

DESIGN AND ANALYSIS OF SINGLE-AXIS CYLINDRICAL
VIBRATORY GYROSCOPE

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VIBRATORY GYROSCOPE**

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ABSTRACT

DESIGN AND ANALYSIS OF SINGLE-AXIS CYLINDRICAL VIBRATORY GYROSCOPE

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Design parameters and mathematical modeling of the Cylindrical Vibratory Gyroscope (CVG) are studied in this thesis. CVG is an inertial sensor that measures angular rate utilizing the effect of Coriolis forces. CVG has a primary motion in the form of vibration. A secondary motion is excited due to Coriolis forces when an angular rate is applied to its sensitive axis. By comparing primary and secondary motions, CVG can measure the angular rate. This sensor stands out with competitive performance despite its lightness and cheapness among the other gyroscopes. The mathematical model of the sensor has been done with the equations of motion of the sensor and state-space estimation of the resonator. This mathematical model has been used for the simulation of the measurement of the angular velocity. The modal analyses of the resonator have been done by using the finite element method (FEM) for different design parameters. Therefore, the effect of design parameters on the natural frequencies of the modes is calculated. In conclusion, a complete mathematical vibrational model of the CVG system is obtained which enables performance and optimization studies for the CVG system, and a new design of the resonator for the CVG system is obtained.

Keywords: Cylindrical Vibratory Gyroscope, Mathematical Modeling, Finite Element Analysis, Inertial Navigation Sensor, Mechanical Design



ÖZ

TEK EKSENLİ SİLİNDİRİK TİTREŞİMLİ JİROSKOP TASARIMI VE ANALİZİ

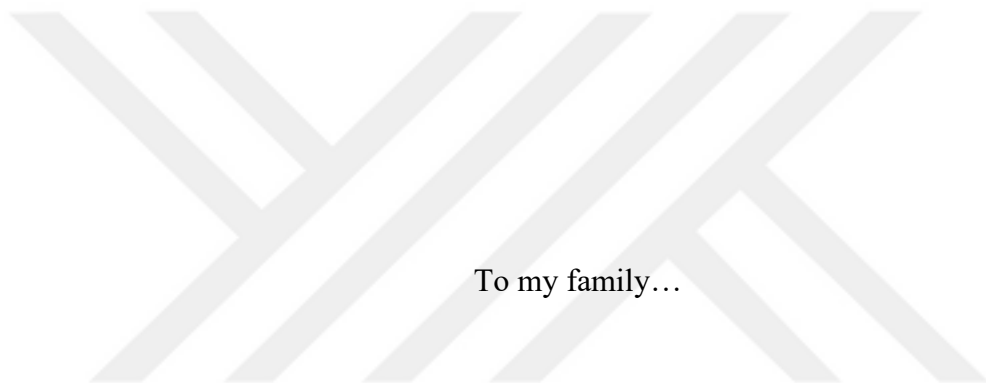
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Bu tezde Silindirik Titreşimli Jiroskopunun (STJ) tasarım parametreleri ve matematiksel modellemesi incelenmiştir. STJ, Coriolis kuvvetlerinin etkisinden yararlanarak açısal hızı ölçen bir sensördür. STJ, titreşim biçiminde birincil bir harekete sahiptir. Hassas eksenine bir açısal hız uygulandığında Coriolis kuvvetleri nedeniyle ikincil bir hareket uyarılır. Birincil ve ikincil hareketleri karşılaştırarak, STJ açısal hızı ölçebilir. Bu sensör, diğer jiroskoplar arasında hafifliği ve ucuzluğuna rağmen rekabetçi performansı ile öne çıkmaktadır. Sensörün matematiksel modeli, sensörün hareket denklemleri ve rezonatörün durum-uzay kestirimi ile yapılmıştır. Bu matematiksel model, açısal hız ölçümünün simülasyonu için kullanılmıştır. Rezonatörün modal analizleri, farklı tasarım parametreleri için sonlu elemanlar metodu (SEM) kullanılarak yapılmıştır. Bu sonuçlarla, tasarım parametrelerinin modların doğal frekansları üzerindeki etkisi hesaplanmıştır. Sonuç olarak, STJ sisteminin performans ve optimizasyon çalışmalarına olanak sağlayan tam bir matematiksel titreşim modeli elde edilmiş ve STJ sistemi için yeni bir rezonatör tasarımı elde edilmiştir.

Anahtar Kelimeler: Silindirik Titreřimli Jiroskop, Matematiksel Modelleme, Sonlu Elemanalar Analizi, Navigasyon Sensörleri, Mekanik Tasarım





To my family...

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LIST OF ABBREVIATIONS

ABBREVIATIONS

IMU	Inertial Measurement Unit
INS	Inertial Navigational System
CVG	Cylindrical Vibratory Gyroscope
ARW	Angular Random Walk
MTBF	Mean Time Between Failures
FEM	Finite Element Method

CHAPTER 1

INTRODUCTION

Navigation has been a vital tool for humankind since the start of history. One desire is always to know the path. For this cause, different reference points and tools are used as guidance. Humankind used the sun, stars, shadows, compasses, and satellites. In recent years, humanity needed one more thing: knowing his path and position without guidance to avoid interference. Accelerometers and gyroscopes have been designed to calculate initial position, acceleration, and rotation. These designed sensors are used in the Inertial Measurement Unit (IMU). The Inertial Measurement Unit contains three accelerometers and three gyroscopes, forming an orthogonal triad. The Inertial Measurement Unit is used in the Inertial Navigation System (INS) along with the gravity model and initial conditions such as starting position and orientation [1].

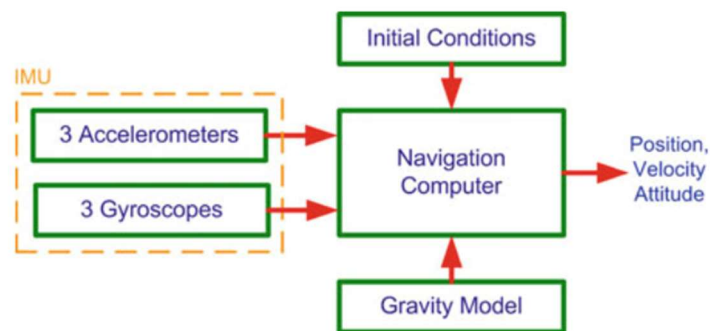


Figure 1.1. Schematic of INS [1]

1.1 Inertial Navigation System Technologies

The main principles of Inertial Navigation rely on Newton's first and second laws. INS unit has three accelerometers which detect acceleration for each axis and gyroscopes which detect rotational velocity for each axis. INS can calculate initial

position, velocity, and orientation by continuously integrating these inertial measurements. The main advantage of the INS is its independence from any external references to operate. Only starting position and orientation are enough [1]. INS can be used in many different areas, such as rockets, spacecraft, aerospace, cameras, electronic stability controllers, etc. Requirements change according to operating areas. Therefore, different INS systems are designed with various specifications.

1.1.1 Inertial Measurement Units

There are two types of Inertial Measurement Units: gimbaled and strap-down systems, as shown in Figure 1.2.

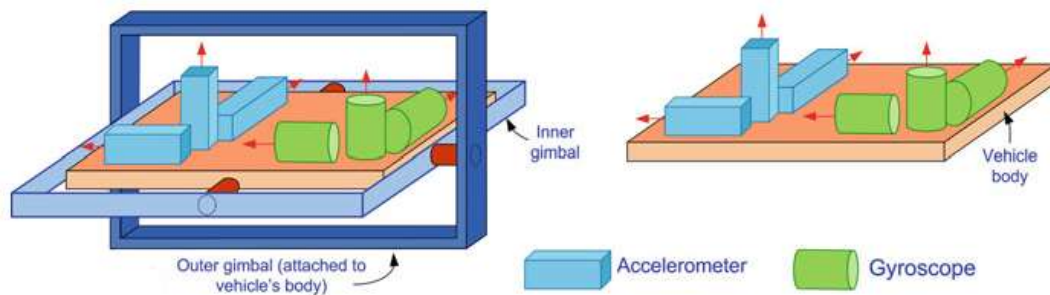


Figure 1.2. Types of IMU [1]

1.1.1.1 Gimbaled IMU

In this IMU type, gimbals work as a stabilization platform. This is achieved by utilizing a group of torque motors. Gyroscopes sense the angular movement, and these torque motors use this measurement. The main advantage of this type of IMU is that the measurement of the accelerometers is always aligned with the navigation frame. However, this type of IMU is mechanically complex and expensive.

1.1.1.2 Strap-down IMU

In this IMU type, computers do the same similar work as gimbal. In this computer simulation, the navigation frame is continuously updated by the gyroscope measurement. Accelerometers are calculated according to the navigation frame of the INS. The main advantages of this type of IMU are reliability, cost, and flexibility.

1.2 Inertial Sensors

A typical IMU has two inertial sensors: the accelerometer and the gyroscope.

1.2.1 Accelerometers

There are many different types of accelerometers, depending on usage, as shown in Figure 1.3.

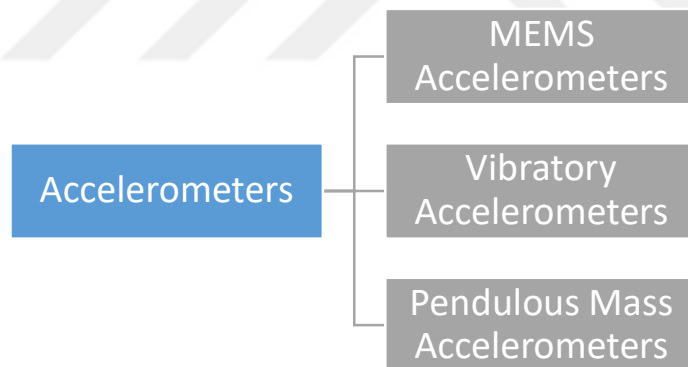


Figure 1.3. Types of Accelerometers

The main principle of the working principle of accelerometers is shown in Figure 1.4. This accelerometer's sensitive axis is the horizontal axis. A proof mass is displaced according to the force on the sensor. This displacement is sensed by some sensing element such as capacitive pick-off, and according to this sensing, acceleration on the sensor can be found. This relationship can be explained briefly as,

$$F = ma \quad (1.1)$$

Where F is the force on the sensor, m is the mass of the proof mass, and a is the acceleration. Displacement and velocity can be found from the acceleration. [1]

$$r = \int v dt = \iint a dt dt \quad (1.2)$$

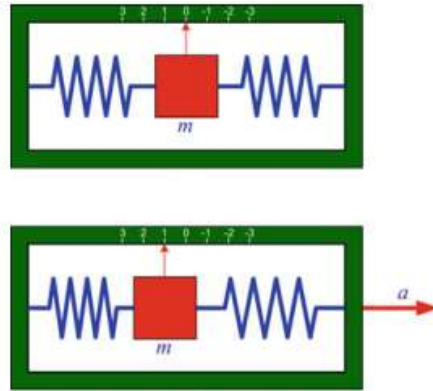


Figure 1.4. Main Work Principle of Accelerometers [1]

1.2.2 Gyroscopes

Many gyroscopes have been designed according to different working principles shown in Figure 1.5.

