YEE'S BIJECTIVE PROOF OF BOUSQUET-MÉLOU AND ERIKSSON'S REFINEMENT OF THE LECTURE HALL PARTITION THEOREM

by

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Abstract

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of a positive integer N is a lecture hall partition of length n if it satisfies the condition

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \cdots \leq \frac{\lambda_n}{n}.$$

Lecture hall partitions are introduced by Bousquet-Mélou and Eriksson, while studying Coxeter groups and their Poincaré series. Bousquet-Mélou and Eriksson showed that the number of lecture hall partitions of length n where the alternating sum of the parts is k equals to the number of partitions into k odd parts which are less than $2n$ by a combinatorial bijection.

Then, Yee also proved the fact by combinatorial bijection which is differently defined for one of the bijections that were suggested by Bousquet-Mélou and Eriksson. In this thesis we give Yee's proof with details and further possible problems which arise from a paper of Corteel et al.

Amfi Parçalanış Teoreminin Bousquet-Mélou ve Eriksson İnceltmesi için Yee'nin Bire Bir Eşlemeli Kanıtı

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Özet

Eğer bir pozitif tamsayı N'in bir tam sayı parçalanışı olan $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \cdots \leq \frac{\lambda_n}{n}$$

koşulunu sağlyorsa λ n uzunluğunda bir amfi parçalanışıdır.

Amfi parçalanışları ilk olarak Bousquet-Mélou ve Eriksson tarafından Coxeter gruplar ve onların Poincaré serileri araştırılırken tanımlanmıştır. Bousquet-Mélou ve Eriksson N'in n uzunluğundaki kısımlarının alternatif toplamı k olan amfi parçalanışlarının sayısıyla $2n$'den küçük k tane tek kısımdan oluşan parçalanışlarının sayısının eşitliğini bir kombinatorik eşleme ile göstermişlerdir.

Daha sonra Yee bu özdeşliği Bousquet-Mélou ve Eriksson'ın önerdiği kombinatorik eşlemelerden birinin farklı tanımlayarak kanıtlamıştır. Bu tezde detaylarıyla Yee'nin kanıtı ve ayrıca Corteel ve diğerleri tarafından yazılan bir makaleden ortaya çıkan olası problemler verilmiştir.

For Deniz,

You tried so hard and get so far
but in the end...

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CHAPTER 1

Introduction

An *integer partition* is a way to split an integer into integer parts. By the definition in [4], the reordering of the parts does not change the partition, hence we can order the parts from the smallest to the largest. More precisely, a partition of length n of a positive integer N is a finite nondecreasing sequence of positive integers, so that the sum over the elements of the sequence is equal to N. The elements of the sequence are called *parts*. For example, $\lambda = (1, 1, 3, 4, 5)$ is a partition of $N = 14$ of length $n = 5$ since

$$\sum_{i=1}^{5} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_5 = 1 + 1 + 3 + 4 + 5 = 14 = N$$

Note that all parts of the partition are positive integers. Sometimes we loosen this requirement and allow zeros. For example in lecture hall partitions we can have a sequence of zeros as smaller parts of the partition.

$$\lambda_1 \lambda_2 \ldots \lambda_k \underbrace{\lambda_{k+1} \ldots \lambda_n}_{\substack{\ell(\lambda) \\ \text{number of positive parts}}}$$

where $\lambda_i = 0$ for $i = 1, 2, \ldots, k$. Hence our concept of "*length of partition*" becomes "*number of non-zero parts*" in the context of the lecture hall partitions.

$p(N)$ is *the partition function* which counts the number of partitions of N, for example $p(N) = 7$ for $N = 5$ since 1+1+1+1+1, 1+1+1+2, 1+2+2 1+1+3, 2+3, 1+4 and 5 are the partitions of $N = 5$. This function appears with a condition on parts mostly, i.e., $p(n|\textit{condition})$. For example, $p(N|\textit{odd parts})$ is the number of partitions of N into odd parts.

The equalities between the number of partitions of different types are called *identities*. In the theory of partitions, Euler proved the first partition identity in 1748[23].

Theorem 1.0.1 (Euler) *The number of partitions of N into odd parts, $p(N|\textit{odd parts})$, is equal to the number of partitions of N into distinct parts, $p(N|\textit{distinct parts})$.*

Example 1.0.1 *Let N=5. Then 1+1+1+1+1, 1+1+3 and 5 are the partitions of 5 into odd parts hence, $p(5|\textit{odd parts}) = 3$. Also, 1+4, 2+3 and 5 are the partitions of 5* *into distinct parts so* $p(5|\textit{distinct parts}) = 3$. *Therefore*

$$p(5|\textit{odd parts}) = 3 = p(5|\textit{distinct parts}).$$

Euler's identity can be proven by constructing a bijection between the set of partitions of given types. The procedure is basic merging (from odds to distincts) and splitting (from distincts to odds) process. When we find two identical parts, we merge them until all parts are distinct and for inverse we are splitting all even parts into half until we have no even parts. We can take the partitions in the previous example.

$$\begin{aligned} &1+1+1+1+1 \rightarrow 2+2+1 \rightarrow 4+1 \\ &1+1+3 \rightarrow 2+3 \\ &5 \rightarrow 5 \end{aligned}$$

As you can see from the example above, we can find a one to one correspondence between partitions into odd parts and partition into distinct parts.

Euler's identity has the following q-series version

$$\begin{aligned} \textstyle\sum_{n\geq 0} p(n|\textit{odd parts})q^n &= \textstyle\prod_{n\geq 1} \frac{1}{1-q^{2n-1}} \\ &= \textstyle\prod_{n\geq 1} \frac{1-q^{2n}}{1-q^n} \\ &= \textstyle\prod_{n\geq 1}(1+q^n) \\ &= \textstyle\sum_{n\geq 0} p(n|\textit{distinct parts})q^n \end{aligned}$$

In [28], a refined version of Euler's theorem has been proven by J.J. Sylvester.

Theorem 1.0.2 (Sylvester) *Let* $A_k(N)$ *be the number of partitions of* N *using exactly* k *different odd parts (repetitions are allowed), and* $B_k(N)$ *the number of partitions of* N *into* k *separate sequence of consecutive integers. Then,*

$$A_k(N) = B_k(N).$$

Note that this theorem is a refined version of Euler's theorem since

$$p(n|\textit{odd parts}) = \sum_{k=0}^{\infty} A_k(N) \quad \textit{and} \quad p(n|\textit{distinct parts}) = \sum_{k=0}^{\infty} B_k(N).$$

Example 1.0.2 *Let* $N = 15$ *and* $k = 3$. *Then we have the following lists ;*

the number of partitions of $N = 15$ using exactly $k = 3$ different odd part sizes:

11+3+1,
9+5+1,
9+3+1+1+1,
7+5+3,
7+5+1+1+1,
7+3+1+1+1+1+1,
7+3+3+1+1,
5+5+3+1+1,
5+3+3+3+1,
5+3+3+1+1+1+1,
5+3+1+1+1+1+1+1+1,

the number of partitions of $N = 15$ into $k = 3$ separate sequence of consecutive integers:

11 +3 +1,
10 +4 +1,
9 +5 +1,
9 +4 +2,
8 +6 +1,
8 +5 +2,
8 +4 +2+1,
7 +5 +3,
7 +5 +2+1,
7 +4+3 +1,
6+5 +3 +1.

Note that the number of partitions in both lists are 11.

Not all partition identities come from purely combinatorial or partition theoretic concerns. Theory of partitions is enriched by interactions between different areas of mathematics. The lecture hall partitions is an example of this. We will give some basic definitions and notions of Coxeter group theory in order to give deeper understanding for background of lecture hall partitions.

C_n, *finite Coxeter group*, is the finite Euclidean reflection group (A group generated by a set of reflections of a finite dimensional Euclidean space.). $\tilde{C}_n$, *affine Coxeter group*, are not finite themselves, but each contains a normal Abelian subgroup such that the corresponding quotient group is finite. By *Poincaré series* of these groups, one means the length generating functions.

$$C_n(q) = \sum_{\pi \in C_n} q^{\ell(\pi)}$$

and

$$\tilde{C}_n(q) = \sum_{\pi \in \tilde{C}_n} q^{\ell(\pi)}.$$

In [15], Bott gave a generalization of Poincaré series of affine Coxeter groups such that

$$\tilde{C}_n(q) = \frac{C_n(q)}{(1-q)(1-q^3)\dots(1-q^{2n-1})}.$$

After realizing the similarity of the denominator and the generating function of a partition function, Bousquet-Mélou and Eriksson gave a combinatorial proof of the equivalence of Bott's generalization and lecture hall theorem. They proved

$$\sum_{\pi \in \tilde{C}_n/C_n} q^{\ell(\pi)} = \frac{1}{(1-q)(1-q^3)\dots(1-q^{2n-1})}.$$

In order to conclude that two theorems are equivalent it is necessary to find a bijection between $\tilde{C}_n/C_n$ and $\mathcal{L}_n$, which is the set of lecture hall partitions of length n, such that $\ell(\pi) = |\lambda(\pi)|$. Their candidate was

$$\lambda_i = \sum_{j=1}^{i} I_{i,j}(\pi)$$

where $I_{i,j}(\pi)$ is the number of $(i, \langle j \rangle)$-class inversions, which satisfies $\ell(\pi) = \lambda_1 + \lambda_2 + \dots + \lambda_n = \lambda(\pi)$. They showed that λ is a bijection between $\tilde{C}_n/C_n$ and $\mathcal{L}_n$ by using the properties of $I_{i,j}(\pi)$ such that

(*i*) $I_{i,j} \geq I_{i-1,j}$ with equality if i lies directly to the right of $i-1$ in π,

(*ii*) If the member of $\langle j \rangle$ in the window of π containing i is to the left of i then $I_{i,j} = I_{i,i}$ must hold, and otherwise $I_{i,j} = I_{i,i} - 1$.

Note that by using (*i*) and (*ii*) in order, we get

$$\frac{\lambda_i}{i} = \frac{\sum_{j=1}^{i} I_{i,j}}{i} \geq \frac{I_{i,i} + \sum_{j=1}^{i-1} I_{i-1,j}}{i} \geq \frac{\sum_{j=1}^{i-1} I_{i-1,j}}{i-1} = \frac{\lambda_{i-1}}{i-1}.$$

Notice that λ satisfies lecture hall condition, as desired. After lecture hall partitions were introduced by Bousquet-Mélou and Eriksson, it is also shown that the number of lecture hall partitions of length n of a positive integer N whose alternating sum, $|\lambda|_a = \lambda_n - \lambda_{n-1} + \dots + (-1)^{n-1}\lambda_1$, is k equals to the number of partitions of N into k odd parts less than $2n$. In [29], Yee proved this partition identity by a combinatorial bijection. The aim of this thesis is redoing Yee's proof in detail.

