To Yavuz...

To my precious sons Ahmet, Kerem, and Yusuf

ANALYSIS OF COOPERATIVE BEHAVIOR WHEN UTILITY IS SEMI TRANSFERABLE

The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

YASEMİN DEDE

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May 2018

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

ANALYSIS OF COOPERATIVE BEHAVIOR WHEN UTILITY IS SEMI-TRANSFERABLE

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The primary purpose of this study is to analyze both cooperative and noncooperative games under semitransferable utility. In game theory literature, utility seems so far to have been assumed "not to be transferable at all" in noncooperative games, while both "fully transferable" and "nontransferable" utility are considered in the context of cooperative games. There are, however, an abundance of real life situations, where utility is partially transferable. Here we introduce the notion of semitransferable utility, which encompasses "full-transferability" and "nontransferability" as its two extreme special subcases. We explore and exemplify what changes some well-known equilibrium notions undergo when one allows utility to be only partially transferable. In particular, we relate core allocations in a convex cooperative transferable utility (TU) game to their counterparts in a corresponding strategic context, to show that, for each core allocation of a given TU game, there is a strategic form game, where that allocation survives, while almost all other allocations are eliminated.

Keywords: α-Core, β-Core, Cooperative Games, Core, Noncooperative Games

ÖZET

FAYDANIN YARI AKTARILABİLİR OLDUĞU DURUMLARDA İŞBİRLİKLİ DAVRANIŞIN İNCELENMESİ

Dede, Yasemin

Doktora, Ekonomi Bölümü

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Bu çalışmanın birincil amacı, hem işbirlikli, hem de işbirliksiz oyunların yarı aktarılabilir yarar altında çözümlenmesidir. Oyunlar kuramı yazınında, şimdiye kadar, işbiliksiz oyunlarda yararın "hiç aktarılamaz" olduğu varsayımı yapılırken, işbirlikli oyunlarda yararın hem "tam akatarılabilir", hem de "hiç aktarılamaz" olduğu durumların ele alındığı görülmektedir. Oysa gerçek yaşamda, yararın kısmen aktarılabilir olduğu sayısız durum söz konusudur. Bu çalışmada, yararın "tam aktarılabilirliği" ile "aktarılamazlığını" özel iki uç hali olarak kabul eden "yarı aktarılabilirlik" kavramını tanımlıyoruz. İyi bilinen bazı denge kavramlarının, yararın yalnızca kısmen aktarılabilir olmasına izin verildiği zaman, ne tür bir değişime uğradıklarını inceliyor ve örnekliyoruz. Özel olarak, dışbükey işbirlikli aktarılabilir yarar (AY) oyunlarının çekirdek dağılımlarını stratejik bir ortamdaki karşılıkları ile ilişkilendirerek, verili bir AY oyununun her çekirdek dağılımı için, o dağılımın denge olarak varlığını sürdürüp, geri kalan dağılımların hemen hemen hepsinin denge olmaktan çıktığı bir stratejik biçim oyununun varolduğunu gösteriyoruz.

Anahtar Kelimeler: α-Çekirdek, β-Çekirdek, Çekirdek, İşbirlikli Oyun, İşbirliksiz Oyun

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CHAPTER 1

INTRODUCTION

In game theory literature, coalitional behavior has so far been analyzed under two different assumptions concerning transferability of utility. Utility is either assumed to be fully transferable (as in transferable utility (TU) games), or not transferable at all (as in nontransferable utility (NTU) games). In a TU game, the characteristic function assigns a value to each coalition in the absence of any kind of strategic structure. Coalitional rationality is based on the value of coalitions, and any utility allocation that gives a coalition a total payoff less than its value is regarded as unstable. Different cooperative solution concepts such as core (Gillies, 1953), Shapley value (Shapley, 1953), kernel (Davis and Maschler, 1965) and nucleolus (Scmeidler, 1969) pertaining to TU games have been defined and the corresponding existence theorems have been proven.

In an NTU game, on the other hand, every coalition is assigned a set of utility allocations for its members, representing what allocations the coalition can secure for its members. Stability of an allocation is now based on the fact that no coalition is able to improve its members' individual welfares by materializing an allocation in the set assigned to itself in the NTU game in question. In a cooperative NTU game, a strategic background rendering the set of allocations assigned to a coalition achievable is again hidden and thus not visible.

The counterparts of some cooperative solution concepts in a strategic context, where coalitions cooperate via agreeing upon joint strategies, are exemplified by strong Nash equilibrium (Aumann, 1959), α- and β-core (Aumann, 1961), and coalition-proof Nash equilibrium (Bernheim et al., 1987). The global assumption here is again that utility is not transferable at all.

There are, however, a large variety of real life situations, where utility is partially transferable. In a fund transfer through a bank, for instance, the amount the recipient receives is less than the sender sends due to transaction costs, how all this gets reflected to the decrease and increase in the utilities of the sender and the recipient, respectively, still being a separate manner. In case a rich man transfers 100 dollars to a poor guy, the increase in the latter's utility is for sure to far exceed the decrease in the former's utility. A complete model for utility transfers would actually require the specification of the rate of loss or gain for any ordered pair of individuals, where the rate may also very well depend upon the amount of transfer.

In this study, we consider the simplest case, where there is a constant rate of loss $\gamma \in [0, 1]$ in all transfers between any two agents. The case, where $\gamma = 1$, corresponds to full transferability of utility, while $\gamma = 0$ is almost the same as, but not identical with nontransferability, as it allows self-destruction of utility. As for cooperative games, TU games and NTU games correspond to the two extreme values of γ, which are 1 and 0, respectively. Our general setup, on the other hand, also enables us to consider strategic form games under semitransferability uf utility along with cooperative games.

The core of a TU game, in case it is nonempty, is rarely a singleton set. A nonempty core is an infinite set unless it is a singleton. Marinacci and Montrucchio (2002) and Derya (2015) can be listed among the studies that find conditions for a TU game to have a singleton core. The infinite multiplicity of core allocations is usually considered as "undesirable", since it diminishes the predictive power of this major stability notion in cooperative game theory. Thus, there have been many attempts to refine the core, mainly using normative criteria such as fairness or focality.

Here we associate to each convex TU game and each core allocation thereof a strategic form game, which induces the TU game via both the maxmin and the minmax operators, such that the chosen core allocation survives in both the α- and β-cores of the strategic form game, while almost every other core allocation is rejected, in case we employ full transferability of utility in the strategic context as well. Thus, core allocations are separated strategically among themselves in the sense that each has its own strategic environment, in which it is the only core-consistent outcome.

The notion of a semitransferable utility (STU) game provides a theoretical framework unifying TU and NTU games under the same umbrella. For each $\gamma \in [0, 1]$, as γ-transferability means that the recipient receives only $\gamma\omega$ "units of utility" from a transfer ω, while the decrease in the utility of the agent who makes the transfer is precisely ω units, all noncooperative and cooperative solution concepts along with efficiency notions can now be modified accordingly.

In particular, we introduce the γ-versions of the α- and β-cores of a strategic form game and examine how the α- and β-core change as γ changes. It is clear that any coalition becomes stronger concerning its ability to block outcomes in either the α- and the β-sense, as γ increases. Thus, both cores continue shrinking as the degree of transferability becomes larger. Moreover, for each strategic form game, there is some critical value γ_c such that the α-core [respectively, the β-

core] of the game under γ is the same as its α-core [respectively, β-core] for any $\gamma \in [0,1]$ with $\gamma > \gamma_c$.

The γ-transferability of utility can be used for purposes of design as well as from the angle of positive theory. Some famous dilemmas or paradoxes in game theory, mainly reflecting a severe tension between stability and efficiency, disappear under semi-transferability for appropriate values of $\gamma \in [0,1]$. In case transfers are observable by the designer, each transfer may be "taxed" by the designer in accordance with some $\gamma \in [0,1]$, so that the players find themselves playing the game under γ-transferability. Pareto improvements thereby obtained are exemplified in this study by Prisoners' Dilemma (Luce and Raiffa, 1957), Tragedy of Commons (Hardin, 1968) and Centipede Game (Rosenthal, 1981).

Making pre-declared transfers a part of the players' strategies clearly transforms a given strategic form game into one, where the strategy sets become much richer than before. The players are now endowed with additional tools possibly enabling them to induce certain best responses on the part of their rivals, which might be more desirable from their viewpoint than the best responses in the "naked game". Formally, we construct a two-stage game, inspired by the notion of "predonations" introduced by Sertel (1992), where one deals with one-sided predonations in a bargaining setting. In our model, the first stage consists of simultaneous declarations of commitments to feasible transfer vectors by each of the players at every joint strategy of the underlying game, where the transferability degree γ is commonly known. In the second stage, the players resolve the (transformed) strategic game a la Nash.

For the extensive form game thus constructed, we employ a modified version of subgame perfect equilibrium, referred to as "quasi subgame perfect equilibrium". As there is no further restriction than feasibility to transfer commitments in the first stage, in the second stage there easily arise some (irrelevant) subgames that have no Nash equilibrium in pure strategies. The quasi subgame perfect equilibrium concept requires the restriction of a joint strategy to a proper subgame to be a Nash equilibrium of that subgame only in case that subgame possesses at least one Nash equilibrium.

Which players gain more power from the introduction of the transfer declaration stage turns out to strongly depend on the structure of the initial strategic form game. Although there is almost always (except for some trivial cases) at least one player benefitting from the introduction of this additional pre-stage, no societal improvement in comparison to the outcome of the naked game is guaranteed.

This two-stage construct also turns out to be related to "games of pretension" introduced by Koray and Sertel (1983), where the players in a strategic form game are allowed to pretend to have any preferences whatsoever in the first stage, provided that they perform accordingly in the second stage, while they enjoy their true utilities at the arising final outcome. As transfers do not allow a player to pretend to have any kind preferences, but rather provide a tool to influence other players' preferences, the two approaches seem to be "dual", rather than similar. The information the two approaches respectively reveal about the structure of the underlying game, however, seems yet to be worthy of a more detailed comparison.

The rest of the study is organized as follows. In section 2, basic definitions of TU and NTU games, necessary tools for adoption of γ-transferability of utility and the relation between equilibrium notions of TU, STU and NTU games are presented. In section 3, γ-TU normal form games are introduced where strategy space is extended with simultaneous choice of transfers and actions to be played, and Nash Equilibria of new game form is analyzed. Section 4 introduces a two stage game and a new equilibrium notion, namely quasi subgame perfect equillibria (QSPE), and its application to Prisoners' Dilemma and Centipede Game to a better understanding of the concept. In Section 5 properties of QSPE is explored, such as existence of an improvement in payoffs with respect to Nash equilibrium, achieving welfare maximizing outcomes as an equilibrium as γ increses, and properties of γ values for a certain outcome when it constitutes QSPE. In Section 6 a direct consequence of incorporation of strategy space to TU games shows that core selections need verifications with respect to strategical background otherwise they are not reliable. In lack of any strategic information other than the value of coalitions any two allocations of the core are equally valuable. Section 7 concludes.

CHAPTER 2

SEMI-TRANSFERABLE UTILITY

Game theory is the branch of mathematics that analyze competitive situations where the outcome and agents' payoffs depend on participants' strategic actions. Coalitional behavior of individuals is also possible and analyzed in the literature under very different two assumptions, where utility is either considered to be non-transferable or fully transferable. To unite NTU and TU games under one framework we introduce semi-transferable utility games. Firstly we provide basic definitions of NTU and TU games and adapting them to the case of γ-transferable utility where $\gamma \in [0, 1]$.

Definition 1. $g = (N, X, u)$ is a *normal form game*, where $N = \{1, 2, \dots, n\}$ is the set of agents with $|N| = n$, $X = \times_{i \in N} X_i$ is the strategy space and $u = (u_1, u_2, \dots, u_n)$ is the utility profile of the agents where $u_i : X \to \mathbb{R}$ for $i \in \{1, 2, \dots, n\}$.

Definition 2. *The associated side payment (transferable utility -TU) game of a normal form game* $g = (N, X, u)$ is a function $v : S \to \mathbb{R}$ for each $S \in 2^N$ with $v(\varnothing) = 0$ that associates a real valued payoff $v(S)$ to each coalition that coalition members can distribue among themselves.

We define two special associated side payment games of a given normal form game as follows:

i. $v_\alpha^g(S) = \max_{x_S \in X_S} \min_{x_{N\setminus S} \in X_{N\setminus S}} \sum_{i\in S} u_i(x_S, x_{N\setminus S})$ and

ii. $v_\beta^g(S) = \min_{x_{N\setminus S} \in X_{N\setminus S}} \max_{x_S \in X_S} \sum_{i\in S} u_i(x_S, x_{N\setminus S})$.

v_α^g corresponds to the case where coalition S can achieve for itself no matter what the complementary coalition does. v_β^g corresponds to the case where the coalition S can somehow anticipate the strategy of the nonmembers and act accordingly.

Definition 3. *Core of a side payment game v* is the set of allocations that no coalition can improve upon. $Core(v) = \{(y_1, y_2, \ldots, y_n) : \sum_{i\in N} y_i = v(N)$ and for all $S \subseteq N$ $\sum_{i\in S} y_i \geq v(S)\}$.

Definition 4. Let $g = (N, X, u)$ be a normal form game. *α-Core of normal form game g*, denoted $C_\alpha(g)$, consists of strategy profiles $x^* \in X$ such that there is no coalition $S \subseteq N$ such that $\exists \bar{x}_S \in X_S$ $\forall x_{N\setminus S} \in X_{N\setminus S}$:

$[\forall i \in S : u_i(\bar{x}_S, x_{N\setminus S}) \geq u_i(x^*)$ and $\exists j \in S : u_j(\bar{x}_S, x_{N\setminus S}) > u_j(x^*)]$.

Definition 5. Let $g = (N, X, u)$ be a normal form game. *β-Core of normal form game g*, denoted $C_\beta(g)$, consists of strategy profiles $x^* \in X$ such that there is no coalition $S \subseteq N$ such that $\forall x_{N\setminus S} \in X_{N\setminus S}$ $\exists x_S \in X_S$:

$[\forall i \in S : u_i(x_S, x_{N\setminus S}) \geq u_i(x^*)$ and $\exists j \in S : u_j(x_S, x_{N\setminus S}) > u_j(x^*)]$.

Notation 1. Given $x, y \in \mathbb{R}^n$,

$x \geq y$ if and only if $\forall i \in \{1, \ldots, n\} : x_i \geq y_i$;

$x > y$ if and only if $x \geq y$ and $x \neq y$;

$x >> y$ if and only if $\forall i \in \{1, \ldots, n\} : x_i > y_i$.

For any nonempty set K, $\mathbb{R}_+^{|K|} = \left\{x \in \mathbb{R}^{|K|} : x \geq 0\right\}$.

Up to this point we have presented some of the the existing definitions in the literature of cooperative game theory. As we have mentioned in the introduction, these definitions are either related to NTU games or TU games. Considering these two type of games as extreme cases of transferability of utility, now we will introduce semi-transferable utility as follows:

Definition 6. *Utility transfer degree* $\gamma \in [0,1]$ is the rate of unlost utility transferred among agents.

Let $g = (N, X, u)$ be given, $x \in X$, $t_i(x) > 0$ and $t_j(x) = 0$. Then payoffs of agents i and j at outcome x with transfers for agents i and j will be $u_i(x) - t_i(x)$ and $u_j(x) + \gamma t_i(x)$ respectively. Agent i loses $t_i(x)$ amount of utility whereas agent j gets only a γ portion of the transferred utility.

Definition 7. Given a finite normal form game $g = (N, X, u)$, where $u : X \rightarrow R^n_+$ with γ-transferable utility, $\gamma \in [0,1]$, $t_S(x) \in R^{|S|}$ is a *feasible transfer vector for coalition* $S \subseteq N$ *at outcome* $x \in X$ where $t_S(x) = (t_i(x))_{i \in S}$ such that

i. $t_S(x) = (t_i(x))_{i \in S}$ such that $t_i(x)$ is the total amount of transfer that agent i proposes to give to other members of coalition where $0 \leq t_i(x) \leq u_i(x)$.

ii. $t_{ij}(x)$ is the amount of transfer than agent i proposes to transfer to agent j at outcome x, where $t_i(x) = (t_{ij}(x))_{j \in S}$ with $\sum_{j \in S} t_{ij}(x) = t_i(x)$, $0 \leq t_{ij}(x)$ and $t_{ii}(x) = 0$.

The set of feasible transfer vectors for coalition $S \subseteq N$ at outcome $x \in X$ is denoted as $T_S(x)$.

Note that as transfers are defined on R^n_+ throughout the study we will be working with normal form games with $u : X \rightarrow R^n_+$ where $R^n_+ = \{y \in R^n : y_i \geq 0 \; \forall i \in N\}$.

Notation 2. Note that given the utility transfer degree γ, the payoff of agent i at outcome x when $t_S(x)$ realizes is $\overline{u}^{\gamma}_i(x, t_S(x)) = u_i(x) - t_i(x) + \gamma \sum_{j \in S \setminus \{i\}} t_{ji}(x)$. We denote $\overline{u}^{\gamma}(x, t_S(x)) = (\overline{u}^{\gamma}_i(x, t_S(x)))_{i=1}^{|N|}$.

Now with proposed definitions above, given a feasible transfer vector, $t_j(x)$ is the amount of utility transfer that agent $j \in S$ is proposing to give to any other member in a given coalition $S \subseteq N$ at an outcome $x \in X$. $t_j(x) = 0$ for some $j \in S$, implies that agent j is a potential transfer reciever from other members of the coalition S. Therefore a coalition $S \subseteq N$ mainly consists of two disjoint groups at an outcome $x \in X$ with respect to $t(x)$. One group S' for which $t_i(x) > 0 \ \forall i \in S'$, the group that is proposing to make utility transfers, and other group transfer takers $S'' = S \backslash S'$ such that $t_i(x) = 0 \ \forall i \in S''$.

The restriction of $t_i(x) \leq u_i(x)$ on feasible transfer vectors is mainly the budget constraint. At an outcome $x \in X$ an agent is prohibited to propose an amount of transfer that is more than his payoff at outcome x.

Moreover, when a coalition S is formed to cooperate, utility transfer is a part of the cooperative behavior, thus feasible transfer vectors are defined with respect to each coalition together with the strategy space of each coalition.

Definition 8. Given a nonempty finite set S where $|S| = s$ and $x, y \in \mathbb{R}^s_+$, we say that y *is reachable from* x *under* γ if, for each $i \in S$, there is some $(\bar{t}_{ij})_{j \in S} \in \mathbb{R}^s_+$ with $\sum_{j \in S} \bar{t}_{ij} \leq x_i$ such that $\forall k \in S : y_k = x_k - \sum_{j \in S} \bar{t}_{kj} + \sum_{j \in S} \gamma \bar{t}_{jk}$.

Definition 9. We define *the correspondence* $\widetilde{\gamma}_S : \mathbb{R}^s_+ \to \mathbb{R}^s_+$ by $\widetilde{\gamma}_S(x) = \{y \in \mathbb{R}^s_+ : y \text{ is reachable from } x \text{ under } \gamma\}$ at each $x \in \mathbb{R}^s_+$.

Definition 10. Given $x, y \in \mathbb{R}^n_+$, we say that

i) x *Pareto dominates* y if $x > y$.

ii) x is *Pareto efficient* if there is no y that Pareto dominates x.

Definition 11. Given $x, y \in \mathbb{R}^n_+$, we say that

i) x *γ-dominates* y *for* S, and write $x \succ^{\gamma}_S y$, if there is some $x' \in \widetilde{\gamma}_S(x)$ such that $x' > y$.

ii) x *γ-dominates* y, and write $x \succ^{\gamma} y$, if there is some $x' \in \widetilde{\gamma}_N(x)$ such that $x' > y$.

iii) $x \in X$ is *γ-efficient* if there is no $y \in \mathbb{R}^n_+$ such that y γ-dominates x.

Remark 1. Pareto efficiency coincides with 0-efficiency.

Remark 2. Given a finite normal form game $g = (N, X, u)$, $u(x)$ is 1-efficient if and only if $u(x) \in argmax \sum_{i \in N} u_i(x)$. Thus we see that as γ increases, γ efficiency is compatible with an increase in welfare such that for $\gamma = 1$ the efficient outcomes are the ones that maximize total welfare.

Lemma 1. Given $x \in \mathbb{R}^n_+$, if x is not γ-efficient for some $\overline{\gamma} \in [0, 1]$, then it is not γ-efficient for all $\gamma \geq \overline{\gamma}$.

Proof: Let $x \in \mathbb{R}^n_+$ be given and suppose that x is not γ-efficient for some $\overline{\gamma} \in [0, 1]$. Then $\exists y \in \mathbb{R}^n_+$ such that $\exists y' \in \widetilde{\overline{\gamma}}_N(y)$ such that $y' \succ^\gamma x$. But then as for any $\gamma \geq \overline{\gamma}$ one has $y' \in \widetilde{\gamma}_N(y)$, x is not γ-efficient for all $\gamma \geq \overline{\gamma}$. $\square$

Definition 12. Given $\varnothing \neq B \subset \mathbb{R}^s_+$, the set $P_\gamma(B) = \{w \in B \mid \nexists w' \in B : w' \succ^\gamma_S w\}$ is called the *γ-Pareto frontier of* B.

Definition 13. Let $g = (N, X, u)$ be given, for any $\varnothing \neq S \subset N$, define the *associated side payment games under γ* as

$v^\alpha_\gamma(S) = P_\gamma(\{w \in \mathbb{R}^s \mid \exists x_S \in X_S \ \forall x_{N \backslash S} \in X_{N \backslash S} \ \exists w' \in \widetilde{\gamma}_S(u_S(x_S, x_{N \backslash S}))$ with $w' \geq w\})$

$v^\beta_\gamma(S) = P_\gamma(\{w \in \mathbb{R}^s \mid \forall x_{N \backslash S} \in X_{N \backslash S} \ \exists x_S \in X_S \ \exists w' \in \widetilde{\gamma}_S(u_S(x_S, x_{N \backslash S}))$ with $w' \geq w\})$.

Definition 14. We say that $S \in 2^N \backslash \{\varnothing\}$ *γ-blocks $x \in X$ in the α [β] sense* if there exists $w \in v^\alpha_\gamma(S)$ [$v^\beta_\gamma(S)$] such that $w > (u_i(x))_{i \in S}$.

Definition 15. Let $g = (N, X, u)$ be a finite normal form game. *The α-core [β-core] of g under γ*, denoted $C^\gamma_\alpha(g)$ [$C^\gamma_\beta(g)$], is defined as

$C^{\gamma}_{\alpha}(g)$ $[C^{\gamma}_{\beta}(g)] = \{x \in X \mid$ there is no $S \in 2^N$ such that S γ-blocks x in the α-sense [β-sense]$\}$.

Proposition 1. Let $g = (N, X, u)$ be given. For any $\gamma, \gamma' \in [0, 1]$ such that $\gamma' > \gamma$ one has $C^{\gamma'}_{\alpha}(g) \subset C^{\gamma}_{\alpha}(g)$ $[C^{\gamma'}_{\beta}(g) \subset C^{\gamma}_{\beta}(g)]$.

Proof: Let $g = (N, X, u)$ be a finite normal form game. Let $x \in X$ and $\gamma, \gamma' \in [0, 1]$ are such that $\gamma' > \gamma$, if there exists $x' \succ^{\gamma}_{S} x$, then $x' \succ^{\gamma'}_{S} x$, hence for any $x \notin C^{\gamma}_{\alpha}(g)$ $[C^{\gamma}_{\beta}(g)]$ one has $x \notin C^{\gamma'}_{\alpha}(g)$ $[C^{\gamma'}_{\beta}(g)]$.

Now let $x \in C^{\gamma'}_{\alpha}(g)$ $[C^{\gamma'}_{\beta}(g)]$, assume on the contrary that $x \notin C^{\gamma}_{\alpha}(g)$ $[C^{\gamma}_{\beta}(g)]$, then there exists a coalition $S \in 2^N$ such that S γ-blocks x in the α-sense [β-sense]. That is there exists $w \in v^{\alpha}_{\gamma}(S)$ $[v^{\beta}_{\gamma}(S)]$ such that $w > (u_i(x))_{i \in S}$. But for any $w' \in v^{\alpha}_{\gamma'}(S)$ $[v^{\beta}_{\gamma'}(S)]$ one has $w' \succ^{\gamma'}_{S} w$ as $\gamma' > \gamma$, which implies S γ-blocks x in the α-sense [β-sense] for γ'. Then $x \notin C^{\gamma'}_{\alpha}(g)$ $[C^{\gamma'}_{\beta}(g)]$, a contradiction. Thus $x \in C^{\gamma}_{\alpha}(g)$ $[C^{\gamma}_{\beta}(g)]$ and $C^{\gamma'}_{\alpha}(g) \subset C^{\gamma}_{\alpha}(g)$ $[C^{\gamma'}_{\beta}(g) \subset C^{\gamma}_{\beta}(g)]$. $\square$

Proposition 2. Let $g = (N, X, u)$ be a finite normal form game. Now $u(C^1_{\alpha}(g)) \subset Core(v^g_{\alpha})$ $[u(C^1_{\beta}(g)) \subset Core(v^g_{\beta})]$.

Proof: Let $g = (N, X, u)$ be a normal form game.

If $C^1_{\alpha}(g) = \varnothing$ the statement holds voidly.

Suppose that $C^1_{\alpha}(g)$ $[C^1_{\beta}(g)] \neq \varnothing$. Suppose that $x^* \in C^1_{\alpha}(g)$ $[C^1_{\beta}(g)]$, assume on the contrary that $u(x^*) \notin Core(v^g_{\alpha})$ $[Core(v^g_{\beta})]$. Then there exists $S' \subseteq N$ such that $v^g_{\alpha}(S') = \max_{x_S \in X_S} \min_{x_{N \setminus S} \in X_{N \setminus S}} \sum_{i \in S} u_i(x_S, x_{N \setminus S}) > \sum_{i \in S'} u_i(x^*)$ $[v^g_{\beta}(S') = \min_{x_{N \setminus S} \in X_{N \setminus S}} \max_{x_S \in X_S} \sum_{i \in S} u_i(x_S, x_{N \setminus S})]$. Then for

$$
\begin{aligned}
y &= \max_{x_{S'} \in X_{S'}} \min_{x_{N \setminus S'} \in X_{N \setminus S'}} \sum_{i \in S'} u_i(x_{S'}, x_{N \setminus S'}) \\
[y &= \min_{x_{N \setminus S'} \in X_{N \setminus S'}} \max_{x_{S'} \in X_{S'}} \sum_{i \in S'} u_i(x_{S'}, x_{N \setminus S'})]
\end{aligned}
$$

one has $(u_i(y))_{i\in S'} > (u_i(x^*))_{i\in S'}$, which implies that y γ-blocks $x^* \in X$ in the α sense [β sense]. Contradiction with the fact that $x^* \in C^1_\alpha(g)$ [$C^1_\beta(g)$].Thus $u(x^*) \in Core(v^g_\alpha)$ [$Core(v^g_\beta)$], and $u(C^1_\alpha(g)) \subset Core(v^g_\alpha)$ [$u(C^1_\beta(g)) \subset Core(v^g_\beta)$]. $\square$

Proposition 1 states that as γ increases the set of $C^\gamma_\alpha(g)$ [$C^\gamma_\beta(g)$] shrinks. Hence there exists games such that $C^0_\alpha(g) \neq \varnothing$ [$C^0_\beta(g) \neq \varnothing$] whereas $C^1_\alpha(g) = \varnothing$ [$C^1_\beta(g) = \varnothing$]. As transferability of utility is continuously defined by $\gamma \in [0,1]$ there exists a critical (intermediate) value $\gamma_c \in [0,1]$ such that $C^\gamma_\alpha(g) = \varnothing$ [$C^\gamma_\beta(g) = \varnothing$] for $\forall\gamma > \gamma_c$.

Now the following intermediate value theorem proves that there exists a critical value of utility transfer degree such that the coalitions have same power in terms of α-blocking [β-blocking] as if the utility is transferable without any loss. This critical value of utility transfer degree is game specific. And if we consider the γ to be exogeneous, in games with $\gamma > \gamma_c(g)$, analysis of the associated TU game and its core will be sufficient for analysis of stable allocations in the STU game. Whereas for normal form games that have $\gamma \leq \gamma_c(g)$, neither NTU game nor TU game will give us a accurate interpretation about the equilibrium strategies. Only in these cases the equilibrium of analysis of the extremes that exist in the literature will suffice.

Theorem 1. Let g be a finite normal form game with $C^0_\alpha(g) \neq \varnothing$ and $C^1_\alpha(g) \neq C^0_\alpha(g)$. There exists a critical value of transfer degree, say γ_c, such that $C^\gamma_\alpha(g) = C^1_\alpha(g)$ where utility is transferable of degree $\gamma > \gamma_c$.

Proof: Let $g = (N, X, u)$ be a finite normal form game and let the associated TU game to be $v^g_\alpha(S)$. Assume that $C^0_\alpha(g) \neq \varnothing$ and $C^0_\alpha(g) \neq C^1_\alpha(g)$. Proposition 1 implies that there exists $x^* \in C^0_\alpha(g)\backslash C^1_\alpha(g)$. As $x^* \notin C^1_\alpha(g)$ there exists a coalition S that γ-blocks $u(x^*)$ in the α-sense for $\gamma = 1$. Then there exists $\bar{x}_S \in X_S$ such that for any $x_{N\setminus S} \in X_{N\setminus S}$ one has $\sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)$. Note that S is not a blocking coalition for the α-core of NTU game g, so S can be written as union of disjoint sets as $S = S'(x_{N\setminus S}) \cup S''(x_{N\setminus S})$ such that $S'(x_{N\setminus S}) = \{i \in S : u_i(\bar{x}_S, x_{N\setminus S}) \geq u_i(x^*)\}$ and $S''(x_{N\setminus S}) = \{j \in S : u_j(\bar{x}_S, x_{N\setminus S}) < u_j(x^*)\}$.

Now the following algorithm ensures to find the minimal critical value of utility transfer degree $\gamma_{S,\bar{x}_S} \in [0,1]$ such that coalition S γ-blocks $u(x^*)$ in the α-sense whenever utility transfer degree is greater than $\gamma_{S,\bar{x}_S}$.

Step 1: Fix some $x_{N\setminus S} \in X_{N\setminus S}$.

Step 2: Compute the amount of transfer to be made among the members of S as follows: For each agent $i \in S'(x_{N\setminus S})$ compute the maximum value to be transferred to the rest of the coalition by taking the $u_i(x^*)$ as his reservation value. That is, $t_i = u_i(\bar{x}_S, x_{N\setminus S}) - u_i(x^*) > 0$, therefore the total maximal amount of utility that can be transferred among coalition members is $M = \sum_{i\in S'(x_{N\setminus S})} t_i = \sum_{i\in S'(x_{N\setminus S})}[u_i(\bar{x}_S, x_{N\setminus S}) - u_i(x^*)]$.

Step 3: Compute the total amount of compensation of utility needed among members of S as follows: Compute the compensation amount of utility for each $j \in S''(x_{N\setminus S})$ by considering $u_j(x^*)$ as his reservation value, $t_j = u_j(x^*) - u_j(\bar{x}_S, x_{N\setminus S})$. Total compensation amount is

$$C = \sum_{j\in S''(x_{N\setminus S})} t_j = \sum_{j\in S''(x_{N\setminus S})} [u_j(x^*) - u_j(\bar{x}_S, x_{N\setminus S})].$$

Step 4: Minimal critical value of transfer degree $\gamma_{S,x_{N\setminus S}}$ that depends on $x_{N\setminus S}$ is found by $\gamma_{S,x_{N\setminus S}} = \frac{C}{M}$.