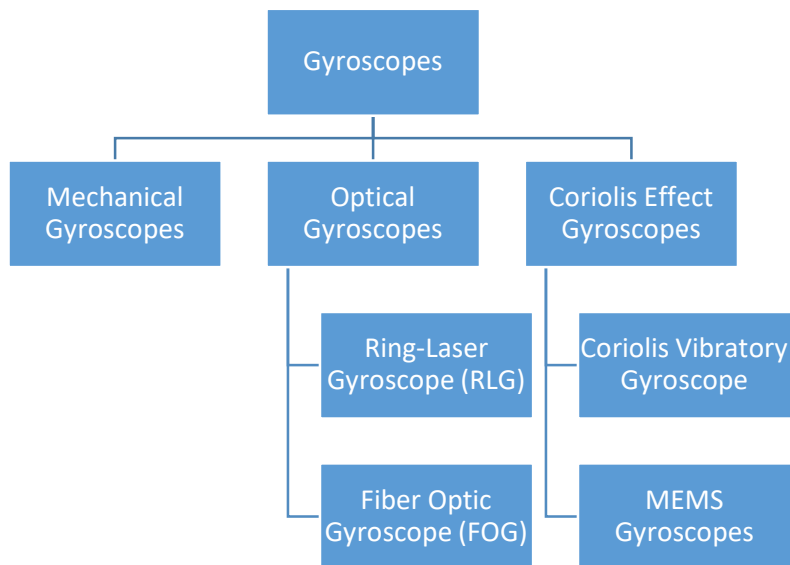


Figure 1.5. Types of Gyroscopes

1.2.2.1 Mechanical Gyroscopes

Mechanical gyroscopes were invented in 1852 by physicist Léon Foucault [2]. Mechanical gyroscopes were one of the first gyroscopes and is shown in Figure 1.6. The working principle is based on the conservation of angular momentum [2]. When the centered mass begins to spin, it tends to keep its angular position when there are no external forces. Therefore, when the gyroscope starts to turn around on the z-axis, there will be a precession force that is perpendicular to the rotation axis, which causes an output angle. This output angle can be converted to the turning input by knowing the spring constant, rotation of the disk, and the inertia of the disk.

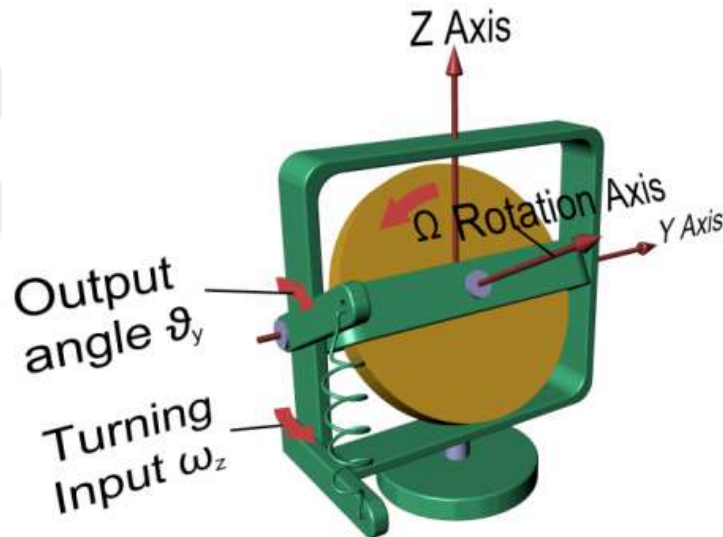


Figure 1.6. Mechanical Gyroscope [2]

1.2.2.2 Optical Gyroscopes

Optical gyroscopes use the Sagnac effect to calculate the angular velocity. Two main sensors use this effect: RLG and FOG. The Sagnac effect is the phase difference between two light beams traveling in opposite directions in a reference frame under the angular velocity [3]. As an example, one can see it in Figure 1.7.

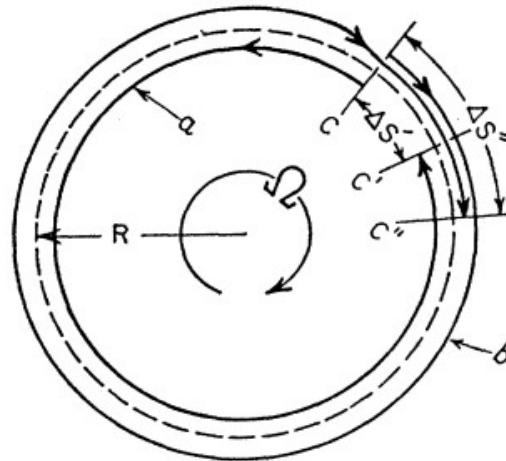


Figure 1.7. Simplified Sagnac Effect [4]

RLG is a high-precision gyroscope, mainly at higher angular velocities. Laser beams travel short distances in RLG. Therefore, RLG uses the frequency difference of the laser beams to calculate angular velocity. However, there is a disadvantage called “lock-in.” [5]. This phenomenon happens at low angular rates, and to overcome this disadvantage, dither mechanisms are used. Additionally, RLG is mechanically compact but complicated, as seen in Figure 1.8.

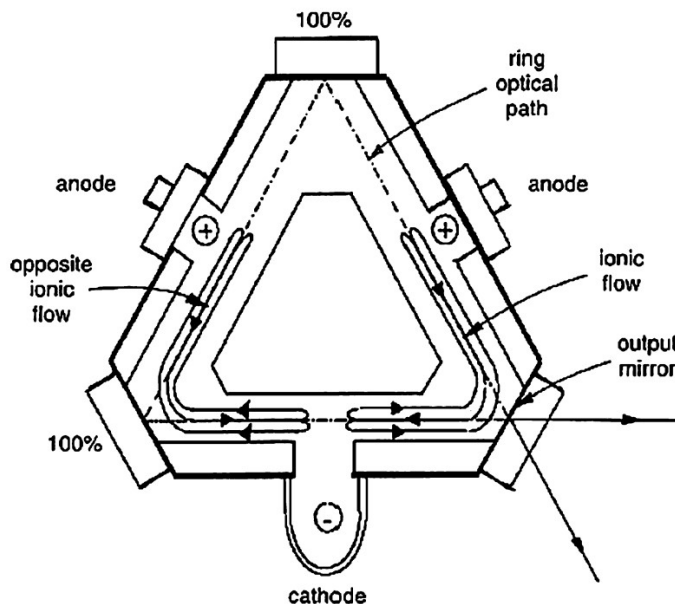


Figure 1.8. Ring-Laser Gyroscope [5]

In recent years, the FOG sensor has surpassed the RLG in terms of stability and low bias noise. FOG technology allows the measurement of phase shifts in all angular velocities without any “lock-in” mechanism. Durability and reliability are lower than RLG. However, it compensates for it with lower complicated designs, as seen in Figure 1.9 [5]. Laser beams travel longer distances in FOG than RLG [5]. Therefore, FOG uses the phase difference of the laser beams to calculate angular velocity.

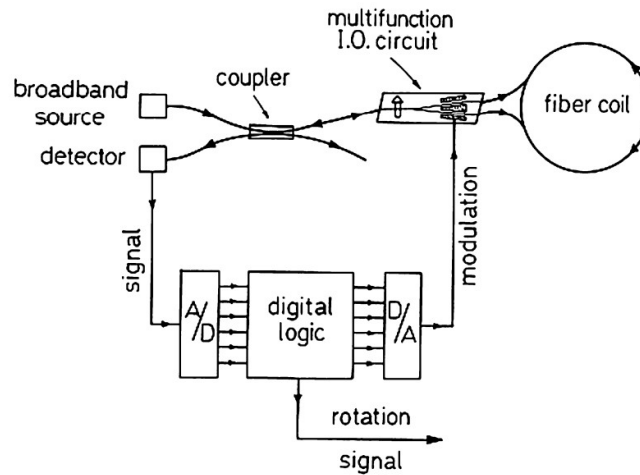


Figure 1.9. Configuration of FOG [5]

1.2.2.3 Coriolis Effect Gyroscopes

Coriolis effect gyroscopes use the Coriolis force to measure the angular velocity. It has a proof mass with two dynamically coupled vibration modes and the mass vibrates in one direction. When the sensor experiences an angular velocity, the perpendicular Coriolis force happens. This leads to a deflection in the second direction. This deflection can be observed by piezoelectric or capacitive elements [6]. A simplified model is given in Figure 1.10.

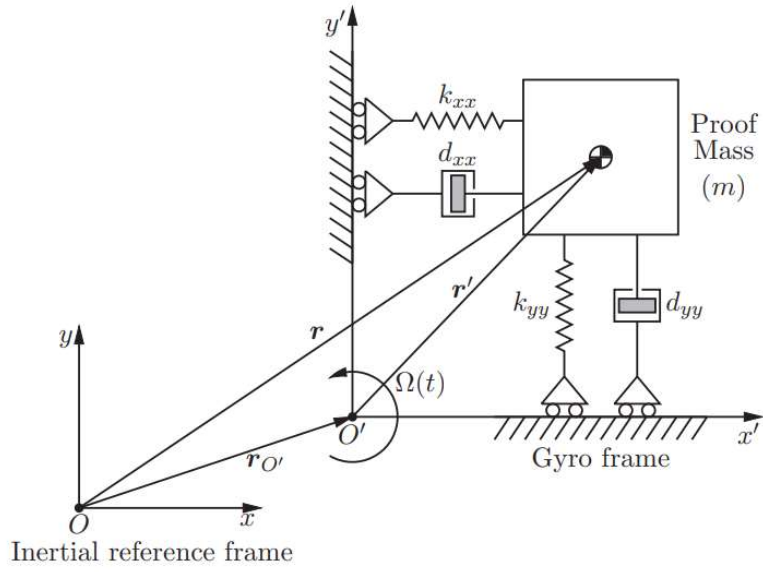


Figure 1.10. Simplified Lumped-Element Model of a Coriolis Effect Gyroscope [6]

The proof mass of the Coriolis effect gyroscopes can be MEMS or resonator. Accordingly, they can be named either MEMS or Coriolis Vibratory Gyroscope sensors. Despite its cheapness and lightness, the Coriolis Vibratory Gyroscope stands out as a high-performance gyroscope.