CHAPTER 2

Preliminaries

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *of length* n of a positive integer N is a finite nondecreasing sequence of nonnegative integers that sums up to N. Although the usual convention is to arrange parts in non-increasing order [1, 16, 17, 18], writing parts in non-decreasing order will be more suitable for our studies. The point is that reordering parts does not give a new partition. A *lecture hall partition* of length n is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ that satisfies the following condition

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq ... \leq \frac{\lambda_n}{n}. \tag{2.1}$$

For a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, with some λ_i possibly equal to 0, we say that

(i) λ_i and λ_{i+1} satisfy *lecture hall condition* if they satisfy the following inequality

$$\frac{\lambda_i}{i} \leq \frac{\lambda_{i+1}}{i+1}.$$

(ii) λ satisfies the lecture hall condition if λ_i and λ_{i+1} satisfy the lecture hall condition for all $i = 1, 2, \ldots, n-1$.

Lecture hall partitions were first defined in [16] and some properties of lecture hall partitions are examined in [17, 18]. Let $\mathcal{L}_n$ be the set of lecture hall partitions of length n. Bousquet-Melóu and Eriksson in [16] showed that the generating function of the set $\mathcal{L}_n$ is

$$\sum_{\lambda \in \mathcal{L}_n} t^{|\lambda|_a} q^{|\lambda|} = \prod_{i=0}^{n-1} \frac{1}{1 - tq^{2i+1}} \tag{2.2}$$

where $|\lambda|_a$ is the *alternating sum* of parts, i.e., $|\lambda|_a = \lambda_n - \lambda_{n-1} + \cdots + (-1)^{n-1}\lambda_1$ and $|\lambda|$ is the *weight* of the partition, i.e., $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. By taking the limit when n approaches to infinity, Sylvester's refinement of Euler's identity [28] can be obtained. Note that, as n tends to infinity, the right hand side of the identity (2.2) is the generating function of the set of partitions into odd parts and counting the number of parts, as well. On the left hand side since we have lecture hall partitions, the parts will satisfy the inequality (2.1). Let's consider the following limit:

$$\lim_{n \to \infty} \left(\frac{\lambda_{n-1}}{n-1} \leq \frac{\lambda_n}{n} \right).$$

It is same as the following one:

$$\lim_{n\to\infty}\left(\frac{\lambda_{n-1}}{\lambda_n}\leq\frac{n-1}{n}\right).$$

Hence as n tends to infinity the last limit gives us

$$0\leq\lim_{n\to\infty}\left(\frac{\lambda_{n-1}}{\lambda_n}\right)<1$$

and this inequality shows that the fraction between the consecutive parts will be strictly less than 1 therefore they must be distinct. Also, we necessarily have a "*long*" list of zeros at the beginning of the partition, so it is sensible to discard the zeros altogether when taking the limits, and focus on the non-zero parts.

$$\sum_{\mu\in\mathcal{D}} t^{|\mu|_a} q^{|\mu|} = \sum_{\mu\in\mathcal{O}} t^{\ell(\mu)} q^{|\mu|} \tag{2.3}$$

Above, $\mathcal{D}$ is the set of partitions into distinct parts, $\mathcal{O}$ is the set of partitions into odd parts and $\ell(\mu)$ is the number of parts in the partition μ. The identity in (2.3) is proven combinatorally by Sylvester[28], Bessenrodt[14] and Kim and Yee[25] but these proofs are not applicable for the finite version.

The aim of this thesis is redoing Yee's proof of the identity that guarantees the equality of number of lecture hall partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of length n, where $|\lambda|_a = k$ and the number of partitions of N into k odd parts which are less than $2n$, in detail.

In Chapter 3 we will define a deletion map and an insertion map, then define two bijections which are defined recursively by the deletion and the insertion maps. Throughout the proof we will check the consistency of our definitions, show the properties of our defined maps and we will end up with the fact that the two bijections are inverses of each other. The purposes of lemmas, corollaries and theorems are given after the proofs of them.

Finally in Chapter 4, we will mention the guidelines for partition analysis that are given by Corteel, Lee and Savage in [21]. This is because we want to find other bijections, and this paper is a fruitful source of identities such that many of them lack bijective proofs. Also, the five guidelines for partition analysis is an algorithmic approach to systematically produce lecture hall type identities.

CHAPTER 3

The Bijection

If we define $\mathcal{O}_n$ as the set of partitions into odd parts which are less than $2n$, then the identity can be written as follows:

$$\sum_{\lambda \in \mathcal{L}_n} t^{|\lambda|_a} q^{|\lambda|} = \sum_{\sigma \in \mathcal{O}_n} t^{\ell(\sigma)} q^{|\sigma|}. \tag{3.1}$$

For a combinatorial proof of the identity (3.1) where $|\lambda|_a$ is alternative sum of the parts, $\ell(\sigma)$ is the number of parts in σ and $|\lambda|$(or $|\sigma|$) is the positive integer partioned as λ (or σ), we will define two bijections namely $\Psi_n : \mathcal{L}_n \to \mathcal{O}_n$ which takes a lecture hall partition of length n and gives an odd partition whose parts are less than $2n$, and inverse of Ψ_n, $\Phi_n : \mathcal{O}_n \to \mathcal{L}_n$ which takes an odd partition whose parts are less than $2n$ gives a lecture hall partition of length n.

3.1. Definition of Maps

3.1.1 Definition of the Map Ψ_n

Given a partition $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{L}_n$, we need a deletion map first. We begin with a lecture hall partition and construct smaller lecture hall partitions by deleting odd number of cells one by one starting from the largest part until we reach the lecture hall partition consisting of n zeros. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{L}_k$, $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_k)$ is a *smaller* lecture hall partition of length k if $\lambda_i \geq \lambda'_i$ $\forall\, i = 1, 2, \ldots, k$. For example $\lambda' = (0, 1, 3, 4, 5)$ is a smaller partition than $\lambda = (0, 1, 3, 5, 7)$ since $\lambda_i \geq \lambda'_i$ for $i = 1, 2, 3, 4, 5$.

First we define the deletion map ψ_n. The deletion map ψ_n takes $\tau \in \mathcal{L}_n$ and deletes one cell from all large parts of τ until

- we have enough number of deleted cells,

or

- we meet a pair (τ_i, τ_{i+1}) such that the pair $(\tau_i - m, \tau_{i+1} - m - 1)$ also satisfies the lecture hall condition.

For defining ψ_n, we need to know two things at the beginning, which are the starting point of deletion, the part of the given lecture hall partition where we start deletion (b) and the number of cells we will delete (k).

Let $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{L}_n$ with the convention that $\tau_0 = 0$, $\ell(\tau)$ be the number of nonzero parts of τ and k_j be the smallest positive integer $k > j$ such that, for all $0 \le j < \lceil \ell(\tau)/2 \rceil$,

$$\frac{\tau_{n-2j-1} - (k-j-1)}{n-2j-1} \le \frac{\tau_{n-2j} - (k-j)}{n-2j}.$$

For example, for $\tau = (0,1,9,12)$, $n = 4$ and $\ell(\tau) = 3$. Then for $j = 0$, $\frac{\tau_3 - k + 1}{3} \le \frac{\tau_4 - k}{4}$, where $\tau_3 = 9$ and $\tau_4 = 12$, implies that $4 \le k$ and $k_0 = 4$. For $j = 1$, $\frac{\tau_1 - k + 2}{1} \le \frac{\tau_2 - k + 1}{2}$, where $\tau_1 = 0$ and $\tau_2 = 1$, implies that $2 \le k$ and $k_1 = 2$.

Also note that writing

$$\tau_i = i \left\lceil \frac{\tau_i}{i} \right\rceil - (i - r_i)$$

where r_i is the remainder of the corresponding part and $1 \le r_i \le i$ transforms the lecture hall condition into

$$\left\lceil \frac{\tau_i}{i} \right\rceil = \left\lceil \frac{\tau_{i+1}}{i+1} \right\rceil \text{ and } r_i < r_{i+1}$$

or

$$\left\lceil \frac{\tau_i}{i} \right\rceil < \left\lceil \frac{\tau_{i+1}}{i+1} \right\rceil.$$

Let us define the following set

$$A = \left\{ 0 \le j < \lfloor \ell(\tau)/2 \rfloor : \left\lceil \frac{\tau_{n-2j-1}}{n-2j-1} \right\rceil = \left\lceil \frac{\tau_{n-2j}}{n-2j} \right\rceil \text{ and } r_{n-2j-1} + 1 = r_{n-2j} \right\}.$$

Therefore, if $j \notin A$, then $k_j = j + 1$. In this case,

$$\left\lceil \frac{\tau_{n-2j-1}}{n-2j-1} \right\rceil < \left\lceil \frac{\tau_{n-2j}}{n-2j} \right\rceil$$

and so

$$\frac{\tau_{n-2j-1}}{n-2j-1} < \frac{\tau_{n-2j}}{n-2j}.$$

If $j \in A$, then $k_j = r_{n-2j} + j$. In this case,

$$\left\lceil \frac{\tau_{n-2j-1}}{n-2j-1} \right\rceil = \left\lceil \frac{\tau_{n-2j}}{n-2j} \right\rceil \text{ and } r_{n-2j} = r_{n-2j-1} + 1.$$

So our final choices are $k = \min\{k_j : j < \lceil \ell(\tau)/2 \rceil\}$ and $b = \min\{j : k_j = k\}$. For $\tau = (1,2,8,11)$, $A = \{0,1\}$ and $k_0 = r_4 = 3, k_1 = r_2 + 1 = 3$. This implies that $k = 3, b = 0$.

