DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

COGALOIS GROUPS OF COVERS FOR SOME QUIVERS

by
Canan ÖZEREN

August, 2024
İZMİR

COGALOIS GROUPS OF COVERS FOR SOME QUIVERS

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M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled "**COGALOIS GROUPS OF COVERS FOR SOME QUIVERS**" completed by **CANAN ÖZEREN** under supervision of **ASSOC. PROF. DR. SALAHATTİN ÖZDEMİR** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

....

Assoc. Prof. Dr. Salahattin ÖZDEMİR

Supervisor

....

Prof. Dr. Noyan Fevzi ER

Jury Member

....

Prof. Dr. Engin BÜYÜKAŞIK

Jury Member

Prof. Dr. Okan FISTIKOĞLU
Director
Graduate School of Natural and Applied Sciences

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Canan ÖZEREN

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ABSTRACT

Torsion free covers exist for abelian groups (in fact, for modules over an integral domain) (see Enochs (1963)). Let $\varphi : C \to A$ be a torsion free cover of an abelian group A. As a dual notion of (absolute) Galois groups, the *coGalois group* of A was introduced in Enochs et al. (2000) as the group of automorphisms $\sigma : C \to C$ such that $\varphi\sigma = \varphi$, denoted by $G(\varphi)$ or $G(A)$ (since a torsion free cover is unique up to isomorphism). A complete characterization of abelian groups having a trivial coGalois group was given in Enochs & Rada (2005). After, coGalois groups have been studied for a pair of abelian groups and characterized when they are trivial in Hill (2008). Actually, coGalois groups have been studied in the category of representations of the quiver (i.e., a directed graph) $q_2 : \bullet \to \bullet$ there. Because, the coGalois group is definable for any category with a covering class, and the torsion free covers exist for the category $(q_n, \mathbb{Z}\text{-Mod})$ of representations of the line quiver $q_n : \bullet \to \bullet \to ... \to \bullet$ with $n-1$ arrows and n vertices (see Wesley (2005)), we define and study coGalois groups in that category. We give some properties of torsion free covers and coGalois groups of objects similar to those given for abelian groups, and characterize the objects in $(q_n, \mathbb{Z}\text{-Mod})$ having a trivial coGalois group, in terms of trivial coGalois groups of abelian groups.

Keywords: coGalois group, torsion free cover, line quiver

BAZI KUİVERLER İÇİN ÖRTÜLERİN COGALOIS GRUPLARI

ÖZ

Abel gruplar için (ayrıca bir bölge üzerine modüller için) burulmasız örtülerin varlığı kanıtlandı (bkz. Enochs (1963)). Bir A abel grubunun burulmasız örtüsü, $\varphi : C \to A$ olsun. Enochs et al. (2000)'de, (tam) Galois grubun bir dual kavramı olarak, A'nın coGalois grubu, $\varphi\sigma = \sigma$ şartını saglayan $\sigma : C \to C$ otomorfizmalarının grubu olarak tanımlandı ve $G(\varphi)$ ya da $G(A)$ olarak gösterildi (burulmasız örtüler isomorfizmaya göre tek olduğundan). coGalois grubu sadece birim elemandan oluşan abel gruplarının tam bir sınıflandırması Enochs & Rada (2005)'de verildi. Sonra, Hill (2008)'de, bir abel gruplar çifti için coGalois group çalışıldı ve ne zaman sadece birim elemandan oluştuğu sınıflandırıldı. Aslında, $q_2 : \bullet \to \bullet$ kuiverinin (yani, bir yönlü graf) temsil kategorisinde coGalois group çalışılmış oldu. Örtü sınıfına sahip olduğumuz herhangi bir kategoride coGalois grup tanımlanabilir ve $n-1$ oklu ve n noktalı $q_n : \bullet \to \bullet \to ... \to \bullet$ doğru kuiverinin temsil kategorisi olan $(q_n, \mathbb{Z}$-Mod$)$'da burulmasız örtüler her zaman var olduğundan (bkz. Wesley (2005)), bu kategoride coGalois grubu tanımladık ve çalıştık. Abel gruplar için verilenlere benzer şekilde, $(q_n, \mathbb{Z}$-Mod$)$ kategorisndeki objeler için de burulmasız örtülerin ve coGalois grubun bazı özelliklerini verdik ve bu kategoride coGalois grubu sadece birim elemandan oluşan objeleri, coGalois grupları sadece birim elemandan oluşan abel gruplarının yardımıyla sınıflandırdık.

Anahtar kelimeler: coGalois grup, burulmasız örtü, doğru kuiver

CONTENTS

LIST OF SYMBOLS

R	an integral domain (or a commutative domain) unless otherwise stated
a module M	a left R-module M
$N \leq M$	N is a submodule of M
$\mathbb{Z}$	the ring of integers
$\mathbb{Z}_{p^\infty}$	the Prüfer group for the prime p
$\mathbb{Q}$	the field of rational numbers
R-Mod	the category of modules
$t(A)$	the torsion subgroup of an abelian group A
$A(p)$	p-primary subgroup of an abelian group A for the prime p
r-prime p	relevant prime p of an abelian group
TFC	a torsion free cover
$G(A)$	the coGalois group of an abelian group A
$C(M)$	the torsion free cover of a module M
ker f	the kernel of the map f
im f	the image of the map f
Hom(M, N)	all module homomorphisms from M to N
$\mathcal{A}b$	the category of abelian groups
q_n	the line quiver $\bullet \to \bullet \to \ldots \to \bullet$ with $n-1$ arrows and n vertices
$(q_n, \text{R-Mod})$	the category of representations by modules of the line quiver q_n

CHAPTER ONE
INTRODUCTION

Throughout this thesis, R will be an integral domain (or a commutative domain), and by a module we mean a left R-module, unless otherwise stated. In fact, R will mostly be the ring of integers $\mathbb{Z}$, and so the modules will be abelian groups. We will write 'TFC' for short instead of a 'torsion free cover' of a module.

We refer the reader to Enochs & Jenda (2011), Fuchs (1970), Kasch (1982) and Kaplansky (1969) for any undefined notion and for further information about the concepts studied in this thesis.

In this chapter, we will give some ideas to motivate the problem of our thesis in Section 1.1. There are some known facts about triviality of the coGalois groups in the category $\mathcal{A}b$ of abelian groups and in the category of representations of the quiver (i.e., a directed graph) $q_2 : \bullet \to \bullet$. The main research topic of this thesis is to generalise them to the category $(q_n, \mathbb{Z}\text{-Mod})$ of representations of the line quiver $q_n : \bullet \to \bullet \to ... \to \bullet$ with $n-1$ arrows and n vertices. To explain our problems, we will give some needed basic definitions and fundamental tools in Section 1.2.

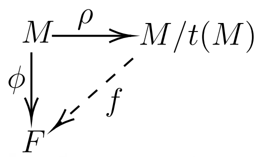
1.1 Motivation

In this section, we will give the motivating ideas for our main results of this thesis. We will see some universal concepts and related groups. See Enochs & Jenda (2011) for the definitions and more information about 'covers and envelopes'.

1.1.1 Envelopes and Galois Groups

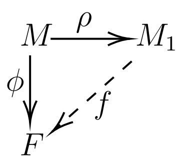
Let M be a module. Then M is called *torsion free* if $rm =$ for $r \in R$, $m \in M$ implies that $r = 0$ or $m = 0$, while it is *torsion* if, for every $m \in M$, there exists a nonzero $r \in R$ such that $rm = 0$, that is, every element of M is trosion. The set of all torsion elements of M is a submodule of M, called a *torsion submodule* of M, and denoted by $t(M)$.

Note that the canonical map $\rho : M \to M/t(M)$ is universal in the sense that for any linear map $\phi : M \to F$ where F is a torsion free module, there is a unique linear map $f : M/t(M) \to F$ such that $f\rho = \phi$, that is, the following diagram commutatives:



We state and prove this universal property in the following lemma, which was stated as a well-known fact without a proof in Enochs (1963). This motivated the definition of a torsion free cover (TFC for short) as a kind of its a dual version.

Lemma 1.1.1. *Let M be a module. Then there is always a torsion free module M_1 with an epimorphism $\rho : M \to M_1$ such that for any linear map $\phi : M \to F$ where F is torsion free, there is a unique linear map $f : M_1 \to F$ such that $f\rho = \phi$. This means the following diagram commutes:*



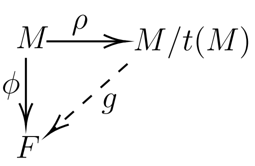
Proof. Observe that it suffices to prove the statement for $M_1 = M/t(M)$, where $t(M)$ is the torsion part of M.

(Existence) Firstly, we will show that $M/t(M)$ is torsion free. Say $r(m+t(M)) = t(M)$ where $r \in R$ and $m + t(M) \in M/t(M)$. Then $rm \in t(M)$. Since $t(M)$ is torsion, there is a nonzero $s \in R$ such that $s(rm) = 0$. Since M is an R-module, we have that $0 = s(rm) = (sr)m$. It means that m is an torsion element. Hence $m \in t(M)$, and so $m + t(M) = t(M)$. Secondly, we show that the canonical mapping $\rho : M \to M/t(M)$ satisfies the 'preenvelope' condition on the statement of this lemma.

Define a linear map $f : M/t(M) \to F$ with $m + t(M) \mapsto \phi(m)$. We know that f is well-defined because $t(M) \le \ker\phi$ since F is torsion free, or say $m + t(M) = m' + t(M)$, and so $m - m' \in t(M)$. Then, there is an element $m'' \in t(M)$ such that $m - m' = m''$. Then, $\phi(m - m') = \phi(m'')$ since ϕ is well-defined. Since ϕ is a linear mapping, we have that $\phi(m'') = \phi(m - m') = \phi(m) - \phi(m')$. Then $\phi(m) = \phi(m'') + \phi(m') = \phi(m'' + m')$. Clearly, $m + t(M) = m' + t(M) = m' + m'' + t(M) = m'' + m' + t(M)$ since $m'' \in t(M)$. Then $f(m + t(M)) = \phi(m) = \phi(m'' + m') = f(m'' + m' + t(M)) = f(m' + t(M))$. To check that the diagram is commutative, i.e., $f\rho = \phi$. Taking $m \in M$, we get $f\rho(m) = f(m + t(M)) = \phi(m)$. To say that f is an linear mapping, we have that:

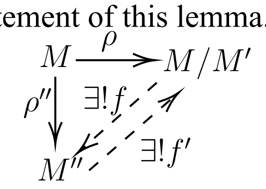
$$
\begin{aligned}
f(r(m + t(M)) + s(m' + t(M)) &= f(rm + sm' + t(M)) \\
&= \phi(rm + sm') \\
&= r\phi(m) + s\phi(m') \\
&= rf(m + t(M)) + sf(m' + t(M)).
\end{aligned}
$$

(Uniqueness of f) Next, we prove the uniqueness of the linear map f. Say there is another linear mapping $g : M/t(M) \to F$ such that the following diagram commutes:



Take $m \in M$. Then we have $\phi(m) = f\rho(m) = f(m+t(M))$ and $\phi(m) = g\rho(m) = g(m + t(M))$. It means that for all $m + t(M) \in M/t(M)$, we have $f(m + t(M)) = g(m + t(M))$, and so $f = g$.

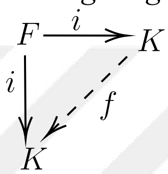
(Uniqueness up to isomorphism) Let $\rho'' : M \to M''$ be another mapping that satisfies the condition on the statement of this lemma. So we have



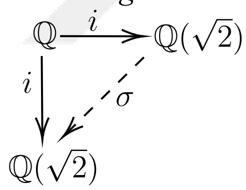
So $f\rho = \rho''$ and $f'\rho'' = \rho$. Easily, we can see that $\rho = f'\rho'' = f'f\rho$ and $\rho'' = f\rho = ff'\rho''$. It means that $f'f = id_{M/t(M)}$ and $ff' = id_{M''}$ are identity maps by the uniqueness of f and f'. Hence $M/t(M) \cong M''$. □

We recall below the definition of a famous group, the Galois group.

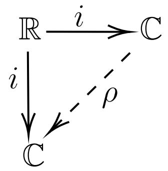
Definition 1.1.1. *The Galois group $Gal(K/F)$ of a field extension K/F is the group of all F-automorphisms of K (i.e. the automorphisms of K which fix F) under the operation of the composition. In other words, the Galois group contains the automorphisms of K such that the following diagram commutes:*



Example 1. *$Gal(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) = \{1, \sigma\}$ where $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ with $\sigma(\sqrt{2}) = -\sqrt{2}$.*
So, we have the following commutative diagram:

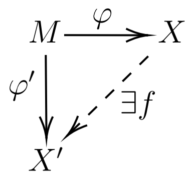


Example 2. *$Gal(\mathbb{C}/\mathbb{R}) = \{1, \rho\}$ where $\rho : \mathbb{C} \to \mathbb{C}$ is the conjugation map with $\rho(a + bi) = a - bi$. And we have*

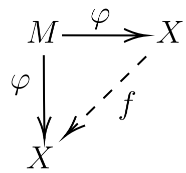


Definition 1.1.2. *Let $\varkappa$ be a class of modules closed under isomorphisms. Let M be a module, and $X \in \varkappa$. A linear map $\varphi : M \to X$ is an $\varkappa$-envelope of M if the following two conditions hold:*

1. *for any linear map $\varphi' : M \to X'$ with $X' \in \varkappa$, there is a linear map $f : X \to X'$ with $\varphi' = f\varphi$. This means the following diagram commutes:*



2. *If an endomorphism $f : X \to X$ is such that $\varphi = f\varphi$, then f must be an automorphism. So, we have that f is an automorphism in the following commutative diagram:*



If a module satisfies the condition (1) and may not satisfy the condition (2), then it is called an $\varkappa-$preenvelope. And, $\varkappa$ is called an *enveloping class* if every module admits an $\varkappa$-envelope.

To give an example, we need the following theorem. Actually, it is entirely same to (17.31) Corollary in Isaacs (1994).

Theorem 1.1.2. *Let F be any field and let E_1 and E_2 be algebraic closures for F. Then E_1 and E_2 are F-isomorphic, i.e., isomorphic and the isomorphism fixes F.*

The following example exhibits a relationship between Galois groups and envelopes.

Example 3. *The class $\varkappa$ of algebraic closed fields is an enveloping class, that is, any field has an $\varkappa$-envelope. In fact, the algebraic closure $\overline{F}$ of a field F is its $\varkappa$-envelope.*

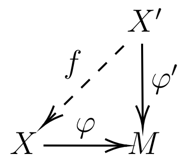
Therefore, the (absolute) Galois group Gal$(\overline{F}/F)$ is actually the group of all automorphisms from $\varkappa$-envelopes. So, the notion of an (absolute) Galois group can be defined in any category where we have an enveloping class.

1.1.2 Covers and coGalois Groups

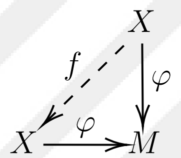
In this section, we will define a dual version of the envelope.

Definition 1.1.3. *Let $\varkappa$ be a class of modules closed under isomorphisms. Let M be a module and $X \in \varkappa$. A linear map $\varphi : X \to M$ is an $\varkappa$-cover of M, if the following two conditions hold:*

1. *For every module $X' \in \varkappa$ and a homomorphism $\varphi' : X' \to M$, there is a homomorphism $f : X' \to X$ such that $\varphi' f = \varphi$, that is, the following diagram commutes:*



2. *If an endomorphism $f : X \to X$ is such that $\varphi f = \varphi$, then f must be an automorphism. That is, any homomorphism f in the following commutative diagram must be an automorphis:*



If a module satisfies the condition (1) and may not satisfy the condition (2), then it is called an $\varkappa-$precover. And, $\varkappa$ is called a *covering class* if every module admits an $\varkappa$-cover.

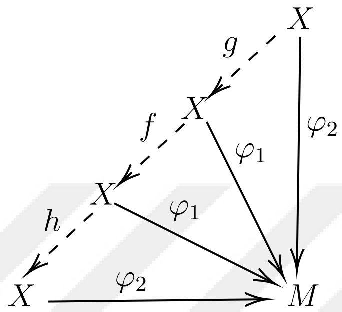
Example 4. *The class $\varkappa$ of torsion free abelian groups is an covering class, that is, any abelian group has an $\varkappa$-cover (i.e., a TFC). An equivalent definition of this cover will be given in Definition 1.1.5.*

In Example 3, we saw that the notion of (absolute) Galois groups is related to the notion of envelopes, and it can be defined in any category where we have an enveloping class. Motivated by this relation, as a dual notion, the coGalois group of an abelian group was first defined in Enochs et al. (2000) as the group of all automorphisms from the TFC of that group. However, coGalois groups can be defined in any category where we have a covering class, not only for TFCs.

Definition 1.1.4. *Let $\varphi : X \to M$ be an $\varkappa$-cover. The group of all automorphisms $f : X \to X$ such that $\varphi f = \varphi$ is called the coGalois group of φ (or M), denoted by*

$G(\varphi)$ *or* $G(M)$.

It is easy to see that an $\varkappa-$cover is unique up to isomorphism, that is, the coGalois group does not depend on φ. So, the coGalois group $G(\varphi)$ of an $\varkappa$-covering map $\varphi : X \to M$ can be also denoted by $G(M)$. More precisely, let $\varphi_1 : X \to M$ and $\varphi_2 : X \to M$ be two $\varkappa-$covering maps of M. Then we have the following commutative diagram:

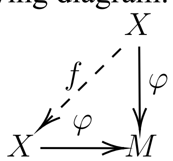


Moreover, we know that f, hfg, hg and gh are all automorphisms, and so g and h are both automorphisms. So we can take g^{-1} instead of h. So, we have an isomorphism

$$\Phi : G(\varphi_1) \to G(\varphi_2) \text{ with } \Phi(f) = f^g$$

where $f^g = g^{-1}fg$ and g is any automorphism such that $\varphi_1 g = \varphi_2$.

