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ABOUT MARTINGALES

M.S. Degree Thesis

by

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SUMMARY

This master of science research study on "Martingales" Consists of the following four sections.

In section one, the necessary fundemantel concepts which is hoped to be useful to define martingale is given.

In section two, the condition of stochastic process to be a martingale and the related definitions, theorems, Doob's martingale and Radon-Nikodyn derivatives are investigated.

Section three deals with, sub and supermartingales and the related definitions and theorems and Markov Times.

In section four, applications of martingales are given.

I hope this research study will be useful to my collegues who are interested in the theory and the application of martingales.

ÖZET

Martingaller üzerine yapılan bu yüksek lisans çalışması dört bölümden oluşmaktadır.

Birinci bölümde, martingallerin tanıtımında faydalı olacağı umulan gerekli temel kavramlar verilmiştir.

İkinci bölümde ise bir stokkastik sürecin martingal olma koşulları ele alınmış ve martingaller ile ilgili tanımlar, teoremler, Doob martingali ve Radon-Nihodyn türevleri incelenmiştir.

Üçüncü bölümde, Sub ve Supmartingaller ve onlarla ilgili tanım ve teoremler, Markov zamanları üzerinde durulmuştur.

Dördüncü bölümde de, Martingallerle ilgili uygulamalara yer verilmiştir.

Bu çalışmanın stokastik süreçlerin martingaller konusunda çalışan meslektaşlarıma yararlı olacağına umarım.

PREFACE

Probability Theory which was initially based on fortune games forms a basis for the theory of "Stochastic Processes". All the research and studies of scientists on the sequence of events, relating to the probability, helped to the formation of the Stochastic Processes. Later on, various applications of stochastic processes were made use of in the other fields of physics engineering, medicine, biology and mathematics. At this stage, the idea of "Martingal" was needed in order to analyze and have a better understanding of the concepts relating to stochastic processes. Conditions necessary to accept a stochastic process of Yn, { n: 0,1,...} as martingal were investigated. Martingal idea found applications in many areas like fortune games, urn problem .... etc.

I am grateful to my respectable supervisor Prof.Dr. Süeda MORALI for her invaluable support and assistance, which she so generously extended to me, during my on "Martingales". Without her guidance and help this work could not have been possible.

Bornova, 1991 — Yaşar GÖKÇE

CONTENTS

Page

CHAPTER III

CHAPTER IV

HISTORICAL DEVELOPMENT

Fortune games were widespread, espacially amongst the rich, during the sixteenth and seventeenth centuries. The aforementioned people had to employ the best strategy to raise their earnings to the highest level; Therefore, on matters related to the probability they used to seek the advice of well-known mathematicians of that time, like Cardano, Galileo, Pascal, Bernoulli and to the scientists who conducted research on probability theory like Lagrange, Simpson, Bernoulli, Bayes, Stirlings, Euler-Laplace and D'Alembert.

The intensive studies of scientists on probability theory contributed to the formation of the "Stochastic Processes".

Following the works of scientists, like D. Bernoulli (1700-1782), L.Euer (1707-1783), Gnedenko, Lamberti on probability theory, the addition of independent random variables, the expected value, the urn problem and Bernoulli Processes, great improvements were observed on the subject of stochastic processes.

In 1812 J.L. Lagrange (1736-1818), and P.C. Laplace (1749-1827) analyzed the urn problem which was introduced by Bernoulli S.D. Possion (1781-1840), Khinchine Watanabe and Reny (1574-1642) made studies on "Poission Processes".

Markov (1856-1922) provided the first application and solution for the urn problem which was introduced by Bernoulli in 1907 and analyzed by P.C. Laplace. He also conducted studies on "Markov Processes" which were named after him. Markov introduced and demonstrated the idea of the limits of transition probabilities, finite state of space, and Markov Chains.

Scientists such as Galton, Watson, Orey and Kendall conducted studies on the application of "Markov Chains". Galton-Watson introduced a new process called branching process, and showed various applications of birth-death process by doing studies on the process.

Feller approved of the accuracy of these applications which were based on the Markov Chains.

In 1933 Chapman and Kolmogrov contributed greatly to the subject of "Stochastic Processes" through their studies on the random variables and finite state of space.

The aforementioned studies and researches helped the idea of martingal to come into being. A lot of studies were made on the martingal concept which means fair gambling. Conditions necessary to declare a 'stochastic process' as martingal were investigated. Scientists like Doob, Radon, Nikodyn conducted studies and researches on martingal.

The unexpected applications in mathematics, in parallel with demands arising from scientists and engineers, helped these improvements to be realized.

CHAPTER I

BASIC CONCEPTIONS

Definition 1.1. (Sample Space)

The set of points representing the possible outcomes of an experiment is called the sample space of the experiment. A subset of sample space S is also called an event.
See [9].

Definition 1.2. (Conditional Probability)

Let A and B be two events of sample space. The conditional probability of B occuring, given that A has occured (Written P(B/A)) is $P(B/A) = P(B \cap A)/P(A)$, if $P(A) \neq 0$.
See [1], [8].

Definition 1.3. (Multiplication Rule)

If A_i (i=1,2,...,n) are events of sample space, then ;

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1).P(A_2/A_1)P(A_3/A_1 \cap A_2)\ldots P(A_n/A_1 \cap A_2 \ldots A_{n-1}).$$

See [3].

Definition 1.4. (Formula of Total Probability)

Let A be an event of sample space S and let B_i (i=1,2,...,n) show division of S. That is, $B_1 \cup B_2 \cup \ldots \cup B_n = S$ and $B_1 \cap B_2 = \Phi$, ... $B_1 \cap B_n = \Phi$, $B_2 \cap B_1 = \Phi$, ..., $B_2 \cap B_n = \Phi$, ..., Otherwise $A = A \cap B_1 \cup A \cap B_2 \cup \ldots \cup A \cap B_n$.
Therefore, the formula of total probability :

$$P(A) = P(A \cap B_1) + \ldots + P(A \cap B_n) = P(B_1).\ P(A/B_1) + \ldots + P(B_n).P(A/B_n).$$

See [3].