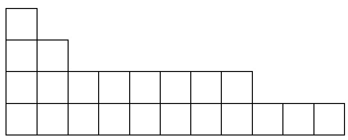
Now we can define the deletion map ψ_n from τ. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be the sequence defined by

$$\begin{aligned}
&\mu_{n-i} = \tau_{n-i} - 1 \qquad \text{for } 0 \leq i \leq 2b-1;\\
&\mu_{n-2b} = \tau_{n-2b} - (k-b);\\
&\mu_{n-2b-1} = \tau_{n-2b-1} - (k-b-1);\\
&\mu_{n-i} = \tau_{n-i} \qquad\quad \text{for } 2b+2 \leq i < n.
\end{aligned}$$

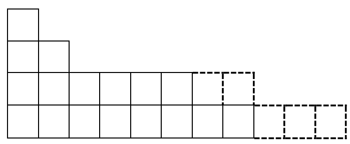
Then define $\psi_n(\tau) = (\mu, 2k-1)$. Note that for $b = 0$ the first line becomes vacuous. However, the μ values can be fully determined by starting to calculate from the second line of the sequence.

Example 3.1.3 *For $\tau = (1, 2, 8, 11)$, $\ell(\tau) = 4$ and $A = \{0, 1\}$. We need to find k and b values. For $j = 0$, since $0 \in A$, $k_0 = r_4 = 3$ and by the same reasoning $k_1 = r_2 + 1 = 3$. Therefore, $k = 3$ and $b = 0$. The sequence defined above will give us the following:*

$$\begin{aligned}
&\mu_{4-2.0} = \mu_4 = \tau_4 - (3-0) = 8\\
&\mu_{4-2.0-1} = \mu_3 = \tau_3 - (3-0-1) = 6\\
&\mu_{4-2} = \mu_2 = \tau_2 = 2\\
&\mu_{4-3} = \mu_1 = \tau_1 = 1
\end{aligned}$$



$\tau = (1, 2, 8, 11)$, $n = 4$, $k = 3$, $b = 0$



$\psi_4(\tau) = ((1, 2, 6, 8), 5)$

Table 3.1: Young diagrams for τ and $\psi_4(\tau)$

The dashed cells will be deleted and $\psi_4(1, 2, 8, 11) = ((1, 2, 6, 8), 5)$.

Iteration of the deletion map ψ_n, until reaching the partition of length n with all parts equal to zero, will give us the definition of bijection Ψ_n from $\mathcal{L}_n$ to $\mathcal{O}_n$. For a given lecture hall partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{L}_n$, let $\lambda^{(0)} = \lambda$, then for each $i = 1, 2, \ldots, |\lambda|_a$, we recursively define $\lambda^{(i)}$ and σ_i by $\psi_n(\lambda^{(i-1)}) = (\lambda^{(i)}, \sigma_i)$. By iteration of ψ_n we will end up with lecture hall partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{|\lambda|_a}$ and odd integers $\sigma_1, \sigma_2, \ldots, \sigma_{|\lambda|_a}$. Then we define

$$\Psi_n(\lambda) = \sigma_1 \sigma_2 \ldots \sigma_{|\lambda|_a}$$

Example 3.1.4 *In Example 3.1.1, we have found that* $\psi_4(1,2,8,11) = ((1,2,6,8),5)$*. Then iteration of the deletion function will give us* $\psi_4(1,2,6,8) = ((0,0,5,7),5)$*,* $\psi_4(0,0,5,7) = ((0,0,3,4),5)$ *and* $\psi_4(0,0,3,4) = ((0,0,0,0),7)$*. Therefore,* $\Psi_4(1,2,8,11) = (5,5,5,7)$*.*

3.1.2 Definition of the Map Φ_n

Since Ψ_n is defined by iteration of the deletion map ψ_n, first we need to define the inverse of the deletion map. Recall that the deletion map takes a lecture hall partition and gives us a smaller lecture hall partition and an odd integer, which is the number of the deleted cells. Hence, the inverse of ψ_n must take a pair $(\mu, 2k-1)$, where $\mu \in \mathcal{L}_n$ and k is an positive integer less than $n+1$, and insert $2k-1$ cells to μ.

For defining the insertion map, ϕ_n , we need to decide on the part of the given lecture hall partition μ to start insertion. As in the deletion map, ϕ_n will add one cell each to the largest parts of μ until we run out of cells or we meet a pair $(\mu_{n-2c-1}, \mu_{n-2c})$ such that

$$\frac{\mu_{n-2c-1}}{n-2c-1} = \frac{\mu_{n-2c}}{n-2c}.$$

If we meet a pair as above, we add $(k-c-1)$ and $(k-c)$ cells to the pair $(\mu_{n-2c-1}, \mu_{n-2c})$, respectively.

Note that c is the corresponding value for the starting point of deletion (b) with suitable k_j values. Let c be the minimum of the set

$$\left\{ j : \frac{\mu_{n-2j-1}}{n-2j-1} = \frac{\mu_{n-2j}}{n-2j}, 0 \le j < \lfloor n/2 \rfloor \right\} \cup \{k-1\}.$$

Let $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ be the sequence defined by

$$\tau_{n-i} = \mu_{n-i} + 1 \qquad \text{for } 0 \le i \le 2c-1;$$

$$\tau_{n-2c} = \mu_{n-2c} + (k-c);$$

$$\tau_{n-2c-1} = \mu_{n-2c-1} + (k-c-1);$$

$$\tau_{n-i} = \mu_{n-i} \qquad \text{for } 2c+2 \le i < n$$

Then define $\phi_n(\mu, 2k-1) = \tau$.

Example 3.1.5 *For* $\mu = (0,0,1,5,7)$*,* $n = 5$ *and* $k = 2$*. So we need to find the* c *value. Note that* $\mu_1/1 \ne \mu_2/2$ *and* $\mu_3/3 \ne \mu_4/4$*. So* $c = k-1 = 1$*.*

The sequence defined above will give us the following:

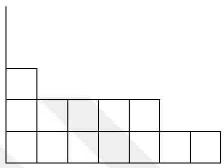
$$\tau_{5-0} = \tau_5 = \mu_5 + 1 = 8$$

$$\tau_{5-1} = \tau_4 = \mu_4 + 1 = 6$$

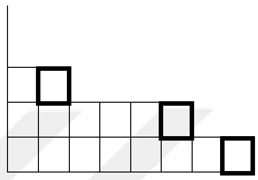
$$\tau_{5-2.1} = \tau_3 = \mu_3 + (2-1) = 2$$

$$\tau_{5-2.1-1} = \tau_2 = \mu_2 + (2-1-1) = 0$$

$$\tau_{5-4} = \tau_1 = \mu_1 = 0$$



$\mu = (0,0,1,5,7), n = 5, k = 2, c = 1$



$\phi_4(\mu, 2k-1) = (0,0,2,6,8)$

Table 3.2: Young diagrams for μ and $\phi_4(\mu, 2k-1)$

So $\phi_4(\mu, 2k-1) = (0,0,2,6,8)$.

Now we can define Φ_n, which is the inverse of Ψ_n, recursively. Let $\lambda^{(0)}$ be the partition that consists of n zeros. If $\lambda^{(i)} \in \mathcal{L}_n$, then we recursively define $\lambda^{(i+1)} \in \mathcal{L}_n$ for $i = 0, 1, \ldots, \ell-1$ by $\lambda^{(i+1)} = \phi_n(\lambda^{(i)}, \sigma_{\ell-i})$ where $\sigma = \sigma_1\sigma_2\ldots\sigma_\ell \in \mathcal{O}_n$. By iteration of ϕ_n, we can define $\Phi_n : \mathcal{O}_n \to \mathcal{L}_n$ as

$$\Phi_n(\sigma) = \lambda^{(\ell)}.$$

Example 3.1.6 *For given* $\sigma = 1, 3, 3, 5, 5$, *we will start with* $\lambda^{(0)} = (0,0,0,0,0)$. *Then*

$$\lambda^{(1)} = \phi_5(\lambda^{(0)}, \sigma_5 = 5) = (0,0,0,2,3)$$

$$\lambda^{(2)} = \phi_5(\lambda^{(1)}, \sigma_4 = 5) = (0,1,2,3,4)$$

$$\lambda^{(3)} = \phi_5(\lambda^{(2)}, \sigma_3 = 3) = (0,1,3,4,5)$$

$$\lambda^{(4)} = \phi_5(\lambda^{(3)}, \sigma_2 = 3) = (0,1,3,5,7)$$

$$\lambda^{(5)} = \phi_5(\lambda^{(4)}, \sigma_1 = 1) = (0,1,3,5,8)$$

Therefore $\Phi_5(\sigma) = (0,1,3,5,8)$.

3.2. Properties of the Deletion and Insertion Maps

Lemma 3.2.1 *For a given lecture hall partition $\tau \in \mathcal{L}_n$, let $\psi_n(\tau) = (\mu, 2k-1)$. Then μ is a lecture hall partition of length n.*

Proof: Let b be the starting point of deletion. We want to show that the pair (μ_{n-h}, μ_{n-h+1}) satisfies the lecture hall condition for $h = 1, 2, \ldots, n-1$. We examine the pairs depending on the values of h, therefore we have three cases as follows: (i) $1 \leq h \leq 2b-1$, (ii) $2b+3 \leq h$, (iii) $2b \leq h \leq 2b+2$.

Note that the cases look like

$$\underbrace{\mu_1\ \mu_2 \cdots \mu_{n-2b-3}}_{(ii)} \Bigg| \underbrace{\mu_{n-2b-2}\ \mu_{n-2b-1}\ \mu_{n-2b}}_{(iii)} \Bigg| \underbrace{\mu_{n-2b+1} \cdots \mu_n}_{(i)}$$

Case(i): $1 \leq h \leq 2b-1$

Since $\tau \in \mathcal{L}_n$, it satisfies the lecture hall condition. Also by definition of the deletion map, $\mu_{n-h} = \tau_{n-h} - 1$ for the chosen values of h. Therefore, since

$$\frac{\tau_{n-h}}{n-h} \leq \frac{\tau_{n-h+1}}{n-h+1},$$

we have that

$$\frac{\tau_{n-h}-1}{n-h} \leq \frac{\tau_{n-h+1}-1}{n-h+1}$$

and this implies

$$\frac{\mu_{n-h}}{n-h} \leq \frac{\mu_{n-h+1}}{n-h+1},$$

the pair (μ_{n-h}, μ_{n-h+1}) satisfies the lecture hall condition.

Case(ii): $2b+3 \leq h$

We know that the pair $(\tau_{n-h}, \tau_{n-h+1})$ satisfies the lecture hall condition and by definition of ψ_n, $\tau_{n-h} = \mu_{n-h}$ for $2b+2 \leq h$. So it is already true for $2b+3 \leq h$. Hence (μ_{n-h}, μ_{n-h+1}) satisfies the lecture hall condition.

