183.044	252.657	201.521	274.963	214.912	291.130
	0.006	182.590	252.065	201.106	274.409	214.548	290.631
10^{-5}	0.001	202.654	278.018	218.614	297.789	229.363	310.960
	0.003	190.396	261.363	208.122	283.778	220.614	298.955
	0.006	186.491	256.567	204.646	279.077	217.632	294.861

10%-10%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
	c_3	10		13		16	
c_1	c_2						
10^{-7}	0.001	179.840	254.680	198.128	277.308	211.414	293.748
	0.003	179.660	254.436	197.963	277.079	211.269	293.541
	0.006	179.615	254.375	197.921	277.021	211.232	293.489
10^{-6}	0.001	182.192	257.862	200.268	280.280	213.280	296.422
	0.003	180.464	255.524	198.697	278.099	211.912	294.462
	0.006	180.019	254.922	198.292	277.536	211.557	293.954
10^{-5}	0.001	199.694	280.965	215.427	301.330	226.022	314.671
	0.003	187.673	264.244	205.159	287.048	217.480	302.437
	0.006	183.843	259.430	201.756	282.217	214.568	298.266

**Table A.3 : Defuzzified interval of terminal wealth in quadratic programming model
(continued)**

10%-15%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	177.272	257.533	195.317	280.431	208.426	297.067
	0.003	177.095	257.285	195.155	280.198	208.286	296.856
	0.006	177.051	257.223	195.115	280.140	208.250	296.803
10^{-6}	0.001	179.577	260.767	197.410	283.455	210.248	299.791
	0.003	177.883	258.391	195.874	281.236	208.913	297.795
	0.006	177.448	257.779	195.477	280.663	208.567	297.277
10^{-5}	0.001	196.734	283.920	212.240	304.871	222.682	318.383
	0.003	184.950	267.125	202.195	290.256	214.346	305.918
	0.006	181.197	262.293	198.866	285.349	211.505	301.670

10%-20%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	174.704	260.386	192.506	283.554	205.439	300.386
	0.003	174.531	260.135	192.348	283.317	205.302	300.172
	0.006	174.487	260.072	192.308	283.257	205.267	300.117
10^{-6}	0.001	176.963	263.672	194.553	286.629	207.216	303.160
	0.003	175.303	261.258	193.051	284.373	205.914	301.127
	0.006	174.876	260.637	192.663	283.790	205.576	300.599
10^{-5}	0.001	193.774	286.869	209.053	308.412	219.341	322.095
	0.003	182.228	270.006	199.231	293.414	211.212	309.398
	0.006	178.550	265.157	195.977	288.484	208.441	305.073

15%-5%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	160.600	272.717	171.514	296.330	179.443	313.486
	0.003	164.437	272.458	179.723	296.087	189.761	313.265
	0.006	167.788	272.394	183.272	296.025	194.760	313.209
10^{-6}	0.001	162.051	276.087	172.847	299.491	180.620	316.343
	0.003	164.776	273.611	179.861	297.172	190.606	314.250
	0.006	167.917	272.973	183.689	296.573	195.132	313.706
10^{-5}	0.001	172.848	299.350	182.330	321.879	189.206	335.843
	0.003	170.235	282.382	184.448	306.237	194.760	322.770
	0.006	170.450	277.503	186.034	301.276	197.217	318.312

**Table A.3 : Defuzzified interval of terminal wealth in quadratic programming model
(continued)**

15%-10%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	156.833	275.570	167.747	299.454	175.676	316.805
	0.003	161.846	275.308	176.845	299.206	186.669	316.580
	0.006	165.198	275.242	180.387	299.144	191.675	316.523
10^{-6}	0.001	158.284	278.992	169.080	302.666	176.853	319.712
	0.003	162.181	276.478	176.998	300.309	187.491	317.582
	0.006	165.325	275.830	180.787	299.700	192.032	317.029
10^{-5}	0.001	169.080	302.305	178.540	325.416	185.266	339.553
	0.003	167.220	285.262	181.515	309.386	191.634	326.253
	0.006	167.807	280.366	183.139	304.409	194.125	321.715

15%-15%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	153.066	278.385	163.980	302.532	171.909	320.075
	0.003	159.256	278.135	173.618	302.301	183.584	319.869
	0.006	162.586	278.073	177.501	302.242	188.589	319.817
10^{-6}	0.001	154.517	281.638	165.313	305.533	173.086	322.737
	0.003	159.586	279.248	174.075	303.331	184.374	320.786
	0.006	162.731	278.632	177.887	302.762	188.931	320.280
10^{-5}	0.001	165.313	305.251	174.779	326.795	181.354	340.910
	0.003	164.485	288.144	178.583	312.386	188.507	328.728
	0.006	165.183	283.229	180.244	307.541	191.032	324.574

15%-20%		Worst-case	Best-case	Worst-case	Best-case	Worst-case	Best-case
		c_3	10	13		16	
c_1	c_2						
10^{-7}	0.001	149.299	281.195	160.213	305.606	168.142	323.340
	0.003	156.666	280.961	170.361	305.393	180.497	323.155
	0.006	159.590	280.902	174.616	305.339	185.504	323.108
10^{-6}	0.001	150.750	284.258	161.546	308.370	169.319	325.727
	0.003	156.991	282.008	171.041	306.342	181.258	323.978
	0.006	159.989	281.429	174.986	305.818	185.831	323.523
10^{-5}	0.001	161.546	307.032	171.014	327.951	177.387	342.027
	0.003	161.824	291.020	175.653	314.687	185.381	331.100
	0.006	162.559	286.092	177.349	310.292	187.939	327.375

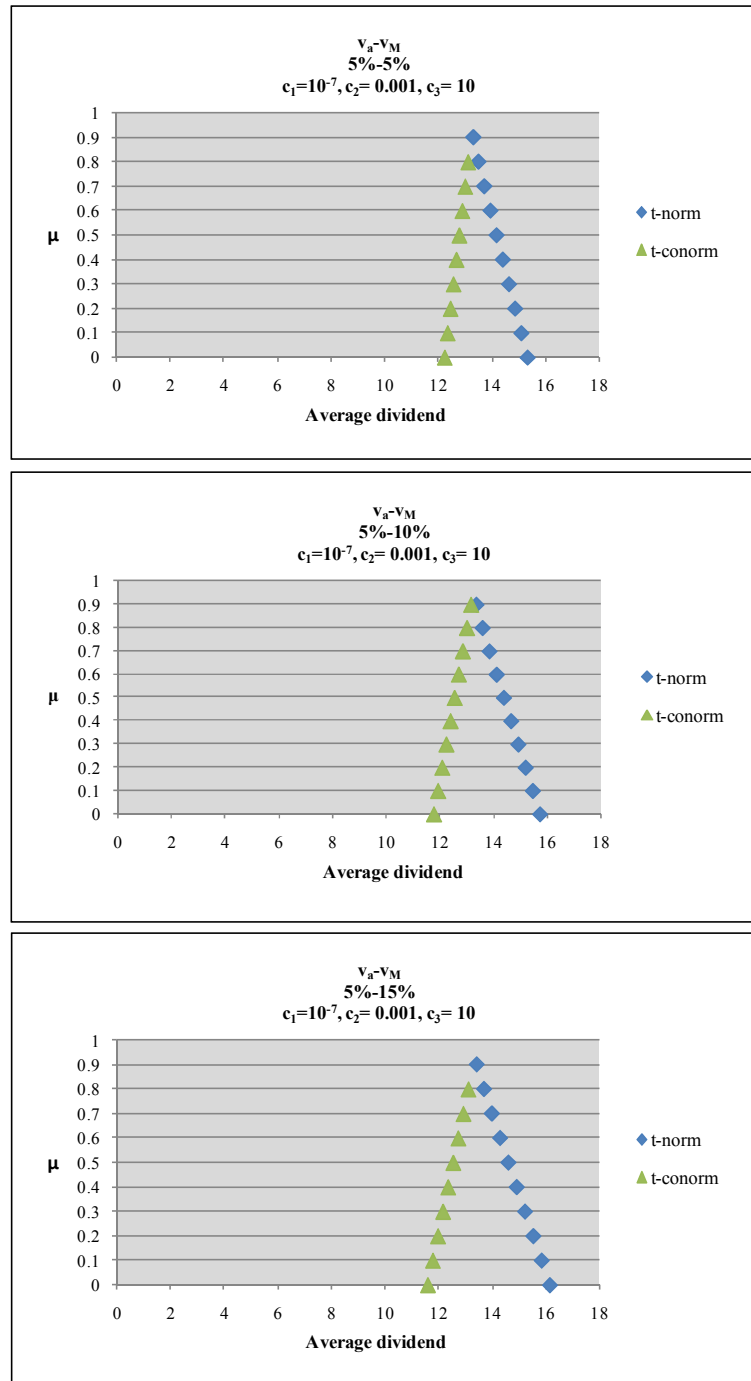


Figure A.4 : Fuzzy numbers representing average dividend in quadratic programming model

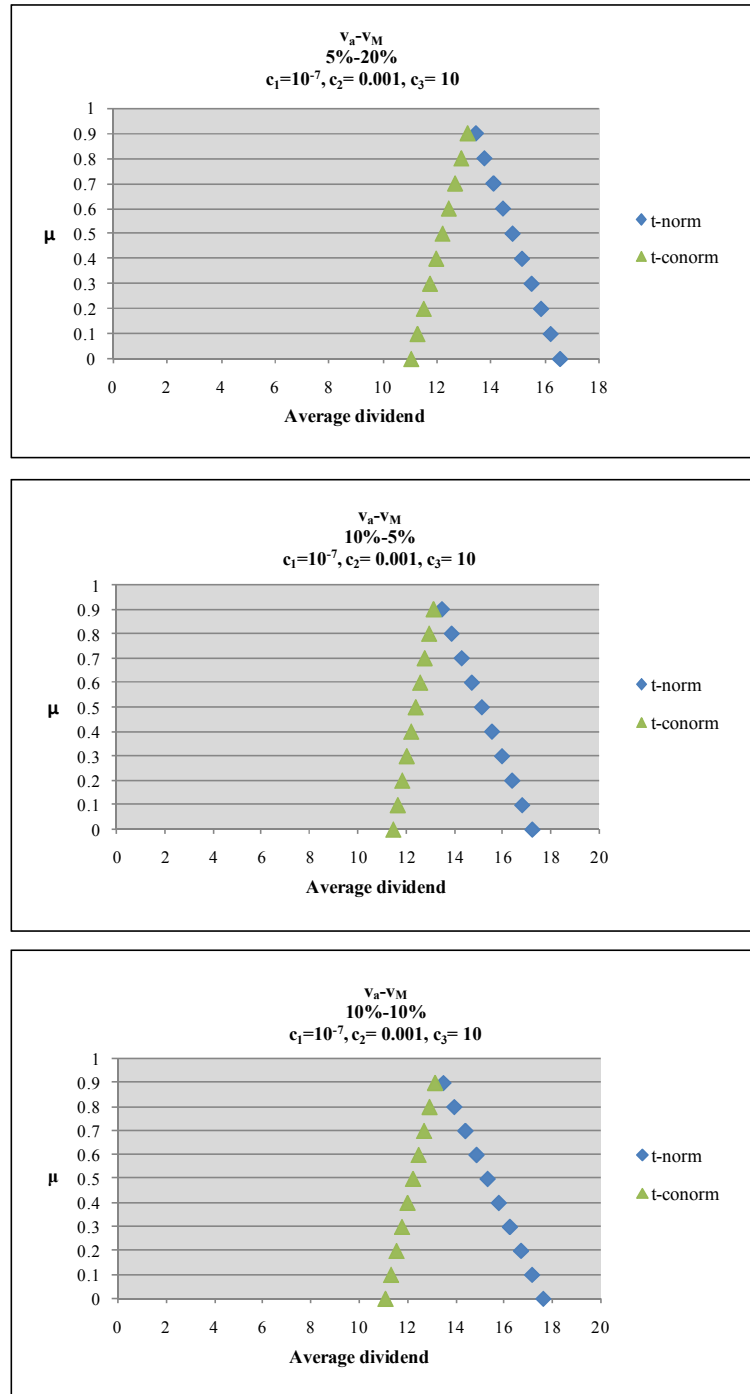


Figure A.4 : Fuzzy numbers representing average dividend in quadratic programming model (continued)

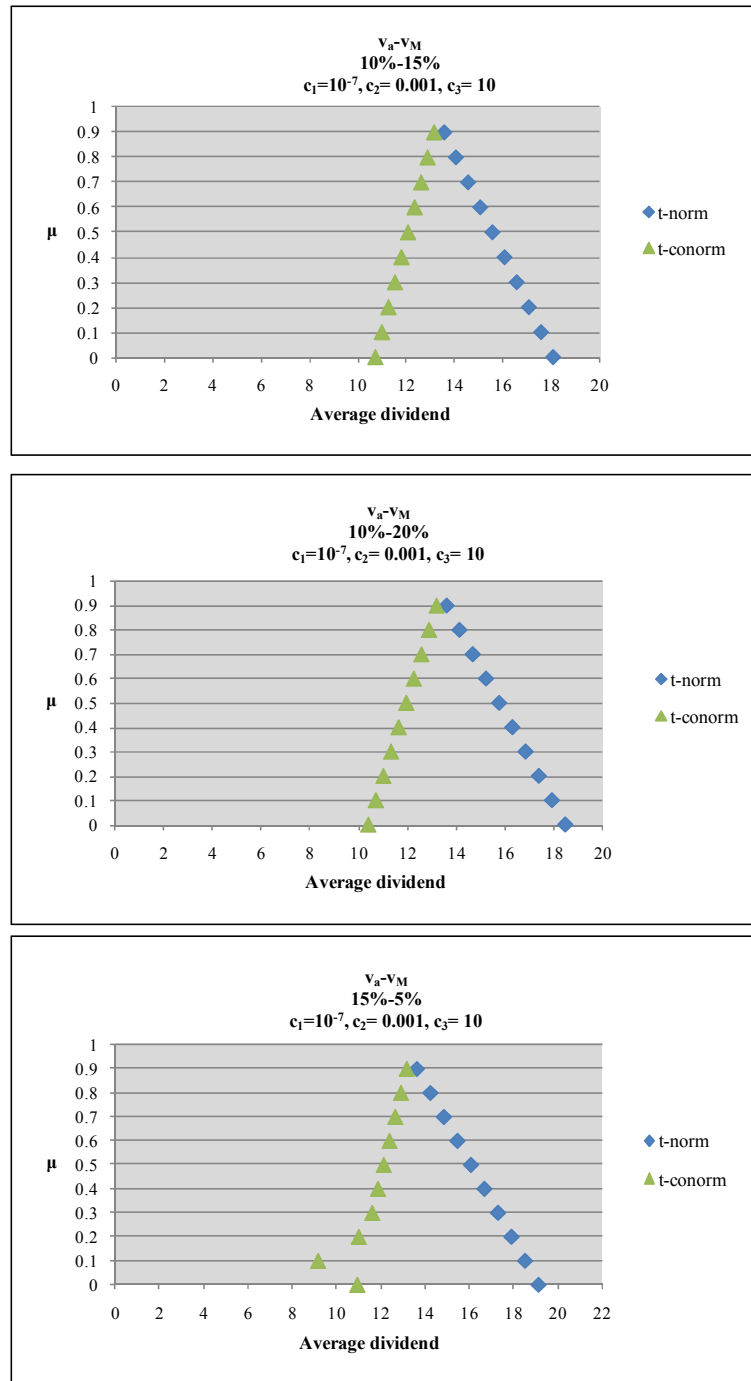


Figure A.4 : Fuzzy numbers representing average dividend in quadratic programming model (continued)

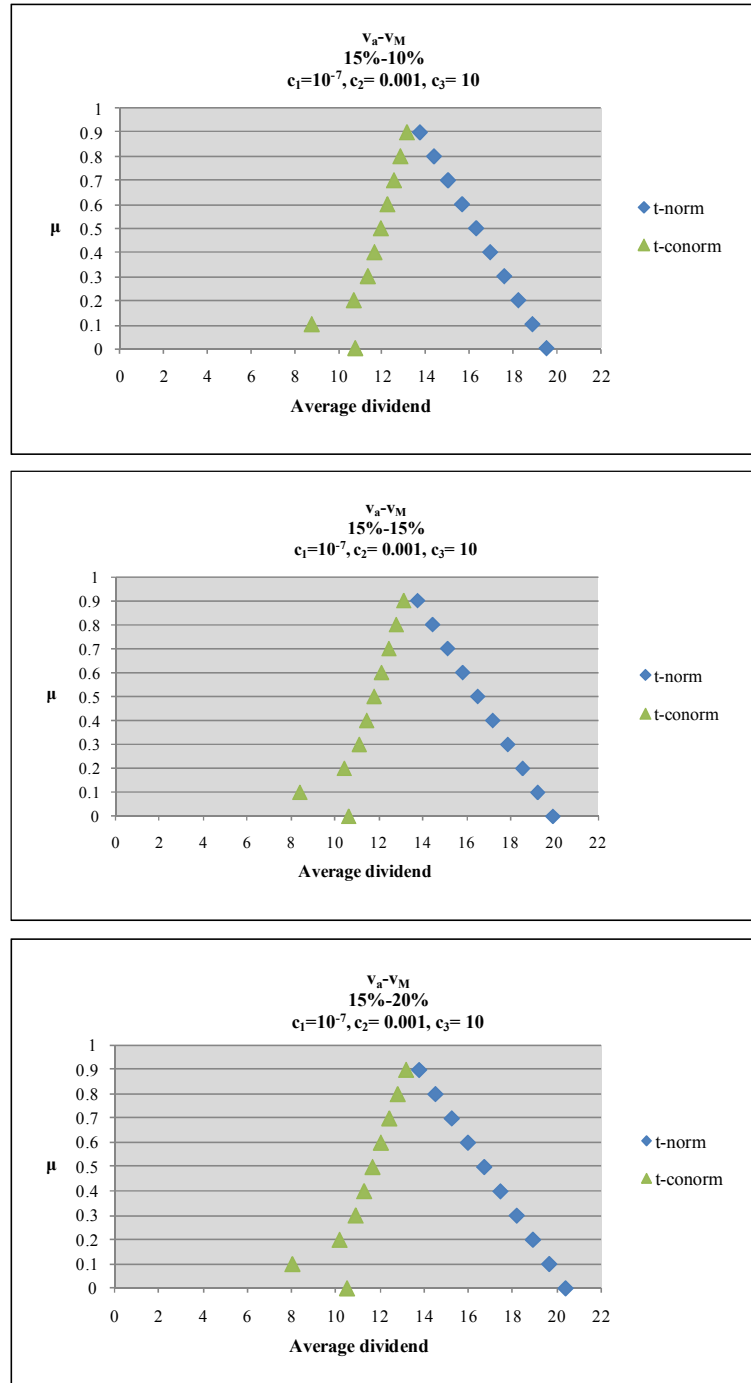


Figure A.4 : Fuzzy numbers representing average dividend in quadratic programming model (continued)