Step 5: Repeat steps 1 to 4 for each $x_{N\setminus S} \in X_{N\setminus S}$ to find the set of $\{\gamma_{S,x_{N\setminus S}} : x_{N\setminus S} \in X_{N\setminus S}\}$.

Step 6: Compute the minimal critical value of utility transfer degree $\gamma_{S,\bar{x}_S} = max_{x_{N\setminus S} \in X_{N\setminus S}}\{\gamma_{S,x_{N\setminus S}}\}$ as coalition S has no control over the strategy to be played by coalition $N\setminus S$.

Step 7: Repeat steps 1 to 6 for each $\bar{x}_S \in X_S$ such that $\forall x_{N\setminus S} \in X_{N\setminus S}$: $\sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)$ in order to find the set $\{\gamma_{S,\bar{x}_S} : \bar{x}_S \in X_S$ such that $\forall x_{N\setminus S} \in X_{N\setminus S} : \sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)\}$.

Step 8: Compute the minimal critical value of utility transfer degree sufficient for S to γ-block $u(x^*)$ in the α-sense since one blocking strategy is sufficient for coalition S to γ-block $u(x^*)$ in the α-sense whenever utility is transferable of degree $\gamma > \gamma_S$.

$$\gamma_S = min_{\bar{x}_S \in X_S}\{\gamma_{S,\bar{x}_S} : \bar{x}_S \in X_S \text{ such that } \forall x_{N\setminus S} \in X_{N\setminus S} :$$

$$\sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)\}$$

Now note that there may exist more than one $S \subset N$ such that $\sum_{i\in S} u_i(x^*) < v_\alpha^g(S)$. Repeating the above process for all such $S \subset N$ will give the set $\{\gamma_S : S \subset N$ such that $\sum_{i\in S} u_i(x^*) < v_\alpha^g(S)\}$.

As existence of one coalition $S \subseteq N$ such that S γ-blocks $u(x^*)$ in the α-sense whenever utility is transferable of degree γ is sufficient to have $x^* \notin C_\alpha^\gamma(g)$ where $\gamma > \gamma_{x^*}^c$, $\gamma_{x^*}^c = min\{\gamma_S : S \subseteq N$ such that $\sum_{i\in S} u_i(x^*) < v_\alpha^g(S)\}$.

Now if there exists more than one $x^* \in C_\alpha^0(g)\setminus C_\alpha^1(g)$ for game g where utility is not transferable (i.e. transferable of degree 0), we will carry out the previous algorithm for each $x^* \in C_\alpha^0(g)\setminus C_\alpha^1(g)$ to find all such critical values $\gamma_{x^*}^c$. $C_\alpha^\gamma(g) = C_\alpha^1(g)$ where utility is transferable of degree $\gamma > \gamma_c$ where $\gamma_c = max\{\gamma_{x^*}^c : x^* \in C_\alpha^0(g)\setminus C_\alpha^1(g)\}$. $\square$

Theorem 2. Let g be a finite normal form game, let $C_\beta^0(g) \neq \varnothing$, whereas

$C^1_\beta(g) \neq C^0_\beta(g)$ there exists a critical value of transfer degree , γ_c, such that $C^\gamma_\beta(g) = C^1_\beta(g)$ where utility is transferable of degree $\gamma > \gamma_c$.

Proof: Let $g = (N, X, u)$ be a finite normal form game and let the associated TU game to be $v^g_\beta(S)$. Assume that $C^0_\beta(g) \neq \varnothing$ and $C^0_\beta(g) \neq C^1_\beta(g)$. Proposition 1 implies that $\exists x^* \in C^0_\beta(g) \backslash C^1_\beta(g)$. As $x^* \notin C^1_\beta(g)$ $\exists S \in 2^N \backslash \{\varnothing\}$: S γ-blocks $u(x^*)$ in the β-sense for $\gamma = 1$. Then $\forall x_{N\backslash S} \in X_{N\backslash S}$ $\exists \bar{x}_S \in X_S$: $\sum_{i\in S} u_i(\bar{x}_S, x_{N\backslash S}) > \sum_{i\in S} u_i(x^*)$. Note that S is not a blocking coalition for the β-core of NTU game g, so S can be written as union of disjoint sets as $S = S'(x_{N\backslash S}) \cup S''(x_{N\backslash S})$ such that $S'(x_{N\backslash S}) = \{i \in S : u_i(\bar{x}_S, x_{N\backslash S}) \geq u_i(x^*)\}$ and $S''(x_{N\backslash S}) = \{j \in S : u_j(\bar{x}_S, x_{N\backslash S}) < u_j(x^*)\}$.

Now the following algorithm ensures to find the minimal critical value of utility transfer degree $\gamma_{S,\bar{x}_S} \in [0, 1]$ such that coalition S γ-blocks $u(x^*)$ in the β-sense whenever utility transfer degree is greater than $\gamma_{S,\bar{x}_S}$.

Step 1: Fix some $x_{N\backslash S} \in X_{N\backslash S}$.

Step 2: Compute the amount of transfer to be made among the members of S as follows: For each agent $i \in S'(x_{N\backslash S})$ compute the maximum value to be transferred to the rest of the coalition by taking the $u_i(x^*)$ as his reservation value. That is, $t_i = u_i(\bar{x}_S, x_{N\backslash S}) - u_i(x^*) > 0$, therefore the total maximal amount of utility that can be transferred among coalition members is $M = \sum_{i\in S'(x_{N\backslash S})} t_i = \sum_{i\in S'(x_{N\backslash S})} [u_i(\bar{x}_S, x_{N\backslash S}) - u_i(x^*)]$.

Step 3: Compute the total amount of compensation of utility needed among members of S as follows: Compute the compensation amount of utility for each $j \in S''(x_{N\backslash S})$ by considering $u_j(x^*)$ as his reservation value, $t_j = u_j(x^*) - u_j(\bar{x}_S, x_{N\backslash S})$. Total compensation amount is

$$C = \sum_{j \in S''(x_{N\setminus S})} t_j = \sum_{j \in S''(x_{N\setminus S})} [u_j(x^*) - u_j(\bar{x}_S, x_{N\setminus S})].$$

Step 4: Minimal critical value of transfer degree $\gamma_{S,\bar{x}_S}$ that depends on $\bar{x}_S$ is found by $\gamma_{S,\bar{x}_S} = \frac{C}{M}$.

Step 5: Repeat steps 1 to 4 for each $\bar{x}_S \in X_S$ such that for $\forall x_{N\setminus S} \in X_{N\setminus S}$ $\exists \bar{x}_S \in X_S : \sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)$ in order to find the set $\{\gamma_{S,\bar{x}_S} : \forall x_{N\setminus S} \in X_{N\setminus S}\ \exists \bar{x}_S \in X_S : \sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)\}$.

Step 6: Compute the minimal critical value of utility transfer degree sufficient for S to γ-block $u(x^*)$ in the β-sense.

$\gamma_S = min_{\bar{x}_S \in X_S}\{\gamma_{S,\bar{x}_S} : \forall x_{N\setminus S} \in X_{N\setminus S}\ \exists \bar{x}_S \in X_S : \sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)\}$ since one blocking strategy is sufficient for coalition S to γ-block $u(x^*)$ in the β-sense whenever utility is transferable of degree $\gamma > \gamma_S$.

Now note that there may exist more than one $S \subseteq N$ such that $\sum_{i\in S} u_i(x^*) < v^g_\beta(S)$. Repeating the above process for all such $S \subseteq N$ will give the set $\{\gamma_S : S \subseteq N \text{ such that } \sum_{i\in S} u_i(x^*) < v^g_\beta(S)\}$.

As existence of one coalition $S \subseteq N$ such that S γ-blocks $u(x^*)$ in the β-sense whenever utility is transferable of degree γ is sufficient to have $x^* \notin C^\gamma_\beta(g)$ where $\gamma > \gamma^c_{x^*}$, $\gamma^c_{x^*} = min\{\gamma_S : S \subseteq N \text{ such that } \sum_{i\in S} u_i(x^*) < v^g_\beta(S)\}$.

Now if there exists more than one $x^* \in C^0_\beta(g)\backslash C^1_\beta(g)$ for game g where utility is not transferable (i.e. transferable of degree 0), we will carry out the previous algorithm for each $x^* \in C^0_\beta(g)\backslash C^1_\beta(g)$ to find all such critical values $\gamma^c_{x^*}$. $C^\gamma_\beta(g) = C^1_\beta(g)$ where utility is transferable of degree $\gamma > \gamma_c$ where $\gamma_c = max\{\gamma^c_{x^*} : x^* \in C^0_\beta(g)\backslash C^1_\beta(g)\}$. $\square$

Up to this point, it is assumed that transfer degree, γ, is homogeneous in the society. To mimic the real life phenomena, the heterogeneity of utility transfer degree in the society can be increased . It is useful to replicate the results about the critical values of utility transfer degree for a heterogeneous society such that there exists a γ value for each member of the society. So we will assume that there exists a different γ value for each transfer taker while we will keep the γ value for transfer givers to be one. This corresponds to the case that when a transfer giver transfers 1 util, he will lose 1 util whereas the transfer taker can get the γ portion of the 1 util where the value of γ may vary with respect to each transfer taker due to wealth level of the transfer taker. In the extreme case one may also assign different values of γ for transfer givers that may be different than 1 which represents that loss of utility also depends on the wealth level of the transfer giver. One may also consider a $\gamma_i > 1$ for some transfer takers. All of these situations can be modeled by inserting γ to the model not as a scalar but as a vector. And the intermediate value theorem can be replicated models that include γ as a vector.

Definition 16. Let $g = (N, X, u)$ be a normal form game where *utility transfer vector* is the vector of $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \in R^n$ with $\gamma_i \in [0, 1]$ is the unlost utility defined for each agent during transfer taking as follows: Consider coalition $\{i, j, k\} \in 2^N$ with $t_i(x) = (t_{ii} = 0, t_{ij} > 0, t_{ik} > 0)$, $t_j(x) = 0$ and $t_k(x) = 0$ at outcome $x \in X$. Then $\overline{u}_i^{\gamma}(x) = u_i(x) - t_i(x)$, $\overline{u}_j^{\gamma}(x) = u_j(x) + \gamma_j t_{ij}(x)$ and $\overline{u}_k^{\gamma}(x) = u_k(x) + \gamma_k t_{ik}(x)$.

The following example elaborates the idea how to use existing results in the case that utility transfer degree is a vector instead of a scalar.

Example 1. Consider $g = (N, X, u)$ where $N = \{1, 2, 3\}$, $X_1 = \{A, B\}$, $X_2 = \{a, b\}$, $X_3 = \{L, R\}$ and payoff vectors for the outcome space are given by the below tables:

Player 2

Player 1	a	b
A	$1,1,1$	$1,2,1$
B	$1,1,1$	$1,1,1$

Player 3 plays L

Player 2

Player 1	a	b
A	$\frac{25}{10},\frac{5}{10},\frac{9}{10}$	$\frac{25}{10},\frac{5}{10},\frac{9}{10}$
B	$1,1,1$	$1,1,1$

Player 3 plays R

$(A,b,L) \in C_\alpha^{\vec{\gamma}}(g)$ where $\vec{\gamma} = (0,0,0)$, whereas $(A,b,L) \notin Core_\alpha^{\vec{\gamma}'}(g)$ where $\vec{\gamma}' = (1,1,1)$ as $S = \{1,3\}$ is a blocking coalition with strategy AR.

Let $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$, $t = (t_1, 0, t_3)$, ($t_3 = 0$ as player 3 is transfer taker)

$u_1(AR,\gamma) = \frac{25}{10} - \gamma_1 t_1$, $u_3(AR,\gamma) = \frac{9}{10} + \gamma_3 t_1$

$\gamma_1 t_1 = \frac{15}{10}$, $\gamma_3 t_1 = \gamma_3 \frac{15}{10}\frac{1}{\gamma_1} = \frac{1}{10}$

$\frac{\gamma_3}{\gamma_1} = \frac{1}{15}$

$\vec{\gamma}_c = \{\gamma \in R^3 : \frac{\gamma_3}{\gamma_1} = \frac{1}{15}, \gamma_1 \in [0,1], \gamma_3 \in [0,1]\}$.

In example 1 $S' = \{1,3\}$ is the only γ-blocking coalition in the α-sense. Note that in general the set of all critical utility transfer vectors is actually a superset of $\vec{\gamma}_c$ as there may exist other coalitions that can γ-block $u(x^*)$ in the α-sense.

Theorem 3. For a given n-person finite normal form game $g = (N,X,u)$, let $C_\alpha^{\overrightarrow{0}}(g) \neq \varnothing$, whereas $C_\alpha^{\overrightarrow{1}}(g) \neq C_\alpha^{\overrightarrow{0}}(g)$, there exists a set of critical utility transfer degree vectors $\vec{\gamma}_c \in [0,1]^n$ such that $C_\alpha^{\vec{\gamma}}(g) = C_\alpha^{\overrightarrow{1}}(g)$ if $\vec{\gamma} > \vec{\gamma}_c$.

Proof. Let $g = (N,X,u)$ be an n-person finite normal form game and let the associated TU game be $v_\alpha^g(S)$. Assume that $C_\alpha^{\overrightarrow{0}}(g) \neq \varnothing$, and $C_\alpha^{\overrightarrow{1}}(g) \neq C_\alpha^{\overrightarrow{0}}(g)$. The proof will be carried out as in the proof of Theorem 1. The steps to find critical transfer vectors will be carried as follows: Let. $\exists x^* \in C_\alpha^{\overrightarrow{0}}(g) \backslash C_\alpha^{\overrightarrow{1}}(g)$. As $x^* \notin C_\alpha^{\overrightarrow{1}}(g)$, $\exists S \in 2^N \backslash \{\varnothing\} : S$ γ-blocks $u(x^*)$ in the α-sense for $\overrightarrow{\gamma} = \overrightarrow{1}$. Then there exists at least one $\bar{x}_S \in X_S$ such that $\forall x_{N\setminus S} \in X_{N\setminus S} : \sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)$. Note that S is not a blocking coalition for the α-core of NTU game g, so S can be written as union of disjoint sets as $S = S'(x_{N\setminus S}) \cup S''(x_{N\setminus S})$ such that $S'(x_{N\setminus S}) = \{i \in S : u_i(\bar{x}_S, x_{N\setminus S}) \geq u_i(x^*_S, x_{N\setminus S})\}$ and $S''(x_{N\setminus S}) = \{j \in S : u_j(\bar{x}_S, x_{N\setminus S}) < u_j(x^*_S, x_{N\setminus S})\}$.

Now the following algorithm ensures to find the minimal critical value of utility transfer degree $\overrightarrow{\gamma}_{S,\bar{x}_S} \in [0,1]$ such that coalition S γ-blocks $u(x^*)$ in the α-sense the strategy profile x^* whenever utility transfer degree $\overrightarrow{\gamma}$ is such that $\overrightarrow{\gamma} > \overrightarrow{\gamma}_{S,\bar{x}_S}$.

Step 1: Fix some $x_{N\setminus S} \in X_{N\setminus S}$.

Step 2: Compute the amount of transfer to be made among the members of S as follows: For each agent $i \in S'(x_{N\setminus S})$ compute the maximum value to be transferred to the rest of the coalition by taking the $u_i(x^*)$ as his reservation value. That is, $t_i(\bar{x}_S, x_{N\setminus S}) = \frac{u_i(\bar{x}_S, x_{N\setminus S}) - u_i(x^*)}{\gamma_i} > 0 \; \forall i \in S'(x_{N\setminus S})$.

Step 3: Compute the total amount of compensation of utility needed among members of S as follows: Compute the compensation amount of utility for each $j \in S''(x_{N\setminus S})$ by considering $u_j(x^*)$ as his reservation value, $t_j(\bar{x}_S, x_{N\setminus S}) = \frac{u_j(x^*) - u_j(\bar{x}_S, x_{N\setminus S})}{\gamma_j} \; \forall j \in S''(x_{N\setminus S})$.

Step 4: Minimal critical utility transfer vector $\overrightarrow{\gamma}_{S,x_{N\setminus S}}$ that depends on $x_{N\setminus S}$ is found by the following equation

$$\sum_{i\in S'(x_{N\setminus S})} \frac{u_i(\bar{x}_S, x_{N\setminus S}) - u_i(x^*)}{\gamma_i} = \sum_{\forall j\in S''(x_{N\setminus S})} \frac{u_j(x^*) - u_j(\bar{x}_S, x_{N\setminus S})}{\gamma_j}.$$

Step 5: Repeat steps 1 to 4 for each $x_{N\setminus S} \in X_{N\setminus S}$ to find the set of

$$\widetilde{\gamma}_{S,x_{N\setminus S}} = \cup\{\overrightarrow{\gamma} \in \mathbb{R}^{|S|} :$$
$$\sum_{i\in S'(x_{N\setminus S})} \frac{u_i(\bar{x}_S, x_{N\setminus S}) - u_i(x^*)}{\gamma_i} = \sum_{\forall j\in S''(x_{N\setminus S})} \frac{u_j(x^*) - u_j(\bar{x}_S, x_{N\setminus S})}{\gamma_j}\}.$$

Be careful to choose the minimal vectors that satisfy the equation on step 4. That is there should be no $\overrightarrow{\gamma'}, \overrightarrow{\gamma} \in \widetilde{\gamma}_{S,x_{N\setminus S}}$ such that $\overrightarrow{\gamma} > \overrightarrow{\gamma'}$ or $\overrightarrow{\gamma} = k\overrightarrow{\gamma'}$ for some $k \in \mathbb{N}$[1].

Step 6: Find the set of minimal critical utility transfer vector

$\widetilde{\gamma}^c_{S,\bar{x}_S} = \cup_{x_{N\setminus S}\in X_{N\setminus S}}\{\overrightarrow{\gamma} \in \widetilde{\gamma}_{S,x_{N\setminus S}} : \nexists\overrightarrow{\gamma'} \in \widetilde{\gamma}_{S,x_{N\setminus S}}$ such that $\exists\overrightarrow{\gamma} \in \widetilde{\gamma}_{S,x'_{N\setminus S}}$ for some $x'_{N/S} \in X_{N\setminus S}$ such that $\overrightarrow{\gamma} > \overrightarrow{\gamma'}$ or $\overrightarrow{\gamma} = k\overrightarrow{\gamma'}$ for some $k \in \mathbb{N}\}$ as coalition S has no control over the strategy to be played by coalition $N\setminus S$.

Step 7: Repeat steps 1 to 6 for each $\bar{x}_S \in X_S$ such that $\forall x_{N\setminus S} \in X_{N\setminus S}$: $\sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)$ in order to find the set $\{\widetilde{\gamma}_{S,\bar{x}_S} : \bar{x}_S \in X_S$ such that $\forall x_{N\setminus S} \in X_{N\setminus S} : \sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)\}$.

Step 8: Compute the minimal critical value of utility transfer degree sufficient for S to γ-block $u(x^*)$ in the α-sense.

$\overrightarrow{\gamma}_S = \{\overrightarrow{\gamma} \in \widetilde{\gamma}_{S,\bar{x}_S} : \nexists\overrightarrow{\gamma'} \in \widetilde{\gamma}_{S,\bar{x}_S}$ such that $\exists\overrightarrow{\gamma} \in \widetilde{\gamma}_{S,x'_{N\setminus S}}$ for some $x'_{N/S} \in X_{N\setminus S}$ such that $\overrightarrow{\gamma} > \overrightarrow{\gamma'}$ or $\overrightarrow{\gamma} = k\overrightarrow{\gamma'}$ for some $k \in \mathbb{N}\}$ since one blocking strategy is sufficient for coalition S to γ-block $u(x^*)$ in the α-sense whenever utility is transferable of degree $\overrightarrow{\gamma} > \overrightarrow{\gamma}_S$.□

Theorem 4. Let $g = (N, X, u)$ be given with $C^{\overrightarrow{0}}_\beta(g) \neq \varnothing$, and $C^{\overrightarrow{1}}_\beta(g) \neq C^{\overrightarrow{0}}_\beta(g)$, there exists a set of critical utility transfer degree vectors $\vec{\gamma}_c \in [0,1]^n$ such that $C^{\vec{\gamma}}_\beta(g) = C^{\overrightarrow{1}}_\beta(g)$ if $\vec{\gamma} > \vec{\gamma}_c$.

Proof. Let $g = (N, X, u)$ be an n-person finite normal form game and let the associated TU game be $v^g_\beta(S)$. Assume that $C^{\overrightarrow{0}}_\beta(g) \neq \varnothing$, and $C^{\overrightarrow{1}}_\beta(g) \neq C^{\overrightarrow{0}}_\beta(g)$.

[1]$\mathbb{N}$ denotes the set of natural numbers.

The proof will be carried out as in the proof of Theorem 1. The steps to find critical transfer vectors will be carried as follows: Let. $\exists x^* \in C_\beta^{\overrightarrow{0}}(g) \backslash C_\beta^{\overrightarrow{1}}(g)$. As $x^* \notin C_\beta^{\overrightarrow{1}}(g)$, $\exists S \in 2^N \backslash \{\varnothing\} : S$ γ-blocks $u(x^*)$ in the β-sense for $\overrightarrow{\gamma} = \overrightarrow{1}$. Then $\forall x_{N \backslash S} \in X_{N \backslash S}$ $\exists \bar{x}_S \in X_S$: $\sum_{i \in S} u_i(\bar{x}_S, x_{N \backslash S}) > \sum_{i \in S} u_i(x^*)$. Note that S is not a blocking coalition for the β-core of NTU game g, so S can be written as union of disjoint sets as $S = S'(x_{N \backslash S}) \cup S''(x_{N \backslash S})$ such that $S'(x_{N \backslash S}) = \{i \in S : u_i(\bar{x}_S, x_{N \backslash S}) \geq u_i(x_S^*, x_{N \backslash S})\}$ and $S''(x_{N \backslash S}) = \{j \in S : u_j(\bar{x}_S, x_{N \backslash S}) < u_j(x_S^*, x_{N \backslash S})\}$.

Now the following algorithm ensures to find the minimal critical value of utility transfer degree $\overrightarrow{\gamma}_{S, \bar{x}_S} \in [0, 1]$ such that coalition S γ-blocks $u(x^*)$ in the β-sense the strategy profile x^* whenever utility transfer degree $\overrightarrow{\gamma}$ is such that $\overrightarrow{\gamma} > \overrightarrow{\gamma}_{S, \bar{x}_S}$.

Step 1: Fix some $x_{N \backslash S} \in X_{N \backslash S}$.

Step 2: Compute the amount of transfer to be made among the members of S as follows: For each agent $i \in S'(x_{N \backslash S})$ compute the maximum value to be transferred to the rest of the coalition by taking the $u_i(x^*)$ as his reservation value. That is, $t_i(\bar{x}_S, x_{N \backslash S}) = \frac{u_i(\bar{x}_S, x_{N \backslash S}) - u_i(x^*)}{\gamma_i} > 0$ $\forall i \in S'(x_{N \backslash S})$.

Step 3: Compute the total amount of compensation of utility needed among members of S as follows: Compute the compensation amount of utility for each $j \in S''(x_{N \backslash S})$ by considering $u_j(x^*)$ as his reservation value, $t_j(\bar{x}_S, x_{N \backslash S}) = \frac{u_j(x^*) - u_j(\bar{x}_S, x_{N \backslash S})}{\gamma_j}$ $\forall j \in S''(x_{N \backslash S})$.

Step 4: Minimal critical utility transfer vector $\overrightarrow{\gamma}_{S, \bar{x}_S}$ that depends on $\bar{x}_S$ is found by the equation

$$\sum_{i \in S'(x_{N \backslash S})} \frac{u_i(\bar{x}_S, x_{N \backslash S}) - u_i(x^*)}{\gamma_i} = \sum_{\forall j \in S''(x_{N \backslash S})} \frac{u_j(x^*) - u_j(\bar{x}_S, x_{N \backslash S})}{\gamma_j}$$

$\widetilde{\gamma}_{S, \bar{x}_S} = \cup\{\overrightarrow{\gamma} \in \mathbb{R}^{|S|}$:

$\sum_{i\in S'(x_{N\setminus S})} \frac{u_i(\bar{x}_S, x_{N\setminus S}) - u_i(x^*)}{\gamma_i} = \sum_{\forall j \in S''(x_{N\setminus S})} \frac{u_j(x^*) - u_j(\bar{x}_S, x_{N\setminus S})}{\gamma_j}\}$.

Be careful to choose the minimal vectors that satisfy the equation on step 4. That is there should be no $\overrightarrow{\gamma'}, \overrightarrow{\gamma} \in \widetilde{\gamma}_{S,\bar{x}_S}$ such that $\overrightarrow{\gamma} > \overrightarrow{\gamma'}$ or $\overrightarrow{\gamma} = \frac{k}{k}\overrightarrow{\gamma'}$ for some $k \in \mathbb{N}$.

Step 5: Repeat steps 1 to 4 for each $\bar{x}_S \in X_S$ such that $\forall x_{N\setminus S} \in X_{N\setminus S}$ $\exists \bar{x}_S \in X_S$ such that $\sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)$ in order to find the set $\{\widetilde{\gamma}_{S,\bar{x}_S} : \bar{x}_S \in X_S$ such that $\forall x_{N\setminus S} \in X_{N\setminus S} : \sum_{i\in S} u_i(\bar{x}_S, x_{N\setminus S}) > \sum_{i\in S} u_i(x^*)\}$.

Step 6: Compute the minimal critical value of utility transfer degree sufficient for S to γ-block $u(x^*)$ in the β-sense.

$\overrightarrow{\gamma}_S = \{\overrightarrow{\gamma} \in \widetilde{\gamma}_{S,\bar{x}_S} : \nexists \overrightarrow{\gamma'} \in \widetilde{\gamma}_{S,\bar{x}_S}$ such that $\exists \overrightarrow{\gamma} \in \widetilde{\gamma}_{S,x'_{N\setminus S}}$ for some $x'_{N/S} \in X_{N\setminus S}$ such that $\overrightarrow{\gamma} > \overrightarrow{\gamma'}$ or $\overrightarrow{\gamma} = k\overrightarrow{\gamma'}$ for some $k \in \mathbb{N}\}$ since one blocking strategy is sufficient for coalition S to $\overrightarrow{\gamma}$-block $u(x^*)$ in the β-sense whenever utility is transferable of degree $\overrightarrow{\gamma} > \overrightarrow{\gamma}_S$. $\square$

CHAPTER 3

$\gamma-$TU NORMAL FORM GAMES

In this section we insert semi-transferability of utility to normal form games with smallest change as possible. Transfer vector is embeded to the strategy space of individuals whereas simultaneity of choice of strategies are conserved. We question whether outcomes other than Nash equilibriua of the original game can be achieved via transfers when utility is semi-transferable.

Definition 17. A strategy profile $x^* \in X$ is a *Nash Equilibrium (NE)* if no unilateral deviation in strategy by any single player is profitable for the deviating player, i.e. $\forall i \in N$, $x_i \in X_i : u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$.

Notation 3. Given a 2-person finite normal form game $g = (X, u)$ we denote the best response relation of agent i for a strategy $x_j \in X_j$ as $BR_i(x_j)$ where for any $\overline{x}_i \in BR_i(x_j)$ one has $u_i(\overline{x}_i, x_j) \geq u_i(x_i, x_j)$

Example 2. Consider the following 2 person normal form game where $X_1 = \{A, B\}$, $X_2 = \{a, b\}$ and payoff vectors are given in the below table where $u(x) = (u_1(x), u_2(x))$[2]:

	a	b
A	4,0	0,1
B	1,1	2,1

[2]Throughout the study in two person games row player is the first player and column player is the second player, and the payoffs at each outcome are shown as $(u_1(x), u_2(x))$.

Let $v(S) = \max_{x_S \in X_S} \min_{x_{N\setminus S} \in X_{N\setminus S}} u_S(x_S, x_{N\setminus S})$. One has $v(1) = 1, v(2) = 1, v(12) = 4$.

Note that for $\gamma > \frac{1}{2}$, when transfer vector $t(A,a) = (t_{12} = \frac{1}{\gamma}, t_{21} = 0)$ is realized, $\bar{u}_1^\gamma(A,a) = 4 - \frac{1}{\gamma} > 2$, $\bar{u}_2^\gamma(A,a) = 0 + \frac{1}{\gamma}\gamma = 1$, new payoff vector γ-dominates $u(B,b) = (2,1)$.

Proposition 1 suggests that α-core [β-core] shrinks as utility transfer degree increases. This is due to the fact that α or β blocking an outcome becomes easier with the increase in utility transfer degree. Note that as utility transfer degree increases, coalitions not only become stronger in blocking, but they also become stronger in utility sharing. As utility transfer degree increases given that binding agreements on transfers exists, an outcome can now become a candidate for γ-α-core by transfers as the example above suggests. Thus transfer giving proposes opportunuties to cooperate and reach more efficient outcomes as equilibria. The outcomes that can be blocked in the α-sense [β-sense] when $\gamma = 0$ can become a member of the α core [β-core] by a rearrangement of the payoff vector at that outcome. To include both of those two effects into consideration, we expand the strategy space of agents by feasible transfers as follows:

Definition 18. Given a finite normal form game $g = (N, X, u)$ and $\gamma \in [0,1]$, a γ-*TU normal form game* is defined as $\bar{g} = (N, (X \times T), \bar{u}^\gamma)$, where $N = \{1, 2, \cdots, n\}$ is the set of agents, $(X \times T)$ is the strategy space where $X = \prod_{i \in N} X_i$ and $T = (t_i^k)_{i \in N, k \in \{S \subset N : 1 \leq |S| \leq n-1\}}$ where $t_i^k : X \to (\mathbb{R}_+^{|k|})_{k \in \{S \subset N : 1 \leq |S| \leq n-1\}}^{2^{n-1}-1}$ such that $t_i^S(x) = (t_{ij})_{j \in S\setminus\{i\}} \in \mathbb{R}_+^S$ with $\sum_{j \in S\setminus\{i\}} t_{ij} \leq u_i(x)$, and $\bar{u}^\gamma = (\bar{u}_1^\gamma, \bar{u}_2^\gamma, \cdots, \bar{u}_n^\gamma)$ is the utility profile of the agents where $\bar{u}_i^\gamma : X \times T \to \mathbb{R}_+$ for $i \in \{1, 2, \cdots, n\}$ where $\bar{u}_i^\gamma(x', t_S(x')) = u_i(x') - \sum_{j \in S\setminus\{i\}} t_{ij}(x') + \gamma \sum_{j \in S} t_{ji}(x')$.

Definition 19. Given a finite γ-TU normal form game $\bar{g} = (N, (X \times T), \bar{u}^\gamma)$, $(x^*, t(x^*)) \in C_\alpha^\gamma(\bar{g})$ (α *core of* γ-*TU normal form game*) if there is no coalition S :

$\exists x_S \in X_S$ $\forall x_{N\backslash S} \in X_{N\backslash S}$ such that $\exists t(x_S, x_{N\backslash S}) \in T_S(x_S, x_{N\backslash S}) : \bar{u}_i^{\gamma}(x^*, t(x^*)) \leq \bar{u}_i^{\gamma}((x_S, x_{N\backslash S}), t(x_S, x_{N\backslash S}))$ for all $i \in S$ and $\exists j \in S$ such that $\bar{u}_j^{\gamma}(x^*, t(x^*)) < \bar{u}_j^{\gamma}((x_S, x_{N\backslash S}), t(x_S, x_{N\backslash S}))$.