1.3 Classification of the Coriolis Vibratory Gyroscope

There are different types of sensitive elements in the Coriolis Vibratory Gyroscope [7]. The primary motion, mass distribution, and number of masses can classify these sensitive elements, as seen in Figure 1.11. Sensitive element changes the performance and the complexity of the sensor. This thesis mainly focuses on Cylindrical Coriolis Vibratory Gyroscope sensors which are Cylindrical Vibratory Gyroscopes (CVG).

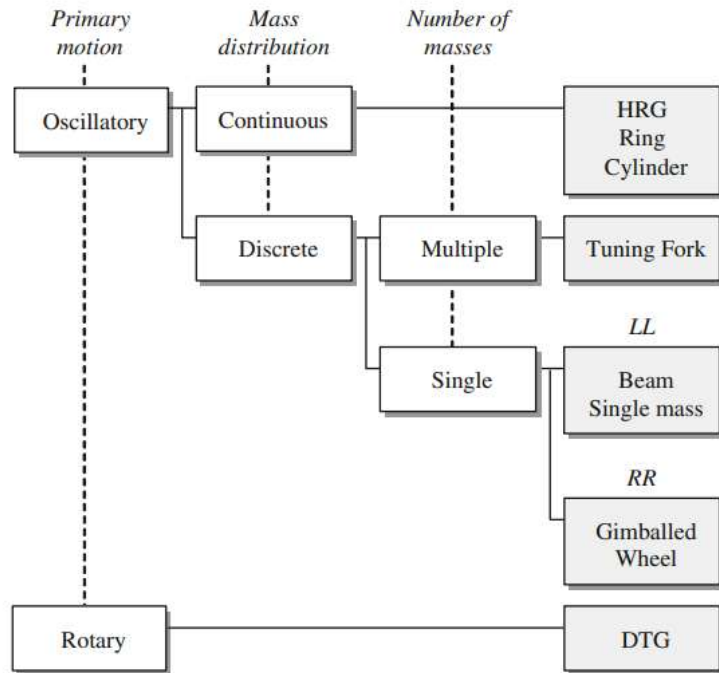


Figure 1.11. Types of Coriolis Vibratory Gyroscope [7]

1.4 The Working Principle of CVG

The resonator is surrounded evenly by eight piezoelectric elements. There are two main axes: primary and secondary axes, in other words, excitation and detection axes. The primary motion is excited by the piezoelectric at the primary axis. This excitation produces the primary or excitation mode. The primary mode has the same natural frequency as the secondary mode, which is 45° apart from each other [8]. In Figure 1.12, one can see these modes.

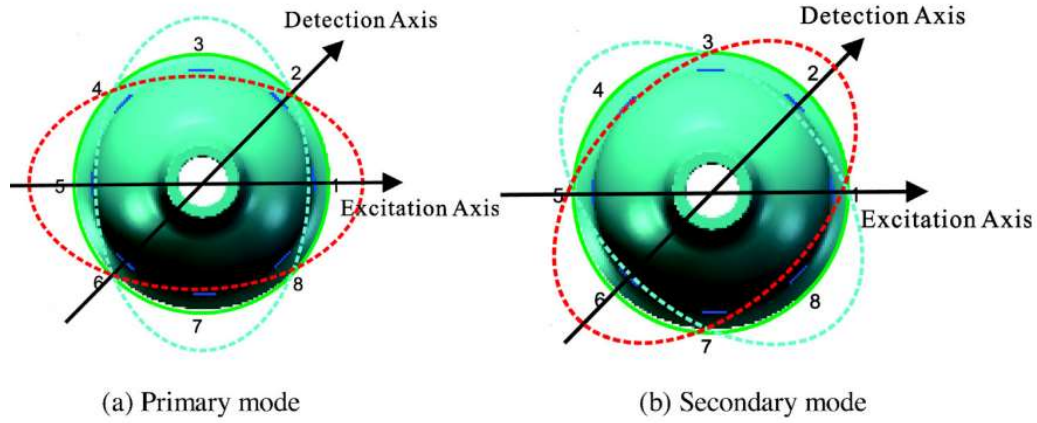


Figure 1.12. Working Principle of CVG [8]

The amplitude of the primary mode is based on the excitation force. Primary and secondary modes are coupled with the Coriolis forces. Under the angular velocity, the standing wave shifts, and the secondary mode occurs, as seen in Figure 1.13. The amplitude of this mode is proportional to excitation force and angular velocity [7].

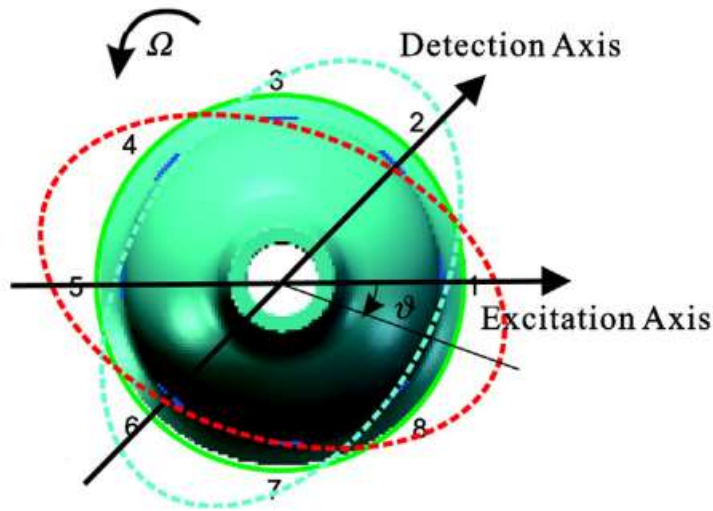


Figure 1.13. Working Principle of CVG with Angular Velocity [8]

1.5 Objective of the Study

This study's main objective is to understand the CVG capabilities from previous works and to make a new, improved sensor design. To create a new design, one

should be confident about the design concepts used in the industry. Finite element method (FEM) analysis will be done on these resonator designs. Using the model order reduction method by state-space estimation, FEM analysis solutions will be implemented in the mathematical model of CVG. Therefore, a complete mathematical vibrational model of CVG will be obtained. Using this model, an optimization study will be conducted on resonators, and a new resonator design will be obtained.

1.6 Outline of the Thesis

This thesis is organized into different sections. Initially, the literature survey about the CVG that was conducted, and it is presented in Section 2. In Section 3, the constructed mathematical model will be explained. This mathematical model is capable of simulating the measured angular velocity of the sensor. In Section 4, the modal analysis of the resonator of the CVG is given. The finite element method (FEM) is used in the modal analysis. Hypotheses and the modeling of the resonator are given in this section. Additionally, the implementation of the modal analysis results in the mathematical model is presented in this section. In section 5, the most common CVG resonators in the industry are presented and compared under different scenarios. In the comparison, analysis parameters are assigned to compare the designs primarily. At the end of the section, the different scenario results of the resonator are given. In section 6, a newly designed resonator is proposed. Design criteria and the design parameters of the resonator are presented in this section. Finally, in Section 7, the conclusion of the thesis is given. The conclusion summarizes the thesis and underlines the important parts of the thesis.

CHAPTER 2

LITERATURE REVIEW

CVGs have garnered considerable attention for their several application areas and significant advantages such as compactness, low power consumption, reliability, and most importantly performance. Several researchers have been working on enhancing the performance and compactness of the CVG. In this section, the past and current technology of the CVG, advancements, and research for the performance aspects of the CVG, and the future of the CVG are discussed.

2.1 History of the CVG

The physics and working principle of the CVG is based on the Coriolis forces. This effect was first described by Gaspard-Gustave Coriolis in 1835. He described the forces that arise from the motion of objects in a rotating reference frame [9]. However, the idea of making this effect as a sensor was founded in 1889 by the physicist G. H. Bryan [10]. He discovered that the tone of the wineglass changes when it is rotated about its stem.

However, it nearly took seventy-five years to make a practical implementation. In 1965, Dr. David Lynch and several PhD students investigated the sensor. Moreover, the first quartz-tuning fork sensor has been made in this research [9]. An example of this type of sensor can be seen in Figure 2.1.

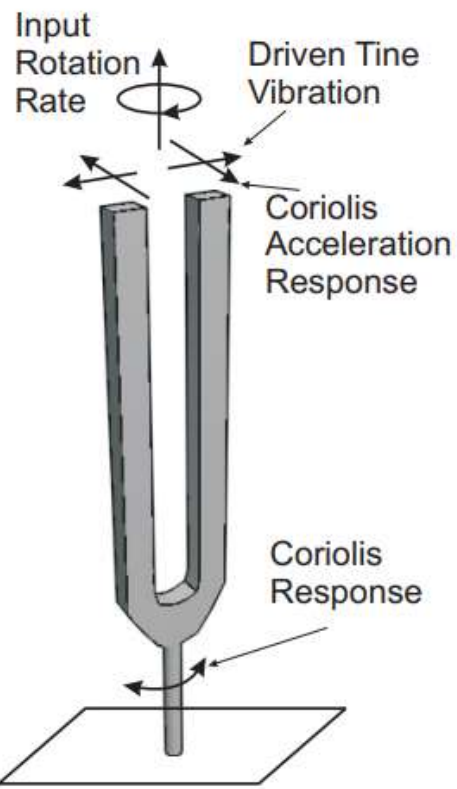


Figure 2.1. Tuning Fork Gyroscope [11]

Nowadays several companies and researchers are working on this subject. Different types of CVGs have been developed till 1965.

2.2 Components and Designs of the CVG

Like every sensor, CVG has different components in its assembly. There are three main components: the base, resonator, and cap which can be seen in Figure 2.2. The resonator is fixed on the base with a screw [12]. The piezoelectric elements, which are important for the working principle of the CVG, are in the resonator.