Definition 1.5. (Independent Events)

Two events, A and B, are independent if and only if $P(A \cap B) = P(A).\ P(B)$.

Definition 1.6. (Bayes' Formula)

If E is an event of sample space, $P(A_i) \neq 0$, $i=1,2,\ldots,n$, $A_i (i=1,2,\ldots,n)$ are disjoint events and their unions are sample space S with conditional $P(E) \neq 0$; then

$$P(A_i | E) = \frac{P(A_i \cap E)}{\sum_{i=1}^{n} P(A_i \cap E)}$$

See [3] .

Definition 1.7. (Random Variable)

A random variable X is a real-valued function of the elements of a sample space S. If the number of possible values of a random variable X is finite or countable infinite, X is a discontinious ramdom variable. If its possible values are from an interval or a collection of intervals, X is a continious random variable.

Definition 1.8. (Expected Value)

If the possible values of a discrete variable X are $x_i (i = 1,2,\ldots,n)$ and its density function $f(x_i)$, then the expected value of the X is $E[X] = \sum_{i=1}^{n} x_i . f(x_i)$. If X is a continious random variable, then its value is

$$E[X] = \int_{-\infty}^{\infty} x.f(x), \; -\infty < x < \infty.$$

Definition 1.9. (Variance of a Random Variable)

(i) Variance of a discrete random variable X,

$$\sigma_X^2 = Var(X) = E[x-E(x)]^2 = \sum_{i=1}^{n} (x_i - \mu)^2 . f(x_i)$$

μ= expected value of X, $f(x_i)$ = Probability function of X..

(ii) If X is a continious random variable, then

$$\sigma_X^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 . f(x).$$

Definition 1.10. (Generation Function)

Let P(x) be the density function of X and $|S| \leq 1$. If F(s) is differentiable with respect to S and is shown as $F(s) = \sum_{x=0} P(x).s^x$, then F(s) is called a generation function.
See [7].

At s=1, the derivative value of this function gives the expected value of X random variable.

That is, $\frac{dF(s)}{ds} = F'(s) = \sum_{x=0}^{\infty} x.p(x).s^{x-1}$, $F'(1) = \sum_{x=0}^{\infty} x.P(x)=E(X)$

Definition 1.11. (Stochastic Process)

Let T be parameter space and time teT. If x(t) shows the value of X(t), then the random variables X form a set { X(t), teT } is called a stochastic process.

Definition 1.12. (State Space)

For each h > 0 and teT, if Pr [x-h < X(t) < x+h] > 0, then X is called a possible value or state of stochastic process{X(t),teT}. All states form a set called state space.
See [2].

Definition 1.13. (Markow Process)

For the n number time points in the parameter space and $t_1 < t_2 < \dots < t_{n-1} < t_n$, let the values $X(t_n)$ takes be $X(t_1)$, $X(t_2)$,..., $X(t_n)$. If the distribution with conditional of $X(t_n)$ only depends on the values of $X(t_{n-1})$, Process {X(t), teT} is called Markov Process. That is,

$$Pr\ \{X(t_n) = x_n \mid X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1}\} = P_r\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}$$

See [2] .

Definition 1.14. (Branching Process or Galton-Watson Process)

In the set of individuals producing individuals from its Own kind, if X_n is the number of people in the nth generation, the number of people in the (n+1) th generation is equal to the sum of progeny of the people in the nth generation.

That is, $X_{n+1} = \sum_{i=1}^{X_n} Y_i$, Y_i : the number of the i th person's progeny.

In a branching process :

(i) If X_n is known, then the probabilities in the (n+1) th generation are independent from the growth of the generations before the n th. That is, $\{X_n, n=0,1,\ldots\}$ is a Markov Process. See [5] .

(ii) Each person produces his progeny independently. And the number of his progeny is independent from the number of other people's progeny.

Definition 1.15. (The recurrent state in a Markov Chain)

When the chain is at the state i, and time n, then the probability of the first returning is shown as :

$f_{ii}^{(n)}$. Where, $f_{ii}^{(n)} = P_r\ \{X_r \neq i,\ r=1,2,\ldots,\ n-1,\ x_n = i \mid x_o = i\}$.

When the chain is at state i, the probability of returning to state i is $f_i = \sum_{n=1}^{\infty} f_{ii}^{(n)}$.

The expected value of the first returning time is $T_i = \sum_{n=1}^{\infty} n.\ f_{ii}^{(n)}$.

When the chain is at the state i, if the returning of the chain to i is certain, then this state is called the returning state. If $f_i = 1$ and $T_i < \infty$, then the state is a positive returning state.

See [4] .

CHAPTER II

MARTİNGALES

Definition 2.1. If a Stochastic process $\{x_n ; n = 0,1,\ldots,\}$ is a martingale, then for $n = 0,1,\ldots,$

i) $E[|X_n|] < \infty$

ii) $E[X_{n+1} | X_o,\ldots,X_n] = X_n.$

See [4] .

Here X_n is a player's fortune at stage n of a game.
And $E[X_n]$ is expectation value of X_n

The martingale property captures one notion of a game being fair in that the player's fortune on the next play is, on the average, his current fortune and is not otherwise affected by the previous history.