We can say the coGalois group comes from the cover because every elements of the coGalois group commutes the following diagram:



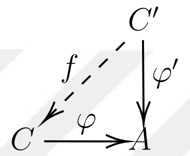
Clearly, it can be seen that ker φ is an invariant.

In previous section, we saw that the Galois group comes from the envelope. Now, we see that the cover is the dual version of the envelope. Because of that, the group that comes from the cover is named the coGalois group. It is sometimes called the absolute coGalois group, because it is actually a dual of the absolute Galois group. See Example 3.

The following definition was given for modules over an integral domain in Enochs (1963), and the motivation can be seen finding out the answer of the dual version of Lemma 1.1.1.

Definition 1.1.5. *For an abelian group A, a torsion free abelian group C is called a torsion free cover of A, written TFC for short, if there is a homomorphism $\varphi : C \to A$ such that the followings hold:*

1. *For every torsion free abelian group C' and a homomorphism $\varphi' : C' \to A$ there is a homomorphism $f : C' \to A$ such that $\varphi' f = \varphi$. So we have the following commutative diagram:*



2. *If* $\ker \varphi$ *has no nonzero pure subgroup of C, where a subgroup P of C is pure in C if $nP = nC \cap P$ for all integer n.*

In Enochs (1963), the condition (1) was called the torsion free factor property, and if a morphism satisfies the torsion free factor property and may not satisfy the condition (2), then it is called a torsion free precover or a precover. It is known that this is equivalent to the definition of the cover with the class of torsion free abelian groups (see (Enochs, 1963, Theorem 2)).

In Enochs & Rada (2005), coGalois groups of TFCs of abelian groups have been studied, and a classification for coGalois groups to be trivial were given. In Hill (2008), an equivalent version of this classification were used. This version will be given in Theorem 1.1.3 after some definitions.

Definition 1.1.6. *Let M be a module. An element $m \in M$ is said to be divisible by $r \in R$, if there exists an $m' \in M$ with $m = rm'$. A nonzero non-unit $p \in R$ is called a prime element of R if, whenever p divides a product ab, then p divides a or p divides b. And, M is called p-divisible for a prime element $p \in R$, if $pM = M$, and is called divisible if $rM = M$ for all nonzero $r \in R$.*

Definition 1.1.7. *Let A be an abelian group. Then A is said to be p-primary or primary* *for a prime p if all elements have order a power of p. A prime number p is called a relevant prime of A, written 'r-prime' for short, if there is an element of order p in A, that is, the p-primary part $A(p)$ of this group is nonzero.*

An abelian group has no element that has a power of p if and only if it has no element with order p. So, whenever we say that for some abelian group has no element with order p, it means that it has no element with order a power of p, that is, the p-primary part is zero. Note that the p-primary part $A(p)$ of an abelian group A is a subgroup.

Theorem 1.1.3. *(Enochs & Rada (2005), Theorem 2.8.) An abelian group A has a trivial coGalois group if and only if A is p-divisible for each of its r-prime p.*

It is easy to see the fact that the coGalois group of a divisible group is trivial. However, there is abelian groups that is not divisible and has a trivial coGalois group. See (Enochs & Rada (2005), Example 2.9).

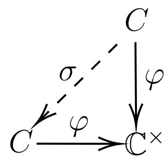
Example 5. *We are familiar to see the nonzero complex numbers $\mathbb{C}^{\times}$ as a multiplicative group. Let $\varphi : C \to \mathbb{C}^{\times}$ be the TFC of $\mathbb{C}^{\times}$.*

We will say the coGalois group of $\mathbb{C}^{\times}$ is trivial with saying that the torsion subgroup $t(\mathbb{C}^{\times})$ is divisible and the quotient group $\mathbb{C}^{\times}/t(\mathbb{C}^{\times})$ is p-divisible for every prime p such that $t(\mathbb{C}^{\times})(p) \neq 0$ by (Enochs & Rada (2005),Theorem 2.8).

We know that the group $\mathbb{C}^{\times}$ is divisible and the torsion subgroup $t(\mathbb{C}^{\times}) \cong \left(\bigoplus_{p \in P} \mathbb{Z}_{p^{\infty}}\right)$ is also divisible where P is the set of all positive primes by Theorem 1.2.1. Moreover, the torsion elements are roots of unity.

Using the fact that a direct sum of abelian groups is divisible if and only if every summand is divisible, we get that $\mathbb{C}^{\times}/t(\mathbb{C}^{\times})$ is divisible. Hence, the coGalois group $G(\mathbb{C}^{\times})$ of $\mathbb{C}^{\times}$ is trivial.

So, we have that there is unique homomorphism σ such that the following diagram commutes



without finding the TFC C of $\mathbb{C}^{\times}$.

The coGalois group is definable for any category with a covering class.

The TFCs exist for the category $(q_n, \text{R-Mod})$ of representations of the line quiver $q_n : \bullet \to \bullet \to \ldots \to \bullet$ with $n-1$ arrows and n vertices (see Wesley (2005)). See Chapter 2 for more details about this category. We also give some results of torsion free precovers of objects in $(q_n, \text{R-Mod})$.

In Hill (2008), coGalois groups have been studied and characterized when they are trivial in the category $(q_2, \mathbb{Z}\text{-Mod})$.

Theorem 1.1.4. *(Hill (2008), Theorem 5.1) The coGalois group of the object* $A_1 \xrightarrow{f_1} A_2$ *in* $(q_2, \mathbb{Z}\text{-Mod})$ *is trivial if and only if the following conditions are satisfied.*

1. *The coGalois group of* A_2 *is trivial.*
2. *The coGalois group of* $\ker f_1$ *is trivial.*
3. $\operatorname{im} f_1$ *is* p*-divisible for each r-prime* p *of* $\ker f_1$.
4. $\operatorname{im} f_1(p) = A_2(p)$ *for each r-prime* p *of* $\ker f_1$.

Motivated by these studies above, we consider the notion of coGalois groups in the category $(q_n, \mathbb{Z}\text{-Mod})$, and we characterize the objects $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} A_n$ in this category having a trivial coGalois group (see Chapter 3). So, our main results generalize the above theorem from the category q_2 to the category q_n.

1.2 Preliminaries

In this section, we will give some fundamental tools, and an important lemma about the pullback diagram, which will be used in Chapter 3.

1.2.1 Divisibility and Purity

In this subsection, we will give some facts about modules over an integral domain R. Clearly, all of them are satisfied for abelian groups since we can see an abelian group as a $\mathbb{Z}$-module. See, for example, Fuchs (1970) or Kaplansky (1969) for the preliminary results of abelian groups given in this section.

The following is not the usual definition of a pure submodule, it is due to Enochs (1963).

Definition 1.2.1. *A submodule M' of a module M is called pure in M if $rM' = rM \cap M'$ for all $r \in R$. So, an abelian subgroup $H \leq G$ is pure in G if $nH = nG \cap H$ for every integer n.*

It is easy to see that $rM' \subseteq rM \cap M'$ is always true, so we will only need to check that $rM' \supseteq rM \cap M'$ for purity. This means that each element of M' is divisible by $r \in R$ in M' whenever it is divisible by r in M.

In the following theorem, we collect some known properties of abelian groups.

Theorem 1.2.1. *For an abelian group A, the followings hold:*

1. *Every direct summand of A is pure in A.*
2. *The torsion subgroup $t(A)$ is pure in A.*
3. *If A is divisible, then every pure subgroup of A is divisible.*
4. *Every divisible subgroup of A is pure in A.*

The converse of (1) in the previous theorem is not true, in other words, a pure subgroup of an abelian group may not be a direct summand. As a counter example, we can give an example of the torsion subgroup of an abelian group that is not a direct summand.

The following well-known facts can be found in Kaplansky (1969). We will prove only the first condition.

Proposition 1.2.2. *The followings hold for abelian groups:*

1. *A direct sum of abelian groups is divisible if and only if each component is divisible.*
2. *A divisible subgroup of an abelian group is a direct summand.*
3. *A divisible abelian group is a direct sum of groups each isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}_{p^\infty}$ for various primes p.*

Proof. (1) ($\Rightarrow$) Let $G = H_1 \oplus H_2$. Take a nonzero element $h \in H_1$. Since G is divisible, there is an element $g \in G$ such that $h = ng$ for any nonzero integer n. But we can write that $g = h_1 + h_2$ where $h_i \in H_i$. Moreover, we have $h = ng = n(h_1 + h_2) = nh_1 + nh_2$ So, we have $nh_2 \in H_1 \cap H_2$, and so $nh_2 = 0$. Hence, $h = nh_1$.

($\Leftarrow$) Let $G = H_1 \oplus H_2$. Take an element $g \in G$ and a nonzero integer n. We have $g = h_1 + h_2$ where $h_i \in H_i$. Since every summand is divisible, we have that there is $h_i' \in H_i$ such that $h_i = nh_i'$. So we can write $g = h_1 + h_2 = nh_1' + nh_2' = n(h_1' + h_2')$.

The same holds for infinite direct sums. □

We give the following some basic but important properties of p-divisible abelian groups, which are needed in our study. Most of them can be found in, for example, Kaplansky (1969), but without proof. So, we will prove them. It is a well-known fact that an epimorphic image of a divisible group is divisible. Similar fact is also true for p-divisible groups:

Theorem 1.2.3. *Let p be a prime number. Then the followings hold for abelian groups.*

1. *An epimorphic image of a p-divisible group is also p-divisible.*
2. *Let G be a p-divisible abelian group, and H an abelian group that has no nonzero p-divisible subgroup. Then, there is no nonzero homomorphism from G into H.*
3. *An abelian group is divisible if and only if it is p-divisible for every prime p.*
4. *A p-group is divisible if and only if it is p-divisible.*

Proof. (1) Let G be a p-divisible group with the homomorphism $\varphi : G \to H$. We will show that the image $\operatorname{im}\varphi$ of φ is p-divisible. Take an element $h \in \operatorname{im}\varphi$, so there is $g \in G$ such that $\varphi(g) = h$. Since G is p-divisible, there is an element $g' \in G$ such that $g = pg'$. So, we have $h = \varphi(g) = \varphi(pg') = p\varphi(g')$. Clearly, $\varphi(g') \in \operatorname{im}\varphi$, and so h is divisible by p in $\operatorname{im}\varphi$.

(2) By the condition (1).

(3) ($\Rightarrow$) Trivial.

($\Leftarrow$) Let G be a p-divisible abelian group for every prime p. Take an element $g \in G$ and a nonzero integer n. So, we can write that $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ where p_i's are primes by the Fundamental Theorem of Arithmetic. We know that G is p_i-divisible for all primes p_i. Then there is an element g_1 such that $g = p_1^{e_1} g_1$. Assume that $g = p_1^{e_1} p_2^{e_2} ... p_j^{e_j} g_j$ for some element g_j of G. But we also have $g_j = p_{j+1}^{e_{j+1}} g_{j+1}$ for some element g_{j+1} of G. So, $g = p_1^{e_1} p_2^{e_2} ... p_k^{e_k} g_k = n g_k$ for some element g_k of G.

(4) ($\Rightarrow$) Trivial.

($\Leftarrow$) Let G be a p-divisible p-group. Take a nonzero element $g \in G$ and an integer n. Let the order of g is p^k since G is a p-group. Say $n = p^m n'$ with $p \nmid n'$. Firstly, we will show that g can be divided by n'. By Bézout's Lemma, we have that there are some integers u and v such that $up^k + vn' = 1$ since $gcd(p^k, n') = 1$. Moreover, $p^k \nmid v$ since $p^k \nmid 1$. So, we have $g = (up^k + vn')g = up^k g + vn'g = vn'g = n'(vg)$. But, G is also p-divisible, so there is an element g' in G such that $vg = p^m g'$. Hence $g = n'(vg) = n'(p^m g') = ng'$. □

Definition 1.2.2. *A subgroup H of an abelian group G is p-pure in G if $pH = pG \cap H$ for a prime p.*

Lemma 1.2.4. *Let G be a torsion free abelian group. A subgroup H of G is pure in G if and only if H is p-pure in G for every prime number p.*

Proof. ($\Rightarrow$) Clear.

(⇐) Take an element $h \in H$ such that $h = ng$, where $g \in G$. We can write that $n = p_1p_2...p_m$ where p_i's are prime numbers that are not necessary to be distinct. So, we have $h = p_1(p_2...p_mg)$. Since H is p_1-pure, we have $h = p_1h_2$, where $h_2 \in H$. Since G is torsion free, we get $h_2 = p_2...p_mg$. Using the same ideas, we have $h_i = p_i...p_mg$, where $h_i \in H$ for all $i \leq m$. Finally, we have $h_m = p_mg$. Since H is p_m-pure, we have $h_m = p_mh_{m+1}$ where $h_{m+1} \in H$. Therefore, we have $p_mg = p_mh_{m+1}$, and so $g = h_{m+1}$ since G is torsion free. Hence, we get $h = p_1p_2...p_mh_{m+1} = nh_{m+1}$ where $h_{m+1} \in H$, as desired. □

Lemma 1.2.5. *Let M be a module, $M' \leq M$ a submodule and $p \in R$ a prime element. If M' and M/M' are p-divisible, then M is also p-divisible.*

Proof. Take an element m of M. If $m \in M'$, then there is nothing to prove.
Assume $m \in M \setminus M'$. Then $m + M' \neq M'$, and so $m + M'$ is a nonzero element of M/M'. Since M/M' is p-divisible, there is an element $m' + M'$ such that $m + M' = p(m'+M') = pm'+M'$. So $m-pm' \in M'$. Since M' is p-divisible, there is an element $m'' \in M'$ such that $m - pm' = pm''$. Hence, $m = pm' + pm'' = p(m' + m'')$. □

Lemma 1.2.6. *Let M and K be modules and $M' \leq M$. If* $\text{Hom}(M', K) = 0 = \text{Hom}(M/M', K)$*, then* $\text{Hom}(M, K) = 0$.

Proof. Take a homomorphism $\varphi : M \to K$. Since $\varphi|_{M'} \in \text{Hom}(M', K)$, we know that $\varphi|_{M'} = 0$ by assumption. So $M' \leq \ker\varphi$. Then we can define $\tilde{\varphi} : M/M' \to K$ with $\tilde{\varphi}(m + M') = \varphi(m)$. But $\tilde{\varphi} \in \text{Hom}(M/M', K)$. So $\tilde{\varphi}$ must be zero by assumption. Hence $\varphi(m) = 0$ for all $m \in M$. This means that $\varphi = 0$, as desired. □

We give some useful features about pure submodules over an integral domain in the following theorem.

Theorem 1.2.7. *The following conditions hold for pure submodules.*

1. *If M is a torsion free module, then a submodule M' of M is pure in M if and only if M/M' is torsion free.*
2. *The union of a chain of pure submodules of a module is still a pure submodule.*

3. *If $M_2 \subset M_1$, are submodules of M such that M_2 is pure in M and M_1/M_2 is pure in M/M_2 then M_1 is pure in M.*
4. *If M is a torsion free R-module, M_1 a submodule of M, and M_2, a pure submodule of M then $M_1 \cap M_2$ is a pure submodule of M_1.*
5. *Let M be a torsion free R-module, M_1 a submodule of M, and M_2 a pure submodule of M. If $M_2 \leq M_1$, then M_2 is pure in M_1.*

Proof. (1) ($\Rightarrow$) Take an element $m + M'$ of M/M' and a nonzero element r of R. Say $r(m + M') = M'$. Then $rm \in rM \cap M'$. By purity, there is an element m' such that $rm = rm'$. So, we have $r(m - m') = 0$. Since M is torsion free, we get that $m - m' = 0$, and so $m = m'$. It says that $m + M' = m' + M' = M'$, as desired.

($\Leftarrow$) Take a nonzero $m' = rm \in rM \cap M'$. Then $r(m + M') = rm + M' = m' + M' = M'$. Since M/M' is torsion free, we know that $m + M' = M'$. So, we get $m \in M'$. Therefore, we conclude $m' = rm \in rM'$, as desired.

(2) Let I be a totally ordered index set. Take a chain $\mathcal{C} = (M_i)_{i \in I}$ of pure submodules of a module M. We will show that $\bigcup_{i \in I} M_i$ is a pure submodule of M. Take $rm \in rM \cap \bigcup_{i \in I} M_i$ where $m \in M$. Since $rm \in \bigcup_{i \in I} M_i$, we have that $rm \in M_j$ for some $j \in I$. By purity, there is an element $m_j \in M_j$ such that $rm = rm_j$. Then, $rm = rm_j \in rM_j \subseteq r \bigcup_{i \in I} M_i$.

(3) Take $m_1 = rm \in rM \cap M_1$. Then, $m_1 + M_2 = rm + M_2$. By purity, there is an element $m_1' + M$ in M_1/M_2 such that $m_1 + M_2 = rm + M_2 = rm_1' + M_2$. So $r(m - m_1') \in M_2$. Say $r(m - m_1') = m_2$. By purity, there is an element $m_2' \in M_2$ such that $r(m - m_1') = m_2 = rm_2'$. Then, $m_1 - rm_1' = rm - rm_1' = rm_2'$. Hence, we have $m_1 = r(m_1' + m_2')$ where $m_1' + m_2' \in M_1 + M_2 = M_1$.

Moreover, there is an easier way to prove by using (1) with saying M/M_2 and $(M/M_2)/(M_1/M_2) \cong M/M_1$ are both torsion free.

(4) Take $m_1 \in M_1 \cap M_2$ and say $m_1 = rm_1'$, where $r \in R$ and $m_1' \in M_1 \leq M$. By purity of M_2, we have that there is an element $m_2 \in M_2$ such that $m_1 = rm_2$. Since

M is torsion free, we have that $m_1' = m_2 \in M_1 \cap M_2$, as desired.

(5) It follows by (4). $\square$

1.2.2 Pullback and Pushout Diagrams

In category theory, a pullback (resp. a pushout) can be seen as the limit (resp. colimit) of a diagram consisting of two morphisms $f : X \to Z$ and $g : Y \to Z$ (resp. $f : Z \to X$ and $g : Z \to Y$). The pullback of two morphisms f and g may not exist. If the pullback exists, it is uniquely determined by these two morphisms as with many universal construction.