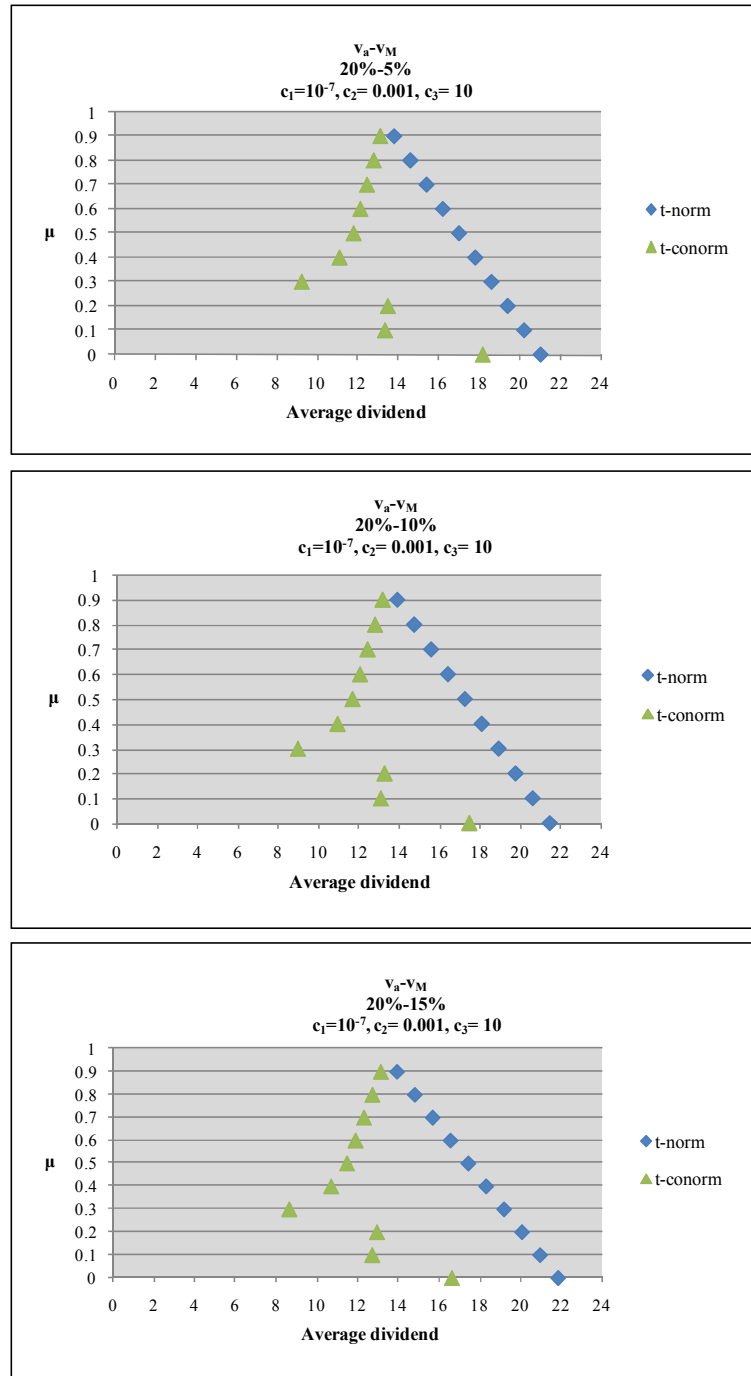


Figure A.4 : Fuzzy numbers representing average dividend in quadratic programming model (continued)

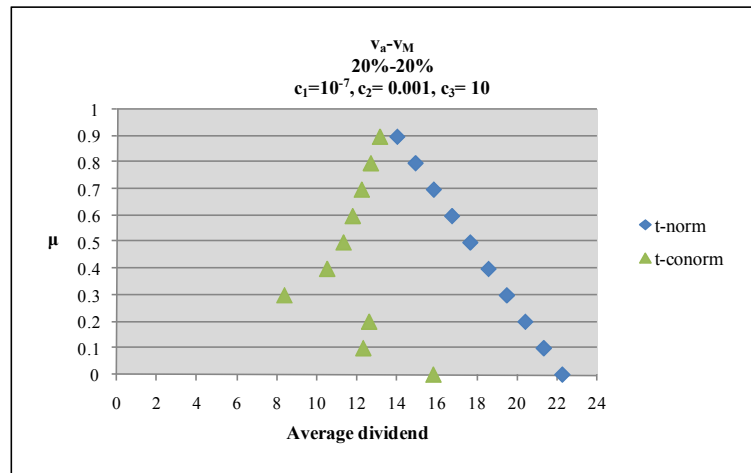


Figure A.4 : Fuzzy numbers representing average dividend in quadratic programming model (continued)

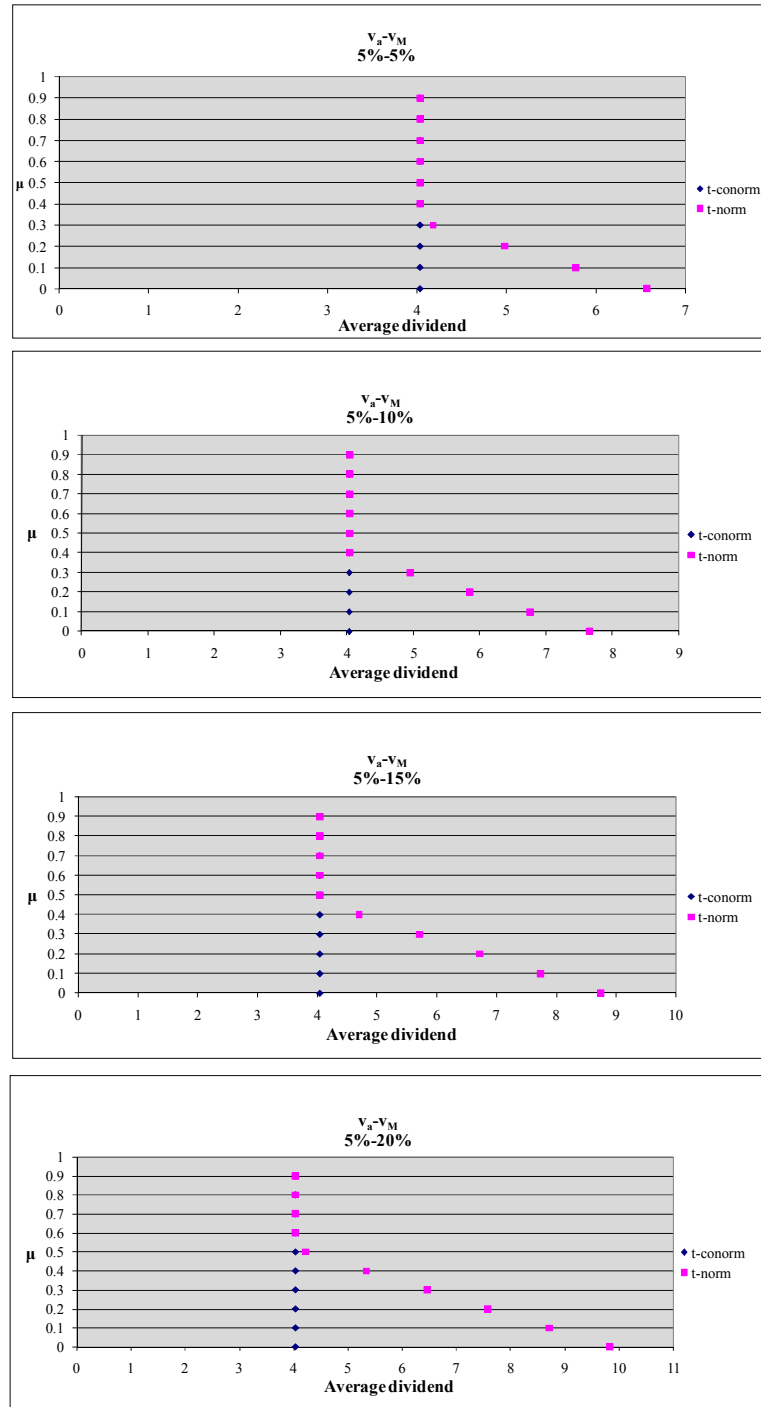


Figure A.5 : Fuzzy numbers representing average dividend in LP model

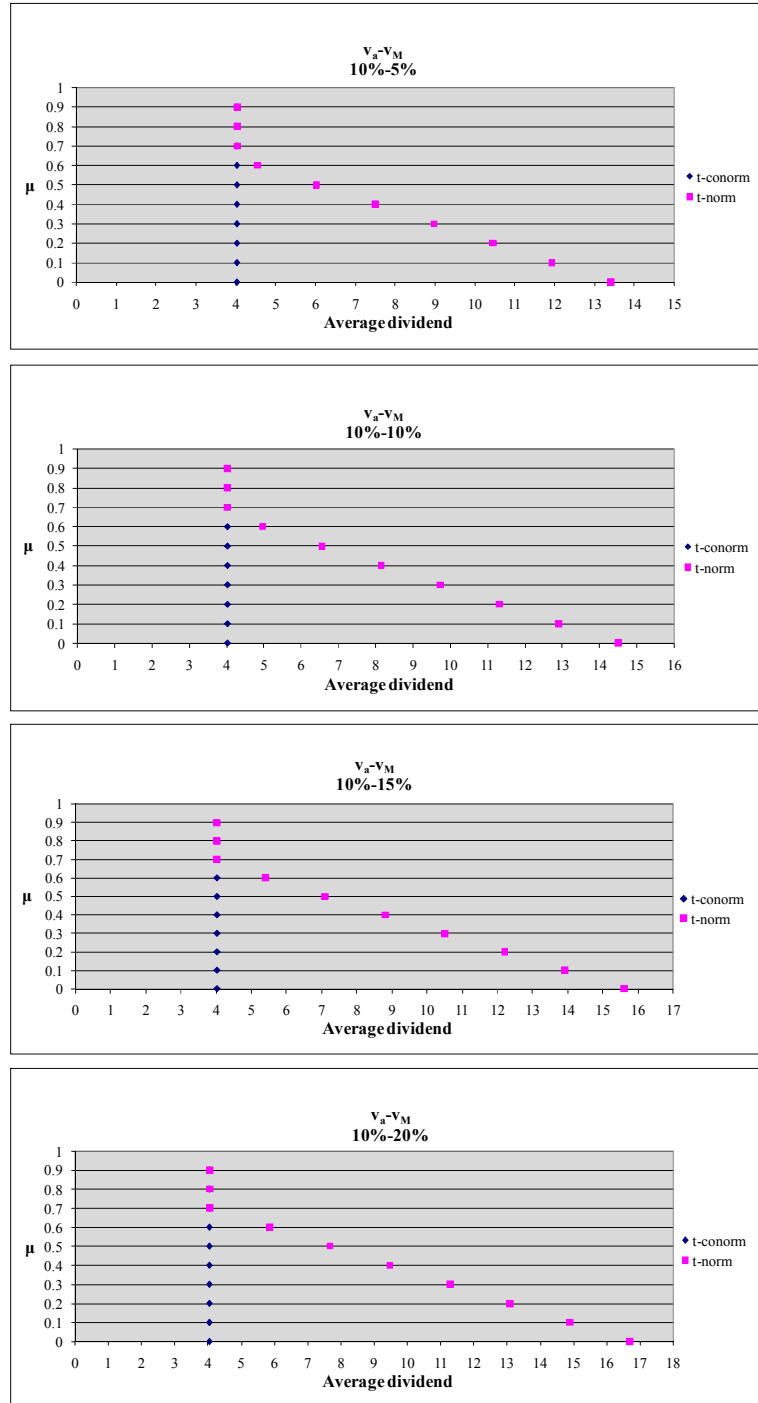


Figure A.5 : Fuzzy numbers representing average dividend in LP model (continued)

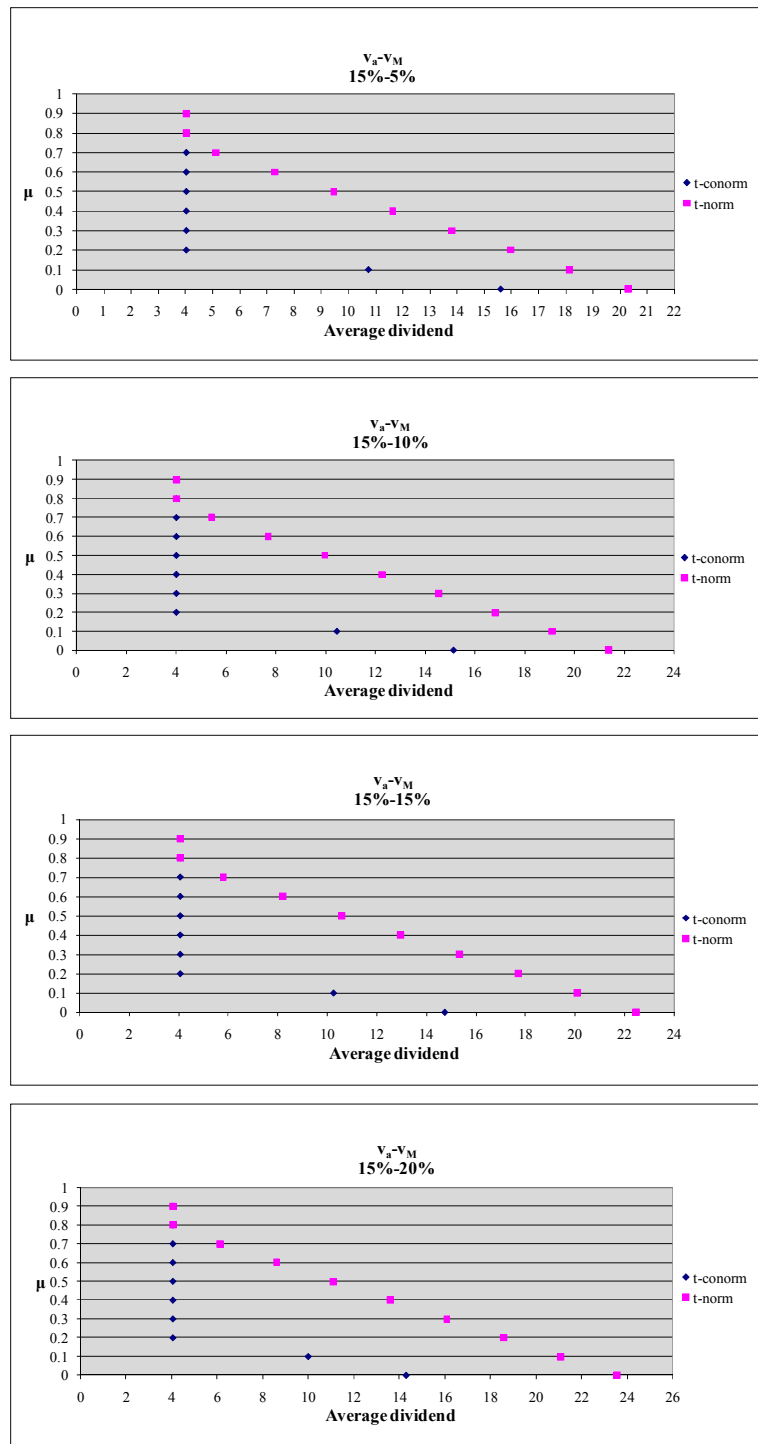


Figure A.5 : Fuzzy numbers representing average dividend in LP model (continued)