Definition 20. Given a finite γ-TU normal form game $\overline{g} = (N, (X \times T), \bar{u}^{\gamma})$, $(x^*, t(x^*)) \in C_{\beta}^{\gamma}(\overline{g})$ (β *core of* γ*-TU normal form game*) if there is no coalition S : $\forall x_{N\backslash S} \in X_{N\backslash S}$ $\exists x_S \in X_S$ such that $\exists t(x_S, x_{N\backslash S}) \in T_S(x_S, x_{N\backslash S}) : \bar{u}_i^{\gamma}(x^*, t(x^*)) \leq \bar{u}_i^{\gamma}((x_S, x_{N\backslash S}), t(x_S, x_{N\backslash S}))$ for all $i \in S$ and $\exists j \in S$ such that $\bar{u}_j^{\gamma}(x^*, t(x^*)) < \bar{u}_j^{\gamma}((x_S, x_{N\backslash S}), t(x_S, x_{N\backslash S}))$.[3]

Now when we turn back to example 2, $((B, b), \overrightarrow{0}) \in C_{\alpha}^{\gamma}(\overline{g})$ for $\gamma \in [0, \frac{1}{2}]$, $((A, a), (t_{12} = \frac{1}{\gamma}, t_{21} = 0)) \in C_{\alpha}^{\gamma}(\overline{g})$ for $\gamma \in [\frac{1}{2}, 1]$.

Proposition 3. Given a 2 person finite normal form game $g = (X, u)$, $(x^*, t(x^*)) \in C_{\beta}^{\gamma}(\overline{g})$ $[C_{\alpha}^{\gamma}(\overline{g})]$ if and only if

$$\begin{aligned} \overline{u}_i^{\gamma}(x^*, t(x^*)) &\geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j) \\ [\overline{u}_i^{\gamma}(x^*, t(x^*)) &\geq max_{x_i \in X_i} min_{x_j \in X_j} u_i(x_i, x_j)] \end{aligned}$$

for $\forall i \in N$ and $\overline{u}^{\gamma}(x^*, t(x^*))$ is γ-efficient.

Proof: Let $g = (X, u)$ be a 2 person finite normal form game and $(x^*, t(x^*)) \in C_{\beta}^{\gamma}(\overline{g})$ $[C_{\alpha}^{\gamma}(\overline{g})]$, then for $\forall i \in N$

$$\overline{u}_i^{\gamma}(x^*, t(x^*)) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)\ [max_{x_i \in X_i} min_{x_j \in X_j} u_i(x_i, x_j)],$$

because if there exists $i \in N$ such that

$$\overline{u}_i^{\gamma}(x^*, t(x^*)) < min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)\ [max_{x_i \in X_i} min_{x_j \in X_j} u_i(x_i, x_j)],$$

[3] Definitions of α- and β-core defined here are compatible with the definitions provided in Chapter 2.

coalition $S = \{i\}$ γ-blocks $\overline{u}^\gamma(x^*, t(x^*))$ in the β-sense [α-sense] by playing (x_i, t_i) where $t_i(x) = \vec{0}$,

$$x_i = \arg\{min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)\}[\arg\{max_{x_i \in X_i} min_{x_j \in X_j} u_i(x_i, x_j)\}].$$

Contradiction with $(x^*, t(x^*)) \in C^\gamma_\beta(\overline{g})$ $[C^\gamma_\alpha(\overline{g})]$. Now assume on the contrary $\overline{u}^\gamma(x^*, t(x^*))$ is γ-dominated. Then $\exists(x', t(x')) \in X \times T$ such that $\forall i \in N$, $\bar{u}^\gamma_i((x^*, t(x^*)) \leq \bar{u}^\gamma_i(x', t(x'))$ $\forall i \in N$ and $\exists j \in N$ $\bar{u}^\gamma_j((x^*, t(x^*)) < \bar{u}^\gamma_j(x', t(x'))$. Contradicting with the fact that $(x^*, t(x^*)) \in C^\gamma_\beta(\overline{g})$ $[C^\gamma_\alpha(\overline{g})]$. Hence $\overline{u}^\gamma(x^*, t(x^*))$ is not γ-dominated.

Now given a 2 person finite normal form game $g = (X, u)$, let the outcome $(x^*, t(x^*))$ be such that $\forall i \in N$ $\bar{u}^\gamma_i(x^*, t(x^*)) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)$ $[max_{x_i \in X_i} min_{x_j \in X_j} u_i(x_i, x_j)]$ and $\overline{u}^\gamma(x^*, t(x^*))$ is not γ-dominated. Assume on the contrary that $(x^*, t(x^*)) \notin C^\gamma_\beta(\overline{g})$ $[C^\gamma_\alpha(\overline{g})]$. As

$$\bar{u}^\gamma_i(x^*, t(x^*)) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)[max_{x_i \in X_i} min_{x_j \in X_j} u_i(x_i, x_j)],$$

$S = \{i\}$ is not a γ blocking coalition in the β-sense [α-sense] as an agent can not transfer herself any utility from outside resources. Thus it should be the grand coalition that γ blocks $\overline{u}^\gamma(x^*, t(x^*))$ in the β-sense [α-sense]. Therefore there exists $(x', t') \in X \times T$ such that $\bar{u}^\gamma_i(x', t(x')) \geq \bar{u}^\gamma_i(x^*, t(x^*))$ $\forall i \in N$ and $\exists j \in N$ such that $\bar{u}^\gamma_j(x', t(x')) > \bar{u}^\gamma_j(x^*, t(x^*))$. Then $\bar{u}^\gamma(x', t(x'))$ γ-dominates $\bar{u}^\gamma(x^*, t(x^*))$, a contradiction. Hence $(x^*, t(x^*)) \in C^\gamma_\beta(\overline{g})$ $[C^\gamma_\alpha(\overline{g})]$.$\square$

Now, theorem below shows the close relation between the core of the associated side payment game v^g_β $[v^g_\alpha]$ and $C^1_\beta(\overline{g})$ $[C^1_\alpha(\overline{g})]$.

Theorem 5. Let $g = (N, X, u)$ be a finite normal form game, if $(x^*, t(x^*)) \in C^1_\beta(\overline{g})$ $[C^1_\alpha(\overline{g})]$ then $\overline{u}^\gamma(x^*, t(x^*)) \in Core(v^g_\beta)$ $[Core(v^g_\alpha)]$. And if $y = (y_1, \cdots, y_{|N|})$

$\in Core(v^g_\beta)$ $[Core(v^g_\alpha)]$ then $\exists(x^*, t(x^*)) \in C^1_\beta(\overline{g})$ $[C^1_\alpha(\overline{g})]$ such that $\overline{u}^\gamma(x^*, t(x^*))$ $= y$.

Proof: Let $g = (N, X, u)$ be a finite normal form game. Suppose that $(x^*, t(x^*)) \in C^1_\beta(\overline{g})$ $[C^1_\alpha(\overline{g})]$, thus there is no coalition $S \subseteq N$ such that

$\forall x_{N\setminus S} \in X_{N\setminus S}$ $\exists x_S \in X_S$ such that

$$\forall i \in S \; \exists t(x_S, x_{N\setminus S}) \in T_S(x_S, x_{N\setminus S}) : \bar{u}^\gamma_i(x^*, t(x^*)) \leq \bar{u}^\gamma_i((x_S, x_{N\setminus S}), t(x_S, x_{N\setminus S}))$$

$$\text{and } \exists j \in S \text{ such that } \bar{u}^\gamma_j(x^*, t(x^*)) < \bar{u}^\gamma_j((x_S, x_{N\setminus S}), t(x_S, x_{N\setminus S})).$$

As $S = N$ is also possible as well, one has $x^* \in argmax_{x\in X} \sum_{i\in N} u_i(x)$. That is $\sum_{i\in N} u_i(x^*) = v^g_\beta(N)$ $[v^g_\alpha(N)]$. Assume on the contrary that $\bar{u}^\gamma(x^*, t(x^*)) \notin Core(v^g_\beta)$ $[Core(v^g_\alpha)]$. Then $\exists S' \subseteq N$ such that $v^g_\beta(S')$ $[v^g_\alpha(S')] > \sum_{i\in S'} \bar{u}^\gamma_i(x^*, t(x^*))$. Then by playing strategy

$$\arg\{\min_{x_j\in X_j} \max_{x_i\in X_i} u_i(x_i, x_j)\}[\arg\{\max_{x_i\in X_i} \min_{x_j\in X_j} u_i(x_i, x_j)\}],$$

coalition $S' \subseteq N$ γ-blocks $\bar{u}^\gamma(x^*, t(x^*))$ in the β-sense [α-sense]. Contradiction with $(x^*, t(x^*)) \in C^1_\beta(\overline{g})$ $[C^1_\alpha(\overline{g})]$.

Now let $y = (y_1, \cdots, y_{|N|}) \in Core(v^g_\beta)$ $[Core(v^g_\alpha)]$. Now as $v^g_\beta(N)$ $[v^g_\alpha(N)] = \sum_{i\in N} y_i$ and utility is transferable of degree 1, there exists an outcome $x^* \in X$ such that $x^* \in \arg\max \sum_{i\in N, x\in X} u_i(x)$ and allocation y can be regenerated from $u(x^*)$ via transfers such that $\overline{u}^\gamma(x^*, t(x^*)) = y$. Now suppose that $(x^*, t(x^*)) \notin C^1_\beta(\overline{g})$ $[C^1_\alpha(\overline{g})]$, then $\exists S \subset N : \forall x_{N\setminus S} \in X_{N\setminus S}$ $\exists x_S \in X_S$ such that $\exists t(x_S, x_{N\setminus S}) \in T_S(x_S, x_{N\setminus S}) : \bar{u}^\gamma_i(x^*, t(x^*)) \leq \bar{u}^\gamma_i((x_S, x_{N\setminus S}), t(x_S, x_{N\setminus S}))$ for all $i \in S$ and $\exists j \in S$ such that $\bar{u}^\gamma_j(x^*, t(x^*)) < \bar{u}^\gamma_j((x_S, x_{N\setminus S}), t(x_S, x_{N\setminus S}))$ [$\exists x_S \in X_S \forall x_{N\setminus S} \in X_{N\setminus S}$ such that $\exists t(x_S, x_{N\setminus S}) \in T_S(x_S, x_{N\setminus S}) : \bar{u}^\gamma_i(x^*, t(x^*)) \leq \bar{u}^\gamma_i((x_S, x_{N\setminus S}), t(x_S, x_{N\setminus S}))$ for all $i \in S$ and $\exists j \in S$ such that $\bar{u}^\gamma_j(x^*, t(x^*)) < \bar{u}^\gamma_j((x_S, x_{N\setminus S}), t(x_S, x_{N\setminus S}))$]. As utility is 1-transferable, summing up inequalities side by side one has $(y_j)_{j\in S} = \bar{u}_S^\gamma(x^*, t(x^*)) < \bar{u}_S^\gamma((x_S, x_{N\setminus S}), t(x_S, x_{N\setminus S}))$. Contradiction with the fact that $y \in Core(v_\beta^g)$ $[Core(v_\alpha^g)]$. □

Lemma 2. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$. For any $\gamma \in [0, 1]$, if $((x_1, t_1), (x_2, t_2)) \in NE(\overline{g})$, then $(x_1, x_2) \in NE(g(t_1, t_2))$.

Proof. Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$ $\forall x \in X$. Suppose $((x_1', t_1), (x_2', t_2)) \in NE(\overline{g})$ but $(x_1', x_2') \notin NE(g(t_1, t_1))$. Then $\exists i \in N$ such that $x_i' \notin BR_i(x_j')$ in game $g(t_1, t_1)$. Hence $\overline{u}_i^\gamma((BR_i(x_j'), t_i), (x_j', t_j)) > \overline{u}_i^\gamma((x_i', t_i), (x_j', t_2))$, hence agent i is better off if he changes his strategy to $(BR_i(x_j'), t_i)$, a contradiction with the fact that $((x_1', t_1), (x_2', t_2)) \in NE(\overline{g})$.□

Lemma 3. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$. For any $\gamma \in [0, 1]$, if $((x_1', t_1), (x_2', t_2)) \in NE(\overline{g})$, then $x_i' \in BR_i(x_j')$ $\forall i \in N$ in game g.

Proof. Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$ $\forall x \in X$. Suppose $((x_1', t_1), (x_2', t_2)) \in NE(\overline{g})$ but $\exists i \in N$ such that $x_i' \notin BR_i(x_j')$ in game g. As x_j' is not a function of t_i agent i is better off by changing his strategy to $t_i'(x_i, x_j) = \{0$ if $(x_i, x_j) = (x_1', x_2')$, $t_i(x_i, x_j)$ otherwise$\}$ and $x_i' = BR_i(x_j')$ as $\overline{u}_i^\gamma((BR_i(x_j'), t_i'), (x_j', t_j)) > \overline{u}_i^\gamma((x_i', t_i), (x_j', t_2))$. A contradiction with the fact that $((x_1', t_1), (x_2', t_2)) \in NE(\overline{g})$.□

Theorem 6. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$ $\forall x \in X$. Let $\gamma \in [0, 1]$ be given, $((t_1, x_1'), (t_2, x_2')) \in NE(\overline{g})$ if and only if $(x_1', x_2') \in NE(g)$.

Proof. Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$ $\forall x \in X$.

Suppose $((x'_1, t_1), (x'_2, t_2)) \in NE(\overline{g})$ but $(x'_1, x'_2) \notin NE(g)$, $\exists i \in N$ such that $x'_i \notin BR_i(x'_j)$ in game g, then contradiction with Lemma 3. Hence $(x'_1, x'_2) \in NE(g)$.

Let $(x'_1, x'_2) \in NE(g)$. Consider the following transfer vector: $t_i(x) \in T\ \forall i \in N$ such that $t_i(x'_1, x'_2) = 0$. For any $(\widetilde{x}_i, \widetilde{x}_j)$ such that $[\widetilde{x}_i \in BR_i(\widetilde{x}_j)$ and $u_j(\widetilde{x}_i, \widetilde{x}_j) > u_j(x'_1, x'_2)]$, let $t_i(\widetilde{x}_i, \widetilde{x}_j) = u_i(\widetilde{x}_i, \widetilde{x}_j)$ and $\forall x_i \in X_i\ t_i(x_i, \widetilde{x}_j) > 0$ such that $\exists \widetilde{\widetilde{x}}_i \in X_i$ such that $\widetilde{\widetilde{x}}_i \in \arg\min_{x_i} u_j(x_i, x_j)$ and $\overline{u}_i^{\gamma}((\widetilde{x}_i, t_i), (\widetilde{x}_j, t_j)) < \overline{u}_i^{\gamma}((\widetilde{\widetilde{x}}_i, t_i), (\widetilde{x}_j, t_j))$.

Note that $\exists \widetilde{\widetilde{x}}_i \in X_i$ as $u_i(x'_1, x'_2) \geq max_{x_i \in X_i} min_{x_j \in X_j} u_i(x_i, x_j)$. For any other $x \in X$ let $t_i(x) = 0$. By construction $(x'_1, x'_2) \in NE(g(t_1, t_2))$, hence deviations in strategies in game $g(t_1, t_2)$ are not beneficial.

Note that $\forall i \in N\ \overline{u}_i^{\gamma}((x'_i, t_i), (x'_j, t_j)) = u_i(x'_i, x'_j) \geq \overline{u}_i^{\gamma}((x^*_i, t'_i), (x^*_j, t_j))$ for any $t'_i \in T$ and $\forall (x^*_j, x^*_j) \in NE(g(t'_i, t_j))$ with $(x^*_j, x^*_j) \neq (x'_1, x'_2)$ as change in t'_i does not effect the strategy $x^{'}_j$ and as agent j propose to play the min strategy for $u_i(x)$ except for the outcome (x'_1, x'_2). Hence $((x'_1, t_1), (x'_2, t_2)) \in NE(\overline{g})$. □

Corollary1. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \rightarrow R^2_+$, and $u(x) > 0$. For any $\gamma \in [0, 1]$ given, let $(x'_1, x'_2) \in C_{\alpha}(g)$, $\exists (t_1, t_2) \in T_1 \times T_2$ such that $((x'_1, t_1), (x'_2, t_2)) \in NE(\overline{g})$ only if $(x'_1, x'_2) \in NE(g)$.

Corollary2. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \rightarrow R^2_+$, and $u(x) > 0$. Let $\gamma \in [0, 1]$ given, let $(x'_1, x'_2) \in C_{\beta}(g)$, $\exists (t_1, t_2) \in T_1 \times T_2$ such that $((x'_1, t_1), (x'_2, t_2)) \in NE(\overline{g})$ only if $(x'_1, x'_2) \in NE(g)$.

CHAPTER 4

$\gamma-$TU PREDONATION GAMES

In this chapter following the idea of Sertel, we will study the semi-transferable utility games when it is possible to make predonations. To insert a predonation mechanism to normal form games when utility is semi-transferable we construct a two stage game to assure that the announced transfers will realize. In our analysis we will preserve the characteristics of the normal form game like the strategy space and simultaneity of the actions, and analyze how introduction of predonation mechanisms will effect the wellfare of the players.

Sertel (1992) shows that in case of bargaining problems, predonation mechanisms yield an increase in the total welfare of the agents. Now, given a normal form game, it is clear that if only one agent has the right to make transfers, this agent will not get worse off with respect to Nash Equilibrium outcomes of the normal form game as he always have the opportunity to choose his strategy to be making no transfers and play the original game. Whereas in the case where all agents have the right to make transfers, it is ambigious whether total welfare will increase in stable outcomes.

Definition 21. $\Gamma = (N, H, P, \gtrsim_i)$ is an *extensive game with perfect information* where

N is the set of players,

H is set of sequences (finite or infinite) that satisfies three properties as

i. The empty sequence $\varnothing$ is a member of H,

ii. If $(a_k)_{k=1,\ldots,K} \in H$ and $L < K$ then $(a_k)_{k=1,\ldots,L} \in H$ (where K may be infinite).

iii. If an infinite sequence $(a_k)_{k=1}^{\infty}$ satisfies $(a_k)_{k=1,\ldots,L} \in H$ for every positive integer L then $(a_k)_{k=1}^{\infty} \in H$.

Each member of H represents a history where each component of a history shows an action taken by a player during the play of the game. A history $(a_k)_{k=1,\ldots,K} \in H$ is terminal if it is infinite or if there is no a $K+1$ such that $(a_k)_{k=1,\ldots,K+1} \in H$. The set of terminal histories is denoted Z.

Function P assigns to each nonterminal history (each member of $H \backslash Z$) a member of N.

$\succsim_i$ is the preference relation of player i on Z.

After any nonterminal history h player $P(h)$ chooses an action from the set $A(h) = \{a : (h, a) \in H\}$.

Definition 22. A strategy of player $i \in N$ in an extensive game with perfect information $\Gamma = (N, H, P, (\succsim)_i)$ is a function that assigns an action in $A(h)$ to each nonterminal history $h \in H \backslash Z$ for which $P(h) = i$.

Definition 23. A Nash equilibrium of an extensive game with perfect information $\Gamma = (N, H, P, (\succsim)_i)$ is a strategy profile s^* such that for every player $i \in N$ we have $O(s^*_{-i}, s^*_i) \succsim_i O(s^*_{-i}, s_i)$ for every strategy s_i of player i. $(O(s^*_{-i}, s^*_i)$ denote the outcome that is realized when strategy profile s^* is chosen. Notation (s^*_{-i}, s_i) denotes the case that the players except than i stick to their strategies at s^* whereas agent i deviates and plays s_i.

Definition 24. *A subgame perfect equilibrium of an extensive game with perfect information* $\Gamma = (N, H, P, (\succsim)_i)$ is a strategy profile s^* such that for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $i \in P(H)$ we have

$O(s^*_{-i}\mid_h, s^*_i\mid_h) \succsim_i O(s^*_{-i}\mid_h, s_i)$ where $s_i|_h$ denotes the strategy that s_i induces in the subgame $\Gamma(h)$.

A subgame perfect equilibrium is a strategy profile s^* in Γ for which for any history h the strategy profile $s^*\mid_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

Given a 2 person finite normal form game $g = (X, u)$ where $u : X_1 \times X_2 \to R^2_+$, we define the associated game with transfers as a two stage game as follows:

In the first stage, the agents announce a transfer at each outcome, where $t_i : X_1 \times X_2 \to R$ such that $0 \leq t_i(x_1, x_2) \leq u_i(x_1, x_2)$ for any $(x_1, x_2) \in X_1 \times X_2$.

In the second stage, the agents play the normal form game with transfers $g(t_1, t_2) = (X, \tilde{u})$, where $\tilde{u}_i : X_1 \times X_2 \to R$ such that $\tilde{u}_i(x_1, x_2) = u_i(x_1, x_2) - t_i(x_1, x_2) + \gamma t_j(x_1, x_2)$ for any $(x_1, x_2) \in X_1 \times X_2$, where $i, j \in \{1, 2\}$.

Definition 25. Given a 2 person finite normal form game $g = (X, u)$ where $X = X_1 \times X_2, u : X_1 \times X_2 \to R^2_+$, *the associated game with transfers*, $v(G)$, is a two stage extensive form game where $v(G) = (Y_1, Y_2, \bar{u}^\gamma_1, \bar{u}^\gamma_2)$, where $Y_i = T_i \times X_i^{T_i \times T_j}, T_i = \{t_i : X_1 \times X_2 \to R$ such that $0 \leq t_i(x_1, x_2) \leq u_i(x_1, x_2)$ for any $(x_1, x_2) \in X_1 \times X_2\}$ and

$$\bar{u}^\gamma_i((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) = \bar{u}^\gamma_i(x_1(t_1, t_2), x_1(t_1, t_2))$$

$$= u_i(x_1(t_1, t_2), x_2(t_1, t_2)) - t_i(x_1(t_1, t_2), x_2(t_1, t_2)) + \gamma t_j(x_1(t_1, t_2), x_2(t_1, t_2)).$$

Definition 26. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R^2_+$, let $v(G)$ be the associated 2 stage game.

$((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2)))$ is a *Quasi Subgame Perfect Equilibrium* if and only if

i. $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2)))$ is a NE of $v(G) = (Y_1, Y_2, \bar{u}^\gamma_1, \bar{u}^\gamma_2)$ and

ii. $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ is a Nash Equilibrium of $g(t'_1, t'_2)$ with $NE(g(t'_1, t'_2))$

$\neq \varnothing$.

Given utility transfer degree γ, we will denote the set of Quasi Subgame Perfect Equilibria as $QSPE^{\gamma}$, and we say that an outcome $x \in X$ *is reachable as a QSPE outcome* if there exists $x_i(t'_i, t'_j)$ and $x_j(t'_i, t'_j)$ and $t_i \in T_i, t_j \in T_j$ such that $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^{\gamma}$.

The notion of quasi subgame perfect equilibrium is very similar to the notion of subgame perfect equilibrium. Note that when t_1, t_2 are announced it is possible to have $NE(g(t'_1, t'_2)) = \varnothing$, then QSPE allows players to announce any strategy function for the second stage. To understand the concept we will investigate $QSPE^{\gamma}$ of Prisoner's Dilemma.

Example 3. Prisoners' Dilemma

Let $g = (N, X_1 \times X_2, u)$ be such that $X_1 = X_2 = \{C, D\}$ and u be defined by the following table where agent 1 is the row player and agent 2 is the column player where $u(x) = (u_1(x), u_2(x))$.

	C	D
C	$1, 1$	$3, 0$
D	$0, 3$	$2, 2$

When the game is transformed to 2-stage game $v(G)$, the game $g(t_1, t_2)$ is as follows:

	C	D
C	$\tilde{u}_1(C, C), \tilde{u}_2(C, C)$	$\tilde{u}_1(C, D), \tilde{u}_2(C, D)$
D	$\tilde{u}_1(D, C), \tilde{u}_2(D, C)$	$\tilde{u}_1(D, D), \tilde{u}_2(D, D)$

where

$(\tilde{u}_1(C,C), \tilde{u}_2(C,C)) = (1 - t_1(C,C) + \gamma t_2(C,C), 1 - t_2(C,C) + \gamma t_1(C,C))$

$(\tilde{u}_1(C,D), \tilde{u}_2(C,D)) = (3 - t_1(C,D), \gamma t_1(C,D))$

$(\tilde{u}_1(D,C), \tilde{u}_2(D,C)) = (\gamma t_2(D,C), 3 - t_2(D,C))$

$(\tilde{u}_1(D,D), \tilde{u}_2(D,D)) = (2 - t_1(D,D) + \gamma t_2(D,D), 2 - t_2(D,D) + \gamma t_1(D,D))$

$(C,C) \in NE(g(t_1,t_2))$ if and only if

$1 - t_1(C,C) + \gamma t_2(C,C) \geq \gamma t_2(D,C)$ (1) and

$1 - t_2(C,C) + \gamma t_1(C,C) \geq \gamma t_1(C,D)$. (2)

$(C,D) \in NE(g(t_1,t_2))$ if and only if

$3 - t_1(C,D) \geq 2 - t_1(D,D) + \gamma t_2(D,D)$ (3) and

$\gamma t_1(C,D) \geq 1 - t_2(C,C) + \gamma t_1(C,C)$. (4)

$(D,C) \in NE(g(t_1,t_2))$ if and only if

$\gamma t_2(D,C) \geq 1 - t_1(C,C) + \gamma t_2(C,C)$ (5) and

$3 - t_2(D,C) \geq 2 - t_2(D,D) + \gamma t_1(D,D)$. (6)

$(D,D) \in NE(g(t_1,t_2))$ if and only if

$2 - t_1(D,D) + \gamma t_2(D,D) \geq 3 - t_1(C,D)$ (7) and

$2 - t_2(D,D) + \gamma t_1(D,D) \geq 3 - t_2(D,C)$. (8)

Note that the above conditions are necessary to have an outcome to be reachable as a QSPE since the outcome should belong to $NE(g(t_1,t_2))$. $QSPE^{\gamma}$ for $\gamma \in [0,1]$ are as follows:

Case1: $\gamma \in [0, \frac{1}{3})$

Let $t_1(x) = \vec{0}$ and $t_2(x) = \vec{0}$ and $x_1(t'_1,t'_2) = \{D$ if $[t'_2(D,C) \geq \frac{1}{\gamma}$ and $t'_1(C,D) \geq \frac{1}{\gamma}], C$ otherwise$\}$, $x_2(t'_1,t'_2) = \{D$ if $[t'_2(D,C) \geq \frac{1}{\gamma}$ and $t'_1(C,D) \geq \frac{1}{\gamma}], C$ otherwise$\}$ Equations (1) and (2) are satisfied thus $(x_1(t_1,t_2), x_2(t_1,t_2)) = (C,C) \in NE(g(t_1,t_2))$. Condition (ii) of definition 27 is satisfied.

Now we will check condition (i) of definition 27. $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in NE(v(G))$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$ as follows: Since agents are symmetric, it is sufficient to show that one sided deviation is not profitable for agent 1. Agent 1 can change game $g(t_1, t_2)$ to $g(t_1', t_2)$ shown below:

	C	D
C	$1 - t_1'(C, C)$, $1 + \gamma t_1'(C, C)$	$3 - t_1'(C, D)$, $\gamma t_1'(C, D)$
D	0, 3	$2 - t_1'(D, D)$, $2 + \gamma t_1'(D, D)$

For any $t_1'(C, C) > 0$ agent 1 only worse off thus $t_1'(C, C) = 0$ is dominant strategy. For any $0 < t_1'(C, D) \leq 3$, $t_1'(C, D) < \frac{1}{\gamma}$ thus $x_2(t_1', t_2) = C$. Similarly there is no use to change $t_1'(D, D) > 0$. Changing t_1' such that $NE(g(t_1', t_2)) = \varnothing$ is not profitable since outcome (C, C) hence payoff vector does not change. Agent 1 cannot unilaterally deviate and become better off, by symmetry neither can agent 2, hence $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in NE(v(G))$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$. Thus $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^\gamma$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$.

$(D, D))$ is reachable as a QSPE outcome as follows: Keeping $x_1(t_1', t_2')$ and $x_2(t_1', t_2')$ same as above, let $t_1(x) = \{3$ if $x = (C, D), 0$ otherwise$\}$, $t_2(x) = \{3$ if $x = (D, C), 0$ otherwise$\}$. Now $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D) \in g(t_1, t_2)$ and clearly $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in NE(v(G))$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D)$. Thus $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^\gamma$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D)$.

$(C, D))$ is not reachable as a QSPE outcome as follows:

Let $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ be such that if game $g(t_1', t_2')$ has Nash Equilibria, $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ constitute a Nash Equilibrium, if game $g(t_1', t_2')$ does not have any Nash Equilibrium $x_1(t_1', t_2') = x_2(t_1', t_2') = C$. Let $\exists t_1 \in T_1$ and $t_2 \in T_2$ such that $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D) \in NE(g(t_1, t_2))$. Agent 2 can always unilaterally change $t_2'' = \vec{0}$, then equation (1) is satisfied for any $t_1(C, C) \in [0, 1]$, and equation (2) is satisfied for any $t_1(C, D) \in [0, 3]$, $t_1(C, C) \in [0, 1]$, hence $(C, C) \in NE(g(t_1, t_2'))$ and agent 2 is better off as

$$\bar{u}_2^\gamma((t_1, x_1(t_1', t_2'')), (t_2', \tilde{x}_2(t_1', t_2''))) = \bar{u}_2^\gamma((t_1, C), (t_2'', C)) =$$

$$= 1 + \gamma t_1(C, C) > \gamma t_1(C, D) = \bar{u}_2^\gamma((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) =$$

$$= \bar{u}_2^\gamma((t_1, C), (t_2, D)).$$

Thus for any $t_1 \in T_1$ and $t_2 \in T_2$, $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \notin QSPE^\gamma$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D)$.

As agents are symmetric, similar arguments as above show that for any $t_1 \in T_1$ and $t_2 \in T_2$, $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \notin QSPE^\gamma$ where
$(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, C)$.

Case 2: $\gamma \in [\frac{1}{3}, \frac{1}{2})$

$(C, C))$ is reachable as a QSPE outcome as follows: Let $x_1(t_1', t_2') = \{D$ if $[t_2'(D, C) \geq \frac{1}{\gamma}$ and $t_1'(C, D) \geq \frac{1}{\gamma}], C$ otherwise$\}$, $x_2(t_1', t_2') = \{D$ if $[t_2'(D, C) \geq \frac{1}{\gamma}$ and $t_1'(C, D) \geq \frac{1}{\gamma}], C$ otherwise$\}$.$((\vec{0}, C), (\vec{0}, C))$ constitute a QSPE as profitable unilateral deviations are not possible. Agent 1 is not able to profitably deviate as follows: As $\gamma \in [\frac{1}{3}, \frac{1}{2}), (C, D) \notin NE(g(t_1', t_2))$ as equation (4) requires $t_1'(C, D) = \frac{1}{\gamma}$ which implies $\bar{u}_1^\gamma(C(t_1', t_2), D(t_1', t_2)) < 1$, so agent 1 is worse off. For a profitable deviation with an equilibrium at outcome (D, D) one needs $\bar{u}_2^\gamma(D(t_1', t_2), D(t_1', t_2)) \geq 3$. As $t_2(D, D) = 0$, one needs $t_1(D, D) = \frac{1}{\gamma}$, then

$\bar{u}_1^{\gamma}(D(t_1', t_2), D(t_1', t_2)) < 1 = \bar{u}_1^{\gamma}(C(t_1, t_2), C(t_1, t_2))$. Agent 2 is symmetric, thus for $\gamma \in [\frac{1}{3}, \frac{1}{2})$, $((\vec{0}, x_1(t_1', t_2')), (\vec{0}, x_2(t_1', t_2'))) \in QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$.

$(D, D))$ is reachable as a QSPE outcome as follows: Similar as in case 1, $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D)$, where $x_1(t_1', t_2')$ and $x_2(t_1', t_2')$ same as above, and $t_1(x) = \{3$ if $x = (C, D), 0$ otherwise$\}$, $t_2(x) = \{3$ if $x = (D, C), 0$ otherwise$\}$.