It is well-known fact that the pullbacks exist in the category of modules over a fixed ring, and so in the category of abelian groups. We just use pullback diagrams for modules over an integral domain, as we stated before in the introduction.

Definition 1.2.3. *Let M, M' and N be modules with morphisms $g : M \to N$ and $f : M' \to N$. A module P with morphisms $\rho : P \to M$ and $\rho' : P \to M'$ is called the pullback of f and g if the following conditions are satisfied.*

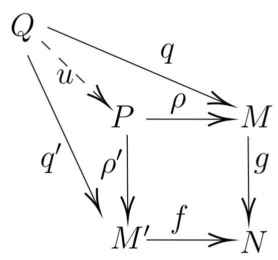
1. *The following diagram commutes:*

$$\begin{array}{ccc} P & \xrightarrow{\rho} & M \\ {\scriptstyle \rho'}\downarrow & & \downarrow{\scriptstyle g} \\ M' & \xrightarrow{f} & N \end{array} .$$

2. *If the following diagram commutes for a module Q and morphisms q and q',*

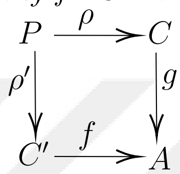
$$\begin{array}{ccc} Q & \xrightarrow{q} & M \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle g} \\ M' & \xrightarrow{f} & N \end{array}$$

then there must be a unique morphism $u : Q \to P$ such that $\rho u = q$ and $\rho u = q$. It can be illustrated as the following commutative diagram with emphasizing the uniqueness of u.



Next lemma has too much significance for our main Theorem 3.3.6, but we put it here because of that the pullback is a fundamental tool.

Lemma 1.2.8. *Let P be the pullback of $f : C' \to A$ and $g : C \to A$*



in a commutative diagram of abelian groups. Then,

1. $\operatorname{im}\rho = g^{-1}(\operatorname{im} f)$

2. $\ker\rho \cong \ker f$*;*

3. For a subgroup C'_1 of C', the pullback of $f|_{C'_1} : C'_1 \to A$ and $g : C \to A$ is the complete inverse image $(\rho')^{-1}(C'_1)$ of C'_1 under $\rho' : P \to C'$ with the pullback maps are the restriction maps $\rho|_{(\rho')^{-1}(C'_1)} : (\rho')^{-1}(C'_1) \to C$ and $\rho'|_{(\rho')^{-1}(C'_1)} : (\rho')^{-1}(C'_1) \to C'_1$.

4. For a subgroup M of $\ker f$*, the pullback of $f : C'/M \to A$ and $g : C \to A$ is the quotient group $P/(\rho')^{-1}(M)$.*

Proof. (1) Take an element $c \in \operatorname{im}\rho$. So, there is an element $(c', c) \in P$ such that $\rho(c', c) = c$. So, $f(c') = g(c)$ by definition of the pullback P. However, $f(c') \in \operatorname{im} f$, and so $c \in g^{-1}(\operatorname{im} f)$. Take an element $c \in g^{-1}(\operatorname{im} f)$, and so for some element $f(c') \in \operatorname{im} f$, we have that $g(c) = f(c')$. By definition of the pullback $(c', c) \in P$, and so $\rho(c', c) = c \in \operatorname{im}\rho$.

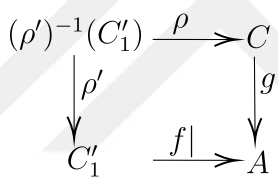
(2) We will interest in the restricted map $\rho'|_{\ker\rho} : \ker\rho \to C'$. Clearly, this map is an injection since $\ker(\rho'|_{\ker\rho}) = \ker\rho \cap \ker\rho' = 0$ by the definition of pullback. So, $\ker\rho \cong \operatorname{im}(\rho'|_{\ker\rho})$. Now, we must prove $\operatorname{im}(\rho'|_{\ker\rho}) = \ker f$. Let $c' \in \operatorname{im}(\rho'|_{\ker\rho})$.

Then, for some $p \in \ker \rho$, we have that $\rho'(p) = c'$. And we also have $\rho(p) = 0$ since $p \in \ker \rho$. Since P is pullback and $p \in P$, we have that $f(c') = g(0) = 0$, and so $c' \in \ker f$. Let $c' \in \ker f$. So $f(c') = 0$. Then, $(c', 0) \in P$ since $f(c') = 0 = g(0)$. Clearly, $\rho(c', 0) = 0$, and so $(c', 0) \in \ker \rho$. However, we have $\rho'(c', 0) = c' \in \ker f$. Also, we can see that

$$\ker \rho = \{(c', 0) : c' \in \ker f\}.$$

So it is easy to get the isomophism between $\ker \rho$ and $\ker f$. From now on, we will not notice any difference between $\ker \rho$ and $\ker f$. For example, we will use the isomorphism theorem as the follows. $P/\ker f \cong \operatorname{im} \rho$ or we will say that P/M is a quotient group of P where $M \leq \ker f$.

(3) It is not ambiguous to notate the restriction maps with ρ, ρ' and $f|$. So, we have the following commutative diagram:

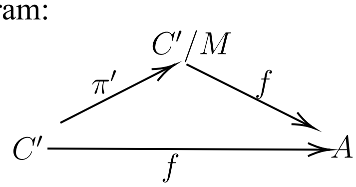


Because, for some element $(c_1', c) \in (\rho')^{-1}(C_1')$ is also in P, and so we have that $\rho(c_1', c) = c$ and $\rho'(c_1', c) = c_1'$ with $f|(c_1') = f(c_1') = g(c)$. It is also easy to see that

$$(\rho')^{-1}(C_1') = \{(c_1', c) : f|(c_1') = f(c_1') = g(c), c_1' \in C_1', c \in C\} \leq P$$

by the definition of the complete inverse image, and so it is also easy to see that $(\rho')^{-1}(C_1')$ is the desired pullback of $f|_{C_1'} : C_1' \to A$ and $g : C \to A$ by the definition of the pullback for abelian groups.

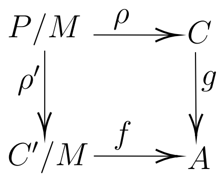
(4) We know that $f : C'/M \to A$ is well-defined with $f(c' + M) = f(c')$ since $M \leq \ker f$. Let π' be the conanical surjection from C' into C'/M. Then, we have the following commutative diagram:



It is easy to recognise that $f(C'/M) = f(C')$. By definition, we have

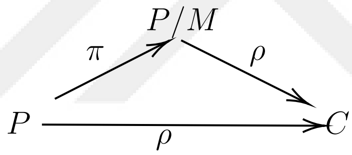
$$P/M = \{(c' + M, c) : f(c' + M) = f(c') = g(c)\} \leq C'/M \oplus C.$$

Define $\rho : P/M \to C$ with $\rho(c' + M, c) = c$, and so $\rho(P/M) = \rho(P)$ since $f(C/M) = f(C)$. Similarly, define $\rho' : P/M \to C/M$ with $\rho'(c' + M, c) = c' + M$. And so, the following diagram:

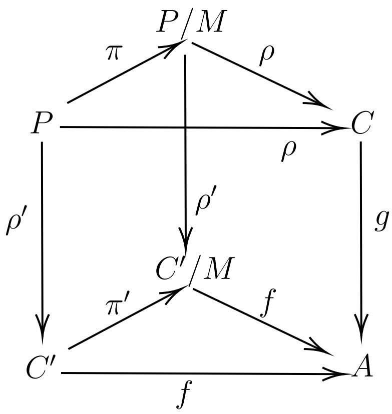


commutes. So, P/M is the pullback of $f : C'/M \to A$ and $g : C \to A$ by the definition of P/M and the definition of the pullback for abelian groups.

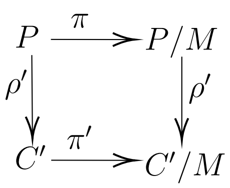
Define the conanical surjection π from P into P/M. Then we have the following commutative diagram:



since M is also in ker ρ ($\cong$ ker f) by Condition (2). To see all in one graph,



To say the previous diagram is commutative, we only need to say the following diagram commutes.



But, we can easily see that

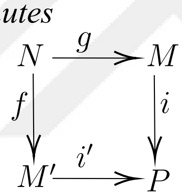
$$\rho'\pi(c, c') = \rho'(c' + M, c) = c' + M = \pi'(c') = \pi'\rho'(c', c).$$

□

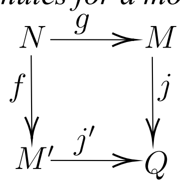
As a dual consept of a pullback, we have the following definition for a pushout.

Definition 1.2.4. *Let M, M' and N be modules over an integral domain R with morphisms $g : N \to M$ and $f : N \to M'$. An R-module P with two morphisms $i : M \to P$ and $i' : M' \to P$ is called the pushout of f and g if the following conditions are satisfied.*

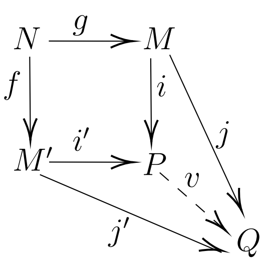
1. *The following diagram commutes*



2. *If the following diagram commutes for a module Q with the morphisms j and j',*



then there must be a unique morphism $v : P \to Q$ such that $vi = j$ and $vi' = j'$. It can be illustrated as the following commutative diagram with emphasizing the uniqueness of v.



CHAPTER TWO
TORSION FREE COVERS OF LINE QUIVERS

In this chapter, we will give our results for a torsion free precover of an object in the category $(q_n, \text{R-Mod})$ or just q_n. M. Dunkum proved the existence of TFCs in q_n, and gave a construction for the TFC of an object in q_n (see Wesley (2005)). See also Özdemir (2011) for the existence of TFCs in the category $(Q, \text{R-Mod})$ for a wide class of quivers (including line quivers) under certain conditions on the ring R.

2.1 Line Quivers

A *quiver* is a directed graph whose edges are called arrows. Usually, a quiver is denoted by Q understanding that $Q = (V, E)$ where V is the set of vertices (dots) and E is the set of arrows. An arrow of a quiver from a vertex v_1 to a vertex v_2 is denoted by $a : v_1 \to v_2$ or $v_1 \xrightarrow{a} v_2$.

A representation by modules of a given quiver Q is determined by giving a module $X(v)$ to each vertex v and a homomorphism $X(a) : X(v_1) \to X(v_2)$ to each arrow $a : v_1 \to v_2$ of Q. A morphism μ between two representations X and Y is a natural transformation, so it will be a family $\{\mu_v\}_{v \in V}$ of module homomorphisms such that $Y(a)\mu_{v_1} = \mu_{v_2}X(a)$ for every arrow $a : v_1 \to v_2$ of Q, that is, the following diagram commutes for every arrow $a : v_1 \to v_2$ of Q:

$$\begin{array}{ccc} X(v_1) & \xrightarrow{X(a)} & X(v_2) \\ \downarrow{\scriptstyle \mu_{v_1}} & & \downarrow{\scriptstyle \mu_{v_2}} \\ Y(v_1) & \xrightarrow{Y(a)} & Y(v_2) \end{array}$$

Note that the representations by modules of a quiver Q over a ring R form a (functor) category, denoted by $(Q, \text{R-Mod})$. Therefore, it is an abelian category by (Stenström, 1975, Chapter IV, Proposition 7.1). So, we notice that the *kernels*, *cokerneles*, *products* and *sums* are constructed 'componentwise' in this category. For the sake of simplicity, we just say a 'representation' or an 'object' in this category for a representation by modules of a quiver over a ring R. See, for example, Schiffler (2014) for more details about quiver representations. See also Özdemir (2011) for some research on relative homological algebra in this category for any ring R and any quiver Q (not necessarily finite).

In this thesis, we are interested in this category only for finite line quivers. Namely, for the quivers q_n of the form

$$q_n : v_1 \xrightarrow{a_1} v_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} v_n$$

We denote the category $(q_n, \text{R-Mod})$ by q_n for short.

Note that the objects in q_n are of the form $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$, where M_i's are modules and f_j's are module homomorphisms. So, a morphism between two objects in this category will consist of a n-tuple of maps $(\alpha_1, \alpha_2, \cdots, \alpha_n) : (M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n) \to (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} N_n)$ such that $\alpha_i : M_i \to N_i$ with $g_j\alpha_j = \alpha_{j+1}f_j$ for all $j \in \{1, 2, \cdots, n-1\}$, that is, the following diagram commutes:

$$\begin{array}{ccccccccccc}
M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & M_{n-1} & \xrightarrow{f_{n-1}} & M_n \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_n} \\
N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & N_{n-1} & \xrightarrow{g_{n-1}} & N_n
\end{array}$$

2.2 Torsion Free Covers of Line Quivers

Firstly, we will define what is the torsion free precover and the TFC of an object in the category q_n, where R is an integral domain.

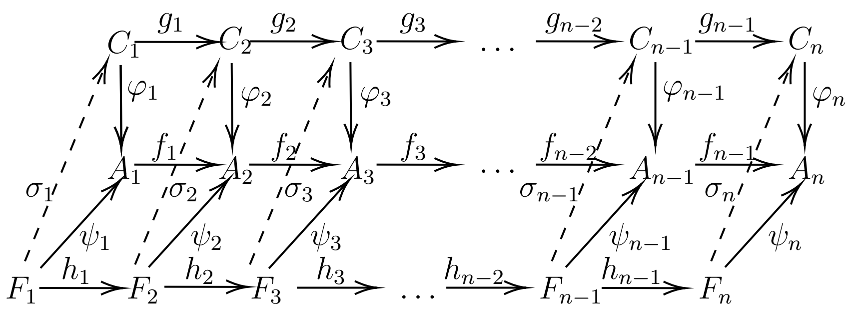
Definition 2.2.1. *An object* $N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} N_n$ *in* q_n *is said to be a subobject of an object* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ *if* $N_i \leq M_i$ *and* $f_j|_{N_i} = g_j$ *for all* $i =$

$1, 2, \cdots, n$ *and* $j = 1, 2, \cdots, n-1$.

Definition 2.2.2. *An object* $T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} T_n$ *in* q_n *is called torsion if all* T_i*'s are torsion modules. And, an object* $F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} F_n$ *in* q_n *is called torsion free if all* F_i*'s are torsion free modules.*

Definition 2.2.3. *A subobject* $N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} N_n$ *of* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ *in* q_n *is said to be a pure subobject of* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ *if* N_i *is a pure submodule of* M_i *for all* $i = 1, 2, \cdots, n$*. In other words,* $rN_i = rM_i \cap N_i$ *for all* $r \in R$ *and* $i = 1, 2, \cdots, n$.

Definition 2.2.4. *If* $C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n$ *is a torsion free object in* q_n*, then the morphism*
$(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *is said to be a torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$*, if, for every morphism*
$(\psi_1, \psi_2, \cdots, \psi_n) : (F_1 \xrightarrow{h_1} F_2 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} F_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *where* $F_1 \xrightarrow{h_1} F_2 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} F_n$ *is torsion free, there is a morphism* $(\sigma_1, \sigma_2, \cdots, \sigma_n) : (F_1 \xrightarrow{h_1} F_2 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} F_n) \to (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n)$ *such that* $(\varphi_1, \varphi_2, \cdots, \varphi_n) \circ (\sigma_1, \sigma_2, \cdots, \sigma_n) = (\psi_1, \psi_2, \cdots, \psi_n)$*, that is, the following diagram commutes:*



Definition 2.2.5. *The above morphism* $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *is said to be a TFC, if the following conditions are satisfied:*

1. $(\varphi_1, \varphi_2, \cdots, \varphi_n)$ *is the torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$*, and*

2. *If P_i is a pure submodule of C_i contained in* $\ker \varphi_i$*, P_{i+1} is a pure submodule of C_{i+1} contained in* $\ker \varphi_{i+1}$*, and if $g_i(P_i) \subseteq P_{i+1}$, then $P_i = P_{i+1} = 0$ for all $i = 1, 2, \cdots, n-1$.*

Note that the second condition is equivalent to the fact that the kernel of the covering mapping has no nonzero pure subobject where the kernel in q_n is defined as

$$\ker(\varphi_1, \varphi_2, \cdots, \varphi_n) := \ker \varphi_1 \to \ker \varphi_2 \to \cdots \to \ker \varphi_n.$$

Remark 1. *It is known that every object in q_n has a TFC when R is an integral domain (see (Wesley, 2005, Theorem 6.5-Example 5.5)). When proving the existence of a TFC in q_n, M. Dunkum used the induction method. Indeed, to construct the cover of the object $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ in q_n, she first assumed that the morphism $(\varphi_2, \varphi_3, \cdots, \varphi_n) : (C_2 \xrightarrow{g_2} C_3 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n)$ was a TFC of $A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n$ in q_{n-1}, and after that $g_1 : C_1 \to C_2$ and $\varphi_1 : C_1 \to A_1$ were found out with a construction to say that $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ is the desired TFC.*

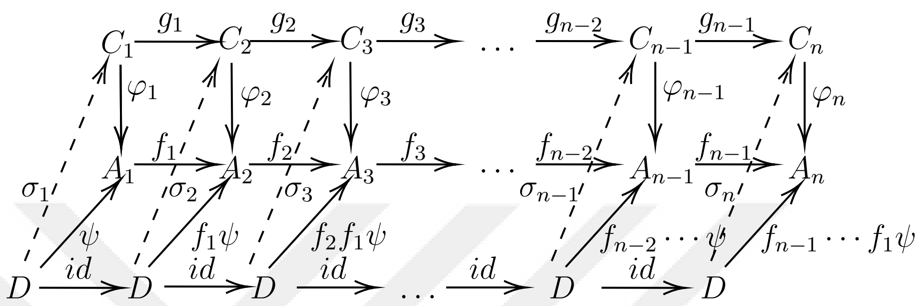
Motivated by this method of proof, we introduce a reduction of an object as follows.