Definition 2.2. Let $\{X_n ; n = 0,1,\ldots\}$ and $\{Y_n ; n = 0,1,\ldots\}$ be stochastic processes. We say $\{X_n\}$ is a martingale with respect to $\{Y_n\}$ if, $n = 0,1,\ldots,$

(i) $E[|X_n|]<\infty$,

(ii) $E[X_{n+1}| Y_o,\ldots,Y_n] = X_n.$

Otherwise, from the property of conditional expactation $E[g(Y_o,\ldots,Y_n) | Y_o,\ldots,Y_n] = g(Y_o,\ldots,Y_n)$, (2.1.i)
We infer that

$$E[X_n| Y_o,\ldots,Y_n] = X_n,$$

and using the law of total probability, now we obtain

$$\begin{aligned} E[X_{n+1}] &= E\{E[X_{n+1} | Y_o,\ldots, Y_n]\} \\ &= E[X_n], \text{ (The property of martingale)} \\ &\vdots \\ &= E[X_o], \quad \text{for all n.} \end{aligned}$$

Theorem 2.1. (Sums of Independent Random Variables)

Let $Y_o = 0$ and $Y_1, Y_2, \ldots$ be independent random variables with $E[|Y_n|] < \infty$ and $E[Y_n] = 0$ for all n. If $X_o = 0$ and $X_n = Y_1 + \ldots + Y_n$ for $n \geq 1$, then $\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

Proof: From (2.1) ;

(i) $E[|X_n|] = E[|Y_1 + Y_2 + \ldots + Y_n|]$

$$\leq E[|Y_1|] + E[|Y_2|] + \ldots + E[|Y_n|] < \infty$$

(ii) $E[X_{n+1} | X_o, \ldots, Y_n] = E[X_n + Y_{n+1} | Y_o, \ldots, Y_n]$

$$= [E\ |X_n\ | Y_o, \ldots, Y_n] + E[Y_{n+1} | X_o, \ldots, Y_n]$$

$= X_n + E[Y_{n+1}]$ (because of the independence assumption on $\{Y_i\}$)

$= X_n$. (Since $E[Y_m] = 0$ by stipulation)

Teorem 2.2. (About More General Sums)

Let $z_i = g_i\ (Y_o, \ldots, Y_i)$ be for arbitrary sequences of random variables Y_i and functions g_i. Let f be a function for which $E[|f(z_k)|] < \infty$. For $k=0,1,\ldots$ And let a_k be a bounded function of k real variables. Then $X_n = \sum_{k=o}^{n} \{f(z_k) - E[f(z_k) | Y_o, \ldots, Y_{k-1}]\} a_k(Y_o, \ldots, Y_{k-1})$ defines a martingale with respect to $\{Y_n\}$.

Proof: Since a_k is bounded, it is written $|a_k\ (Y_o, \ldots, Y_{k-1})| < A_k$, for all $y_o, \ldots, y_{k-1}$.

n according to definition (2.1) :

(i) $E\left[|X_n|\right] = E\left[\left|\sum_{k=o}^{n} \{f(z_k) - E[f(z_k) \mid Y_o,\dots,Y_{k-1}]\, a_k(Y_o,\dots,Y_{k-1})\right|\right]$

$$\leqslant \sum_{k=o}^{n} E\{|f(z_k) + E[f(z_k) \mid Y_o,\dots,Y_{k-1}]\} E[|a_k(Y_o,\dots,Y_{k-1})|]$$

$$\leqslant \sum_{k=o}^{n} 2E\,|\,|f(z_k)|\,|.\ E|A_k|\ ,\ (E\,[A_k] = A_k)$$

$$= 2\sum_{k=o}^{n} A_k.E[|f(z_k)|] < \infty$$

(ii) $E\{X_{n+1} \mid Y_o,\dots,Y_n\} = E\{\sum_{k=o}^{n} \{f(z_k) - E[f(z_k)|Y_o,\dots,Y_{k-1}]\} a_k(Y_o,\dots,Y_{k-1}) | Y_o,\dots,Y_n\}$

$$+ E\ \{f(z_{n+1}) - E\left[f(z_{n+1}) | Y_o,\dots,Y_n\} | Y_o,\dots,Y_n\right]$$

$$= E\{X_n \mid Y_o,\dots,Y_n\} + E\left[f(z_n) - f(z_n)\right],\ \text{from (2.1.1)}$$

$$= X_n + 0\ ,\quad \text{from}\quad (2.1.1)$$

$$= X_n .$$

Theorem 2.3. (About the Variance of a Sum a Sum as a Martingale)

For $n=0,1,2,\dots,$ Let $\{Y_n\}$ be independent identically distributed random variables with $E[Y_k] = 0$ and $E[Y_k^2] = \sigma^2$, $k=1,2,\dots,$ Then the process $\{X_n\}$, $X_n = (\sum_{k=1}^{n} Y_k)^2 - n\sigma^2$ is a martingale with respect to $\{Y_n\}$.

Proof: (i) $E[|X_n|] = E[|\,(\sum_{k=1}^{n} Y_k)^2 - n\sigma^2|]$

$$\leqslant E[|\ (Y_1 + Y_2 + \dots + Y_n)^2 + n\sigma^2|]$$

$$= E[|\ Y_1^2|] + E[|Y_2^2] + \dots + E[|Y_n^2|] + 2E[|Y_1|].E[|Y_2|] + \dots + n\sigma^2$$

$$= n\sigma^2 + n\sigma^2$$

$$= 2n\sigma^2 < \infty$$

(ii) $E\left[X_{n+1}|Y_o,\dots,Y_n\right] = E\{\left[\left(\sum_{k=1}^{n+1} Y_k\right)^2-(n+1)\sigma^2\right]|Y_o,\dots,Y_n\}$

$$= E\{\left[\left(Y_{n+1}+\sum_{k=1}^{n} Y_k\right)^2-(n+1)\sigma^2\right]|Y_o,\dots,Y_n\}$$

$$= E\{\left[Y_{n+1}^2+2Y_{n+1}\sum_{k=1}^{n} Y_k+\left(\sum_{k=1}^{n} Y_k^2\right)-n\sigma^2-\sigma^2\right]|Y_o,\dots,Y_n\}$$

$$= \sigma^2+0-\sigma^2+E\{\left[\left(\sum_{k=1}^{n} Y_k\right)^2-n\sigma^2\right]|Y_o,\dots,Y_n\}$$

$$= E\left[X_n \mid Y_o,\dots,Y_n\right] = X_n.$$

Theorem 2.4. (About Markov Chains and Reguler Sequence as Martingele)

Let the Process $\{Y_n\}$ represent a Markov chain process governed by the transition Probability matrix $P=\left[|P_{ij}|\right]$. $f(i)$ is nonnegative and satisties $f(i)=\sum_j P_{ij}\, f(j)$, then the process $\{x_n\}$ with $X_n=f(Y_n)$ is a martingale.