For any $t_i \in T_i, ((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \notin QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D)$ as follows. Let $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ be such that if game $g(t_1', t_2')$ has Nash Equilibria, $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ constitute a Nash Equilibrium, if game $g(t_1', t_2')$ does not have any Nash Equilibrium $x_1(t_1', t_2') = x_2(t_1', t_2') = C$. Let $\exists t_1 \in T_1$ and $t_2 \in T_2$ such that $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D) \in g(t_1, t_2)$. If $\bar{u}_2^{\gamma}((t_1, C), (t_2, D) < 1$, agent 2 can always profitably unilaterally change $t_2'' = \vec{0}$ as explained in case 1. If $\bar{u}_2^{\gamma}((t_1, C), (t_2, D) \geq 1$ one has $\bar{u}_1^{\gamma}((t_1, C), (t_2, D)) < 1$, then agent 1 can unilaterally deviate and play $t_1'' = \vec{0}$ and become better off as follows: If $NE(g(t_1'', t_2)) \neq \varnothing$, then agent 1 gets a payoff greater than or equal to 1 as $max_{x_1(\vec{0}, t_2')} min_{x_2(\vec{0}, t_2')} \bar{u}_1^{\gamma}(x_1(\vec{0}, t_2'),\ x_2(\vec{0}, t_2')) = 1$. If $NE(g(t_1'', t_2)) = \varnothing$, then outcome (C, C) will be realized with payoff $\bar{u}_1^{\gamma}(x_1(\vec{0}, t_2'), x_2(\vec{0}, t_2')) = 1 + \gamma t_2(C, C) \geq 1 > 1 = \bar{u}_1^{\gamma}((t_1, C), (t_2, D)$.

As agents are symmetric, similar arguments as above show that for any $t_1 \in T_1$ and $t_2 \in T_2$ $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \notin QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, C)$.

Case 3: $\gamma \in [\frac{1}{2}, 1)$

For $\gamma = \frac{1}{2}$, $x_1(t_1', t_2') = \{D$ if $[t_2'(D, C) \geq \frac{1}{\gamma}$ and $t_1'(C, D) \geq \frac{1}{\gamma}]$,$C$ otherwise$\}$, $x_2(t_1', t_2') = \{D$ if $[t_2'(D, C) \geq \frac{1}{\gamma}$ and $t_1'(C, D) \geq \frac{1}{\gamma}]$,$C$ otherwise$\}$. $((\vec{0}, x_1(t_1', t_2'))$,

$(\vec{0}, x_2(t'_1, t'_2))) \in QSPE^{\frac{1}{2}}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$ as profitable unilateral deviations are not possible.

For $\gamma > \frac{1}{2}$, for any $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \notin QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$ as follows: Agent 1 can unilaterally profitably deviate by announcing $t''_1(C, D) = \frac{1}{\gamma}, t''_1(D, D) = 2, t''_1(x) = 0$ if $x \in \{(C, C), (D, C)\}$. $(C, D) \in NE(g(t''_1, t_2))$ as

$$\bar{u}^{\gamma}_2(C(t''_1, t_2), D(t''_1, t_2)) = 1 \geq 1 - t_2(C, C) = \bar{u}^{\gamma}_2(C(t''_1, t_2), C(t''_1, t_2)) \text{ and}$$

$$\bar{u}^{\gamma}_1(C(t''_1, t_2), D(t''_1, t_2)) = 3 - \frac{1}{\gamma} > 1 \geq \gamma t_2(D, D) = \bar{u}^{\gamma}_1(D(t''_1, t_2), D(t''_1, t_2)).$$

Agent 1 is better off as

$\bar{u}^{\gamma}_1(C(t''_1, t_2), D(t''_1, t_2)) = 3 - \frac{1}{\gamma} > 1 \geq \bar{u}^{\gamma}_1(C(t_1, t_2), C(t_1, t_2))$.

(Note that in above cases we have shown that if $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$, $\bar{u}^{\gamma}_i((t_1, C), (t_2, C)) = 1$.)

For any $t_i \in T_i$, $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \notin QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D)$ as follows. Let $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ be such that if game $g(t'_1, t'_2)$ has Nash Equilibria, $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ constitute a Nash Equilibrium, if game $g(t'_1, t'_2)$ does not have any Nash Equilibrium $x_1(t'_1, t'_2) = x_2(t'_1, t'_2) = C$. Let $\exists t_1 \in T_1$ and $t_2 \in T_2$ such that $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D) \in g(t_1, t_2)$. If $\bar{u}^{\gamma}_2((t_1, C), (t_2, D) < 1$, agent 2 can always profitably unilaterally change $t''_2 = \vec{0}$ as explained in case 1, thus $\bar{u}^{\gamma}_2((t_1, C), (t_2, D) \geq 1$. Then $\bar{u}^{\gamma}_1((t_1, C), (t_2, D) < 2$ as $\frac{1}{2} < \gamma < 1$. If $t_2(D, C) \geq \frac{1}{\gamma}$ agent 1 can unilaterally deviate and play $t''_1(x) = \{0$ if $x \in \{(C, C), (D, C), (D, D)\}, 3$ if $x = (C, D)\}$ and become better off as $(D, D) \in g(t''_1, t_2)$ and $\bar{u}^{\gamma}_1((t''_1, D), (t_2, D)) = 2 > \bar{u}^{\gamma}_1((t_1, C), (t_2, D)$. If $t_2(D, C) < \frac{1}{\gamma}$ agent 2 can unilaterally deviate to $t''_2(x) = \{0$ if $x \in \{(C, C), (C, D)\}, \frac{1}{\gamma}$ if $x = (D, C), 2$ if $x = (D, D)\}$ and become better off as $(D, C) \in g(t_1, t''_2)$ and $\bar{u}^{\gamma}_1((t''_2, D), (t_2, D)) > 1 = \bar{u}^{\gamma}_2((t_1, C), (t_2, D)$. Hence for any $t_i \in T_i$, $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \notin$

$QSPE^\gamma$ where $(x_1(t_1,t_2),x_2(t_1,t_2)) = (C,D)$ for $\frac{1}{2} < \gamma < 1$.

As agents are symmetric, similar arguments as above show that for any $t_1 \in T_1$ and $t_2 \in T_2$ $((t_1, x_1(t_1',t_2')),(t_2,x_2(t_1',t_2'))) \notin QSPE^\gamma$ where $(x_1(t_1,t_2),$ $x_2(t_1,t_2)) = (D,C)$ for $\frac{1}{2} < \gamma < 1$.

Whereas for $\gamma = \frac{1}{2}$, $((t_1, x_1(t_1',t_2')),(t_2,x_2(t_1',t_2'))) \in QSPE^\gamma$ where $(x_1(t_1,t_2),$ $x_2(t_1,t_2)) = (C,D)$ and $(x_1(t_1',t_2'),x_2(t_1',t_2'))$ are such that if game $g(t_1',t_2')$ has Nash Equilibria, $(x_1(t_1',t_2'),x_2(t_1',t_2'))$ constitute a Nash Equilibrium, if game $g(t_1',t_2')$ does not have any Nash Equilibrium $x_1(t_1',t_2') = x_2(t_1',t_2') = C$, $t_1(x) = \{0$ if $x \in \{(C,C),(D,C)\}$, 2 if $x \in \{(C,D),(D,D)\}\}$, $t_2(x) = \vec{0}$. And $((t_1, x_1(t_1',t_2')),$ $(t_2,x_2(t_1',t_2'))) \in QSPE^\gamma$ where $(x_1(t_1,t_2),x_2(t_1,t_2)) = (C,D)$ and $t_1(x) = \vec{0}$, $t_2(x) = \{0$ if $x \in \{(C,C),(C,D)\}$, 2 if $x \in \{(D,C),(D,D)\}\}$.

Similar as in case 1, $((t_1, x_1(t_1',t_2')),(t_2,x_2(t_1',t_2'))) \in QSPE^\gamma$, where $(x_1(t_1,t_2),$ $x_2(t_1,t_2)) = (D,D)$ and $x_1(t_1',t_2')$ and $x_2(t_1',t_2')$ are such that if game $g(t_1',t_2')$ has Nash Equilibria, $(x_1(t_1',t_2'),x_2(t_1',t_2'))$ constitute a Nash Equilibrium, if game $g(t_1',t_2')$ does not have any Nash Equilibrium $x_1(t_1',t_2') = x_2(t_1',t_2') = C$, and

$$t_1(x) = \{3 \text{ if } x = (C,D), 0 \text{ otherwise}\}, t_2(x) = \{3 \text{ if } x = (D,C), 0 \text{ otherwise}\}.$$

Case 4: $\gamma = 1$.

Similar arguments as in case 3 show that for any $t_1 \in T_1$ and $t_2 \in T_2$, $((t_1, x_1(t_1',t_2')),(t_2,x_2(t_1',t_2'))) \notin QSPE^1$ where $(x_1(t_1,t_2),x_2(t_1,t_2)) = (C,C)$.

Similar as in case 1, $((t_1, x_1(t_1',t_2')),(t_2,x_2(t_1',t_2'))) \in QSPE^1$ where $(x_1(t_1,t_2),$ $x_2(t_1,t_2)) = (D,D)$ and $x_1(t_1',t_2') = \{D$ if $[t_2'(D,C) \geq \frac{1}{\gamma}$ and $t_1'(C,D) \geq \frac{1}{\gamma}]$, C otherwise$\}$, $x_2(t_1',t_2') = \{D$ if $[t_2'(D,C) \geq \frac{1}{\gamma}$ and $t_1'(C,D) \geq \frac{1}{\gamma}]$, C otherwise$\}$ and $t_1(x) = \{3$ if $x = (C,D), 0$ otherwise$\}$, $t_2(x) = \{3$ if $x = (D,C), 0$ otherwise$\}$.

For any $t_i \in T_i$, $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \notin QSPE^1$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D)$ as follows. Let $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ be such that if game $g(t_1', t_2')$ has Nash Equilibria, $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ constitute a Nash Equilibrium, if game $g(t_1', t_2')$ does not have any Nash Equilibrium $x_1(t_1', t_2') = x_2(t_1', t_2')$ $= C$. Let $\exists t_1 \in T_1$ and $t_2 \in T_2$ such that $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, D) \in g(t_1, t_2)$. Now if $t_1(C, D) < 1, \bar{u}_2^\gamma((t_1, C), (t_2, D) < 1$ and agent 2 can always profitably unilaterally change $t_2'' = \vec{0}$ as explained in case 1, thus $t_1(C, D) \geq 1$ and $\bar{u}_2^\gamma((t_1, C), (t_2, D)) \geq 1$. If $t_1(C, D) > 1, \bar{u}_1^\gamma((t_1, C), (t_2, D) < 2$ and agent 1 can unilaterally deviate to $t_1''(x) = \{t_1(C, D) - \varepsilon > 1$ if $x = (C, D), t_1(x)$ otherwise$\}$ and become better off as $\bar{u}_1^\gamma((t_1'', C), (t_2, D)) = 3 - t_1(C, D) + \varepsilon > 3 - t_1(C, D) = \bar{u}_1^\gamma((t_1, C), (t_2, D))$. (Note that this rule is generally true for a QSPE, that is the transfers given should be minimal possible.) Therefore $t_1(C, D) = 1$, then agent 2 can unilaterally deviate to $t_2''(x) = \{0$ if $x \in \{(C, C), (C, D)\}, 3$ if $x = (D, C), t_1(D, D)$ if $x = (D, D)\}$ and become better off as follows: $(D, D) \in g(t_1, t_2'')$ since $\bar{u}_1^\gamma((t_1, C), (t_2, D)) = u_1(C, D) - t_1(C, D) = 3 - t_1(C, D) = 3 - 1 = 2 \leq 2 - t_1(D, D) + t_2''(D, D) = 2 - t_1(D, D) + t_1(D, D) = \bar{u}_1^\gamma((t_1, D), (t_2'', D)$ and $\bar{u}_2^\gamma((t_1, D), (t_2'', C) = 0 < 2 - t_2''(D, D) + t_1(D, D) = 2 = \bar{u}_2^\gamma((t_1, D), (t_2'', D)$. As $\bar{u}_2^\gamma((t_1, D), (t_2'', D) = 2 > 1 = \bar{u}_2^\gamma((t_1, C), (t_2, D))$ agent 2 is better off. Thus for any $t_1 \in T_1$ and $t_2 \in T_2$ $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \notin QSPE^1$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (C, C)$.

As agents are symmetric, similar arguments as above show that for any $t_1 \in T_1$ and $t_2 \in T_2$ $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \notin QSPE^1$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, C)$.

For $\gamma \in [0, \frac{1}{2}), (D, D)$ is reachable as a QSPE outcome with $t_1(x) \in T_1$, $t_2(x) \in T_2$ such that $t_1(C, D) - t_1(D, D) + \gamma t_2(D, D) = 1$ and $t_2(D, C) - t_2(D, D) + \gamma t_1(C, D) = 1$ where $t_1(C, D) \in [1, 2]$ and $t_2(D, C) \in [1, 2]$ as follows: With given $t_i(x) \in T_i$ one has $\bar{u}_1^\gamma((t_1, D), (t_2, D) = \bar{u}_1^\gamma((t_1, C), (t_2, D)$ and

$\bar{u}_2^{\gamma}((t_1, D), (t_2, D) = \bar{u}_2^{\gamma}((t_1, D), (t_2, C)$. Hence $(D, D) \in g(t_1, t_2)$. Equations (7) and (8) being tight with $t_1(C, D) \in [1, 2]$ and $t_2(D, C) \in [1, 2]$ implies the following: Firstly $\bar{u}_i^{\gamma}((t_1, D), (t_2, D) \geq 1, \forall i \in N$. Secondly agent i cannot unilaterally deviate from t_i and become better off since only outcome that agent i could become better off is outcome (D, D) as $\gamma \in [0, \frac{1}{2})$. Thus (D, D) is rachable as a QSPE outcome with $\bar{u}^{\gamma}((t_1, D), (t_2, D)) = (2 - t_1(D, D) + \gamma t_2(D, D), 2 - t_2(D, D) + \gamma t_1(D, D)) \gg (1, 1)$.

For $\gamma \in [\frac{1}{2}, 1]$, (D, D) is reachable as a QSPE outcome with $t_1(x) \in T_1, t_2(x) \in T_2$ such that $t_1(C, D) - t_1(D, D) + \gamma t_2(D, D) = 1$ and $t_2(D, C) - t_2(D, D) + \gamma t_1(C, D) = 1$ where $t_1(C, D) \in [1, \frac{1}{\gamma}]$ and $t_2(D, C) \in [1, \frac{1}{\gamma}]$ as follows: With given $t_i(x) \in T_i$ one has $\bar{u}_1^{\gamma}((t_1, D), (t_2, D) = \bar{u}_1^{\gamma}((t_1, C), (t_2, D)$ and $\bar{u}_2^{\gamma}((t_1, D), (t_2, D) = \bar{u}_2^{\gamma}((t_1, D), (t_2, C)$. Hence $(D, D) \in g(t_1, t_2)$. Equations (7) and (8) being tight with $t_1(C, D) \in [1, \frac{1}{\gamma}]$ and $t_2(D, C) \in [1, \frac{1}{\gamma}]$ implies the following: Firstly $\bar{u}_i^{\gamma}((t_1, D), (t_2, D) \geq 3 - \frac{1}{\gamma}, \forall i \in N$. Secondly agent i cannot unilaterally deviate from t_i and become better off since agent can maximally get $3 - \frac{1}{\gamma}$ at an outcome different than (D, D). And it is not possible to make $(D, D) \in NE(g(t_i'', t_j))$ where $\bar{u}_i^{\gamma}(D(t_i', t_j), D(t_i', t_j))$ as equations (7) and (8) are tight. Thus (D, D) is reachable as a QSPE outcome with $u^{\gamma}((t_1, D), (t_2, D)) = (2 - t_1(D, D) + \gamma t_2(D, D), 2 - t_2(D, D) + \gamma t_1(D, D)) \gg (3 - \frac{1}{\gamma}, 3 - \frac{1}{\gamma})$.

Example 4. Centipede game

<table>
<tr><td></td><td>1</td><td>C</td><td>2</td><td>C</td><td>1</td><td>C</td><td>2</td><td>C</td><td>1</td><td>C</td><td>2</td><td>C</td><td></td></tr>
<tr><td></td><td>.</td><td>-</td><td>.</td><td>-</td><td>.</td><td>-</td><td>.</td><td>-</td><td>.</td><td>-</td><td>.</td><td>-</td><td>(6,5)</td></tr>
<tr><td>S</td><td>|</td><td>S</td><td>|</td><td>S</td><td>|</td><td>S</td><td>|</td><td>S</td><td>|</td><td>S</td><td>|</td><td></td><td></td></tr>
<tr><td></td><td>(1,0)</td><td></td><td>(0,2)</td><td></td><td>(3,1)</td><td></td><td>(2,4)</td><td></td><td>(5,3)</td><td></td><td>(4,6)</td><td></td><td></td></tr>
</table>

The corresponding strategic normal form game is given by the below table where player 1 and 2 are row and column players respectively.

$X_1 = X_2 = \{SSS, SSC, SCS, SCC, CSS, CSC, CCS, CCC\}$, and

$u(x_1, x_2) = (u_1(x_1, x_2),\ u_2(x_1, x_2))$ are given in the below table.

	SSS	SSC	SCS	SCC	CSS	CSC	CCS	CCC
SSS	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
SSC	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
SCS	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
SCC	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
CSS	0,2	0,2	0,2	0,2	3,1	3,1	3,1	3,1
CSC	0,2	0,2	0,2	0,2	3,1	3,1	3,1	3,1
CCS	0,2	0,2	0,2	0,2	2,4	2,4	5,3	5,3
CCC	0,2	0,2	0,2	0,2	2,4	2,4	4,6	6,5

$\beta_1 = \min_{x_2} \max_{x_1} u_1(x_1, x_2) = 1.$

$\beta_2 = \min_{x_1} \max_{x_2} u_2(x_1, x_2) = 0.$

i. $x = (x_1, x_2) \in \{SSS, SSC, SCS, SCC\} \times X_2$ where $u(x) = (1, 0)$.

For $\gamma < 0.15$,

$((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^\gamma$ where $(x_1(t_1, t_2)\ , x_2(t_1, t_2)) = x$.

Let $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ are such that if game $g(t_1', t_2')$ has Nash Equilibria, $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ constitute a Nash Equilibrium, if game $g(t_1', t_2')$ does not have any Nash Equilibrium $x_1(t_1', t_2') = x_2(t_1', t_2') = SSS$. $t_1(x) = \{u_1(x)$ if $u_1(x) > 1$, 0 otherwise$\}$, $t_2(x) = \{u_2(x)$ if $u_2(x) \neq 2$, 0 otherwise$\}$.

Firstly note that $x \in NE(g(t_1, t_2))$. Agent 1 cannot profitably deviate as follows:

$3 + \gamma - \frac{2}{\gamma} > 1$ requires $\gamma > 0.73$

$2 + 4\gamma - \frac{2}{\gamma} > 1$ requires $\gamma > 0.59$

$5 + 3\gamma - \frac{2}{\gamma} > 1$ requires $\gamma > 0.25$

$4 + 6\gamma - \frac{2}{\gamma} > 1$ requires $\gamma > 0.38$

$6 + 5\gamma - \frac{2}{\gamma} > 1$ requires $\gamma > 0.31$

Agent 2 cannot profitably deviate as follows:

$1 + 3\gamma - \frac{1}{\gamma} > 0$ requires $\gamma > 0.43$

$4 + 2\gamma - \frac{1}{\gamma} > 0$ requires $\gamma > 0.22$

$3 + 5\gamma - \frac{1}{\gamma} > 0$ requires $\gamma > 0.24$

$6 + 4\gamma - \frac{1}{\gamma} > 0$ requires $\gamma > 0.15$

$5 + 6\gamma - \frac{1}{\gamma} > 0$ requires $\gamma > 0.17$

Note that in a game $g(t_1', t_2')$ agent 1 and 2 individually can guarantee a total payoff

$5 + 3\gamma - \frac{4}{\gamma} + 6 + 4\gamma - \frac{5}{\gamma}$ and $11 + 7\gamma - \frac{9}{\gamma} > 1$ whenever $\gamma > 0.63$, , note that $\frac{4}{\gamma} < 5$ whenever $\gamma > 0.8$ and $\frac{5}{\gamma} < 6$ whenever $\gamma > 0.83$ hence $x = (x_1, x_2) \in \{SSS, SSC, SCS, SCC\} \times X_2$ where $u(x) = (1, 0)$ is not a part of any QSPE for $\gamma > 0.83$.

ii. $x = (x_1, x_2) \in \{CSS, CSC, CCS, CCC\} \times \{SSS, SSC, SCS, SCC\} = \widetilde{X}_1 \times \widetilde{X}_2$ where $u(x) = (2, 0)$.

Suppose $\exists (t_1, t_2) \in T$ such that $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^\gamma$ for some $\gamma \in [0, 1]$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = x \in \widetilde{X}_1 \times \widetilde{X}_2$.

Note that if x is a part of a QSPE, agent 1 cannot get a payoff of 1 due to proposition and β_1.

As $\beta_2 = 0$ one should have $2 - \frac{1}{\gamma} > 0$ implying $\gamma > \frac{1}{2}$. (Note also that $2 - \frac{1}{\gamma} > 1$ if $\gamma > 1$).Now consider $x_2' = CSS$. If in game $g(t_1, t_2)$ $BR_1(CSS) \in \{CSS, CSC, CCS, CCC\}$ then agent 2 should better deviate as $2 - \frac{1}{\gamma} < 1$ as $\gamma < 1$ and $2 - \frac{1}{\gamma} < 4$. Similar arguments imply that the payoff vector (2,4) must be destructed by agent 1. Suppose this destruction is at maximal level, i.e. $\overline{u}^\gamma(x_1', CSS) \in \{(1, 0), (0, 1 + 3\gamma), (0, 4 + 2\gamma)\}$. But $4 + 2\gamma - \frac{1}{\gamma} > 2 - \frac{1}{\gamma}$ as $2 + 2\gamma > 0$, and $1 + 3\gamma - \frac{1}{\gamma} > 2 - \frac{1}{\gamma}$ as $\gamma > \frac{1}{2}$. Therefore $\nexists (t_1, t_2) \in T$ such that $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^\gamma$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = x \in \widetilde{X}_1 \times \widetilde{X}_2$ for any $\gamma \in [0, 1]$.

iii. $x = (x_1, x_2) \in \{CSS, CSC\} \times \{CSS, CSC, CCS, CCC\} = \overline{X}_1 \times \overline{X}_2$ where $u(x) = (3, 1)$.

For $\gamma < 0.4$,

$((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^\gamma$ where $(x_1(t_1, t_2),\ x_2(t_1, t_2)) = x \in \overline{X}_1 \times \overline{X}_2$.

Let $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ are such that if game $g(t'_1, t'_2)$ has Nash Equilibria, $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ constitute a Nash Equilibrium, if game $g(t'_1, t'_2)$ does not have any Nash Equilibrium $x_1(t'_1, t'_2) = x_2(t'_1, t'_2) = SSS$. $t_1(x) = \{6$ if $u_1(x) = 6$, 2 if $u_1(x) = 2$, 5 if $u_1(x) = 5$, 4 if $u_1(x) = 4$, 0 otherwise$\}$, $t_2(x) = \{6$ if $u_2(x) = 6$, 5 if $u_2(x) = 5$, 4 if $u_2(x) = 4$, 3 if $u_2(x) = 3$, 0 otherwise$\}$.

Firstly note that $x \in NE(g(t_1, t_2))$. Agent 1 cannot profitably deviate as follows:

$2 + 4\gamma - \frac{2}{\gamma} > 3$ requires $\gamma > 0.84$

$5 + 3\gamma - \frac{2}{\gamma} > 3$ requires $\gamma > 0.55$

$4 + 6\gamma - \frac{2}{\gamma} > 3$ requires $\gamma > 0.5$

$6 + 5\gamma - \frac{2}{\gamma} > 3$ requires $\gamma > 0.4$

Agent 2 cannot profitably deviate as follows:

$4 + 2\gamma - \frac{3}{\gamma} > 1$ requires $\gamma > 0.68$

$3 + 5\gamma - \frac{3}{\gamma} > 1$ requires $\gamma > 0.96$

$6 + 4\gamma - \frac{3}{\gamma} > 1$ requires $\gamma > 0.44$

$5 + 6\gamma - \frac{3}{\gamma} > 1$ requires $\gamma > 0.45$

Note that agent 1 and 2 individually can guarantee a total payoff

$5 + 3\gamma - \frac{4}{\gamma} + 6 + 4\gamma - \frac{5}{\gamma}$ and $11 + 7\gamma - \frac{9}{\gamma} > 4$ whenever $\gamma > 0.74$, note that $\frac{4}{\gamma} < 5$ whenever $\gamma > 0.8$ and $\frac{5}{\gamma} < 6$ whenever $\gamma > 0.83$ hence $x = (x_1, x_2) \in$

$\{SSS, SSC, SCS, SCC\} \times X_2$ where $u(x) = (3, 1)$ is not a part of any QSPE for $\gamma > 0.83$.

iv. $x = (x_1, x_2) \in \{CCS, CCC\} \times \{CSS, CSC\}$ where $u(x) = (2, 4)$.

For $\gamma < 0.58$,

$((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = x$.

Let $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ are such that if game $g(t'_1, t'_2)$ has Nash Equilibria, $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ constitute a Nash Equilibrium, if game $g(t'_1, t'_2)$ does not have any Nash Equilibrium $x_1(t'_1, t'_2) = x_2(t'_1, t'_2) = SSS$. $t_1(x) = \{6$ if $u_1(x) = 6$, 5 if $u_1(x) = 5$, 0 otherwise$\}$, $t_2(x) = \{6$ if $u_2(x) = 6$, 5 if $u_2(x) = 5$, 3 if $u_2(x) = 3$, 0 otherwise$\}$.

Firstly note that $x \in NE(g(t_1, t_2))$. Agent 1 cannot profitably deviate as follows:

$3 - \frac{1}{\gamma} > 2$ requires $\gamma > 1$

$5 + 3\gamma - \frac{4}{\gamma} > 2$ requires $\gamma > 0.76$

$4 + 6\gamma - \frac{4}{\gamma} > 2$ requires $\gamma > 0.68$

$6 + 5\gamma - \frac{4}{\gamma} > 2$ requires $\gamma > 0.58$

Agent 2 cannot profitably deviate as follows:

$6 + 4\gamma - \frac{3}{\gamma} > 4$ requires $\gamma > 0.65$

$5 + 6\gamma - \frac{3}{\gamma} > 4$ requires $\gamma > 0.63$

Note that agent 1 and 2 individually can guarantee a total payoff

$5 + 3\gamma - \frac{4}{\gamma} + 6 + 4\gamma - \frac{5}{\gamma} = 11 + 7\gamma - \frac{9}{\gamma}$ and $11 + 7\gamma - \frac{9}{\gamma} > 6$ whenever $\gamma > 0.83$, note that $\frac{4}{\gamma} < 5$ whenever $\gamma > 0.8$ and $\frac{5}{\gamma} < 6$ whenever $\gamma > 0.83$, hence $x = (x_1, x_2) \in \{CCS, CCC\} \times \{CSS, CSC\}$ where $u(x) = (2, 4)$ is not reachable as a QSPE for $\gamma > 0.83$.

v. $x = (x_1, x_2) \in \{CCS\} \times \{CCS, CCC\}$ where $u(x) = (5, 3)$.

For $\gamma < 0.54$,

$((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^{\gamma}$ where $(x_1(t_1, t_2),\ x_2(t_1, t_2)) = x$.

Let $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ are such that if game $g(t'_1, t'_2)$ has Nash Equilibria, $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ constitute a Nash Equilibrium, if game $g(t'_1, t'_2)$ does not have any Nash Equilibrium $x_1(t'_1, t'_2) = x_2(t'_1, t'_2) = SSS$. $t_1(x) = \{4$ if $u_1(x) = 4$, 6 if $u_1(x) = 6$, 0 otherwise$\}$, $t_2(x) = \{5$ if $u_2(x) = 5$, 6 if $u_2(x) = 6$, 0 otherwise$\}$.

Firstly note that $x \in NE(g(t_1, t_2))$. Agent 1 cannot profitably deviate as follows:

$4 + 6\gamma - \frac{4}{\gamma} > 5$ requires $\gamma > 0.90$

$6 + 5\gamma - \frac{4}{\gamma} > 5$ requires $\gamma > 0.54$

Agent 2 cannot profitably deviate as follows:

$4 - \frac{1}{\gamma} > 3$ requires $\gamma > 1$

$6 + 4\gamma - \frac{5}{\gamma} > 3$ requires $\gamma > 0.80$

$5 + 6\gamma - \frac{5}{\gamma} > 3$ requires $\gamma > 0.76$

Note that for $\gamma > 0.94$ agent 1 and 2 individually can guarantee a total payoff

$5 + 3\gamma - \frac{4}{\gamma} + 6 + 4\gamma - \frac{5}{\gamma}$ and $11 + 7\gamma - \frac{9}{\gamma} > 8$ whenever $\gamma > 0.94$, note that $\frac{4}{\gamma} < 5$ whenever $\gamma > 0.8$ and $\frac{5}{\gamma} < 6$ whenever $\gamma > 0.83$, hence $x = (x_1, x_2) \in \{CCS, CCC\} \times \{CSS, CSC\}$ where $u(x) = (5, 3)$ is not a part of any QSPE for $\gamma > 0.94$.

vi. $x = (x_1, x_2) = (CCC, CCS)$ where $u(x) = (4, 6)$.

For $\gamma < 0.91$,

$((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^{\gamma}$ where $(x_1(t_1, t_2),\ x_2(t_1, t_2)) = x$.

Let $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ are such that if game $g(t'_1, t'_2)$ has Nash Equilibria, $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ constitute a Nash Equilibrium, if game $g(t'_1, t'_2)$ does not have any Nash Equilibrium $x_1(t'_1, t'_2) = x_2(t'_1, t'_2) = SSS$. $t_1(x) = \{1$ if $u_1(x) = 5$, 0 otherwise$\}$, $t_2(x) = \{5$ if $u_2(x) = 5$, 0 otherwise$\}$.

Firstly note that $x \in NE(g(t_1,t_2))$. Agent 1 cannot profitably deviate as follows:

$5 - \frac{4}{\gamma} > 4$ requires $\gamma > 1$

$6 + 5\gamma - \frac{6}{\gamma} > 4$ requires $\gamma > 0.91$ and $\frac{6}{\gamma} \leq 6$ requires $\gamma \geq 1$

Agent 2 cannot profitably deviate as she is getting her maximal payoff in the centipede game.

Note that for $\gamma = 1$ agent 1 guarantees payoff of $6 + 5\gamma - \frac{6}{\gamma} = 5$ and agent 2 guarantees payoff of $6+4\gamma - \frac{5}{\gamma} = 5$, suppose t_1', t_2' are such that $\overline{u}^\gamma(CCC, CCS) = (5,5)$ and $(CCC, CCS) \in NE(g(t_1', t_2'))$, note that agent 1 can always unilaterally deviate to $t_1''(x) = \{(5+\varepsilon) - (5 - t_2'(CCC, CCC))$ if $x = (CCC, CCC)$, $t_1'(x)$ otherwise$\}$ where $\varepsilon > 0$ (infinitely small),where $(CCC, CCC) \in NE(g(t_1'', t_2'))$ and $\overline{u}_1^\gamma((CCC, t_1''), (CCC, t_2')) > 5$, this profitable deviation implies that (CCC, CCC) cannot be reachable as aQSPE when $\gamma = 1$.

vii. $x = (x_1, x_2) = (CCC, CCC)$ where $u(x) = (6, 5)$.

For $\gamma \in [0, 1]$,

$((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^\gamma$ where $(x_1(t_1, t_2),\ x_2(t_1, t_2)) = x$.

