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GRAVITY AS A GAUGE THEORY

M.Sc. THESIS

Orhan TUNCA

Department of Physics Engineering

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Orhan TUNCA
(509161113)

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Thesis Advisor: Doç. Dr. Mehmet ÖZKAN

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AYAR TEORİSİ OLARAK KÜTLEÇEKİMİ

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Orhan TUNCA
(509161113)

Fizik Mühendisliği Anabilim Dalı

Fizik Mühendisliği Programı

Tez Danışmanı: Doç. Dr. Mehmet ÖZKAN

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Orhan TUNCA, a M.Sc. student of ITU Graduate School of Science Engineering and Technology 509161113 successfully defended the thesis entitled "GRAVITY AS A GAUGE THEORY", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor : **Doç. Dr. Mehmet ÖZKAN** ....
Istanbul Technical University

Jury Members : **Doç. Dr. A. Savaş ARAPOĞLU** ....
Istanbul Technical University

Doç. Dr. V. Nefer Şenoğuz ....
Mimar Sinan Fine Arts University

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To my family and friends,

FOREWORD

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ABBREVIATIONS

GR	**:** General Relativity
NG	**:** Newtonian Gravity
SR	**:** Special Relativity
QM	**:** Quantum Mechanics

SYMBOLS

$\mu, \nu, \rho, \ldots,$	**:** Coordinate indices
$a, b, c, \ldots$	**:** Local Lorentz indices
∂_μ	**:** Partial derivative
∇_μ	**:** Covariant derivative

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GRAVITY AS A GAUGE THEORY

SUMMARY

In this thesis, we summarize Einstein's theory of General relativity and examine its symmetries. We develop a systematic way to deal with symmetries as gauge theories. Approaching from the gauge perspective, we show how to construct Einstein's theory again.

General relativity (GR) has been proven to be in line with many observations where Newtonian gravity fails. Some of them include precessions of Mercury's perihelion, expansion of the universe and gravitational waves. Although its great achievements, it is in contradiction to quantum mechanics which explains remaining interactions very well. At the intersection of these theories, there are black holes that require both theories to give a consistent description. However, the unification of these two theories yields infinities. One of the solution attempts to this problem is adding more symmetries to GR. To deal with symmetries, gauge theories are very useful as we explain in chapter 3.

In the second chapter, we present that special relativity and Newtonian gravity are in contradiction. The solution to this inconsistency gives birth to GR by Einstein. It is needed to learn and gain experience with geometrical tools of GR. Thus, we summarize them here. Manifolds are given by a simple intuitive definition to match them curved spaces. Tensors are explained to investigate how objects transform under coordinate transformations. Metric is presented as an object that specifies the structure of spacetime. New derivatives are defined to deal with correct transformation properties. Other objects such as curvature tensor which is related to metric and spacetime structure are presented. Eventually, with the usage of correct tools, Einstein's gravitation theory is given.

In the third chapter, we develop a systematic way to examine symmetries. Lie algebra of the Poincare group is investigated. The infinitesimal transformations of quantities are found. Then, we present how to localize symmetries. This procedure is known as gauging and requires new objects called gauge fields. To preserve symmetries, one needs to change derivatives of the theory with the covariant ones. At this point, gauge fields come into play with particular transformation properties.

In the subsequent chapter, we show how the objects of gauge theories match the GR's with some constraints. One can construct Einstein's theory from gauge theories. We take Poincare algebra and the gauging procedure gives us GR with one independent field.

In the last chapter, we first show how Newton's gravity can be formulated as geometrically known as Newton-Cartan theory. Newton's gravity can be embedded into spacetime as in GR, but it requires 2 degenerate metrics with 2 conditions called Trautman and Ehler.

As in GR, the Newton-Cartan theory can be reformulated as a gauge theory. Gauging of Bargmann algebra with some extra conditions again gives us the Newton-Cartan theory.

AYAR TEORİSİ OLARAK KÜTLEÇEKİMİ

ÖZET

Bu tez çalışmasında, Einstein'ın kütleçekimi teorisi özetlenmiş ve bu teorinin simetrileri detaylı bir şekilde incelenmiştir. Daha sonra, simetrilerle çalışabilmek için, sistematik bir yöntem, ayar teorisi adı altında verilmiştir. Ayar teorisi perspektiviyle, Einstein'ın kütleçekim teorisinin nasıl elde edilebileceği gösterilmiştir.

Genel görelilik bir çok yönden gözlemlerle uyum içinde çıkmış ve Newton'un kütleçekim teorisinin açıklayamadığı bir çok gözlemi açıklamayı başarabilmiştir. Bunlardan bazıları, fizikçileri uzun süre meşul eden Merkür'ün perihelon problemi, evrenin genişlemesi ve kütleçekim dalgalarıdır. Yine de büyük başarılarına rağmen genel görelilik , modern fiziğin bir diğer başarısı olan ve yerçekimi dışındaki diğer bütün etkileşimleri, üzerine kurulu olan Standard model ile açıklamada epey başarılı olan kuantum mekaniğiyle uyum içinde değildir. Bir teori, basitçe açıklamak gerekirse, atomik ölçeklerde çok başarılı olurken, diğeri astronomik ölçeklerde gayet iyi çalışmaktadır. Bu iki teorinin kesişim noktasında kara delikler bulunmaktadır. Kara deliklerin fiziğinin anlaşılması kuantum mekaniğiyle genel göreliliğin birleşimini gerektirmektedir. Fakat, bu iki teorinin birleştirimesi için denenen bir çok yöntem istenmeyen sonsuzluklara yol açmaktadır.

Tezin ikinci bölümünde özel görelilik ile Newton'un kütleçekim teorisinin çelişkisi açıklanmaktadır. Özel görelilik hiç bir şeyin ışıktan hızlı ilerleyemeyeceğini söylerken, Newton denkleminin çözümleri kütleçekimsel alanın değişimlere anlık olarak tepki verdiğini söylemektedir. Bu çelişkinin çözümü Einstein'ın kütleçekim teorisinin doğumuna yol açmaktadır. Genel görelilik uzayzamanı bir bütün olarak ele almakta ve yerçekimi algısının yerine uzayzamanın eğriliğini yerleştirmektedir. Bu görevi yapabilmek ve teoriyi iyi anlayabilmek için, eğri uzay zamanlarla alakalı pek çok kişi için yeni olan bazı geometrik araçları tanımak gerekmektedir. Bu amaçla eğri uzayları açıklayan manifoldlar ilk başta tanıtılmıştır. Daha sonra, farklı gözlemcileri birbirine bağlayan koordinat dönüşümleri için faydalı bir araç olan tensorler tanıtılmıştır. Uzayzamanın yapısı hakkında bilgi edinebileceğimiz metrik kavramı sunulmuştur. Objelerin uygun dönüşebilmesi için yeni türevler tanıtılmıştır. Metrik ile ilgili olan ve uzayzamanın yapısı hakkında bilgi elde edebileceğimiz bir diğer obje olan eğrilik tensörüde sunulmuştur. Ayrıca, metriğin Veilbein'ler ile ifade edilebileceği de gösterilmiştir. Bir metrik için birden fazla Vielbein çözümleri bulunmakta ve bu çözümler birbirlerine Lorentz dönüşüm matrisleriyle bağlanmaktadır. Vielbeinler bize ayrıca Lorentz tensörleri tanımlama imkanı da vermektedir. Bu tensorler koordinat dönüşümlerinden etkilenmeyip, Vielbein tercihlerine göre dönüşmektedirler. Lorentz tensörleri de kendilerine ait bağlantı ve kovaryant türevlere sahiptirler. Daha sonra bağlantılar tanımlanıp, aralarındaki ilişki Vielbeinler ile kurulmuştur. Simetrik bağlantılar genel görelilikte kulanılan özel bir bağlantıya yol açmaktadır. En sonunda, doğru objelerin kullanılmasıyla, Einstein'ın kütleçekim teorisi tamamlanıp sunulmuştur.

Üçüncü bölümde simetrileri çalışabilmek için sistematik bir yöntem geliştirilmiştir. Poincare grubunun Lie cebiri elde edilmiştir. Değişik objelerin sonsuz küçük dönüşüm kuralları verilmiştir. Daha sonra evrensel simetrilerin nasıl yerelleştireleceği anlatılmıştır. Bu prosedür ayarlama olarak bilinir ve ayar alanlarını gerektirmektedir. Yerelleşen simetrileri korumak için türevlerin kovaryant türevlerle değiştirilmesi gerekmektedir. Bu aşamada ayar alanları devreye girmekte ve özel dönüşüm kurallarıyla simetrileri korumaktadır. Ayar teorileri için eğrilik objesinin tanımı yapılmış ve dönüşüm kuralları verilmiştir. Ayar teorilerinde her bir simetri için bir de eğrilik vardır. Bu eğrilik Bianchi eşitliğini sağlamaktadır.

Takip eden bölümde, ayar teorisinin genel görelilik teorisiyle nasıl eşleştiği anlatılmıştır. Ayar teorilerinin simetrilerini, genel koordinat dönüşümlerine bağlayan önemli bir denklik sunulmuştur. Bu görevi yapabilmek için tabi ki de bazı koşullar uygulanması gerekmektedir. Daha sonra ayar teorilerinden Einstein'ın kütleçekim teorisinin nasıl elde edildiği gösterilmiştir. Ayar teorisinden Einstein'ın kütle çekim teorisini elde etmek için öteleme simetrisinin eğriliğinin sıfıra eşitlenmesi gerekmektedir. Bu eşitlik, geometrik olarak karşımıza çıkan Cartan yapı denklemiyle aynı denklemi vermekte ve ayar teorisindeki iki ayar alanını, genel görelilikteki Vielbein ve spin bağlantı alanının birbirine bağlandığı gibi biribirine bağlamaktadır. Bu eşitlik ayrıca bize bir tane bağımsız ayar alanı bırakmaktadır. Daha sonra bu ayar alanının dönüşüm özellikleri incelenerek aslında Vielbein olduğu gösterilmeye çalışılmıştır. Bunların uygulanmasıyla genel görelilikteki tüm objeler Poincare cebrinin ayar edilmesiyle elde edilebilmektedir. Daha sonre Einstein hareket denkleminin öne sürülmesiyle teori tamamlanmaktadır.