Case(iii): $2b \leq h \leq 2b+2$

We have three different values for h: $2b$, $2b+1$ and $2b+2$.

$\underline{h = 2b}$: Recall that τ_{n-2b} and τ_{n-2b+1} satisfy the lecture hall condition, and $k-b \geq 1$. By definition of ψ_n,

$$\mu_{n-2b} = \tau_{n-2b} - (k-b) \text{ and } \mu_{n-2b-1} = \tau_{n-2b-1} - 1.$$

Since the pair $(\tau_{n-2b}, \tau_{n-2b+1})$ satisfies the lecture hall condition, we have

$$\frac{\tau_{n-2b}}{n-2b} \leq \frac{\tau_{n-2b+1}}{n-2b+1},$$

and since $k-b \geq 1$ we have

$$\frac{\tau_{n-2b}-(k-b)}{n-2b} \leq \frac{\tau_{n-2b+1}-1}{n-2b+1}.$$

By definition of μ_{n-2b} and μ_{n-2b+1}, and by the last inequality above the pair $(\mu_{n-2b}, \mu_{n-2b+1})$ satisfy the lecture hall condition.

$\underline{h = 2b+1}$: By definition of the deletion map

$$\mu_{n-2b-1} = \tau_{n-2b-1} - (k-b-1) \text{ and } \mu_{n-2b} = \tau_{n-2b} - (k-b).$$

Since we have

$$\frac{\tau_{n-2b-1}}{n-2b-1} \leq \frac{\tau_{n-2b}}{n-2b} \text{ and } k-b \geq 1,$$

$(\mu_{n-2b-1}, \mu_{n-2b})$ satisfies the lecture hall condition

$$\frac{\tau_{n-2b-1}-(k-b-1)}{n-2b-1} \leq \frac{\tau_{n-2b}-(k-b)}{n-2b}.$$

$\underline{h = 2b+2}$: Since

$$\mu_{n-2b-2} = \tau_{n-2b-2} \quad \text{and} \quad \mu_{n-2b-1} = \tau_{n-2b-1} - (k-b-1),$$

we need to examine two cases as $b = k-1$ and $b < k-1$.

If $b = k-1$, then we have

$$\mu_{n-2b-2} = \tau_{n-2b-2} \text{ and } \mu_{n-2b-1} = \tau_{n-2b-1}.$$

Since $(\tau_{n-2b-2}, \tau_{n-2b-1})$ satisfies the lecture hall condition, $(\mu_{n-2b-2}, \mu_{n-2b-1})$ satisfies it, too.

If $b < k-1$, then we have two cases to consider: $b+1 \in A$ or not. Let $b+1 \in A$. Then $k_{b+1} = r_{n-2b-2} + b + 1$ then by minimality of k we get $r_{n-2b-2} + b + 1 \geq r_{n-2b} + b$ which implies that $r_{n-2b-2} \geq r_{n-2b} - 1 = r_{n-2b-1}$ where r_h is the remainder of τ_h. Hence

$$\left\lceil \frac{\tau_{n-2b-2}}{n-2b-2} \right\rceil < \left\lceil \frac{\tau_{n-2b-1}}{n-2b-1} \right\rceil$$

since τ is a lecture hall partition.

Now let $b+1 \notin A$. Then $k_{b+1} = b+2$ and $b+2 \geq r_{n-2b} + b$, so $2 \geq r_{n-2b}$ and $2 = r_{n-2b} = r_{n-2b-1} + 1$. Since $\mu_{n-2b-2} = \tau_{n-2b-2}$ and we assume that $b < k-1$,

$$\mu_{n-2b-1} = (n-2b-1)\left(\left\lceil \frac{\tau_{n-2b-2}}{n-2b-2} \right\rceil - 1\right).$$

Recall that τ is lecture hall partition which satisfies

$$\lceil\frac{\tau_{n-2b-2}}{n-2b-2}\rceil < \lceil\frac{\tau_{n-2b-1}}{n-2b-1}\rceil.$$

Hence, $\frac{\mu_{n-2b-2}}{n-2b-2} \leq \frac{\mu_{n-2b-1}}{n-2b-1}$ is obtained.

□

By Lemma 3.2.1, we have that we can iterate ψ_n, since it takes a lecture hall partition, τ , and returns the pair $(\mu, 2k-1)$ where μ is a lecture hall partition and $k \in \mathbb{Z}^+$.

Example 3.2.7 *Let $\tau = (0, 1, 3, 4, 5)$. Then $\ell(\tau) = 4$ (Recall that $\ell(\tau)$ is defined as number of non-zero parts in τ.) so we need to check if $j = 0$ and $j = 1$ are in the set A or not. Since*

$$\left\lceil\frac{\tau_4}{4}\right\rceil = \left\lceil\frac{4}{4}\right\rceil = \left\lceil\frac{5}{5}\right\rceil = \left\lceil\frac{\tau_5}{5}\right\rceil \text{ and } r_4 + 1 = 4 + 1 = 5 = r_5,$$

$j = 0 \in A$. By similar reasoning $j = 1 \notin A$. (Note that $\lceil\frac{\tau_2}{2}\rceil = \lceil\frac{1}{2}\rceil = \lceil\frac{3}{3}\rceil = \lceil\frac{\tau_3}{3}\rceil$ but $r_2 + 1 = 1 + 1 \neq 3 = r_3$.)

Now we need to calculate the corresponding k_j values. By definition of k_j, for $j = 0 \in A$ $k_0 = r_5 = 5$ and for $j = 1 \notin A$ $k_1 = 1 + 1 = 2$. Recall that

$$k = \min\left\{k_j : j < \left\lceil\frac{\ell(\tau)}{2}\right\rceil\right\},$$

then $k = \min\{k_0, k_1\} = \min\{5, 2\} = 2$. Also by definition of b, for this example $b = 1$.

By considering the the determined values of k and b, it is possible to write the following sequence,

$$\mu_{5-0} = \mu_5 = \tau_5 - 1 = 4$$

$$\mu_{5-1} = \mu_4 = \tau_4 - 1 = 3$$

$$\mu_{5-2.1} = \mu_3 = \tau_3 - (2-1) = 2$$

$$\mu_{5-2.1-1} = \mu_2 = \tau_2 - (2-1-1) = 1$$

$$\mu_{5-4} = \mu_1 = \tau_1 = 0$$

(i) From the sequence $(\mu_4, \mu_5) = (3, 4)$, so $\lceil\frac{3}{4}\rceil = 1 = \lceil\frac{4}{5}\rceil$ and $r_4+1 = 3+1 = 4 = r_5$. Thus (μ_4, μ_5) satisfy the lecture hall condition.

(ii) For $(\mu_0, \mu_1) = (0, 0)$ we can easily conclude that the pair satisfies the lecture hall condition.

(iii) If h=2, then we have the pair (μ_3, μ_4). Recall that (τ_3, τ_4) satisfies the lecture hall condition then since $\mu_3 = \tau_3 - (2-1)$ and $\mu_4 = \tau_4 - 1$, (μ_3, μ_4) also satisfies the lecture hall condition.

If $h = 3$, then we have the pair (μ_2, μ_3). We have that $\mu_2 = \tau_2 - (2 - 1 - 1)$ and $\mu_3 = \tau_3 - (2 - 1)$ so the pair (μ_2, μ_3) satisfies the lecture hall condition.

Finally, if $h = 4$, the we need to examine the pair (μ_1, μ_2). Since we are in the case $b = k - 1$ (recall that $k = 2$ and $b = 1$) then we have $\mu_1 = \tau_1 = 0$ and $\mu_2 = \tau_2 = 1$. The inequality $\lceil \frac{0}{1} \rceil = 0 < 1 = \lceil \frac{1}{2} \rceil$ implies that the pair (μ_1, μ_2) satisfies the lecture hall condition.

Thus, μ is a lecture hall partition since (μ_i, μ_{i+1}) satisfies the lecture hall condition for all $i = 0, 1, 2, 3$ and 4.

Lemma 3.2.2 *For a given lecture hall partition $\tau \in \mathcal{L}_n$ such that $|\tau|_a > 1$, let $\psi_n(\tau) = (\mu, 2k - 1)$ and $\psi_n(\mu) = (\rho, 2m - 1)$. Then $k \leq m$.*

Proof: Let

$$A = \left\{ j < \left\lfloor \frac{\ell(\tau)}{2} \right\rfloor \ : \left\lceil \frac{\tau_{n-2j-1}}{n-2j-1} \right\rceil = \left\lceil \frac{\tau_{n-2j}}{n-2j} \right\rceil \text{ and } r_{n-2j-1} + 1 = r_{n-2j} \right\}$$

and

$$B = \left\{ 0, 1, 2, \ldots, \left\lceil \frac{\ell(\tau)}{2} \right\rceil \right\} \setminus A.$$

Also let A' and B' be the respective sets for μ, and r'_{n-h} be the remainder of μ_{n-h}. Let b and b' be the starting point of deletion from τ and μ, respectively.

By definition of the number of cells that will be deleted,

$$k = \min\bigl(\{r_{n-2j} + j : j \in A\} \cup \{j + 1 : j \in B\}\bigr) \text{ and}$$

$$m = \min\bigl(\{r'_{n-2j} + j : j \in A'\} \cup \{j + 1 : j \in B'\}\bigr).$$

Then we have three cases depending on the relation between b and b'.

For $h \geq n - 2b + 1$, $\tau_h - 1 = \mu_h$ and $r'_{n-2j} = r_{n-2j} - 1$, by the definition of the deletion map. By minimality of k, for all $j < b$, we have that $r_{n-2j} + j > k$ and hence $r'_{n-2j} \geq k$. The last inequality proves that $k \leq m$, if $b > b'$.

Now, for $h \leq n - 2b - 2$, $\tau_h = \mu_h$ and $r'_{n-2j} = r_{n-2j}$. By minimality of k, for all $j > b$, we have $k \leq j + 1$ if $j \in B$ and $k \leq r_{n-2j} + j$ if $j \in A$. Similarly for all $j > b$, we have $k \leq j + 1$ if $j \in B'$ and $k \leq r'_{n-2j} + j$ if $j \in A'$. This shows that $k \leq m$ if $b < b'$.

Finally, if $b = b'$ and $k = b + 1$, then $k \leq m$. If $b = b'$ and $k = r_{n-2b} + b$, then $b \in A'$ and $m = r'_{n-2b} + b = (n - 2b) + b \geq k$.