Let $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ are such that if game $g(t_1', t_2')$ has Nash Equilibria, $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ constitute a Nash Equilibrium, if game $g(t_1', t_2')$ does not have any Nash Equilibrium $x_1(t_1', t_2') = x_2(t_1', t_2') = SSS$. $t_1(x) = \overrightarrow{0}$, $t_2(x) = \{6$ if $u_2(x) = 6$, 0 otherwise$\}$.

Firstly note that $x \in NE(g(t_1,t_2))$. Agent 1 cannot profitably deviate as follows:

$4 + 6\gamma - \frac{5}{\gamma} > 6$ is not possible as $\gamma > 0$ and $\frac{5}{\gamma} > 4$

Agent 2 cannot profitably deviate as

$6 - \frac{1}{\gamma} > 5$ requires $\gamma > 1$

Note that for $\gamma = 1$, the only outcome that are reachable as a QSPE is the one which maximizes total welfare.

Note also that as γ increases, the outcomes that are reachable as a QSPE are the ones which are more efficient with respect to total welfare.

We introduced semi transferable utility where utility can assume non-integral values between 0 and 1 allowing players to utilize their advantages in terms of payoffs in the outcome space. We added utility transfer to strategic form games in such a way that players do not have means of coordination when announcing transfers or choosing strategies, yet it turns out that outcomes that are reachable by coordination now become stable equilibria. For the games where Nash equilibrium gives players their minmax values and self interested players cannot improve upon these payoffs in spite of existence of superior outcomes for both agents' best interests, without having means of cooperation there is an improvement in efficiency of stable outcomes in STU games. Depending on the payoff and strategy structure of a given game one can achieve welfare maximizing outcome to be the only equilibrium as γ increases. Introduction of STU might have interesting results depending on variations of the associated STU games such as agents not moving simultaneously or having right of not aacepting the transfers given by the opponent. These points will motivate future studies.

CHAPTER 5

FURTHER STUDIES ON QSPE

In previous chapter we introduced a two stage game given a normal form game, and we showed that more efficient outcomes can become stable outcomes in famous examples like Prisoners' Dilemma and Centipede Game as γ increases. In this chapter we will question whether this is valid for every game by questioning properties of QSPE. For a better understanding of the QSPE we analyze cases where choice of transfers and actions are simultaneous or made in an order in order to utilize the classification of 2 person games of pretension in Koray and Sertel (1983).

5.1. Simultaneous Choice of Transfers and Actions

As introduced in previous chapter, two stage game is played such that firstly the choice of transfers and consequtively choice of actions are made simultaneously by all players. Following remark states that given γ, tension between stability and efficiency prevails for QSPE.

Remark 5. A quasi subgame perfect equilibrium is not necessarily γ-efficient.

We have seen in example 1, Prisoners' Dilemma, for $\gamma \in (0, \frac{1}{3})$ (C, C) is reachable as a quasi subgame perfect equilibrium which is Pareto dominated by

(D, D), i.e. γ-dominated for $\gamma = 0$, hence by Lemma 1, γ-dominated $\forall \gamma \in (0, \frac{1}{3})$ by outcome (D, D).

In Prisoners' Dilemma it is the case that as γ increases QSPE outcomes offer a better payoff to one agent and the other agent is not worse off with respect to the Nash Equilibrium payoff of the original game. Can we argue in general that a QSPE outcome is not Pareto dominated at least by one Nash Equilibrium of the original game? Formally, given a finite 2-person normal form game g let $(t^*, x(t)) \in QSPE^\gamma$ where $x(t) = x^*$ for some $\gamma \in [0, 1]$ it is not necessarily true that $\exists y^* \in NE(g)$ such that x^* is not Pareto dominated by y^*. Consider the following example.

Example 5. Consider the following 2-person finite normal form game g:

	A	B	C
a	$1, 1$	$\frac{3}{4}, \frac{3}{4}$	$\frac{1}{2}, \frac{1}{4}$
b	$\frac{3}{4}, \frac{3}{4}$	$\frac{3}{4}, 0$	$\frac{1}{4}, 0$
c	$\frac{1}{4}, \frac{1}{2}$	$\frac{3}{4}, 0$	$0, 0$

(a, A) is the only Nash Equilibrium of game g.

Let $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ be such that if game $g(t'_1, t'_2)$ has Nash Equilibria, $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ constitute a Nash Equilibrium, if game $g(t'_1, t'_2)$ does not have any Nash Equilibrium $x_1(t'_1, t'_2) = c$, $x_2(t'_1, t'_2) = C$.

Let $\gamma = \frac{1}{2}$ be given. $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^\gamma$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (b, A)$ where $t_1(x_1, x_2) = \{1$ if $(x_1, x_2) = (a, A)$, 0 otherwise$\}$ and $t_2(x_1, x_2) = \{1$ if $(x_1, x_2) = (a, A)$, 0 otherwise$\}$.

$g(t_1, t_2)$ is as follows:

	A	B	C
a	$\frac{1}{2},\frac{1}{2}$	$\frac{3}{4},\frac{3}{4}$	$\frac{1}{2},\frac{1}{4}$
b	$\frac{3}{4},\frac{3}{4}$	$\frac{3}{4},0$	$\frac{1}{4},0$
c	$\frac{1}{4},\frac{1}{2}$	$\frac{3}{4},0$	$0,0$

$(b,A) \in NE(g(t_1,t_2))$ and we will check whether agent 1 can unilaterally deviate by changing $t_1(x_1,x_2)$. Agent 1 can transform game to $g(t_1',t_2)$ as follows:

	A	B	C
a	$\widetilde{u}_1(a,A),\widetilde{u}_2(a,A)$	$\widetilde{u}_1(a,B),\widetilde{u}_2(a,B)$	$\widetilde{u}_1(a,C),\widetilde{u}_2(a,C)$
b	$\widetilde{u}_1(b,A),\widetilde{u}_2(b,A)$	$\widetilde{u}_1(b,B),\widetilde{u}_2(b,B)$	$\widetilde{u}_1(b,C),\widetilde{u}_2(b,C)$
c	$\widetilde{u}_1(c,A),\widetilde{u}_2(c,A)$	$\widetilde{u}_1(c,B),\widetilde{u}_2(c,B)$	$\widetilde{u}_1(c,C),\widetilde{u}_2(c,C)$

$(\widetilde{u}_1(a,A),\widetilde{u}_2(a,A)) = (\frac{3}{2} - t_1^{'}(a,A), \gamma t_1^{'}(a,A))$

$(\widetilde{u}_1(a,B),\widetilde{u}_2(a,B)) = (\frac{3}{4} - t_1^{'}(a,B), \frac{3}{4} + \gamma t_1^{'}(a,B))$

$(\widetilde{u}_1(a,C),\widetilde{u}_2(a,C)) = (\frac{1}{2} - t_1^{'}(a,C), \frac{1}{4} + \gamma t_1^{'}(a,C))$

$(\widetilde{u}_1(b,A),\widetilde{u}_2(b,A)) = (\frac{3}{4} - t_1^{'}(b,A), \frac{3}{4} + \gamma t_1^{'}(b,A))$

$(\widetilde{u}_1(b,B),\widetilde{u}_2(b,B)) = (\frac{3}{4} - t_1^{'}(b,B), \gamma t_1^{'}(b,B))$

$(\widetilde{u}_1(b,C),\widetilde{u}_2(b,C)) = (\frac{1}{4} - t_1^{'}(b,C), \gamma t_1^{'}(b,C))$

$(\widetilde{u}_1(c,A),\widetilde{u}_2(c,A)) = (\frac{1}{4} - t_1^{'}(c,A), \frac{1}{2} + \gamma t_1^{'}(c,A))$

$(\widetilde{u}_1(c,B),\widetilde{u}_2(c,B)) = (\frac{3}{4} - t_1^{'}(c,B), \gamma t_1^{'}(c,B))$

$(\widetilde{u}_1(c,C),\widetilde{u}_2(c,C)) = (0,0)$

Only outcome that agent 1 possibly becomes better off is (a,A). $(a,A) \in NE(g(t_1',t_2))$ requires $\frac{3}{2} - t_1'(a,A) > \frac{3}{4}$ and $\gamma t_1'(a,A) \geq \frac{3}{4}$. Second inequality implies $t_1'(a,A) \geq \frac{3}{4}\gamma = \frac{3}{2}$. Then $\frac{3}{2} - t_1'(a,A) \leq 0$. Thus agent 1 can not beneficially unilaterally deviate. Note that in case of a game such that $NE(g(t_1',t_2)) = \varnothing$, agent 1 can get at most $\frac{1}{2} < \frac{3}{4}$, hence a beneficial unilateral deviation for agent 1 such that $NE(g(t_1',t_2)) = \varnothing$ is not possible.

Similarly in game $g(t_1, t_2')$ agent 2 can only become better off at (a, A). $(a, A) \in NE(g(t_1, t_2'))$ requires $\frac{3}{2} - t_2'(a, A) > \frac{3}{4}$ and $t_2'(a, A) \geq \frac{3}{4}$. Second inequality implies $t_2'(a, A) \geq \frac{3}{4}\gamma = \frac{3}{2}$. Then $\frac{3}{2} - t_2'(a, A) \leq 0$. Thus agent 2 cannot profitably unilaterally deviate. Note that in case of a game such that $NE(g(t_1, t_2')) = \varnothing$, agent 2 can get at most $\frac{1}{2} < \frac{3}{4}$, hence a beneficial unilateral deviation for agent 2 such that $NE(g(t_1, t_2')) = \varnothing$ is not possible.

Thus $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^{\frac{1}{2}}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (b, A)$ and outcome $(a, A) \in NE(g)$ Pareto dominates (b, A).

It is clear that if the right of transfer giving belongs to only one agent in the game, that agent will always become better off or at least he is not worse off at a QSPE outcome with respect to Nash Equilibria of game g, as he always has the opportunity to play $t = \vec{0}$ and play the original game g. When all agents have the opportunity to transfer the game into another via transfer making, and they do it simultaneously; as they do not communicate, it is possible that both agent might have changed the game to another such that restoring the original game by unilateral deviations via transfers is not possible.

In Prisoners' Dilemma we have seen that the 1-efficient outcome where social welfare is maximized is reachable by predonation games even when $\gamma = 0$, and for $\gamma > \frac{1}{2}$ 1-efficient outcome is the only QSPE outcome, which can be interpreted as an improvement in the wellfare. Given a finite normal form game g, is it always possible to reach the 1-efficient outcome as a QSPE outcome for some $\gamma \in [0, 1]$? Following example shows that it is not necessarily true.

Example 6. Consider the following 2 person normal form game:

	C	D
C	$0, 0$	$\frac{11}{2}, \frac{1}{100}$
D	$\frac{1}{100}, 1$	$6, 0$

(D, D) is the only 1-efficient outcome.

Let $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ be such that if game $g(t'_1, t'_2)$ has Nash Equilibria, $(x_1(t'_1, t'_2), x_2(t'_1, t'_2))$ constitute a Nash Equilibrium, if game $g(t'_1, t'_2)$ does not have any Nash Equilibrium $x_1(t_1, t_2) = x_2(t_1, t_2) = C$.

For any $t_i \in T_i$, $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \notin QSPE^0$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D)$, as $(D, D) \in NE(g(t_1, t_2))$ requires that $t_2(D, C) = 1$ where $\bar{u}_2^{\gamma}(D(t_1, t_2), D(t_1, t_2)) = 0$. Agent 2 is better off if she deviates and plays $t''_2 = \vec{0}$ as $(D, C) \in NE(g(t_1, t''_2))$ for any $t_1 \in T_1$ and $\bar{u}_2^{\gamma}(D(t_1, t''_2), C(t_1, t''_2)) = 1 > 0 = \bar{u}_2^{\gamma}(D(t_1, t_2), D(t_1, t_2))$.

For any $\gamma > 0$ and for any $t_i \in T_i$, $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \notin QSPE^{\gamma}$ where $(x_1(t_1, t_2),\ x_2(t_1, t_2)) = (D, D)$. Let $\exists t_1 \in T_1$ and $t_2 \in T_2$ such that $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D) \in g(t_1, t_2)$. If $\bar{u}_2^{\gamma}(D(t_1, t_2),\ D(t_1, t_2)) < 1$ as explained above agent 2 can always deviate to $t''_2 = \vec{0}$ and get better off. $\bar{u}_2^{\gamma}(D(t_1, t_2), D(t_1, t_2)) \geq 1$ implies that $t_1(D, D) \geq \frac{1}{\gamma} \geq 1$ as $\gamma \in (0, 1]$. Thus $\bar{u}_1^{\gamma}(D(t_1, t_2), D(t_1, t_2)) = 6 - t_1(D, D) \leq 6 - \frac{1}{\gamma} \leq 5$. Note that agent 1 is better off if he deviates and plays $t''_1(x) = \{1 \text{ if } x = (D, D), 0 \text{ otherwise}\}$ as $(C, D) \in NE(g(t''_1, t_2))$ for any $t_2 \in T_2$, and $\bar{u}_1^{\gamma}(C(t''_1, t_2), D(t''_1, t_2)) = \frac{11}{2} > 1 \geq \bar{u}_1^{\gamma}(D(t_1, t_2), D(t_1, t_2))$. Thus for any $t_i \in T_i$, $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \notin QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D)$.

Hence for $\forall t_i \in T_i$ and $\forall \gamma \in [0, 1]$, $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \notin QSPE^{\gamma}$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (D, D)$.

Now in example 3 the 1-efficient outcome where social welfare is maximized is never reachable as a QSPE outcome for any $\gamma \in [0, 1]$.

If reachability of 1-efficient outcome as a QSPE is considered as a measure of tension between stability and efficiency, we can say that Prisoner's Dilemma has tension 0, whereas the game in example 6 has tension infinity.

Remark 6. Example 6 also shows that it is not necessarily true that given $\gamma \in [0,1]$, if $(x_1', x_2') \in argmax \sum_{i \in \{1,2\},(x_1,x_2) \in X} u_i(x_1, x_2)$ is not reachable as a QSPE outcome, then $QSPE^\gamma = \varnothing$.

Now following remark states that if we pick an outcome $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is it necessarily true that the set of γ such that $(\bar{x}_1, \bar{x}_2)$ is reachable as a QSPE outcome is connected.

Remark 7. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R^2_+$, let $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ and $I(\bar{x}_1, \bar{x}_2) = \{\gamma \in [0,1] : \exists t_i \in T_i$ such that $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in QSPE^\gamma$ where $(x_1(t_1, t_2), x_2(t_1, t_2)) = (\bar{x}_1, \bar{x}_2)\}$. I is not necessarily connected. The following constitutes an example:

Example 7. Consider the following 2 person normal form game:

	C	D
C	$\frac{1}{100}, \frac{1}{100}$	$3, 0$
D	$\frac{1}{200}, \frac{1}{100}$	$2, 2$

Let $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ be such that if game $g(t_1', t_2')$ has Nash Equilibria, $(x_1(t_1', t_2'), x_2(t_1', t_2'))$ constitute a Nash Equilibrium, if game $g(t_1', t_2')$ does not have any Nash Equilibrium $x_1(t_1', t_2') = x_2(t_1', t_2') = C$.

For $\gamma = 0$, let $t_1(x) = \{1$ if $x = (C, D), 0$ otherwise$\}$, and $t_2(x) = \vec{0}$. $(D, D) \in NE(g(t_1, t_2))$. $u_2^\gamma(D(t_1, t_2), D(t_1, t_2)) = 2$ is the maximal utility that agent 2 can achieve in the given game, hence agent 2 has no incentive to deviate. Can agent 1 become better of by deviating? Only outcome that agent 1 can become better off is (C, D) where agent 2 gets a payoff of zero, $(C, D) \notin NE(g(t_1', t_2)) \; \forall t_1' \in T_1$. Thus agent 1 cannot profitably deviate, and $((t_1, x_1(t_1', t_2')), (t_2, x_2(t_1', t_2'))) \in$

$NE(v(G))$ where $x_1(t_1,t_2) = x_2(t_1,t_2) = D$. Thus $((t_1, x_1(t'_1,t'_2)), (t_2, x_2(t'_1,t'_2))) \in QSPE^0$ where $x_1(t_1,t_2) = x_2(t_1,t_2) = D$.

Now for $\gamma = \frac{1}{4}$, agent 1 can guarantee himself a payoff of $3 - \frac{4}{100}$ as playing $t'_1(x) = \{\frac{4}{100}$ if $x = (C,D), 0$ otherwise$\}$, he can always have $(C,D) \in NE(t'_1,t_2)$ $\forall t_2 \in T_2$. Assume that $\exists(t_1,t_2) \in T_1 \times T_2$ such that $((t_1, x_1(t'_1,t'_2)), (t_2, x_2(t'_1,t'_2))) \in QSPE^{\frac{1}{4}}$ where $x_1(t_1,t_2) = x_2(t_1,t_2) = D$. By previous argument it should be the case that $\bar{u}_1^{\gamma}(D(t_1,t_2), D(t_1,t_2)) \geq 3 - \frac{4}{100}$, that is $2 + \gamma t_2(D,D) \geq 3 - \frac{4}{100}$, hence $t_2(D,D) \geq 4(1 - \frac{4}{100}) > 2$, a contradiction. Hence $\forall(t_1,t_2) \in T_1 \times T_2$, $((t_1, x_1(t'_1,t'_2)), (t_2, x_2(t'_1,t'_2))) \notin QSPE^{\frac{1}{4}}$ where $x_1(t_1,t_2) = x_2(t_1,t_2) = D$.

For $\gamma = 1$, let $t_1(x) = \{\frac{1}{100}$ if $x = (C,D), 0$ otherwise$\}$, and $t_2(x) = \{1 - \frac{1}{100}$ if $x = (D,D), 0$ otherwise$\}$. $(D,D) \in NE(g(t_1,t_2))$. $u_2^{\gamma}(D(t_1,t_2), D(t_1,t_2)) = 1 + \frac{1}{100}$ and agent 2 cannot get better off at any game $g(t_1,t'_2)$, thus agent 2 has no incentive to deviate. Agent 1 cannot become better of by deviating as $\bar{u}_1^{\gamma}(D(t_1,t_2), D(t_1,t_2)) = 3 - \frac{1}{100}$ and agent 1 cannot have a strictly greater payoff at (C,D) which is the only outcome that he can probably get better off at any game $g(t'_1,t_2)$. Thus $((t_1, x_1(t'_1,t'_2)), (t_2, x_2(t'_1,t'_2))) \in QSPE^1$ where $x_1(t_1,t_2) = x_2(t_1,t_2) = D$. Therefore $0, 1 \in I(D,D)$ whereas $\frac{1}{4} \notin I(D,D)$, hence $I(D,D)$ is not connected.

Proposition 4. Given a 2 person finite normal form game $g = (X,u)$, where $u : X_1 \times X_2 \rightarrow R_+^2$, and $u(x) > 0$ $\forall x \in X$. If $((t_1, x_1(t'_1,t'_2)), (t_2, x_2(t'_1,t'_2))) \in QSPE^{\gamma}$ where $x_i(t_1,t_2) = \bar{x}_i$, one has

$$\bar{u}_i^{\gamma}(\bar{x}_i(t_1,t_2), \bar{x}_j(t_1,t_2)) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j).$$

Proof: Let $g = (X,u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \rightarrow R_+^2$, and $u(x) > 0$ $\forall x \in X$. Let $((t_1, x_1(t'_1,t'_2)), (t_2, x_2(t'_1,t'_2))) \in QSPE^{\gamma}$ where $x_i(t_1,t_2) = \bar{x}_i$, but assume on the contrary that $\exists i \in N$ such that

$\bar{u}_i^\gamma(\bar{x}_i(t_1,t_2),\bar{x}_j(t_1,t_2)) < min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)$. Then agent i can deviate to strategy $t_i'(x) = \vec{0}$, now there are 3 cases:

Case 1: $NE(g(t_i',t_j)) \neq \varnothing$ and $(\tilde{x}_i,\tilde{x}_j) \in NE(g(t_i',t_j))$ where $\tilde{x}_i \in BR_i(\tilde{x}_j)$ then $\bar{u}_i^\gamma(\tilde{x}_i(t_i',t_j),\tilde{x}_j(t_i',t_j)) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)$ as $t_i' = \vec{0}$.

Case 2: $NE(g(t_i',t_j)) \neq \varnothing$ and $(\tilde{x}_i,\tilde{x}_j) \in NE(g(t_i',t_j))$ where $\tilde{x}_i \notin BR_i(\tilde{x}_j)$. As $t_i'(x) = \vec{0}$, it should be the case that $t_j(\tilde{x}_i,\tilde{x}_j) > 0$ such that $\bar{u}_i^\gamma(\tilde{x}_i(t_i',t_j),\tilde{x}_j(t_i',t_j)) \geq u_i(BR_i(\tilde{x}_j),\tilde{x}_j)$. Thus $\tilde{x}_i$ constitutes a best response strategy to $\tilde{x}_j$ in game $g(t_i',t_j)$, hence $\bar{u}_i^\gamma(\tilde{x}_i(t_i',t_j),\tilde{x}_j(t_i',t_j)) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)$.

Case 3: $NE(g(t_i',t_j)) = \varnothing$. Then agents are free in choosing their strategies, and agent i can choose his deviation strategy as $x_i^{'} \in BR_i(x_j^{'})$ where agent 2 proposes to play $x_j^{'}$ in case $NE(g(t_i,t_j)) = \varnothing$. Thus again one has $\bar{u}_i^\gamma(x_i^{'}(t_i',t_j),x_j^{'}(t_i',t_j)) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)$.

Proposition 5: Given a 2 person finite normal form game $g = (X,u)$, where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0\ \forall x \in X$. Let $C_\beta^0(\overline{g}) = \varnothing$ and $\exists \bar{\gamma} \in [0,1]$ such that $\forall x \in X\ \forall t(x) \in T\ \exists i \in N$ such that, $\bar{u}_i^\gamma(\bar{t},\bar{x}) < min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)$ for all $\gamma \leq \bar{\gamma}$, then $QSPE^\gamma = \varnothing$ for $\gamma \leq \bar{\gamma}$.

Proof: Let $g = (X,u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0\ \forall x \in X$. Let $\gamma = 0$ and suppose that $C_\beta^0(\overline{g}) = \varnothing$, then by proposition 3 there are 2 cases:

Case 1: $\forall x \in X\ \exists i \in N$ such that $u_i(x) < min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)$. Then $\forall (t,x) \in T \times X$ one has $\bar{u}_i^\gamma(t,x) \leq u_i(x) < min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)$ for $\forall t \in T$. By proposition 4 $QSPE^0 = \varnothing$.

Case 2: If $\exists x \in X$ such that $u_i(x) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i,x_j)\ \forall i \in N$ but $x \notin C_\beta^0(\overline{g})$, x is Pareto dominated by some $\tilde{x} \in X$. If $\tilde{x}$ is not Pareto dominated, by proposition 3 $\exists(\widetilde{t}, \tilde{x}) \in C^0_\beta(\overline{g})$, contradicting $C^0_\beta(\overline{g}) = \varnothing$. Then there must be an outcome $x' \in X$ such that x' that Pareto dominates $\tilde{x}$, similar as above if x' is not Pareto dominated, by proposition 3 which leads to $(t^{'}, x') \in C^0_\beta(\overline{g})$. As the game is finite this sequence will last and contradict with $C^0_\beta(\overline{g}) = \varnothing$. Thus case 2 is not possible.

And if $\exists\bar{\gamma} \in [0,1]$ such that $\forall x \in X$ $\forall t(x) \in T$ $\exists i \in N$ such that , $\bar{u}^\gamma_i(\bar{t}, \bar{x}) < min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)$ for all $\gamma \leq \bar{\gamma}$, then $QSPE^\gamma = \varnothing$ for $\gamma \leq \bar{\gamma}$ by proposition 4. □

Proposition 6. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R^2_+$, and $u(x) > 0$ $\forall x \in X$. Let $C^0_\beta(\overline{g}) = \varnothing$ and $\exists\bar{\gamma} \in [0,1]$ such that $C^\gamma_\beta(\overline{g}) = \varnothing$ for $\gamma \leq \bar{\gamma}$, then $QSPE^\gamma = \varnothing$ for $\gamma \leq \bar{\gamma}$.

Proof: In proof of proposition 5 we see that if $C^\gamma_\beta(\overline{g}) = \varnothing$ for $\gamma \leq \bar{\gamma}$ where $\bar{\gamma} \in [0,1]$, then the 2 cases in proof of proposition 5 suggests that $\forall x \in X$ $\exists i \in N$ such that $\forall t \in T, \bar{u}^\gamma_i(\bar{t}, \bar{x}) < min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)$ for all $\gamma \leq \bar{\gamma}$. Then by Proposition 4 $QSPE^\gamma = \varnothing$ for $\gamma \leq \bar{\gamma}$.□

Proposition 7. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R^2_+$, and $u(x) > 0$ $\forall x \in X$. If $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^\gamma$ where $x_1(t'_1, t'_2) = \bar{x}_1$, $x_2(t'_1, t'_2) = \bar{x}_2$, then $\bar{u}^\gamma_i(\bar{x}_i(t_i, t_j), \bar{x}_j(t_i, t_j)) \leq u_i(BR_i(\bar{x}_j), \bar{x}_j), \forall i \in N$.

Proof: Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \to R^2_+$, and $u(x) > 0$ $\forall x \in X$. Suppose that $((t_1, x_1(t'_1, t'_2)), (t_2, x_2(t'_1, t'_2))) \in QSPE^\gamma$ where $x_1(t'_1, t'_2) = \bar{x}_1, x_2(t'_1, t'_2) = \bar{x}_2$. Assume on the contrary that $\exists i \in N$ such that $\bar{u}^\gamma_i(\bar{x}_i(t_i, t_j), \bar{x}_j(t_i, t_j)) > u_i(BR_i(\bar{x}_j), \bar{x}_j)$.

Now if $\bar{x}_i \in BR_i(\bar{x}_j)$ in game g, as $\bar{u}^\gamma_i(\bar{x}_i(t_i, t_j), \bar{x}_j(t_i, t_j)) > u_i(BR_i(\bar{x}_j), \bar{x}_j)$ it must be the case that $t_j(\bar{x}_i, \bar{x}_j) > 0$. As $(\bar{x}_i, \bar{x}_j) \in NE(g(t_1, t_2))$ one has

$\bar{u}_j^\gamma(\bar{x}_i(t_i,t_j),\bar{x}_j(t_i,t_j)) \geq \bar{u}_j^\gamma(\bar{x}_i(t_i,t_j),x_j(t_i,t_j))\ \forall x_j \in X_j$. But as $\bar{x}_i \in BR_i(\bar{x}_j)$ agent j can deviate to $t'_j(x) = \{\frac{1}{\gamma}[u_i(\bar{x}_i,\bar{x}_j) - t_i(\bar{x}_i,\bar{x}_j))]$ if $x = (\bar{x}_i,\bar{x}_j), t_j(x_i,x_j)$ otherwise$\}$ and become better off as $(\bar{x}_i,\bar{x}_j) \in NE(g(t_i,t'_j))$ and $\forall x_j \in X_j$

$$\bar{u}_j^\gamma(\bar{x}_i(t_i,t_j^{'}),\bar{x}_j(t_i,t_j^{'})) > \bar{u}_j^\gamma(\bar{x}_i(t_i,t_j),\bar{x}_j(t_i,t_j)) \geq \bar{u}_j^\gamma(\bar{x}_i(t_i,t_j),x_j(t_i,t_j)).$$

Because with the new transfer vector $\bar{u}_i^\gamma(\bar{x}_i(t_i,t_j^{'}),\bar{x}_j(t_i,t_j^{'})) = u_i(BR_i(\bar{x}_j),\bar{x}_j)$ hence $t'_j(\bar{x}_i,\bar{x}_j) < t_j(\bar{x}_i,\bar{x}_j)$. Contradicting with $((t_1,x_1(t'_1,t'_2)),(t_2,x_2(t'_1,t'_2))) \in QSPE^\gamma$ where $x_1(t'_1,t'_2) = \bar{x}_1, x_2(t'_1,t'_2) = \bar{x}_2$.

If $\bar{x}_i \notin BR_i(\bar{x}_j)$ in game g, as $\bar{u}_i^\gamma(\bar{x}_i(t_i,t_j),\bar{x}_j(t_i,t_j)) > u_i(BR_i(\bar{x}_j),\bar{x}_j)$ it must be the case that $t_j(\bar{x}_i,\bar{x}_j) > 0$. As $(\bar{x}_i,\bar{x}_j) \in NE(g(t_1,t_2))$ one has $\forall x_j \in X_j$

$$\bar{u}_j^\gamma(\bar{x}_i(t_i,t_j),\bar{x}_j(t_i,t_j)) \geq \bar{u}_j^\gamma(\bar{x}_i(t_i,t_j),x_j(t_i,t_j))$$

But as $\bar{u}_i^\gamma(\bar{x}_i(t_i,t_j),\bar{x}_j(t_i,t_j)) > u_i(BR_i(\bar{x}_j),\bar{x}_j)$, agent j can deviate to

$$t'_j(x) = \{\frac{1}{\gamma}[u_i(BR_i(\bar{x}_j),\bar{x}_j) - (u_i(\bar{x}_i,\bar{x}_j) - t_i(\bar{x}_i,\bar{x}_j))] \text{ if } x = (\bar{x}_i,\bar{x}_j),\ t_j(x) \text{ otherwise}\}$$

and become better off as $(\bar{x}_i,\bar{x}_j) \in NE(g(t_i,t'_j))$ and $\bar{u}_j^\gamma(\bar{x}_i(t_i,t_j^{'}),\bar{x}_j(t_i,t_j^{'})) > \bar{u}_j^\gamma(\bar{x}_i(t_i,t_j),\bar{x}_j(t_i,t_j))$. Because with the new transfer vector

$$\bar{u}_i^\gamma(\bar{x}_i(t_i,t_j^{'}),\bar{x}_j(t_i,t_j^{'})) = u_i(BR_i(\bar{x}_j),\bar{x}_j)$$

hence $t'_j(\bar{x}_i,\bar{x}_j) < t_j(\bar{x}_i,\bar{x}_j)$.

Contradiction with $((t_1,x_1(t'_1,t'_2)),(t_2,x_2(t'_1,t'_2))) \in QSPE^\gamma$ where $x_1(t'_1,t'_2) = \bar{x}_1, x_2(t'_1,t'_2) = \bar{x}_2$. □

Theorem 7. Given a 2 person finite normal form game $g = (X,u)$, where $u : X_1 \times X_2 \rightarrow R_+^2$ where $u_i(x) > 0\ \forall x \in X, \forall i \in \{1,2\}$.If $(\bar{x}_1,\bar{x}_2) \in C_\beta(g)$ then

$\exists \bar{t}_i \in T_i$ such that $((\bar{t}_1, \bar{x}_1), (\bar{t}_2, \bar{x}_2)) \in QSPE^0$.