Aynı Einstein'ın kütleçekimini uzay zamanın geometrisinin içine yerleştirdiği gibi, klasik olarakta, Newton'un kütleçekim teorisi uzay zaman geometrisinin içine yerleştirilebilmektedir. Bu yeni formulasyonu ilk olarak E. Cartan 1920'li yıllarda ortaya sunmustur. Bu teori Newton-Cartan teorisi ismiyle bilinmektedir.

Genel göreliliğin aksine, bu prosedür iki tane dejenere metrik gerektirmektedir. Bu iki metrik, metrik uygunluk koşullarıyla ve simetrik bağlantı şartıyla bize tek bir bağlantı vermemektedir. Bu şartlar genel görelilikte tek bir bağlantı için yeterli olmaktadır.

Yukarıda bahsedilen şartlar altında, bulunan en genel bağlantının üzerine iki koşul verilerek, Newton'un teorisine ulaşılabilmektedir. Bu koşullardan biri Trautman diğeri de Ehler koşuludur.

İki dejenere metrik ve Ehler ile Trautman koşullarıyla, Newton'un kütleçekim teorisi uzay zamanın içine gömülebilmektedir.

Bu koşullar altında, özel bir koordinat dönüşümüyle, hem geodezik denklemi hem de Poisson denklemi elde edilip, Newton-Cartan teoreminin, klasik Newton yerçekimine eşitliği gösterilebilmektedir.

Genel göreliliğin ayar teorisi olarak tekrar yazılabildiği gibi, Newton-Cartan teorisi de ayar teorisi olarak elde edilebilmektedir. Poincare cebrinin ayar edilmesiyle gene görelilik, Bargmann cebrinin ayar edilmesi ile de Newton-Cartan teorisi elde edilebilmektedir.

Bargmann cebri, Galilei cebrinin bir merkezcil üreteçle genişletilmiş halidir. Bargmann cebrinin ayar edilmesi ve üstüne Ehler ve Trautman koşullarının uygulanması bize Newton-Cartan teorisini vermektedir. Yalnız, Newton-Cartan teorisinin aksine, burda ayar etme prosedürü bize tek bir bağlantı vermektedir. Hareket denkleminin empoze edilmesiyle teori tamamlanmaktadır.

1. INTRODUCTION

General Relativity (GR) is Einstein's successful theory of gravity such that replacing our perception of gravity with the curvature of spacetime.

Many observations of the Solar system and astronomy were very well explained by Newtonian Mechanics except the precession of Mercury's perihelion. One of the first achievements of GR was to explain the deviation of the precession from the Newtonian gravity prediction.

GR does not only explain the motions of planets. It also describes many other phenomena such as the evolution of the universe. It was shown that Einstein's equations allow solutions for the expanding universe. Thus, Edwin Hubble's discovery of the expansion of the universe was accepted as a direct confirmation of GR by many scientists.

GR also predicts the existence of gravitational waves. According to it, the merger of two astronomical objects such as black holes or neutron stars can lead to gravitational waves. The existence of gravitational waves has been recently confirmed by the Laser Interferometer Gravitational-Wave Observatory (LIGO).

Although GR has been successful to explain many observations, we still have some gaps in our knowledge. Our current understanding of physics separates into two categories, GR and quantum mechanics (QM). Simply put, GR explains one of the four known fundamental interactions today, gravity. Other remaining interactions are explained by Standard Model (SM) which is built on QM. At the intersection of these two theories, there are black holes that have been directly observed by the Event Horizon Telescope (EHT) recently. In order to understand black holes properly, it is needed to unify QM and GR. However, many attempts for this unification gives unwanted infinities.

There are several possibles routes in order to combine GR and QM. One of them is adding more symmetries to Einstein's theory. Those include conformal symmetry, supersymmetry, and superconformal symmetry. Here, considering gravity as a gauge theory has been very successful to construct gravity models. Moreover, the quantization of the gauge theories is a well-understood topic. This is one of the main motivations for us to write gravity as a gauge theory. Furthermore, non-relativistic gauge theories have been contributed to understanding of many phenomena such as quantum fractional Hall effect.

The notion of gauge transformation was first brought to physics by H.Weyl in 1921 [1]. After that, it became a significant tool in modern physics. Subsequently, it is discovered that quantum electrodynamics is a gauge invariant theory [2–4]. Then, with the introduction of non-Abelian gauge field theories by Yang and Mills [5], electroweak theory and quantum chromodynamics was built on gauge field theories [6–8]. Later, they became the most successful theories in physics.

It was shown by E.Cartan that Newton's gravity can be embedded into spacetime known as Newton-Cartan theory [9]. These geometrical view of gravity then improved later [10–14]. It is investigated and showed that GR in classical limit leads to the Newton-Cartan theory [13, 15–17].

An important motivation for studying the Newton-Cartan theory is the Newton-Cartan cosmology. Some complicated problems of GR get easier in the Newtonian approximation such as structure formation in the early universe and cosmic no-hair theorem [18, 19].

2. GENERAL RELATIVITY AS A GEOMETRICAL THEORY

I will mention briefly Newtonian gravity (NG) and why it fails. Eventually, "gravity" will show itself as a thing related to spacetime structure, but not a force, in general relativity (GR).

2.1 Newtonian Gravity

NG can be summarized in two equations. First, how particles behave under gravitational potential Φ,

$$\mathbf{a} = -\boldsymbol{\nabla}\Phi, \tag{2.1}$$

i.e, the second law of Newtonian mechanics, where **a** stands for acceleration. Second, how matter, i.e., mass, generates gravitational potential,

$$\nabla^2\Phi = 4\pi G\rho, \tag{2.2}$$

where G and ρ are Newton's gravitational constant and mass density, respectively. This equation is known as Poisson's equation for gravity.

2.2 Why Newtonian Gravity Fails

Special relativity (SR) contradicts NG. According to the first postulate of SR, physics laws must be the same in all inertial reference frames. However, Poisson's equation is not Lorentz-invariant.

Furthermore, the solutions of Poisson's equations allow the gravitational field propagate instantaneously, which conflicts with the second postulate of SR, nothing can travel faster than light.

In NG, mass is seen as the "amount of matter" and it is responsible for gravitational interaction. However, SR says that mass is a kind of energy. It is known that hot gases weigh heavier than cold gases, rotating objects also weigh more than steady ones. Moreover, gravity bends light trajectory. All of those observations suggest that rather than mass, energy and motion are responsible for gravity as Einstein's equation states in GR.

2.3 Equivalence Principle

It is known that all bodies behave in the same way independent of its composition. Also, Einstein realized freely falling bodies feel weightless. Thus, he concluded there is no experiment to distinguish uniform acceleration from the gravitational field in short distances for observers. Short distances are required to keep gravitational field uniform. Gravity actually behaves like fictitious force. This is a sign that it is a different thing rather than force. The same thing is not true for electromagnetic force because of electromagnetic charge. Thus inertial frames are updated as "freely falling" frames. Consequently, this led to Einstein think that, in small regions of spacetime, physics must reduce to SR laws, in other words, physics in the "freely falling" frames are the same in the absence of gravity, known as Equivalence Principle. This principle shows itself in GR as "metric" of spacetime must be reducible to the "Minkowski metric" at every point of spacetime.

2.4 Differential Geometry

2.4.1 Manifolds

N dimensional manifolds are simply spaces that you can find one to one maps from a point on them to coordinates $(x^1, x^2, \ldots, x^N)$. Those maps are called coordinate charts. A point can be represented by more than one chart. Coordinate transformations $x'^\mu(x^\nu)$ must be invertible and analytic.

2.4.2 Tensors

Tensors are objects transforming with a particular rule under general coordinate transformations such as,

$$f'(x') = f(x),$$
$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x),$$
$$w'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} V_{\nu}(x),$$
$$T'^{\mu}{}_{\nu\rho}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\rho}} T^{\alpha}{}_{\beta\gamma}(x). \tag{2.3}$$

Those objects, $f, V^{\mu}, w_{\nu}, T^{\mu}{}_{\nu\rho}$, are called scalar, contravariant vector, covariant vector, and mixed $(1,2)$ type tensor, respectively. They have properties like product of two tensors,with type (p,q) and (r,s), yields a type $(p+q, r+s)$ tensor. Also, contraction (i.e. summation) of indices gives a related type tensor. Einstein summation convention will be used for summation of indices throughout this thesis.

2.4.3 Lie derivatives

Lie derivative is an operator such that when applied to a tensor, it gives a same type tensor. Its definition requires a vector field.