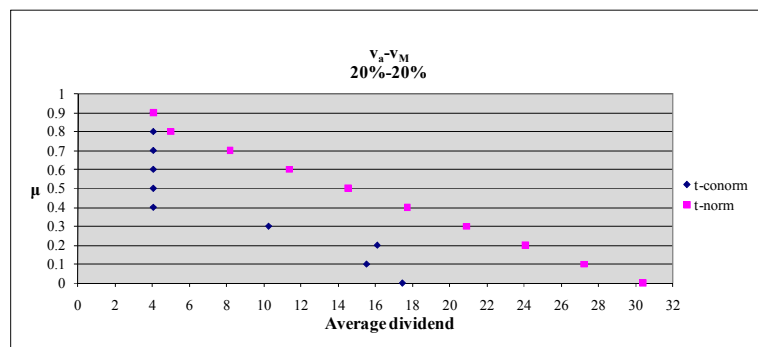
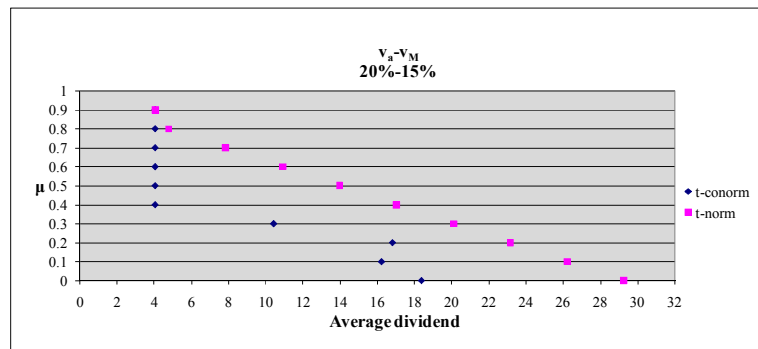
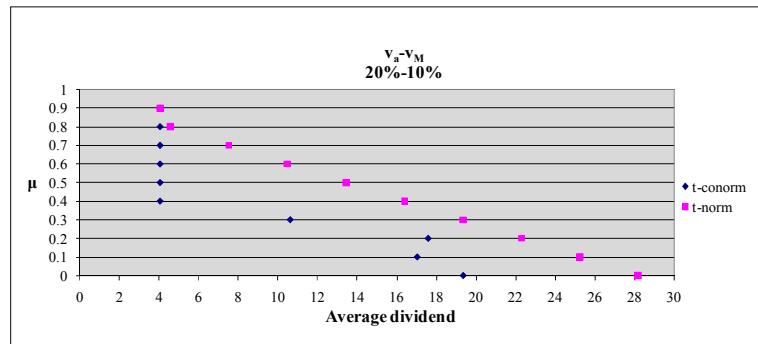
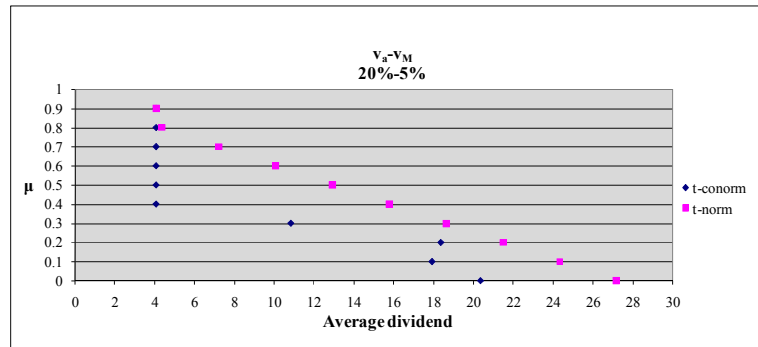


Figure A.5 : Fuzzy numbers representing average dividend in LP model (continued)

Table A.4 : Defuzzified average dividend in quadratic programming model

5%-5%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.088	11.860	10.242
	0.003	14.111	11.882	10.261
	0.006	14.117	11.887	10.266
10^{-6}	0.001	13.167	11.062	9.547
	0.003	13.376	11.255	9.718
	0.006	13.430	11.305	9.762
10^{-5}	0.001	11.042	9.198	7.956
	0.003	12.519	10.461	9.023
	0.006	12.984	10.879	9.386

5%-10%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.079	11.873	10.535
	0.003	14.102	11.895	10.556
	0.006	14.108	11.901	9.164
10^{-6}	0.001	12.311	10.439	9.997
	0.003	12.512	10.621	10.152
	0.006	12.564	10.668	6.485
10^{-5}	0.001	10.308	8.696	8.733
	0.003	11.704	9.873	9.519
	0.006	12.141	10.266	6.232

5%-15%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.201	11.977	10.692
	0.003	14.225	11.999	10.713
	0.006	14.231	12.005	9.266
10^{-6}	0.001	12.308	10.434	10.039
	0.003	12.509	10.616	10.235
	0.006	12.561	10.663	6.523
10^{-5}	0.001	10.310	8.695	8.595
	0.003	11.714	9.870	9.482
	0.006	12.145	10.262	6.268

Table A.4 : Defuzzified average dividend in quadratic programming model
(continued)

5%-20%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.320	12.077	10.849
	0.003	14.343	12.100	10.870
	0.006	14.349	12.105	9.368
10^{-6}	0.001	12.306	10.430	10.198
	0.003	12.506	10.612	10.416
	0.006	12.558	10.659	6.567
10^{-5}	0.001	10.311	8.695	8.490
	0.003	11.724	9.867	9.566
	0.006	12.149	10.259	6.307

10%-5%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.608	12.320	11.110
	0.003	14.632	12.343	11.133
	0.006	14.639	12.349	9.538
10^{-6}	0.001	12.682	10.752	10.350
	0.003	12.892	10.943	10.574
	0.006	12.946	10.993	6.649
10^{-5}	0.001	10.609	8.926	8.600
	0.003	12.091	10.159	9.705
	0.006	12.528	10.576	6.382

10%-10%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.722	12.416	11.281
	0.003	14.747	12.440	11.305
	0.006	14.753	12.445	9.641
10^{-6}	0.001	12.682	10.749	10.536
	0.003	12.891	10.940	10.759
	0.006	12.945	10.989	6.704
10^{-5}	0.001	10.641	8.927	8.570
	0.003	12.105	10.160	9.863
	0.006	12.535	10.579	6.432

Table A.4 : Defuzzified average dividend in quadratic programming model
(continued)

10%-15%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.834	12.511	11.506
	0.003	14.859	12.534	11.534
	0.006	14.865	12.540	9.744
10^{-6}	0.001	12.682	10.747	10.692
	0.003	12.891	10.937	10.900
	0.006	12.945	10.986	6.763
10^{-5}	0.001	10.673	8.928	8.621
	0.003	12.118	10.165	10.032
	0.006	12.541	10.583	6.486
10%-20%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	14.943	12.602	12.015
	0.003	14.968	12.626	12.072
	0.006	14.974	12.632	9.848
10^{-6}	0.001	12.683	10.744	10.791
	0.003	12.891	10.933	10.982
	0.006	12.944	10.983	6.825
10^{-5}	0.001	10.705	8.928	8.739
	0.003	12.132	10.174	10.162
	0.006	12.548	10.587	6.544
15%-5%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	15.395	12.835	7.852
	0.003	15.253	12.942	11.146
	0.006	15.156	12.820	10.019
10^{-6}	0.001	14.007	12.470	10.928
	0.003	13.358	11.612	10.890
	0.006	12.957	11.118	6.936
10^{-5}	0.001	12.477	10.405	8.889
	0.003	12.691	10.896	10.189
	0.006	12.590	10.748	6.645

Table A.4 : Defuzzified average dividend in quadratic programming model
(continued)

15%-10%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	15.525	13.000	11.011
	0.003	15.356	13.034	11.317
	0.006	15.255	12.904	10.123
10^{-6}	0.001	14.076	12.530	10.933
	0.003	13.339	11.593	10.836
	0.006	12.938	11.099	7.007
10^{-5}	0.001	12.556	10.520	9.030
	0.003	12.680	10.884	10.154
	0.006	12.577	10.733	6.710
15%-15%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	15.656	13.149	11.333
	0.003	15.467	13.123	11.433
	0.006	15.351	12.986	10.227
10^{-6}	0.001	14.164	12.611	10.930
	0.003	13.333	11.582	10.778
	0.006	12.922	11.083	7.084
10^{-5}	0.001	12.631	10.828	9.362
	0.003	12.669	10.883	10.172
	0.006	12.564	10.719	6.810
15%-20%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	15.786	13.288	11.496
	0.003	15.578	13.210	11.529
	0.006	15.445	13.067	10.332
10^{-6}	0.001	14.255	12.692	10.908
	0.003	13.335	11.572	10.722
	0.006	12.906	11.068	7.164
10^{-5}	0.001	12.796	11.127	9.615
	0.003	12.658	10.940	10.186
	0.006	12.551	10.734	6.918

Table A.4 : Defuzzified average dividend in quadratic programming model
(continued)

20%-5%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	17.392	14.649	12.655
	0.003	19.055	14.665	12.744
	0.006	16.085	12.886	10.504
10^{-6}	0.001	15.762	13.093	11.319
	0.003	16.160	13.478	11.788
	0.006	15.206	13.410	7.307
10^{-5}	0.001	13.629	11.649	10.141
	0.003	15.428	12.680	11.280
	0.006	14.894	13.112	7.164

20%-10%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	17.664	14.847	12.824
	0.003	30.341	14.877	13.064
	0.006	16.220	13.318	10.608
10^{-6}	0.001	16.095	13.219	11.501
	0.003	16.266	13.687	11.929
	0.006	15.220	13.450	7.394
10^{-5}	0.001	14.069	11.794	10.434
	0.003	15.652	13.008	11.521
	0.006	14.938	13.191	7.286

20%-15%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	18.053	15.041	12.996
	0.003	12.751	15.125	13.675
	0.006	16.348	13.581	10.713
10^{-6}	0.001	16.374	13.466	11.717
	0.003	16.353	13.904	12.035
	0.006	15.232	13.483	7.483
10^{-5}	0.001	14.565	12.143	10.826
	0.003	15.820	13.347	11.722
	0.006	14.981	13.263	7.410

Table A.4 : Defuzzified average dividend in quadratic programming model
(continued)

20%-20%				
	c_3	10	13	16
c_1	c_2	Average Dividend		
10^{-7}	0.001	18.900	15.248	13.174
	0.003	15.837	15.526	13.173
	0.006	16.478	13.760	12.725
10^{-6}	0.001	16.562	13.776	11.687
	0.003	16.325	14.088	11.985
	0.006	15.311	13.533	12.085
10^{-5}	0.001	15.022	12.616	11.015
	0.003	15.921	13.639	11.707
	0.006	15.099	13.355	11.953

Table B.1 : Xpress-MP Code for robust optimization approach to capital rationing

```
model "Capital Rationing Problem"
uses "mmxprs"

declarations
J: integer !number of projects
N: integer !number of time periods
B: integer !number of time periods with uncertain budget
end-declarations

initializations from 'rationing.txt'
J N B
end-initializations

declarations
PROJECT = 1..J
TIME = 0..N
TIME_1=0..B
TIME_2=B+1..N
GAMMA_A=26..30
GAMMA_M=10
M: array(TIME) of real
A: array(TIME,PROJECT) of real
AUNCERTAINTY: array(TIME,PROJECT) of real
MUNCERTAINTY: array(TIME_1) of real
DISCOUNT: array(TIME) of real
NPV: array (PROJECT) of real
CHOICE: array(PROJECT) of mpvar
P_A: array(TIME,PROJECT) of mpvar
P_M: array(TIME_1) of mpvar
Z_A: array(TIME)of mpvar
Z_M: array(TIME_1)of mpvar
end-declarations

initializations from 'rationing22.txt'
M A AUNCERTAINTY MUNCERTAINTY DISCOUNT NPV
end-initializations

forall(a in GAMMA_A) do

!Define Objective function
Objective:= sum (j in PROJECT)NPV(j)*CHOICE(j)-
sum (i in TIME) ((sum (j in PROJECT)P_A(n,j)+(a/10)*Z_A (n))*DISCOUNT (n))
```

Table B.1 : Xpress-MP Code for robust optimization approach to capital rationing
(continued)

```

!Constraints
forall (n in TIME_1)
    sum (j in PROJECT) -A(n,j)*CHOICE(j)+ sum (j in PROJECT) P_A(n,j)+
(a/10)*Z_A (n)+P_M (n)+(GAMMA_M/10)*Z_M (n) <= M(n)

forall (n in TIME_2)
    sum (j in PROJECT) -A(n,j)*CHOICE(j)+ sum (j in PROJECT) P_A(n,j)+
(a/10)*Z_A (n) <= M(n)

forall (n in TIME, j in PROJECT)
    Z_A(n)+P_A(n,j)>= AUNCERTAINTY(n,j)*CHOICE(j)

forall (n in TIME_1)
    Z_M(n)+P_M(n)>=MUNCERTAINTY(n)

forall (n in TIME, j in PROJECT)do
    CHOICE(j)<=1
    CHOICE(j)>=0
    P_A(n,j)>= 0
    Z_A(n)>= 0
end-do

forall (n in TIME_1)do
    P_M(n)>=0
    Z_M(n)>=0
end-do

maximize(Objective)

! Solution printing
forall(j in PROJECT) writeln(getsol(CHOICE(j)))
forall(n in TIME, j in PROJECT) writeln(getsol(P_A (n,j)))
forall(n in TIME) writeln(getsol(Z_A (n)))
writeln(getobjval)
end-do
end-model

```

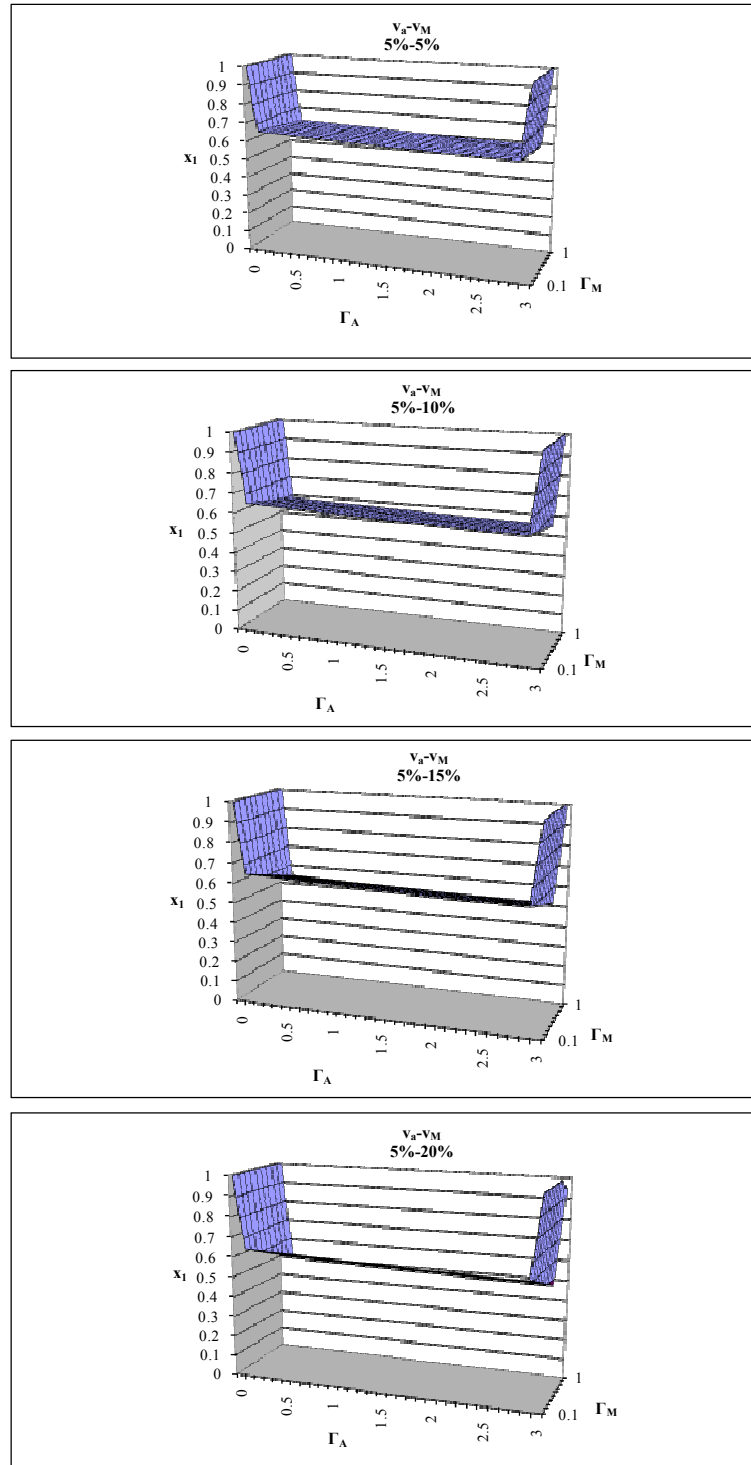


Figure B.1 : Level of investment in Project 1

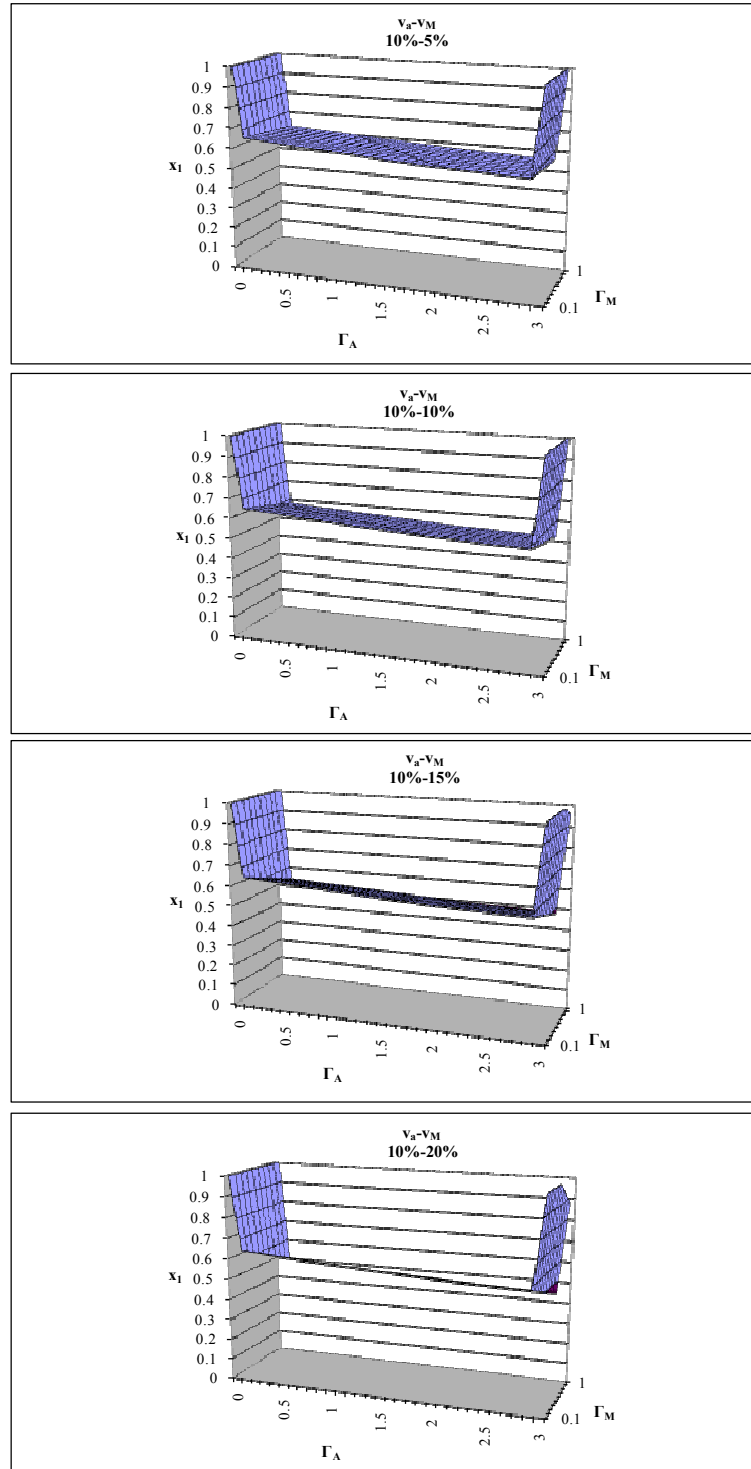


Figure B.1 : Level of investment in Project 1 (continued)