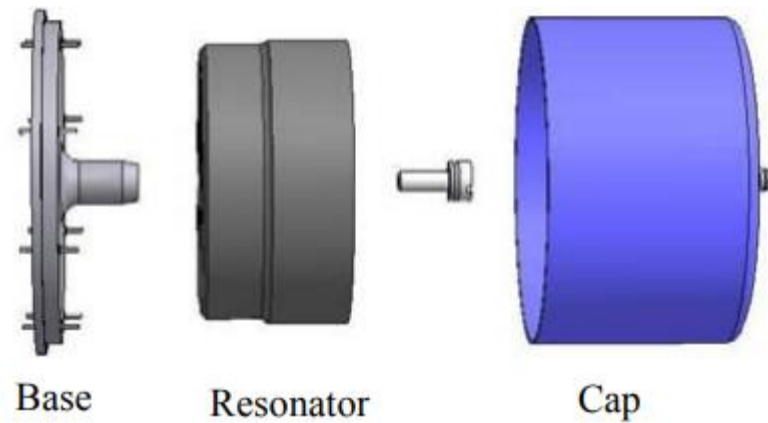


Figure 2.2. Assembly of the CVG [12]

The resonator and the piezoelectric element can be different. Therefore, there are several designs in the literature. Rozelle [9] designed the wineglass and mushroom resonators, Shkel [11] designed the MEMS resonator designs, Xi et al. [13] designed the multiple-shell resonator, Liu et al., [8], [14], [15] designed the bell-shaped resonator design, Tao et al. [16] designed the ring resonator, and Luo et al. [17] designed the cylindrical resonator. The common part of these designs is the axisymmetric feature of the resonator which is required for the working principle of the sensor.

2.3 General Performance of the CVG

2.3.1 Performance Criteria

Several criteria are important for the performance of the CVG which can be seen from the IEEE specification for the CVG [18]. The model equation of the CVG can be seen in Table 2.1.

Table 2.1. The Model Equation of CVG [18]

$\frac{V_0}{S_0} = (I + D_F + D_R)$		
$V_0 = \text{Output [V]}$	$I = \text{Inertial Input [°/h]}$	$D_R = \text{Random Errors [°/h]}$
$S_0 = \text{Scale factor [V/(°/h)]}$		$D_F = \text{Bias Error [°/h]}$

The scale factor is one of the most important performance criteria of the CVG. It helps convert the output voltage to measured angular velocity. One can find scale factors very easily by knowing the given angular velocity to the sensor. The scale factor can be expressed as in Figure 2.3.

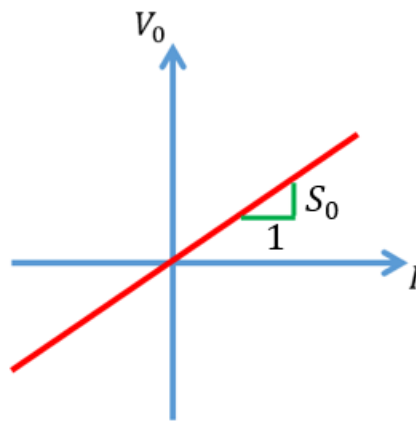


Figure 2.3. Scale Factor

The bias error of the sensor is a very important criterion too. Bias error is the constant error in the sensor. It can be found with a similar test as the scale factor. The bias error can be expressed as in Figure 2.4.

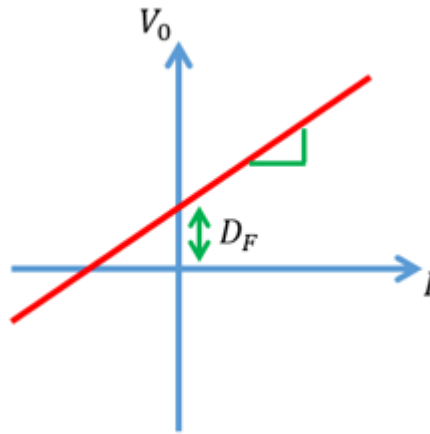


Figure 2.4. Bias Error

The random errors are a little bit more complicated and not certain like the scale factor and the bias error. Random errors come from mainly the electronics of the sensor. Random errors can be found with the Allan-Variance method. For the Allan-variance method, firstly one should gather data from the sensor in a quiet place. After that, the data is divided into different clusters, and average data is found in these clusters. Finally, the standard deviation of these clusters is found and the graph is constructed with these standard deviations [19]. The graph can be seen below.

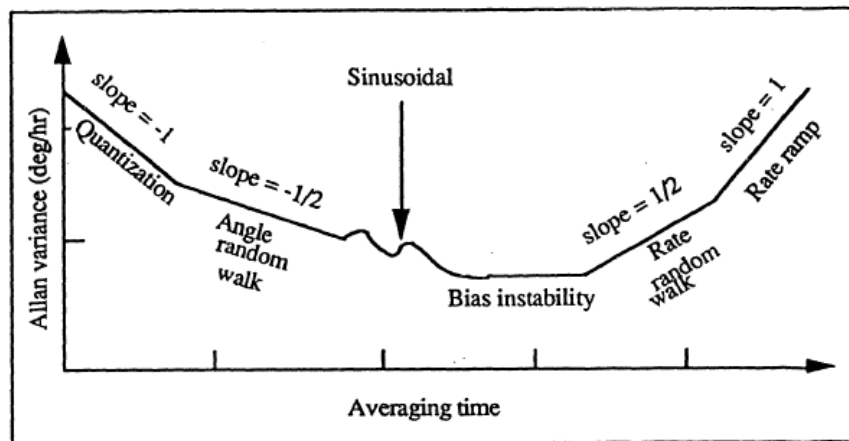


Figure 2.5. Allan-Variance Method [19]

Quantization error is strictly due to the discrete nature of a sensor's output.

Angle random walk (ARW) is the small varying drift with increasing variance in time. This error type has low bandwidth. It is one of the most important errors in the attitude control systems.

Bias instability is caused by the electronics or other components that are susceptible to random flickering. It is the other important random error.

Rate random walk is the noise because of the rate accumulates random fluctuations over time.

A rate ramp is an error that is deterministic rather than random. It represents a drift in rate over time.

2.3.2 Comparison to Other Gyroscopes

Beitia and Kowaltschek [20] stated that CVG is the lowest-cost gyroscope and its mean time between failures (MTBF) is the highest among other gyroscopes which can be seen in Figure 2.6. Additionally, they stated that the bias instability value is less than 0.02 deg/hr and ARW is less than 0.002 deg/ $\sqrt{\text{hr}}$.

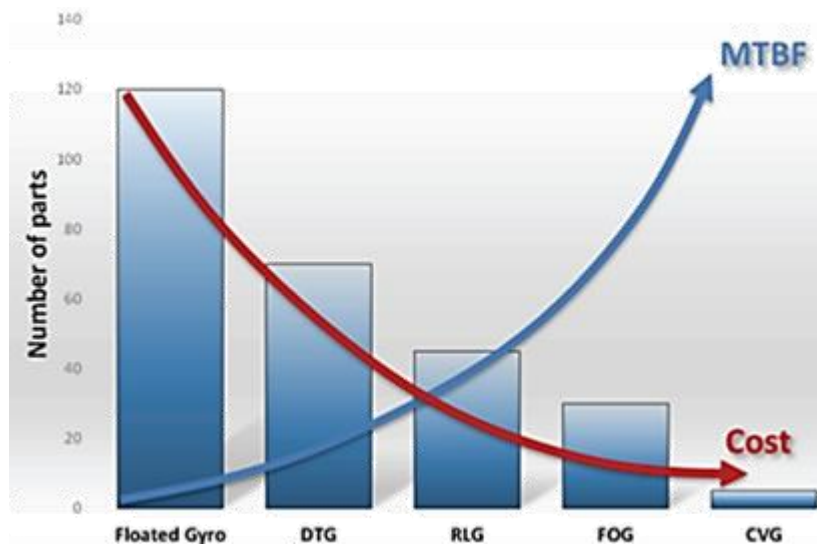


Figure 2.6. Comparison to Other Gyroscopes by Cost and MTBF [20]

Chikovani [21] compared a Ukrainian CVG and a Russian FOG which can be seen in Table 2.2. Chikovani stated that CVG is better than most of the performance parameters other than bandwidth.

Table 2.2. CVG and FOG Parameters Comparison [21]

Parameter	FOG	CVG
Angle rate range, deg/s	± 80	$\pm 200, \pm 400$
Scale Factor Instability, %	0.1	0.03
Bias Instability deg/hr	0.03	0.01
Bandwidth, Hz	400	100-150
Temperature range, °C	-30, +70	-40, +75
Shock Resistance	40g, 1ms	400g, 2ms
Vibroresistance, RMS amplitude, frequency	2g, 20-500 Hz	15g, 20-3000Hz

In another publication, Chikovani [22] compared the random errors of CVG, FOG, and RLG. Chikovani stated that CVG is better than FOG. Moreover, CVG is competitive enough with RLG. CVG has a main advantage over RLG which is the cost of the sensor. The comparison values are shown in Table 2.3.

Table 2.3. RLG, FOG, and CVG Comparison in terms of Random Errors [22]

Parameters	RLG	FOG	CVG
Quantization Error, arcsec	0.65	1.60	0.72
ARW, deg/ $\sqrt{\text{hr}}$	0.0018	0.01	0.003
Bias Instability, deg/h	0.0022	0.09	0.03

2.4 Q-Factor of CVG

The Q factor is an important factor for the performance of the CVG. Q factor helps the sustainability of the vibration amplitude. A high Q factor means low damping in

the resonator [23]. A high Q factor is desirable in CVGs in order to minimize the power needed to maintain the constant vibration [24]. Fused silica resonators have nearly 800,000 Q factor [25], and Metallic resonators used in this thesis have nearly 10,000-40,000 Q factor [26], [27]. The Q factor of the resonator is affected by the geometry and material of the sensor, the unbalance in the resonator, and the atmospheric gas in the environment. The axisymmetric feature of the CVG resonator is important for the Q factor. If the damping ratio is different than the others in one direction, it will cause drift error in the measurement [24]. Additionally, the amplitude response of the system will change with the Q factor. In Figure 2.7, one can see different amplitude responses for different damping ratios (Q factor).