For scalars, it is defined as

$$\mathscr{L}_V \phi(x) \equiv V^{\mu}(x) \partial_{\mu} \phi(x) \tag{2.4}$$

Its definition for relevant type tensors goes as follow,

$$\mathscr{L}_V U^{\mu} \equiv V^{\lambda} \partial_{\lambda} U^{\mu} - (\partial_{\lambda} V^{\mu}) U^{\lambda},$$
$$\mathscr{L}_V W_{\mu} \equiv V^{\lambda} \partial_{\lambda} W_{\mu} + (\partial_{\mu} V^{\lambda}) W_{\lambda},$$
$$\mathscr{L}_V T^{\mu}_{\nu} \equiv V^{\lambda} \partial_{\lambda} T^{\mu}_{\nu} - (\partial_{\lambda} V^{\mu}) T^{\lambda}_{\nu} + (\partial_{\nu} V^{\lambda}) T^{\mu}_{\lambda}. \tag{2.5}$$

For the coordinate transformations $x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$, the infinitesimal transformations of tensors can be calculated. They are given by Lie derivatives

$$\delta\phi(x) = \phi'(x) - \phi(x) = \mathscr{L}_{\xi}\phi,$$
$$\delta U^{\mu}(x) = U'^{\mu}(x) - U^{\mu}(x) = \mathscr{L}_{\xi} U^{\mu},$$
$$\delta V_{\mu}(x) = V'_{\mu}(x) - V_{\mu}(x) = \mathscr{L}_{\xi} V_{\mu},$$
$$\delta T^{\mu}_{\nu}(x) = T'^{\mu}_{\nu}(x) - T^{\mu}_{\nu}(x) = \mathscr{L}_{\xi} T^{\mu}_{\nu} \tag{2.6}$$

2.4.4 Metric

Metric is a symmetric, type (0,2), non-degenerate tensor denoted by $g_{\mu\nu}$. It is related to its inverse $g^{\mu\nu}$ by relation with kronocker delta δ^{μ}_{ν} due to non-degeneracy as follows,

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}. \tag{2.7}$$

This property allow us to raise or lower indices by defining objects like $V^{\mu} \equiv g^{\mu\nu} V_{\nu}$ for any type of tensors.

Metric gives us information about the infinitesimal distance, ds, between two neighboring points on a manifold by the relation,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\,. \tag{2.8}$$

Since it is symmetric, it can be diagonalized by orthogonal transformation, which will allow us to define "frame fields". Its signature, i.e signs of eigenvalues, are invariant under coordinate transformations. We will use metric signature $-++\cdots+$.

2.4.5 Frame field

Since our metric $g^{\mu\nu}$ is symmetric, it can be brought a special form, Minkowski metric $\eta_{ab} = \mathrm{diag}(-1,1,\ldots,1)$, at a point on manifold by an orthogonal transformation with the help of matrices called frame fields $e^{a}_{\mu}(x)$,

$$g_{\mu\nu}(x) = e^{a}_{\mu}(x)\eta_{ab} e^{b}_{\nu}(x). \tag{2.9}$$

Given a metric, there are more than one solution at a point for equation (2.9). Other solutions can be found by local Lorentz transformation matrices $\Lambda^{a}{}_{b}(x)$. With our convention and given a frame field $e^{a}_{\mu}(x)$, satisfying (2.9), they become,

$$e'^{a}_{\mu}(x) = \Lambda^{-1a}{}_{b}(x) e^{b}_{\mu}(x). \tag{2.10}$$

From (2.9), it can be shown that frame fields transform under coordinate transformations as covariant vector,

$$e'^{a}_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} e^{a}_{\nu}(x). \tag{2.11}$$

Since frame fields are orthogonal, they are invertible with inverse matrices e^{μ}_{a}, which satisfy relations,

$$e^{\mu}_{a} e^{b}_{\mu} = \delta^{b}_{a}, \quad \text{and} \quad e^{\mu}_{a} e^{a}_{\nu} = \delta^{\mu}_{\nu}. \tag{2.12}$$

As a result, its local Lorentz indices (i.e. Latin indices) are raised or lowered by Minkowski metric η_{ab}, where coordinate indices (Greek indices) are handled by general metric $g_{\mu\nu}$.

2.4.6 Covariant derivatives and connections

Usually, ordinary partial differentiation ∂_μ does not produce a tensor when applied to another tensor. In order to achieve a covariant differentiation ∇_μ, we have to define objects transforms in a particular way, which we will call affine connections $\Gamma^\mu_{\nu\rho}$. Its transformation property will compensate the term coming from the differentiation of metric.

Covariant derivative for covariant and contravariant vectors are given as,

$$\begin{aligned} \nabla_\mu U^\nu &= \partial_\mu U^\nu + \Gamma^\nu_{\mu\rho} U^\rho, \\ \nabla_\mu W_\nu &= \partial_\mu W_\nu - \Gamma^\rho_{\mu\nu} W_\rho, \end{aligned} \tag{2.13}$$

with transformation rule for affine connection,

$$\Gamma'^\rho_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \Gamma^\sigma_{\alpha\beta}. \tag{2.14}$$

The rule for covariant differentiation of arbitrary type tensor can be deduced from following example,

$$\nabla_\mu T^{\nu\rho}{}_{\sigma\lambda} = \partial_\mu T^{\nu\rho}{}_{\sigma\lambda} + \Gamma^\nu_{\mu\alpha} T^{\alpha\rho}{}_{\sigma\lambda} + \Gamma^\rho_{\mu\alpha} T^{\nu\alpha}{}_{\sigma\lambda} - \Gamma^\alpha_{\mu\sigma} T^{\nu\rho}{}_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} T^{\nu\rho}{}_{\sigma\alpha}. \tag{2.15}$$

There also objects that transforming with local Lorentz transformation $\Lambda^{-1a}{}_b(x)$, which we call Lorentz tensors, as in the case with Latin indices of (2.10). Latin indices will be used to indicate them. However, not all objects with those Latin indices will be Lorentz tensors. Any type of coordinate tensors like $T^\mu{}_\nu$ can be converted to Lorentz tensors using frame fields, such as $T^\mu{}_\nu \equiv e^\mu_a e^b_\nu T^a{}_b$.

Ordinary partial derivative ∂_μ will also change the transformation properties of Lorentz tensors. In order them to be remain same, we will introduce this time spin connection $\omega_\mu{}^a{}_b$ with a particular transformation property under local Lorentz transformations,

$$\omega'_\mu{}^a{}_b = \Lambda^{-1a} c \partial_\mu \Lambda^c{}_b + \Lambda^{-1a}_c \omega_\mu{}^c{}_d \Lambda_\mu{}^d{}_b. \tag{2.16}$$

Then we can define covariant differentiation for Lorentz tensors D_μ. For covariant and contravariant Lorentz vectors it is given as

$$D_\mu U^a = \partial_\mu U^a + \omega_\mu{}^a{}_b U^b,$$
$$D_\mu V_a = \partial_\mu V_a - \omega_\mu{}^b{}_a V_b. \tag{2.17}$$

For arbitrary type tensors, rule goes as follow,

$$D_\mu T^{ab}{}_{cd} = \partial_\mu T^{ab}{}_{cd} + \omega_\mu{}^a{}_f T^{fb}{}_{cd} + \omega_\mu{}^b{}_f T^{af}{}_{cd} - \omega_\mu{}^f{}_c T^{ab}{}_{fd} - \omega_\mu{}^f{}_d T^{ab}{}_{cf}. \tag{2.18}$$

We can define spin connection's raised or lowered indices forms of Latin indices with Minkowski metric η_{ab}. It is obvious from (2.16) that its Latin indices do not behave like Lorentz tensors. It will be imposed that its Greek index transforms as a coordinate tensor and it is antisymmetric in its Latin indices (i.e. $\omega_\mu{}^a{}_b = -\omega_{\mu b}{}^a$).

Now we can relate two covariant derivative by the relation $\nabla_\mu V^\nu \equiv e^\nu_a D_\mu V^a$. This equation allow us to determine relation between affine and spin connection with frame fields,

$$\Gamma^\rho_{\mu\nu} = e^\rho_a(\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu), \tag{2.19}$$

or, in another form,

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho = 0. \tag{2.20}$$

Here, above, covariant derivative for mixed type tensors are written with both connections. We can also show identity $\nabla_\mu g_{\nu\rho} = 0$ using (2.19).

Since we have just a transformation rule for affine connection, we have many of them. Given a connection $\Gamma^\mu_{\nu\rho}$, $\Gamma^\mu_{\rho\nu}$ will also transform properly, as it can be seen from (2.14).

The difference of two connections, $\bar{\Gamma}^\mu_{\nu\rho}$ and $\Gamma^\mu_{\nu\rho}$, will also transform as a tensor. Thus,given a connection, we can define a new tensor known as Torsion tensor $T^\mu{}_{\nu\rho}$,

$$T^\mu{}_{\nu\rho} \equiv \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu} = 2\Gamma^\mu_{[\nu\rho]}. \tag{2.21}$$

If we add property that affine connection is symmetric in its last two indices (this will make torsion tensor vanish), we can find a unique connection from a metric. Using $\nabla_\mu g_{\nu\rho} = 0$ with that symmetry, one can show that,

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}). \tag{2.22}$$

This connection is known as Christoffel connection.