Proof: (i) Since f is a bounded sequence $E\left[|X_n|\right]=E\left[|f(Y_n)|\right]<\infty$

(ii) $E\left[X_{n+1}|Y_o,\dots,Y_n\right] = E\left[f(Y_{n+1})|Y_o,\dots,Y_n\right]$

$$= E\left[f(Y_{n+1} \mid Y_n\right], \text{ (The property of Markov)}$$

$$= E\left[f(Y_{n+1})|\, Y_n=i\right]$$

$$= \sum_j P_{Y_{n},j}\, f(j) = f(Y_n) = X_n.$$

Theorem 2.5. (About Eigenvectors of Transition Matrix as Martingale)

Let Process $\{Y_n\}$ be a Markov chain having transition probability matrix $\underline{P}=\left[|P_{ij}|\right]$. Let λ be the eigenvalue and f a right eigenvector of $\underline{P}$. If $\lambda f(i) = \sum_j P_{ij}\, f(j)$ is, then the process $\{X_n\}$ with $X_n=\lambda^{-n} f(Y_n)$ is a martingale.

Proof : (i) $E\left[|X_n|\right] = |\lambda^{-n}| . E\left[|f(Y_n)|\right] < \infty$, (f is an eigenvector)

$$
\begin{aligned}
\text{(ii) } E\left[X_{n+1} \mid Y_o,\dots,Y_n\right] &= E \mid \lambda^{-(n+1)}.f(Y_{n+1}) \mid Y_o,\dots,Y_n\right] \\
&= \lambda^{-(n+1)}. E\left[f(Y_{n+1}) \mid Y_o,\dots,Y_n\right] \\
&= \lambda^{-(n+1)}. E\left[f(Y_{n+1}) \mid Y_n = i\right], \text{ (The Property of Markov)} \\
&= \lambda^{-(n+1)}.\lambda\, f(Y_n) \\
&= \lambda^{-n}.f(Y_n) \\
&= X_n.
\end{aligned}
$$

Theorem 2.6. (About a Branching Process)

Let $\{Y_n\}$ be a branching process and suppose that the mean of the progeny distribution in the nth genaration is $m < \infty$. Then the process $\{X_n\}$ with $X_n = m^{-n}. Y_n$ is a martingale.

Proof : (i) $E\left[|X_n|\right] = |m^{-n}| . E\left[|Y_n|\right] < \infty$

$$
\begin{aligned}
\text{(ii) } E\left[X_{n+1} \mid Y_o,\dots,Y_n\right] &= E\left[m^{-(n+1)}.Y_{n+1} \mid Y_o,\dots,Y_n\right] \\
&= m^{-(n+1)}. E\left[Y_{n+1} \mid Y_o,\dots,Y_n\right] \\
&= m^{-(n+1)}. E\left[Y_{n+1} \quad Y_n\right]
\end{aligned}
$$

Here, Let $z_{(i)}^{(n)}$ be the number of progeny produced by the ith existing parent in the nth genaration. The number of progeny in (n+1) th generation is $Y_{n+1} = \sum_{i=1}^{Y_n} z_{(i)}^{(n)}$.

Then,

$$E[X_{n+1} | Y_0,\ldots,Y_n] = m^{-(n+1)}. E[Y_{n+1} | Y_n]$$

$$= m^{-(n+1)}.\{E[z^{(n)}_{(1)}] + E[z^{(n)}_{(2)}] + \ldots + E[z^{(n)}_{(Y_n)}]\}$$

$$= m^{-(n+1)}. Y_n.m$$

$$= m^{-n}. Y_n$$

$$= X_n.$$

Theorem 2.7. (Doob's Theorem)

Let $Y_o, Y_1, \ldots$ be an arbitrary sequence of random variables and suppose X is a random variable satisfying $E[|X|]<\infty$
Then $X_n = E[X | Y_o,\ldots,Y_n]$ is a martingale with respect to $\{Y_n\}$ and Called Doob's process.

Proof: (i) $E[|X_n|] = E\{| E[X | Y_o,\ldots,Y_n|]\}$

$$< E\{ E[|X|] Y_o,\ldots,Y_n|\}$$

$$= E[|X|]<\infty$$

(ii) $E[X_{n+1}| Y_o,\ldots,Y_n] = E\{E[X | Y_o,\ldots,Y_{n+1}] | Y_o,\ldots,Y_n\}$

$$= E[X| Y_o,\ldots,Y_n],$$ (by the law of the total probability)

$$= X_n.$$

Theorem 2.8. (About Radon-Nikodym Derivatives)

Let z be an uniformly distributed random variable on $[0,1)$.
Let's define the random variables as $Y_n = \frac{k}{2^n}$ (the unique k depends

on n and z) that satisfies $\frac{k}{2^n} \leqslant z < \frac{k+1}{2^n}$. Suppose f be a bounded function on $[0,1]$. Then the process $\{X_n\}$ with $X_n = 2^n\{f(Y_n + 2^{-n}) - f(Y_n)\}$ is a martingale with respect to $\{Y_n\}$.

Proof: Z, conditional on $Y_0, \ldots, Y_n$, has a uniform distribution on $[Y_n, Y_n + 2^{-n})$. Therefore, Y_{n+1} is equally likely to be Y_n or $Y_n + 2^{-(n+1)}$.

(i) $E[|X_n|] < \infty$, from f is a bounded function on $[0,1]$.

(ii) $E[X_{n+1} | Y_0, \ldots, Y_n] = 2^{n+1}.E[f(Y_{n+1} + 2^{-(n+1)}) - f(Y_{n+1}) | Y_0, \ldots, Y_n]$

$$= 2^{n+1}.\{ \frac{1}{2} [f(Y_n + 2^{-(n+1)}) - f(Y_n)]$$

$$+ \frac{1}{2} [f(Y_n + 2^{-(n+1)} + 2^{-(n+1)}) - f(Y_n + 2^{-n})]\}$$

$$= 2^{n+1}\{\frac{1}{2}[f(Y_n + 2^{-(n+1)}) - f(Y_n) + f(Y_n + 2^{-n}) - f(Y_n + 2^{-n})]\}$$

$$= 2^n. [f(Y_n + 2^{-(n+1)}) - f(Y_n)]$$

$$= X_n.$$

Note : Here, $X_n = \dfrac{f(Y_n + 2^{-(n+1)}) - f(Y_n)}{2^{-n}}$ is approximately the derivative of f at z.