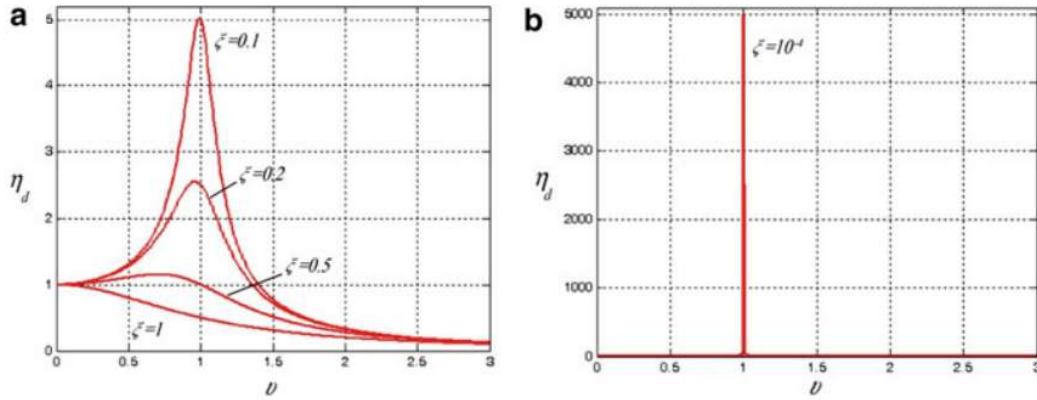


Figure 2.7. Q Factor Influence on the Amplitude in Frequency Domain. (a) Low Q Factor - High Damping Ratio (b) High Q Factor - Low Damping Ratio [27]

2.5 Errors of the CVG

The CVG has a specific error type due to its axisymmetric feature. This error is the frequency mismatch error. According to the work principle of the CVG, frequencies must be the same in the two operating modes. Even a small mismatch can have a significant effect. This frequency mismatch causes a phase difference between the two operating modes and results in an error in the measured angular velocity [28].

The frequency split is the most popular research area for the CVG. Many researchers are working in this area. For example, frequency split is one of the most important

areas for the SAGEM company [24]. In Figure 2.8, one can see the drift error by the frequency mismatch and control accuracy (ϵ) according to SAGEM.

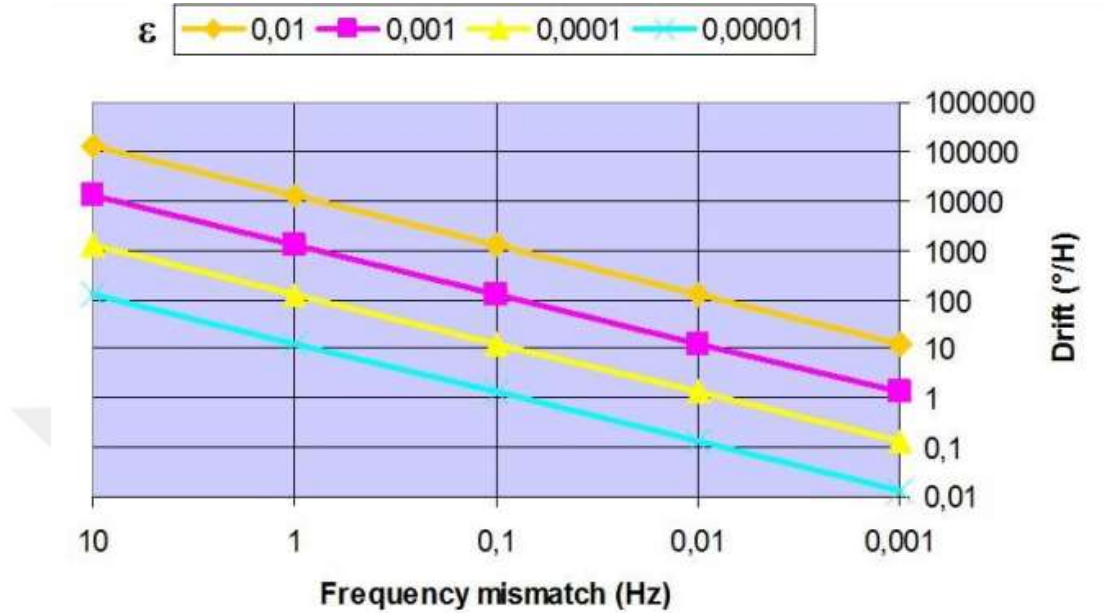


Figure 2.8. Caused Drift Error by Frequency Mismatch [24]

Choi and Kim [29] developed an analytical model to estimate the frequency split in a hemispherical shell. Firstly, they study in perfect shell to verify its mathematical model. Then they used the Rayleigh method to determine the split of natural frequencies. Xiang Xi et al. [30] studied frequency split due to geometric errors and inhomogeneous material for ring resonators. It is stated that geometric accuracy is not enough to eliminate frequency split, homogeneous properties of the material are more important for the frequency split. Vakhlyarsky et al. [31] represent a formula to estimate frequency split due to defects in the resonator. In their research, cylindrical and bell-shaped resonators are used.

2.6 Balancing and Tuning of the CVG

Due to the manufacturing processes of the CVG, it may not be possible to have the frequencies very close. To eliminate this frequency mismatch, balancing and tuning processes have been found. This balancing can be made as a mass removal or mass adding from the resonator [24].

Rourke et al. [32] analyzed the trimming process for eliminating the frequency split. This work has found a new numerical method to predict the trimming process. Lin et al. [33] proposed a new approach to eliminate the frequency split. They proposed adding a balancing mass to the resonator. In the article, theoretical and experimental approaches are used. Additionally, their approach can be used in nearly every axisymmetric resonator. Since MEMS gyroscopes are much smaller than resonator gyroscopes, eliminating the frequency split is a harder process. Hu et al. [34] proposed a new electrostatic mode tuning for the MEMS gyroscopes. Experimental tests show that the 1.5 Hz frequency split can be lowered to 6mHz by using this method. Hu et al. [35] introduced the laser trimming method in the article. In the trimming process, micro-holes are used by drilling with a laser. By this method, approximately 2 Hz frequency split can be reduced to 0.1 Hz. Drilled holes can be seen in Figure 2.9.

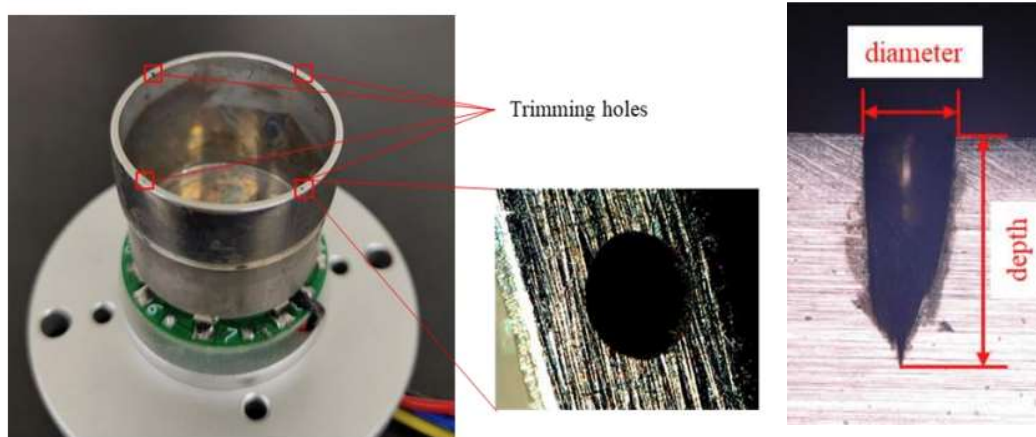


Figure 2.9. Drilled Holes in the Resonator [35]

2.7 Temperature Effects in the CVG

The temperature changes in the environment or the resonator can affect the characteristics of the CVG. Therefore, the temperature effect can cause variations in the output. Important vibration characteristics which were explained before, resonant frequency, frequency mismatch, and Q factor, can change with the temperature [36].

In MEMS gyroscope, Lin et al. [37] experimented with MEMS gyroscopes on the vibration characteristics. Experiments were done in 60 °C, 25 °C, and -10 °C. Vibration characteristics have been found with the frequency sweeping in different temperatures. The results can be seen in Table 2.4.

Table 2.4. Parameters Measured by Frequency Sweeping for MEMS Gyroscope [37]

Parameters	60 °C	25 °C	-10 °C
Quality Factor	6493	7413	8107
Resonant Frequency, Hz	1812.87	1814.98	1816.63

In CVG resonators, Xiao et al. [36] did several experiments on the vibration characteristics of the fused quartz cylindrical resonator with a 10^6 Q factor. Experiments were done between 253.15 K and 353.15 K. According to these experiments, the Q factor increases from 253.15 K to 333.15 K, the frequency mismatch doesn't change in this temperature interval, and the resonant frequency increases linearly with the temperature. Results can be seen in Figure 2.10, Figure 2.11.

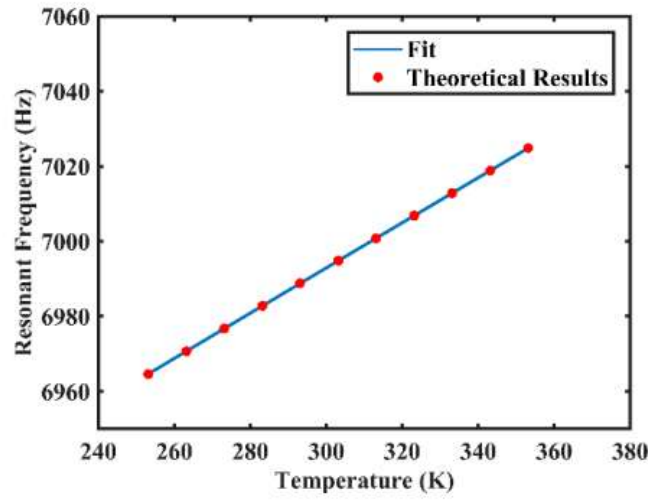


Figure 2.10. The Fitted Value of Resonant Frequency Varying with Temperature [36]

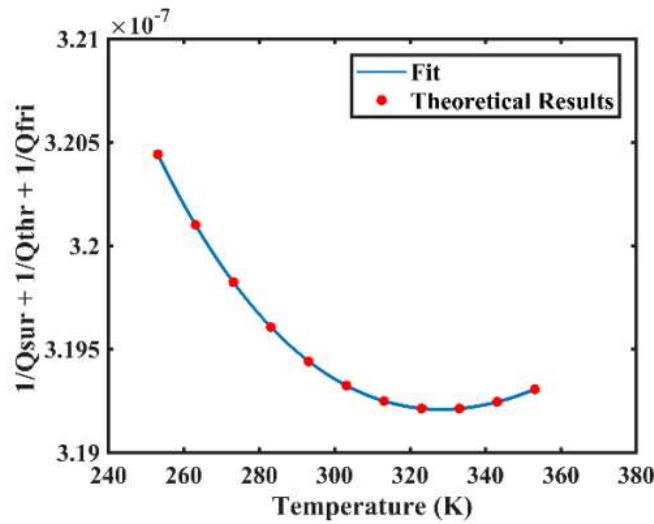


Figure 2.11. The Q-Factor (Sum of Q-Factor Effects) Varies with Temperature [36]

In both gyroscopes, vibration characteristics change with the temperature. These vibration characteristics result in errors in the scale factor and measurements. For example in the MEMS gyroscope, Zotov et al. [38] investigated scale factor change of a frequency-modulated MEMS gyroscope. From this experiment, one can see that the scale factor changes with the temperature and the quality factor. The results of the experiment can be seen in Figure 2.12.