2.4.7 Cartan structure equation

Furthermore, making the connection torsionless gives us Cartan structure equation. We present it here for future purposes. From the equation (2.19), we make antisymmetric part of the connection vanish $\Gamma^{\rho}_{[\mu\nu]} = 0$, then we have

$$\partial_{[\mu} e^{a}_{\nu]} + \omega_{[\mu}{}^{a}{}_{b} e^{b}_{\nu]} = 0. \tag{2.23}$$

Besides, Cartan structure equation make us able to solve spin connection in terms of frame fields and and their derivatives;

$$\omega_{\mu}{}^{ab}(e, \partial e) = 2e^{\lambda[a} \partial_{[\lambda} e^{b]}_{\mu]} + e_{\mu c} e^{\lambda a} e^{\rho b} \partial_{[\lambda} e^{c}_{\rho]}. \tag{2.24}$$

2.4.8 Parallel transport

Covariant derivative gives us a way for "parallel transport" a tensor. For example, given a vector V^{μ} at a point on our manifold, if we wanted to move it, keeping the vector constant on a curve parametrized by λ, we just would equate its derivative to zero and then, solve the differential equation $\mathrm{d}V^{\mu}/\mathrm{d}\lambda = \mathrm{d}x^{\nu}/\mathrm{d}\lambda \times \partial_{\nu} V^{\mu} = 0$. However, it is not a tensor equation so we just simply replace partial derivative with the covariant one. Now, we can define directional covariant derivative as

$$\frac{\mathrm{D}}{\mathrm{d}\lambda} \equiv \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda} \nabla_{\nu}. \tag{2.25}$$

Then, the equation for parallel transport, for arbitrary rank tensor, just becomes

$$\frac{\mathrm{D}V^{\mu\nu\dots}{}_{\rho\sigma\dots}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\lambda} \nabla_{\alpha} V^{\mu\nu\dots}{}_{\rho\sigma\dots} = 0. \tag{2.26}$$

2.4.9 Riemann curvature tensor, Ricci tensor and Ricci scalar

As we learned in the previous section, covariant derivative is related to parallel transporting a vector. Thus we can extract information about the "curvature" of spacetime using it. The commutator of two covariant derivative ∇_{μ} and ∇_{ν} will give us difference of transporting a vector field, to the same point, using two different way. Then, we find

$$\begin{aligned} [\nabla_{\mu}, \nabla_{\nu}] V^{\rho} &= (\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}) V^{\rho} \\ &= (\partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}) V^{\sigma} - 2\Gamma^{\lambda}_{[\mu\nu]} \nabla_{\lambda} V^{\rho}. \end{aligned} \tag{2.27}$$

Here, the second term is torsion tensor, so the term in the parenthesis must be a tensor, and it is defined as Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$,

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{2.28}$$

Now, if we use Christoffel connection, we can find following identities,

$$\begin{aligned}
R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma}, \quad \text{antisymmetry in first pair,} \\
R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}, \quad \text{antisymmetry in second pair,} \\
R_{\mu\nu\rho\sigma} &= R_{\sigma\rho\mu\nu}, \quad \text{symmetry between second and first pair,} \\
R_{\mu\nu\rho\sigma} + R_{\mu\nu\rho\sigma} + R_{\mu\nu\rho\sigma} &= 0, \quad \text{cyclic symmetry of last three indices.}
\end{aligned} \tag{2.29}$$

The last identity can be written in a more compact form as $R_{\mu[\nu\rho\sigma]} = 0$. There is also an identity known as Bianchi,

$$\nabla_{[\mu}R_{\nu\rho]\sigma\lambda} = 0. \tag{2.30}$$

Contraction of the first and third indices of Riemann tensor gives us Ricci tensor $R_{\mu\nu}$,

$$R_{\mu\nu} \equiv R^{\rho}{}_{\mu\rho\nu}. \tag{2.31}$$

Using identities of Riemann tensor, its symmetry can be shown,

$$R_{\mu\nu} = R_{\nu\mu}. \tag{2.32}$$

Trace of Ricci tensor is known as Ricci scalar R,

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}. \tag{2.33}$$

Einstein tensor $G_{\mu\nu}$ is defined using Ricci tensor and Ricci scalar,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{2.34}$$

It satisfies the relation,

$$\nabla^{\mu}G_{\mu\nu} = 0. \tag{2.35}$$

2.5 Towards GR

In GR, spacetime is a four-dimensional manifold with the signature $-+++$. Spacetime is "locally flat" in the sense that metric can be made equal to the Minkowski metric at every point. Spacetime structure can be deduced from the metric. Thus, as in the Poisson's equation, we need an equation to govern the spacetime structure. In GR, not only mass, but also all forms of energy and motion are responsible for determining the spacetime structure, as we indicated before.

2.6 Energy-Momentum Tensor

Suppose we have N particles, with four momentum for nth particle p_n^μ, and word-lines $y_n^\mu(\lambda)$ parametrized by λ. The energy-momentum tensor $T^{\mu\nu}(x^\rho)$ is given by

$$T^{\mu\nu}(x^\rho) = \sum_{n=1}^{N} \int \frac{p_n^\mu p_n^\nu}{p_n^0} \frac{\mathrm{d}x^0}{\mathrm{d}\lambda} \frac{\delta^{(4)}(x^\rho - y_n^\rho(\lambda))}{\sqrt{-g}} \mathrm{d}\lambda\,, \tag{2.36}$$

where g is determinant of $g_{\mu\nu}$ and $\delta^{(4)}(x^\mu)$ is four-dimensional Dirac Delta function. This definition ensures that it is symmetric (i.e. $T^{\mu\nu} = T^{\nu\mu}$)

It can be shown that the 00 component of energy-momentum tensor gives energy density. The $0i$ components give momentum density and the ij components give momentum flux density in the direction i projected on j.

2.7 Einstein's Equation

Now, we can present Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.37}$$

where G is Newton's gravitational constant.

It is analog of the Poisson's equation in Newtonian Gravity. The equation allows us to determine the spacetime structure in the presence of energy and motion. It is a set of non-linear differential equations for metric $g_{\mu\nu}$ as function with variables as spacetime coordinates x^μ.

2.8 Einstein-Hilbert Action

As in most of the theories, Einstein's equation can be derived from an action,

$$S_{EH} = \int \mathrm{d}^4x \sqrt{-g}R. \tag{2.38}$$

where R and g are Ricci scalar and determinant of the metric, respectively. This action is called Einstein-Hilbert action. Its variation with respect to metric gives,

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{2.39}$$

Of course, since Einstein-Hilbert action includes second order derivatives of metric, in order to get (2.39), one must also set variations of first order derivatives of metric as well as its variation, at the boundary. Or, that term can be eliminated by adding another term to the action.

By setting the variation (2.39) to zero, one can get Einstein's equation in vacuum. In order to get full equation, we must add a term S_M to become responsible for matter part of the Einstein's equation. Now we can introduce action in the following form

$$S = \frac{1}{16\pi G}S_{EH} + S_M. \tag{2.40}$$

Taking variation of it, we can simply show that for the right equation, variation of matter term must be,

$$\frac{\delta S_M}{\delta g^{\mu\nu}} = -\frac{1}{2}T_{\mu\nu}\sqrt{-g}. \tag{2.41}$$

2.9 Geodesic Equation

Once we solved Einstein's equation of motion, we know the structure of spacetime and can determine the particles paths (worldlines) using geodesic equation for free particles. It is the analog of Newton's second equation in Gr. We present it;

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0, \tag{2.42}$$

where λ is a scalar wordline parameter and $\Gamma^\mu_{\nu\rho}$ is the symmetric Christoffel connection.

It can be extended to include Lorentz Force.

3. NON-ABELIAN GAUGE THEORIES

Gauge theories are the field theories having local symmetries. Promoting global symmetries to local ones require objects called gauge fields. Since localized parameters cannot escape derivatives anymore, we have to change derivatives with "covariant" ones with gauge fields having particular transformation properties. Non-abelian is the property that the symmetry group is not commutative.

3.1 Global Symmetries

Global symmetries are symmetries that transformation parameters ε^A's do not depend on the spacetime coordinates. The letters $A,B,C,\dots$ denotes the related symmetries. As in our case, linear infinitesimal symmetry transformations δ_ε are given as

$$\delta_\varepsilon = \varepsilon^A T_A, \tag{3.1}$$

where T_A's are symmetry generators. T_A's act on the space of fields.

We will develop the algebra using internal symmetries but the formalism can also be applied spacetime symmetries. First, we have matrices t_A's as Lie algebra representation of operators T_A's, with the following commutation relations

$$[t_A, t_B] = f_{AB}{}^C t_C. \tag{3.2}$$

$f_{AB}{}^C$'s are called structure constants, and they are obviously antisymmetric in the indices A and B.

Assume we have fields ϕ^is, transform under this representation as

$$T_A\phi^i = -t_A{}^i{}_j\phi^j. \tag{3.3}$$

This relation ensures the identity

$$[T_A, T_B] = f_{AB}{}^C T_C. \tag{3.4}$$

For successive transformations, transformations acts on the dynamical fields in our convention. Then, we can find the commutator of two infinitesimal transformation,

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\phi^i = \varepsilon_2^B \varepsilon_1^A f_{AB}{}^C T_C \phi^i = \delta_{\varepsilon_3}\phi^i, \tag{3.5}$$

where $\varepsilon_3^C = \varepsilon_2^B \varepsilon_1^A f_{AB}{}^C$. Both results (3.4) and (3.5) can be extended to include spacetime symmetries, i.e. translations and Lorentz transformations.

3.2 Translations and Lorentz Group

Our convention for Lorentz transformations with the transformation matrices $\Lambda^\mu{}_\nu$'s goes as

$$x'^\mu = \Lambda^{-1\mu}{}_\nu x^\nu. \tag{3.6}$$

Lorentz transformation preserve the Minkowski norm of any vector. Then, we have

$$\Lambda^{-1\mu}{}_\rho \eta_{\mu\nu} \Lambda^{-1\nu}{}_\sigma = \eta_{\rho\sigma}. \tag{3.7}$$

This relation defines the Lorentz transformations.