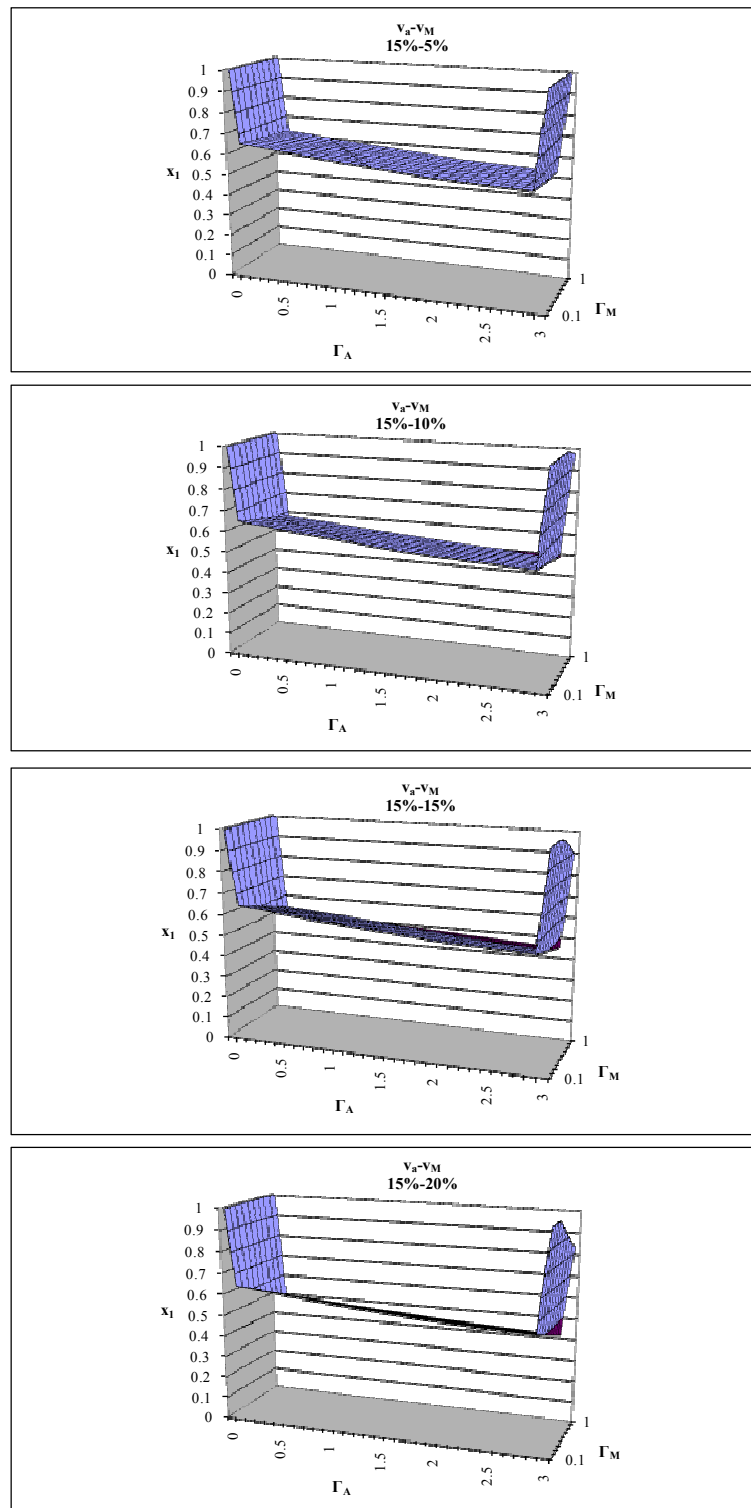


Figure B.1 : Level of investment in Project 1 (continued)

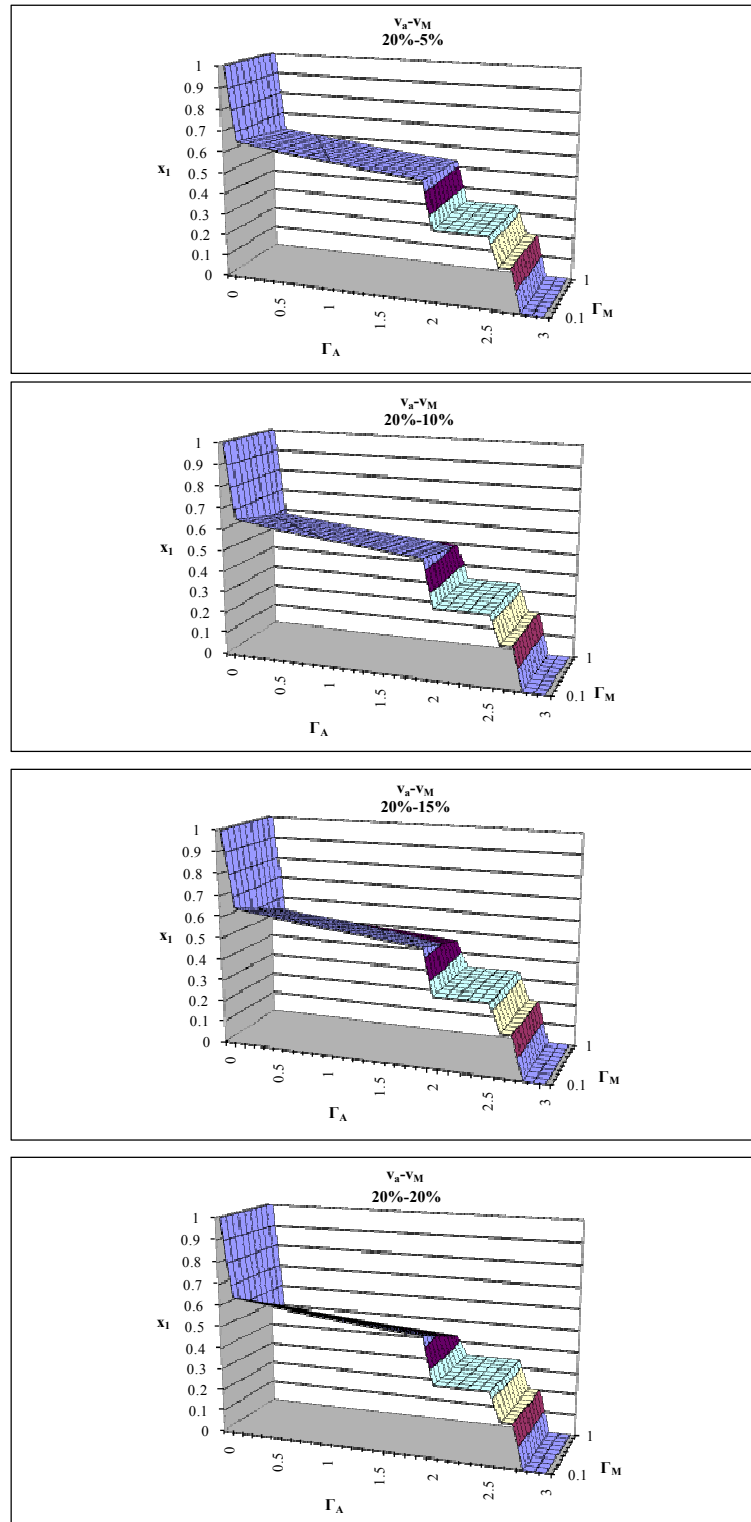


Figure B.1 : Level of investment in Project 1 (continued)

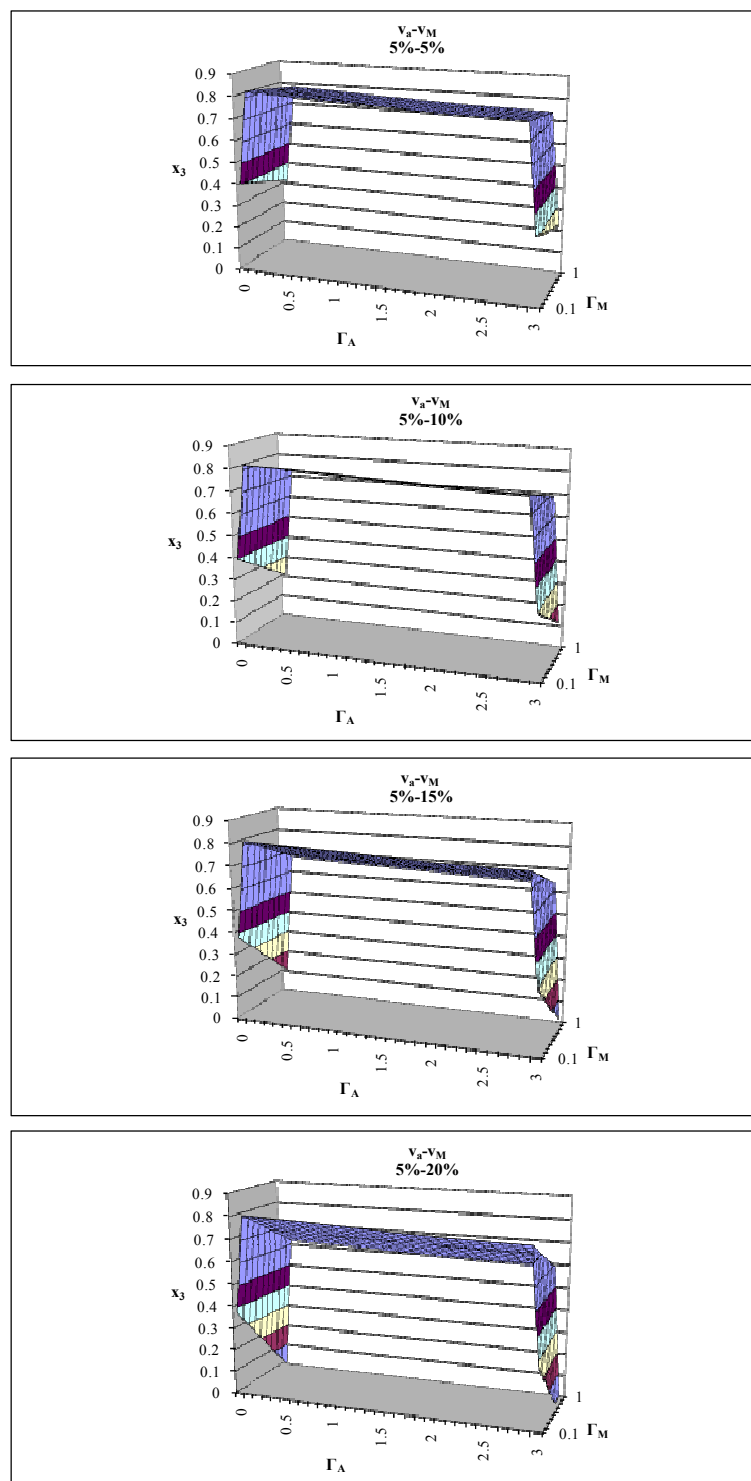


Figure B.2 : Level of investment in Project 3

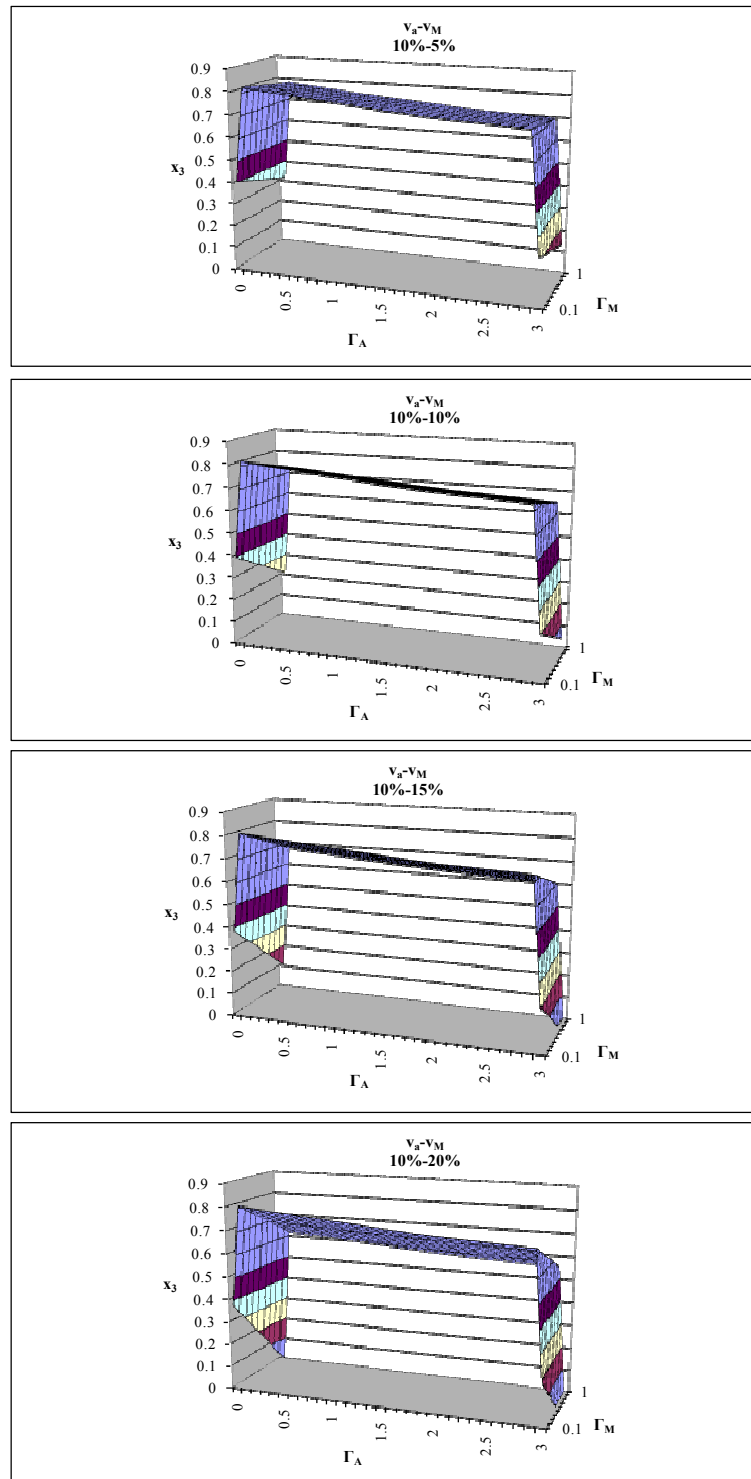


Figure B.2 : Level of investment in Project 3 (continued)