□

Lemma 3.2.2 shows that parts of σ are weakly increasing, since iteration of ψ_n gives us the parts of σ. So we have showed that Ψ_n is given by the iteration of the deletion map ψ_n by Lemma 3.2.1, and the parts of the partition constructed by $\Psi_n(\lambda)$ are weakly decreasing by Lemma 3.2.2. Hence we have the following theorem.

Theorem 3.2.3 *For a given $\lambda \in \mathcal{L}_n$, $\Psi_n(\lambda)$ is a partition whose parts are odd integers less than $2n$, and so Ψ_n is well-defined.*

Lemma 3.2.4 *For a given* $\sigma_1\sigma_2\ldots\sigma_\ell \in \mathcal{O}_n$ *such that* $\sigma_1 = 2k-1$*, suppose that* $\phi_n(\lambda^{(i)}, \sigma_{\ell-i}) = \lambda^{(i+1)}$ *is a lecture hall partition for all* $i = 0,1,\ldots,\ell-1$*, where* $\lambda^{(0)} = (0,0,\ldots,0) \in \mathcal{L}_n$*. Then, for all* $j = 0,1,\ldots,k-2$*,*

$$\left\lceil \frac{\lambda^{(\ell)}_{n-2j-1}}{n-2j-1} \right\rceil = \left\lceil \frac{\lambda^{(\ell)}_{n-2j}}{n-2j} \right\rceil \text{ and } r_{n-2j-1}+1 = r_{n-2j} \tag{3.2}$$

where r_h *is the remainder of* $\lambda^{(\ell)}_h$*.*

Proof: We prove the statement by induction on ℓ. As a basis step if $\ell = 0$, then there is nothing to insert and since $\lambda^{(0)} = (0,0,\ldots,0)$ is lecture hall partition, it satisfies the conditions in (5).

Suppose that the statement is true for $\ell-1$. Let c be the starting point of the ℓ^{th} insertion. Note that $c < k$ and σ is weakly increasing. Firstly it must be shown that, for any j, $0 \le j \le c-1$, $\lambda^{(\ell-1)}_{n-2j-1}$ is a multiple of $(n-2j-1)$ if and only if $\lambda^{(\ell-1)}_{n-2j}$ is a multiple of $(n-2j)$. Assume that $\lambda^{(\ell-1)}_{n-2j-1} = q(n-2j-1)$ where $q \in \mathbb{Z}^+$. Then $r_{n-2j-1} = n-2j-1$. The induction hypothesis implies that $\lambda^{(\ell-1)}_{n-2j} = q(n-2j)$ and $r_{n-2j} = n-2j = r_{n-2j-1}+1$. similarly, it can be shown that if $\lambda^{(\ell-1)}_{n-2j}$ is a multiple of $(n-2j)$, then $\lambda^{(\ell-1)}_{n-2j-1}$ is a multiple of $(n-2j-1)$. By minimality of c, $\lambda^{(\ell-1)}_{n-2j-1}$ and $\lambda^{(\ell-1)}_{n-2j}$ cannot be multiple of $(n-2j-1)$ and $(n-2j)$, respectively. Thus for $0 \le j \le c-1$,

$$\left\lceil \frac{\lambda^{(\ell)}_{n-2j-1}}{n-2j-1} \right\rceil = \left\lceil \frac{\lambda^{(\ell-1)}_{n-2j-1}}{n-2j-1} \right\rceil \text{ and } r_{n-2j-1}+1 = r'_{n-2j}$$

where r'_h is the remainder of $\lambda^{(\ell)}_h$. Hence for $0 \le j \le c-1$, the statement is true for $\lambda^{(\ell)}$ by the induction hypothesis.

For $c \le j < k-1$, we need to consider two cases as $c = k-1$ and $c < k-1$. If $c = k-1$, then the conclusion is trivial. Assume that $c < k-1$, so the pair $\left(\lambda^{(\ell-1)}_{n-2c-1}, \lambda^{(\ell-1)}_{n-2c}\right)$ is critical. Hence the pair is multiple of the pair $(n-2c-1, n-2c)$, respectively. This implies that there is an integer q such that $\lambda^{(\ell-1)}_{n-2c-1} = q(n-2c-1)$ and $\lambda^{(\ell-1)}_{n-2c} = q(n-2c)$. By definition of the insertion map, we have that $\lambda^{(\ell)}_{n-2c-1} = q(n-2c-1)+(k-c-1)$ and $\lambda^{(\ell)}_{n-2c} = q(n-2c)+(k-c)$. Hence $\lambda^{(\ell)}$ is a lecture hall partition. This implies that $k-c \le n-2c$, so that $r_{n-2c} = r_{n-2c-1}+1 = k-c$. So, for $j = c$ the condiditons (5) holds. For $h \ge 2(c+1)$, $\lambda^{(\ell)}_{n-h} = \lambda^{(\ell-1)}_{n-h}$ by definition of the insertion map.

□

Recall the definition of set A from the deletion map, ψ_n. Lemma 3.2.4 guarantees that the partitions which are obtained by the insertion map, ϕ_n, can be an input for the deletion map ψ_n.

Corollary 3.2.5 *Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_\ell \in \mathcal{O}_n$. Assume that $\phi_n(\lambda^{(i)}, \sigma_{\ell-i}) = \lambda^{(i+1)}$ is a lecture hall partition for all $i < \ell$. Let c be the starting point of the insertion for $\lambda^{(\ell)}$. Then for $0 \leq h \leq 2c-1$, $\lambda^{(\ell)}_{n-h}$ is not a multiple of $(n-h)$.*

Note that proof of the Corollary 3.2.5 is given in the proof of the Lemma 3.2.4 and the result will be used in the upcoming properties.

Lemma 3.2.6 *For a given $\sigma = \sigma_1\sigma_2 \ldots \sigma_\ell \in \mathcal{O}_n$ such that $\sigma_1 = 2k-1$, suppose that $\phi_n(\lambda^{(i)}, \sigma_{\ell-i}) = \lambda^{(i+1)}$ is a lecture hall partition for all $i = 0, 1, ..., \ell-1$, where $\lambda^{(0)} = (0, 0, \ldots, 0) \in \mathcal{L}_n$. For all $j = 0, 1, ..., k-2$, if $\lambda^{(\ell)}_{n-2j} > 0$, then $r_{n-2j-1} \geq k-j-1$ and $r_{n-2j} \geq k-j$, where r_h is the remainder of $\lambda^{(\ell)}_h$.*

Proof: We prove the statement by induction on ℓ. As a basis step, if $\ell = 0$, similar to the proof of the previous lemma, since there is nothing to insert, remainders will satisfy the given conditions.

Assume that the statement is true for $\ell - 1$. Let c be the starting point of the ℓ^{th} insertion. So we have three different cases depending on the values of h: (i) $h \geq 2(c+1)$, (ii) $h < 2c$, (iii) $h = 2c, 2c+1$.

Note that the cases look like

$$\underbrace{\lambda^{(\ell-1)}_1 \; \lambda^{(\ell-1)}_2 \cdots \lambda^{(\ell-1)}_{n-2c-2}}_{(i)} \;\Big|\; \underbrace{\lambda^{(\ell-1)}_{n-2c-1} \; \lambda^{(\ell-1)}_{n-2c}}_{(iii)} \;\Big|\; \underbrace{\lambda^{(\ell-1)}_{n-2c+1} \cdots \lambda^{(\ell-1)}_{n}}_{(ii)}$$

Case(i): $h \geq 2(c+1)$

By definition of the insertion map we have that $\lambda^{(\ell)}_{n-h} = \lambda^{(\ell-1)}_{n-h}$. Since σ is weakly increasing, the induction hypothesis implies that the statement is true for $c < j < k-1$.

Case(ii): $h < 2c$

By Corollary 3.2.5, $\lambda^{(\ell-1)}_{n-h}$ is not multiple of $n-h$ for chosen values of h. This implies that $r'_{n-h} < n-h$, where r'_{n-h} is the remainder of $\lambda^{(\ell-1)}_{n-h}$, and after ℓ^{th} insertion $r_{n-h} = r'_{n-h} + 1$. By the induction hypohesis, and the fact that σ is weakly increasing, the statement is true.

Case(iii): $h = 2c, 2c+1$

If $c = k-1$, then we have $r'_{n-2c-1} \geq 0$ and $r_{n-2c} \geq 1$ and we are done. So we need to prove that the pair $(\lambda^{(\ell)}_{n-2c-1}, \lambda^{(\ell)}_{n-2c})$ satisfies the statement. Since $\lambda^{(\ell)}_{n-2c-1} = \lambda^{(\ell-1)}_{n-2c-1} + (k-c-1)$, $\lambda^{(\ell)}_{n-2c} = \lambda^{(\ell-1)}_{n-2c} + (k-c)$ and $\lambda^{(\ell)}$ is a lecture hall partition, $r_{n-2c-1} = (k-c-1)$ and $r_{n-2c} = (k-c)$.

□

Lemma 3.2.7 *For a given $\sigma = \sigma_1\sigma_2 \ldots \sigma_\ell \in \mathcal{O}_n$ such that $\sigma_1 = 2k-1$, suppose that $\phi_n(\lambda^{(i)}, \sigma_{\ell-i}) = \lambda^{(i+1)}$ is a lecture hall partition for all $i = 0, 1, ..., \ell-1$, where*

$\lambda^{(0)} = (0, 0, \ldots, 0) \in \mathcal{L}_n$. *Let* $\phi_n(\lambda^{(\ell-1)}, \sigma_1) = \lambda^{(\ell)}$. *Then for all* $h = 1, 2, \ldots, n$,

$$\left\lceil \frac{\lambda_h^{(\ell)}}{h} \right\rceil \leq \left\lceil \frac{\lambda_h^{(\ell-1)}}{h} \right\rceil + 1$$

Proof: Let c be the starting point of the ℓ^{th} insertion. By Corollary 3.2.5, we have that $\lambda_h^{(\ell-1)}$ cannot be a multiple of h for $h \geq n - 2c + 1$. Since the definition of the insertion map implies that $\lambda_h^{(\ell)} = \lambda_h^{(\ell-1)} + 1$, we have the inequality in the statement already. So by the definition of the insertion map we have that

$$\begin{cases} \left\lceil \frac{\lambda_h^{(\ell)}}{h} \right\rceil \geq \left\lceil \frac{\lambda_h^{(\ell-1)}}{h} \right\rceil & \text{if } h = n-2c-1, n-2c \\ \left\lceil \frac{\lambda_h^{(\ell)}}{h} \right\rceil = \left\lceil \frac{\lambda_h^{(\ell-1)}}{h} \right\rceil & \text{if } h \leq n-2c-2 \end{cases}$$

Therefore the statement holds for $h \leq n - 2c - 2$ and we need to prove for $h = n - 2c - 1, n - 2c$. Assume on the contrary that

$$\left\lceil \frac{\lambda_{n-2c}^{(\ell)}}{n-2c} \right\rceil > \left\lceil \frac{\lambda_{n-2c}^{(\ell-1)}}{n-2c} \right\rceil + 1.$$

This assumption implies that $k - c \geq 2$. So we get $k - c > n - 2c$. Let us consider r'_{n-2c+2} which is the remainder of $\lambda_{n-2c+2}^{(\ell-1)}$. By Lemma 3.2.6 we have $r'_{n-2c+2} \geq k-c+1$. Now, we compute the difference between $n - 2c + 2$ and r'_{n-2c+2}:

We have the inequality

$$n - 2c + 2 - r'_{n-2c+2} \leq n - 2c + 2 - (k - c + 1)$$

since $r'_{n-2c+2} \geq k - c + 1$ by Lemma 3.2.6.