In order to find the Lie algebra of Lorentz transformations, we must expand the $\Lambda^\mu{}_\nu$'s around identity matrix,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon m^\mu{}_\nu + \dots. \tag{3.8}$$

If we put (3.8) in (3.7), we can find the relation

$$m_{\mu\nu} = -m_{\nu\mu}. \tag{3.9}$$

The indices of $m^\mu{}_\nu$ are raised or lowered by Minkowski metric. The identity (3.9) gives us $D(D+1)/2$ constraints for the matrix elements $m^\mu{}_\nu$'s. Thus we have $D(D-1)/2$ independent matrix elements. For this reason, we have $D(D-1)/2$ independent parameters $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$ with basis matrices $m_{\mu\nu}{}^\rho{}_\sigma = -m_{\nu\mu}{}^\rho{}_\sigma$. The indices of $\lambda^{\mu\nu}$ are also raised or lowered by Minkowski metric. Here, the first two indices indicate the name of matrices while remaining two their matrix elements. A useful basis are given as

$$m_{\mu\nu}{}^\rho{}_\sigma = \delta_\mu^\rho \eta_{\sigma\nu} - \delta_\nu^\rho \eta_{\mu\sigma}. \tag{3.10}$$

Now, we can introduce finite Lorentz transformation by applying infinite times the infinitesimal ones,

$$\Lambda = e^{\frac{1}{2}\lambda^{\mu\nu} m_{\mu\nu}}. \tag{3.11}$$

The $1/2$ factor are inserted to prevent double counting.

Now we can derive the commutation relations for matrices $m_{\mu\nu}$ and its structure constant using (3.10),

$$\left[m_{\mu\nu}, m_{\rho\sigma}\right] = 4\eta_{[\rho[\nu} m_{\mu]\sigma]}. \tag{3.12}$$

Thus, the structure constant becomes

$$f_{[\mu\nu][\rho\sigma]}{}^{[\alpha\beta]} = 8\eta_{[\rho[\nu}\delta^{[\alpha}_{\mu]}\delta^{\beta]}_{\sigma]}. \tag{3.13}$$

If the symmetry indices in (3.4) are replaced by antisymmetric indices, then one must insert $1/2$ factor to prevent double counting as we did finding the structure constant above.

We will be interested in active transformations, so will let the fields change not coordinates. For the scalar fields, infinitesimal Lorentz transformation generator $M_{\mu\nu}$ is given as

$$M_{\mu\nu} = 2x_{[\mu}\partial_{\nu]}. \tag{3.14}$$

Its commutators are isomorphic to (3.12). It can be easily verified that generator $M_{\mu\nu}$'s correspond to infinitesimal Lorentz transformations,

$$\delta_\lambda\phi = \phi'(x) - \phi(x) = -\frac{1}{2}\lambda^{\mu\nu}M_{\mu\nu}\phi = -\lambda^{\mu\nu}x_\mu\partial_\nu\phi. \tag{3.15}$$

One can also show that Lorentz transformation generators for covariant and contravariant vector field are given as

$$\begin{aligned} J_{\rho\sigma}V^\mu(x) &\equiv (M_{\rho\sigma}\delta^\mu_\nu + m_{\rho\sigma}{}^\mu{}_\nu)V^\nu(x), \\ J_{\rho\sigma}W_\nu(x) &\equiv (M_{\rho\sigma}\delta^\mu_\nu + m_{\rho\sigma\nu}{}^\mu)W_\mu(x), \end{aligned} \tag{3.16}$$

with infinitesimal transformations

$$\begin{aligned} \delta_\lambda V^\mu &= -\frac{1}{2}\lambda^{\rho\sigma}J_{\rho\sigma}V^\mu = -\lambda^{\rho\sigma}x_\rho\partial_\sigma V^\mu - \lambda^\mu{}_\nu V^\nu, \\ \delta_\lambda W_\mu &= -\frac{1}{2}\lambda^{\rho\sigma}J_{\rho\sigma}W_\mu = -\lambda^{\rho\sigma}x_\rho\partial_\sigma W^\mu - \lambda_\mu{}^\nu W_\nu. \end{aligned} \tag{3.17}$$

For space time translations $x'^\mu = x^\mu - a^\mu$, corresponding generators are $P_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}$ with parameters a^μ. Since partial derivatives commute, its structure constant vanishes.

Finally, the Poincare group is defined by the composite transformation

$$x'^\mu = \Lambda^{-1\mu}{}_\nu(x^\nu - a^\nu). \tag{3.18}$$

Then, one can find the following infinitesimal transformation rules for the Poincare transformation,

$$\begin{aligned} \delta_{\lambda,a}\phi &= (a^\nu - \lambda^{\mu\nu}x_\mu)\partial_\nu\phi, \\ \delta_{\lambda,a}V^\mu &= (a^\sigma - \lambda^{\rho\sigma}x_\rho)\partial_\sigma V^\mu - \lambda^\mu{}_\nu V^\nu, \\ \delta_{\lambda,a}W_\mu &= (a^\sigma - \lambda^{\rho\sigma}x_\rho)\partial_\sigma W^\mu - \lambda_\mu{}^\nu W_\nu. \end{aligned} \tag{3.19}$$

The Lie algebra for Poincare group includes $D(D+1)/2$ generators with the following commutators

$$\begin{aligned} \left[J_{\mu\nu}, J_{\rho\sigma}\right] &= 4\eta_{[\rho[\nu}J_{\mu]\sigma]}, \\ \left[J_{\rho\sigma}, P_{\mu}\right] &= 2\eta_{\mu[\sigma}P_{\rho]}, \\ \left[P_{\mu}, P_{\nu}\right] &= 0. \end{aligned} \tag{3.20}$$

3.3 Local Symmetries

In local symmetries, parameters become coordinate dependent $\varepsilon^A(x)$. In order to compensate terms coming from derivatives of parameters, one needs to introduce new covariant derivatives with gauge fields B_μ^A's. For this reason, we will define covariant quantities and special transformation rules for gauge fields to preserve symmetries.

A covariant quantity is defined as a local function that transforms under all local symmetries without the derivatives of the transformation parameter.

Here, we define covariant derivative D_μ in the following way

$$D_\mu \equiv \partial_\mu - \delta_{B_\mu}, \tag{3.21}$$

where δ_{B_μ} means compute all infinitesimal gauge transformations with parameters replaced by B_μ^A.

The right transformation rule for gauge fields B_μ^A is

$$\delta_\varepsilon B_\mu^A = \partial_\mu \varepsilon^A + \varepsilon^C B_\mu^B f_{BC}{}^A. \tag{3.22}$$

It can be easily shown for scalar field set ϕ^i, with localized internal symmetry having generators T_A, that the quantity $D_\mu \phi^i$ transforms without the derivative of transformation parameter $\varepsilon^A(x)$.

One must include all symmetries in the prescriptions above.

Now, we can introduce parameters and gauge fields for translations, Lorentz transformations and internal symmetries in the table (3.1).

We will also introduce commutators, structure constants and third parameters for our gauge symmetries in the table (3.2).

Table 3.1 : Gauge parameters and fields.

gauge symmetry T_A	paremeter ε^A	gauge field B^A_μ
translations P_a	ξ^a	e^a_μ
Lorentz Transformations M_{ab}	λ^{ab}	${\omega_\mu}^{ab}$
internal symmetry T_A	θ^A	A^A_μ

Table 3.2 : Commutators, structure constants and third parameters.

commutators	structure constants	third parameters
$[M_{ab}, M_{cd}] = 4\eta_{[a[c}M_{d]b]}$	${f_{[ab][cd]}}^{[ef]} = 8\eta_{[c[b}\delta^{[e}_{a]}\delta^{f]}_{d]}$	$\lambda_3^{ab} = -2\lambda_1{}^{[a}{}_c\lambda_2{}^{b]c}$
$\left[P_a, M_{[bc]}\right] = 2\eta_{a[b}P_{c]}$	${f_{a,[bc]}}^d = 2\eta_{a[b}\delta^d_{c]}$	$\xi_3^a = -\lambda_2^{ab}\xi_{1b} + \lambda_1^{ab}\xi_{2b}$
$[P_a, P_b] = 0$		

3.4 Local Poincare Transformations

We have seen that Poincare transformation for scalar fields are given by $\delta_{\lambda,a}\phi = (a^\mu + \lambda^{\mu\nu}x_\nu)\partial_\mu\phi$. While localizing symmetry, we can put the effect of both parameters $a^\mu(x)$ and $\lambda^{\mu\nu}(x)$ into one parameter $\xi^\mu(x) = a^\mu + \lambda^{\mu\nu}x_\nu$. Then local translations for scalar fields are given by

$$\delta_\xi\phi(x) = \xi^\mu(x)\partial_\mu\phi(x). \tag{3.23}$$

This exactly corresponds to the general coordinate transformations with infinitesimal parameter $\xi^\mu(x)$.

Including local Lorentz transformation $\lambda^{ab}(x)$ with local translations $\xi^\mu(x)$, we define the local Poincare transformations. Since scalar fields have no Lorentz indices, its transformation property remains same,

$$\delta_{\xi,\lambda}\phi(x) = \xi^\mu(x)\partial_\mu\phi(x). \tag{3.24}$$

For contravariant vector fields V^μ, local Poincare transformation is given by only Lie derivative (as we did in the previous chapter) since they have no Lorentz indices,

$$\delta_{\xi,\lambda}V^\mu = \mathscr{L}_\xi V^\mu = \xi^\nu\partial_\nu V^\mu - (\partial_\nu\xi^\mu)V^\nu. \tag{3.25}$$

For covariant vector fields W^μ, local Poincare transformation is given by only Lie derivative again for the same reason,

$$\delta_{\xi,\lambda}W_\mu = \mathscr{L}_\xi W_\mu = \xi^\nu\partial_\nu W_\mu + (\partial_\mu\xi^\nu)W_\nu. \tag{3.26}$$

Frame field transforms with both Lie derivative and local Lorentz,

$$\delta_{\xi,\lambda} e^a_\mu = \xi^\rho \partial_\rho e^a_\mu + (\partial_\mu \xi^\rho) e^a_\rho - \lambda^a{}_b e^b_\mu. \tag{3.27}$$

Infinitesimal transformation of the spin connection can be deduced from (2.16) (also not forgetting its Greek index),

$$\delta_{\xi,\lambda} \omega_\mu{}^{ab} = \xi^\nu \partial_\nu \omega_\mu{}^{ab} + (\partial_\mu \xi^\nu) \omega_\nu{}^{ab} + \partial_\mu \lambda^{ab} - \lambda^{ac} \omega_{\mu c}{}^b + \omega_\mu{}^a{}_c \lambda^{cb}. \tag{3.28}$$

Frame vectors are related to the coordinate vectors, $V^\mu = e^\mu_a V^a$. Thus we can find their transformations using (3.25), (3.26) and (3.27). They are given as

$$\begin{aligned} \delta_{\xi,\lambda} V^a &= \xi^\mu \partial_\mu V^a - \lambda^a{}_b V^b, \\ \delta_{\xi,\lambda} W_a &= \xi^\mu \partial_\mu W_a - \lambda_a{}^b W_b. \end{aligned} \tag{3.29}$$

3.5 Curvatures

In gauge theories, curvatures are defined using the covariant derivatives,

$$[D_\mu, D_\nu] = -\delta_{R_{\mu\nu}}, \tag{3.30}$$

$$R_{\mu\nu}{}^A = 2\partial_{[\mu} B_{\nu]}{}^A + B_\nu{}^C B_\mu{}^B f_{BC}{}^A. \tag{3.31}$$

As in the covariant derivatives, $\delta_{R_{\mu\nu}}$ means compute gauge transformations of the quantities with transformation parameters replaced by $R_{\mu\nu}{}^A$.