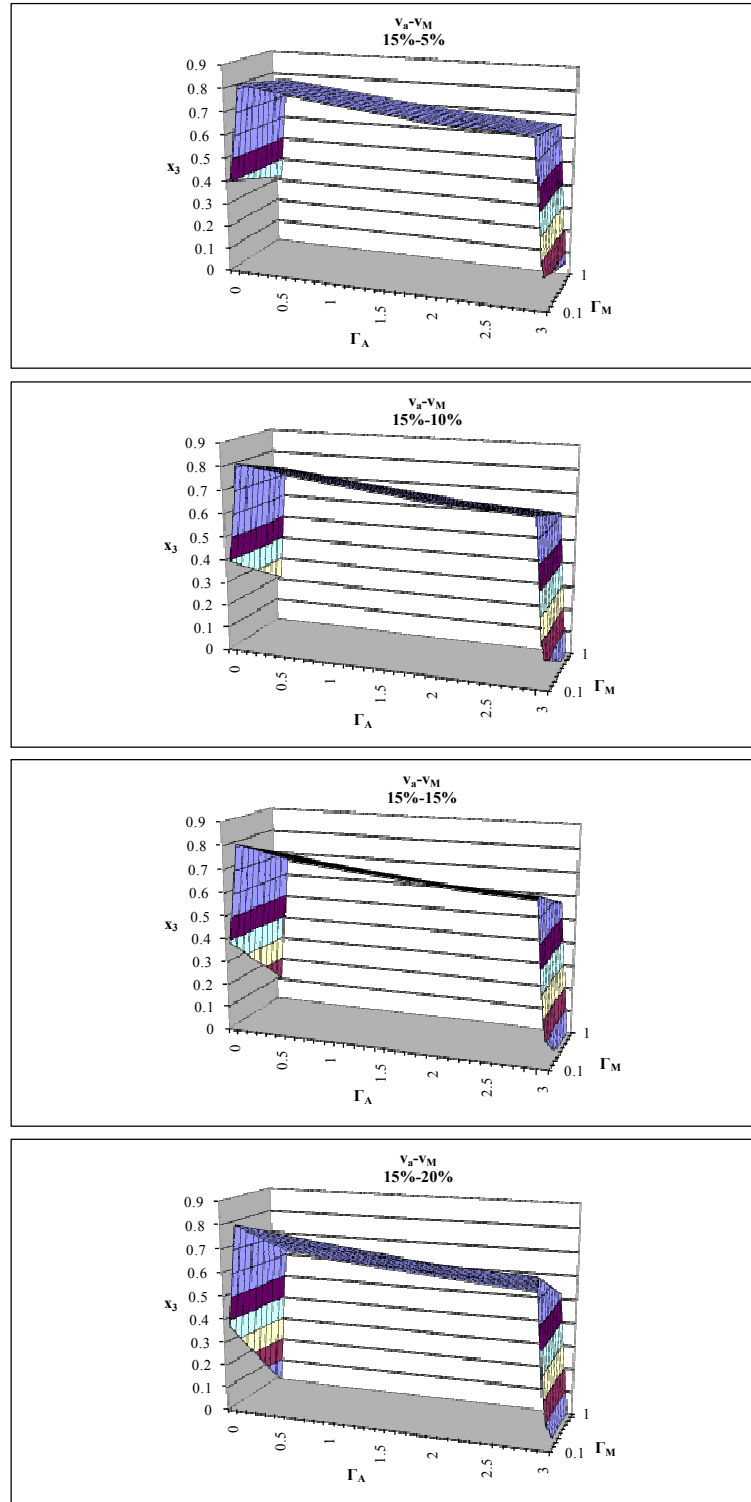


Figure B.2 : Level of investment in Project 3 (continued)

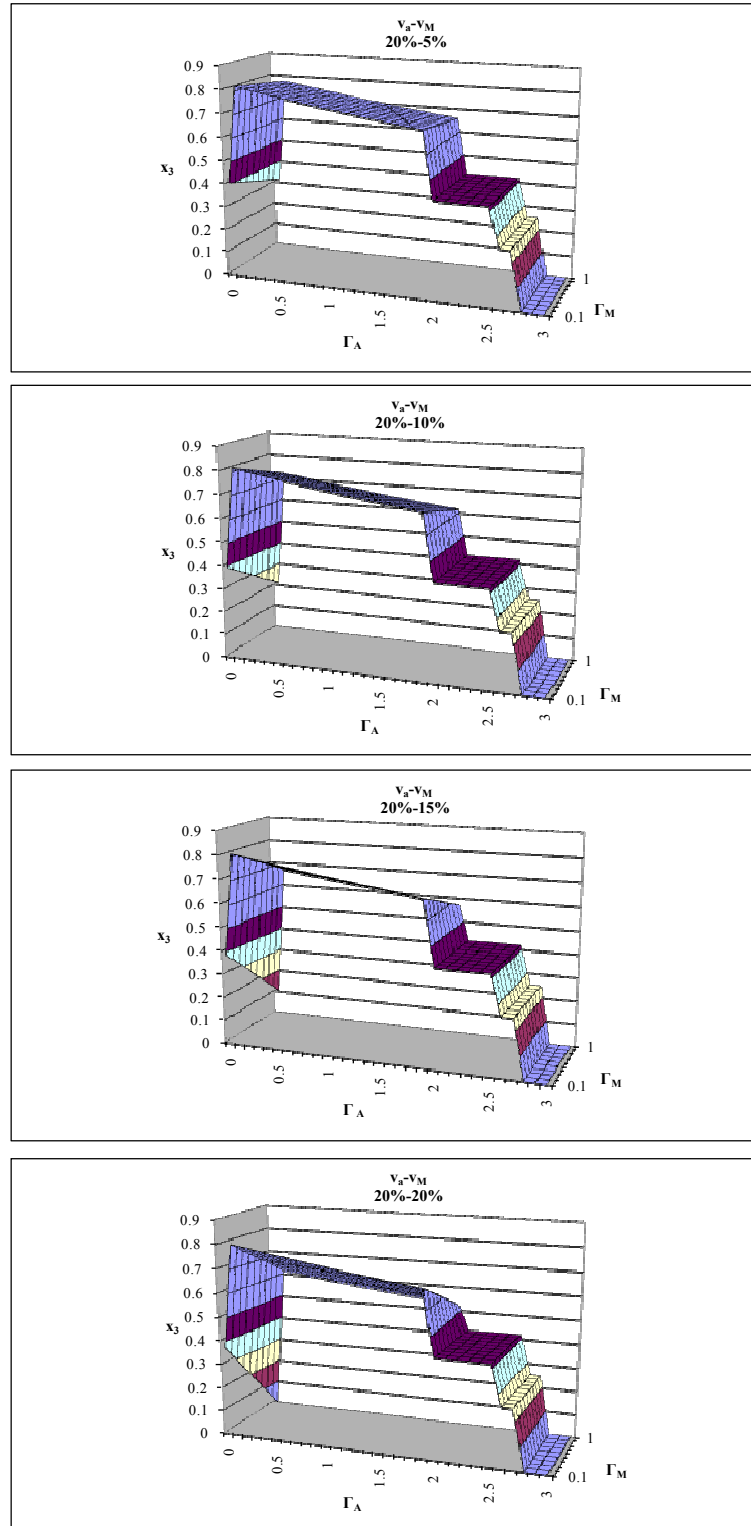


Figure B.2 : Level of investment in Project 3 (continued)

Table B.2 : Xpress-MP Code for robust optimization approach to horizon capital budgeting

```

model "Capital Budgeting Problem"
uses "mmxprs"

declarations
J: integer !number of projects
N: integer !number of time periods
rl:real
rb:real
B: integer !number of time periods with uncertain budget
end-declarations

initializations from 'budgeting.txt'
J N rl rb B
end-initializations

declarations
TIME = 0..N
TIME_1=1..B
TIME_2=B+1..N
PROJECT = 1..J
GAMMA_A=21..30
GAMMA_M=10
M: array(TIME) of real
A: array(TIME, PROJECT) of real
AUNCERTAINTY: array(TIME,PROJECT) of real
MUNCERTAINTY: array(TIME_1) of real
HORIZON: array(PROJECT) of real
CHOICE: array(PROJECT) of mpvar
P_A: array(TIME,PROJECT) of mpvar
Z_A: array(TIME)of mpvar
P_M: array(TIME_1) of mpvar
Z_M: array(TIME_1)of mpvar
v: array(TIME)of mpvar
w: array(TIME)of mpvar
end-declarations

initializations from 'budgeting.txt'
M A AUNCERTAINTY MUNCERTAINTY HORIZON
end-initializations

forall(a in GAMMA_A) do

```

Table B.2 : Xpress-MP Code for robust optimization approach to horizon capital budgeting

```

!Constraints
    sum (j in PROJECT) (-A(0,j)*CHOICE(j)+P_A(0,j))+ a/10*Z_A(0)+v(0)-w(0)
    <= M(0)
forall (n in TIME_1)
    sum (j in PROJECT) (-A(n,j)*CHOICE(j)+P_A(n,j))+
    a/10*Z_A(n)+P_M(n)+GAMMA_M/10*Z_M (n)+v(n)-(1+rl)*v(n-1)+(1+rb)*w(n-
    1)-w(n) <= M(n)
forall (n in TIME_2)
    sum (j in PROJECT) (-A(n,j)*CHOICE(j)+P_A(n,j))+ a/10*Z_A(n)+v(n)-
    (1+rl)*v(n-1)+(1+rb)*w(n-1)-w(n) <= M(n)

forall (n in TIME, j in PROJECT) do
    Z_A(n)+P_A(n,j)>= AUNCERTAINTY(n,j)*CHOICE(j)
    P_A(n,j)>= 0
    Z_A(n)>= 0
    CHOICE(j)<=1
    CHOICE(j)>=0
    v(n)>=0
    w(n)>=0
end-do

forall (n in TIME_1)do
    Z_M(n)+P_M(n)>=MUNCERTAINTY(n)
    Z_M(n)>=0
    P_M(n)>=0
end-do

!Define Objective function
Objective:= sum(j in PROJECT) HORIZON(j)*CHOICE(j)+v(N)-w(N)
maximize(Objective)

! Solution printing
forall(j in PROJECT) writeln(getsol(CHOICE(j)))
forall(i in TIME) writeln(getsol(v(i)))
forall(i in TIME) writeln(getsol(w(i)))
writeln(getobjval)
end-do

end-model

```

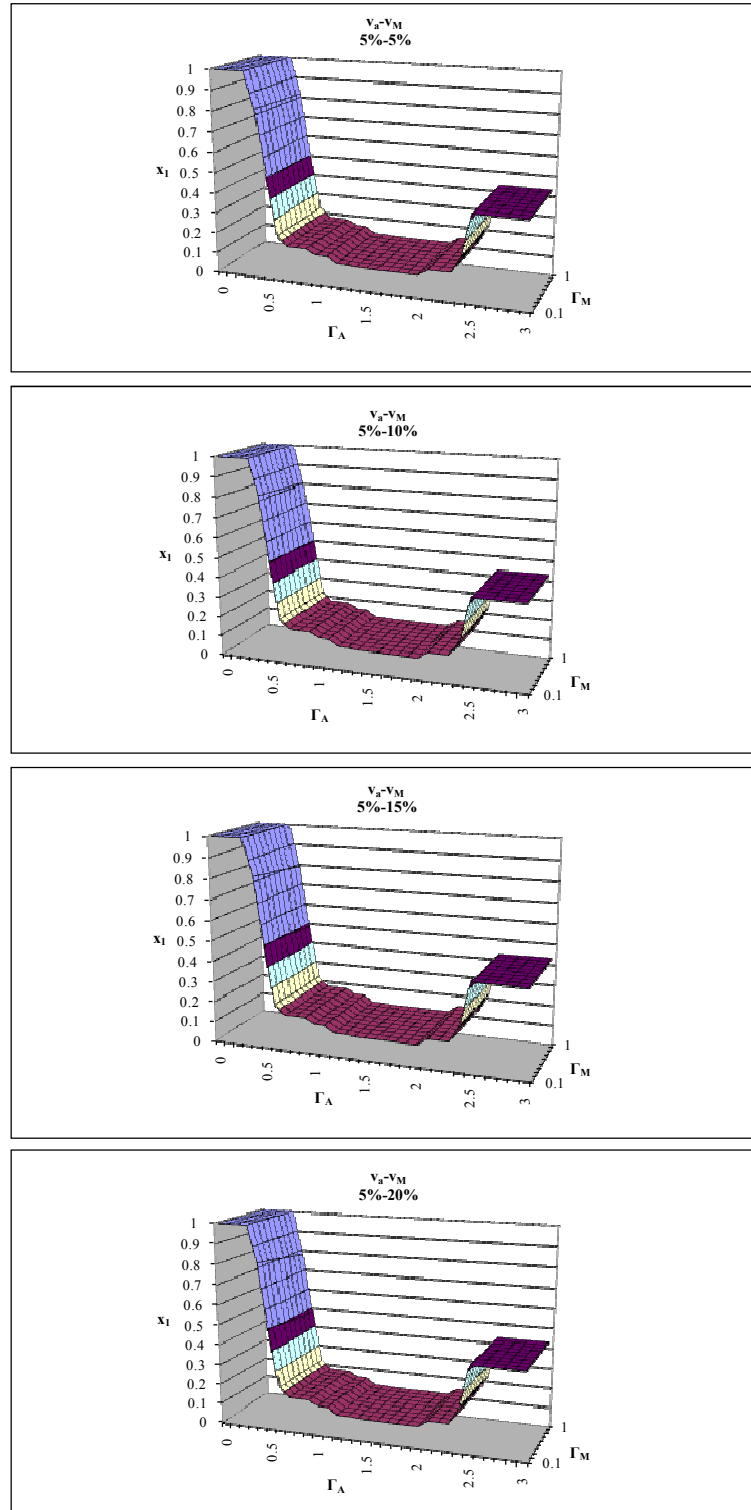


Figure B.3 : Level of investment in Project 1

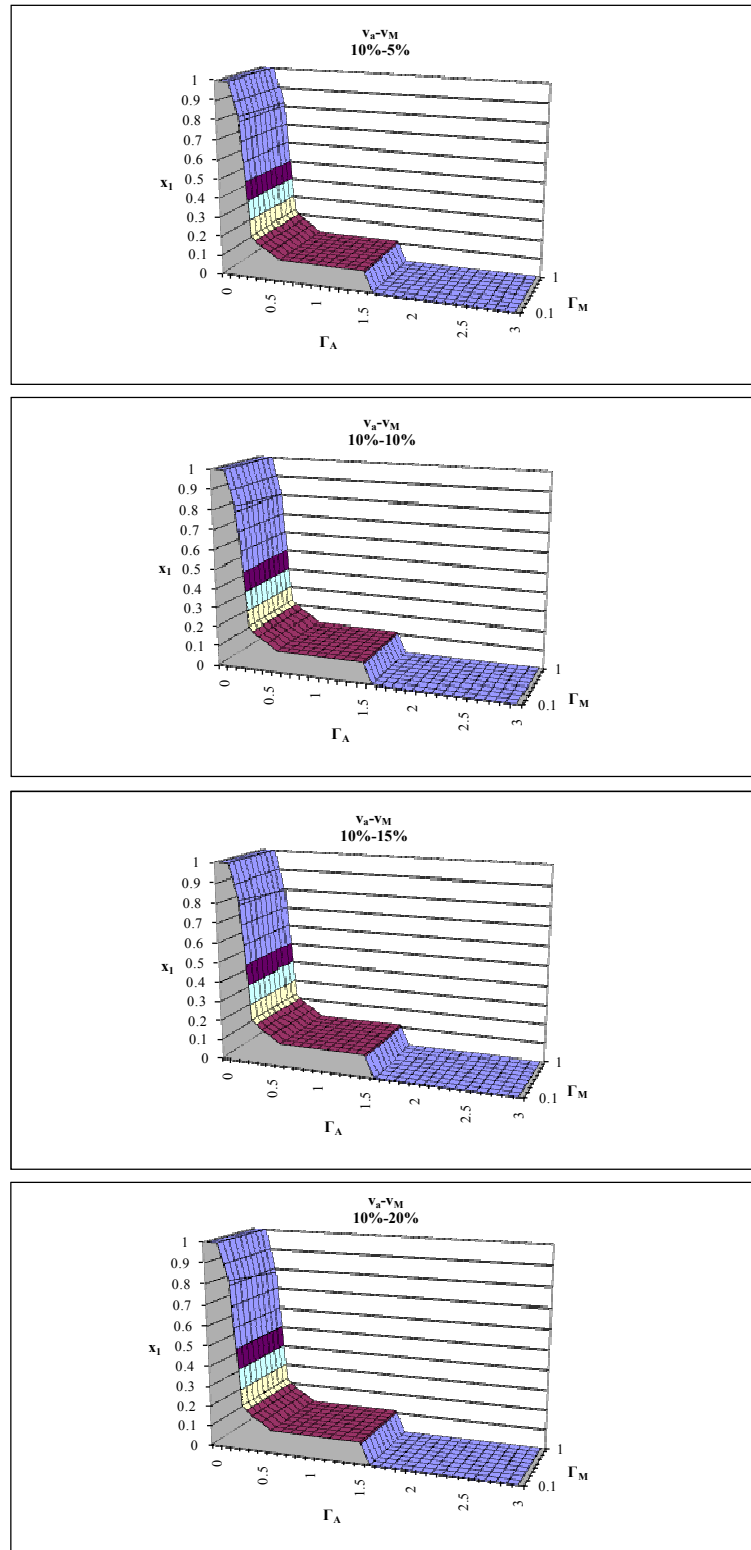


Figure B.3 : Level of investment in Project 1 (continued)