Theorem 2.9. (About Generalization of the Eigenvector's Martingale)

Let $Y_0, Y_1, \ldots$ be arbitrary random variables satisfying $E[|Y_n|] < \infty$. For $n = 0, 1, 2, \ldots$, Let's suppose $E[Y_{n+1} | Y_0, \ldots, Y_n] = a_n + b_n Y_n$, $b_n \neq 0$. Let $\ell_{n+1}(y) = a_n + b_n y$ be the linear function, whose inverse is $\ell^{-1}_{n+1}(y) = \frac{y - a_n}{b_n}$, and let

$L_n(y) = \ell^{-1}_1(\ell^{-1}_2(\ldots(\ell^{-1}_n(y)\ldots)))$. Then, for any constant k, $X_n = k.L_n(Y_n)$ is a Martingale.

Proof: (i) $E[|X_n|] = |k| . E[|L_n(Y_n)|] < |k| . L_n\{E[|Y_n|]\} < \infty$

(ii) $E[Y_{n+1} | Y_o, \dots, Y_n] = a_n + b_n Y_n$, Let's take $a_n = 0$.

$$\ell_{n+1}(y) = b_n y \;, \quad \ell_{n+1}^{-1}(y) = \frac{y}{b_n}$$

$$L_n(y) = \ell_1^{-1}(\ell_2^{-1}(\dots \ell_n^{-1}(y)\dots)))$$

for example, $L_2(y) = \ell_1^{-1}(\ell_2^{-1}(y)) = \ell_1^{-1}(\frac{y}{b_1}) = \frac{y/b_1}{b_o} = \frac{y}{b_o b_1}$.

Smilarly,

$$L_n(y) = \frac{y}{b_o b_1 \dots b_{n-1}} \;, \quad L_{n+1}(y) = \frac{y}{b_o b_1 \dots b_n}$$

Otherwise,

$$L_{n+1}(\ell_{n+1}(y)) = L_{n+1}(b_n y)$$

$$= \frac{b_n/y}{b_o b_1 \dots b_{n-1} . b_n} = \frac{y}{b_o b_1 \dots b_{n-1}}$$

Thus, we can show that $L_{n+1}(\ell_{n+1}(y)) = L_n(y)$.

Then,

$$E[X_{n+1} | Y_o, \dots, Y_n] = k . L_{n+1}\{E[Y_{n+1} | Y_o, \dots, Y_n]\}$$

$$= k . L_{n+1}(\ell_{n+1}(Y_n))$$

$$= k . L_n(Y_n)$$

$$= X_n .$$

Definition 2.3. (Upon Likelihood Ratios)

Let Y_o, $Y_1, \ldots$ be independent, identically distrubuted random variables and Let f_o and f_1 be two probability density functions. A stochastic process of fundamental importance in the theory of testing statistical hypotheses is the sequence of likelihood ratios

$$X_n = \frac{f_1(Y_o).f_1(Y_1)\ldots f_1(Y_n)}{f_o(Y_o).f_o(Y_1)\ldots f_o(Y_n)}, \quad n = 0,1 \ldots.$$

Since $Y_o, Y_1, \ldots$ are independent,

$$E\left[X_{n+1} \mid Y_o,\ldots,Y_n\right] = E\left[X_n . \frac{f_1(Y_{n+1})}{f_o(Y_{n+1})} \mid Y_o,\ldots,Y_n\right]$$

$$= X_n . E\left[\frac{f_1(Y_{n+1})}{f_o(Y_{n+1})}\right].$$

When the common distribution of the Y_k's is f_o, then $\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

To prove, we need only to verify

$$E\left[\frac{f_1(Y_{n+1})}{f_o(Y_{n+1})}\right] = 1. \quad \text{But} \quad E\left[\frac{f_1(Y_{n+1})}{f_o(Y_{n+1})}\right] = \int \frac{f_1(y)}{f_o(y)} . f_o(y)\, dy$$

$$= \int f_1(y)\, dy = 1,$$

as desired.

CHAPTER III

SUPERMARTINGALES AND SUBMARTINGALES

For many purposes, Martingales may be expressed by inequalities For this reason it will be usefull to consider supmartingales and submartingales.

Definition 3.1. Let $\{X_n, n=0,1,...\}$ and $\{Y_n, n=0,1,...\}$ be stochastic processes. Then, $\{X_n\}$ is called a supermartingale with respect to $\{Y_n\}$ if, for all n, then

(i) $E[X_n^-] > -\infty$, where $x^- = \min\{x,0\}$

(ii) $E[X_{n+1} | Y_0,..., Y_n] \leq X_n$,

(iii) X_n is a function of $\{Y_0,...,Y_n\}$.

$\{X_n\}$ is a submartingale with respect to $\{Y_n\}$ if, for all n,

(i) $E[X_n^+] < \infty$, where $X^+ = \max\{0,X\}$

(ii) $E[X_{n+1} | Y_0,...,Y_n] \geq X_n$

(iii) X_n is a function of $\{Y_0,...,Y_n\}$.

Theorem 3.1. Let $\{Y_n\}$ be a Markov chain with $P = [|P_{ij}|]$. If f is a superregular sequence for P (a nonnegative sequence satisfying $\sum_j P_{ij} f(j) \leq f(i)$ for all i), then $X_n = f(Y_n)$ defines a supermartingale with respect to $\{Y_n\}$.