It can be shown that curvatures transform as

$$\delta_\varepsilon R_{\mu\nu}{}^A = \varepsilon^c R_{\mu\nu}{}^B f_{BC}{}^A. \tag{3.32}$$

Thus, curvatures are covariant quantities.

They also satisfy the Bianchi identities,

$$D_{[\mu} R_{\nu\rho]}{}^A = 0. \tag{3.33}$$

4. GENERAL RELATIVITY AS A GAUGE THEORY

In this chapter, we will show how GR arises as gauging the Poincare algebra.

The following identity can be proven to hold for gauge theories

$$0 = \delta_{gct}(\xi^\lambda)B_\mu^A + \xi^\lambda R_{\mu\nu}{}^A - \sum_{\{C\}} \delta(\xi^\lambda B_\lambda^C)B_\mu^A. \tag{4.1}$$

where parenthesis show transformation parameters. Thus for P transformation we have

$$\delta_{gct}(\xi^\lambda)e_\mu^a = \delta_P(\xi^\lambda e_\lambda^b)e_\mu^a - \xi^\lambda R_{\mu\nu}{}^a(P) - \delta_M(\xi^\lambda \omega_\lambda{}^{bc})e_\mu^a. \tag{4.2}$$

Now, relating the parameters ξ^λ and ξ^a by

$$\xi^\lambda = e_a^\lambda \xi^a, \tag{4.3}$$

and making curvature of P transformations zero, we find

$$\delta_{gct}(\xi^\lambda)e_\mu^a = \delta_P(\xi^b)e_\mu^a - \delta_M(\xi^\lambda \omega_\lambda{}^{bc})e_\mu^a. \tag{4.4}$$

Hence, we found that if a gauge field transforms under P transformations, we can relate it to the general coordinate transformations with other gauge symmetries of the algebra, by making its curvature equal to zero.

4.1 Frame Fields and Gauge Fields

We want to show the gauge fields for P transformations e_μ^a and M transformations $\omega_\mu{}^{ab}$ are actually frame field and spin connection, respectively. This is the reason why we used the same symbols for them in the previous chapter. We will show it calculating their infinitesimal transformations.

Using the prescription (3.22), we can find the gauge transformation of the gauge field e_μ^a,

$$\delta e_\mu^a = \partial_\mu \xi^a + \xi^b \omega_\mu{}^a{}_b - \lambda^a{}_b e_\mu^b. \tag{4.5}$$

We will also assume the gauge field e_μ^a is invertible in the way that $e_a^\mu e_\mu^b = \delta_a^b$ and $e_a^\mu e_\nu^a = \delta_\nu^\mu$.

Here, we present curvature for translations $R_{\mu\nu}(P^a)$ using (3.30),

$$R_{\mu\nu}(P^a) = 2\partial_{[\mu} e^a_{\nu]} + 2\omega_{[\mu}{}^a{}_b e^b_{\nu]}. \tag{4.6}$$

Making it equal zero, we get the first Cartan structure equation and, thus, connect the gauge field e^a_μ and $\omega_\mu{}^a{}_b$ in the same way as we did geometrically. Thus, if we show now that the gauge field e^a_μ transforms in the same way as frame field, we also prove that the gauge field $\omega_\mu{}^a{}_b$ is actually the spin connection.

Now, we can calculate the transformation of the frame field under the translations and local Lorentz transformation. Since the frame field transforms as a coordinate tensor in its Greek index, it will transform with the Lie derivative under translations. There will be also a term coming from local Lorentz transformation. Thus, we have

$$\delta e^a_\mu = \xi^\rho \partial_\rho e^a_\mu + (\partial_\mu \xi^\rho) e^a_\rho - \lambda^a{}_b e^b_\mu. \tag{4.7}$$

Playing with this equation and defining $\lambda'^a{}_b + \xi^\rho \omega_\rho{}^a{}_b$, we can show it is equal to

$$\delta e^a_\mu = \partial_\mu \xi^a + \xi^b \omega_\mu{}^a{}_b - \lambda'^a{}_b e^b_\mu + 2\xi^\rho R_{\rho\mu}(P^a). \tag{4.8}$$

The last term is zero from the Cartan structure equation. Thus, the two equations, (4.5) and (4.7), are equal to each other. We have just proved our assertion.

4.2 Curvatures

Now we will show that curvature for the M transformation $R_{\mu\nu}(M^{ab})$ is closely related to the Riemann curvature tensor.

We begin by presenting the curvature $R_{\mu\nu}(M^{ab})$ using the (3.30),

$$R_{\mu\nu}(M^{ab}) = 2\partial_{[\mu} \omega_{\nu]}{}^{ab} + 2\omega_{[\mu c}{}^{[a} \omega_{\nu]}{}^{b]c}. \tag{4.9}$$

For our purpose, we give the following quantity;

$$e^\sigma_b e^\rho_a R_{\mu\nu}(M^{ab}) = 2e^\sigma_b e^\rho_a \partial_{[\mu} \omega_{\nu]}{}^{ab} + 2e^\sigma_b e^\rho_a \omega_{[\mu c}{}^a \omega_{\nu]}{}^{bc}. \tag{4.10}$$

Now, we need to state Riemann tensor in terms of frame fields and spin connections. For this reason, we present the two identity from previous chapters,

$$R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{4.11}$$

$$\Gamma^\rho_{\mu\nu} = e^\rho_a \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^\rho_a e^b_\nu. \tag{4.12}$$

If we insert (4.12) into the (4.11), we find

$$R_{\mu\nu}{}^{\rho\sigma} = 2e^{\sigma}_{b}e^{\rho}_{a}\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2e^{\sigma}_{b}e^{\rho}_{a}\omega_{[\mu c}{}^{a}\omega_{\nu]}{}^{bc}. \tag{4.13}$$

This is the same equation as (4.10). Thus, we have just found that curvature for M transformations in our gauge theory appears the Riemann curvature tensor,

$$R_{\mu\nu}{}^{\rho\sigma} = e^{\rho}_{a}e^{\sigma}_{b}R_{\mu\nu}(M^{ab}). \tag{4.14}$$

4.3 Γ Connection

Now, we impose Vielbein postulate to solve for Γ connection from our gauge theory

$$\nabla_{\mu}e^{a}_{\nu} \equiv \partial_{\mu}e^{a}_{\nu} - \Gamma^{\rho}_{\nu\mu}e^{a}_{\rho} + \omega_{\mu}{}^{a}{}_{b}e^{b}_{\nu} = 0. \tag{4.15}$$

Using this, we find

$$\Gamma^{\rho}_{\nu\mu} = e^{\rho}_{a}(\partial_{\mu}e^{a}_{\nu} + \omega_{\mu}{}^{a}{}_{b}e^{b}_{\nu}). \tag{4.16}$$

This is the same connection as we found geometrically. Furthermore, it is symmetric due to curvature for P transformations giving us Cartan structure equation. Cartan equation also makes our gauge field e^{a}_{μ} only independent field.

We can continue building elements of GR by defining metric and its inverse by relations,

$$g_{\mu\nu} = e^{a}_{\mu}\eta_{ab}e^{b}_{\nu}, \quad \text{and} \quad g^{\mu\nu} = e^{\mu}_{a}\eta^{ab}e^{\nu}_{b}. \tag{4.17}$$

Eventually, we have all the elements of GR. Thus, we related our gauge theory to the GR. Now, we can finish our gauge theory formulation by just imposing Einstein's equations of motion.

5. NEWTON-CARTAN GRAVITY

Newton's gravity can be embedded into spacetime geometry as in GR. The equation of motion of a particle in the absence of other forces are given by

$$\ddot{x}^i + \partial^i \phi = 0. \tag{5.1}$$

This equation can be converted to geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}t^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} \frac{\mathrm{d}x^\rho}{\mathrm{d}t} = 0, \tag{5.2}$$

provided

$$\Gamma^i_{00} = \delta^{ij} \partial_j \phi \tag{5.3}$$

is the only non-zero connection coefficients and $x^0 = t$. From this connection, we can find following non-zero curvature terms

$$R^i{}_{0j0} = \delta^{ik} \partial_k \partial_j \phi. \tag{5.4}$$

Hence, we have only one non-vanishing Ricci tensor term R_{00}. If we impose $R_{00} = 4\pi G\rho$, we get Poisson's equation. Thus, we must write a covariant equation of motion and show that it reduces to Poisson's equation with a proper metric.