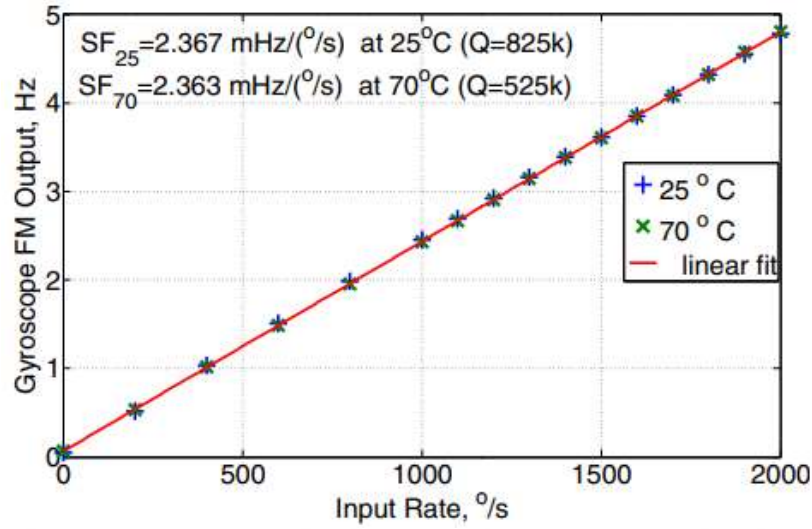


Figure 2.12 Measurement of the MEMS Gyroscope in Different Temperatures [38]

Different compensation techniques can be used for the temperature sensitivity of the sensors. However, in this thesis, the main scope will be on the design and frequency split of the resonator. Therefore, the temperature effects are neglected throughout the thesis.

2.8 Control of the Sensor

There are two main operational modes for the CVG. Amplitude-modulated operational mode and Frequency-modulated operational mode. In an amplitude-modulated gyroscope, one of the modes excited with constant amplitude at resonance. Under the Coriolis effect, the angular rate can be measured by the rebalance force used in the control mode or the amplitude of the second mode [39]. Rebalance force is created in the Force-To-Rebalance mode. Force-to-rebalance mode, the standing wave is fixed in a position [40]. Additionally, in this mode, angular rate information is related to the feedback voltage [41].

Ning, Zhong, and Hong [42] developed a novel control loop method, which uses a force-to-rebalance control mode with a linear demodulator. This mode has different control styles, including amplitude and frequency control. The linear demodulator

helps calculate the amplitude and phase in the loops as a preprocess. A diagram of the control mode can be seen in Figure 2.13.

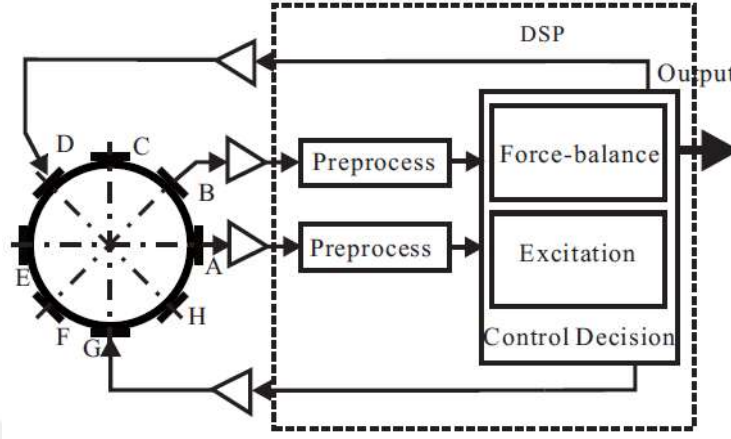


Figure 2.13. Diagram of the Control Mode [42]

He et al. [43] contributed to the literature with a new novel force-to-rebalance control mode. There is no demodulation and modulation in the control loop. The advantage of this control method is that the Coriolis force and quadrature force are rebalanced simultaneously in only one control loop. This control loop can be seen in Figure 2.14.

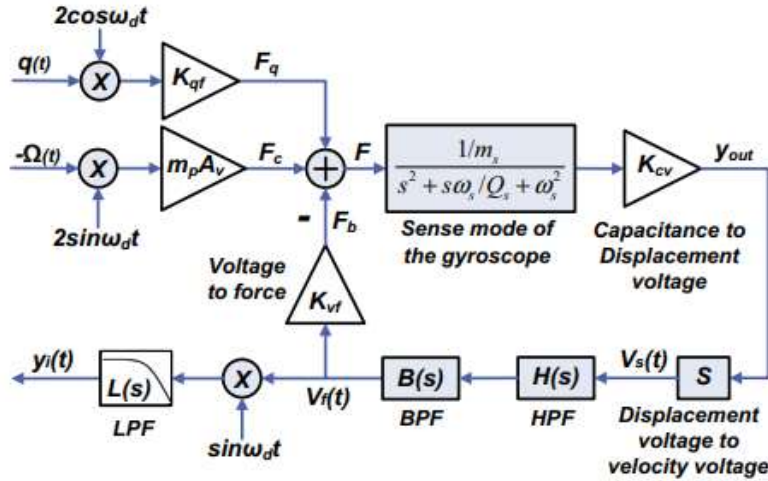


Figure 2.14. Diagram of the novel Force-to-Rebalance Control Mode [43]

However, amplitude-modulated gyroscopes have disadvantages in signal amplitudes, which cause large noise and limited bandwidth, high sensitivity to

temperature and electrical noise [39], [44]. Additionally, Zhang et al. [45] claim that high Q-factor CVG has poor dynamic characteristics in the force balance mode which can be improved with the alternative operational mode.

Therefore, an alternative operational mode has developed which is the Frequency-modulated operational mode. It offers high-scale-factor stability and low-temperature independence [39]. In this operational mode, the standing wave of a vibrating pattern is allowed to move freely around the resonator, and angular velocity is calculated from the frequency difference between the eigenfrequencies of the two operating modes. The difference between the two main operational modes is given in Table 2.5.

Table 2.5. Comparison of AM and FM Gyroscopes [39]

	AM	FM
Readout	Amplitude	Frequency
Scale Factor	Temperature Dependent	Temperature Independent
Bandwidth	Limited	Almost Unlimited
Resolution	Better	Worse

2.9 Novelty of this Thesis

Apostolyuk [7], a mathematical model has been performed for the CVG sensors. Wu et al. [27] explain modal analysis for the CVG resonators. Using a mathematical model and modal analysis, the CVG sensors' performance is analyzed from different perspectives. The mathematical model and the modal analysis results are coupled with the state-space matrices and natural frequency of the resonator. Therefore, a complete mathematical vibrational model has been constructed within this thesis. Additionally, different types of CVGs are investigated in this thesis using this model. Based on the analyses, the best-performing resonator has been found, and a new improved design has been done based on the best-performing resonator. Therefore, this thesis offers practical insights for engineers and designers in this field.

CHAPTER 3

MATHEMATICAL MODEL OF CYLINDRICAL VIBRATORY GYROSCOPE

CVG contains various elements to measure the angular velocity along the sensitive axis. It is necessary to model the CVG and integrate these different various aspects into a mathematical model. This section introduces the dynamic equations of the Cylindrical vibratory gyroscope and signal processing.

3.1 Dynamic Equations of the CVG

One of the most essential parts of the analysis is deriving equations of motion. In all types of CVGs, the mechanical part at the center is the sensitive element. Additionally, sensitive elements should be capable of two mechanical oscillations: primary and secondary motions. In this part, a single set of equations containing two motion equations has been derived.

In Figure 3.1, the translational sensitive element of the CVG has been given. The sensitive element contains proof mass " m_2 ", decoupling frame " m_1 ", and two springs. Primary oscillations are excited along the Y-axis, followed by secondary oscillations along the X-axis, making the Z-axis the designated sensitive axis.

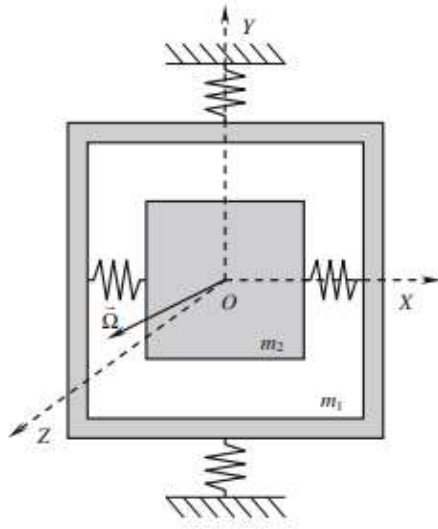


Figure 3.1. Translational Sensitive Element [7]

Position $\vec{X}_1$ of the decoupling frame and position $\vec{X}_2$ of the proof mass in the reference frame OXYZ is defined as

$$\vec{X}_1 = \{0, x_1, 0\} \quad (3.1)$$

$$\vec{X}_2 = \{x_2, x_1, 0\} \quad (3.2)$$

The displacement y-axis represents the movement of the decoupling frame to the fixed base, and the displacement x-axis refers to the movement of the proof mass relative to the decoupling frame.

An arbitrary angular velocity vector $\vec{\Omega}$ represents the sensitive element rotation.

$$\vec{\Omega} = \{\Omega_x, \Omega_y, \Omega_z\} \quad (3.3)$$

The Lagrange equation is used to derive the motion equations of the sensitive element.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \left(\frac{\partial L}{\partial x_i} \right) = Q_i \quad (3.4)$$

$$L = E_k - E_p \quad (3.5)$$

The definition of energies, E_k which is the total kinetic energy of the sensitive element, and E_p which is the total potential energy of the sensitive element. Q_i is the generalized force acting on the sensitive element of the sensor, and the subscript “i” is named 1 for the primary motion and 2 for the secondary motion.