Proof: Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \rightarrow R^2_+$ where $u_i(x) > 0 \ \forall x \in X, \forall i \in \{1, 2\}$. Let $(\bar{x}_1, \bar{x}_2) \in C_\beta(g))$, then $u_i(\bar{x}_1, \bar{x}_2) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)$ for $\forall i \in N$. We will construct $(\bar{t}_1, \bar{t}_2) \in T$ such that $(\bar{x}_1, \bar{x}_2) \in NE(g(\bar{t}_1, \bar{t}_2))$ as follows:

As $(\bar{x}_1, \bar{x}_2) \in C_\beta(g)$ it is not Pareto dominated, that is if $u_i(x_1, x_2) > u_i(\bar{x}_1, \bar{x}_2)$ then $u_j(x_1, x_2) < u_j(\bar{x}_1, \bar{x}_2)$.Let $\bar{t}_i(x) = \{u_i(x_1, x_2)$ if $u_i(x_1, x_2) > u_i(\bar{x}_1, \bar{x}_2)$, $u_i(x_1, x_2)$ if $u_j(x_1, x_2) > u_j(\bar{x}_1, \bar{x}_2)$, 0 otherwise$\}$ $\forall i \in N$. $(\bar{x}_1, \bar{x}_2) \in NE(g(\bar{t}_1, \bar{t}_2))$ as $\bar{t}_i(x) = u_i(x_1, x_2)$ if $u_i(x_1, x_2) > u_i(\bar{x}_1, \bar{x}_2)$, $((\bar{t}_1, \bar{x}_1), (\bar{t}_2, \bar{x}_2)) \in NE(v(G))$ as agent i cannot unilaterally deviate and become better off since $\bar{t}_i(x) = u_i(x_1, x_2)$ if $u_i(x_1, x_2) > u_i(\bar{x}_1, \bar{x}_2)$, and $\bar{t}_i(x) = u_i(x_1, x_2)$ if $u_j(x_1, x_2) > u_j(\bar{x}_1, \bar{x}_2)$ for every $i \in N$. □

Corollary 3. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \rightarrow R^2_+$ where $u_i(x) > 0 \ \forall x \in X, \forall i \in \{1, 2\}$. If $\exists (\bar{x}_1, \bar{x}_2) \in C_\beta(g)$ such that $(\bar{x}_1, \bar{x}_2) \in argmax \sum_{i \in N, x \in X} u_i(x)$ then $\exists \bar{t}_i \in T_i$ such that $((\bar{t}_1, \bar{x}_1), (\bar{t}_2, \bar{x}_2)) \in QSPE^0$.

Proposition 8 . Given a 2 person finite normal form game $g = (X, u)$, any $(x_1^*, x_2^*) \in X$ satisfying $u_i(x_1^*, x_2^*) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)$ for all $i \in N$, is reachable as a QSPE for $\gamma = 0$.

Proof. Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \rightarrow R^2_+$, and $u(x) > 0 \ \forall x \in X$. Let $(x_1^*, x_2^*) \in X$ be such that $u_i(x_1^*, x_2^*) \geq min_{x_j \in X_j} max_{x_i \in X_i} u_i(x_i, x_j)$. Consider the transfer vector $(t_1^*, t_2^*) \in T$ such that $\forall i \in N$ $t_i^*(x)$ is chosen as follows:

If $x_i^* \in BR_i(x_j^*)$ then $t_i^*(x_i^*, x_j^*) = 0$.

If $x_i^* \notin BR_i(x_j^*)$ then $t_i^*(x_i, x_j^*) > 0$ such that $\overline{u}_i^{\gamma}((t_i^*, x_i^*), (t_j^*, x_j^*)) > \overline{u}_i^{\gamma}((t_i^*, x_i), (t_j^*, x_j^*))$ $\forall x_i \in X_i$. This is possible as $u_i(x) > 0$ $\forall x \in X, \forall i \in \{1, 2\}$.

For any $(\widetilde{x}_i, \widetilde{x}_j)$ such that $u_j(\widetilde{x}_i, \widetilde{x}_j) > u_j(x_i^*, x_j^*)$.

i) if $\widetilde{x}_i \notin BR_i(\widetilde{x}_j)$ then $t_i^*(\widetilde{x}_i, \widetilde{x}_j) = 0$,

ii) if $\widetilde{x}_i \in BR_i(\widetilde{x}_j)$ then $t_i^*(\widetilde{x}_i, \widetilde{x}_j) > 0$ such that $\exists \widetilde{\widetilde{x}}_i \in X_i$ such that if $\widetilde{x}_i \in BR_i(\widetilde{x}_j)$ then $t_i^*(\widetilde{x}_i, \widetilde{x}_j) > 0$ such that $\exists \widetilde{\widetilde{x}}_i \in X_i$ such that $\overline{u}_i^{\gamma}((t_i^*, \widetilde{x}_i), (t_j^*, \widetilde{x}_j)) < \overline{u}_i^{\gamma}((t_i^*, \widetilde{\widetilde{x}}_i), (t_j^*, \widetilde{x}_j))$ where $u_j(\widetilde{\widetilde{x}}_i, \widetilde{x}_j) \leq u_j(x_i^*, x_j^*)$. Suppose that $\exists \widetilde{\widetilde{x}}_i \notin X_i$ such that $u_j(\widetilde{\widetilde{x}}_i, \widetilde{x}_j) \leq u_j(x_i^*, x_j^*)$. So $\forall x_i \in X_i$ $u_j(x_i, \widetilde{x}_j) > u_j(x_i^*, x_j^*)$. Then

$$\begin{aligned} min_{x_i \in X_i} max_{x_j \in X_j} u_j(x_i, x_j) &> u_j(x_i, \widetilde{x}_j) \\ &> u_j(x_i^*, x_j^*) \\ &\geq min_{x_i \in X_i} max_{x_j \in X_j} u_j(x_i, x_j) \end{aligned}$$

which implies that

$min_{x_i \in X_i} max_{x_j \in X_j} u_j(x_i, x_j) > min_{x_i \in X_i} max_{x_j \in X_j} u_j(x_i, x_j)$, a contradiction. Thus $\exists \widetilde{\widetilde{x}}_i \in X_i$ such that

$\overline{u}_i^{\gamma}((t_i^*, \widetilde{x}_i), (t_j^*, \widetilde{x}_j)) < \overline{u}_i^{\gamma}((t_i^*, \widetilde{\widetilde{x}}_i), (t_j^*, \widetilde{x}_j))$ where $u_j(\widetilde{\widetilde{x}}_i, \widetilde{x}_j) \leq u_j(x_i^*, x_j^*)$.

For any other $x \in X$ choose $t_i^*(x) = 0$.

Now by construction $(x_1^*, x_2^*) \in NE(g(t_1^*, t_2^*))$. Thus one sided deviations in strategies are not profitable.

Note that the $t_i^*(x)$ are chosen in such a way that for any $\widetilde{x} \in X$ that Pareto dominates (x_1^*, x_2^*), both agents have destructed their utilities, thus one sided deviations in strategywise or transferwise cannot ensure $\widetilde{x} \in g(t_i', t_j^*)$ $\forall i \in N$.

One sided deviations in transfers is not profitable as follows: Transfer vectors correspond to destruction of utility of an agents own payoff, and the agent j cannot effect the best response relation of the opponent by a change in $t_j \in T_j$.

As t_i^* is chosen in such a way that $\forall (x_i, x_j) \in X$ such that $\exists j \in N$ $u_j(x_i, x_j) > u_j(x_1^*, x_2^*)$ one has $x_i \notin BR_i(x_j)$. Thus agent j cannot become better off in any game $g(t_1^*, t_j)$ $\forall t_j \in T_j$.

Hence $((t_1^*, x_1^*), (t_2^*, x_2^*)) \in QSPE^0$.$\square$

Definition 27. Let $g = (N, X, u)$ be a normal form game, x^* is said to be a strong equilibrium of g if and only if, for any $S \in 2^N \backslash \{\emptyset\}$ and $x_S \in X_S$, one has $u_i(x^*) \geq u_i(x_S, x^*_{N \backslash S})$ for all $i \in S$.

Corollary 4. Given a 2 person finite normal form game $g = (X, u)$. Then the following holds:

i. If $x \in NE(g)$ then x is reachable as a QSPE for $\gamma = 0$.

ii. If $x \in SNE(g)$ then x is reachable as a QSPE for $\gamma = 0$.

iii.If $x \in Core_\beta(g)$ then x is reachable as a QSPE for $\gamma = 0$.

Proof: Due to proposition 8.

Above results imply that x belonging to $Core_\alpha(g)$ is not necessarily reachable as a QSPE for $\gamma = 0$.

Lemma 4. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$ $\forall x \in X$. If $((\bar{t}_1, x_1(t_1', t_2')), (\bar{t}_2, x_2(t_1', t_2'))) \in QSPE^\gamma$ then $\bar{u}_i^\gamma(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2)) \geq \mu_i(\gamma)$ where

$$\mu_i(\gamma) = max_{x_i \in X_i}\{max_{x_j \in X_j}\{u_i(x_i, x_j) - \frac{(u_j(x_i, BR_j(x_i)))}{\gamma} + \gamma u_j(x_i, x_j), 0\}\}$$

Proof: Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$ $\forall x \in X$. Suppose $((\bar{t}_1, x_1(t_1', t_2')), (\bar{t}_2, x_2(t_1', t_2'))) \in$

$QSPE^{\gamma}$ whereas there exists $i \in N$ such that $\bar{u}_i^{\gamma}(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2)) < \mu_i(\gamma)$ where $\mu_i(\gamma) = max_{x_i \in X_i}\{\max_{x_j \in X_j}\{u_i(x_i, x_j) - (\frac{u_j(x_i, BR_j(x_i)))}{\gamma} + \gamma u_j(x_i, x_j), 0\}\}$.

As $((\bar{t}_1, x_1(t_1', t_2')), (\bar{t}_2, x_2(t_1', t_2'))) \in QSPE^{\gamma}$ one has $(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2)) \in NE(g(\bar{t}_1, \bar{t}_2))$. Let $(x_i', x_j') \in argmax_{x_i \in X_i}\{max_{x_j \in X_j}\{0, u_i(x_i, x_j) - (\frac{u_j(x_i, BR_j(x_i)))}{\gamma} + \gamma u_j(x_i, x_j)\}\}$. Agent i can deviate to strategy $t_i'(x) = \{\bar{t}_i(\bar{x}_i, \bar{x}_j) + \varepsilon$ if $x = (\bar{x}_i, \bar{x}_j), \frac{u_j(x_i', BR_j(x_i')) - (u_j(x_i', x_j') - t_j(x_i', x_j'))}{\gamma}$ if $x = (x_i', x_j'), u_i(x_i, x_j)$ if $x = (x_i, x_j') \; \forall x_i \in X_i, \bar{t}_i(x_i, x_j)$ otherwise$\}$ if $t_i(\bar{x}_i, \bar{x}_j) < u_i(\bar{x}_i, \bar{x}_j)$. If $\bar{t}_i(\bar{x}_i, \bar{x}_j) = u_i(\bar{x}_i, \bar{x}_j)$ agent i can deviate to strategy

$$t_i'(x) = \{\bar{t}_i(\bar{x}_i, \bar{x}_j) + \varepsilon \text{ if } x = (\bar{x}_i, \bar{x}_j),$$

$$\frac{u_j(x_i', BR_j(x_i')) - (u_j(x_i', x_j') - t_j(x_i', x_j'))}{\gamma} \text{ if } x = (x_i', x_j'),$$

$$u_i(x_i, x_j) \text{ if } x = (x_i, x_j') \forall x_i \in X_i,$$

$$\bar{t}_i(x_i, x_j) \text{ otherwise}\}.$$

Agent i becomes better off as follows: $(x_i', x_j') \in NE(g(t_i', t_j))$ as $x_j' \in BR_j(x_i')$ and $x_i' \in BR_i(x_j')$ in game $g(t_i', t_j)$ and $\bar{u}_i^{\gamma}(x_i'(t_i', \bar{t}_2), x_j'(t_i', \bar{t}_2)) \geq \mu_i(\gamma) > \bar{u}_i^{\gamma}(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2))$. Contradiction with $((\bar{t}_1, x_1(t_1', t_2')), (\bar{t}_2, x_2(t_1', t_2'))) \in QSPE^{\gamma}$. Thus one must have $\bar{u}_i^{\gamma}(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2)) \geq \mu_i(\gamma)$ if $((\bar{t}_1, \bar{x}_1), (\bar{t}_2, \bar{x}_2)) \in QSPE^{\gamma}$. □

Corollary 5. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \rightarrow R_+^2$, and $u(x) > 0 \; \forall x \in X$. If $((\bar{t}_1, x_1(t_1', t_2')), (\bar{t}_2, x_2(t_1', t_2'))) \in QSPE^{\gamma}$ then $\bar{u}_i^{\gamma}(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2)) \geq \max\{\mu_i(\gamma), \min_{x_j \in X_j} \max_{x_i \in X_i} u_i(x_i, x_j)\}$.

Proof. Due to Proposition 4 and Lemma 4. □

Proposition 9. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \rightarrow R_+^2$, and $u(x) > 0$. $(x^*, t^*) \in QSPE^1$ then the following conditions hold:

i. $\bar{u}_i^\gamma(x_1^*(t_1^*, t_2^*), x_2^*(t_1^*, t_2^*)) \geq \max\{\mu_i(1), \min_{x_j \in X_j} \max_{x_i \in X_i} u_i(x_i, x_j)\}$.

ii. $u_i(BR_i(x_j^*), x_j^*) \geq \max\{\mu_i(1), \min_{x_j \in X_j} \max_{x_i \in X_i} u_i(x_i, x_j)\}\ \forall i \in N$.

Proof: Due to Proposition 7 and corollary 5.

Note that given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$. Then the following holds:

i. If $x \in NE(g)$ then $\exists t \in T$ such that $(t, x) \in QSPE^1$ only if

$$\bar{u}_i^\gamma(x_1(t_1, t_2), x_2(t_1, t_2)) \geq \mu_i(1) \text{ and}$$

$$\exists j \in N \text{ such that } \bar{u}_j^\gamma(x_i(t_1, t_2), BR_j(x_i)(t_1, t_2)) \geq \mu_j(1).$$

ii. If $x \in SNE(g)$ then $\exists t \in T$ such that $(t, x) \in QSPE^1$ only if

$$\bar{u}_i^\gamma(x_1(t_1, t_2), x_2(t_1, t_2)) \geq \mu_i(1) \text{ and}$$

$$\exists j \in N \text{ such that } \bar{u}_j^\gamma(x_i(t_1, t_2), BR_j(x_i)(t_1, t_2)) \geq \mu_j(1).$$

iii.If $x \in Core_\beta(g)$ then $\exists t \in T$ such that $(t, x) \in QSPE^1$ only if

$$\bar{u}_i^\gamma(x_1(t_1, t_2), x_2(t_1, t_2)) \geq \mu_i(1) \text{ and}$$

$$\exists j \in N \text{ such that } \bar{u}_j^\gamma(x_i(t_1, t_2), BR_j(x_i)(t_1, t_2)) \geq \mu_j(1).$$

Proposition 10. Given a 2 person finite normal form game $g = (X, u)$, where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$. If $\sum_{i \in N} \mu_i(\gamma) > max_{x \in X} \sum_{i \in N} u_i(x)$ for some $\gamma \in [0, 1]$, then $QSPE^{\gamma'} = \varnothing$ for all $\gamma' \geq \gamma$.

Proof: Let $g = (X, u)$ be a 2 person finite normal form game where $u : X_1 \times X_2 \to R_+^2$, and $u(x) > 0$ and $\sum_{i \in N} \mu_i(\gamma) > max_{x \in X} \sum_{i \in N} u_i(x)$ for some $\gamma \in [0, 1]$. Assume that $((\bar{t}_1, x_1(t_1', t_2')), (\bar{t}_2, x_2(t_1', t_2'))) \in QSPE^\gamma$. Now then for any $x \in X$ hence for $(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2))$ one has $\exists i \in N$ such that $\mu_i(\gamma) < \bar{u}_i^{\gamma}(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2))$. By Lemma 4 $((\bar{t}_1, x_1(t_1', t_2')), (\bar{t}_2, x_2(t_1', t_2'))) \notin QSPE^{\gamma}$. As $\mu_i(\gamma) < \bar{u}_i^{\gamma}(x_1(\bar{t}_1, \bar{t}_2), x_2(\bar{t}_1, \bar{t}_2))$ for any $\in X$, $QSPE^{\gamma} = \varnothing$.

Note that $\mu_i(\gamma)$ is nondecreasing in γ as $\frac{\partial \mu_i(\gamma)}{\partial \gamma} = -\frac{u_j(x_i, BR_j(x_i))}{\gamma^2} + u_j(x_i, x_j) \leq 0$ as $u_j(x_i, BR_j(x_i)) \geq u_j(x_i, x_j)$ and $\gamma \in [0, 1]$. Thus $QSPE^{\gamma'} = \varnothing$ for all $\gamma' \geq \gamma$. □

5.2. Ordered Choice of Transfers and Actions

Up to this point we have considered the case that two stage game is played such that in the first stage agents choose their transfer vectors simultaneously, and in the second stage agents choose their actions simultaneously. This led to inefficiency in QSPE as shown in example 5, where agents could end up in a situation where everybody is worse off when compared to the Nash equilibria of the original normal form game.

In study of Sertel and Koray (1983) "Games of Pretension", they analyze their two stage game and with a richer structure where they also consider the cases where agents can play the first and second stage of the game in an order. By adapting their notation we will denote 2 stage game as Γ_j^i such that $i, j \in \{0, 1, 2\}$, where superscript i denotes that agent i plays first in the in the first stage where transfers are chosen, and subscript j denotes that agent j plays first in the second stage where actions are chosen. Thus Γ_2^1 is the two stage game where agent 1 is the stackelberg leader in choosing transfer vector, and agent 2 is the stackelberg leader in choosing the actions from the set X_2. If i or j equals 0, agents are choosing their strategies in the first or second stage simultaneously respectively that coincides with subsection 5.1.

We will denote the set of strategy profiles (t, x) which constitute a solution for the game Γ_j^i as $\overline{\Gamma}_j^i$.

Now we see that for $\gamma = 0$, $\overline{\Gamma}_0^0$ is equal to the solution of game of pretension where $g = (X = X_1 \times X_2, (u, v))$ where u and v are such that $u(x) > 0$ and $v(x) > 0$ for every $x \in X$. Having a strictly positive payoffs is compatible with games of pretension where agents are free to choose any type of robot in the first stage. But note that in games of pretension agents are free to pretend to have any preference relation but eventually with the chosen robots they will get the payoffs in the original game. Whereas in our model in 2 stage game with transfers if agents announce to give positive transfers at an outcome, that transfer realizes and the game played eventually is $g(t_1, t_2)$, not g. For $\gamma = 0$, as transfer vector of any agent only effects the preference relation of the agent himself, but not the opponent, we have that if $(t, x) = ((t_1, t_2), (x_1, x_2)) \in \overline{\Gamma}_j^i$, then $t_1(x_1, x_2) = t_2(x_1, x_2) = 0$. Thus $\overline{\Gamma}_0^0$ is equal to the solution of game of pretension where $g = (X = X_1 \times X_2, (u, v))$ where u and v are such that $u(x) > 0$ and $v(x) > 0$ for every $x \in X$ for $\gamma = 0$. But note that this is not generally true for the case $\gamma > 0$.

Now we will analyze the $\overline{\Gamma}_j^i$ where $i, j \in \{1, 2\}$.

Assumption 1. Given $\overline{\Gamma}_j^i$ where $i, j \in \{1, 2\}$,*minimal public spirit* is the requirement that the follower who is indifferent between two choices of a strategy in response to a choice of strategy by the leader should not be malovelent to the leader and choose the strategy that will please the leader the most.

5.2.1. Ordered Choice of Transfers and Actions where $\gamma = 0$

In our study we allow γ to have a value in the interval $[0, 1]$, For the ease of analysis firstly we will assume $\gamma = 0$.

Proposition 11. $\overline{\Gamma}^i_j$ where $i, j \in \{1, 2\}$ has a unique value in terms of payoff functions. That is if $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)), ((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}^i_j$ then $u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) = u_i(\widetilde{x}_1, \widetilde{x}_2) - \widetilde{t}_i(\widetilde{x}_1, \widetilde{x}_2)$ $\forall i \in N$.

Proof. Assume on the contrary $u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) \neq u_i(\widetilde{x}_1, \widetilde{x}_2) - \widetilde{t}_i(\widetilde{x}_1, \widetilde{x}_2)$ (*). Without loss of generality assume $u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) > u_i(\widetilde{x}_1, \widetilde{x}_2) - \widetilde{t}_i(\widetilde{x}_1, \widetilde{x}_2)$. Let $i = 1$, as $\bar{t}_2$ is a BR to $\bar{t}_1$, agent 1 becomes better off if he chooses $\bar{t}_1$. So one has $u_1(\overline{x}_1, \overline{x}_2) - \bar{t}_1(\overline{x}_1, \overline{x}_2) = u_1(\widetilde{x}_1, \widetilde{x}_2) - \widetilde{t}_1(\widetilde{x}_1, \widetilde{x}_2)$, then we have the equation (*) for agent 2. That is $u_2(\overline{x}_1, \overline{x}_2) - \bar{t}_2(\overline{x}_1, \overline{x}_2) > u_2(\widetilde{x}_1, \widetilde{x}_2) - \widetilde{t}_2(\widetilde{x}_1, \widetilde{x}_2)$, but then minimal public spirit rules out $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}^1_2$. thus (*) cannot hold $\forall i \in N$. When we change $i, j \in \{1, 2\}$, the above proof is valid $\forall \overline{\Gamma}^i_j$.□

Lemma 5. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^i_j$ where $i, j \in \{1, 2\}$, then $\bar{t}_i(\overline{x}_1, \overline{x}_2) = 0$ $\forall i \in N$.

Proof. Note that when $\gamma = 0$, transfers can only change best response relation of the transfer making agent himself not the opponent. Thus if $\exists i \in N$ such $t_i(\overline{x}_1, \overline{x}_2) > 0$, deviating to

$t'_i(x) = \{0$ if $x = (\overline{x}_1, \overline{x}_2)$, $t_i(x)$ otherwise$\}$ makes agent i better off as best response relation of agent j does not change and $((t'_i, \bar{t}_j), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^i_j$. Thus $\bar{t}_i(\overline{x}_1, \overline{x}_2) = 0$ $\forall i \in N$.□

Lemma 6. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^i_j$ where $i, j \in \{1, 2\}$, then $u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) \geq \max_{x_i} \min_{x_j} u_i(x_1, x_2)$ $\forall i \in N$.

Proof. Assume not, let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^i_j$ where $i, j \in \{1, 2\}$ suppose $u_j(\overline{x}_1, \overline{x}_2) < \max_{x_j} \min_{x_i} u_j(x_1, x_2)$. Let $t'_j(x) = \overrightarrow{0}$. Then even $\bar{t}_i(x)$ is such that $BR_i(x_j) = \min_{x_i} u_j(x_1, x_2)$ in game $g(\bar{t}_i, t'_j)$, agent j can choose her strategy x'_j such that $u_j(x'_i, x'_j) \geq \max_{x_j} \min_{x_i} u_j(x_1, x_2)$ where $((\bar{t}_i, t'_j), (x'_i, x'_j)) \in \overline{\Gamma}^i_j$. A contradiction.

Then it must be the case that $u_i(\overline{x}_1, \overline{x}_2) < \max_{x_i} \min_{x_j} u_i(x_1, x_2)$. Now let $t'_i(x) = \overrightarrow{0}$. For any $t_j \in T_j$, one has $((t'_i, t_j), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}^i_j$ such that $u_i(\widetilde{x}_1, \widetilde{x}_2) \in \arg\max_{x_j} u_j(x'_i, x_j)$ where $x'_i \in \{x^*_i \in X_i : u_i(x^*_i, x_j) \geq u_i(x_i, x_j)\ \forall x_j\}$. Thus $u_i(\widetilde{x}_1, \widetilde{x}_2) \geq \max_{x_i} \min_{x_j} u_i(x_1, x_2)$. Contradicting Proposition 11.

Hence $u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) \geq \max_{x_i} \min_{x_j} u_i(x_1, x_2)\ \forall i \in N.\square$

Proposition 12. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^i_j$ where $i, j \in \{1, 2\}$, then $(u_1(\overline{x}_1, \overline{x}_2), u_2(\overline{x}_1, \overline{x}_2))$ is Pareto efficient, i.e. 0-efficient.

Proof. Let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^i_j$ where $i, j \in \{1, 2\}$, assume on the contrary that $(u_1(\overline{x}_1, \overline{x}_2), u_2(\overline{x}_1, \overline{x}_2))$ is not Pareto efficient.

Suppose that $\exists (x'_1, x'_2) \in X_1 \times X_2$ such that $u(x'_1, x'_2)$ is Pareto dominates $(\overline{x}_1, \overline{x}_2)$.

Let $i \neq j$, $t_j(x)$ be such that $BR_j(x_i) = \{x'_j$ if $x_i = x'_i$, $x_j \in \arg\min_{x_j} u_i(x_1, x_2)$ if $x_i \neq x'_i\}$ and $t_j(x_i, x'_j) = u_j(x_i, x'_j)$ where $u_i(x_i, x'_j) > u_i(x'_i, x'_j)$ at game $g(t'_1, t_2)$ $\forall t_2$. This is possible as $u_j(x) > 0\ \forall x \in X$. Now as agent j is first mover in the second stage to announce x'_j.

Let $t_i(x)$ where $BR_i(x_j) = x'_i\ \forall x_j \in X_j$. $((t_i, t_j), (x'_i, x'_j)) \in \overline{\Gamma}^i_j$ where $u_i(x'_i, x'_j) > u_i(\overline{x}_1, \overline{x}_2)$ leads to a contradiction by Proposition 11.

Now let $i = j$, be as above. For any $t_i(x)$ one has $x'_i \in \arg\max_{x_i} u_i(x_i, x_j)$ for any $x_j \in \arg\max u_{ij}(x_i, x_j)\ \forall x_i \in X_i$. Thus the above solution $((t_i, t_j), (x'_i, x'_j)) \in \overline{\Gamma}^i_j$ where $u_i(x'_i, x'_j) > u_i(\overline{x}_1, \overline{x}_2)$, a contradiction.

Thus $(u_1(\overline{x}_1, \overline{x}_2), u_2(\overline{x}_1, \overline{x}_2))$ is Pareto efficient, i.e. 0-efficient.$\square$

Proposition 13. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^1_2$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}^2_2$ then $u_1(\overline{x}_1, \overline{x}_2) \geq u_1(\widetilde{x}_1, \widetilde{x}_2)$.

Proof. Let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_2^2$, assume on the contrary that $u_1(\overline{x}_1, \overline{x}_2) < u_1(\widetilde{x}_1, \widetilde{x}_2)$. Let $t_1'(x)$ be such that $BR_1(x_2) = \{\widetilde{x}_1 \text{ if } x_2 = \widetilde{x}_2,\ x_1 \in \arg\min u_2(x_1, x_2)\ \forall x_2 \neq \widetilde{x}_2\}$ where $t_1'(x) = u_1(x)$ where $u_2(x) > u_2(\widetilde{x}_1, \widetilde{x}_2)$ at game $g(t_1', t_2)\ \forall t_2$. Now let $t_2'(x)$ be such that $BR_2(x_1) = \widetilde{x}_2$ $\forall x_1 \in X_1$, is compatible with $((t_1', t_2'), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_2^1$ where $u_1(\widetilde{x}_1, \widetilde{x}_2) > u_1(\overline{x}_1, \overline{x}_2)$, contradicting Proposition 11.

Thus $u_1(\overline{x}_1, \overline{x}_2) \geq u_1(\widetilde{x}_1, \widetilde{x}_2)$. □

Proposition 14. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_1^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_1^2$ then $u_1(\overline{x}_1, \overline{x}_2) \geq u_1(\widetilde{x}_1, \widetilde{x}_2)$.

Proof. Let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_1^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_1^2$. Assume on the contrary that $u_1(\widetilde{x}_1, \widetilde{x}_2) > u_1(\overline{x}_1, \overline{x}_2)$. By proposition 13 we know that $u_2(\widetilde{x}_1, \widetilde{x}_2) \geq u_2(\overline{x}_1, \overline{x}_2)$. But these two inequalities together imply that $(\overline{x}_1, \overline{x}_2)$ is not Pareto efficient which contradicts with Proposition 12. □

Proposition 15. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_1^1$ then $u_1(\overline{x}_1, \overline{x}_2) \geq u_1(\widetilde{x}_1, \widetilde{x}_2)$.

Proof. Let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_1^1$. Assume on the contrary that $u_1(\widetilde{x}_1, \widetilde{x}_2) > u_1(\overline{x}_1, \overline{x}_2)$. Let $t_1'(x) = \{0 \text{ if } x = (\widetilde{x}_1, \widetilde{x}_2),\ u_1(x_1, \widetilde{x}_2)$ if $x = (x_1, \widetilde{x}_2),\ u_1(x_1, x_2)$ if (x_1, x_2) is such that $u_2(x_1, x_2) > u_2(\widetilde{x}_1, \widetilde{x}_2)\}$. Now let $t_2'(x)$ be such that $BR_2(x_1) = \widetilde{x}_2\ \forall x_1 \in X_1$, is compatible with $t_1'(x)$ where $((t_1', t_2'), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_2^1$ where $u_1(\widetilde{x}_1, \widetilde{x}_2) > u_1(\overline{x}_1, \overline{x}_2)$ contradicting Proposition 11. Thus $u_1(\overline{x}_1, \overline{x}_2) \geq u_1(\widetilde{x}_1, \widetilde{x}_2)$.□

Proposition 16. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^2$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_1^2$ then $u_1(\overline{x}_1, \overline{x}_2) \geq u_1(\widetilde{x}_1, \widetilde{x}_2)$.

Proof. Similar to proof of Proposition 14, with use of Proposition 15 and Proposition 12.□

5.2.2. Ordered Choice of Transfers and Actions where $\gamma = 1$

In previous section we studied the set of equilibria where choice of transfers and actions are made in an order for the case $\gamma = 0$, now we will study the set of equilibria for the other extreme case where $\gamma = 1$ in order to improve our understanding for the general case where $\gamma \in [0, 1]$.

Proposition 17. $\overline{\Gamma}_j^i$ where $i, j \in \{1, 2\}$ has a unique value in terms of payoff functions. That is if $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)), ((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_j^i$ then $u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) = u_i(\widetilde{x}_1, \widetilde{x}_2) - \widetilde{t}_i(\widetilde{x}_1, \widetilde{x}_2)$ $\forall i \in N$.

Proof. Similar to case $\gamma = 0$. □

Proposition 18. $\overline{\Gamma}_j^i$ where $i, j \in \{1, 2\}$ is incentive compatible.

Proof. Assume not, let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_j^i$ but $u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) + \bar{t}_j(\overline{x}_1, \overline{x}_2) < \max_{x_i} \min_{x_j} u_i(x_i, x_j)$. Let $t'_i(x) = \overrightarrow{0}$, now as agent i is second mover in second stage, agent j will maximize her utility from choosing among best responses of agent i. For $\forall t_j \in T_j$, clearly at the solution $((t'_i, t_j), (x'_1, x'_2))$ one has $u_i(x'_1, x'_2) + t_j(x'_1, x'_2) \geq \max_{x_i} \min_{x_j} u_i(x_i, x_j)$. So agent i should play $t'_i(x) = \overrightarrow{0}$, and $((t'_i, t_j), (x'_1, x'_2)) \in \overline{\Gamma}_j^i$, contradicting Proposition 17 as $u_i(x'_1, x'_2) + t_j(x'_1, x'_2) > u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) + \bar{t}_j(\overline{x}_1, \overline{x}_2)$.

Now assume that $u_j(\overline{x}_1, \overline{x}_2) - \bar{t}_j(\overline{x}_1, \overline{x}_2) + \bar{t}_i(\overline{x}_1, \overline{x}_2) < \max_{x_j} \min_{x_i} u_j(x_i, x_j)$. Let $t'_j(x) = \overrightarrow{0}$, $\forall t_i \in T_i$ even if t_i is such that best response relation of agent i chooses $\min_{x_i} u_j(x_i, x_j)$ $\forall x_j \in X_j$ at game $g(t_i, t'_j)$ agent j xan guarantee a payoff $u_j(x'_1, x'_2) + t_j(x'_1, x'_2) \geq \max_{x_j} \min_{x_i} u_j(x_i, x_j)$. A contradiction.

The above case was for $i \neq j$, but works also for $i = j$ with same strategy that the agent who gets less than his maxmin value will play $t'(x) = \overrightarrow{0}$. □

Proposition 19. $\overline{\Gamma}_j^i$ where $i, j \in \{1, 2\}$ is 1-efficient.