The connection (5.3) can not be obtained from a non-degenerate metric. It can be seen from that Riemann tensor, defined by that connection, does not satisfy non-degenerate Riemann tensor symmetry properties. Besides, Newtonian limit of Minkowski metric ($c \to \infty$) leads degenerate metrics;

$$\eta_{\mu\nu}/c^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1_3/c^2 \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} -1/c^2 & 0 \\ 0 & 1_3 \end{pmatrix}. \tag{5.5}$$

Thus, we have two degenerate metric $\tau_{\mu\nu}$ with three zero eigenvalues , and $h^{\mu\nu}$ with one zero eigenvalues. Since $\tau_{\mu\nu}$'s rank is 1, it is effectively 1×1 matrix. Hence, we can decompose it $\tau_{\mu\nu} = \tau_\mu \tau_\nu$. We can see from (5.5) that $h^{\mu\nu}\tau_{\nu\rho} = 0$ so we have $h^{\mu\nu}\tau_\nu = 0$.

Next step is to impose metric compatibility conditions.

$$\nabla_\mu h^{\nu\rho} = 0, \quad \text{and} \quad \nabla_\mu \tau^\nu = 0. \tag{5.6}$$

The second equation implies

$$\tau_\mu = \partial_\mu f(x) \tag{5.7}$$

where $f(x)$ is a scalar field.

In GR metric compatibility conditions with a symmetry requirement gives us a unique connection. However, here, the situation is a little different.

The connection's shift

$$\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + h^{\rho\lambda} K_{\lambda(\mu}\tau_{\nu)}, \tag{5.8}$$

with an arbitrary two form $K_{\mu\nu}$, still satisfies (5.6).

We can write most general connection compatible with (5.6) using $K_{\mu\nu}$. For this reason, we introduce inverse metrics $h_{\mu\nu}$ and τ^μ by relations

$$\begin{aligned} h_{\mu\nu}h^{\nu\rho} = \delta^\rho_\mu - \tau^\rho \tau_\mu, \qquad \tau^\mu \tau_\mu = 1, \\ h_{\mu\nu}\tau^\nu = 0, \qquad h^{\mu\nu}\tau_\nu = 0. \end{aligned} \tag{5.9}$$

From these relations, we find

$$\Gamma^\rho_{\mu\nu} = \tau^\rho \partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}(\partial_\nu h_{\sigma\mu} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu}) + h^{\sigma\lambda}K_{\lambda(\mu}\tau_{\nu)}. \tag{5.10}$$

This connection is rather different than the one (5.3) that we want to reach. In order to achieve that we must impose two extra conditions.

At first, we must use adapted coordinates where $f = t$ in (5.7). Then, from relations (5.9), we have

$$\begin{aligned} \tau_\mu = \delta^0_\mu, \quad \tau^\mu = (1, \tau^i), \\ h^{\mu 0} = 0, \quad h_{\mu 0} = -h_{\mu i}\tau^i. \end{aligned} \tag{5.11}$$

The relations above are preserved under the coordinate transformations,

$$\begin{aligned} &x^0 \to x^0 + \zeta^0, \\ &x^i \to x^i + \xi^i(x^\mu). \end{aligned} \tag{5.12}$$

In adapted coordinates, connection coefficients become

$$\begin{aligned}
\Gamma^i_{00} &= h^{ij}(\partial_0 h_{j0} - \frac{1}{2}\partial_j h_{00} + K_{j0}) \equiv h^{ij}\Phi_j, \\
\Gamma^i_{0j} &= h^{ik}(\partial_0 h_{jk} + \partial_{[j}h_{k]0} - \frac{1}{2}K_{jk}) \equiv h^{ik}(\frac{1}{2}\partial_0 h_{jk} + \Omega_{jk}), \\
\Gamma^i_{jk} &= \begin{Bmatrix} i & \\ j & k \end{Bmatrix}, \\
\Gamma^0_{\mu\nu} &= 0,
\end{aligned} \tag{5.13}$$

where $\begin{Bmatrix} i & \\ j & k \end{Bmatrix}$ are the Christoffel connections with metric h_{ij}.

Now, we introduce equations of motion

$$R_{\mu\nu} = 4\pi G \tau_\mu \tau_\nu. \tag{5.14}$$

In adapted coordinates, only non-zero component of Ricci tensor is R_{00}. $R_{ij} = 0$ implies flat spatial surfaces. Then we can choose $h^{ij} = \delta^{ij}$ and $h_{ij} = \delta_{ij}$.

The usage of flat metric reduces allowed coordinate transformations to

$$x^0 \to x^0 + \zeta^0, \quad x^i \to A^i{}_j(t)x^j + \xi^i(t), \tag{5.15}$$

where $A^i{}_j \in \mathrm{SO}(3)$, and make connections,

$$\begin{aligned}
\Gamma^i_{0j} = h^{ik}\Omega_{jk} \quad &\to \quad \Omega_{ij} = h_{k[j}\Gamma^k_{i]0}, \\
\Gamma^i_{00} = h^{ij}\Phi_j \quad &\to \quad \Phi_i = h_{ij}\Gamma^j_{00}.
\end{aligned} \tag{5.16}$$

Now, we can impose our first condition, Trautman condition, to be able to derive Poisson's equation,

$$h^{\sigma[\lambda}R^{\mu]}{}_{(\nu\rho)\sigma}(\Gamma) = 0. \tag{5.17}$$

This implies in adapted coordinates that

$$\partial_0 \Omega_{ji} - \partial_{[j}\Phi_{i]} = 0, \quad \text{and} \quad \partial_{[i}\Omega_{jk]} = 0. \tag{5.18}$$

These equations are covariant under (5.15). They can be also expressed as

$$\partial_{[\rho}K_{\mu\nu]} = 0 \quad \to \quad K_{\mu\nu} = 2\partial_{[\mu}m_{\nu]}, \tag{5.19}$$

where $m_\mu = \partial_\mu f$ and f is a some scalar field.

Our second requirement is that Ω_{ij} must depend on only time [20]. This can be achieved by one of the three Ehler conditions below,

$$h^{\rho\lambda}R^{\mu}{}_{\nu\rho\sigma}(\Gamma)R^{\nu}{}_{\mu\lambda\alpha}(\Gamma) = 0, \quad \text{or}$$
$$\tau_{[\lambda}R^{\mu}{}_{\nu]\rho\sigma}(\Gamma) = 0, \quad \text{or} \quad h^{\sigma[\lambda}R^{\mu]}{}_{\nu\rho\sigma}(\Gamma) = 0. \tag{5.20}$$

After that, we can set $\Omega'_{ij} = 0$ by a time dependent rotation $x'^i = A^i{}_j(t)x^j$. Hence, in this new coordinate system, (5.18) gives us $\partial'_{[i}\Phi'_{j]} = 0$, so $\Phi'_i = \partial' i\Phi$ where Φ is some scalar field. It follows from (5.16) that

$$\Gamma'^i_{00} = \delta^{ij}\partial'_j\Phi. \tag{5.21}$$

Thus, we have, from the equations of motion (5.14),

$$R_{00} = \partial_i\Gamma^i_{00} = \delta^{ij}\partial_j\Phi = 4\pi G\rho. \tag{5.22}$$

Hence, we reached Poisson's equation.

Geodesic equation can be obtained also by the time dependent rotation above using adapted coordinates,

$$\ddot{x}'^0(t) = 0, \quad \text{and} \quad \ddot{x}'^i(t) + \partial'^i\Phi = 0. \tag{5.23}$$

Finally, our proof is complete for the Newton-Cartan gravity.

5.1 Newton-Cartan Gravity as a Gauge Theory

It is known that GR reduces to the Newton-Cartan theory in the nonrelativistic limit. We will apply the same procedure to obtain Newton-Cartan gravity as in GR.

Gauging Bargmann algebra gives us the Newton-Cartan gravity. Bargmann algebra is actually Galilean algebra with a central generator M,

$$\begin{aligned}
\left[J_{ij}, J_{kl}\right] &= 4\delta_{[i[k}J_{l]j]}, \\
\left[J_{ij}, P_k\right] &= -2\delta_{k[i}P_{j]}, \\
\left[J_{ij}, G_k\right] &= -2\delta_{k[i}G_{j]}, \\
\left[G_i, H\right] &= -P_i, \\
\left[G_i, P_j\right] &= -\delta_{ij}M.
\end{aligned} \tag{5.24}$$

Table 5.1 : Generators, parameters and gauge fields

	generators	parameters	gauge fields
spatial translations	P	ζ^i	e^i_μ
temporal translations	H	$\zeta^0 \equiv \tau$	$e^0_\mu \equiv \tau_\mu$
Galilean boosts	G	λ^{i0}	$\omega_\mu{}^{i0}$
spatial rotations	J	λ^{ij}	$\omega_\mu{}^{ij}$
mass	M	σ	m_μ

Now, we should gauge it as in the case of GR. Its parameters and gauge fields, splitting temporal and spatial parts, can be summarized as in the table (5.1).

Transformations properties of gauge fields are;

$$\begin{aligned}
\delta\tau_\mu &= \partial_\mu \tau, \\
\delta e^i_\mu &= D_\mu \zeta^i + \lambda^{ij} e^j_\mu + \lambda^{i0}\tau_\mu - \tau\omega_\mu{}^{i0}, \\
\delta\omega_\mu{}^{i0} &= D_\mu \lambda^{i0} + \lambda^{ij}\omega_\mu{}^{j0}, \\
\delta\omega_\mu{}^{ij} &= D_\mu \lambda^{ij}, \\
\delta m_\mu &= \partial_\mu \sigma - \zeta^i \omega_\mu{}^{i0} + \lambda^{i0} e^i_\mu .
\end{aligned} \tag{5.25}$$

The derivative D_μ represents covariant derivative with connection $\omega_\mu{}^{ab}$.