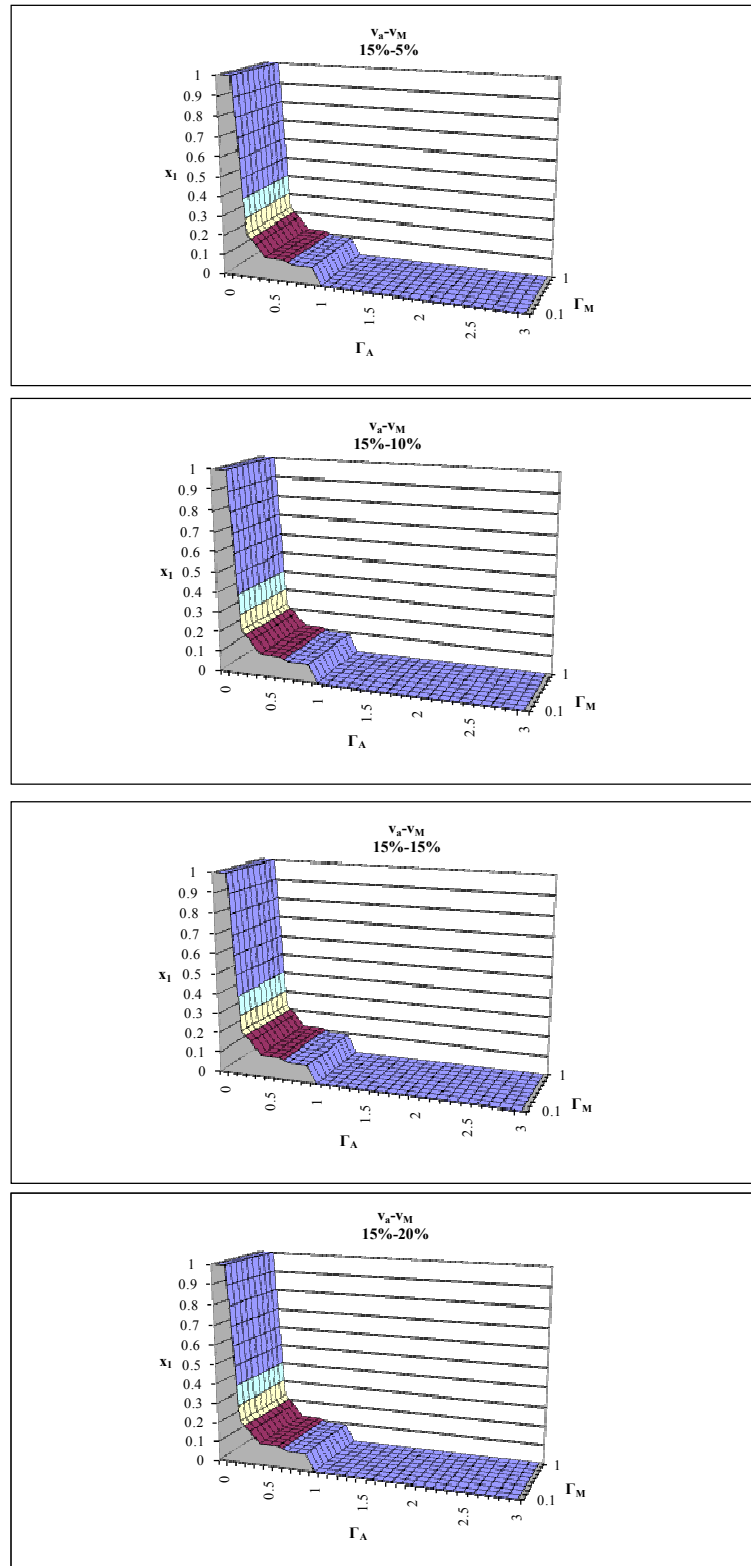


Figure B.3 : Level of investment in Project 1 (continued)

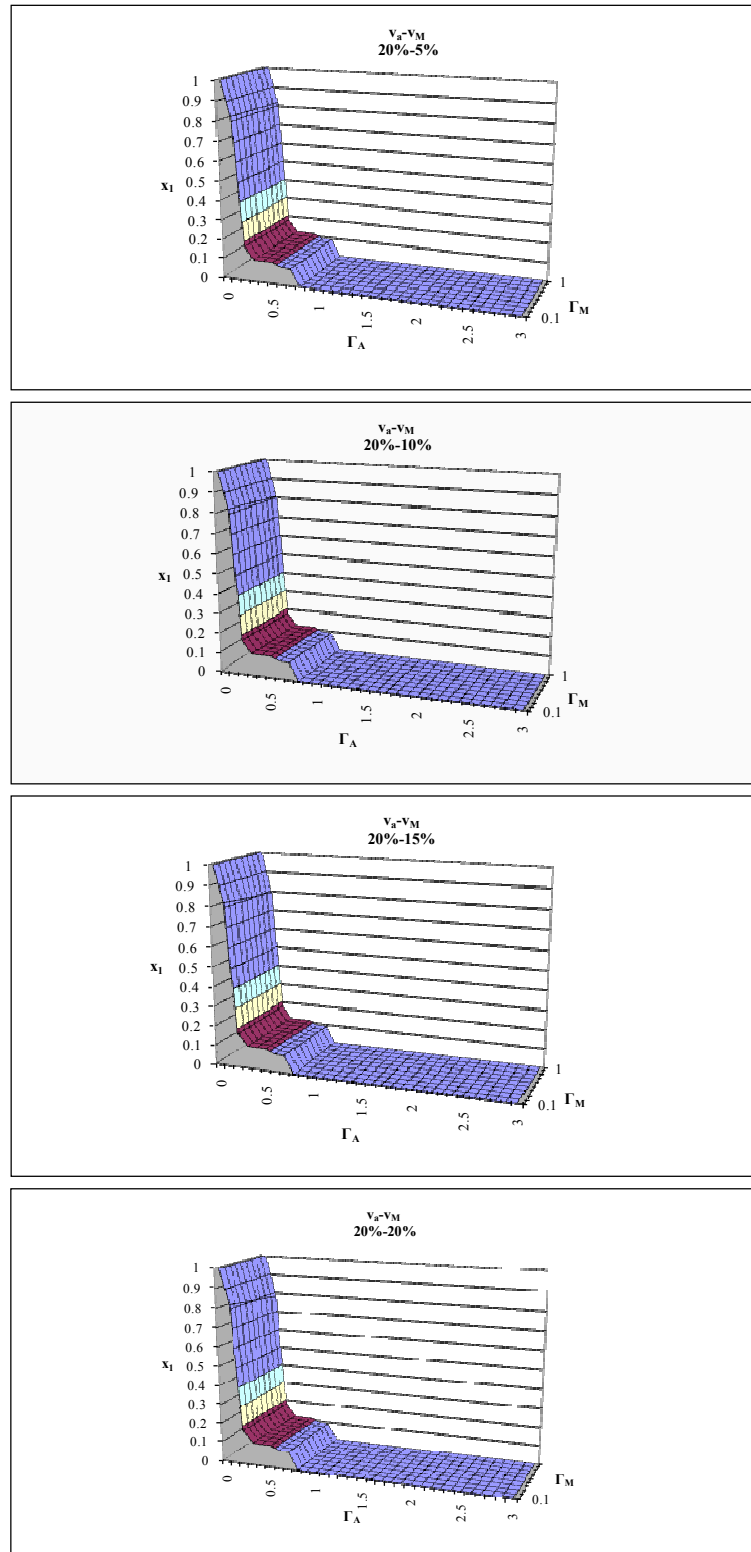


Figure B.3 : Level of investment in Project 1 (continued)

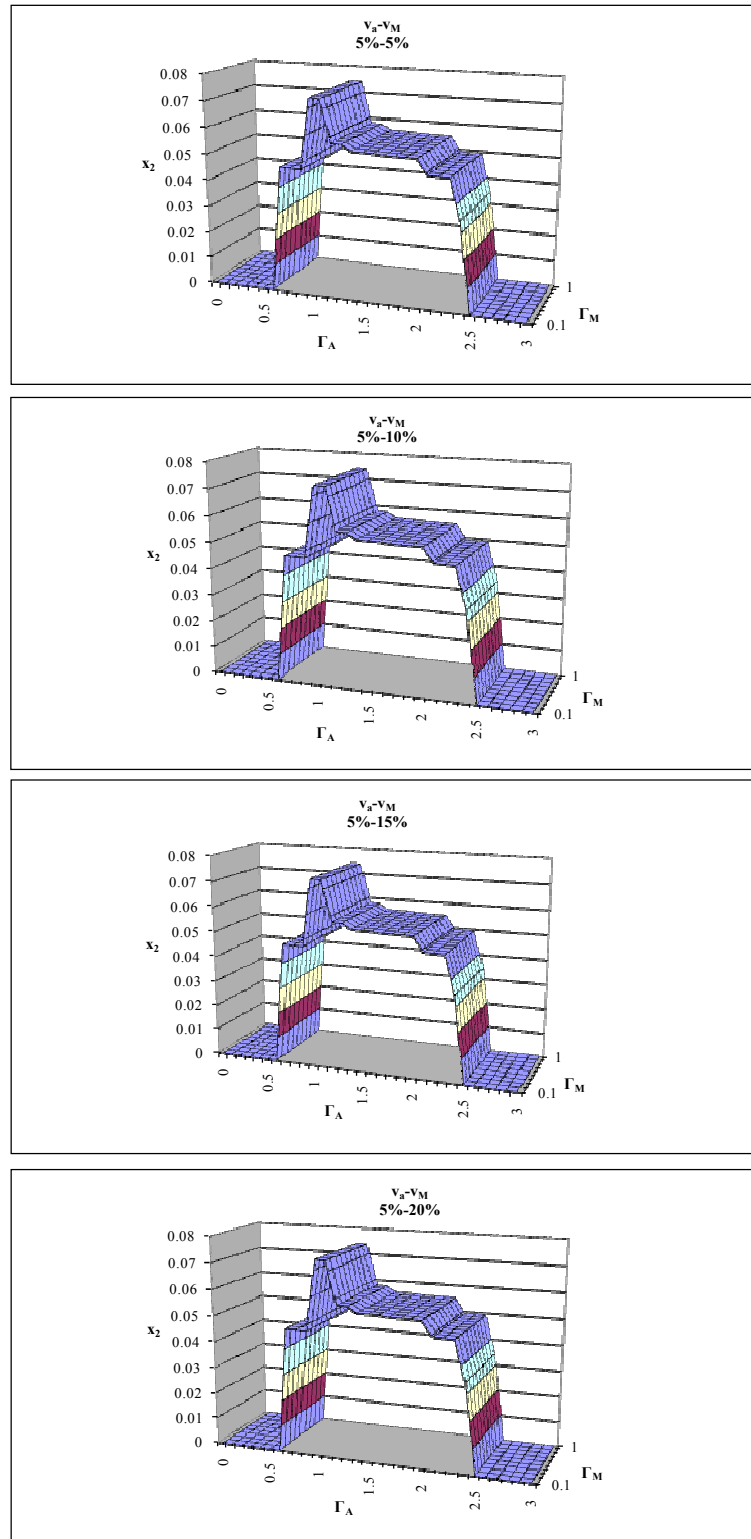


Figure B.4 : Level of investment in Project 2

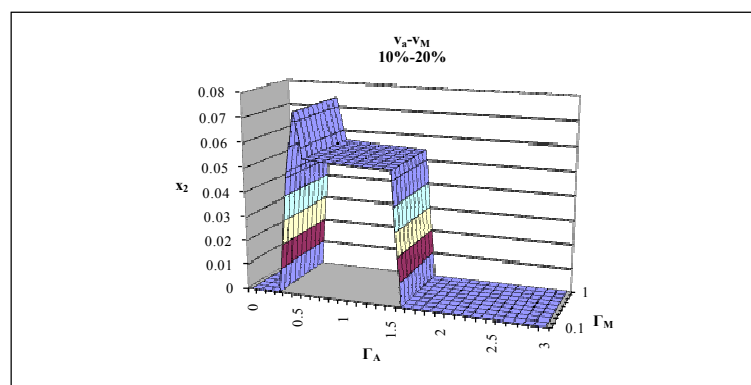
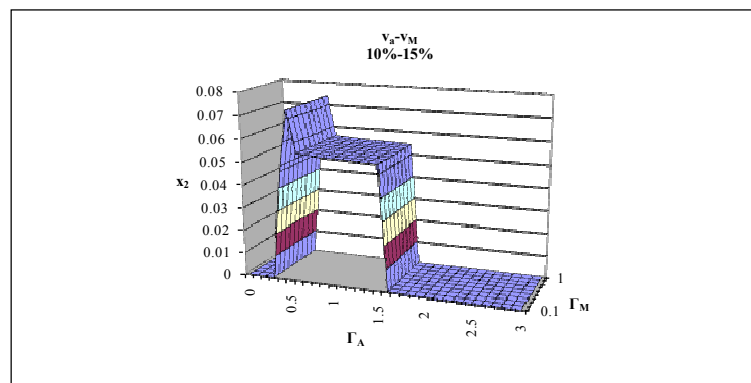
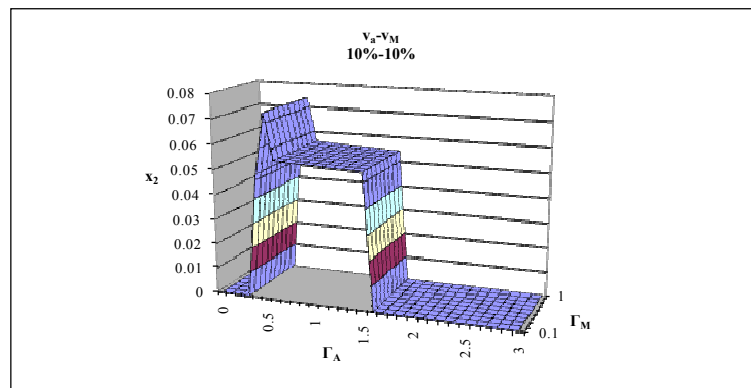
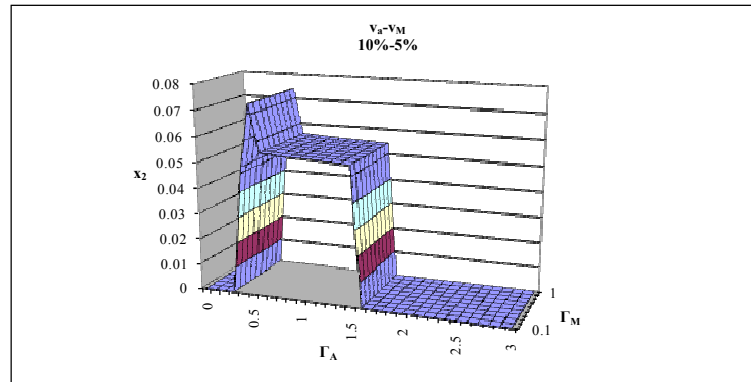


Figure B.4 : Level of investment in Project 2 (continued)

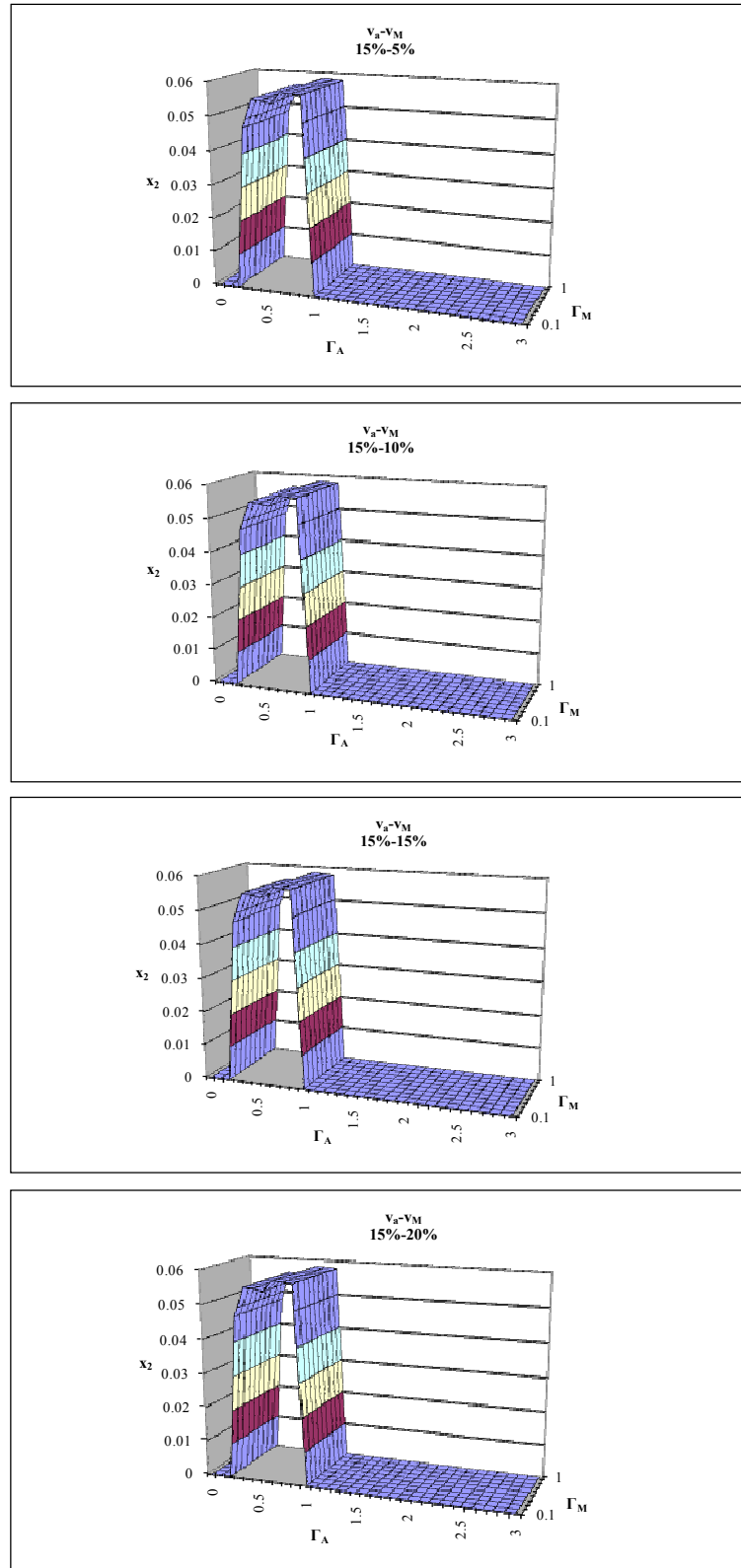


Figure B.4 : Level of investment in Project 2 (continued)