Kinetic energy can be written as

$$E_k = \frac{m}{2} (\vec{V} \cdot \vec{V}) \quad (3.6)$$

Which

$$\vec{V} = \dot{\vec{X}} = \frac{d}{dt} \vec{X} + \vec{\Omega} \times \vec{X} \quad (3.7)$$

$\vec{V}$ represents the absolute velocity vector, and $\vec{X}$ is the displacement of the point mass.

According to Equation (3.1), (3.2), (3.6), and (3.7), the kinetic energy of the masses can be found as

$$\vec{V}_1 = \{-x_1\Omega_z, -\dot{x}_1, x_1\Omega_x\} \quad (3.8)$$

$$\vec{V}_2 = \{-x_1\Omega_z + \dot{x}_2, x_2\Omega_z + \dot{x}_1, x_1\Omega_x - x_2\Omega_y\} \quad (3.9)$$

$$E_{k1} = \frac{m_1}{2} [\dot{x}_1^2 + x_1^2(\Omega_x^2 + \Omega_z^2)] \quad (3.10)$$

$$E_{k2} = \frac{m_2}{2} [(\dot{x}_1 + x_2\Omega_z)^2 + (\dot{x}_2 - x_1\Omega_z)^2 + (x_1\Omega_x - x_2\Omega_y)^2] \quad (3.11)$$

$$E_k = E_{k1} + E_{k2} \quad (3.12)$$

Total potential energy can be written as

$$E_p = \frac{k_1}{2} x_1^2 + \frac{k_2}{2} x_2^2 \quad (3.13)$$

By using Equations (3.4), (3.5), (3.12), and (3.13), the equation set of the motion for the CVG is derived, which can be seen below.

$$\begin{aligned}
(m_1 + m_2)\ddot{x}_1 + [k_1 - (m_1 + m_2)(\Omega_x^2 + \Omega_z^2)]x_1 \\
+ m_2[(\Omega_x\Omega_y + \dot{\Omega}_z)x_2 + 2\Omega_z\dot{x}_2] = Q_1 \\
m_2\ddot{x}_2 + [k_2 - m_2(\Omega_y^2 + \Omega_z^2)]x_2 \\
+ m_2[(\Omega_x\Omega_y - \dot{\Omega}_z)x_1 + 2\Omega_z\dot{x}_1] = Q_2
\end{aligned} \tag{3.14}$$

This equation set is simplified by dividing the whole equations with the coefficient of the highest-order derivative term.

$$\begin{aligned}
\ddot{x}_1 + (\omega_1^2 - \Omega_x^2 - \Omega_z^2)x_1 + d(\Omega_x\Omega_y + \dot{\Omega}_z)x_2 + 2d\Omega_z\dot{x}_2 = q_1 \\
\ddot{x}_2 + (\omega_2^2 - \Omega_y^2 - \Omega_z^2)x_2 + (\Omega_x\Omega_y - \dot{\Omega}_z)x_1 - 2\Omega_z\dot{x}_1 = q_2
\end{aligned} \tag{3.15}$$

Where $d = m_2/(m_1 + m_2)$ is the dimensionless inertia asymmetry factor. Generalized accelerations from different external forces that act along respective axes are the $q_1 = Q_1/(m_1 + m_2)$, $q_2 = Q_2/m_2$. $\omega_1^2 = k_1/(m_1 + m_2)$ and $\omega_2^2 = k_2/m_2$ are the squared natural frequencies of the primary and secondary motions.

Additionally, damping forces will be added, and the equation set becomes as follows,

$$\begin{aligned}
\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + (\omega_1^2 - \Omega_x^2 - \Omega_z^2)x_1 + d(\Omega_x\Omega_y + \dot{\Omega}_z)x_2 + 2d\Omega_z\dot{x}_2 = q_1 \\
\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + (\omega_2^2 - \Omega_y^2 - \Omega_z^2)x_2 + (\Omega_x\Omega_y - \dot{\Omega}_z)x_1 - 2\Omega_z\dot{x}_1 = q_2
\end{aligned} \tag{3.16}$$

ζ_1 and ζ_2 are the dimensionless damping factors corresponding to the primary and secondary motions of the sensitive element.

In Equation (3.3), the angular velocity vector contains all the axes. However, CVG is a single-axis gyroscope. Therefore, only the z-axis term is not zero.

$$\vec{\Omega} = \{0, 0, \Omega\} \tag{3.17}$$

The new angular velocity vector simplified the equation set of motion of the CVG, which can be seen below,

$$\begin{aligned}
\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + (\omega_1^2 - \Omega^2)x_1 = q_1 - d\dot{\Omega}x_2 - 2d\Omega\dot{x}_2 \\
\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + (\omega_2^2 - \Omega^2)x_2 = q_2 + \dot{\Omega}x_1 + 2\Omega\dot{x}_1
\end{aligned} \tag{3.18}$$

The equations indicate that angular velocity couples the primary and secondary motions. This implies that when there aren't any external forces along the secondary direction, forced displacement will occur only with angular velocity in this direction.

Therefore, one can use this equation set to analyze the CVG dynamics to find the angular velocity.

The previous case uses lumped mass to sense the angular velocity. Additionally, distributed mass shapes can be used as a sensitive element of CVG. One of the leading examples, also the main shape of the sensitive element in this thesis, is given in Figure 3.2.

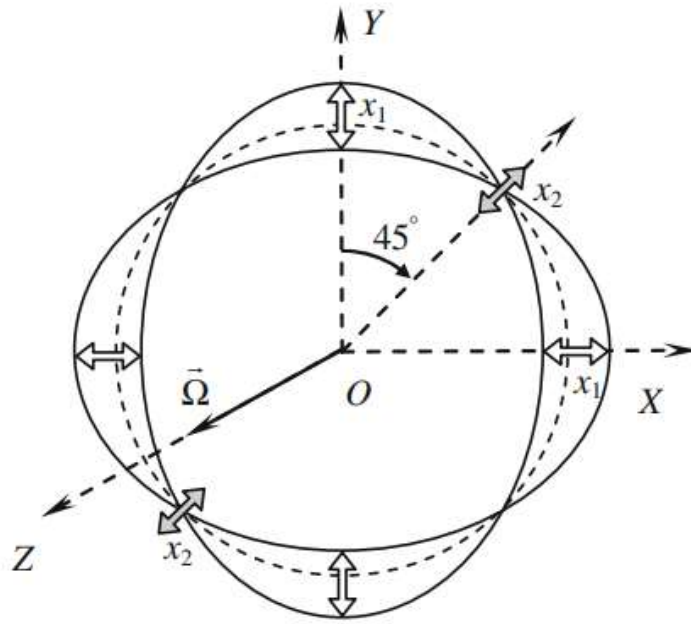


Figure 3.2. Cylinder-Shaped or Ring-Shaped CVG Sensitive Element [7]

Primary and secondary motions are along the X and Y axes. When a ring-shaped sensitive element is set to oscillate with the sinus force in the X-axes, only x_1 will occur. These oscillations could be accepted as a standing wave. In the absence of the angular velocity, the primary wave has four nodes, which are separated by 45 degrees from the X and Y axes.

When there is an angular velocity in the Z-axis, the standing wave starts to shift to the secondary direction due to the Coriolis force. Therefore, nodes will start to oscillate, which helps with the measurement of the angular velocity.

The equation set of motion of the Ring-Shaped CVG sensitive element can be seen below,

$$\begin{aligned} \ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + (\omega_1^2 - \Omega^2)x_1 &= q_1 - c\dot{\Omega}x_2 - 2c\Omega\dot{x}_2 \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + (\omega_2^2 - \Omega^2)x_2 &= q_2 + c\dot{\Omega}x_1 + 2c\Omega\dot{x}_1 \end{aligned} \quad (3.19)$$

c is the Bryan coefficient. Bryan coefficient is the ratio of the angular rate of the standing wave to the angular rate of the vibrating body. This coefficient varies with geometric and material properties [46]. This coefficient can be assumed as 0.8 for the cylinder resonators (sensitive element) [47].

From the Equation (3.19) the additional forces other than excitation forces can be explained below [27]:

- Ω^2x_1, Ω^2x_2 are the centripetal convective acceleration due to the angular velocity of the rotating coordinate system.
- $c\dot{\Omega}x_2, c\dot{\Omega}x_1$ are the tangential convective acceleration due to the angular acceleration of the rotating coordinate system.
- $2c\Omega\dot{x}_2, 2c\Omega\dot{x}_1$ are the Coriolis acceleration.

Equation (3.19) can be rewritten as below, and the coupling effect is clearly seen.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & 2c\Omega \\ -2c\Omega & 2\zeta_2\omega_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 - \Omega^2 & c\dot{\Omega} \\ -c\dot{\Omega} & \omega_2^2 - \Omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad (3.20)$$

The coupling effect comes with two different forces, Coriolis acceleration and tangential convective acceleration. When there is only constant angular velocity, the coupling effect comes from only the Coriolis acceleration. However, if the angular velocity is not constant, one more coupling effect comes from the tangential convective acceleration force. The analytical solutions given in this thesis are based on the constant angular velocity to solve the equation based on the steady-state values and simplify the solutions. However, the mathematical model includes the effect of the angular velocity rate. Analytical solutions and mathematical models are shown in later sections.

3.2 Solution of the Dynamic Equations

Dynamic equations are found in the previous part. As per the working principle of the CVG, external force only occurs in the first mode, which is named as the excitation force. Therefore, there are no external forces in the second mode ($q_2 = 0$). Additionally, when there is no external angular velocity ($\Omega = 0$), this equation set becomes uncoupled, as can be seen below.

$$\begin{aligned}\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2x_1 &= q_1 \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2x_2 &= 0\end{aligned}\tag{3.21}$$

An excitation force can be modeled as a harmonic force that creates oscillations. Therefore, equations transform as below.

$$\begin{aligned}\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2x_1 &= q_{10} \sin(\omega t) \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2x_2 &= 0\end{aligned}\tag{3.22}$$

By solving Equation (3.22), the solutions below are obtained.

$$\begin{aligned}x_1(t) &= C_1 e^{-\zeta_1 t} \sin(t\omega_1 \sqrt{1-\zeta_1^2} + \varphi_1) + \frac{q_{10}}{\sqrt{(\omega_1^2 - \omega^2)^2 + 4\omega_1^2 \zeta_1^2 \omega^2}} \sin(\omega t + \gamma) \\ x_2(t) &= 0\end{aligned}\tag{3.23}$$

The settled oscillations are mainly important for measuring the angular velocity. Therefore, the amplitude of the $x_1(t)$ is the second part of the solution.

$$A_{10} = \frac{q_{10}}{\sqrt{(\omega_1^2 - \omega^2)^2 + 4\omega_1^2 \zeta_1^2 \omega^2}}\tag{3.24}$$

Higher amplitude will lead to more reliable measurements. Therefore, one can see the linear relationship between the amplitude of the external force and the amplitude of the first mode. Additionally, the amplitude of the first mode is higher if the excitation force has the exact value of the first natural frequency [7].