We present curvatures below;

$$\begin{aligned}
R_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]}, \\
R_{\mu\nu}{}^i(P) &= 2(D_{[\mu} e^i_{\nu]} - \omega_{[\mu}{}^{i0}\tau_{\nu]}), \\
R_{\mu\nu}{}^{ij}(J) &= 2(\partial_{[\mu}\omega_{\nu]}{}^{ij} - \omega_{[\mu}{}^{ki}\omega_{\nu]}{}^{jk}), \\
R_{\mu\nu}{}^{i0}(G) &= 2D_{[\mu}\omega_{\nu]}{}^{i0}, \\
R_{\mu\nu}(M) &= 2(\partial_{[\mu} m_{\nu]} + e^j_{[\mu}\omega_{\nu]}{}^{j0}).
\end{aligned} \tag{5.26}$$

Using the prescription in (4.1), we will relate P and H transformations to the general coordinate transformation. First, we define inverse gauge fields e^μ_i and τ^μ by relations

$$\begin{aligned}
e^i_\mu e^\mu_j &= \delta^i_j, \\
e^i_\mu e^\nu_i &= \delta^\nu_\mu - \tau_\mu \tau^\nu, \\
\tau^\mu e^i_\mu &= 0, \\
\tau_\mu e^\mu_i &= 0, \\
\tau^\mu \tau_\mu &= 1.
\end{aligned} \tag{5.27}$$

These equations are compatible with (5.9) with the realization of

$$h_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}, \quad \text{and} \quad h^{\mu\nu} = e^\mu_i e^\nu_j \delta^{ij}. \tag{5.28}$$

Now, we can relate P and H transformations with general coordinate transformations of space and time by setting

$$\xi^\mu = e^\mu_i \zeta^i + \tau^\mu \tau. \tag{5.29}$$

The fields e^i_μ, τ_μ and m_μ transform under P and H transformations. Thus, following the same reasoning in (4.1), we must make their curvatures equal to zero

$$R_{\mu\nu}{}^i(P) = R_{\mu\nu}(H) = R_{\mu\nu}(M) = 0. \tag{5.30}$$

Hence, their gauge fields e^i_μ, τ_μ and m_μ will remain independent.

$R_{\mu\nu}(H) = 0$ implies $\partial_{[\mu}\tau_{\nu]} = 0$. Thus, we can take it as in Newton-Cartan theory. From the remaining relations we can express $\omega_\mu{}^{ij}$ and $\omega_\mu{}^{i0}$ with independent gauge fields,

$$\begin{aligned}\omega_\mu{}^{kl} &= \partial_{[\mu} e^k_{\nu]} e^{\nu l} - \partial_{[\mu} e^l_{\nu]} e^{\nu k} + e^i_\mu \partial_{[\nu} e^i_{\rho]} e^{\nu k} e^{\rho l} - \tau_\mu e^{\rho[k} \omega_\rho{}^{l]0}, \\ \omega_\mu{}^{i0} &= e^{\nu i} \partial_{[\mu} m_{\nu]} + e^{\nu i} \tau^\rho e^j_\mu \partial_{[\nu} e^j_{\rho]} + \tau_\mu \tau^\nu e^{\rho i} \partial_{[\nu} m_{\rho]} + \tau^\nu \partial_{[\mu} e^i_{\nu]}.\end{aligned} \tag{5.31}$$

Next, we impose vielbein postulates to find a Γ connection,

$$\begin{aligned}\partial_\mu e^i_\nu - \omega_\mu{}^{ij} e^j_\nu - \omega_\mu{}^{i0} \tau_\nu - \Gamma^\rho_{\nu\mu} e^i_\rho &= 0, \\ \partial_\mu \tau_\nu - \Gamma^\lambda_{\nu\mu} \tau_\lambda &= 0.\end{aligned} \tag{5.32}$$

Then we can solve for $\Gamma^\rho_{\mu\nu}$,

$$\Gamma^\rho_{\mu\nu} = \tau^\rho \partial_{(\mu} \tau_{\nu)} + e^\rho_i (\partial_{(\mu} e^i_{\nu)} - \omega_{(\mu}{}^{ij} e^j_{\nu)} - \omega_{(\mu}{}^{i0} \tau_{\nu)}). \tag{5.33}$$

This connection satisfies metric compatibility conditions in the Newton-Cartan gravity. However, this gives Γ connection uniquely unlike that case. Comparing this with (5.10) we find

$$K_{\mu\nu} = 2\omega_{[\mu}{}^{i0} e^i_{\nu]}. \tag{5.34}$$

Then, expanding dependent field $\omega_\mu{}^{i0}$ we find

$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]}, \tag{5.35}$$

which relates m_μ to the Newton-Cartan gravity via (5.19).

Now, from Bianchi identities for our gauge theory we have

$$R_{[\lambda\mu}{}^{ij}(J)e^{j}_{\nu]} = -R_{[\lambda\mu}{}^{i0}(G)\tau_{\nu]}, \tag{5.36}$$

which is equal to the Trautman conditions one can show explicitly.

Besides, each one of the Ehler conditions corresponds to the

$$R_{\mu\nu}{}^{ij}(J) = 0. \tag{5.37}$$

Finally, we can show, with all these constraints, Riemann curvature tensor $R^{\mu}{}_{\nu\rho\sigma}(\Gamma)$, following from the connection (5.33), becomes

$$R^{\mu}{}_{\nu\rho\sigma}(\Gamma) = -e^{\mu}_{i}\tau_{\nu}R_{\rho\sigma}{}^{i0}(G). \tag{5.38}$$

One can show that only non-zero component of this curvature, with the help of (5.36) is

$$\delta^{k(j}R^{i)}_{0k0}(\Gamma) = \tau^{\mu}e^{\nu(i}R_{\mu\nu}{}^{j)0}, \tag{5.39}$$

which is exactly the same non-zero components as in the Newton-Cartan theory.

Eventually, we found the same connection and metrics. By just imposing the same equation of motion in the Newton-Cartan theory, we can have the same theory.

REFERENCES

[1] **H.Weyl** (1918). *Gravitation and Elektrizitat*, Springer-Verlag.

[2] **V.Fock** (1927). *Zeit. f. Physik*, *39*, 226.

[3] **H.Weyl** (1929). *Zeit. f. Physik*, *56*, 330.

[4] **W.Pauli**. *Handbuch der physik, Geiger and scheel, 2, Aufl.*, *24*(1), 83–272.

[5] **C.N.Yang, R.** (1954). *Phys Rev*, *96*, 191.

[6] **S.Glashow** (1961). *Nucl Phys*, *22*, 579.

[7] **S.Weinberg** (1967). *Phys Rev Lett*, *19*, 1264.

[8] **A.Salam** (1968). *Elementary Particle Theory*.

[9] **Cartan, E.** (1923). Sur les variétés à connexion affine et la théorie de la relativité generalisée (suite), *Ann. Ècole Norm. Sup.*, *40*, 325–412.

[10] **Havas, P.** (1964). Four-Dimensional Formulations of Newtonian Mechanics and their Relation to the Special and the General Theory of Relativity, *Rev. Mod. Phys.*, *36*, 938.

[11] **Anderson, J.L.** (1967). *Principles of Relativity Physics*, Academic Press.

[12] **Trautman, A.** (1965). *”Theories of Space, Time and Gravitation” in Lectures on General Relativity*, Prentice-hall.

[13] **Künzle, H.** (1972). Galilei and Lorentz Structures on Space-Time: Comparison of the Corresponding Geometry and Physics, *Ann. Inst. Henry Poincaré*, *42*, 337.

[14] **Kuchař, K.** (1980). Gravitation, Geometry, and Non-Relativistic Quantum Theory, *Phys. Rev*, *22D*(6), 1285.

[15] **Dautcourt, G.** (1964). Die Newtonske Gravitationstheorie als Strenger Grenzfall der Allge- meinen Relativitätheorie, *Acta Phys. Pol.*, *25*(5), 637.

[16] **Ehlers, J.** (1986). *"On Limit Relations between, and Approximative Explanations of, Physical Theories”, in Logic, Methodology and Philosophy of Science*, Elsiever.

[17] **Malament, D.** (1986). *“Newtonian Gravity, Limits, and the Geometry of Space”, in From Quarks to Quasars*, Pittsburgh University Press.

[18] **Ruede, C. and Straumann, N.** (1997). On Newton-Cartan cosmology, *Helv. Phys. Acta*, *70*, 318.

[19] **U. Brauer, A.R. and Reula, O.** (1994). The Cosmic no hair theorem and the nonlinear stability of homogeneous Newtonian cosmological models, *Class. Quant. Grav.*, *11*, 2283.

[20] **Dautcourt, G.** (1990). On the Newtonian limit of General Relativity, *Acta. Phys. Pol. B*, *21*, 765–766.

CURRICULUM VITAE

Name Surname: Orhan Tunca

Place and Date of Birth: Izmir 07.07.1992

E-Mail:tuncaor@gmail.com

EDUCATION:

- **B.Sc.:** 2016, Istanbul Technical University, Faculty of Science and Arts, Physics Engineering
- **M.Sc.:** 2019, Istanbul Technical University, Faculty of Science and Arts, Physics Engineering

PROFESSIONAL EXPERIENCE AND REWARDS:

- 2018 Research Assistant at Istanbul Technical University

PUBLICATIONS, PRESENTATIONS AND PATENTS ON THE THESIS:

- Özdemir, N., Özkan, M., Tunca, 0., Zorba, U. 2019. Extended Newtonian (Super)Gravity, *Journal of High Energy Physics,* `https://doi.org/10.1007/JHEP05(2019)130`, 2019(5), 130.