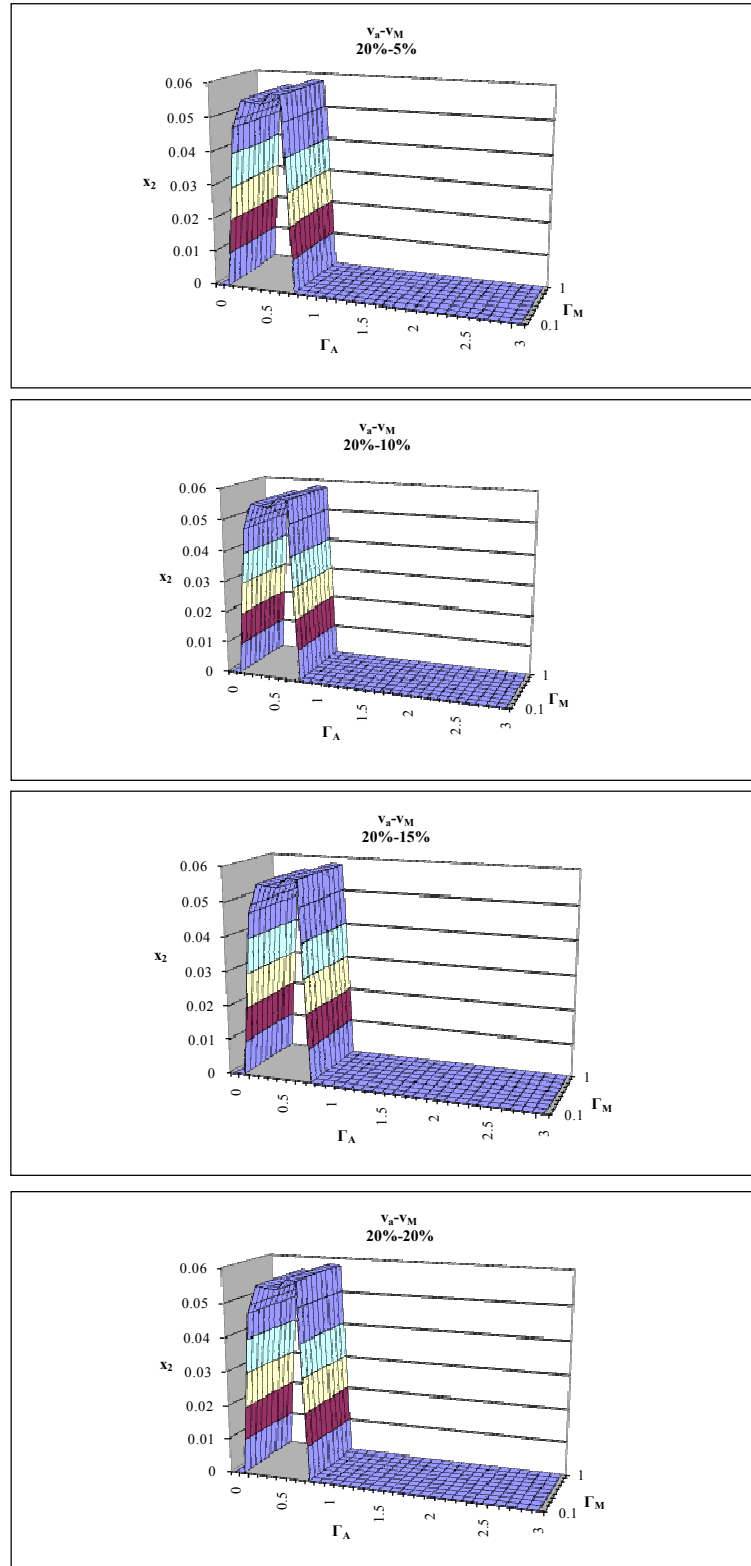


Figure B.4 : Level of investment in Project 2 (continued)

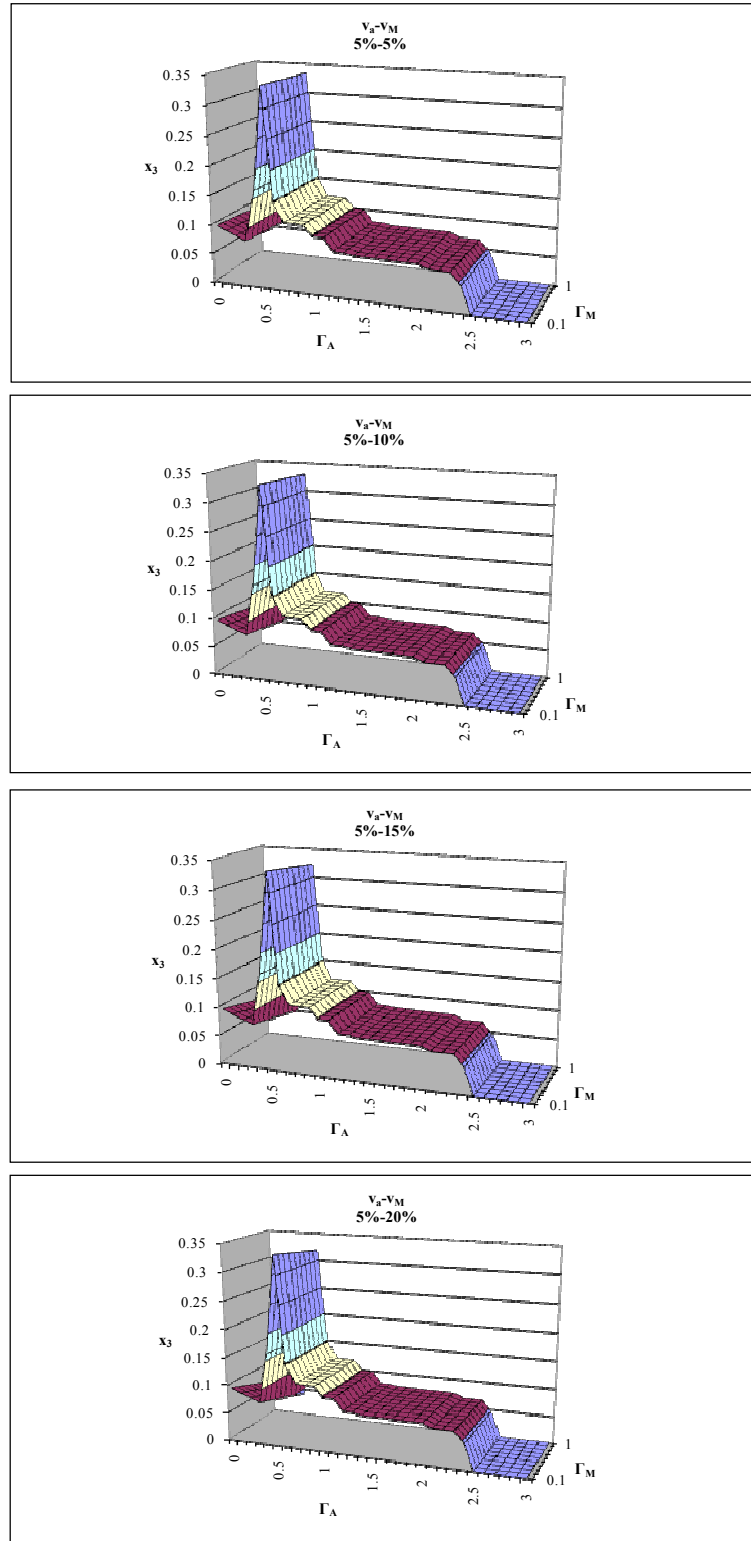


Figure B.5 : Level of investment in Project 3

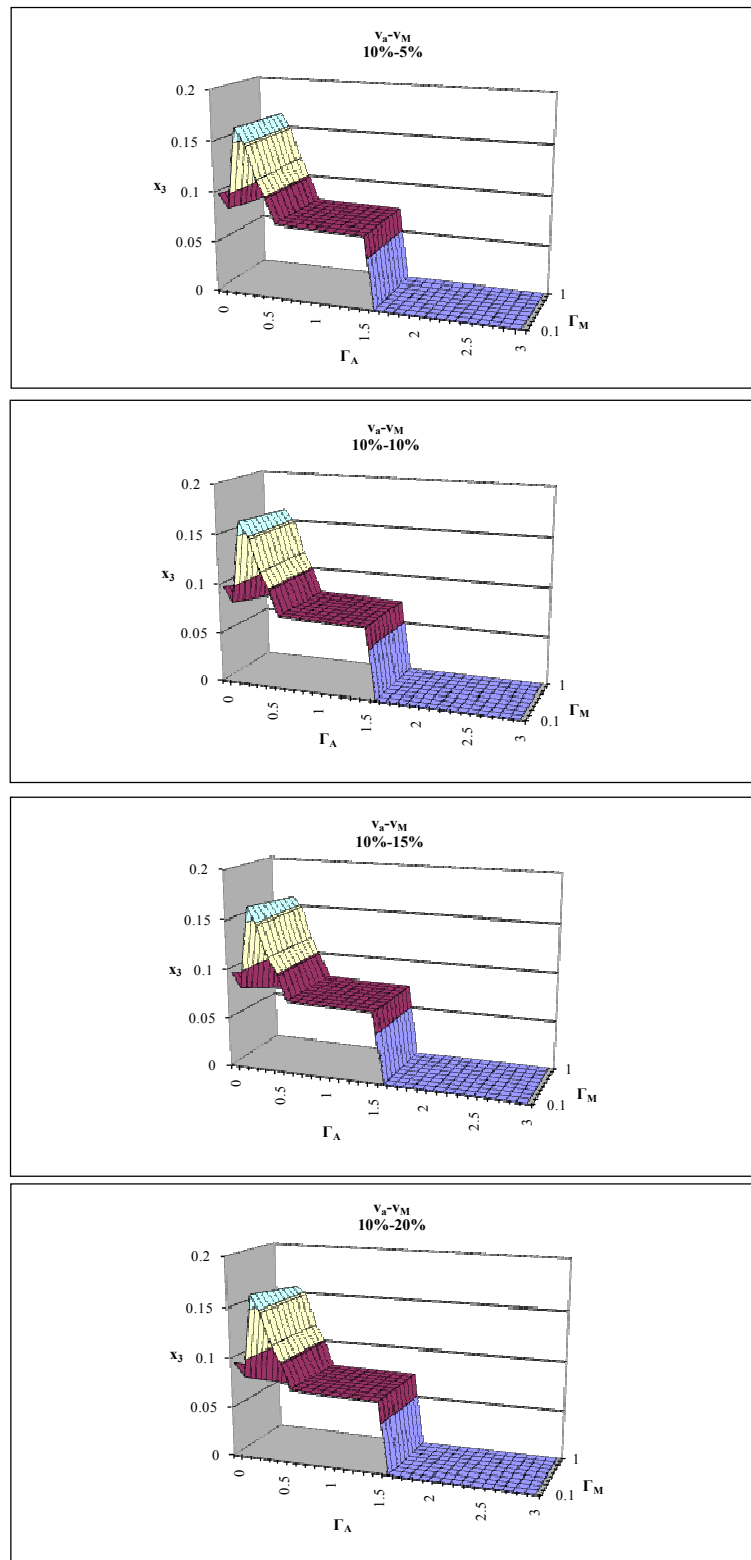


Figure B.5 : Level of investment in Project 3 (continued)

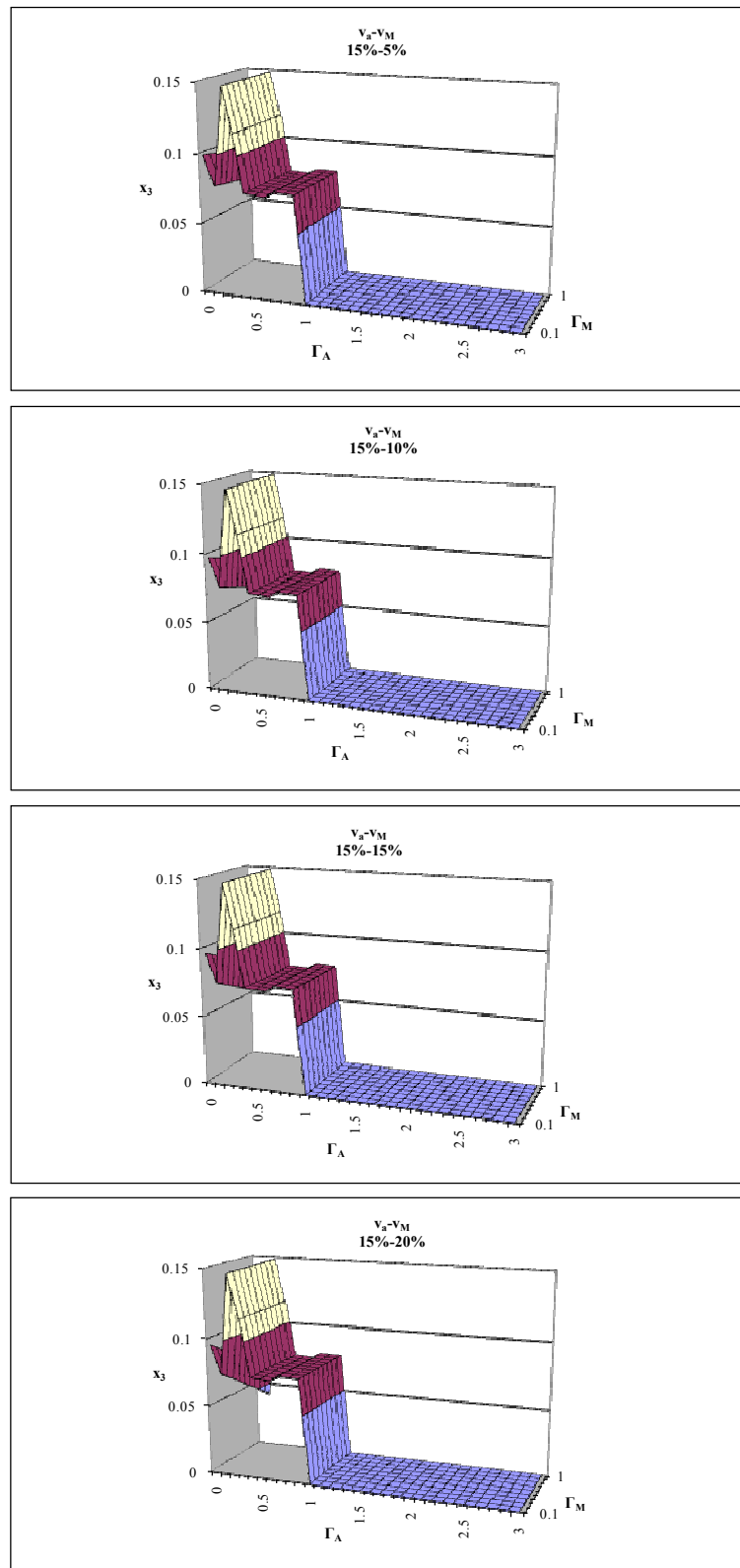


Figure B.5 : Level of investment in Project 3 (continued)