Proof. Suppose not, let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_j^i$ but not 1-efficient and there exists $(\widetilde{x}_1, \widetilde{x}_2)$ which is 1-efficient and $u_1(\widetilde{x}_1, \widetilde{x}_2) + u_2(\widetilde{x}_1, \widetilde{x}_2) > \sum_{j=1}^{2} u_j(\overline{x}_1, \overline{x}_2) - \bar{t}_j(\overline{x}_1, \overline{x}_2) + \bar{t}_i(\overline{x}_1, \overline{x}_2) = u_1(\overline{x}_1, \overline{x}_2) + u_2(\overline{x}_1, \overline{x}_2)$. Now there are three cases:

Case 1: $u_1(\widetilde{x}_1, \widetilde{x}_2) > u_1(\overline{x}_1, \overline{x}_2) - \bar{t}_1(\overline{x}_1, \overline{x}_2) + \bar{t}_2(\overline{x}_1, \overline{x}_2)$
$u_2(\widetilde{x}_1, \widetilde{x}_2) > u_2(\overline{x}_1, \overline{x}_2) - \bar{t}_2(\overline{x}_1, \overline{x}_2) + \bar{t}_1(\overline{x}_1, \overline{x}_2)$.

Now if $\widetilde{x}_i \in BR_i(\widetilde{x}_j)$ under $\bar{t}_i$, this contradicts with the choice of $\overline{x}_i$. Thus we have $\widetilde{x}_i \notin BR_i(\widetilde{x}_j)$ under $\bar{t}_i$. Consider $t'_i(x)$ such that for $x \neq (x_i, \widetilde{x}_j)$ $t'_i(x) = \bar{t}_i(x)$, and $\widetilde{x}_i \in BR_i(\widetilde{x}_j)$. Now agent j is better off at some other x so that $\bar{t}_j$ is not a best response to t'_i. As t'_i is equal to $\bar{t}_i$ only at row $(x_i, \widetilde{x}_j)$, if agent j is better than $u_j(\widetilde{x}_1, \widetilde{x}_2)$ she must change BR of agent i at $\widetilde{x}_j$. But then $\exists(\widetilde{\widetilde{x}}_i, \widetilde{x}_j)$ such that $u_j(\widetilde{\widetilde{x}}_i, \widetilde{x}_j) - t'_j(\widetilde{\widetilde{x}}_i, \widetilde{x}_j) > u_j(\widetilde{x}_i, \widetilde{x}_j)$ and $u_i(\widetilde{\widetilde{x}}_i, \widetilde{x}_j) + t'_j(\widetilde{\widetilde{x}}_i, \widetilde{x}_j) = u_i(\widetilde{x}_i, \widetilde{x}_j)$, which contradicts with the fact that $(\widetilde{x}_i, \widetilde{x}_j)$ is 1-efficient. Thus $((t'_i, \bar{t}_j), (\widetilde{x}_i, \widetilde{x}_j)) \in \overline{\Gamma}_j^i$ with $u_i(\widetilde{x}_i, \widetilde{x}_j) - t'_i(\widetilde{x}_i, \widetilde{x}_j) + t'_j(\widetilde{x}_i, \widetilde{x}_j) > u_i(\overline{x}_i, \overline{x}_j) - \bar{t}_i(\overline{x}_i, \overline{x}_j) + \bar{t}_j(\overline{x}_i, \overline{x}_j)$, contradiction with Proposition 17. Thus case 1 cannot hold.

Case 2: $u_i(\widetilde{x}_1, \widetilde{x}_2) > u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) + \bar{t}_j(\overline{x}_1, \overline{x}_2)$
$u_j(\widetilde{x}_1, \widetilde{x}_2) \leq u_j(\overline{x}_1, \overline{x}_2) - \bar{t}_j(\overline{x}_1, \overline{x}_2) + \bar{t}_i(\overline{x}_1, \overline{x}_2)$.

Note that in case of equality in the second inequality above minimal public spirit solves the problem. Let $t'_i(x)$ be such that same as $\bar{t}_i(x)$ (as in Case 1) at $x \neq (x_i, \widetilde{x}_j)$, only different at $(x_i, \widetilde{x}_j)$ such that $\widetilde{x}_i \in BR_i(\widetilde{x}_j)$ where

$u_j(\widetilde{x}_i, \widetilde{x}_j) + t_i(\widetilde{x}_i, \widetilde{x}_j) = u_j(\overline{x}_i, \overline{x}_j) - \bar{t}_j(\overline{x}_i, \overline{x}_j) + \bar{t}_i(\overline{x}_i, \overline{x}_j)$.

$t'_i(x)$ be same as $\bar{t}_j(x)$ at $x \neq (\widetilde{x}_i, x_j)$ only differs at $(\widetilde{x}_i, x_j)$ such that at $(\widetilde{x}_i, BR_j(\widetilde{x}_i))$ agent j has payoff equal $u_j(\overline{x}_i, \overline{x}_j) - \bar{t}_j(\overline{x}_i, \overline{x}_j) + \bar{t}_i(\overline{x}_i, \overline{x}_j)$. t'_j is a best response to t'_i. As in case 1 agent i or j cannot become better off with respect to the payoff of outcome $(\widetilde{x}_i, \widetilde{x}_j)$, otherwise it contardicts 1-efficiency of $(\widetilde{x}_i, \widetilde{x}_j)$.

Thus $((t'_1, t'_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}^i_j$ where agent i is strictly better off, as 1-efficiency implies first inequality of case 2 has a surplus bigger than the deficiency of payoff of agent j. Contradiction with Proposition 17.

Case 3: $u_i(\widetilde{x}_1, \widetilde{x}_2) \leq u_i(\overline{x}_1, \overline{x}_2) - \bar{t}_i(\overline{x}_1, \overline{x}_2) + \bar{t}_j(\overline{x}_1, \overline{x}_2)$
$u_j(\widetilde{x}_1, \widetilde{x}_2) \leq u_j(\overline{x}_1, \overline{x}_2) - \bar{t}_j(\overline{x}_1, \overline{x}_2) + \bar{t}_i(\overline{x}_1, \overline{x}_2)$.

Similar transfer vectors as in Case 2 works and minimal public spirit implies that given that agent 1 gets the same maximal payoff, he should choose the transfer and strategy such that agent j is better off.□

Proposition 20. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^1_2$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}^2_2$ then $u_1(\overline{x}_1, \overline{x}_2) - t_1(\overline{x}_1, \overline{x}_2) + t_2(\overline{x}_1, \overline{x}_2) \leq u_1(\widetilde{x}_1, \widetilde{x}_2) - t_1(\widetilde{x}_1, \widetilde{x}_2) + t_2(\widetilde{x}_1, \widetilde{x}_2)$.

Proof. Let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^1_2$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}^2_2$. Assume on the contrary that

$u_1(\overline{x}_1, \overline{x}_2) - t_1(\overline{x}_1, \overline{x}_2) + t_2(\overline{x}_1, \overline{x}_2) > u_1(\widetilde{x}_1, \widetilde{x}_2) - t_1(\widetilde{x}_1, \widetilde{x}_2) + t_2(\widetilde{x}_1, \widetilde{x}_2)$ ...(*).

Now there are following cases:

Case 1: $\bar{t}_1(\overline{x}_1, \overline{x}_2) = \bar{t}_2(\overline{x}_1, \overline{x}_2) = 0$.

Then it is the case that $u_1(BR_1(\overline{x}_2), \overline{x}_2) > u_1(\overline{x}_1, \overline{x}_2)$. Consider $t'_1(x)$ such that $BR_1(\overline{x}_2)$ under t'_1 is $\overline{x}_1$ and $BR_1(x_2)$ under t'_1 for any other $x_2 \in X_2$ is $x_1 \in \arg\min_{x_1}(x_1, x_2)$. If agent 1 is second mover in transfer making, this implies that best response t'_2 to t'_1 is such that $((t'_1, t'_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^2_2$. assume that $\exists t''_2$ such that $\exists(x''_1, x''_2)$ such that $u_2(x''_1, x''_2) - t_2(x''_1, x''_2) + t_1(x''_1, x''_2) > u_2(\overline{x}_1, \overline{x}_2)$ and $u_1(x''_1, x''_2) - t_1(x''_1, x''_2) + t_2(x''_1, x''_2) = u_1(\overline{\overline{x}}_1, x''_2)$ where $\overline{\overline{x}}_1 \in \arg\min_{x_1} u_2(x_1, x''_2)$. But then player 2 could play t''_2 instead of $\bar{t}_2$ when agent 1 plays $\bar{t}_1$ and chooses strategy x''_2 which contradicts with $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^1_2$. Thus such a t''_2 with (x''_1, x''_2) does not exist. $\forall t'_2$ which is BR to t'_1 one has $((t'_1, t'_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}^2_2$, where agent 1 is better off by (*), contradiction with Proposition 17.

Case 2: $\bar{t}_1(\overline{x}_1, \overline{x}_2) > 0$ and $\bar{t}_2(\overline{x}_1, \overline{x}_2) = 0$.

Similar as in case 1, agent 1 plays $t'_1(x)$ with $t'_1(\overline{x}_1, \overline{x}_2) = \bar{t}_1(\overline{x}_1, \overline{x}_2)$ and $BR_1(\overline{x}_2)$ under t'_1 is $\overline{x}_1$, and $BR_1(x_2)$ under t'_1 for any other $x_2 \in X_2$ is $x_1 \in \arg\min_{x_1} u_2(x_1, x_2)$. Best response t'_2 to t'_1 results with $((t'_1, t'_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^2$. Equation (*), contradiction with Proposition 17.

Case 3: $\bar{t}_1(\overline{x}_1, \overline{x}_2) = 0$ and $\bar{t}_2(\overline{x}_1, \overline{x}_2) > 0$.

Similar as above $t'_1(x)$ is such that $BR_1(\overline{x}_2)$ under t'_1 is $\overline{x}_1$, $t'_1(\overline{x}_1, \overline{x}_2) = \bar{t}_1(\overline{x}_1, \overline{x}_2)$ and $BR_1(x_2)$ under t'_1 for any other $x_2 \in X_2$ is $x_1 \in \arg\min_{x_1} u_2(x_1, x_2)$ which results with a t'_2 such that $((t'_1, t'_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^2$, equation (*) contradicts with Proposition 17.

Case 4: $\bar{t}_1(\overline{x}_1, \overline{x}_2) > 0$ and $\bar{t}_2(\overline{x}_1, \overline{x}_2) > 0$.

In this case as there is no loss in utility,

if $\bar{t}_1(\overline{x}_1, \overline{x}_2) = \bar{t}_2(\overline{x}_1, \overline{x}_2)$ we are in case 1.

if $\bar{t}_1(\overline{x}_1, \overline{x}_2) > \bar{t}_2(\overline{x}_1, \overline{x}_2)$ we are in case 2.

if $\bar{t}_1(\overline{x}_1, \overline{x}_2) < \bar{t}_2(\overline{x}_1, \overline{x}_2)$ we are in case 3.□

Proposition 21. If $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_1^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_1^2$ then $u_1(\overline{x}_1, \overline{x}_2) - t_1(\overline{x}_1, \overline{x}_2) + t_2(\overline{x}_1, \overline{x}_2) \leq u_1(\widetilde{x}_1, \widetilde{x}_2) - t_1(\widetilde{x}_1, \widetilde{x}_2) + t_2(\widetilde{x}_1, \widetilde{x}_2)$.

Proof. Let $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_1^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_1^2$, by Proposition 20 one has $u_2(\overline{x}_1, \overline{x}_2) - t_2(\overline{x}_1, \overline{x}_2) + t_1(\overline{x}_1, \overline{x}_2) \geq u_2(\widetilde{x}_1, \widetilde{x}_2) - t_2(\widetilde{x}_1, \widetilde{x}_2) + t_1(\widetilde{x}_1, \widetilde{x}_2)$, and by Proposition 19 one has $u_1(\overline{x}_1, \overline{x}_2) - t_1(\overline{x}_1, \overline{x}_2) + t_2(\overline{x}_1, \overline{x}_2) \leq u_1(\widetilde{x}_1, \widetilde{x}_2) - t_1(\widetilde{x}_1, \widetilde{x}_2) + t_2(\widetilde{x}_1, \widetilde{x}_2)$.□

Now in two stage games when played in order it is shown that there is a preference relation on moving first or second with respect to gamma values. While choosing the strategy first is better for $\gamma = 0$, moving second is better for $\gamma = 1$. The question to be answered is whether there exists a monotonicity on this change in preference on moves. In other words can one argue that there exists an intermediate value of gamma where preferences on moves change and continue monotonically. The following example shows that it is not necessarily true that such an intermediate value exists.

Example 8. Consider the following 2 person normal form game:

	a	b	c
A	$4,6$	$1,1$	$\frac{1}{100},\frac{1}{100}$
B	$1,1$	$6,4$	$\frac{1}{100},\frac{1}{100}$
C	$1,2$	$1,7$	$16,\frac{1}{100}$

For $\gamma = 0$, $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^1$ where $\bar{t}_1(x) = \{4$ if $x = (A, a)$, 1 if $x = (C, a)$, 0 otherwise$\}$. $\bar{t}_2(x) = \overrightarrow{0}$. $(\overline{x}_1, \overline{x}_2) = (B, b)$ where $\overline{u}^\gamma((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) = (6, 4)$. $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_2^2$ where $\widetilde{t}_1(x) = \overrightarrow{0}$. $\widetilde{t}_2(x) = \{u_2(x_1, x_2)$ if $x_2 \in \{b, c\}$ 0 otherwise$\}$. $(\widetilde{x}_1, \widetilde{x}_2) = (A, a)$ where $\overline{u}^\gamma((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) = (4, 6)$.

For $\gamma = 0.31$, $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^1$ where $\bar{t}_1(x) = \overrightarrow{0}$ and $\bar{t}_2(x) = \overrightarrow{0}$. $(\overline{x}_1, \overline{x}_2) = (A, a)$ where $\overline{u}^\gamma((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) = (4, 6)$. $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_2^2$ where $\widetilde{t}_1(x) = \{4$ if $x_2 = (A, a)$, 0 otherwise$\}$. $\widetilde{t}_2(x) = \overrightarrow{0}$. $(\widetilde{x}_1, \widetilde{x}_2) = (B, b)$ where $\overline{u}^\gamma((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) = (6, 4)$.

For $\gamma = 0.7$, $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^1$ where $\bar{t}_1(x) = \{4 - \frac{1}{100}$ if $x_2 = (C, c)$, 0 otherwise$\}$ and $\bar{t}_2(x) = \{3$ if $x_2 = (C, b)$, 0 otherwise$\}$. $(\overline{x}_1, \overline{x}_2) = (C, c)$ where $\overline{u}^\gamma((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) = (7.4, 6)$. $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_2^2$ where $\widetilde{t}_1(x) = \{7 - \frac{1}{100}$ if $x_2 = (C, c), 0$ otherwise$\}$. $\widetilde{t}_2(x) = \overrightarrow{0}$. $(\widetilde{x}_1, \widetilde{x}_2) = (C, c)$ where $\overline{u}^\gamma((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) = (6.01, 7)$.

For $\gamma = 1$, by Proposition 20 if $((\bar{t}_1, \bar{t}_2), (\overline{x}_1, \overline{x}_2)) \in \overline{\Gamma}_2^1$ and $((\widetilde{t}_1, \widetilde{t}_2), (\widetilde{x}_1, \widetilde{x}_2)) \in \overline{\Gamma}_2^2$ then $u_1(\overline{x}_1, \overline{x}_2) - t_1(\overline{x}_1, \overline{x}_2) + t_2(\overline{x}_1, \overline{x}_2) \leq u_1(\widetilde{x}_1, \widetilde{x}_2) - t_1(\widetilde{x}_1, \widetilde{x}_2) + t_2(\widetilde{x}_1, \widetilde{x}_2)$. Thus preferences of agents on the order of the game to be played does not necessarily change only once between $\gamma = 0$ and $\gamma = 1$.

CHAPTER 6

A NEW PERSPECTIVE ON REFINING THE CORE

The core of a transferable utility (TU) game consists of imputations that cannot be blocked by any coalition of players. Consequently, the elements of the core shows a stability property such that if any allocation from the core is considered as a solution for the game, no coalition can deviate and improve upon it. However for any other allocation that does not lie in the core, there exists a coalition S whose members can do better by forming S and leaving the grand coalition. Although core has nice properties as stated above, if there exists $x, y \in Core(v)$ such that $x \neq y$, core has infinitely many elements due to the fact that it is convex. It then follows which imputation among core members should be proposed as a solution and implemented for nonsingleton cores. A core selection is a function which for each cooperative game with non-empty core selects a subset, possibly a single element of the core. Core selections provide a concise form of information contained in the TU game. Nucleolus (Schmeidler, 1969), per capita nucleolus (Grotte, 1970), core center (Gonzalez-Diaz and S´anchez-Rodriguez, 2007), and Shapley value for convex games (Shapley, 1953) are among the reported core selections. Interestingly whether or not, the information structure provided by the value functions in choosing from among the set of a core is adequate, remains obscure to date.

Unlike TU games, information structure is richer in normal form games where cooperational behavior and stability is also analyzed. Aumann (1961) introduces core like coalition proof solution concepts to normal form games where utility is not transferable. Solution concepts of Strong Nash Equilibrium (Aumann, 1959), α-core (Aumann, 1961) and β-core (Aumann, 1961) are composed of strategy profiles such that any coalition does not deviate and become better off given specific deviation circumstances related with each solution concept. But unlike from core imputations the information structure provided by α-core and β-core contains strategy spaces and payoff functions. To utilize this information we will adopt the approach suggested by Von Neumann and Morgenstern (1944) in seminal book Theory of Games and Economic Behaviour, where TU games are developed from normal form games. Hence given a normal form game there exists an associated TU game and given a convex TU game there exists a normal form game where one can find the underlying strategy space and utility function for each agent. This approach implies that there is further information about agents and coalitions which is lost during the formation of the TU game from its underlying normal form game.

In this chapter it is shown that in order to make a refinement of the core accurately, the lost information structure of the underlying strategic form game of the TU game is needed. To incorporate the strategic information structure to TU games, the notions of α-core and β-core are generalized to the setting where utility is transferable without any loss. The generalized notion of α-core and β-core for transferable utility is made for any level of transferability of utility, where utility can be γ-transferable for $\gamma \in [0, 1]$. Thus, nontransferable utility and fully transferable utility are special cases of generalized notion of α-core and β-core with γ-transferable utility where $\gamma = 0$ or $\gamma = 1$. Together with the information structure incorporated to TU games via generalized versions of α-core and β-core in case of transferable utility, a new refinement of the core is provided. Examples are presented to show that any refinement of the core that has nice properties and looks "fair" with a limited information of value functions will be considered as "unfair" given further information about the underlying strategy space and utility profiles. Finally it is shown that any two elements of the core is equally valuable given that further information about the underlying strategies and utility profiles are completely lost.

Consider the following normal form game;

Example 9. Prisoners' Dilemma

Let $g = (N, X_1 \times X_2, u)$ be such that $X_1 = X_2 = \{C, D\}$ where C represents "confess" and D is "don't confess". And let $u : X_1 \times X_2 \to \mathbb{R}$ be defined by the following table where agent 1 is the row player and agent 2 is the column player.

	C	D
C	$1,1$	$3,0$
D	$0,3$	$2,2$

This is a tight game as $v^g_\alpha(S) = v^g_\beta(S) = v$ where $v(\{1\}) = 1 = v(\{2\})$ and $v(\{1,2\}) = 4$. $Core(v) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, x_1 + x_2 = 4\}$. So our statement about 1-α-core and 1-β-core both are represented by the example above.

The core of the associated TU game consists of all allocations on the half line between points $(1,3)$ and $(3,1)$. But note that only the allocation $(2,2)$ is implementible as it is the only vector in the core that is compatible with the strategic information structure of the normal form game that v emerges from. The reasoning is as follows: v is defined in the context where utility is completely transferable. An individual can guarantee a payoff of 1, and block any allocation that gives less than 1, whereas if agents become a coalition they can achieve a total utility of 4. However, when normal form game is transformed to the side payment game, strategic information structure is completely lost. The only information is the minimal level of utility that agents and coalitions will not try to change. When strategical information is kept, it is clear that agents together can get the total payoff of 4 by playing (D, D) and get $u(D, D) = (2, 2)$. Any allocation in the core except than $(2, 2)$ requires one of the agents to transfer a positive amount of utility to the opponent. This is not compatible with the incentives as single agent coalitions has value 1, hence any agent has no power to oppose the payoff vector $(2, 2)$. Moreover no agent has the incentive to transfer some of his payoff to the opponent as the opponent can not deviate from the imputation $(2, 2)$ and become better off. Thus the allocation $(2, 2)$ is the only achievable payoff vector among core allocations.

Core refinement of Shapley value, nucleolus and center of the core is $(2, 2)$ which is consistent with the strategy profile $(D, D) \in C^1_\alpha(g)$ $[(D, D) \in C^1_\beta(g)]$.The allocation $(2, 2)$ chosen from the core as a refinement seems as a meaningful core selection due to the fact that agents are not only symmetric in terms of value functions, but they are symmetric with respect to the payoff vectors at outcomes in the normal form game. Now consider second version of Prisoners' Dilemma where $u(C, C) = (1, 1)$, $u(C, D) = (3, 0)$, $u(D, C) = (0, 3)$ and $u(D, D) = (1, 3)$. Note that the new game has the same associated TU game v and agents are still symmetric in terms of value functions. Therefore Shapley value, nucleolus and center of the core still selects $(2, 2)$. But according to the argumentation above $(1, 3)$ is the only achievable payoff vector from among core allocations as (D, D) is the only member of 1-α core [1-β core].

Consider a third version of Prisoners' Dilemma where $u(C, C) = (1, 1)$, $u(C, D)$ $= (2, 0)$ and $u(D, C) = (0, 3)$, $u(D, D) = (1, 3)$. Note that in this example, the associated TU game still does not change. Now although agents are symmetric in terms of value functions, agents are not completely symmetric in the normal form game. Meanwhile (D, D) is the only member of the 1-α core [1-β core], hence $(1, 3)$ is the only core selection that is compatible with the strategy space. Whereas if we selected the refinement of the core allocation with respect to the Shapley value, nucleolus or center of the core, one would never achieve the allocation $(1, 3)$. Choosing the imputation $(1, 3)$ over other core imputations seems unfair with respect to the information provided by the value function v.

Now in a fourth version of Prisoners' Dilemma suppose that $u(C, C) = (1, 1)$, $u(C, D) = (3, 0)$, $u(D, C) = (0, 3)$, $u(D, D) = (4, 0)$. Then $C^1_\alpha(g) = C^1_\beta(g) = \varnothing$. It is the fact that the agents can achieve the total payoff of 4 at (D, D), at outcome (D, D) where agent 2 has to be compansated by agent 1 such that $u_2(D, D) > 1$. as utility is 1-transferable agent 1 will give agent 2 a payoff of 1. Thus whenever $\overline{u}^\gamma((D, D), t(D, D)) = (3, 1)$, agent 1 shares enough utility with agent 2 in order to make her not to deviate, and agent 1 keeps the surplus to himself. At payoff vector $(3, 1)$ agent 2 will not be able to block in the α-sense or β-sense. Clearly giving the least transfer amount is the optimal strategy for agent 1. Now the core refinement compatible with the information structure given by the normal form game is $\overline{u}^\gamma((D, D), t(D, D)) = (3, 1)$. So in this approach, the outcome $x \in \arg\max_{x \in X} \sum_{i \in N} u_i(x)$ such that $\overline{u}^\gamma(x, t(x)) \in \partial Core(v)$ where $\forall i \in S \subset N$ such that $\sum_{i \in S} u_i(x) < v(S)$ one has $\sum_{i \in S} \overline{u}^\gamma_i(x, t(x)) = v(S)$, is proposed as an element of the core refinement.

Core refinement need not be a singleton, any $y \in Core(v)$ such that $\exists x \in C^1_\alpha(g)$ [$\exists x \in C^1_\beta(g)$] such that $u(x) = y$; y is belongs to the set of core refinement. By this approach normal form games are classified into groups such that refinement of the core of the associated TU game is single valued, or multivalued.

Given a normal form game g, whenever $\left|C^1_\beta(g)\right| = 1$ [$|C^1_\alpha(g)| = 1$], then the set of core refinement is singleton, otherwise it is a multivalued set.

The refinement of the core porposed is as follows:

Given a finite normal form game $g = (N, X, u)$, let the associated side payment game be v^g_α $[v^g_\beta]$ and let $|Core(v^g_\alpha)| > 1$ $[|Core(v^g_\beta)| > 1]$

i. if $C^1_\alpha(g) \neq \varnothing$ $[C^1_\beta(g) \neq \varnothing]$, core refinement is the set $CR = \{u(x) : x \in C^1_\alpha(g)$ $[C^1_\beta(g)]\}$,

ii. if $C^1_\alpha(g) = \varnothing$ $[C^1_\beta(g) = \varnothing]$, core refinement is $CR = \{\overline{u}^\gamma(x, t(x)) : x \in \arg\max \sum_{i\in N, x\in X} u_i(x)$ and $|L(x) \cap Core(v^g_\alpha)| = 1$ $[|L(x) \cap Core(v^g_\beta)| = 1]$ where $L(x) = \{y \in \mathbb{R}^2 : y = \theta u(x) + (1-\theta)\overline{u}^\gamma(x, t(x)), \theta \in [0,1]\}$ where $L(x)$ is the line segment between the vectors $u(x)$ and $\overline{u}^\gamma(x, t(x))$. Note that $L(x) \cap Core(v^g_\alpha) = \overline{u}^\gamma(x, t(x))$, and at $\overline{u}^\gamma(x, t(x))$ $\forall i \in S \subset N$ such that $\sum_{i\in S} u_i(x) < v(S)$ one has $\sum_{i\in S} \overline{u}^\gamma_i(x, t(x)) = v(S)$.

On the other hand one may argue that the TU games are defined in the context where the strategical information about the coalitions are not existent except for the value of coalitions. The following theorem shows that if strategical background of a given TU game is completely unknown, it is not possible to distinguish between the elements of the core to make a refinement. This is due to the fact that, given a convex TU game v with a nonempty core, for any element $y \in Core(v)$ there exists a normal form game g that leads to v such that $\exists x \in C^1_\alpha(g)$ $[\exists x \in C^1_\beta(g)]$ such that $u(x) = y$. And if the information structure of v is compatible with g, y should be in the refinement of the $Core(v)$. So any two allocations of the core are equivalent in terms of valuebility.

In previous sections with the introduction of semi-transferable utility we analyzed γ-TU games. In this section we will be focusing on the extreme cases of γ but the origin of the problem is TU games now. We know that given a normal form game g there exists a TU game that can be derived from g. On the other hand, given a TU game v there exists a family of finite normal form games that have v as their associated TU game ($v = v^g_\alpha$ or $v = v^g_\beta$) given that v is convex. Convexity arises naturally as v being the associated game and any agent cannot become worse off by cooperation in the β-sense or α-sense.

Definition 28. *A TU game* is said to be

i) *convex* if $v(S_1)+v(S_2) \leq v(S_1 \cup S_2) - v(S_1 \cap S_2)$ $\forall S_1, S_2 \subseteq N$ with $S_1 \nsubseteqq S_2$ and $S_2 \nsubseteqq S_1$.

ii) *strictly convex* if $v(S_1)+v(S_2) < v(S_1 \cup S_2) - v(S_1 \cap S_2)$ $\forall S_1, S_2 \subseteq N$ with $S_1 \nsubseteqq S_2$ and $S_2 \nsubseteqq S_1$.

Now consider the following 3 agent TU game that which we will construct the strategic coalition formation game. Then given a convex TU game v we will generalize the construction of coalition formation game to n agent case.

Example 10. Given $v(1), v(2), v(3), v(12), v(13), v(23), v(123)$ such that (v, N) is a convex game. Let core of game v be nonempty, a singleton set.(Below in the propositions core need not to be a singleton set). Consider the following associated normal form game $g = (N, X, u)$ where $N = \{1, 2, 3\} = \{i, j, k\}$ where $X_i = \{\varnothing, \{j\}, \{k\}, \{j, k\}\}$. the utility payoffs are given by the following 4 tables where player 1 is the row player, player 2 is the column player and tables represent the payoffs where agent 3 plays $\varnothing$ in the first table, $\{1\}$ in the second table, $\{2\}$ in the third table, $\{1, 2\}$ in the fourth table and $u(x) = (u_1(x), u_2(x), u_3(x))$.