Proof: (i) $E[X_n^-] = E[f(Y_n^-)] > -\infty$, (Because, f is nonnegative sequence)

$$(ii)\ E[X_{n+1} | Y_0,...,Y_n] = E[f(Y_{n+1}) | Y_0,...,Y_n]$$

$$= E[f(Y_{n+1}) | Y_n], \text{ (Markov Property)}$$

$$= E[f(Y_{n+1}) | Y_n = i]$$

$$= \sum_j P_{Y_{n,j}} f(j)$$

$$\leq f(i) = f(Y_n) = X_n.$$

(iii) X_n is a function of $\{Y_o, \ldots, Y_n\}$.

Definition 3.3. (About Jensen's inequality)

If for every $X_1, X_2, \ldots, X_m \in I$ and $\alpha_i \geq 0$, $\sum_{i=1}^{m} \alpha_i' = 1$, $\sum_{i=1}^{m} \alpha_i \Phi(x_i) \geq \Phi(\sum_{i=1}^{m} \alpha_i x_i)$ (3.3.1) ; then a function Φ defined on an interval is said to be convex if and only if $d^2\Phi / dx^2 \geq 0$.

If X is a random variable that takes the value x_i with probability α_i $(i=1,2,\ldots,m)$, then the equation (3.3.1) can be written in the form $E[\Phi(x)] \geq \Phi(E[x])$. And the inequality $E[\Phi(x) | Y_o, \ldots, Y_n] \geq \Phi(E[X | Y_o, \ldots, Y_n])$ is called Jensen's inequality.

THEOREM 3.2. Let $\{X_n\}$ be a martingale with respect to $\{Y_n\}$. For all n, $E[\Phi(x_n)^+] < \infty$ and if Φ is a convex function, then $\{\Phi(x_n)\}$ is a submartingale with respect to $\{Y_n\}$.

Proof: (i) $E[|\Phi(x_n)^+|] < \infty$. (As is given in the theorem)

(ii) $E[\Phi(x_{n+1}) | Y_o, \ldots, Y_n] > \Phi\{E[x_{n+1} | Y_o, \ldots, Y_n]\}$ (from Jensen's inequality)

$= \Phi(x_n)$, (from $\{X_n\}$ is a martingale with respect to $\{Y_n\}$)

(iii) $\Phi(X_n)$ is a function of $\Phi(Y_o, \ldots, Y_n)$.

THEOREM 3.3. Let $\{X_n\}$ be a submartingale with respect to $\{Y_n\}$. If $E[\Phi(X_n)^+] < \infty$ and Φ is a convex and increasing function, then $\Phi(X_n)$ is a submartingale.

Proof: (i) $E[\Phi(X_n)^+] < \infty$.

(ii) $E[\Phi(X_{n+1}) \mid Y_o,\ldots,Y_n] > \Phi(X_n)$, ($\Phi$ is a increasing function)

THEOREM 3.4. If $\{X_n\}$ is a (sup) martingale with respect to $\{Y_n\}$, then for every $k \geqslant 0$, $E[X_{n+k} \mid Y_o,\ldots,Y_n] \ (\leqslant) = X_n$.

Proof: For $k=1$, this theorem is true. For $k \neq 1$:

$$E[X_{n+k+1} \mid Y_o,\ldots,Y_n] = E\{E[X_{n+k+1} \mid Y_o,\ldots,Y_{n+k}] \mid Y_o,\ldots,Y_n\}$$

$$(\leqslant) = E[X_{n+k} \mid Y_o,\ldots,Y_n], \text{ (from } \{X_n\} \text{ is a supermartingale)}$$

$$(\leqslant) = E\{E[X_{n+k} \mid Y_o,\ldots,Y_{n+k-1}] \mid Y_o,\ldots,Y_n\}$$

$$(\leqslant) = E[X_{n+k-1} \mid Y_o,\ldots,Y_n]$$

$$\vdots$$

$$(\leqslant) = E[X_{n+1} \mid Y_o,\ldots,Y_n] \ (\leqslant) = X_n.$$

THEOREM 3.5. For $0 \leqslant k \leqslant n$, if $\{X_n\}$ is a (super) martingale then $E[X_n] \ (\leqslant) = E[X_k] \ (\leqslant) = E[X_o]$.

Proof: in accord with theorem (3.4), $E[X_n \mid Y_o,\ldots,Y_k] \ (\leqslant) = X_k$.

$$E[X_n] = E\{E[X_n \mid Y_o,\ldots,Y_k]\} \ (\leqslant) = E[X_k]$$

$$= E\{E[X_k \mid Y_o,\ldots,Y_{k-1}]\}$$

$$(\leqslant) = E[X_{k-1}]$$

$$\vdots$$

$$(\leqslant) = E[X_o].$$

THEOREM 3.6. If $\{X_n\}$ is a (super) martingale with respect to $\{Y_n\}$ and g is a nonnegative function of $(Y_o,\ldots,Y_n)$, then $E[g(Y_o,\ldots,Y_n). X_{n+k} \mid Y_o,\ldots,Y_n] \ (\leqslant) = g(Y_o,\ldots,Y_n). X_n$.

Proof: $E[g(Y_o,\ldots,Y_n).X_{n+k}|Y_o,\ldots,Y_n](\leqslant) = g(Y_o,\ldots,Y_n)E[X_{n+k}|Y_o,\ldots,Y_n]$

$(\leqslant) = g(Y_o,\ldots,Y_n).X_n.$

(from theorem (3.4)).

MARKOV TIMES

Definition 3.3. A random variable T is called a Markov time with respect to $\{Y_n\}$ if T takes values in $\{0,1,\ldots,\infty\}$ and if, for every $n=0,1,\ldots,$ then the event $\{T=n\}$ is determined by $\{Y_o,\ldots,Y_n\}$.