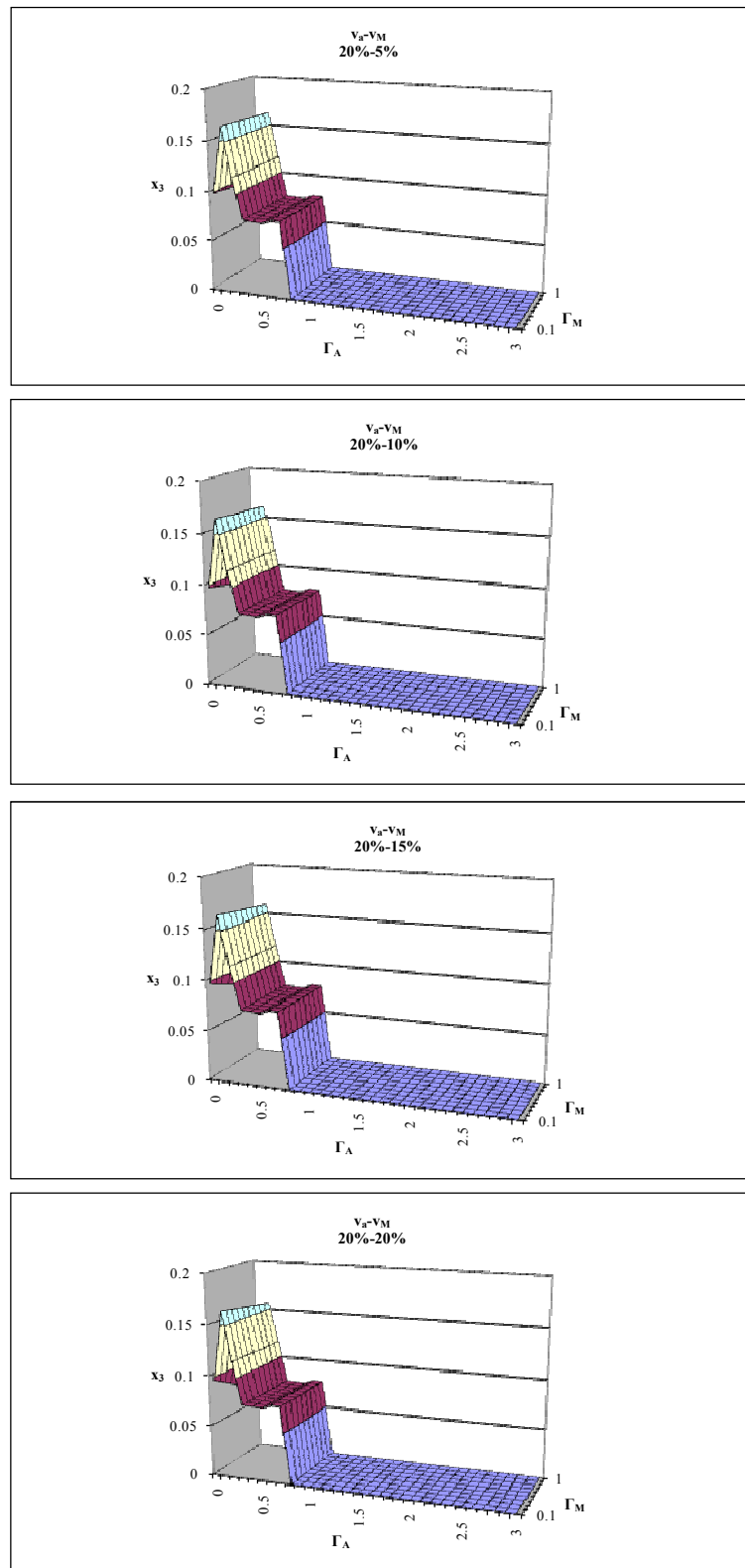


Figure B.5 : Level of investment in Project 3 (continued)

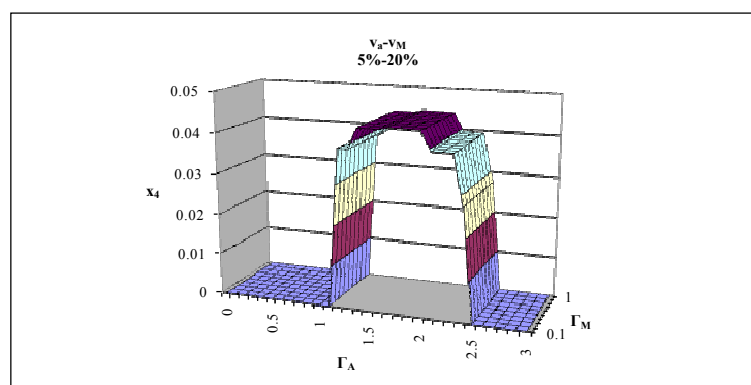
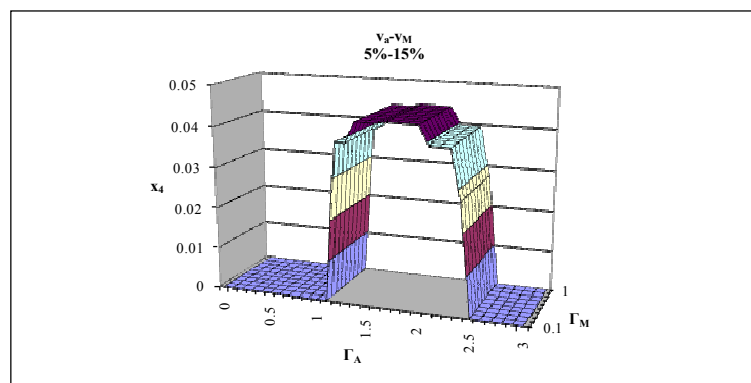
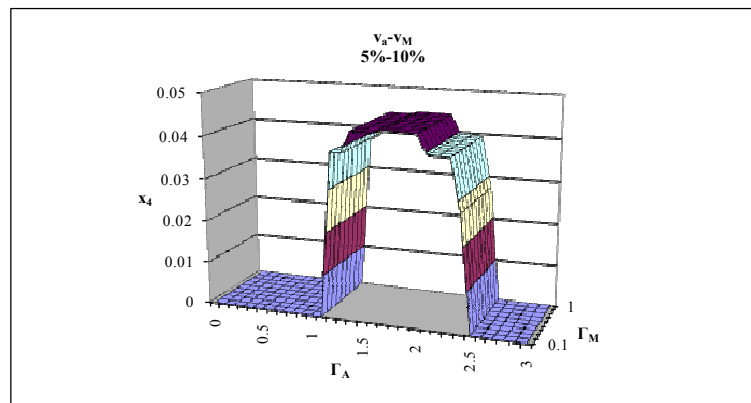
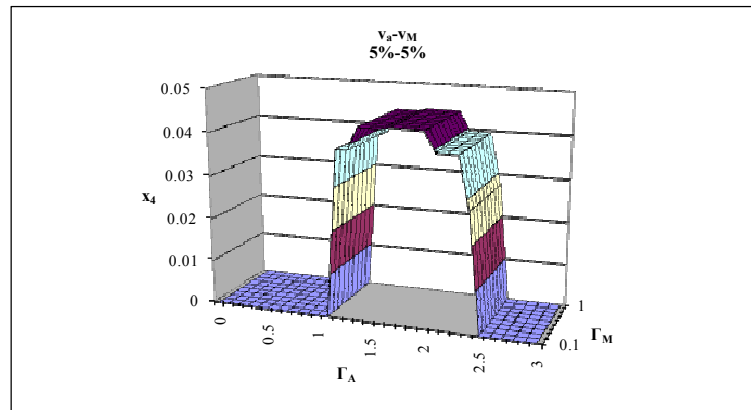


Figure B.6 : Level of investment in Project 4

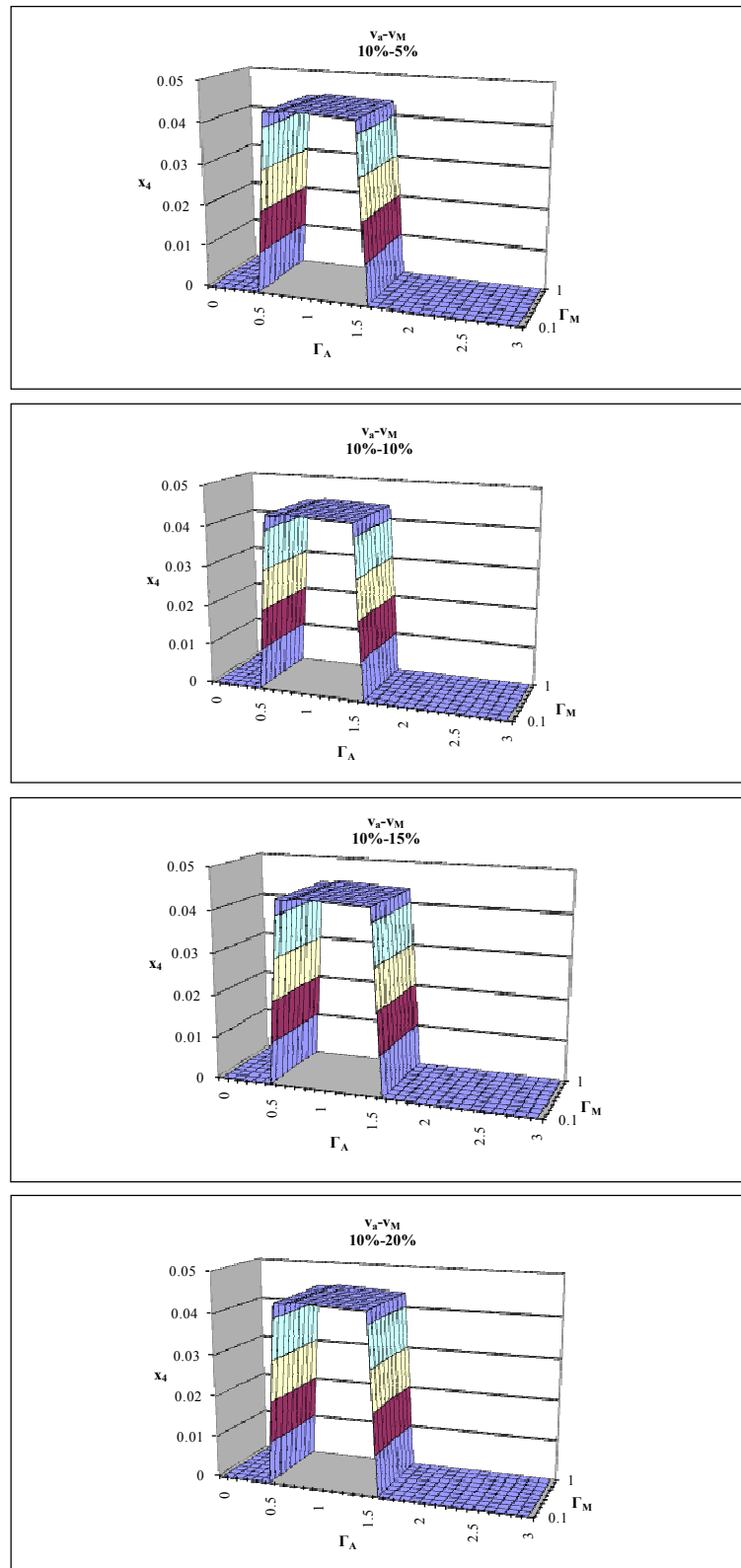


Figure B.6 : Level of investment in Project 4 (continued)

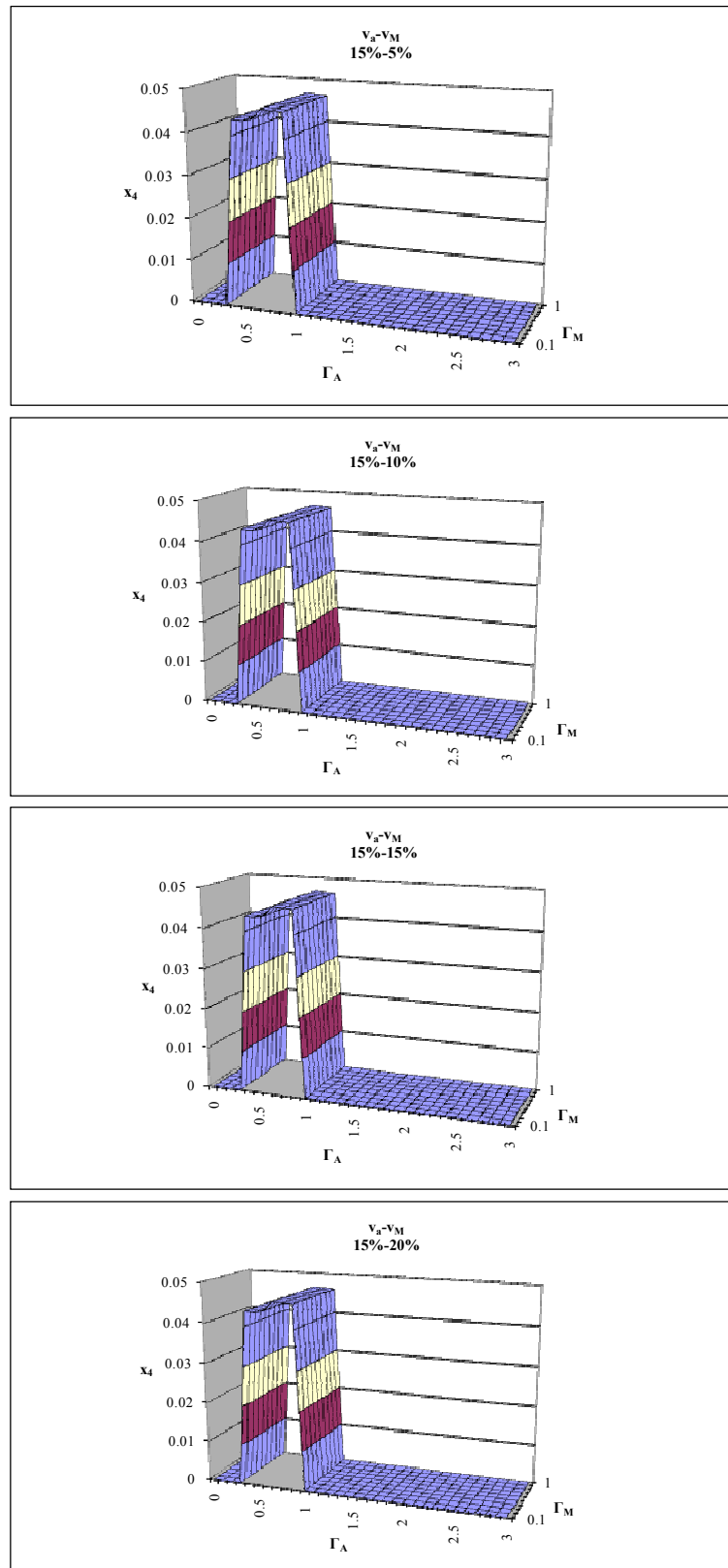


Figure B.6 : Level of investment in Project 4 (continued)

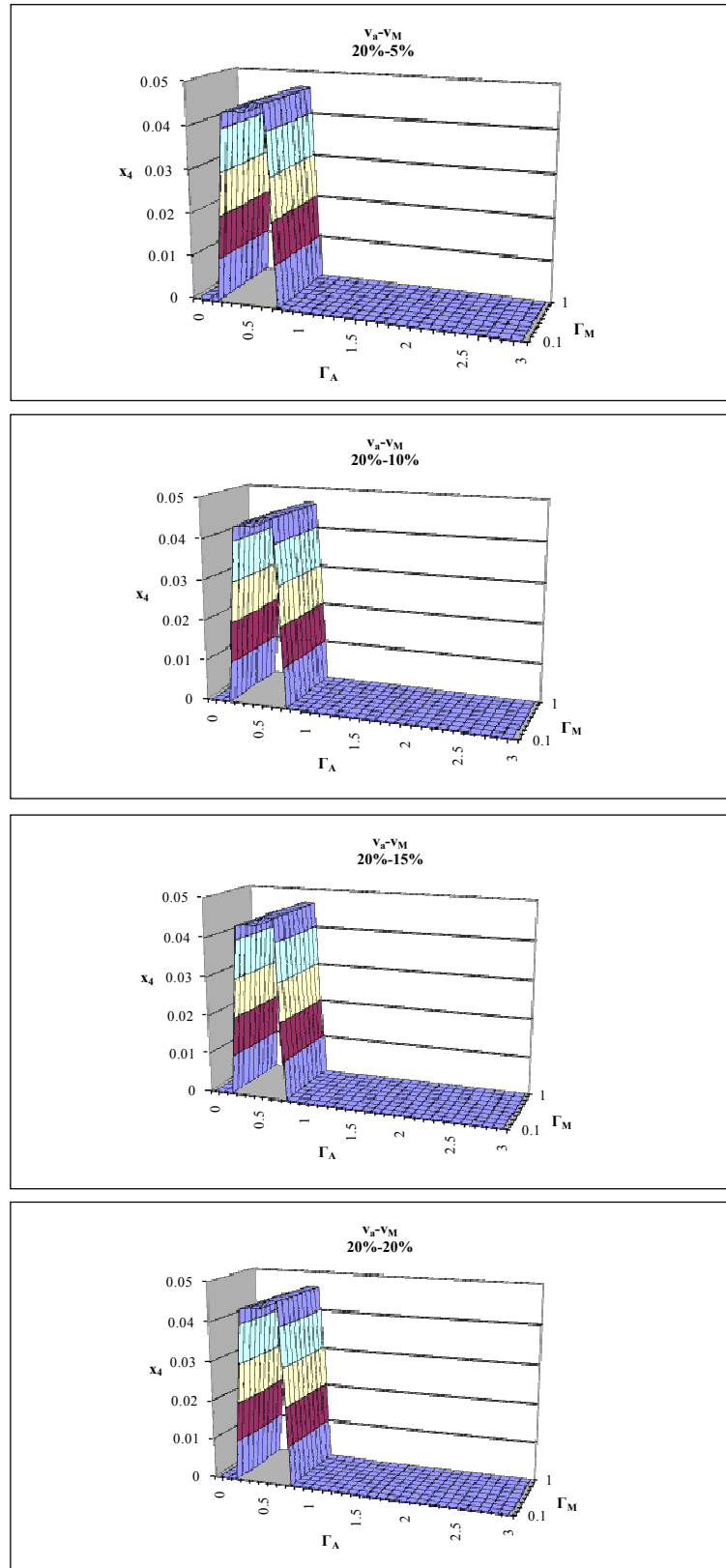


Figure B.6 : Level of investment in Project 4 (continued)

CURRICULUM VITAE

Esra Baş was born in Istanbul. She graduated from Yildiz Technical University as a mechanical engineer. She had an MBA degree from Yeditepe University and a Dipl.-Ing. Degree from Berlin Technical University Global Production Engineering program. During her education in Germany, she was supported by Turkish Education Foundation (TEV)-Deutscher Akademischer Austausch Dienst (DAAD) joint scholarship. She was a Fulbright visiting scholar during 2006-2007 education year at Columbia University Industrial Engineering and Operations Research (IEOR) department. She is currently affiliated with Istanbul Technical University Industrial Engineering department as a Research Assistant in “Engineering Economics”.