Remark 2. *To 'reduce' an object in q_n to an object in q_i where $i \leq n$ will be practical in our study. So, from now on, by a 'reduction' of an object $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ in q_n to an object in q_i, we mean the object $A_{n-i+1} \xrightarrow{f_{n-i+1}} A_{n-i+2} \xrightarrow{f_{n-i+2}} \cdots \xrightarrow{f_{n-1}} A_n$. Therefore, we can easily see that the TFC of the reduction $A_{n-i+1} \xrightarrow{f_{n-i+1}} A_{n-i+2} \xrightarrow{f_{n-i+2}} \cdots \xrightarrow{f_{n-1}} A_n$ is the reduction of the TFC of the object $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$.*

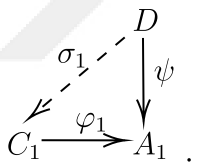
The following lemma will be useful in proving some of our main results.

Lemma 2.2.1. *If $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ is a torsion free precover of $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ in q_n, then $\varphi_i : C_i \to A_i$ is a torsion free precover of A_i in* R-Mod *for all $i = 1, 2, \cdots, n$.*

Proof. By Remark 2, it is enough to prove the statement only for $i = 1$. Suppose that D is any torsion free module and $\psi : D \to A_1$ is a module homomorphism. Clearly, $D \xrightarrow{id} D \xrightarrow{id} \cdots \xrightarrow{id} D$ is a torsion free object in q_n, and we have a morphism $(\psi, f_1\psi, f_2f_1\psi, \cdots, f_{n-2}f_{n-3}\cdots f_2f_1\psi, f_{n-1}f_{n-2}\cdots f_2f_1\psi) : (D \xrightarrow{id} D \xrightarrow{id} \cdots \xrightarrow{id} D) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ in q_n. So we have the following commutative diagram by assumption:



Finally, we find a homomorphism σ_1 which makes the following diagram is commutative:



That is, $\varphi_1 : C_1 \to A_1$ is a torsion free precover of A_1.

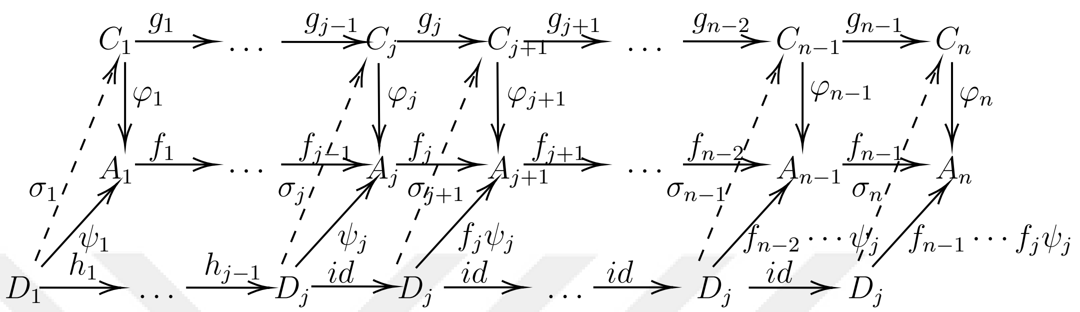
□

More general version of the previous lemma is also true.

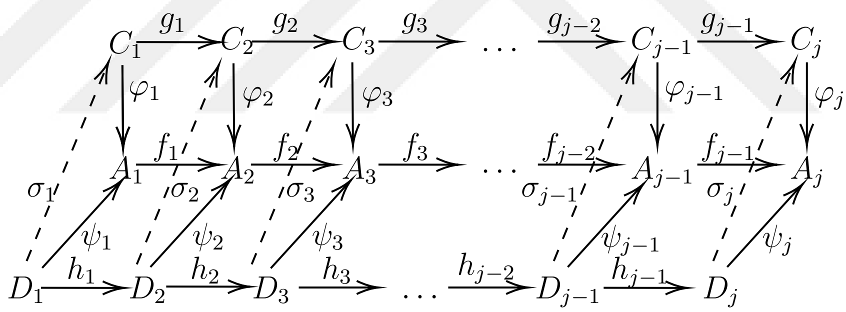
Lemma 2.2.2. *If* $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *is a torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ *in* q_n, *then* $(\varphi_i, \cdots, \varphi_j) : (C_i \xrightarrow{g_i} \cdots \xrightarrow{g_{j-1}} C_j) \to (A_i \xrightarrow{f_j} \cdots \xrightarrow{f_{j-1}} A_j)$ *is a torsion free precover of* $A_i \xrightarrow{f_j} \cdots \xrightarrow{f_{j-1}} A_j$ *in the category* q_{j-i+1}.

Proof. By Remark 2, It is enough to prove the statement for $i = 1$. Suppose that $D_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{j-1}} D_j$ is a torsion free object, and

$(\psi_1, \cdots, \psi_j) : (D_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{j-1}} D_j) \to (A_1 \xrightarrow{f_j} \cdots \xrightarrow{f_{j-1}} A_j)$ is a morphism in q_j. Clearly, $D_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{j-1}} D_j \xrightarrow{id} D_j \xrightarrow{id} \cdots \xrightarrow{id} D_j$ is a torsion free object in q_n, and we have a morphism $(\psi_1, \cdots, \psi_j, f_j\psi_j, f_{j+1}f_j\psi_j, \cdots, f_{n-1}f_{n-2}\cdots f_{j+1}f_j\psi_j)$: $(D_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{j-1}} D_j \xrightarrow{id} D_j \xrightarrow{id} \cdots \xrightarrow{id} D_j) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$. So we have the following commutative diagram by precover:



Hence, we get the morphism $(\sigma_1, \sigma_2, \cdots, \sigma_j)$ such that the following diagram commutes:



□

Remark 3. *Observe that we can find the corresponding torsion free precovers of the objects* $(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} A_i)$ *and* $(A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_{n-1}} A_n)$ *in* q_i *and* q_{n-i}*, respectively if we already know a torsion free precover of the object* $(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *in* q_n*. But the converse is not true in general.*

More precisely, let $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *be a torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ *in* q_n*, and* $(\varphi'_1, \varphi'_2, \cdots, \varphi'_m) : (C'_1 \xrightarrow{g'_1} C'_2 \xrightarrow{g'_2} \cdots \xrightarrow{g_{m-1}'} C'_m) \to (A'_1 \xrightarrow{f'_1} A'_2 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{m-1}} A'_m)$ *is a torsion free precover of* $A'_1 \xrightarrow{f'_1} A'_2 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{n-1}} A'_m$ *in* q_m*. We have an object*

$(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f} A'_1 \xrightarrow{f'_1} A'_2 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{n-1}} A'_m)$ *in* q_{n+m}. *Because of the zero map, we know that there is at least one homomorphism such as* $f : A_n \to A'_1$.

Moreover, we have a composition map $f\varphi_n : C_n \to A'_1$, *and so we have that there is a homomorphism* $g : C_n \to C'_1$ *such that* $f\varphi_n = \varphi'_1 g$ *since* $\varphi'_1 : C'_1 \to A'_1$ *is a torsion free precover by Lemma 2.2.1. So we have a morphism* $(\varphi_1, \varphi_2, \cdots, \varphi_n, \varphi'_1, \varphi'_2, \cdots, \varphi'_m) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n \xrightarrow{g} C'_1 \xrightarrow{g'_1} C'_2 \xrightarrow{g'_2} \cdots \xrightarrow{g_{m-1'}} C'_m) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f} A'_1 \xrightarrow{f'_1} A'_2 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{n-1}} A'_m)$ *in* q_{n+m}. *But this map does not have to be a torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f} A'_1 \xrightarrow{f'_1} A'_2 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{n-1}} A'_m$ *in* q_{n+m}. *As a counter example, take* $n = 1$ *and* $m = 1$.

Lemma 2.2.3. *If* $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *is a torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ *in* q_n, *then* $(\varphi_1, \varphi_n) : (C_1 \xrightarrow{g_{n-1}g_{n-2}\cdots g_1} C_n) \to (A_1 \xrightarrow{f_{n-1}f_{n-2}\cdots f_1} A_n)$ *is a torsion free precover of* $A_1 \xrightarrow{f_{n-1}f_{n-2}\cdots f_1} A_n)$ *in* q_2.

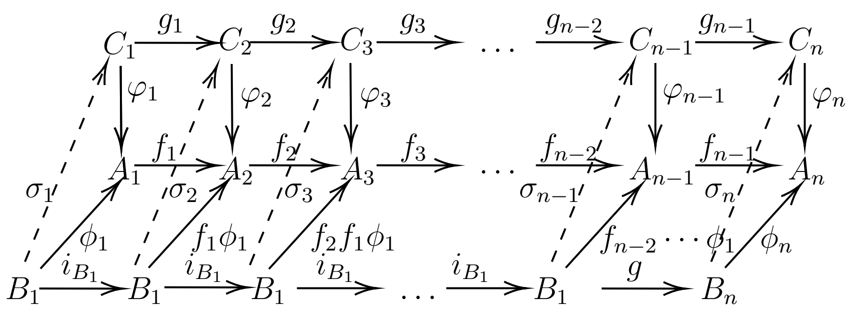
Proof. Let $B_1 \xrightarrow{g} B_n$ be a torsion free object in q_2 with the morphism $(\phi_1, \phi_n) : (B_1 \xrightarrow{g} B_n) \to (A_1 \xrightarrow{f_{n-1}\cdots f_1} A_n)$. So, we have the following commutative diagram:

$$\begin{array}{ccc} B_1 & \xrightarrow{g} & B_n \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_n} \\ A_1 & \xrightarrow{f_{n-1}\cdots f_1} & A_n \end{array}$$

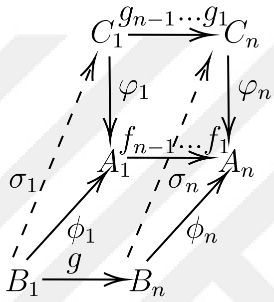
So, we also have the following commutative diagram:

$$\begin{array}{ccccccccccc} B_1 & \xrightarrow{i_{B_1}} & B_1 & \xrightarrow{i_{B_1}} & B_1 & \xrightarrow{i_{B_1}} & \cdots & \xrightarrow{i_{B_1}} & B_1 & \xrightarrow{g} & B_n \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle f_1\phi_1} & & \downarrow{\scriptstyle f_2f_1\phi_1} & & & & \downarrow{\scriptstyle f_{n-2}\cdots f_1\phi_1} & & \downarrow{\scriptstyle \phi_n} \\ A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n \end{array}$$

Now, using the precover of $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$, we get the following commutative diagaram:



Therefore, we have the following commutative diagram, which completes the proof:



□

Note that the previous lemma is true for any pair of morphisms.

Proposition 2.2.4. *If* $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *is a torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ *in* q_n*, then* $(\varphi_i, \varphi_{i+k}) : (C_i \xrightarrow{g_{i+k-1}g_{i+k-2}\cdots g_i} C_{i+k}) \to (A_i \xrightarrow{f_{i+k-1}f_{i+k-2}\cdots f_i} A_{i+k})$ *is a torsion free precover of* $A_i \xrightarrow{f_{i+k-1}f_{i+k-2}\cdots f_i} A_{i+k}$ *in* q_2.

Proof. It is trivial by using both Lemma 2.2.2 and Lemma 2.2.3. □

There is a generalisation of all we do above. We will say that a version of the previous proposition is also true for more than one composition map. For the sake of simplicity, it will be stated without saying composition maps.

Theorem 2.2.5. *If* $(\varphi_1, \varphi_2, \cdots, \varphi_n) : (C_1 \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n)$ *is a torsion free precover of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ *in*

q_n*, then* $(\varphi_i, \varphi_{i+k_1}, \varphi_{i+k_2}, \cdots, \varphi_{i+k_m}) : (C_i \to C_{i+k_1} \to C_{i+k_2} \to \cdots \to C_{i+k_m}) \to (A_i \to A_{i+k_1} \to A_{i+k_2} \to \cdots \to A_{i+k_m})$ *is a torsion free precover of the object* $A_i \to A_{i+k_1} \to A_{i+k_2} \to \cdots \to A_{i+k_m}$ *in* q_{m+1}.

Proof. More clearly, we want to get $(\varphi_i, \varphi_{i+k_1}, \varphi_{i+k_2}, \cdots, \varphi_{i+k_m}) : (C_i \xrightarrow{g_{i+k_1-1}\cdots g_i} C_{i+k_1} \xrightarrow{g_{i+k_2-1}\cdots g_{i+k_1}} C_{i+k_2} \xrightarrow{g_{i+k_3-1}\cdots g_{i+k_2}} \cdots \xrightarrow{g_{i+k_m-1}\cdots g_{i+k_{m-1}}} C_{i+k_m}) \to (A_i \xrightarrow{f_{i+k_1-1}\cdots f_i} A_{i+k_1} \xrightarrow{f_{i+k_2-1}\cdots f_{i+k_1}} A_{i+k_2} \xrightarrow{f_{i+k_3-1}\cdots f_{i+k_2}} \cdots \xrightarrow{f_{i+k_m-1}\cdots f_{i+k_{m-1}}} A_{i+k_m})$
is a torsion free precover of the object
$A_i \xrightarrow{f_{i+k_1-1}\cdots f_i} A_{i+k_1} \xrightarrow{f_{i+k_2-1}\cdots f_{i+k_1}} A_{i+k_2} \xrightarrow{f_{i+k_3-1}\cdots f_{i+k_2}} \cdots \xrightarrow{f_{i+k_m-1}\cdots f_{i+k_{m-1}}} A_{i+k_m})$
in q_{m+1}.

It can be proven by induction on m with using Proposition 2.2.4 as an initial case.

□

2.3 A Construction for Torsion Free Covers of Line Quivers

The following theorem gives a construction for TFCs in q_n, and it is a foundation in this study (see the proof of Theorem 6.5 in Wesley (2005)).

Theorem 2.3.1. *Wesley (2005) Let* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n$ *be an object in* q_n*. Let* $\varphi_i : C_i \to A_i$ *be the TFC of* A_i *in* R-Mod *with* $\ker \varphi_i = K_i$*. Then,*

$$\begin{array}{ccccccccc}
P_1/M_1 & \xrightarrow{\rho_1} & P_2/M_2 & \xrightarrow{\rho_2} & \cdots & \xrightarrow{\rho_{n-2}} & P_{n-1}/M_{n-1} & \xrightarrow{\rho_{n-1}} & C_n \\
\downarrow{\scriptstyle \varphi_1\rho_1'} & & \downarrow{\scriptstyle \varphi_2\rho_2'} & & & & \downarrow{\scriptstyle \varphi_{n-1}\rho_{n-1}'} & & \downarrow{\scriptstyle \varphi_n} \\
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n
\end{array}$$

is a TFC of $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n$ *in* q_n*, where, in the first square,* P_{n-1} *is the pullback of* $f_{n-1}\varphi_{n-1} : C_{n-1} \to A_n$ *and* $\varphi_n : C_n \to A_n$*, where* $\rho_{n-1}' : P_{n-1} \to C_{n-1}$ *and* $\rho_{n-1} : P_{n-1} \to C_n$ *are the pullback maps, and* M_{n-1} *is a maximal pure submodule of* $\ker(f_{n-1}\varphi_{n-1})$ *contained in* K_{n-1}*; and also,* P_i *is the pullback of* $f_i\varphi_i : C_i \to A_{i+1}$ *and* $\varphi_{i+1}\rho_{i+1}' : P_{i+1}/M_{i+1} \to A_{i+1}$*, where* $\rho_i' : P_i \to C_{i-1}$ *and*

$\rho_i : P_i \to C_i$ are the pullback maps, and M_i is a maximal pure submodule of $\ker(f_i\varphi_i)$ *contained in K_i for each $i = 1, 2, \cdots, n-2$.*

Proof. For the case $n = 2$, we refer (Wesley (2005), Lemma 6.1). In the general case, this construction can be found easily by induction on n with using (Wesley (2005), Theorem 6.5). □

Lemma 2.3.2. *The kernel of the covering map in the previous theorem is of the form:*

$$\bigoplus_{j=1}^{n} K_j/M_j \xrightarrow{\rho_1} \bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} K_{n-1}/M_{n-1} \oplus K_n \xrightarrow{\rho_{n-1}} K_n.$$

Proof. By (Wesley (2005), Proposition 6.2) and (Wesley (2005), Theorem 6.4), we know that

$$\ker(\varphi_{n-1}\rho'_{n-1}, \varphi_n) = K_{n-1}/M_{n-1} \oplus K_n \xrightarrow{\rho_{n-1}} K_n.$$

Assume that the kernel of the covering map $(\varphi_2\rho'_2, \cdots, \varphi_{n-1}\rho'_{n-1}, \varphi_n)$: $(P_2/M_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} P_{n-1}/M_{n-1} \xrightarrow{\rho_{n-1}} C_n) \to (A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n)$ is of the form

$$\bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} K_{n-1}/M_{n-1} \oplus K_n \xrightarrow{\rho_{n-1}} K_n.$$

Firstly, we will show that $\ker(\varphi_1\rho'_1, \varphi_2\rho'_2, \cdots, \varphi_{n-1}\rho'_{n-1}, \varphi_n)$ of the precovering map $(\varphi_1\rho'_1, \varphi_2\rho'_2, \cdots, \varphi_{n-1}\rho'_{n-1}, \varphi_n)$: $(P_1 \xrightarrow{\rho_1} P_2/M_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} P_{n-1}/M_{n-1} \xrightarrow{\rho_{n-1}} C_n) \to (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n)$ is of the form

$$K_1 \oplus \bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_1} \bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} K_{n-1}/M_{n-1} \oplus K_n \xrightarrow{\rho_{n-1}} K_n.$$

So, we will just show that

$$\ker(\varphi_1\rho'_1) = K_1 \oplus \bigoplus_{j=2}^{n} K_j/M_j.$$

Assume $(k_1, (k_2, \cdots, k_n)) \to (k_2, \cdots, k_n) \to \cdots \to k_n$ is an element of the kernel

$\ker(\varphi_1\rho_1') \xrightarrow{\rho_1} \ker(\varphi_2\rho_2') \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-1}} \ker\varphi_n$ of the precovering mapping in q_n. So $(k_1,(k_2,\cdots,k_n)) \in \ker(\varphi_1\rho_1')$. This means $\varphi_1\rho_1'(k_1,(k_2,\cdots,k_n)) = \varphi_1(k_1) = 0$, and so $k_1 \in \ker\varphi_1 = K_1$. Moreover, we also have

$$0 = f_1\varphi_1\rho_1'(k_1,(k_2,\cdots,k_n)) = \varphi_2\rho_2'\rho_1(k_1,(k_2,\cdots,k_n)) = \varphi_2\rho_2'(k_2,\cdots,k_n)$$

by using the commutativity of the diagram in Theorem 2.3.1. So, we have

$$(k_2,\cdots,k_n) \in \ker(\varphi_2\rho_2') = \bigoplus_{j=2}^{n} K_j/M_j.$$

Thus, we get

$$(k_1,(k_2,\cdots,k_n)) \in K_1 \oplus \bigoplus_{j=2}^{n} K_j/M_j.$$

Conversely, assume that $(k_1,(k_2,\cdots,k_n)) \to (k_2,\cdots,k_n) \to \cdots \to k_n$ is an element of the object $K_1 \oplus \bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_1} \bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-1}} K_n$ in q_n. Then, we have $(k_1,(k_2,\cdots,k_n)) \in K_1 \oplus \bigoplus_{j=2}^{n} K_j/M_j$. So, we have

$$\varphi_1\rho_1'(k_1,(k_2,\cdots,k_n)) = \varphi_1(k_1) = 0$$

since $k_1 \in K_1 = \ker\varphi_1$. Thus, $(k_1,(k_2,\cdots,k_n)) \in \ker(\varphi_1\rho_1')$. We showed that $\ker(\varphi_1\rho_1' : P_1 \to A_1) = K_1 \oplus \bigoplus_{j=2}^{n} K_j/M_j$.