Player 3 plays $\varnothing$

	$\varnothing$	$\{1\}$	$\{3\}$	$\{1,3\}$
$\varnothing$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{2\}$	$v(1), v(2), v(3)$	$\gamma, v(12) - \gamma, v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{3\}$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{2,3\}$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$

Player 3 plays $\{1\}$

	$\varnothing$	$\{1\}$	$\{3\}$	$\{1,3\}$
$\varnothing$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{2\}$	$v(1), v(2), v(3)$	$\gamma, v(12) - \gamma, v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{3\}$	$\bar{\gamma}, v(2), v(13) - \bar{\gamma}$	$\bar{\gamma}, v(2), v(13) - \bar{\gamma}$	$\bar{\gamma}, v(2), v(13) - \bar{\gamma}$	$\bar{\gamma}, v(2), v(13) - \bar{\gamma}$
$\{2,3\}$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$

Player 3 plays $\{2\}$

	$\varnothing$	$\{1\}$	$\{3\}$	$\{1,3\}$
$\varnothing$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), \tilde{\gamma}, v(23) - \tilde{\gamma}$	$v(1), v(2), v(3)$
$\{2\}$	$v(1), v(2), v(3)$	$\gamma, v(12) - \gamma, v(3)$	$v(1), \tilde{\gamma}, v(23) - \tilde{\gamma}$	$v(1), v(2), v(3)$
$\{3\}$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), \tilde{\gamma}, v(23) - \tilde{\gamma}$	$v(1), v(2), v(3)$
$\{2,3\}$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), \tilde{\gamma}, v(23) - \tilde{\gamma}$	$v(1), v(2), v(3)$

Player 3 plays $\{1,2\}$

	$\varnothing$	$\{1\}$	$\{3\}$	$\{1,3\}$
$\varnothing$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{2\}$	$v(1), v(2), v(3)$	$\gamma, v(12) - \gamma, v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{3\}$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$
$\{2,3\}$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	$v(1), v(2), v(3)$	y_1, y_2, y_3

Let the associated TU game is defined as $v^g_\beta(S)$. Note that

$u_1(x_1, \varnothing, \varnothing) \leq v(1)$

$u_1(x_1, \{1\}, \varnothing) \le \max\{\gamma, v(1)\}$

$u_1(x_1, \{3\}, \varnothing) \le v(1)$

$u_1(x_1, \{1,3\}, \varnothing) \le v(1)$

$u_1(x_1, \varnothing, \{1\}) \le \max\{\bar{\gamma}, v(1)\}$

$u_1(x_1, \{1\}, \{1\}) \le \max\{\bar{\gamma}, \gamma, v(1)\}$

$u_1(x_1, \{3\}, \{1\}) \le \max\{\bar{\gamma}, v(1)\}$

$u_1(x_1, \{1,3\}, \{1\}) \le \max\{\bar{\gamma}, v(1)\}$

$u_1(x_1, \varnothing, \{2\}) \le v(1)$

$u_1(x_1, \{1\}, \{2\}) \le \max\{\gamma, v(1)\}$

$u_1(x_1, \{3\}, \{2\}) \le v(1)$

$u_1(x_1, \{1,3\}, \{2\}) \le v(1)$

$u_1(x_1, \varnothing, \{1,2\}) \le v(1)$

$u_1(x_1, \{1\}, \{1,2\}) \le \max\{\gamma, v(1)\}$

$u_1(x_1, \{3\}, \{1,2\}) \le v(1)$

$u_1(x_1, \{1,3\}, \{1,2\}) \le \max\{y_1, v(1)\}$

Hence $v^g_\beta(1) = v(1)$

$u_2(\varnothing, x_2, \varnothing) \le v(2)$

$u_2(\{2\}, x_2, \varnothing) \le \max\{v(2), v(12) - \gamma\}$

$u_2(\{3\}, x_2, \varnothing) \le v(2)$

$u_2(\{2,3\}, x_2, \varnothing) \le v(2)$

$u_2(\varnothing, x_2, \{1\}) \le v(2)$

$u_2(\{2\}, x_2, \{1\}) \le \max\{v(2), v(12) - \gamma\}$

$u_2(\{3\}, x_2, \{1\}) \le v(2)$

$u_2(\{2,3\}, x_2, \{1\}) \le v(2)$

$u_2(\varnothing, x_2, \{2\}) \le \max\{v(2), v(23) - \tilde{\gamma}\}$

$u_2(\{2\}, x_2, \{2\}) \le \max\{v(12) - \gamma, v(13) - \bar{\gamma}, v(2)\}$

$u_2(\{3\}, x_2, \{2\}) \leq \max\{v(2), v(23) - \tilde{\gamma}\}$

$u_2(\{2,3\}, x_2, \{2\}) \leq \max\{v(2), v(23) - \tilde{\gamma}\}$

$u_2(\varnothing, x_2, \{1,2\}) \leq v(2)$

$u_2(\{2\}, x_2, \{1,2\}) \leq \max\{v(2), v(12) - \gamma\}$

$u_2(\{3\}, x_2, \{1,2\}) \leq v(2)$

$u_2(\{2,3\}, x_2, \{1,2\}) \leq \max\{v(2), y_2\}$

Hence $v^g_\beta(2) = v(2)$

$u_3(\varnothing, \varnothing, x_3) \leq v(3)$

$u_3(\varnothing, \{1\}, x_3) \leq v(3)$

$u_3(\varnothing, \{3\}, x_3) \leq \max\{v(3), v(23) - \tilde{\gamma}\}$

$u_3(\varnothing, \{1,3\}, x_3) \leq v(3)$

$u_3(\{2\}, \varnothing, x_3) \leq v(3)$

$u_3(\{2\}, \{1\}, x_3) \leq v(3)$

$u_3(\{2\}, \{3\}, x_3) \leq \max\{v(3), v(23) - \tilde{\gamma}\}$

$u_3(\{2\}, \{1,3\}, x_3) \leq v(3)$

$u_3(\{3\}, \varnothing, x_3) \leq \max\{v(3), v(13) - \bar{\gamma}\}$

$u_3(\{3\}, \{1\}, x_3) \leq \max\{v(3), v(13) - \bar{\gamma}\}$

$u_3(\{3\}, \{3\}, x_3) \leq \max\{v(23) - \tilde{\gamma}, v(13) - \bar{\gamma}, v(3)\}$

$u_3(\{3\}, \{1,3\}, x_3) \leq \max\{v(3), v(13) - \bar{\gamma}\}$

$u_3(\{2,3\}, \varnothing, x_3) \leq v(3)$

$u_3(\{2,3\}, \{1\}, x_3) \leq v(3)$

$u_3(\{2,3\}, \{3\}, x_3) \leq \max\{v(3), v(23) - \tilde{\gamma}\}$

$u_3(\{2,3\}, \{1,3\}, x_3) \leq \max\{v(3), y_3\}$

Hence $v^g_\beta(3) = v(3)$

$u_{12}(x_1, x_2, \varnothing) \leq v(12)$

$u_{12}(x_1, x_2, \{1\}) \leq \max\{v(12), \bar{\gamma} + v(2)\}$

$u_{12}(x_1, x_2, \{2\}) \leq \max\{v(12), \tilde{\gamma} + v(2)\}$

$u_{12}(x_1, x_2, \{1, 2\}) \leq \max\{v(12), y_1 + y_2\}$

Hence $v^g_\beta(12) = v(12)$

$u_{13}(x_1, \varnothing) \leq v(13)$

$u_{13}(x_1, \{1\}, x_3) \leq \max\{v(13), \gamma + v(3)\}$

$u_{13}(x_1, \{3\}, x_3) \leq \max\{v(13), v(23) - \tilde{\gamma} + v(1)\}$

$u_{13}(x_1, \{1, 3\}, x_3) \leq \max\{v(13), y_1 + y_3\}$

Hence $v^g_\beta(13) = v(13)$

$u_{23}(\varnothing, , x_2, x_3) \leq v(23)$

$u_{23}(\{2\}, x_2, x_3) \leq \max\{v(23), v(12) - \gamma + v(3)\}$

$u_{23}(\{3\}, x_2, x_3) \leq \max\{v(23), v(13) - \bar{\gamma}\}$

$u_{23}(\{2, 3\}, x_2, x_3) \leq \max\{v(23), y_2 + y_3\}$

Hence $v^g_\beta(23) = v(23)$.

Therefore $v^g_\beta = v$.

Also note that if we let

$$v^g_\alpha(S) = \max_{x_S} \min_{x_{N \setminus S}} u_S(x_S, x_{N \setminus S}) = \max_{x_S} \min_{x_{N \setminus S}} \sum_{i \in S} u_i(x_S, x_{N \setminus S}), \forall S \subseteq N,$$

we have the following results:

$u_1(\varnothing, x_2, x_3) \geq v(1)$

$u_1(\{2\}, x_2, x_3) \geq \min\{\gamma, v(1) - 2\varepsilon\}$

$u_1(\{3\}, x_2, x_3) \geq \min\{\bar{\gamma}, v(1) - 2\varepsilon\}$

$u_1(\{2, 3\}, x_2, x_3) \geq v(1) - 2\varepsilon$. Note that we have $x_1 \geq v(1)$

$u_2(x_1, \varnothing, x_3) \geq v(2)$

$u_2(x_1, \{1\}, x_3) \geq \min\{v(12) - \gamma, v(2) - 2\varepsilon\}$

$u_2(x_1, \{3\}, x_3) \geq \min\{\tilde{\gamma}, v(2) - 2\varepsilon\}$

$u_2(x_1, \{1, 3\}, x_3) \geq v(2) - 2\varepsilon$. Note that we have $x_2 \geq v(2)$

$u_3(x_1, x_2, \varnothing) \geq v(3)$

$u_3(x_1, x_2, \{1\}, x_3) \geq \min\{v(13) - \bar{\gamma}, v(3) - 2\varepsilon\}$

$u_3(x_1, x_2, \{2\}, x_3) \geq \min\{v(23) - \tilde{\gamma}, v(2) - 2\varepsilon\}$

$u_3(x_1, x_2, \{1, 2\}, x_3) \geq v(3) - 2\varepsilon$. Note that we have $x_3 \geq v(3)$

$u_{12}(\varnothing\varnothing, x_3) \geq v(1) + v(2)$

$u_{12}(\varnothing\{1\}, x_3) \geq v(1) + v(2) - 2\varepsilon$

$u_{12}(\varnothing\{3\}, x_3) \geq \min\{v(1) + v(2) - 2\varepsilon, v(1) + \tilde{\gamma}\}$

$u_{12}(\varnothing\{1, 3\}, x_3) \geq v(1) + v(2) - 2\varepsilon$

$u_{12}(\{2\}\varnothing, x_3) \geq v(1) + v(2) - 2\varepsilon$

$u_{12}(\{2\}\{1\}, x_3) \geq v(12)$

$u_{12}(\{2\}\{3\}, x_3) \geq \min\{v(1) + v(2), v(1) + \tilde{\gamma}\}$

$u_{12}(\{2\}\{1, 3\}, x_3) \geq v(1) + v(2)$

$u_{12}(\{3\}\varnothing, x_3) \geq \min\{v(1) + v(2) - 2\varepsilon, v(2) + \bar{\gamma}\}$

$u_{12}(\{3\}\{1\}, x_3) \geq \min\{v(1) + v(2), v(2) + \bar{\gamma}\}$

$u_{12}(\{3\}\{3\}, x_3) \geq \min\{v(1) + v(2), v(2) + \bar{\gamma}, v(1) + \tilde{\gamma}\}$

$u_{12}(\{3\}\{1, 3\}, x_3) \geq \min\{v(1) + v(2), v(2) + \bar{\gamma}\}$

$u_{12}(\{2, 3\}\varnothing, x_3) \geq v(1) + v(2) - 2\varepsilon$

$u_{12}(\{2, 3\}\{1\}, x_3) \geq v(1) + v(2)$

$u_{12}(\{2, 3\}\{3\}, x_3) \geq \min\{v(1) + v(2), v(1) + \tilde{\gamma}\}$

$u_{12}(\{2, 3\}\{1, 3\}, x_3) \geq v(1) + v(2)$ as $x_1 + x_2 \geq v(12)$

$u_{13}(\varnothing\varnothing, x_2) \geq v(1) + v(3)$

$u_{13}(\varnothing\{1\}, x_2) \geq v(1) + v(3) - 2\varepsilon$

$u_{13}(\varnothing\{2\}, x_2) \geq \min\{v(1) + v(3) - 2\varepsilon, v(1) + v(23) - \tilde{\gamma}\}$

$u_{13}(\varnothing\{1,2\}, x_2) \geq v(1) + v(3) - 2\varepsilon$

$u_{13}(\{2\}\varnothing, x_2) \geq \min\{v(1) + v(3), v(3) + \gamma\}$

$u_{13}(\{2\}\{1\}, x_2) \geq \min\{v(1) + v(3), v(3) + \gamma\}$

$u_{13}(\{2\}\{2\}, x_2) \geq \min\{v(1) + v(3), v(3) + \gamma\}$

$u_{13}(\{2\}\{1,2\}, x_2) \geq \min\{v(1) + v(3), v(3) + \gamma\}$

$u_{13}(\{3\}\varnothing, x_2) \geq v(1) + v(3) - 2\varepsilon$

$u_{13}(\{3\}\{1\}, x_2) \geq v(13)$

$u_{13}(\{3\}\{2\}, x_2) \geq \min\{v(1) + v(3), v(1) + v(23) - \tilde{\gamma}\}$

$u_{13}(\{3\}\{1,2\}, x_2) \geq v(1) + v(3)$

$u_{13}(\{2,3\}\varnothing, x_2) \geq v(1) + v(3) - 2\varepsilon$

$u_{13}(\{2,3\}\{1\}, x_2) \geq v(1) + v(3)$

$u_{13}(\{2,3\}\{2\}, x_2) \geq \min\{v(1) + v(3), v(1) + v(23) - \tilde{\gamma}\}$

$u_{13}(\{2,3\}\{1,2\}, x_2) \geq v(1) + v(3)$ as $x_1 + x_3 \geq v(13)$

$u_{23}(x_1, \varnothing\varnothing) \geq v(2) + v(3)$

$u_{23}(x_1, \varnothing\{1\}) \geq \min\{v(2) + v(3) - 2\varepsilon, v(2) + v(13) - \bar{\gamma}\}$

$u_{23}(x_1, \varnothing\{2\}) \geq v(2) + v(3) - 2\varepsilon$

$u_{23}(x_1, \varnothing\{1,2\}) \geq v(2) + v(3) - 2\varepsilon$

$u_{23}(x_1, \{1\}\varnothing) \geq \min\{v(2) + v(3) - 2\varepsilon, v(12) + v(3) - \gamma\}$

$u_{23}(x_1, \{1\}\{1\}) \geq \min\{v(2) + v(3), v(12) + v(3) - \gamma, v(2) + v(13) - \bar{\gamma}\}$

$u_{23}(x_1, \{1\}\{2\}) \geq \min\{v(2) + v(3), v(12) + v(3) - \gamma\}$

$u_{23}(x_1, \{1\}\{1,2\}) \geq \min\{v(2) + v(3), v(12) + v(3) - \gamma\}$

$u_{23}(x_1, \{3\}\varnothing) \geq v(2) + v(3) - 2\varepsilon$

$u_{23}(x_1, \{3\}\{1\}) \geq \min\{v(2) + v(3), v(2) + v(13) - \bar{\gamma}\}$

$u_{23}(x_1, \{3\}\{2\}) \geq v(23)$

$u_{23}(x_1, \{3\}\{1,2\}) \geq v(2) + v(3)$

$u_{23}(x_1, \{1,3\}\varnothing) \geq v(2) + v(3) - 2\varepsilon$

$u_{23}(x_1, \{1,3\}\{1\}) \geq \min\{v(2) + v(3), v(2) + v(13) - \bar{\gamma}\}$

$u_{23}(x_1, \{1,3\}\{2\}) \geq v(2) + v(3)$

$u_{23}(x_1, \{1,3\}\{1,2\}) \geq v(2) + v(3)$ as $x_2 + x_3 \geq v(23)$

Thus we have $v_\alpha^g(S) = v$ and $(x_1, x_2, x_3) \in Core_\alpha(g)$ where $u(x_1, x_2, x_3) \in Core(v)$.

Now, given a TU game v we have a normal form game which turns out to have nice properties. This coalition formation game is tight and the outcomes such that the coalitions that form a partition of the set of agents belong to the set of Nash equilibria. And note that any allocation $x \in Core(v)$ can be written as payoff vector of (x_1, x_2, x_3) which implies that $(x_1, x_2, x_3) \in Core_\beta(g)$ $[Core_\alpha(g)]$. Now we have the following propositions that generalize the normal form game we generated for 3 agent TU games to any TU game with finite set of agents.

Let v be a convex TU game and $\omega \in Core(v)$. For each $i \in N$, set $X_i = \{S \in 2^N : i \in S\}$. For any $x \in X = \prod\limits_{i \in N} X_i$, we define a coalitional partition $B(x)$ of N as follows: For any $T \in 2^N$ with $|T| > 1$, we let $T \in B(x)$ if and only if $x_i = T$ for any $i \in T$. For any $j \in N$, we let $\{j\} \in B(x)$ if and only if either $x_j = \{j\}$ or $x_k \neq x_j$ for some $k \in x_j$. We say that $g = (N, X, u)$ is a canonical strategic form game for (v, ω) if and only if u satisfies conditions (a) and (b) below:

Condition (a): If $x \in X$ is such that $x_i = N$ for all $i \in N$, then $u_i(x) = \omega_i$ for each $i \in N$.

Condition (b): If $x \in X$ is such that $x_i \neq N$ for some $i \in N$, then $u_j(x) \geq v(\{j\})$ for all $j \in N$, and $\sum_{j \in T} u_j(x) = v(T)$ for any $T \in B(x)$.

The existence of a canonical strategic form game for any (v, ω) as above follows from the convexity of v.

Theorem 8. Let v be a convex TU game, $\omega \in Core(v)$ and $g = (N, X, u)$ a canonical strategic form game for (v, ω). Also let $x \in X$ be such that $x_i = N$ for all $i \in N$. Now

i) $v_g^\alpha = v_g^\beta = v$.

ii) $x \in C_\alpha^1(g) \cap C_\beta^1(g)$ and $u_i(x)_{i\in N} = \omega$.

Proof: (a) First note that $v_g^\alpha(N) = v_g^\beta(N) = Max_{x\in X} \sum\limits_{i\in N} u_i(x)$ by definition of v_g^α and v_g^β. For any $x \in X\backslash\{x\}$, one has $B(x) \neq \{N\}$, and thus $\sum_{T\in B(x)} v(T) \leq v(N)$ by convexity of v. On the other hand, $\sum\limits_{i\in N} u_i(x) = \sum\limits_{i\in N} \omega_i = v(N)$ by definition of g and since $\omega \in Core(v)$. So, $Max_{x\in X} \sum\limits_{i\in N} u_i(x) = \sum\limits_{i\in N} u_i(x)$, implying that $v_g^\alpha(N) = v_g^\beta(N) = v(N)$.

Now take any $S \in 2^N\backslash\{\emptyset, N\}$. Let $\tilde{x} \in X$ be such that $\tilde{x}_i = S$ for all $i \in S$ and $\tilde{x}_j = N\backslash S$ for all $j \in N\backslash S$. Now $\sum\limits_{i\in S} u_i(\tilde{x}_S, x_{N\backslash S}) = v(S)$ for all $x_{N\backslash S} \in X_{N\backslash S}$ since $B(\tilde{x}_S, x_{N\backslash S}) = \{S, T_1, \ldots, T_l\}$ for some partition $\{T_1, \ldots, T_l\}$ of $N\backslash S$. Thus, we have both $v_g^\alpha(S) \geq v(S)$ and $v_g^\beta(S) \geq v(S)$. On the other hand, for any $x_S \in X_S$, we have $B(x_S, \tilde{x}_{N\backslash S}) = \{N\backslash S, T_1, \ldots, T_k\}$ for some partition $\{T_1, \ldots, T_k\}$ of S. By definition of g, it follows that $\sum\limits_{i\in S} u_i(x_S, \tilde{x}_{N\backslash S}) = \sum\limits_{l=1}^{k} v(T_l) \leq v(S)$ by convexity of v. So, $v_g^\beta(S) \leq v(S)$. As $v_g^\alpha(S) \leq v_g^\beta(S)$, we also conclude that $v_g^\alpha(S) \leq v(S)$. Therefore, we have $v_g^\alpha(S) = v(S) = v_g^\beta(S)$ for all $S \in 2^N\backslash\{\emptyset\}$.

(b) Suppose that $x \notin C_\beta^1(g)$. Now there is some $S \in 2^N\backslash\{\emptyset\}$, which TU-$\beta$ blocks x. In particular, there is some $x_S \in X_S$ such that $\sum\limits_{i\in S} u_i(x_S, \tilde{x}_{N\backslash S}) > \sum\limits_{i\in S} u_i(x) = \sum\limits_{i\in S} \omega_i \geq v(S)$, since $\omega \in Core(v)$. Recall that $B(x_S, \tilde{x}_{N\backslash S}) = \{N\backslash S, T_1, \ldots, T_k\}$ for some partition $\{T_1, \ldots, T_k\}$ of S.

We thus also have $\sum\limits_{i\in S} u_i(x_S, \tilde{x}_{N\backslash S}) = \sum\limits_{l=1}^{k} v(T_l)$ by definition of g, while $\sum\limits_{l=1}^{k} v(T_l) \leq v(S)$ by convexity of v, yielding a contradiction. Thus, $x \in C_\beta^1(g)$. Since clearly $C_\beta^1(g) \subset C_\alpha^1(g)$, we have $x \in C_\alpha^1(g)$ as well. Moreover $u_i(x)_{i\in N} = \omega$ by condition (a). □

Corollary 6. Let v be a strictly convex TU game and $\omega \in Core(v)$. If $g = (N, X, u)$ is a canonical strategic form game for (v, ω), then $C_\alpha^1(g) = C_\beta^1(g) = \{x\}$ for some $x \in X$ with $\omega = u_i(x)_{i\in N}$.

Proof. Suppose that $x, y \in C_\alpha^1(g)$ with $x \neq y$ for some canonical game $g =$

(N, X, u) for (v, ω). By definition of g, we have $B(x) \neq \{N\}$ or $B(y) \neq \{N\}$. Assume without loss of generality that $B(x) = \{T_1, \ldots, T_l\}$ with $l > 1$. Again by construction of g, we have $\sum_{j \in T_k} u_j(x) = v(T_k)$ for each $k \in \{1, \ldots, l\}$. Thus $\sum_{j \in N} u_j(x) = \sum_{k=1}^{l} v(T_k) < v(N)$, where the last strict inequality follows from the strict convexity of v. Then, however, N TU-α blocks x by playing $x \in X$ with $x_i = N$ for each $i \in N$, in contradiction with $x \in C^1_\alpha(g)$. As we already know that $x \in C^1_\alpha(g)$ we conclude that $C^1_\alpha(g) = \{x\}$ with $u_i(x)_{i \in N} = \omega$. Finally since $x \in C^1_\beta(g) \subset C^1_\alpha(g)$, it also follows that $C^1_\beta(g) = \{x\}$. □

Definition 29. Let $g = (N, X, u)$ be a strategic form game and $x \in X$. We say that x is a *TU-strong equilibrium of* g if and only if, for any $S \in 2^N \backslash \{\emptyset\}$ and $x_S \in X_S$, one has $\sum_{i \in S} u_i(x) \geq \sum_{i \in S} u_i(x_S, x_{N \backslash S})$.

Theorem 9. Let v be a convex TU game, $\omega \in Core(v)$ and $g = (N, X, u)$ a canonical strategic form game for (v, ω). Also let $x \in X$ be such that $x_i = N$ for all $i \in N$. Now x is a TU-strong equilibrium of g with $u_i(x)_{i \in N} = \omega$. Moreover, if v is strictly convex, then x is the unique TU-strong equilibrium of g.

Proof. Suppose that there are some $S \in 2^N \backslash \{\emptyset\}$ and $x_S \in X_S$ such that $\sum_{i \in S} u_i(x_S, x_{N \backslash S}) > \sum_{i \in S} u_i(x)$. Then $B(x_S, x_{N \backslash S}) = \{T_1, \ldots, T_l\} \cup \{\{j\} \mid j \in N \backslash S\}$ for some partition $\{T_1, \ldots, T_l\}$ of S. Now, however, $\sum_{i \in S} u_i(x_S, x_{N \backslash S}) = \sum_{k=1}^{l} v(T_k) \leq v(S)$ by definition of g along with the convexity of v. On the other hand, $\sum_{i \in S} u_i(x) = \sum_{i \in S} \omega_i \geq v(S)$ since $\omega \in Core(v)$, yielding a contradiction. Thus, x is a TU-strong equilibrium of g.

Now assume that v is strictly convex. Suppose that there is a TU-strong equilibrium $y \in X$ of g with $y \neq x$. Then, however, $B(y) \neq \{N\}$, i.e., $B(y) = \{T_1, \ldots, T_l\}$ with $l > 1$. But now $\sum_{i \in N} u_i(y) = \sum_{k=1}^{l} v(T_k) < v(N)$ by strict convexity of v, contradicting that y is a TU-strong equilibrium. □

No matter which natural strategic counterparts of the core we take, i.e. TU-α

core, the TU-β core or the TU-strong equilibrium, the family of canonical games we constructed strategically separates different core allocations from each other.

We have so far been interested in strategically separating different core outcomes from each other. The canonical family of strategic form games associated with each pair (v, ω), where v is a convex game and $\omega \in Core(v)$, however, paves also the ground for strategically separating core outcomes from noncore outcomes.

Given a convex TU game v, instead of specifying a particular core allocation of v, we now construct a strategic form game the set of whose TU-strong equilibria turns out to yield all core allocations of v.

Let v be a convex TU game. For each $i \in N$, set $X_i = \{(S, \omega) \in 2^N \times R^S \mid i \in S \text{ and } \sum_{i \in S} \omega_i = v(S)\}$. Given any $x \in X = \prod_{i \in N} X_i$, we write $x_i = (S^i, \omega^i)$ for each $i \in N$. We define, for any $i \in N$,

$u_i(x) = \{\omega_i^i \text{ if } x_j = x_i \text{ for all } j \in S^i,\ v(\{i\}) \text{ otherwise } \}$

at each $x \in X$. We refer to (N, X, u) as the strategic form game induced by v and denote it by g_v.

The game g_v induced by a convex TU game v can also be described by the following scenario. Once every player $i \in N$ picks a strategy $x_i \in X_i$, the joint strategy x leads to a coalition structure and a feasible distribution of the worth of any coalition in this partition among its members. Formally, denoting this "outcome function" by h, we have, for each $x \in X$, $h(x) = \{(T_1, \omega^{T_1}), \ldots, (T_l, \omega^{T_l})\}$, where $B(x) = \{T_1, \ldots, T_l\}$ is a partition of N and $\omega^{T_k} \in \mathbb{R}^{T_k}$ with $\sum_{i \in T_k} \omega_i^{T_k} = v(T_k)$ for each $k \in \{1, \ldots, l\}$ defined as follows in a similar fashion to our canonical games. For any $T \in 2^N$ with $|T| > 1$, we let $T \in B(x)$ if and only if $x_i = (T, \omega^T)$ for all $i \in T$. For any $j \in N$, we let $\{j\} \in B(x)$ if and only if $x_j = (\{j\}, v(\{j\}))$ or $x_k \neq x_j$ for some $k \in S^j$. For nonsingleton coalitions $T \in B(x), \omega^T$ is the allocation agreed upon by the members of T via their declarations. A singleton coalition $\{i\} \in B(x)$ receives $v(\{i\})$. The utility profile u of g_v above summarizes the outcome of this process.

We denote the set of all TU-strong equilibria of a strategic form game g by $SE_{TU}(g)$.

Theorem 10. For any convex TU game v, one has $u(SE_{TU}(g_v)) = Core(v)$.

Proof. Take any $\omega \in Core(v)$. Let $x \in X$ be such that $x_i = (N, \omega)$ for each $i \in N$. Take any $S \in 2^N \backslash \{\emptyset\}$ and $x_S \in X_S$ with $x_S \neq x_S$. Now $B(x_S, x_{N \backslash S}) = \{T_1, \ldots, T_l\} \cup \{\{j\} \mid j \in N \backslash S\}$ for some partition $\{T_1, \ldots, T_l\}$ of S, implying that $\sum_{i \in S} u_i(x_S, x_{N \backslash S}) = \sum_{k=1}^{l} v(T_k) \leq v(S)$ by convexity of v. As $\omega \in Core(v)$, we also have $v(S) \leq \sum_{i \in S} w_i = \sum_{i \in N} u_i(x)$ since clearly $u_i(x) = \omega_i$ by definition of g_v. Thus $\sum_{i \in S} u_i(x_S, x_{N \backslash S}) \leq \sum_{i \in S} u_i(x)$. So $x \in SE_{TU}(g_v)$ with $u_i(x) = \omega$.

Conversely, let $x \in SE_{TU}(g_v)$. Now suppose that there is some $S \in 2^N \backslash \{\emptyset\}$ with $\sum_{i \in S} u_i(x) < v(S)$. Let $\omega \in \mathbb{R}^S$ be such that $\sum_{i \in S} \omega_i = v(S)$, and set $x_i = (S, \omega)$ for each $i \in S$. Now, however, $\sum_{i \in S} u_i(x_S, x_{N \backslash S}) = \sum_{i \in S} \omega_i = v(S) > \sum_{i \in S} u_i(x)$, in contradiction with $x \in SE_{TU}(g_v)$. Thus, $u(x) \in Core(v)$. $\square$

CHAPTER 7

CONCLUSION

In this study we have introduced the notion of semi-transferable utility to complete the analysis of transferable utility where utility is neither nontransferable nor fully transferable. The efficiency notion (Pareto) of NTU games, and the solution concepts α-core and β-core are generalized for any given utility transfer degree $\gamma \in [0, 1]$. With the notions of α-core and β-core where utility is semi-transferable, the relation of α-core and the β-core in the NTU games and the γ-α-core and γ-β-core are provided respectively. Moreover it is shown that the utility profile of 1-α-core and 1-β-core belongs to the the core of the relevant associated TU game. The intermediate value theorems we provide show that there exists a critical value of utility transfer degree such that with high enough utility transfer degree, payoffs of α-stable or β-stable outcomes are in accordance with the Core of the relevant associated TU game. We provide critical vector of utility transfer degree where each agent can have different utility transfer degree which is related to his/her wealth level, which is a generalization of the above result via vectors.

Adding semi-transferable utility to a normal form game involves adding transfers at each outcome as commitments. In this study we propose 2 types of games where transfers are introduces to strategy space of each agent. In the first approach, transfers are added to the strategy space of the agent as a simultaneous choose with the actions of the normal form game. In the second approach we propose a two stage extensive form game where transfers are incorporated to the game as the first stage game. The second stage of the game is a normal form game where transfers proposed in the first stage are realized.

In the game introduced via the first approach, γ-TU normal form game is introduced where agents choose the strategies and outcomes simultaneously. γ-α-core and γ-β-core of the γ-TU normal form game is introduced. Given a 2 person finite normal form game necessary and sufficient conditions for the non-emptiness of γ-α-core and γ-β-core of the γ-TU normal form game is provided. The relation between the 1-α-core or 1-β-core of the γ-TU normal form game and the core of the relevant associated TU game is shown. The relation between the Nash equilibrium, α-core and β-core of a finite normal form game and the Nash equilibrium of the γ-TU normal form game is questioned. It turns out that inserting the semi-transferability of the utility to the model via transfers as simultaneous choices together with the strategies does not help to implement outcomes other than the Nash equilibrium of the original normal form game.

In the second approach, the transfers are incorporated to the finite normal form game as a pre-stage to the normal form game to be played. In the first stage the agents announce the transfers that they are proposing to give at each outcome constrained with a budget constraint; in the second stage the game is played with the transfers realized. As we have changed the setting to an extensive form game, a new equilibrium concept Quasi Subgame Perfect Equilibrium is introduced. Although utility transfer is incorporated to strategic form games in such a way that players do not have means of coordination when announcing transfers or choosing strategies, yet it turns out that outcomes that are reachable by coordination now become stable equilibria in famous examples Prisoners' Dilemma and Centipede Game. It turns out that for the games where Nash equilibrium gives players their minmax values and self interested players cannot improve upon these payoffs in spite of existence of superior outcomes for both agents' best interests, without having means of cooperation semi-transferability of utility with binding agreements on transfers improves efficiency of stable outcomes. More properties of quasi subgame perfect equilibrium such as whether an improvement with respect to Nash equilibrium concept in terms of welfare exists as gamma increase, achievability of welfare maximizing outcomes as stable outcomes are questioned. Relation between γ-β-core of the γ-TU normal form game and quasi subgame perfect equilibrium at γ is presented. It is shown that Nash equilibrium, Strong Nash equilirium, β-core can be implemented as QSPE if utility transfer degree is zero. Necessary conditions for a QSPE for $\gamma > 0$ are characterized. Utilyzing the approach of games of pretension set of equilibria when agents choose their transfers and actions in order are analyzed.

Moreover utilizing the relation between the normal form games and their associated TU games, the study questions the refinement of the core of a TU game given that there is no information about the underlying strategic structure of coalitions. Given a convex TU game with a nonempty core, we construct a canonical normal form game where α-core, β-core and TU strong equilibria coincide. We show that if the strategic information structure is lost, any two allocations that belong to the core are equally valuable as there exists a strategic background respecting them. The strict convexity of a TU game also guarantees strict separation between different core outcomes. Given a convex TU game v along with a particular core allocation ω, the utility profile of the associated coalition formation game depends upon both v and ω. We obtain the mechanism that "implements" the core from our canonical family of coalition formation games by making the payoff profile a function of the joint strategy instead of fixing it contingent upon a given core allocation.

Introduction of semi-transferable utility to study the cases in between the two extreme cases of transferability of utility is the main contribution of this study. It is applicable to both cooperative and noncooperative games and any model that exists in the literature of game theory. As a precedent we study semi-transferable utility for normal form games and mainly α-core and β-core. Assuming utility to be γ-transferable where $\gamma \in [0, 1]$ enables us to find the relation between equilibrium notions of NTU games and TU games. With this approach we not only incorporate strategy structure to TU games, we also embed a utility destruction possibility to strategic space of individuals in NTU games. Given a 2 person normal formal game, to insert semi-transferability of the utility to the game, two different game forms are proposed where transfers are incorporated to the strategy structure. It turns out that incorporating the transfers to the game as simultaneous choose with strategies of the normal form game cannot provide a mechanism to implement different outcomes other than the Nash equilibria of the NTU game since announcement of transfers are not binding agreements. Whereas inserting transfers to a NTU game as a 2 stage transfer mechanism makes it possible to implement outcomes that belong to α-core and the β-core when $\gamma = 0$. Lastly, utilizing the relation between the normal form game and its associated TU game, given a TU game v a strategic canonical coalition formation game is proposed. This canonical game approach shows that without the knowledge of the underlying strategic information structure of a TU game, any two elements of the core is equally valuable. As a refinement of core of a given TU game, utility profiles of outcomes that belong to 1-β-core is proposed.

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