Also

$$n - 2c + 2 - (k - c + 1) < n - 2c + 2 - (n - 2c + 1) = 1$$

by considering $k - c > n - 2c$. Thus,

$$n - 2c + 2 - r'_{n-2c+2} < 1$$

and this strict inequality implies that $n-2c+2 = r'_{n-2c+2}$. Hence $\lambda_{n-2c+2}^{(\ell-1)}$ is a multiple of $n - 2c + 2$ which contradicts the fact we proved in Corollary 3.2.5.

Assume that

$$\left\lceil \frac{\lambda_{n-2c-1}^{(\ell)}}{n-2c-1} \right\rceil > \left\lceil \frac{\lambda_{n-2c-1}^{(\ell-1)}}{n-2c-1} \right\rceil + 1.$$

This assumption implies that $k - c - 1 \geq 1$. So we get $k - c - 1 > n - 2c - 1$. Let us consider r'_{n-2c+1} which is the remainder of $\lambda_{n-2c+1}^{\ell-1}$. By Lemma 3.2.6 we have $r'_{n-2c+1} \geq k - c + 1 - 1$. Now, the difference between $n - 2c + 1$ and r'_{n-2c+1} will give us:

$$n - 2c + 1 - r'_{n-2c+1} \leq n - 2c + 1 - (k - c)$$

since $r'_{n-2c+1} \geq k-c$ by Lemma 3.2.6.

Besides,

$$n-2c+1-(k-c) < n-2c+1-(n-2c) = 1$$

by the fact that $k-c > n-2c$. Hence,

$$n-2c+1-r'_{n-2c+1} < 1$$

and this implies that $n-2c+1 = r'_{n-2c+1}$. Then we have that $\lambda^{(\ell-1)}_{n-2c+1}$ is a multiple of $n-2c+1$ which gives a contradiction.

□

Notice that this lemma gives a bound for the increment in the h^{th} part, which is h, after ℓ^{th} insertion.

Lemma 3.2.8 *For a given $\sigma = \sigma_1\sigma_2\ldots\sigma_\ell \in \mathcal{O}_n$ such that $\sigma_1 = 2k-1 > n$, suppose that $\phi_n(\lambda^{(i)}, \sigma_{\ell-i}) = \lambda^{(i+1)}$ is a lecture hall partition for all $i = 0, 1, ..., \ell-1$, where $\lambda^{(0)} = (0, 0, \ldots, 0) \in \mathcal{L}_n$. Then there is a $j \in \{1, 2\ldots, \lfloor n/2 \rfloor - 1\}$ satisfying that*

$$\frac{\lambda^{(\ell)}_{n-2j-1}}{n-2j-1} = \frac{\lambda^{(\ell)}_{n-2j}}{n-2j}. \tag{3.3}$$

In the definition of starting point of insertion, c, we have the following set

$$\left\{ j : \frac{\mu_{n-2j-1}}{n-2j-1} = \frac{\mu_{n-2j}}{n-2j}, 0 \leq j < \lfloor n/2 \rfloor \right\}.$$

Lemma 3.2.8 guarantess the existence of such j values.

Proof: Recall that in the proof of the Lemma 3.2.1, we showed that $\lambda^{(\ell)}_{n-2j-1}$ is a multiple of $n-2j-1$ if and only if $\lambda^{(\ell)}_{n-2j}$ is a multiple of $n-2j$. Hence the equality (3.3) holds. Now, we need to consider two cases depending on the parity of n.

Case(i): *n is even.*

The equality (3.3) holds for $j = (n/2)-1$ as $\lambda^{(\ell)}_1$ is multiple of 1.

Case(ii): *n is odd.*

If n is odd, then $n-2j-1$ will be even. Now, we consider $\lambda^{(\ell)}_2$. If $\lambda^{(\ell)}_2 = 0$, then $\lambda^{(\ell)}_2$ is multiple of 2, hence the equality (3.3) holds. Assume $\lambda^{(\ell)}_2 > 0$, then by Lemma 3.2.6,

$$r_2 \geq k - \frac{n-3}{2} - 1, \tag{3.4}$$

since for $j = \frac{n-3}{2}$, $n-2j-1 = 2$. By assumption $2k-1 \geq n+1$ but since n is odd we have

$$2k-1 \geq n+2 \tag{3.5}$$

instead. So combining (3.4) and (3.5) implies that

$$r_2 \geq k - \frac{n-3}{2} - 1 \geq k - \frac{2k-6}{2} - 1 = k - k + 3 - 1 = 2.$$

Hence $\lambda_2^{(\ell)}$ is multiple of 2, and the pair $(\lambda_2^{(\ell)}, \lambda_3^{(\ell)})$ satisfies the equation (3.3).

□

Theorem 3.2.9 *For any $\sigma \in \mathcal{O}_n$, $\Phi_n(\sigma)$ is a lecture hall partition.*

Proof: Let $\sigma = \sigma_1\sigma_2\ldots\sigma_\ell$ and $\sigma_1 = 2k-1$ for some $k \in \mathbb{Z}^+$. We will apply induction on ℓ. Suppose that $\Phi_n(\sigma_2\sigma_3\ldots\sigma_\ell) = \lambda^{(\ell-1)}$ is a lecture hall partition and c is the starting point of ℓ^{th} insertion. Then we have the following three cases depending on the values of h, where $\lambda_{n-h}^{(\ell-1)}$, (i) $0 \leq h \leq 2c-1$, (ii) $2c+3 \leq h < n$, (iii) $h = 2c, 2c+1, 2c+2$.

Case(i): By Corollary 3.2.5, for $0 \leq h \leq 2c-1$, $\lambda_{n-h}^{(\ell-1)}$ is not multiple of $n-h$. Thus,

$$\left\lceil \frac{\lambda_{n-h}^{(\ell)}}{n-h} \right\rceil = \left\lceil \frac{\lambda_{n-h}^{(\ell-1)}}{n-h} \right\rceil \text{ and } r'_{n-h} + 1 = r_{n-h}$$

where r'_{n-h} and r_{n-h} are remainders of $\lambda_{n-h}^{(\ell-1)}$ and $\lambda_{n-h}^{(\ell)}$, respectively. Since $\lambda^{(\ell-1)}$ is lecture hall partition and by induction assumption $\lambda_{n-h}^{(\ell)}$ and $\lambda_{n-h+1}^{(\ell)}$ satisfy the lecture hall condition for $1 \leq h \leq 2c-1$.

Case(ii): For $2c+2 \leq h < n$, $\lambda_{n-h}^{(\ell)} = \lambda_{n-h}^{(\ell-1)}$ by definition of the deletion map. Then by induction hypothesis $\lambda_{n-h}^{(\ell)}$ and $\lambda_{n-h+1}^{(\ell)}$ satisfy the lecture hall condition.

Case(iii): For $h = 2c+2$, since $\lambda_{n-2c-2}^{(\ell)} = \lambda_{n-2c-2}^{(\ell-1)}$ and $\lambda_{n-2c-1}^{(\ell)} = \lambda_{n-2c-1}^{(\ell-1)} + (k-c-1)$, the pair $(\lambda_{n-2c-2}^{(\ell)}, \lambda_{n-2c-1}^{(\ell)})$ satisfies the lecture hall condition. (Note that here we have $k > c$.)

For $h = 2c+1$, by Lemma 3.2.7 $\lambda_{n-2c-1}^{(\ell)}$ and $\lambda_{n-2c}^{(\ell)}$ satisfy the lecture hall condition.

Finally, for $h = 2c$, we need to show that $\lambda_{n-2c}^{(\ell)}$ and $\lambda_{n-2c+1}^{(\ell)}$ satisfy the lecture hall condition. By Corollary 3.2.5, $\lambda_{n-2c+1}^{(\ell-1)}$ is not multiple of $n-2c+1$. Then we need to consider the cases that $\lambda_{n-2c}^{(\ell-1)}$ is not a multiple of $n-2c$ and it is.

If $\lambda_{n-2c}^{(\ell-1)}$ is not a multiple of $n-2c$, then $\lambda_{n-2c}^{(\ell)} = \lambda_{n-2c}^{(\ell-1)} + 1$. Also we have $\lambda_{n-2c+1}^{(\ell)} = \lambda_{n-2c+1}^{(\ell-1)} + 1$. So $r_{n-2c} = r'_{n-2c} + 1$ and $r_{n-2c+1} = r'_{n-2c+1} + 1$. Therefore, $\lambda_{n-2c}^{(\ell)}$ and $\lambda_{n-2c+1}^{(\ell)}$ satisfy the lecture hall condition.