Hence, it is easy to see that $\ker(\varphi_1\rho_1' : P_1/M_1 \to A_1) = K_1/M_1 \oplus \bigoplus_{j=2}^{n} K_j/M_j = \bigoplus_{j=1}^{n} K_j/M_j$, as desired. □

CHAPTER THREE
LINE QUIVERS HAVING A TRIVIAL COGALOIS GROUP

In Hill (2008), coGalois groups of TFCs of quiver representations have been studied for q_2 and $R = \mathbb{Z}$. A classification for coGalois groups to be trivial were given in $(q_2, \mathbb{Z}\text{-Mod})$. Using Hill's ideas and results, we generalize (Hill, 2008, Theorem 5.1) to the category $(q_n, \mathbb{Z}\text{-Mod})$. We will use the notations in Theorem 2.3.1 and Lemma 2.3.2, throughout this chapter. Therefore, for an object $\mathbb{A} := A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n$ in q_n, the map $\varphi_i : C_i \to A_i$ will be the TFC of A_i in $\mathbb{Z}$-Mod with $\ker \varphi_i = K_i$ for each i. Moreover, M_i will be maximal in K_i among all pure subgroups of $\ker f_i\varphi_i$.

3.1 coGalois Groups of Line Quivers

For the objects $\mathbb{A} := A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ and $\mathbb{B} := B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} B_n$ in q_n, whenever the following diagram commutes,

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & & & \downarrow{\scriptstyle \alpha_{n-1}} & & \downarrow{\scriptstyle \alpha_n} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n
\end{array}$$

we denote the morphism $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ in q_n from $\mathbb{A}$ to $\mathbb{B}$ shortly by α. We also denote by $\text{Mor}(\mathbb{A}, \mathbb{B})$ the class of all morphisms between the objects $\mathbb{A}$ and $\mathbb{B}$ in q_n.

From now on, let $\varphi : \mathbb{C} \to \mathbb{A}$ be a TFC of $\mathbb{A}$ in q_n, and let $\mathbb{K} = \ker \varphi$. We know exactly what the TFC $\mathbb{C}$ with a covering morphism φ and its kernel $\mathbb{K}$, by Theorem 2.3.1 and Lemma 2.3.2. Note that the TFC of $\mathbb{A}$ is unique up to isomorphism and is actually an $\varkappa$-cover, where $\varkappa$ is a covering class of all torsion free objects in q_n.

Since coGalois groups can be introduced in any category with a covering class, we can define the coGalois group of $\mathbb{A}$ in q_n (see (Enochs et al., 2000, Definition 3.1)).

Definition 3.1.1. *The coGalois group of an object $\mathbb{A}$ in q_n is defined as the group of all automorphsims from the TFC of $\mathbb{A}$, denoted by $G(\mathbb{A})$ or $G(\varphi)$. That is,*

$$G(\mathbb{A}) = \{\sigma : \mathbb{C} \to \mathbb{C} \mid \varphi\sigma = \varphi\}.$$

The following important lemma generalizes (Enochs & Rada, 2005, Lemma 2.2), and the proof is done by a similar way.

Lemma 3.1.1. *The coGalois group of $\mathbb{A}$ is trivial if and only if Mor$(\mathbb{C}, \mathbb{K}) = 0$.*

Proof. ($\Rightarrow$) By assumption, we know that any morphism σ from the cover to itself, such that $\varphi\sigma = \varphi$, is the identity morphism $id_{\mathbb{C}}$. Assume that Mor$(\mathbb{C}, \mathbb{K}) \neq 0$ and take a nonzero morphism $\delta : \mathbb{C} \to \mathbb{C}$ with $\operatorname{im}\delta \leq \mathbb{K}$. We can easily see that $\varphi\delta = 0$. Now, we have a nontrivial object $\delta + id_{\mathbb{C}}$ in the coGalois group, because $\varphi(\delta + id_{\mathbb{C}}) = \varphi\delta + \varphi id_{\mathbb{C}} = 0 + \varphi = \varphi$ and δ is nonzero. It contradicts with $G(\mathbb{A}) = 1$. Thus, Mor$(\mathbb{C}, \mathbb{K}) = 0$.

($\Leftarrow$) Say $G(\mathbb{A}) \neq 1$. Take a nontrivial morphism $\sigma : \mathbb{C} \to \mathbb{C}$ from $G(\mathbb{A})$, and so $\varphi\sigma = \varphi$. Then we get a nonzero morphism $id_{\mathbb{C}} - \sigma : \mathbb{C} \to \mathbb{C}$ with im$(id_{\mathbb{C}} - \sigma) \leq \mathbb{K}$, since we have $\varphi(id_{\mathbb{C}} - \sigma) = \varphi - \varphi\sigma = \varphi - \varphi = 0$. Hence $id_{\mathbb{C}} - \sigma$ is actually a nonzero morphism from $\mathbb{C}$ into $\mathbb{K}$. It means that Mor$(\mathbb{C}, \mathbb{K}) \neq 0$, as desired. □

Lemma 3.1.2. *If the coGalois group of the object $\mathbb{A} := A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ is trivial, then the coGalois group of the abelian group A_n is trivial.*

Proof. Assume that $G(A_n) \neq 1$. Then Hom$(C_n, K_n) \neq 0$ by Lemma 3.1.1, taking $n = 1$. So, let $0 \neq \delta_n \in$ Hom(C_n, K_n). Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
P_1/M_1 & \xrightarrow{\rho_1} & P_2/M_2 & \xrightarrow{\rho_2} & \cdots & \xrightarrow{\rho_{n-2}} & P_{n-1}/M_{n-1} & \xrightarrow{\rho_{n-1}} & C_n \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & & & \downarrow{\scriptstyle \delta_{n-1}} & & \downarrow{\scriptstyle \delta_n} \\
\bigoplus_{i=1}^{n} K_i/M_i & \xrightarrow{\rho_1} & \bigoplus_{i=2}^{n} K_i/M_i & \xrightarrow{\rho_2} & \cdots & \xrightarrow{\rho_{n-2}} & K_{n-1}/M_{n-1} \oplus K_n & \xrightarrow{\rho_{n-1}} & K_n
\end{array}$$

where $\delta_i = \delta_{i+1}\rho_i$ for all $i = 1, 2, \ldots, n-1$. Then $(\delta_1, \delta_2, \ldots, \delta_n)$ is a nonzero map from the TFC of $\mathbb{A}$ to the kernel of the covering map. Hence $G(\mathbb{A}) \neq 1$ by Lemma 3.1.1. $\square$

With a similar proof, more general version is also true.

Lemma 3.1.3. *If the coGalois group of an object $\mathbb{A}$ is trivial, then the coGalois group of each reduction of $\mathbb{A}$ is trivial.*

Proof. Let $A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} A_n$ be the $(n-i+1)$*th* reduction of $\mathbb{A}$. Assume that $G(A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} A_n) \neq 1$. So we have a nontrivial morphism $(\delta_i, \delta_{i+1}, \cdots, \delta_n)$ from the TFC of the reduction to the kernel of the covering map. Then $(\delta_1, \delta_2, \ldots, \delta_{i-1}, \delta_i, \ldots, \delta_n)$ where $\delta_j = \delta_{j+1}\rho_j$ for all $j < i$ is a nontrivial morphism from the TFC of $\mathbb{A}$ to the kernel of the covering map. Hence $G(\mathbb{A})$ is not trivial, as desired. $\square$

So, we see that if $G(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n) = 1$, then $G(A_i \xrightarrow{f_i} \cdots \xrightarrow{f_{n-1}} A_n) = 1$ for any i from 1 to n.

Note that if $G(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n) = 1$, then for any morphism $(\delta_i, \delta_{i+1}, \ldots, \delta_n)$ from $\mathrm{Mor}(P_i/M_i \xrightarrow{\rho_i} P_{i+1}/M_{i+1} \xrightarrow{\rho_{i+1}} \cdots \xrightarrow{\rho_{n-2}} P_{n-1}/M_{n-1} \xrightarrow{\rho_{n-1}} C_n, \bigoplus_{j=i}^{n} K_j/M_j \xrightarrow{\rho_i} \bigoplus_{j=i+1}^{n} K_j/M_j \xrightarrow{\rho_{i+1}} \cdots \xrightarrow{\rho_{n-2}} K_{n-1}/M_{n-1} \oplus K_n \xrightarrow{\rho_{n-1}} K_n)$, where $i \geq 2$, it is true that $\delta_i = \delta_{i+1} = \cdots = \delta_n = 0$. So, we will focus to find the conditions when $\delta_1 = 0$. To illustrate,

$$\begin{array}{ccccccccc}
P_1/M_1 & \xrightarrow{\rho_1} & P_2/M_2 & \xrightarrow{\rho_2} & \cdots & \xrightarrow{\rho_{n-2}} & P_{n-1}/M_{n-1} & \xrightarrow{\rho_{n-1}} & C_n \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & & & \downarrow{\scriptstyle \delta_{n-1}} & & \downarrow{\scriptstyle \delta_n} \\
\bigoplus_{i=1}^{n} K_i/M_i & \xrightarrow{\rho_1} & \bigoplus_{i=2}^{n} K_i/M_i & \xrightarrow{\rho_2} & \cdots & \xrightarrow{\rho_{n-2}} & K_{n-1}/M_{n-1} \oplus K_n & \xrightarrow{\rho_{n-1}} & K_n
\end{array}$$

In fact, δ_1 must be actually a map from P_1/M_1 into K_1/M_1 by the commutative diagram since $\delta_2 = 0$.

So, we have the following result.

Theorem 3.1.4. *If the coGalois group of an object $\mathbb{A}$ is trivial in q_n, then* $\text{Hom}(P_1/M_1, K_1/M_1) = 0$.

Proof. We can easily find a nonzero morphism $(\delta, 0, \cdots, 0)$ from Mor$(P_1/M_1 \xrightarrow{\rho_1} P_2/M_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} P_{n-1}/M_{n-1} \xrightarrow{\rho_{n-1}} C_n, \bigoplus_{j=1}^{n} K_j/M_j \xrightarrow{\rho_1} \bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} {}^{K_{n-1}}/_{M_{n-1}} \oplus K_n \xrightarrow{\rho_{n-1}} K_n)$ for a nonzero homomorphism $\delta \in \text{Hom}(P_1/M_1, K_1/M_1)$. Thus, the statement is proved contrapositively. □

Clearly, we can also get the following theorem.

Theorem 3.1.5. *The coGalois group of an object $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ is trivial in q_n if and only if the coGalois group of the object $A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ is trivial in q_{n-1} and* $\text{Hom}(P_1/M_1, K_1/M_1) = 0$.

Lemma 3.1.6. *The morphism φ_i :* ker $f_i\varphi_i/M_i \to$ ker f_i *is the TFC of* ker f_i *with the kernel K_i/M_i.*

Proof. We know $\varphi_i : {\varphi_i}^{-1}(\text{ker } f_i) \to \text{ker } f_i$ is a precover of ker f_i since $\varphi_i : C_i \to A_i$ is TFC of A_i, and $ker f_i \leq A_i$ by (Enochs (1963), Lemma 1). Moreover, $K_i = ker\varphi_i \leq \text{ker}(f_i\varphi_i) = {\varphi_i}^{-1}(\text{ker } f_i)$, and so the kernel of precovering map of ker f_i is K_i. We know that M_i is a maximal pure subgroup of ker$(f_i\varphi_i)$ which is contained in K_i by Theorem 2.3.1. So ker$(f_i\varphi_i)/M_i$ is the TFC of ker f_i with the covering map φ_i and the kernel is K_i/M_i by Definition 1.1.5. □

Theorem 3.1.7. *If* ker f_i*'s are all torsion free abelian groups (in particular, if f_i's are monic), then the coGalois group of $\mathbb{A}$ is trivial if and only if A_n has trivial coGalois group.*

Proof. ($\Rightarrow$) It follows by Lemma 3.1.2.

($\Leftarrow$) We will show that $(\delta_1, \delta_2, \cdots, \delta_n) = 0$ by induction to say $G(\mathbb{A}) = 1$. By hypothesis, we have $G(A_n) = 1$. It means that $\text{Hom}(C_n, K_n) = 0$. Then $\delta_n = 0$, and so we have the commutative diagram:

$$\begin{array}{ccccccccc}
P_1/M_1 & \xrightarrow{\rho_1} & P_2/M_2 & \xrightarrow{\rho_2} & \dots & \xrightarrow{\rho_{n-2}} & P_{n-1}/M_{n-1} & \xrightarrow{\rho_{n-1}} & C_n \\
\downarrow \delta_1 & & \downarrow \delta_2 & & & & \downarrow \delta_{n-1} & & \downarrow \delta_n = 0 \\
\bigoplus_{i=1}^{n} K_i/M_i & \xrightarrow{\rho_1} & \bigoplus_{i=2}^{n} K_i/M_i & \xrightarrow{\rho_2} & \dots & \xrightarrow{\rho_{n-2}} & K_{n-1}/M_{n-1} \oplus K_n & \xrightarrow{\rho_{n-1}} & K_n
\end{array}$$

We could take the $\delta_n = 0$ as a base step of induction, but we prefer to show one more step to see it easily. Since $\rho_{n-1}\delta_{n-1} = \delta_n\rho_{n-1}$, we get that δ_{n-1} must map into K_{n-1}/M_{n-1}. We know that

$$\ker(f_{n-1}\varphi_{n-1})/K_{n-1} = \ker(f_{n-1}\varphi_{n-1})/\ker\varphi_{n-1} \cong \ker f_{n-1}.$$

Since $\ker f_{n-1}$ is torsion free by assumption, we have $\ker(f_{n-1}\varphi_{n-1})/K_{n-1}$ is torsion free. We also know that $\ker(f_{n-1}\varphi_{n-1})$ is torsion free since $\ker(f_{n-1}\varphi_{n-1}) \leq C_{n-1}$ and C_{n-1} is torsion free. It means that K_{n-1} is a pure subgroup of $\ker(f_{n-1}\varphi_{n-1})$. Then $K_{n-1} = M_{n-1}$ since M_{n-1} is a maximal pure subgroup of $\ker(f_{n-1}\varphi_{n-1})$ that is contained in K_{n-1}. And then $\delta_{n-1} : P_{n-1}/M_{n-1} \to K_{n-1}/M_{n-1}$ is the zero map, because $K_{n-1}/M_{n-1} = 0$. Now, we assume that $\delta_{i+1} = 0$ for some i, then we have the following commutative diagram:

$$\begin{array}{ccccccc}
\dots & \xrightarrow{\rho_{i-1}} & P_i/M_i & \xrightarrow{\rho_i} & P_{i+1}/M_{i+1} & \xrightarrow{\rho_{i+1}} & \dots \\
& & \downarrow \delta_i & & \downarrow \delta_{i+1} = 0 & & \\
\dots & \xrightarrow{\rho_{i-1}} & \bigoplus_{j=i}^{n} K_j/M_j & \xrightarrow{\rho_i} & \bigoplus_{l=i+1}^{n} K_l/M_l & \xrightarrow{\rho_{i+1}} & \dots
\end{array}$$

Similarly, since $\rho_i\delta_i = \delta_{i+1}\rho_i$, the morphism δ_i must be a map into K_i/M_i. We also know that $\ker(f_i\varphi_i)/K_i = \ker(f_i\varphi_i)/\ker\varphi_i \cong \ker f_i$. Since $\ker f_i$ is torsion free by assumption, we have $\ker(f_i\varphi_i)/K_i$ is torsion free. We also know that $\ker(f_i\varphi_i)$ is torsion free since $\ker(f_i\varphi_i) \leq C_i$ and C_i is torsion free. It means that K_i is a pure subgroup of $\ker(f_i\varphi_i)$. Then $K_i = M_i$ since M_i is a maximal pure subgroup of $\ker(f_i\varphi_i)$ that is contained in K_i. And then $\delta_i : P_i/M_i \to K_i/M_i$ is the zero map since $K_i/M_i = 0$. Hence $(\delta_1, \delta_2, \cdots, \delta_n) = 0$, as desired. $\square$

Corollary 3.1.7.1. *If* ker f_i*'s are all torsion free, then the kernel of the covering map of* $\mathbb{A}$ *is the object* $K_n \xrightarrow{id} K_n \xrightarrow{id} \cdots \xrightarrow{id} K_n$ *in* q_n*.*

Corollary 3.1.7.2. *Let* ker f_i*'s be torsion free for all* $i \leq j$*. Then the coGalois group of* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ *is trivial if and only if the coGalois group of* $A_{j+1} \xrightarrow{f_{j+1}} A_{j+2} \xrightarrow{f_{j+2}} \cdots \xrightarrow{f_{n-1}} A_n$ *is trivial.*

Since the Prüfer group $\mathbb{Z}_{p^\infty}$ is divisible, its coGalois group is trivial. The following is an example of an object in q_3 with a non-trivial coGalois group, even though it consists of abelian groups with a trivial coGalois group. See also (Hill (2008), Example).