Definition 3.4. If $I_{\{T=n\}}(Y_o,\ldots,Y_n$ is the indicator function of the event $\{T=n\}$, then

$$I_{\{T=n\}}(Y_o,\ldots,Y_n) = \begin{matrix} 1, & \text{if } T=n, \\ 0, & \text{if } T\neq n. \end{matrix}$$

Definition 3.5. If T is a Markov time, then for every n the events $\{T \leqslant n\}$, $\{T > n\}$, $\{T \geqslant n\}$ and $\{T < n\}$ are also determined by $(Y_o,\ldots,Y_n)$.

where, $I_{\{T\leqslant n\}}(Y_o,\ldots,Y_n) = \sum_{k=o}^{n} I_{\{T=k\}}(Y_o,\ldots,Y_n)$,

$I_{\{T > n\}}(Y_o,\ldots,Y_n) = 1-I_{\{T\leqslant n\}}(Y_o,\ldots,Y_n)$, and so on.

Definition 3.6. T=k (k is a constant) is a Markov time.

That is, for al $Y_o,\ldots,Y_n$, $I_{\{T=n\}}(Y_o,\ldots,Y_n) = \begin{matrix} 1, & n=k, \\ 0, & n\neq k. \end{matrix}$

THEOREM 3.7. If S and T are Markov times, then S+T is a Markov time.

Proof : $I_{\{S+T=n\}}(Y_o,\ldots,Y_n) = \sum_{k=o}^{n} I_{\{S=k\}} . I_{\{T=n-k\}}$.

THEOREM 3.8. The smaller of two Markov times S,T, denoted

$S\wedge T = \min\{S,T\}$, is also a Markov time.

Proof : $I_{\{S \wedge T > n\}}(Y_o,\ldots,Y_n) = I_{\{S > n\}}(Y_o,\ldots,Y_n) \cdot I_{\{T > n\}}(Y_o,\ldots,Y_n)$.

Thus, if T is a Markov time, then $T \wedge n = \min(n,T)$, for any fixed $n=0,1,\ldots,$ is also a Markov time.

THEOREM 3.9. If S and T are Markov times, then the larger $S \vee T = \max\{S,T\}$ is also a Markov time.

Proof: $I_{\{S \vee T \leqslant n\}}(Y_o,\ldots,Y_n) = I_{\{S \leqslant n\}}(Y_o,\ldots,Y_n) \cdot I_{\{T \leqslant n\}}(Y_o,\ldots,Y_n)$.

THEOREM 3.10. If $\{X_n\}$ is a (super) martingale and T a Markov time with respect to $\{Y_n\}$. Then for all $n \geqslant k$, $E[X_n I_{\{T=k\}}] (\leqslant) = E[X_k I_{\{T=k\}}]$.

Proof :
$$E[X_n I_{\{T=k\}}] = E\{E[X_n I_{\{T=k\}}(Y_o,\ldots,Y_k) \mid Y_o,\ldots,Y_k]\}$$
$$= E\{I_{\{T=k\}} E[X_n \mid Y_o,\ldots,Y_k]\}$$
$$(\leqslant) = E[I_{\{T=k\}} X_k].$$

THEOREM 3.11. If $\{X_n\}$ is a martingale and T a Markov time, then for all $n=1,2,\ldots,$ $E[X_o] (\geqslant) = E[X_{T\wedge n}] (\geqslant) = E[X_n]$.

Proof :
$$E[X_{T\wedge n}] = \sum_{k=o}^{n-1} E[X_T I_{\{T=k\}}] + E[X_n \cdot I_{\{T=n\}}]$$
$$= \sum_{k=o}^{n-1} E[X_k I_{\{T=k\}}] + E[X_n I_{\{T \geqslant n\}}], \quad (X_T = X_k \text{ when } T=k)$$
$$(\geqslant) = \sum_{k=o}^{n-1} E[X_n I_{\{T=k\}}] + E[X_n I_{\{T \geqslant n\}}] = E[X_n].$$

Definition 3.7. Let i be recurrent state in a Markov chain $\{Y_n\}$. $P_r\{Y_n$ returns to i at least once$\} = P_r\{T_i < \infty\} = 1$, where , $T_i = \min\{n \geq 1 : Y_n = i\}$ is the time of first return to i.

THEOREM 3.12. Let i be recurrent state in a Markov chain. $P_r\{Y_n$ returns to i at least twice$\} = 1$.

Proof : $P_r\{Y_n$ returns to i at least twice$\}$

$$= P_r\{\text{ returns after } T_i \mid T_i = k\}$$

$$= P_r\{Y_{k+n} = i, \text{ for some } n=0,1,\ldots \mid Y_j \neq i, j=1,\ldots,k-1, \; Y_k = i\}$$

$$= P_r\{Y_{k+n} = i \mid Y_k = i\}, \quad \text{(by the Markov property)}$$

$$= P_r\{Y_n = i \mid Y_o = i\}$$

$$= P_r\{T_i < \infty \mid Y_o = i\}$$

$$= 1.$$

CHAPTER IV

APPLICATIONS

Application 4.1. (About Generalization of the Eigenvector's Martingale)

Let $Y_o, Y_1, \ldots$ be arbitrary random variables having finite absolute mean. Y_n is given, Let Y_o be defined in an interval $[0,1]$, Y_{n+1} defined in an interval $[Y_n, 1]$ with uniform distributions.
Let us show that $X_n = 2^n [1-Y_n]$ is a martingale.