If $\lambda_{n-2c}^{(\ell-1)}$ is a multiple of $n-2c$, then

$$\left\lceil \frac{\lambda_{n-2c}^{(\ell-1)}}{n-2c} \right\rceil < \left\lceil \frac{\lambda_{n-2c+1}^{(\ell-1)}}{n-2c+1} \right\rceil$$

since $\lambda^{(\ell-1)}$ is lecture hall partition. Assume that $\lambda_{n-2c}^{(\ell)}$ and $\lambda_{n-2c+1}^{(\ell)}$ do not satisfy the lecture hall condition, i.e.,

$$\left\lceil \frac{\lambda_{n-2c}^{(\ell)}}{n-2c} \right\rceil > \left\lceil \frac{\lambda_{n-2c+1}^{(\ell)}}{n-2c+1} \right\rceil \text{ and } r_{n-2c} \geq r_{n-2c+1}.$$

From Lemma 3.2.7, we can obtain $r_{n-2c} = k - c$. From Lemma 3.2.6 and the definition of the insertion map implies that $r_{n-2c+1} = r'_{n-2c+1} + 1 \geq m - c + 1$ where $m \in \mathbb{Z}^+$ such that $\sigma_2 = 2m - 1$. Therefore,

$$r_{n-2c} = k - c \geq r_{n-2c+1} = r'_{n-2c+1} + 1 \geq m - c + 1 > m - c \text{ so, } k > m$$

which gives a contradiction since σ is weakly increasing.

□

Lemma 3.2.10 *For any $\sigma = \sigma_1\sigma_2 \dots \sigma_\ell \in \mathcal{O}_n$, where $\mathcal{O}_n$ is the set of partitions whose parts are odd and less than $2n$, such that $\sigma_1 = 2m - 1$, let $\mu = \Phi_n(\sigma)$. Then for any $k = 1, 2, \dots, m$,*

$$\psi_n(\phi_n(\mu, 2k - 1)) = (\mu, 2k - 1).$$

Proof: Let $\tau = \phi_n(\mu, 2k - 1)$ and c be the starting point of insertion. We need to prove that c is the starting point of deletion and ψ_n removes $2k - 1$ cells from τ. Let r_i and r'_i be the remainders of μ_i and τ_i, respectively. By Lemma 3.2.6, for all $\mu_{n-2j} > 0$, where $j = 0, 1, \dots, m - 2$, we have that $r_{n-2j} \geq m - j$. By definition of insertion for $j < c$, we have $\tau_{n-2j} = \mu_{n-2j} + 1$ and also for $j < c$, μ_{n-2j} is not multiple of $n - 2j$ by Corollary 3.2.5. So,

$$\begin{cases} r'_{n-2j} + j = r_{n-2j} + j + 1 > m \geq k & \text{for } j < c, \\ r'_{n-2j} + j = r_{n-2j} + j + 1 \geq m \geq k & \text{for } j > c, \end{cases} \tag{3.6}$$

since $m \geq k$ and by our assumption on c.

By Lemma 3.2.4, for $0 \leq j \leq k - 2$, $\left\lceil \frac{\tau_{n-2j-1}}{n-2j-1} \right\rceil = \left\lceil \frac{\tau_{n-2j}}{n-2j} \right\rceil$, and $r'_{n-2j-1} + 1 = r'_{n-2j}$. Recalling the definition of the set A, gives us that for $j \leq k - 2$, $j \in A$. By definition of k and (3.6), we have that $k_j > k$ for $j < c$ and $k_j \geq k$ for $k - 2 \geq j > c$. If $j \geq k - 1$,i.e., $j \in A$, we have $k_j \geq j + 1 \geq k$.

Now we need to show that $k_c = k$. For showing that consider the following two cases: (i) $c < k - 1$ and (ii) $c = k - 1$.

Case(i): $c < k - 1$

We meet a pair such that we can add one more cell to the larger part than the smaller one $(\mu_{n-2c-1}, \mu_{n-2c})$. By definition of the insertion map we have $\tau_{n-2c-1} = \mu_{n-2c-1} + (k - c - 1)$ and $\tau_{n-2c} = \mu_{n-2c} + (k - c)$. Hence

$$\lceil \frac{\tau_{n-2c-1}}{n - 2c - 1} \rceil = \lceil \frac{\tau_{n-2c}}{n - 2c} \rceil$$

and

$$r'_{n-2c-1} + 1 = r'_{n-2c} = k - c.$$

Thus $c \in A$, $k_c = r_{n-2c} + c = k$.

Case(ii): $c = k-1$

In this case, $\tau_{n-2c-1} = \mu_{n-2c-1}$ and $\tau_{n-2c} = \mu_{n-2c}+1$. By definition of k_c, we have $k_c = c+1 = k-1+1 = k$.

Hence, $k_c = k$ and we have $\psi(\tau) = (\mu, 2k-1)$, in any case.

□

Lemma 3.2.11 *Given any $\tau \in \mathcal{L}_n$, where $\mathcal{L}_n$ is the set of lecture hall partitions of length n, $\phi_n(\psi_n(\tau)) = \tau$.*

Proof: Let $(\mu, 2k-1) = \psi(\tau)$ and b be the starting point of deletion. We need to prove that b is also the starting point of insertion. Let r_i and r'_i be the remainders of μ_i and τ_i, respectively. The following two observations can be made from the definition of b: any $j < b$, $j \in A$ hence

$$\left\lceil \frac{\tau_{n-2j-1}}{n-2j-1} \right\rceil = \left\lceil \frac{\tau_{n-2j}}{n-2j} \right\rceil \text{ and } r'_{n-2j-1}+1 = r'_{n-2j}$$

and also $r'_{n-2j} > 2$. So, for $i \geq n-2b+1$, $r_i = r'_i - 1$ by definition of the deletion map. Hence, $r_{n-2j} = r'_{n-2j} - 1 < n-2j$. Note that $(\mu_{n-2j-1}, \mu_{n-2j})$ is a pair such that we can add one more cell to the larger partition than the smaller one. Now we have two cases: (i) $b = k-1$ and (ii) $b < k-1$

Case(i): $b = k-1$

Then $b = k-1$ must be the starting point of insertion, since we do not meet a pair $(\mu_{n-2j-1}, \mu_{n-2j})$ such that

$$\frac{\mu_{n-2j-1}}{n-2j-1} = \frac{\mu_{n-2j}}{n-2j}.$$

Case(ii): $b < k-1$

Then from the definition of the deletion map

$$\frac{\mu_{n-2b-1}}{n-2b-1} = \frac{\mu_{n-2b}}{n-2b}.$$

So $(\mu_{n-2b-1}, \mu_{n-2b})$ is the pair that we searching for, with $c = b$.

Hence, $\phi(\mu, 2k-1) = \tau$ in all cases.

□

Lemma 3.2.10 and Lemma 3.2.11 showed that the defined deletion and insertion maps are inverses of each other. Also these two lemmas imply the following theorem.

Theorem 3.2.12 *The map $\Psi_n : \mathcal{L}_n \to \mathcal{O}_n$ is the inverse of $\Phi_n : \mathcal{O}_n \to \mathcal{L}_n$.*

Proof: Let $\sigma \in \mathcal{O}_n$.To show that $\Psi_n(\Phi_n(\sigma_1\sigma_2\ldots\sigma_\ell)) = \sigma_1\sigma_2\ldots\sigma_\ell$ we apply induction on ℓ. If $\ell = 1$, then by Lemma 3.2.10

$$\Psi_n(\Phi_n(\sigma_\ell)) = \psi_n(\phi_n(\sigma_\ell)) = \sigma_\ell.$$

Suppose that $\Psi_n(\Phi_n(\sigma_2\ldots\sigma_\ell)) = \sigma_2\ldots\sigma_\ell$. Let

$$\Phi_n(\sigma_2\ldots\sigma_\ell) = \tau \text{ and } \Phi_n(\sigma_1\sigma_2\ldots\sigma_\ell) = \lambda.$$

By Lemma 3.2.10, we have that $\phi_n(\tau, \sigma_1) = \lambda$ and $\psi_n(\lambda) = (\tau, \sigma_1)$. Since

$$\Phi_n(\sigma_1\sigma_2 \ldots \sigma_\ell) = \phi_n(\tau, \sigma_1) = \lambda \text{ and } \Psi_n(\lambda) = \sigma_1\Psi_n(\tau) = \sigma_1\sigma_2 \ldots \sigma_\ell,$$

$\Psi_n \circ \Phi_n(\sigma) = \sigma$. Similarly we can show that $\Phi_n(\Psi_n(\lambda)) = \lambda$. $\square$

Example 3.2.8 *Recall that in Example 3.1.4 we found the $\Phi_5(\sigma)$, for $\sigma = (1, 3, 3, 5, 5)$, by iteration of the insertion map ϕ_5. Now, note that for $\lambda = (0, 1, 3, 5, 8)$,*

$$\psi_5(\lambda^{(0)}) = (\lambda^{(1)}, \sigma_1) = ((0, 1, 3, 5, 7), 1)$$
$$\psi_5(\lambda^{(1)}) = (\lambda^{(2)}, \sigma_2) = ((0, 1, 3, 4, 5), 3)$$
$$\psi_5(\lambda^{(2)}) = (\lambda^{(3)}, \sigma_3) = ((0, 1, 2, 3, 4), 3)$$
$$\psi_5(\lambda^{(3)}) = (\lambda^{(4)}, \sigma_4) = ((0, 0, 0, 2, 3), 5)$$
$$\psi_5(\lambda^{(4)}) = (\lambda^{(5)}, \sigma_5) = ((0, 0, 0, 0, 0), 5)$$

and so $\Psi_5(\lambda) = (1, 3, 3, 5, 5)$ which is exactly the same partition that Φ_5 takes as input in Example 3.1.4.

3.3. Comparison of Weights, Lengths and Alternative Sums

At the beginning our claim was the equality of the number of lecture hall partitions of length n, with $|\lambda|_a = k$ and the number of partitions into k odd parts which are less than $2n$. Note that this claim gives us the following two equalities:

$$|\lambda| = |\sigma| \quad \text{and} \quad \ell(\sigma) = |\lambda|_a.$$

Now, we will show that the defined bijections by deletion and insertion maps preserve the given two equalities.

First note that $|\tau| = |\mu| + 2k - 1$, where $\tau = \phi_n(\mu)$, since for $0 \leq i \leq 2c - 1$ we add one cell to each μ_i, for $i = 2c$ we add $(k - c)$ cells to μ_{2c}, for $i = 2c + 1$ we add $(k - c - 1)$ cells to μ_{2c+1} and for $2c + 2 \leq i < n$ we did not add any cell to any μ_i. Thus, in total we add $2k - 1$ cells to μ. For Φ_n, we are starting with the partition $\lambda^{(0)} = (0, 0, \ldots, 0) \in \mathcal{L}_n$. Therefore, in each insertion we are adding $2k_i - 1$ cells to λ^i for $0 \leq i \leq \ell$ where $2k_i - 1$'s are the parts of $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell) \in \mathcal{O}_n$. After the ℓ^{th} insertion increment in $|\lambda^{(0)}|$ will be $|\sigma|$, hence $|\lambda^{(\ell)}| = |\sigma|$.