Example 6. *Let* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ *be an object in* q_3 *such that* $A_i = \mathbb{Z}_{p^\infty}$ *for each* i *and that* $f_1 = 0 = f_2$*. Let* $\varphi_i : C_i \to A_i$ *be the TFCs of* A_i*. Then, clearly we have* $C_1 = C_2 = C_3$ *and* $K_1 = K_2 = K_3$*. Therefore, we have* $\ker(f_1\varphi_1) = C_1$ *and* $\ker(f_2\varphi_2) = C_2$*. Since there are no nonzero pure subgroups of* C_1 *and* C_2 *contained, respectively, in* K_1 *and* K_2*, it follows that* $M_1 = 0 = M_2$*. And so,* $P_2/M_2 = P_2 = C_2 \oplus K_3$ *and* $P_1/M_1 = P_1 = C_1 \oplus K_2 \oplus K_3$*. Then, we have the commutative diagram:*

$$\begin{array}{ccccc} C_1 \oplus K_2 \oplus K_3 & \xrightarrow{\rho_2} & C_2 \oplus K_3 & \xrightarrow{\rho_1} & C_3 \\ \swarrow id_K \oplus id_K & & \swarrow id_K & & \downarrow 0 \\ K_1 \oplus K_2 \oplus K_3 & \xrightarrow{\rho_2} & K_2 \oplus K_3 & \xrightarrow{\rho_1} & K_3 \end{array}$$

Thus, we obtain nonzero morphisms $(id_K \oplus id_K, id_K, 0)$ *and* $(id_K, 0, 0)$ *from the cover into the kernel of the covering map in* q_3*. Hence* $G(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3) \neq 1$*.*

Inspired by the previous example, we get in general the following results.

Proposition 3.1.8. *Let* $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$ *be an object in* q_3 *such that* $f_1 = 0 = f_2$ *and that* $G(A_1) = G(A_2) = G(A_3) = 1$*. Then* $G(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3) = 1$ *if and only if* $\mathrm{Hom}(K_3, K_2) = 0 = \mathrm{Hom}(K_2 \oplus K_3, K_1)$*.*

Proof. ($\Rightarrow$) Since $G(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3) = 1$, we know that all the morphisms from the cover to the kernel must be zero. So, we get $\mathrm{Hom}(K_2 \oplus K_3, K_1) = \mathrm{Hom}(K_3, K_2) = 0$, for if $\mathrm{Hom}(K_2 \oplus K_3, K_1) \neq 0$ or $\mathrm{Hom}(K_3, K_2) \neq 0$, then we can find a nonzero map $(\delta_1, \delta_2, \delta_3)$, which can be easily seen from the previous example.

($\Leftarrow$) Suppose that $(\delta_1, \delta_2, \delta_3)$ is a morphism from the cover of the object to the kernel of the covering map. We show that is zero. First, since $G(A_3) = 1 = G(A_2)$, we know that $\mathrm{Hom}(C_3, K_3) = 0 = \mathrm{Hom}(C_2, K_2)$, and so $\delta_3 = 0$. Second, by the commutative diagram, we have $\mathrm{Hom}(C_2 \oplus K_3, K_3) = 0$. Moreover, since $\mathrm{Hom}(K_3, K_2) = 0$ by the assumption, it follows that $\mathrm{Hom}(C_2 \oplus K_3, K_2 \oplus K_3) = 0$, and so $\delta_2 = 0$. Finally, since $G(A_1) = 1$ we know that $\mathrm{Hom}(C_1, K_1) = 0$. By the commutative diagram, we get $\mathrm{Hom}(C_3 \oplus K_2 \oplus K_3, K_2 \oplus K_3) = 0$. So the only possibility to get nonzero δ_1 is a nonzero map from $\mathrm{Hom}(K_2 \oplus K_3, K_1)$. But we know that it is not possible since the assumption says that $\mathrm{Hom}(K_2 \oplus K_3, K_1) = 0$. Hence, $\delta_1 = 0$, and so we get $G(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3) = 1$. □

The next result is a more general version of the previous proposition, and it can be proved similarly. Because, we actually said that $G(A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3) = 1$ if and only if $\mathrm{Hom}(K_3, K_2) = \mathrm{Hom}(K_3, K_1) = \mathrm{Hom}(K_2, K_1) = 0$ if and only if $\mathrm{Hom}(K_3, K_2 \oplus K_1) = 0 = \mathrm{Hom}(K_2, K_1)$ under the hypothesis of Proposition 3.1.8.

Proposition 3.1.9. *Let $\mathbb{A}$ be an object in q_n such that $G(A_i) = 1$ for all $i = 1, 2, \ldots, n$ and that $f_j = 0$ for all $j = 1, 2, \ldots, n-1$. Then the coGalois group of $\mathbb{A}$ is trivial if and only if* $\mathrm{Hom}(K_i, \bigoplus_{j=1}^{i-1} K_j) = 0$ *for all $i = 2, 3, \ldots, n$.*

Before continuing, we need to know more about the TFCs in q_n.

Lemma 3.1.10. *The subgroup* $\mathrm{im}(\rho_i \rho_{i-1} \cdots \rho_1 : P_1/M_1 \to P_{i+1}/M_{i+1})$ *of P_{i+1}/M_{i+1} is the pullback of homomorphisms $f_{i+1}|\varphi_{i+1} : \varphi_{i+1}^{-1}(\mathrm{im}\, f_i \cdots f_1)/M_{i+1} \to A_{i+2}$ and $\varphi_{i+2}\rho'_{i+2} : P_{i+2}/M_{i+2} \to A_{i+2}$ with the restricted pullback maps ρ'_{i+1} and ρ_{i+1}:*

$$\begin{array}{ccc} \operatorname{im} \rho_i \rho_{i-1} \cdots \rho_1 & \xrightarrow{\rho_{i+1}} & P_{i+2}/M_{i+2} \\ \Big\downarrow \rho'_{i+1} & & \Big\downarrow \varphi_{i+2}\rho'_{i+2} \\ \varphi_{i+1}^{-1}(\operatorname{im} f_i \cdots f_1)/M_{i+1} & \xrightarrow{f_{i+1}|\varphi_{i+1}} & A_{i+2} \end{array}$$

where $f_{i+1}| : \operatorname{im}(f_i f_{i-1} \cdots f_1) \to A_{i+2}$ *is the restriction of* $f_{i+1} : A_{i+1} \to A_{i+2}$ *to* $\operatorname{im}(f_i f_{i-1} \cdots f_1 : A_1 \to A_{i+1})$ *for all* $i = 1, 2, \cdots, n-2$.

So, we have $\operatorname{im}(\rho_i \rho_{i-1} \cdots \rho_1) / \ker(f_{i+1}|\varphi_{i+1}) \cong \operatorname{im}(\rho_{i+1}\rho_i .. \rho_1)$.

Proof. To avoid an ambiguity, we must remember the projection map $\varphi_i : C_i/M_i \to A_i$ denoted samely with the TFCs $\varphi_i : C_i \to A_i$ of A_i.

Firstly, we must see that $\operatorname{im} f_1 = \operatorname{im}(f_1\varphi_1)$. Consider the following commutative diagram:

$$\begin{array}{ccc} P_1/M_1 & \xrightarrow{\rho_1} & \operatorname{im} \rho_1 \\ \varphi_1\rho'_1 \Big\downarrow & & \Big\downarrow \varphi_2\rho'_2 \\ A_1 & \xrightarrow{f_1} & \operatorname{im} f_1 \end{array}$$

Moreover, we also have that:

$$\begin{aligned} \operatorname{im} \rho_1 &= (\varphi_2\rho'_2)^{-1}(\operatorname{im} f_1) \\ &= (\rho'_2)^{-1}(\varphi_2^{-1}(\operatorname{im} f_1)) \\ &= (\rho'_2)^{-1}(\varphi_2^{-1}(\operatorname{im} f_1)/M_2) \\ &= (\rho'_2)^{-1}(\varphi_2^{-1}(\operatorname{im} f_1))/M_2 \end{aligned}$$

by Lemma 1.2.8 and Theorem 2.3.1. So, we have the following commutative diagram:

$$\begin{array}{ccc} P_1/M_1 & \xrightarrow{\rho_1} & \operatorname{im} \rho_1 \\ \rho'_1 \Big\downarrow & & \Big\downarrow \varphi_2\rho'_2 \\ \varphi_1^{-1}(\operatorname{im} f_1) & \xrightarrow{f_1\varphi_1} & \operatorname{im} f_1 \end{array}$$

By using Lemma 1.2.8 and Theorem 2.3.1, we can get an isomorphism

$$(P_1/M_1)/(\ker f_1\varphi_1/M_1) \cong P_1/\ker f_1\varphi_1 = \operatorname{im}\rho_1.$$

Similarly, it is easy to see that $(\rho_2')^{-1}(\varphi_2^{-1}(\operatorname{im} f_1))$ is the pullback of the composition maps $f_2|\varphi_2 : \varphi_2^{-1}(\operatorname{im} f_1) \to A_3$ and $\varphi_3\rho_3' : P_3/M_3 \to A_3$ since $\varphi_2^{-1}(\operatorname{im} f_1) \leq C_2$ by Lemma 1.2.8. So, we have the following commutative diagrams:

$$\begin{array}{ccc}
(\rho_2')^{-1}(\varphi_2^{-1}(\operatorname{im} f_1)) & \xrightarrow{\rho_2} & P_3/M_3 \\
\downarrow{\scriptstyle \rho_2'} & & \downarrow{\scriptstyle \varphi_3\rho_3'} \\
\varphi_2^{-1}(\operatorname{im} f_1) & \xrightarrow{f_2|\varphi_2} & A_3
\end{array}$$

and

$$\begin{array}{ccc}
(\rho_2')^{-1}(\varphi_2^{-1}(\operatorname{im} f_1))/M_2 & \xrightarrow{\rho_2} & P_3/M_3 \\
\downarrow{\scriptstyle \rho_2'} & & \downarrow{\scriptstyle \varphi_3\rho_3'} \\
\varphi_2^{-1}(\operatorname{im} f_1)/M_2 & \xrightarrow{f_2|\varphi_2} & A_3
\end{array}$$

Similarly, we have

$$\begin{aligned}
\operatorname{im}\rho_1/(\ker(f_2|\varphi_2)/M_2) &= ((\rho_2')^{-1}(\varphi_2^{-1}(\operatorname{im} f_1))/M_2)/(\ker(f_2|\varphi_2)/M_2) \\
&\cong (\rho_2')^{-1}(\varphi_2^{-1}(\operatorname{im} f_1))/\ker(f_2|\varphi_2) \\
&\cong \rho_2(\operatorname{im}\rho_1) \\
&= \operatorname{im}(\rho_2\rho_1).
\end{aligned}$$

Now, assume for some j with $1 \leq j < n-2$ that $\operatorname{im}(\rho_j\rho_{j-1}\cdots\rho_1)$ is the pullback in the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{im}(\rho_j\cdots\rho_1) & \xrightarrow{\rho_{j+1}} & P_{j+2}/M_{j+2} \\
\downarrow{\scriptstyle \rho_{j+1}'} & & \downarrow{\scriptstyle \varphi_{j+2}\rho_{j+2}'} \\
\varphi_{j+1}^{-1}(\operatorname{im}(f_j\cdots f_1))/M_{j+1} & \xrightarrow{f_{j+1}|\varphi_{j+1}} & A_{j+2}
\end{array}$$

It follows that

$$\begin{aligned}
\operatorname{im}(\rho_j \cdots \rho_1)/\ker(f_{j+1}|\varphi_{j+1}) &= ((\rho'_{j+1})^{-1}(\varphi_{j+1}^{-1}(\operatorname{im}(f_j \cdots f_1)))/M_{j+1})/(\ker(f_{j+1}|\varphi_{j+1})/M_{j+1}) \\
&\cong (\rho'_{j+1})^{-1}(\varphi_{j+1}^{-1}(\operatorname{im}(f_j \cdots f_1)))/\ker(f_{j+1}|\varphi_{j+1}) \\
&\cong \rho_{j+1}(\operatorname{im}(\rho_j \cdots \rho_1)) \\
&= \operatorname{im}(\rho_{j+1}\rho_j \cdots \rho_1)
\end{aligned}$$

using the Lemma 1.2.8 and Theorem 2.3.1. We also know that

$$\operatorname{im}(\rho_{j+1}\rho_j..\rho_1) = (\varphi_{j+2}\rho'_{j+2})^{-1}(\operatorname{im}(f_{j+1}f_j \cdots f_1)),$$

since we have $\operatorname{im}(f_{j+1}|\varphi_{j+1}) = f_{j+1}|(\operatorname{im}(f_j f_{j-1} \cdots f_1)) = \operatorname{im}(f_{j+1}f_j \cdots f_1)$.

Since $\operatorname{im}(f_{j+1}f_j \cdots f_1) \leq A_{j+2}$, and so $(\varphi_{j+2})^{-1}(\operatorname{im} f_{j+1}f_j \cdots f_1) \leq C_{j+2}/M_{j+2}$, the complete inverse image $(\varphi_{j+2}\rho'_{j+2})^{-1}(\operatorname{im}(f_{j+1}f_j \cdots f_1)) = \operatorname{im}(\rho_{j+1}\rho_j..\rho_1)$ is the pullback in the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{im}(\rho_{j+1} \cdots \rho_1) & \xrightarrow{\rho_{j+2}} & P_{j+3}/M_{j+3} \\
\Big\downarrow \rho'_{j+2} & & \Big\downarrow \varphi_{j+3}\rho'_{j+3} \\
\varphi_{j+2}^{-1}(\operatorname{im}(f_{j+1} \cdots f_1))/M_{j+2} & \xrightarrow{f_{j+2}|\varphi_{j+2}} & A_{j+3}
\end{array}$$

And so we get that:

$$\begin{aligned}
\operatorname{im}(\rho_j \cdots \rho_1)/\ker(f_{j+1}|\varphi_{j+1}) &= ((\rho'_{j+1})^{-1}(\varphi_{j+1}^{-1}(\operatorname{im} f_j \cdots f_1))/M_{j+1})/(\ker(f_{j+1}|\varphi_{j+1}/M_{j+1})) \\
&\cong (\rho'_{j+1})^{-1}(\varphi_{j+1}^{-1}(\operatorname{im}(f_j f_{j-1} \cdots f_1))/\ker(f_{j+1}|\varphi_{j+1}) \\
&\cong \rho_{j+1}(\operatorname{im}(\rho_j \rho_{j-1} \cdots \rho_1)) \\
&= \operatorname{im}(\rho_{j+1} \cdots \rho_1)
\end{aligned}$$

by Lemma 1.2.8.

□

Under the hypothesis of the previous lemma, we have the following result.

Proposition 3.1.11. $\ker(f_{i+1}|\varphi_{i+1})$ *is pure in* $\operatorname{im}(\rho_i\rho_{i-1}\cdots\rho_1)$.

Proof. We have $\operatorname{im}(\rho_i\cdots\rho_1)/\ker(f_{i+1}|\varphi_{i+1}) \cong \operatorname{im}(\rho_{i+1}\rho_i\cdots\rho_1)$ by Lemma 3.1.10. Moreover, we know that $\operatorname{im}(\rho_i\cdots\rho_1)$ and $\operatorname{im}(\rho_{i+1}\rho_i\cdots\rho_1)$ are torsion free since $\operatorname{im}(\rho_i\cdots\rho_1)$ and $\operatorname{im}(\rho_{i+1}\rho_i\cdots\rho_1)$ are subgroups of the torsion free abelian groups P_{i+1}/M_{i+1} and P_{i+2}/M_{i+2}, respectively. Thus, it is easy to see that $\ker(f_{i+1}|\varphi_{i+1})$ is pure in $\operatorname{im}(\rho_i\rho_{i-1}..\rho_1)$ by using Theorem 1.2.7. □

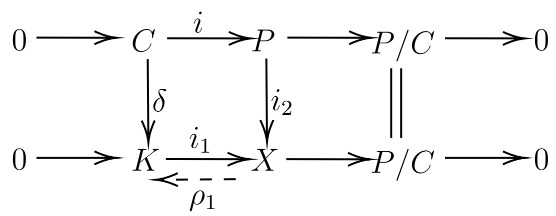
3.2 Some Relations between coGalois Groups and p-divisibility

Before giving the classification of objects that have trivial coGalois group in q_n, we need some features about TFCs and coGalois groups in the category $\mathcal{A}b$ of abelian groups. The aim of this section is to prove that $\operatorname{Hom}(P_1/M_1, K_1/M_1) = 0$ if and only if P_1/M_1 is p-divisible for each r-prime p of ker f_1.

A similar version of the following useful lemma were used in the proof of (Hill, 2008, Theorem 4.1) without proving. So, we will give its proof.

Lemma 3.2.1. *Let* $\varphi : C \to A$ *be a TFC of an abelian group* A *with the kernel* K*. If* C *is pure in a torsion free abelian group* P*, then any homomorphism* $\delta : C \to K$ *can be extended to a homomorphism* $\tilde{\delta} : P \to K$*.*

Proof. Consider the following commutative diagram with exact rows, which is obtained by pushout of δ and the inclusion map $i : C \to P$:



Since C is pure in P, then P/C is a torsion free abelian group. Therefore, the bottom row splits by (Xu, 1996, Lemma 2.1.1) as K is the kernel of a TFC. This means that there exists a map $\rho_1 : X \to K$ such that $\rho_1 i_1 = id_K$. Thus the composition map $\tilde{\delta} = \rho_1 i_2$ extends δ, as desired.