Proof : (i) $E[|X_n|] = 2^n . E[|1-Y_n|]$

$$\leq 2^n . E[|1+Y_n|]$$

$$\leq 2^n \{1+E[|Y_n|]\} < \infty$$

(ii) $E[X_{n+1} | Y_o, \ldots, Y_n] = 2^{n+1} \{1-E[Y_{n+1} | Y_n]\}$

$$= 2^{n+1} \{1 - \frac{1}{2} [1+Y_n]\}$$

$$= 2^n [1-Y_n]$$

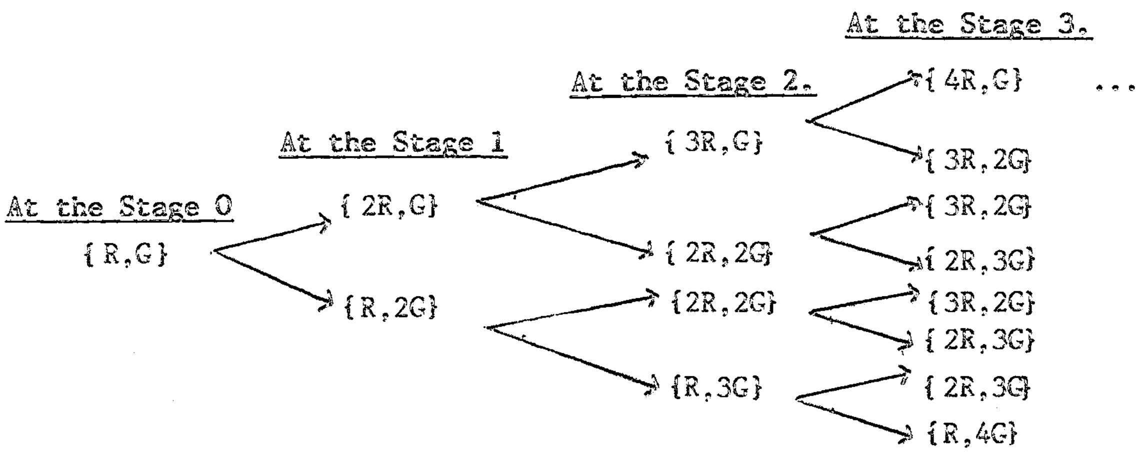
$$= X_n .$$

Application 4.2. (An Urn Scheme)

Let us consider an urn that at stage 0 contains one red and one green ball. A ball is drawn at random from the urn and this one and one more of the same color are returned to the urn. The experiment is repeated indefinetely. Let X_n be the fraction of red balls at stage n, and let $Y_n = (n+2) X_n$ be the number of red balls.

Then $\{X_n\}$ is a martingale with respect to $\{Y_n\}$.

Proof : Let us take $Y_n = k$.



At the stage 1, the number of red balls is $Y_1 = (1+2).\frac{2}{3} = 2.$

At the stage 2, the number of red balls is $Y_2 = (2+2).\frac{3}{4} = 3.$

$\vdots$

At the stage n, the number of red balls is $Y_n = (n+2).\frac{n+1}{n+2} = n+1 = k.$

At the stage (n+1), the number of red balls is $Y_{n+1} = (n+1+2).\frac{n+2}{n+3} = n+2 = k+1.$

That is,

$$Y_{n+1} = \begin{cases} k+1, & \text{with probability } \frac{k}{n+2}, \\ k, & \text{with probability } (1-\frac{k}{n+2}). \end{cases}$$

Hence, $E[Y_{n+1} | Y_n = k] = (k+1).\frac{k}{n+2} + k.(1-\frac{k}{n+2})$

$$= \frac{k(k+1) + k(n+2-k)}{n+2}$$

$$= \frac{k(k+1+n+2-k}{n+2} = \frac{k(n+3)}{n+2} .$$

In that case, $E[Y_{n+1} | Y_n = k] = \frac{n+3}{n+2}.Y_n$. If $\frac{n+3}{n+2} = b_n$,

$E[Y_{n+1} | Y_n] = b_n Y_n.$

Now, Let us show that $\{X_n\}$ is a martingale with respect to $\{Y_n\}$:

(i) $E[X_n] = E[|\frac{1}{n+2}. Y_n|] = \frac{1}{n+2} E[|Y_n|]$, $Y_n = k$

$$= \frac{1}{n+2} .k < \infty ,$$

(ii) $E[X_{n+1} | Y_o, \ldots, Y_n] = E[\frac{1}{n+3} .Y_{n+1} | Y_o, \ldots, Y_n]$

$$= \frac{1}{n+3} . E[Y_{n+1} | Y_n]$$

$$= \frac{1}{n+3} . \frac{n+3}{n+2} .Y_n$$

$$= \frac{1}{n+2} . Y_n = X_n .$$

Application 4.3. (About Likelihood Ratios's Martingale)

Let $Y_o, Y_1, \ldots, Y_n$ be independent, identically distributed random variables. Let f_o, f_1 be normal distributed density functions with mean 0, variance σ^2 and mean μ, variance σ^2.

Then for $n=0,1,\ldots$, $X_n = \frac{f_1(Y_o).f_1(Y_1)\ldots f_1(Y_n)}{f_o(Y_o).f_o(Y_1)\ldots f_o(Y_n)} = \exp \frac{\mu}{\sigma^2}(Y_1+\ldots+Y_n) - \frac{n\mu^2}{2\sigma^2}$.

Proof :

$$X_n = \frac{\frac{1}{\sqrt{2\pi\sigma}} \cdot e^{-\frac{1}{2}\left(\frac{Y_1-\mu}{\sigma}\right)^2} \cdot \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{1}{2}\left(\frac{Y_2-\mu}{\sigma}\right)^2} \cdots \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{1}{2}\left(\frac{Y_n-\mu}{\sigma}\right)^2}}{\frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{1}{2}\left(\frac{Y_1}{\sigma}\right)^2} \cdot \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{1}{2}\left(\frac{Y_2}{\sigma}\right)^2} \cdots \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{1}{2}\left(\frac{Y_n}{\sigma}\right)^2}}$$

$$= e^{\frac{\mu Y_1}{\sigma^2} - \frac{\mu^2}{\sigma^2}} \cdot e^{\frac{\mu Y_2}{\sigma^2} - \frac{\mu^2}{\sigma^2}} \cdots e^{\frac{\mu Y_n}{\sigma^2} - \frac{\mu^2}{2\sigma^2}}$$

$$= e^{\frac{\mu}{\sigma^2}(Y_1 + Y_2 + \ldots + Y_n)} \cdot e^{-\frac{n\mu^2}{2\sigma^2}}$$

$$= \exp\left\{ \frac{\mu}{\sigma^2}(Y_1 + \ldots + Y_n) - \frac{n\mu^2}{2\sigma^2} \right\}$$

CONCLUSION

As it has also been indicated in practice, the Martingal Theory plays an important role, especially on branching processes. It is obvious that there will be further studies in this field. Our aim was only to present the subject into the attention of those interested.

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