To show that $|\tau|_a = |\mu|_a + 1$ we will use the definition of ϕ_n as in the previous case and compute the change in the alternating sum of μ as follows:

$$\left(\sum_{i=0}^{2c-1}(-1)^i 1\right) + (-1)^{2c}(k - c) + (-1)^{2c+1}(k - c - 1) + \sum_{i=2c+2}^{n-1}(-1)^i 0.$$

Notice that the first summand in the above sum will give 0 since we have c many $(-1, +1)$ pairs. Since $2c$ is an even number, the second summand will be equal to $(k-c)$ and by a similar reasoning the third summand will be equal to $(-k+c+1)$. Finally, the fourth summand will be equal to 0. Therefore, increment in the alternating sum of μ is equal to $0+(k-c)+(-k+c+1)+0=1$. Since Φ_n defined as the recursion of ϕ_n, increment in the alternating sum, by one after each insertion, counts the number of insertions which is equal to number of parts in σ, $\ell(\sigma)$.

We need to show that the deletion map ψ_n and the bijection Ψ_n satisfy the same properties. For showing that $|\mu| = |\tau| - 2k + 1$ again we use the definition of ψ_n. Since, for $0 \leq i \leq 2b-1$ we delete one cell from each τ_i, decrement in $|\tau|$ will be $2b$. For $i = 2b$ and $i = 2b+1$ we delete $(k-b)$ and $(k-b-1)$ cells from τ_{2b} and τ_{2b+1}, respectively. Finally for $2b+2 \leq i < n$ we do not change τ_i. Hence, decrement in $|\tau|$, in total, is equal to $2k-1$. For Ψ_n, we are starting with $\tau \in \mathcal{L}_n$ and recursively apply the deletion map ψ_n. Since for one deletion process decrement in the weight of τ is $2k_i - 1$ and $2k_i - 1$'s are the parts of the constructed partition, $\lambda^{(\ell)}$, by Ψ_n,

$$|\tau| = \sum_{i}^{\ell} 2k_i - 1 = |\lambda^{(\ell)}|.$$

Finally, we will show that $|\mu|_a = |\tau|_a - 1$. Similar to the insertion map case we will examine the decrement in the alternating sum of τ. By definition of the deletion map we have the following sum that is equal to decrement in $|\tau|_a$:

$$\left(\sum_{i=0}^{2b-1}(-1)^i 1\right) + (-1)^{2b}(k-b) + (-1)^{2b+1}(k-b-1) + \sum_{i=2b+2}^{n-1}(-1)^i 0$$

Note that the first summand is equal to zero by a similar reasoning in the case for ϕ_n. The second and the third summands are equal to $(k-b)$ and $(-k+b+1)$, respectively, since $2b$ is and even number while $2b+1$ is an odd one. Thus, in total, the decrement in $|\tau|_a$ is equal to $0+(k-b)+(-k+b+1)+0=1$. Since Ψ_n is defined recursively, by ψ_n, the decrement in the altenating sum, by one in each deletion, counts the number of parts in σ, $\ell(\sigma)$.

CHAPTER 4

Partition Analysis and Future Studies

4.1. The Five Guidelines

In the paper of Corteel, Lee, and Savage [21], the *weight* of a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ where each $\lambda_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \dots, n$ is defined the same as our definition of weight of a partition λ, i.e., $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Also a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is called *composition* if all $\lambda_i \in \mathbb{N}$ for $1 \leq i \leq n$, and if $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n-1$, then λ is a *partition*.

For a given $r \times n$ integer matrix $C = [c_{i,j}]$ they define S_C as the set of nonnegative integer sequences λ satisfying the following constraints:

$$c_{i,0} + c_{i,1}\lambda_1 + c_{i,2}\lambda_2 + \cdots + c_{i,n}\lambda_n \geq 0 \text{ for } 1 \leq i \leq r \tag{4.1}$$

They try to find the full generating function

$$F_C(x_1, x_2, \dots, x_n) = \sum_{\lambda \in S_C} x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}. \tag{4.2}$$

Note that

- the coefficient of q^N in $F_C(qx_1, qx_2, \dots, qx_n)$ is a listing of all nonnegative integer solutions to (4.2)
- the coefficient of q^N in $F_C(q, q, \dots, q)$ is the number of solutions in the list

In another paper by Corteel, Savage and Wilf [22], it is shown that for homogeneous systems, if the constraint matrix C is an $n \times n$ invertible matrix, and if all entries of $C^{-1} = B[b_{i,j}]$ are nonnegative integers then

$$F_C(x_1, x_2, \dots, x_n) = \prod_{j=1}^{n} \frac{1}{1 - x_1^{b_{1,j}} x_2^{b_{2,j}} \dots x_n^{b_{n,j}}}$$

This theorem helps the enumeration of the followings and provides bijections as well as generating functions.

- *Hickerson partitions*[24]

  Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, be a partition of N such that for $1 \leq i \leq n-1$, $\lambda_i \geq r\lambda_{i+1}$ where r is a positive integer.

  Example 4.1.9 *For N=12 and r=1, $\lambda = (4,3,3,2)$ is a Hickerson partition, since*

  $$\lambda_1 = 4 \geq 1.3 \qquad \lambda_2 = 3 \geq 1.3 \qquad \lambda_3 = 3 \geq 1.2$$

  If $f(r,n)$ denotes the number of partitions of n of the form $n = b_0 + b_1 + \cdots + b_s$, where, for $0 \leq i \leq s-1$, $b_i \geq rb_{i+1}$, and $g(r,n)$ denotes the number of partitions of n, where each part is of the form $1 + r + r^2 + r^3 + \cdots + r^i$ for some $i \geq 0$, then

  $$f(r,n) = g(r,n).$$

  The combinatorial proof and proof by generating functions of the above identity is given by Hickerson in[24].

- *Santos' interpretation of Euler's family*[26]

  Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of N such that the inequality

  $$\lambda_1 > 2\lambda_2 + \sum_{i \geq 3} \lambda_i$$

  holds.

  Example 4.1.10 *Let $N = 44$. $\lambda = (26, 8, 4, 2, 2, 1)$ will satisfy the given condition since $26 > 2.8 + 4 + 2 + 2 + 1 = 25$.*

  Santos proved that the number of partitions of N into odd parts equals to the number of partitions of N of the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i \geq \lambda_{i+1}$ in which the largest part is at least

  $$n\lambda_n + (n-1)(\lambda_{n-1} - \lambda_n) + (n-2)(\lambda_{n-2} - \lambda_{n-1}) + \cdots + 2(\lambda_2 - \lambda_3).$$

- *Super-concave partitions*[27]

  $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is super-concave if and only if for all positive integers $i < j < k \leq n$

  $$\lambda_i(k-j) + \lambda_j(i-k) + \lambda_k(j-i) \geq 0.$$

  Example 4.1.11 *$\lambda = (7, 5, 3, 1)$, $N = 16$, and $n = 4$. The set of all possible triples (i, j, k) is $\{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$. All inequalities that the* *parts of λ should satisfy are the followings:*

$$\lambda_1 - 2\lambda_2 + \lambda_3 \geq 0$$
$$2\lambda_1 - 3\lambda_2 + \lambda_4 \geq 0$$
$$\lambda_1 - 3\lambda_3 + 2\lambda_4 \geq 0$$
$$\lambda_2 - 2\lambda_3 + \lambda_4 \geq 0$$

From above inequalities the parts of λ should satisfy

$$(\lambda_1 + \lambda_4) - (\lambda_2 + \lambda_3) \geq 0.$$

Hence, choosen λ is a super-concave partition.

Canfield et al [19] have studied partitions with non-negative m^{th} differences. Specializing their results to the case $m = 2$, Snellman and Paulsen conclude: There is a bijection between partitions of n into triangular numbers and super-concave partitions.

However, since this technique can fail in some cases, i.e., either the constraint matrix is not invertible or integer matrix, Corteel et. al.[21] propose five guidelines to compute the full generating function $F_{\mathcal{C}}$ where $\mathcal{C}$ is system of linear diophantine inequalities. Their main goal is using guidelines for derivation of recurrence relations for $F_{\mathcal{C}_n}$, where $\{\mathcal{C}_n | n \geq 1\}$ is an infinite family of constraint systems.

They showed an application of guidelines which is derivation of full generating function of generalization of $\mathcal{L}_n$. For deriving the full generating function of generalization of $\mathcal{L}_n$, they used the *truncated lecture hall partitions*, which are defined as the integer sequences satisfying the following constraints:

$$\mathcal{L}_{n,k} = \left[\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_n}{1} \geq 0\right]$$

In[20], they showed that if

$$\bar{\mathcal{L}}_{n,k} = \left[\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_n}{1} > 0\right]$$

then the generating function is

$$\bar{L}_{n,k}(q) = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1};q)_k}{(-q^{2n-k+1};q)_k} \tag{4.3}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{-n-1})_k (-1)^k q^{nk-k(k-1)/2}}{(q)_k}.$$

Note that putting $k = n$ and dividing by $q^{\binom{n+1}{2}}$ gives

$$\sum_{\lambda_{S_{\mathcal{L}_n}}} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1 - q^{2i-1}}.$$

At last they proved a proposition which states that the generating function for truncated lecture hall partitions satisfies

$$\bar{\mathcal{L}}_{n,k}(x_1, \ldots, x_k) = \frac{x_k \bar{\mathcal{L}}_{n,k-1}(x_1, \ldots, x_{k-1})}{1 - x_k} - \frac{\bar{\mathcal{L}}_{n,k-1}(x_1, \ldots, x_{k-2}, x_{k-1}x_k)}{1 - x_k}$$
$$- \frac{z_{n,k} \bar{\mathcal{L}}_{n,k-1}(x_1, \ldots, x_{k-2}, x_{k-1}x_k)}{1 - z_{n,k}}$$

with $z_{n,k} = x_1^n x_2^{n-1} \ldots x_k^{n-k+1}$.

As it can be seen clearly the guidelines are advantageous for deriving recurrence relations for full generating functions and solving them, however the guidelines do not provide bijective proofs. Since the guidelines for partition analysis do not produce bijective proofs, our goal is searching for bijections or classes of bijections for partition identities provided by the guidelines for partition analysis. This looks feasible after getting sufficient knowledge about partition analysis from the series of papers [2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13] which are written by Andrews, Paule and Riese.

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