□

The following result were used in the proof of (Hill, 2008, Theorem 5.1) without proving. We state it as a lemma here with a proof.

Lemma 3.2.2. *Let $\varphi : C \to A$ be the TFC of an abelian group A with the kernel K. Then, there is no nonzero p-divisible subgroup of C in K for every r-prime p of A.*

Proof. If there is no r-prime of A, then A is torsion free, and so A is a TFC of itself with zero kernel. In this case, there is nothing to prove.

Assume that A is not torsion free. Suppose, the contrary, that there is a nonzero p-divisible subgroup H in K for every r-prime p of A. Then, H is p-pure in C for every r-prime p. Observe that K is $q-$pure for all non-r-prime q of A, for if $k = qc \in qC \cap K$, then we have $0 = \varphi(k) = \varphi(qc) = q\varphi(c)$. This implies that $\varphi(c) = 0$ since A has no nonzero element of order q, that is, $c \in K$.

Let H' be a subset of C defined as $H' = \{c \in C : q^n c \in qC \cap H\}$ for all non-r-primes q of A. Clearly, $H' \subseteq K$. We claim that $H^* =< H, H' >$ is a pure subgroup of C that is contained in K, which will contradict the fact that K is the kernel of the covering map. First, we will show that H^* is p-divisible for every r-prime p, and it suffices to show this for H' since H is already p-divisible. Taking an arbitrary element $c \in H'$, we have $q^n c \in qC \cap H$. Then $q^n c = ph'$ for some $h' \in H$ since H is p-divisible, and so $q^n \mid h'$ as $q \nmid p$, or equivalently, $h' = q^n h''$ for some $h'' \in C$. It follows that $q^n c = q^n(ph'')$, which implies that $c = ph''$ since C is torsion free. Moreover, we have $q^n h'' = h' \in qC \cap H$, and so $h'' \in H'$. Thus, H' is p-divisible, and so is H^* for every r-prime p of A. Finally, it is clear by definition that H^* is also r-pure for every non-r-prime of A. Hence H^* is p-pure in C for all primes p, and so it is pure in C by Lemma 1.2.4, since C is torsion free. □

The importance of the previous lemma is that if G is a p-divisible group for every r-prime p of A, then $\text{Hom}(G, K) = 0$. Because a homomorphic image of a p-divisible group is also p-divisible.

Remark 4. *It is known that the TFC of a (p-) divisible group is (p-) divisible (see (Enochs et al., 2000, Proposition 3.3)). But, the torsion free precover of a divisible group need not be divisible. To give an example, let* $\varphi : C \to A$ *be the TFC of a divisible group* A*. Let* C' *be any torsion free group and consider the zero map* $0 : C' \to A$*. Then* $C \oplus C'$ *is a torsion free precover of* A *with the torsion free precovering map* $\varphi \oplus 0$*. Moreover, the precover* $C \oplus C'$ *quitely depends on our choice for* C'*, and we are free to choose any torsion free group that is not divisible.*

Let's give some notations which we will use in the following two lemmas. Let $\varphi : C \to A$ be a TFC of A with the kernel K, and let $A' \leq A$. Then the complete inverse image $\varphi^{-1}(A')$ of A' is a torsion free precover of A'. Now, we will give some facts about divisibility of the torsion free precover $\varphi^{-1}(A')$. Moreover, the ideas in the proof of (Hill (2008), Theorem 5.1) will be used.

Lemma 3.2.3. *Suppose that the following conditions are satisfied, where* p *is a prime.*

1. *A has a trivial coGalois group.*
2. *A' is p-divisible.*
3. $A'(p) = A(p)$.

Then $\varphi^{-1}(A')$ *is* p*-divisible.*

Proof. Since $G(A) = 1$ by (1), we also know that A is p-divisible for each its r-primes p. This means that the torsion subgroup $t(A)$ of A is divisible, and so a direct summand of A. Then, $A = t(A) \oplus F$, where F is a torsion free subgroup of A. Moreover, F is also p-divisible for each r-prime of A. By the uniqueness of the TFCs, we can get $C \cong \varphi^{-1}(t(A)) \oplus F$ with $K \leq \varphi^{-1}(t(A))$.

As the first case, let $A' = 0$. Since $A(p) = 0$ by (3), we get p is not r-prime of A. Then, we will show that $\varphi^{-1}(0) = \ker \varphi = K$ is p-divisible for all non-r-primes

p of A. By the Fundamental Homomorphism Theorem, we have $C/K \cong A$, and so $(C/K)(p) = A(p) = 0$. So, K is p-pure in C, for if $0 = \varphi(k) = \varphi(pc) = p\varphi(c)$ for some $k \in K \cap pC$, then $\varphi(c) = 0$ as $A(p) = 0$ by (3). That is, $c \in K$ and so $k = pc \in pK$. Therefore, K is p-pure in $\varphi^{-1}(t(A))$, and so it is p-divisible since $\varphi^{-1}(t(A))$ is divisible, which proves the first case.

Now, assume that $A' \neq 0$. Then, using the arguments above, we write again $A = t(A) \oplus F$ where $t(A)$ is divisible and F is p-divisible for each r-prime of A or $t(A)$. Therefore, as a p-group, $A'(p)$ is divisible by the condition (2), and so we have by (3) that

$$A = A'(p) \oplus t(A)^* \oplus F,$$

where $t(A)^*$ has no nonzero element of order p. So, by the uniqueness of TFCs, we have

$$C = C(A'(p)) \oplus C(t(A)^*) \oplus F,$$

where $C(A'(p))$ and $C(t(A)^*)$ are the TFCs of $A'(p)$ and $t(A)^*$, respectively. Since $A'(p)$ and $t(A)^*$ are divisible, we know that $C(A'(p))$ and $C(t(A)^*)$ are also divisible. Now, if we intersect the first equality above by A', we get $A' = A'(p) \oplus t(A')^* \oplus F'$, where $t(A')^*$ has no nonzero element of order p and F' is the torsion free part of A', and so we have $C' = C(A'(p)) \oplus C(t(A')^*) \oplus F'$, where C' and $C(t(A')^*)$ are TFCs of A' and $t(A')^*$, respectively. Besides, if we intersect the second equality above by $\varphi^{-1}(A')$, we get $\varphi^{-1}(A') = C' \oplus L$, with $L \leq K^*$, where K^* is the kernel of $\varphi^* : C(t(A)^*) \to t(A)^*$. As a consequence, to complete the proof it is enough to show that both C' and L are p-divisible. Clearly, we already know that C' is p-divisible since it is a TFC of a p-divisible group A' by (2).

Since $(A/A')(p) = 0$ by the conditions (2) and (3), it follows by the following isomorphisms that $(C/\varphi^{-1}(A'))(p) = 0$.

$$A/A' \cong (C/K)/(\varphi^{-1}(A')/K) \cong C/\varphi^{-1}(A').$$

Thus, $\varphi^{-1}(A')$ is p-pure in C, and so C' and L are p-pure in C as direct summands.

Then $C(t(A')^*)$ and L are p-pure in $C(t(A)^*)$. Hence C' and L are p-divisible since $C(t(A)^*)$ is. □

Lemma 3.2.4. *If A is an abelian group with a trivial coGalois group, and $A' \leq A$, then A' is p-divisible and $A'(p) = A(p)$ if and only if $\varphi^{-1}(A')$ is p-divisible.*

Proof. ($\Rightarrow$) It follows by the Lemma 3.2.3.

($\Leftarrow$) As the first case, let $A' = 0$. It is enough to show that $A(p) = 0$ when $K = \ker\varphi$ is p-divisible. Suppose, the contrary, that $A(p) \neq 0$. Since $G(A) = 1$, we know that A is p-divisible for every its r-prime. So $A(p)$ is divisible as a p-group, and then it contains the Prüfer group $\mathbb{Z}_{p^\infty}$ as a direct summand. So, by the uniqueness of TFCs, C contains the TFC $C(\mathbb{Z}_{p^\infty})$ of $\mathbb{Z}_{p^\infty}$ as a direct summand. Then the kernel K^* of $\varphi^* : C(\mathbb{Z}_{p^\infty}) \to \mathbb{Z}_{p^\infty}$ is contained in K as a direct summand. Since it is known that K^* is not p-divisible by Lemma 3.2.2, we conclude that K is not p-divisible, contradicting the assumption.

Next, assume that $A' \neq 0$. Since A' is an epimorphic image of the p-divisible group $\varphi^{-1}(A')$, it is also p-divisible. Now, assume that $A'(p) \neq A(p)$. Then, both of them cannot be 0, and so $A(p) \neq 0$. Moreover, we already know that A is p-divisible for every its r-prime since $G(A) = 1$. Then we can write $A = \mathbb{Z}_{p^\infty} \oplus A''$, where $A' \leq A''$ since $A(p)$ and $A'(p)$ are both divisible. So, by the uniqueness of TFCs, we get $C = C(\mathbb{Z}_{p^\infty}) \oplus C''$ where C'' is the TFC of A''. Intersecting this equation by $\varphi^{-1}(A')$, yields that $\varphi^{-1}(A') = K^* \oplus (\varphi^{-1}(A') \cap C'')$. Since K^* is not p-divisible, we get that $\varphi^{-1}(A')$ cannot be p-divisible, a contradiction. □

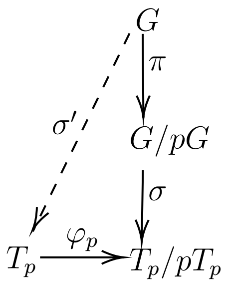
Theorem 3.2.5. *Let $\varphi_i : C_i \to A_i$ be the TFC of an abelian group A_i with the kernel K_i for $i = 1, 2$. Assume that the coGalois group of A_1 is trivial. For a subgroup A' of A_1, if A' is p-divisible and $A'(p) = A_1(p)$ for every r-prime p of A_2, then* $\mathrm{Hom}(\varphi_1^{-1}(A'), K_2) = 0$.

Proof. We know that $\varphi_1^{-1}(A')$ is p-divisible for every r-prime of A_2 by the Lemma 3.2.4. We also know that there is no nonzero p-divisible subgroup in K_2 for every r-prime of A_2 by the Lemma 3.2.2. So it is easy to see that there is no nonzero map from $\varphi_1^{-1}(A')$ into K_2 by the well-known fact that the homomorphic image of a p-divisible group is also p-divisible. □

Theorem 3.2.6. *Let A be an abelian group that has a trivial coGalois group, and p, a r-prime of A. Then there is a nonzero map from a torsion free abelian group G that is not p-divisible into the kernel of the TFC of A.*

Proof. Let $\varphi : C \to A$ be the TFC of A with the kernel K. We know that A is p-divisible for every its r-primes since its coGalois group is trivial. So we have that A has a direct summand that is isomorphic to $\mathbb{Z}_{p^\infty}$. Since the TFC is unique upto isomorphism, we can assume that C contains the TFC $C(\mathbb{Z}_{p^\infty})$ of $\mathbb{Z}_{p^\infty}$ as a direct summand. So the kernel T_p of the covering mapping of $\mathbb{Z}_{p^\infty}$ is contained in K as a direct summand. Let G be a torsion free abelian group that is not p-divisible. We will show that there is nonzero map from G into T_p.

Since G is not p-divisible, the quotient group G/pG is a nonzero vector space over $\mathbb{Z}/p\mathbb{Z}$. Similarly, T_p/pT_p is a nonzero vector space over $\mathbb{Z}/p\mathbb{Z}$. Because, we know that T_p cannot be p-divisible by Lemma 3.2.2. Now, we have a nonzero homomorphism $\sigma : G/pG \to T_p/pT_p$ since both of them are nonzero vector spaces over $\mathbb{Z}/p\mathbb{Z}$. We have the facts that T_p is a direct summand of a product of copies of a completion of localization of the ring $\mathbb{Z}$ at the prime p, and T_p is the TFC of T_p/pT_p with the canonical map φ_p, by (Xu, 1996, Proposition 4.1.6) and (Enochs et al., 2000, Section 3). Since $\varphi_p : T_p \to T_p/pT_p$ is also a torsion free precover of T_p/pT_p, G is torsion free, and there is a nonzero composition map $\sigma\pi : G \to T_p/pT_p$ where $\pi : G \to G/pG$ is the projection map, we know that there is a nonzero map $\sigma' : G \to T_p$ makes the following diagram commutes:



This completes the proof. □

The previous theorem have a significant importance for our main theorem, and although it seems different from (Enochs & Rada (2005), Lemma 2.7), but the proof of this lemma motivated the proof of the previous theorem.

Theorem 3.2.7. *Let A be a torsion abelian group that has trivial coGalois group. Then there is no nonzero homomorphism from a torsion free abelian group G into the kernel of the TFC of A if and only if G is p-divisible for every r-prime p of A.*

Proof. Let $\varphi : C \to A$ be the TFC with the kernel K. Since $G(A) = 1$ and A is torsion, we know that A is divisible.

Since G is torsion free, we know that the identity map $id_G : G \to G$ is the TFC of G. So we have that $\varphi \oplus id_G : C \oplus G \to A \oplus G$ is the TFC of $A \oplus G$ with the kernel K.

Clearly $(A \oplus G)/t(A \oplus G) \cong (A \oplus G)/A \cong G$. Hence, we have that $\mathrm{Hom}(G, K) = 0$ if and only if G is p-divisible for every r-prime p of A by using (Enochs & Rada (2005), Lemma 2.7). □

Proposition 3.2.8. *Let A be an abelian group that has trivial coGalois group. Then there is no nonzero homomorphism from a torsion free abelian group G into the kernel of the TFC of A if and only if G is p-divisible for every r-prime p of A.*

Proof. Let $\varphi : C \to A$ be the TFC with the kernel K. We know that the torsion subgroup $t(A)$ of A is divisible, the homomorphism $\varphi : \varphi^{-1}(t(A)) \to t(A)$ is the TFC of $t(A)$ with the kernel K, and moreover $t(A)$ also has a trivial coGalois group. Hence, the proof is completed by using Theorem 3.2.7. □

Theorem 3.2.9. *Let $\varphi_i : C_i \to A_i$ be the TFC of an abelian group A_i with the kernel K_i for $i = 1, 2$. Assume that $G(A_1) = 1$. For a subgroup A' of A_1, A' is p-divisible and $A'(p) = A_1(p)$ for every r-prime of A_2 if and only if* $\mathrm{Hom}(\varphi_1^{-1}(A'), K_2) = 0$.

Proof. ($\Rightarrow$) By Theorem 3.2.5.

($\Leftarrow$) By using Lemma 3.2.4, it is enough to show that there is a nonzero homomorphism from $\varphi_1^{-1}(A')$ into K_2 with assuming $\varphi_1^{-1}(A')$ is not p-divisible for any r-prime p of A_2. This fact comes with using Theorem 3.2.6. □

The previous theorem can also be proven easily with using Theorem 3.2.5 and Proposition 3.2.8.

3.3 The Classification of Objects in q_n Having a Trivial coGalois Group

Now, we return to give necessary conditions for the coGalois group of an object $\mathbb{A}$ in q_n is trivial.

Theorem 3.3.1. *If the coGalois group of an object $\mathbb{A}$ is trivial, then the coGalois group of an abelian group* ker f_i *is trivial for all i's.*

Proof. Assume $G(\ker f_i) \neq 1$ for any i. We know that $\ker(f_i\varphi_i)/M_i$ is the TFC of $\ker f_i$ with the covering map φ_i and the kernel K_i/M_i by Lemma 3.1.6. By assumption, we have a nonzero map $\delta_i \in \mathrm{Hom}(\ker(f_i\varphi_i)/M_i, K_i/M_i)$. By using Lemma 3.2.1, δ_i can be extended a map from P_i/M_i to K_i/M_i. Then $(\delta_i\rho_{i-1}\cdots\rho_1, \cdots, \delta_i\rho_{i-1}, \delta_i, 0, \cdots, 0)$ is a nonzero morphism from the cover of $\mathbb{A}$ into the kernel of the covering morphism. Hence $G(\mathbb{A}) \neq 1$. □

Proposition 3.3.2. *If the coGalois group of an object $\mathbb{A}$ in q_n is trivial, and $f_{i-1}\cdots f_1 = 0$, then* $\mathrm{Hom}(\bigoplus_{j=i}^{n-1} K_j/M_j, K_1/M_1) = 0$.

Proof. We must recognise that $f_i| : \text{im}(f_{i-1}\cdots f_1) \to A_{i+1}$ is the zero map since $\text{im}(f_{i-1}\cdots f_1) = 0$. So $\ker(f_i|\varphi_i) = K_i/M_i$. Since $\text{im}(\rho_{i-1}\cdots\rho_1)$ is an epimorphic image of P_1/M_1 and $\ker(f_i|\varphi_i)$ is pure in $\text{im}(\rho_{i-1}\cdots\rho_1)$ by Lemma 3.1.10, we can get $\text{Hom}(K_i/M_i, K_1/M_1) = 0$ by using Lemma 3.2.1. And $f_j\cdots f_{i-1}\cdots f_1 = 0$ since $f_{i-1}\cdots f_1 = 0$ where $j \geq i-1$. Because of the same reason, we have $\text{Hom}(K_j/M_j, K_1/M_1) = 0$ for all $j \geq i-1$. Hence, $\text{Hom}(\bigoplus_{j=i}^{n-1} K_j/M_j, K_1/M_1) = 0$, as desired. □

Lemma 3.3.3. *If the coGalois group of an object $\mathbb{A}$ is trivial in q_n, then the following conditions are satisfied for every r-prime p of* $\ker f_1$ *where $i = 2, \ldots, n-1$:*

1. *The coGalois group of the object $A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n$ is trivial in q_{n-1}.*
2. *The coGalois group of the abelian group* $\ker f_1$ *is trivial.*
3. $\ker f_i \cap \text{im}(f_{i-1}\cdots f_1)$ *is p-divisible.*
4. $(\ker f_i \cap \text{im}(f_{i-1}\cdots f_1))(p) = \ker f_i(p)$.
5. $\text{im}(f_{n-1}\cdots f_1)$ *is p-divisible.*
6. $A_n(p) = \text{im}(f_{n-1}\cdots f_1)(p)$.

Proof. The conditions (1) and (2) are satisfied by Lemma 3.1.3 and Theorem 3.3.1, respectively.

For the conditions (3) and (4), first remember that $\varphi_i : (\ker f_i\varphi_i)/M_i \to \ker f_i$ is the TFC of $\ker f_i$ with the kernel K_i/M_i of the covering mapping. Moreover, the coGalois group of $\ker f_i$ is trivial by Theorem 3.3.1. In particular, $\ker f_1\varphi_1/M_1$ is the TFC of $\ker f_1$ with the kernel K_1/M_1, and that $G(\ker f_1) = 1$. Recognise that $f_i| : \text{im}(f_{i-1}\cdots f_1) \to A_{i+1}$ is defined in Lemma 3.1.10, and we know that $\ker(f_i|) = \ker f_i \cap \text{im}(f_{i-1}\cdots f_1) \leq \ker f_i$. We also have that $\varphi_i^{-1}(\ker f_i \cap \text{im}(f_{i-1}\cdots f_1)) = \ker(f_i|\varphi_i : \varphi_i^{-1}(\text{im}(f_{i-1}\cdots f_1)) \to A_{i+1})$ is the complete inverse image of $\ker f_i \cap \text{im}(f_{i-1}\cdots f_1)$. Therefore, using Lemma 3.2.4, we can say that the conditions (3) and (4) are equivalent to the fact that $\ker(f_i|\varphi_i)$ is p-divisible for every r-prime p of $\ker f_1$.

Now, by Lemma 3.2.1 and Proposition 3.1.11, we can see that any nonzero map in $\text{Hom}(\ker(f_i|\varphi_i), K_1/M_1)$ can be extended to a nonzero map in

$\mathrm{Hom}(\mathrm{im}(\rho_{i-1}\cdots\rho_1), K_1/M_1)$. Since $\mathrm{im}(\rho_{i-1}\cdots\rho_1)$ is an epimorphic image of P_1/M_1, we get a nonzero map from P_1/M_1 to K_1/M_1. To see it clearly, take a nonzero homomorphism $\delta \in \mathrm{Hom}(\mathrm{im}(\rho_{i-1}\cdots\rho_1), K_1/M_1)$, then $(\delta\rho_{i-1}\cdots\rho_1, 0, 0, \ldots, 0)$ is a nonzero morphism from the cover of $\mathbb{A}$ into the kernel of the covering map of $\mathbb{A}$, which means that $G(\mathbb{A}) \neq 1$, conradicting the assumption. Thus, $\mathrm{Hom}(\ker(f_i|\varphi_i), K_1/M_1) = 0$, and so $\ker(f_i|\varphi_i)$ is p-divisible for every r-prime of $\ker f_1$, using Theorem 3.2.9, as desired.

For the conditions (5) and (6), we will use similar ideas. Firstly, we must recognise that $\mathrm{im}(\rho_{n-1}\cdots\rho_1) = \varphi_n^{-1}(\mathrm{im}(f_{n-1}\cdots f_1))$ is the complete inverse image of $\mathrm{im}(f_{n-1}\cdots f_1)$ under the TFC $\varphi_n : C_n \to A_n$. We know that $G(A_n) = 1$ by (1) and Lemma 3.1.3. Note that $\ker f_1\varphi_1/M_1$ is the TFC of $\ker f_1$ with the kernel K_1/M_1, and that $G(\ker f_1) = 1$ under the hypothesis by using Theorem 3.3.1. Therefore, using Lemma 3.2.4, we can get that the conditions (5) and (6) are equivalent to the fact that $\varphi_n^{-1}(\mathrm{im}(f_{n-1}\cdots f_1))$ is p-divisible for each r-prime p of $\ker f_1$.

Moreover, $\mathrm{im}(\rho_{n-1}\cdots\rho_1)$ is an epimorphic image of P_1/M_1. This means that for any nonzero homomorphism $\delta \in \mathrm{Hom}(\mathrm{im}(\rho_{n-1}\cdots\rho_1), K_1/M_1)$, we have a nonzero morphism $(\delta\rho_{n-1}\cdots\rho_1, 0, 0, \ldots, 0)$ from the cover of $\mathbb{A}$ into the kernel of the covering map of $\mathbb{A}$, which means that $G(\mathbb{A}) \neq 1$, conradicting the assumption. Thus, $\mathrm{Hom}(\varphi_n^{-1}(\mathrm{im}(f_{n-1}\cdots f_1)), K_1/M_1) = 0$, and so $\varphi_n^{-1}(\mathrm{im}(f_{n-1}\cdots f_1))$ is p-divisible for every r-prime of $\ker f_1$, using Theorem 3.2.9, as desired. □

Proposition 3.3.4. *If the coGalois group of an object $\mathbb{A}$ is trivial, and $f_{n-1}\cdots f_1 = 0$, then* $\mathrm{Hom}(K_n/M_n, K_1/M_1) = 0$.

Proof. Since $f_{n-1}\cdots f_1 = 0$, we have that $\mathrm{im}(\rho_1\rho_2\cdots\rho_{n-1}) = K_n/M_n$. This means that K_n/M_n is an epimorphic image of P_1/M_1. If there is a nontrivial hoomorphism from $\mathrm{Hom}(K_n/M_n, K_1/M_1)$, then we can get a nonzero homomorphism from $Hom(P_1/M_1, K_1/M_1)$. □

Now we can see the following fact with using both of Proposition 3.3.2 and

Proposition 3.3.4:

Proposition 3.3.5. *If the coGalois group of an object $\mathbb{A}$ is trivial, and $f_{i-1}\cdots f_1 = 0$, then* $\operatorname{Hom}(\bigoplus_{j=i}^{n} K_j/M_j, K_1/M_1) = 0$.

Now, we will give the main theorem that is the classification for line quivers that have the trivial coGalois group.

Theorem 3.3.6. *The coGalois group of an object $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$ in q_n is trivial if and only if the following conditions are satisfied for each r-prime p of* $\ker f_1$*, where $i = 2, \ldots, n-1$:*

1. *The coGalois group of the object $A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n$ is trivial in q_{n-1}.*
2. *The coGalois group of the abelian group* $\ker f_1$ *is trivial.*
3. $\ker f_i \cap \operatorname{im}(f_{i-1}\cdots f_1)$ *is p-divisible.*
4. $(\ker f_i \cap \operatorname{im}(f_{i-1}\cdots f_1))(p) = \ker f_i(p)$.
5. $\operatorname{im}(f_{n-1}\cdots f_1)$ *is p-divisible.*
6. $A_n(p) = \operatorname{im}(f_{n-1}\cdots f_1)(p)$.

Proof. ($\Rightarrow$) It follows by Lemma 3.1.3, Theorem 3.3.1, and Lemma 3.3.3.

($\Leftarrow$) We can use Theorem 3.1.5 to say that the proof will be all about showing $\operatorname{Hom}(P_1/M_1, K_1/M_1) = 0$, but we will not use it for completeness.

It suffices to show that $\operatorname{Mor}(P_1/M_1 \xrightarrow{\rho_1} P_2/M_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} P_{n-1}/M_{n-1} \xrightarrow{\rho_{n-1}} C_n, \bigoplus_{j=1}^{n} K_j/M_j \xrightarrow{\rho_1} \bigoplus_{j=2}^{n} K_j/M_j \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} {}^{K_{n-1}}/_{M_{n-1}} \oplus K_n \xrightarrow{\rho_{n-1}} K_n) = 0$, by Lemma 3.1.1. Take a morphism $(\delta_1, \delta_2, \cdots, \delta_n)$ from the cover of $\mathbb{A}$ into the kernel $\mathbb{K}$. So, the following diagram commutes:

$$\begin{array}{ccccccccc}
P_1/M_1 & \xrightarrow{\rho_1} & P_2/M_2 & \xrightarrow{\rho_2} & \cdots & \xrightarrow{\rho_{n-2}} & P_{n-1}/M_{n-1} & \xrightarrow{\rho_{n-1}} & C_n \\
\downarrow{\scriptstyle\delta_1} & & \downarrow{\scriptstyle\delta_2} & & & & \downarrow{\scriptstyle\delta_{n-1}} & & \downarrow{\scriptstyle\delta_n} \\
\bigoplus_{i=1}^{n} K_i/M_i & \xrightarrow{\rho_1} & \bigoplus_{i=2}^{n} K_i/M_i & \xrightarrow{\rho_2} & \cdots & \xrightarrow{\rho_{n-2}} & {}^{K_{n-1}}/_{M_{n-1}} \oplus K_n & \xrightarrow{\rho_{n-1}} & K_n
\end{array}$$

We know that $(\delta_2, \ldots, \delta_n) \in \operatorname{Mor}(P_2/M_2 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} P_{n-1}/M_{n-1} \xrightarrow{\rho_{n-1}}$

$C_n, \bigoplus_{i=2}^{n} K_i/M_i \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-2}} {}^{K_{n-1}}/_{M_{n-1}} \oplus K_n \xrightarrow{\rho_{n-1}} K_n$), and so it is the zero morphism by Condition (1). So, δ_1 must be a homomorphism from P_1/M_1 into K_1/M_1 since $0 = \delta_2\rho_1 = \rho_1\delta_1$, and so $\operatorname{im}\delta_1 \leq \ker\rho_1 = K_1/M_1$. We claim that $\delta_1 : P_1/M_1 \to K_1/M_1$ is the zero homomorphism. This is equivalent to show that P_1/M_1 is p-divisible for every r-prime p of $\ker f_1$ by Proposition 3.2.8. Note that $\ker(f_1\varphi_1)/M_1$ is the TFC of $\ker f_1$ with the kernel K_1/M_1, and that $\ker f_1$ is p-divisible by Condition (2) for every its r-prime, and so $\ker(f_1\varphi_1)/M_1$ is p-divisible. Thus, we have $\operatorname{Hom}(\ker(f_1\varphi_1)/M_1, K_1/M_1) = 0$ by (2). Therefore,

$$(P_1/M_1)/(\ker(f_1\varphi_1)/M_1) \cong P_1/\ker(f_1\varphi_1) \cong \operatorname{im}\rho_1.$$

So, it is enough to show that $\operatorname{im}\rho_1$ is p-divisible for every r-prime of $\ker f_1$.

As trivial cases, we know if $\operatorname{im} f_1 = 0$, then $\operatorname{im}\rho_1 \cong \bigoplus_{i=2}^{n} K_i/M_i$. If $\operatorname{im}(f_{i-1}\cdots f_1) \neq 0$ and $\operatorname{im}(f_i f_{i-1}\cdots f_1) = 0$, then $\operatorname{im}(\rho_i\rho_{i-1}\cdots\rho_1) \cong \bigoplus K_i/M_i$.

By Lemma 3.1.10, we also have

$$\begin{gathered}
\operatorname{im}\rho_1/\ker(f_2|\varphi_2) \cong \operatorname{im}(\rho_2\rho_1) \\
\operatorname{im}(\rho_2\rho_1)/\ker(f_3|\varphi_3) \cong \operatorname{im}(\rho_3\rho_2\rho_1) \\
\cdots \\
\operatorname{im}(\rho_{i-1}\cdots\rho_1)/\ker(f_i|\varphi_i) \cong \operatorname{im}(\rho_i\rho_{i-1}\cdots\rho_1). \\
\cdots \\
\operatorname{im}(\rho_{n-2}\cdots\rho_1)/\ker(f_{n-1}|\varphi_{n-1}) \cong \operatorname{im}(\rho_{n-1}\rho_{n-2}\cdots\rho_1).
\end{gathered}$$

where $f_i| : \operatorname{im}(f_{i-1}\cdots f_1) \to A_{i+1}$ is a restriction map of $f_i : A_i \to A_{i+1}$ for all $i = 2, \ldots, n-1$. So, it is enough to show that $\ker(f_j|\varphi_j)$'s and $\operatorname{im}(\rho_{n-1}\rho_{n-2}\cdots\rho_1)$ are all p-divisible for every r-prime p of $\ker f_1$. Moreover, we have

$$\ker(f_j| : \operatorname{im}(f_{j-1}\cdots f_1) \to A_{j+1}) = \ker f_j \cap \operatorname{im}(f_{j-1}\cdots f_1).$$

First, we know that $G(\ker f_j) = 1$ for all $j = 2, \ldots, n-1$ by Condition (1) and

Lemma 3.3.1. So, by the conditions (3), (4) and Lemma 3.2.3, we have the complete inverse image $\varphi_j^{-1}(\ker f_j|) = \ker f_j|\varphi_j$ of the subgroup $\ker f_j|$ of $\ker f_j$ is p-divisible for every r-prime p of $\ker f_1$.

To be more clear, from Lemma 3.1.10, we can remember that $\mathrm{im}(\rho_{j-1}\cdots\rho_1)$ is the pullback in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{im}(\rho_{j-1}\cdots\rho_1) & \xrightarrow{\rho_j} & P_{j+1}/M_{j+1} \\ \Big\downarrow{\scriptstyle \rho'_j} & & \Big\downarrow{\scriptstyle \varphi_{j+1}\rho'_{j+1}} \\ \varphi_j^{-1}(\mathrm{im}(f_j\cdots f_1)) & \xrightarrow{f_j|\varphi_j} & A_{j+1} \end{array}$$

We assumed that $(\ker f_j \cap \mathrm{im}(f_{j-1}\cdots f_1))(p) = \ker f_j(p)$ and $\ker f_j \cap \mathrm{im}(f_{j-1}\cdots f_1)$ is p-divisible for every r-prime p of $\ker f_1$ by conditions (3) and (4). Then the complete inverse image $\varphi_j^{-1}(\ker f_j \cap \mathrm{im}(f_{i-1}\cdots f_1))$ under the covering mapping of $\ker f_i$ is p-divisible for every r-prime p of $\ker f_1$ by Lemma 3.2.3. It also means $\mathrm{Hom}(\varphi_j^{-1}(\mathrm{im}(f_j\cdots f_1)), K_1/M_1) = 0$. Moreover, we also have:

$$\mathrm{im}(\rho_{j-1}\cdots\rho_1)/\ker(f_j|\varphi_j) \cong \rho_j(\mathrm{im}(\rho_{j-1}\cdots\rho_1)) = \mathrm{im}(\rho_j\rho_{j-1}\cdots\rho_1)$$

by Lemma 3.1.10. So it is enough to show that $\mathrm{Hom}(\mathrm{im}(\rho_j\rho_{j-1}\cdots\rho_1), K_1/M_1) = 0$. Similarly, we know that $\mathrm{im}(\rho_j\rho_{j-1}\cdots\rho_1)$ is the pullback in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{im}(\rho_j\rho_{j-1}\cdots\rho_1) & \xrightarrow{\rho_{j+1}} & P_{j+2}/M_{j+2} \\ \Big\downarrow{\scriptstyle \rho'_{j+1}} & & \Big\downarrow{\scriptstyle \varphi_{j+2}\rho'_{j+2}} \\ \varphi_{j+1}^{-1}(\mathrm{im}(f_{j+1}f_j\cdots f_1)) & \xrightarrow{f_{j+1}|\varphi_{j+1}} & A_{j+2} \end{array}$$

Similarly, we get that $\ker(f_{j+1}|\varphi_{j+1})$ is p-divisible for every r-prime p of $\ker f_1$.

Next, it remains to prove that $\mathrm{im}(\rho_{n-1}\cdots\rho_1)$ is p-divisible for every r-prime p of $\ker f_1$. Now, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{im}(\rho_{n-2}\cdots\rho_1) & \xrightarrow{\rho_{n-1}} & C_n \\ \downarrow{\scriptstyle \rho'_{n-1}} & & \downarrow{\scriptstyle \varphi_n} \\ \varphi_{n-1}^{-1}(\mathrm{im}(f_{n-2}\cdots f_1)) & \xrightarrow{f_{n-1}|\varphi_{n-1}} & A_n \end{array}$$

And so, $\rho_{n-1}(\mathrm{im}(\rho_{n-2}\cdots\rho_1)) = \mathrm{im}(\rho_{n-1}\cdots\rho_1) = \varphi_n^{-1}(\mathrm{im}(f_{n-1}\cdots f_1))$ since $\mathrm{im}(f_{n-1}|\varphi_{n-1}) = \mathrm{im}(f_{n-1}| : \mathrm{im}(f_{n-2}\cdots f_1) \to A_n) = \mathrm{im}(f_{n-1}\cdots f_1)$. We assume $\mathrm{im}(f_{n-1}\cdots f_1)$ is p-divisible and $\mathrm{im}(f_{n-1}\cdots f_1)(p) = A_n(p)$ for every r-prime of ker f_1 by conditions (5) and (6). Clearly, we have $G(A_n) = 1$ by the condition (1) and Lemma 3.1.2.

Hence, it follows that $\mathrm{im}(\rho_{n-1}\cdots\rho_1) = \varphi_n^{-1}(\mathrm{im}(f_{n-1}\cdots f_1)$ is p-divisible for every r-prime of ker f_1 by Lemma 3.2.3. □

CHAPTER FOUR
CONCLUSION

A complete characterization of abelian groups having a trivial coGalois group was given in Enochs & Rada (2005). After, coGalois groups have been studied for a pair of abelian groups and characterized when they are trivial in Hill (2008). Actually, coGalois groups have been studied in the category of representations of the quiver (i.e., a directed graph) $q_2 : \bullet \to \bullet$ there. Because, the coGalois group is definable for any category with a covering class, and the torsion free covers exist for the category $(q_n, \mathbb{Z}\text{-Mod})$ of representations of the line quiver $q_n : \bullet \to \bullet \to \ldots \to \bullet$ with $n-1$ arrows and n vertices (see Wesley (2005)), we define and study coGalois groups in that category. We give some properties of torsion free covers and coGalois groups of objects similar to those given for abelian groups, and characterize the objects in $(q_n, \mathbb{Z}\text{-Mod})$ having a trivial coGalois group, in terms of trivial coGalois groups of abelian groups.

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