When constant angular velocity is present ($\dot{\Omega} = 0$), equations become coupled with only the Coriolis acceleration force, and solutions become more complicated. Since the excitation force is harmonic, it can be represented in a complex form.

$$q_1(t) = \text{Re}\{q_{10}e^{i\omega t}\} \quad (3.25)$$

ω is the excitation frequency, and the phase is assumed as zero. The system's solutions are represented as complex forms, too.

$$\begin{aligned} x_1(t) &= \text{Re}\{A_1e^{i\omega t}\} \\ x_2(t) &= \text{Re}\{A_2e^{i\omega t}\} \end{aligned} \quad (3.26)$$

$$\begin{aligned} A_1 &= A_{10}e^{i\varphi_{10}} \\ A_2 &= A_{20}e^{i\varphi_{20}} \end{aligned} \quad (3.27)$$

$$A_{10} = |A_1| = \sqrt{\text{Re}^2 A_1 + \text{Im}^2 A_1} \quad (3.28)$$

$$\tan \varphi_{10} = \frac{\text{Im} A_1}{\text{Re} A_1} \quad (3.29)$$

From Equation (3.20), (3.25), and (3.26), the following equations can be found.

$$\begin{aligned} (\omega_1^2 - \Omega^2 - \omega^2 + 2\zeta_1\omega_1 i\omega)A_1 + 2ci\omega\Omega A_2 &= q_{10} \\ (\omega_2^2 - \Omega^2 - \omega^2 + 2\zeta_2\omega_2 i\omega)A_2 - 2ci\omega\Omega A_1 &= 0 \end{aligned} \quad (3.30)$$

Before the solution for the above equation, new variables are introduced to simplify the solution. $k = \omega_1$, $\delta k = \omega_2/k$, $\delta\omega = \omega/k$, $\delta\Omega = \Omega/k$. These variables give the ratios of the frequencies. Additionally, measured angular velocity is much lower than the natural or excitation frequency [48]. Therefore, the values of $\delta\Omega$ and $\delta\Omega^2$ can be assumed as zero. Therefore, the solution is given below.

$$A_{10} = \frac{q_{10}}{k^2 \sqrt{(1 - \delta\omega^2)^2 + 4\zeta_1^2 \delta\omega^2}} \quad (3.31)$$

$$A_{20} = \frac{2cA_{10}\delta\omega}{\sqrt{(\delta k^2 - \delta\omega^2)^2 + 4\zeta_2^2 \delta k^2 \delta\omega^2}} \delta\Omega \quad (3.32)$$

$$\varphi_1 = \text{atan}\left(\frac{2\delta\omega\zeta_1}{1 - \delta\omega^2}\right) \quad (3.33)$$

$$\varphi_2 = \text{atan}\left(\frac{\delta k^2 - (1 + 4\zeta_1\zeta_2\delta k + \delta k^2)\delta\omega^2 + \delta\omega^4}{2\delta k\delta\omega(\zeta_2 + \zeta_1\delta k) - 2\delta\omega^3(\zeta_1 + \zeta_2\delta k)}\right) \quad (3.34)$$

An example of the solution of A_{20} is given in Figure 3.3.

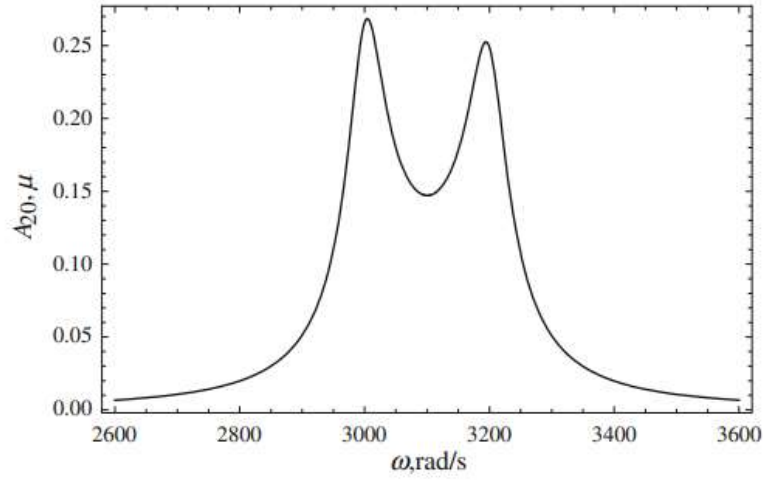


Figure 3.3. Secondary Amplitude vs Excitation Frequency ($\omega_1 = 3000, \omega_2 = 3200, \zeta_1 = \zeta_2 = 0.01, \Omega = 1$) [7]

The figure shows that the amplitude is higher when the excitation force equals one of the natural frequencies. Therefore, the excitation force must be equal to the natural frequency of the operating modes.

Based on the research that has been done, a mathematical model for the sensitive element dynamics has been constructed. The main equation that has been used is given below.

$$\begin{aligned} \ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2x_1 &= q_1 - c\dot{\Omega}x_2 - 2c\Omega\dot{x}_2 \\ \ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2x_2 &= c\dot{\Omega}x_1 + 2c\Omega\dot{x}_1 \end{aligned} \quad (3.35)$$

The mathematical model constructed in Simulink software has been given below in Figure 3.4.

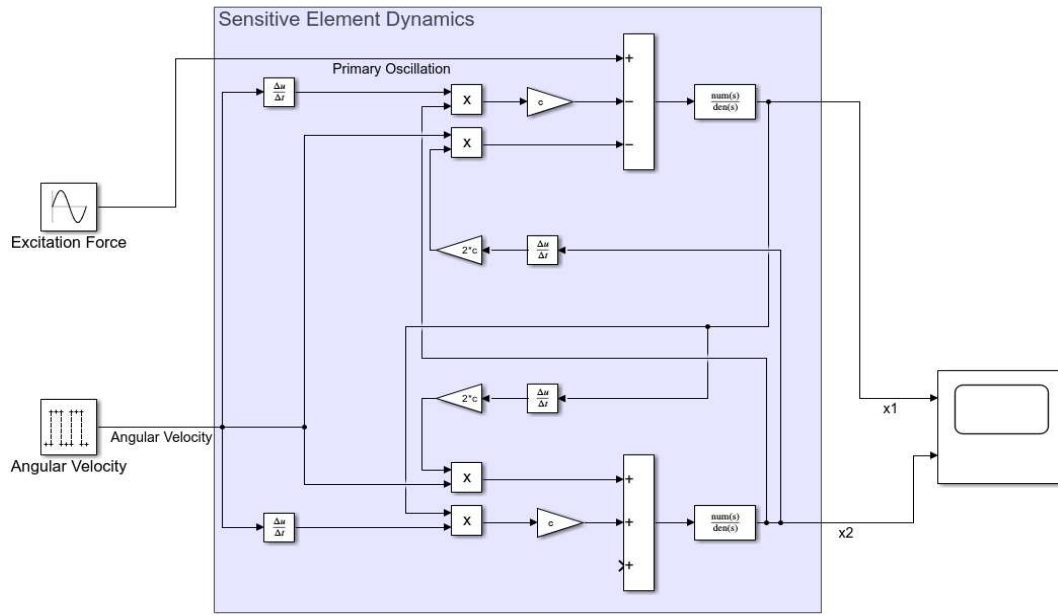


Figure 3.4. The Mathematical Model of Sensitive Element Dynamics

The above mathematical model calculates the displacements, which are primary and secondary motions (x_1 and x_2). The first case has a 20 rad/s angular velocity which can be seen in Figure 3.5. In the second case, there is a 200 rad/s angular velocity which can be seen in Figure 3.7. Two different scenario results have been given in Figure 3.6 and Figure 3.8. The resonator has 5000 Hz for the natural frequency of primary and secondary modes.

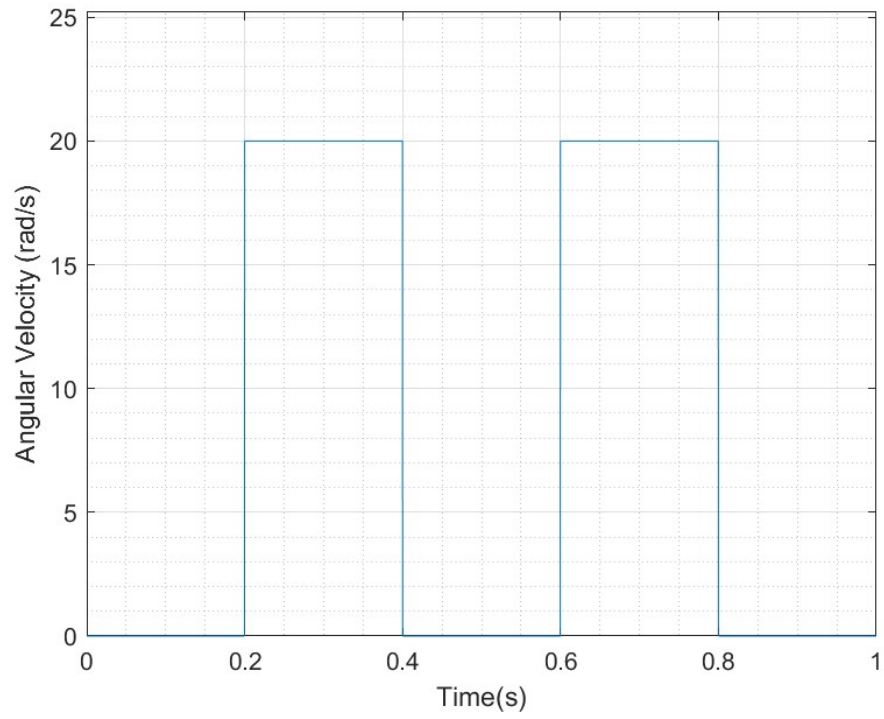


Figure 3.5. Input Angular Velocity for 20 rad/s Angular Velocity Scenario

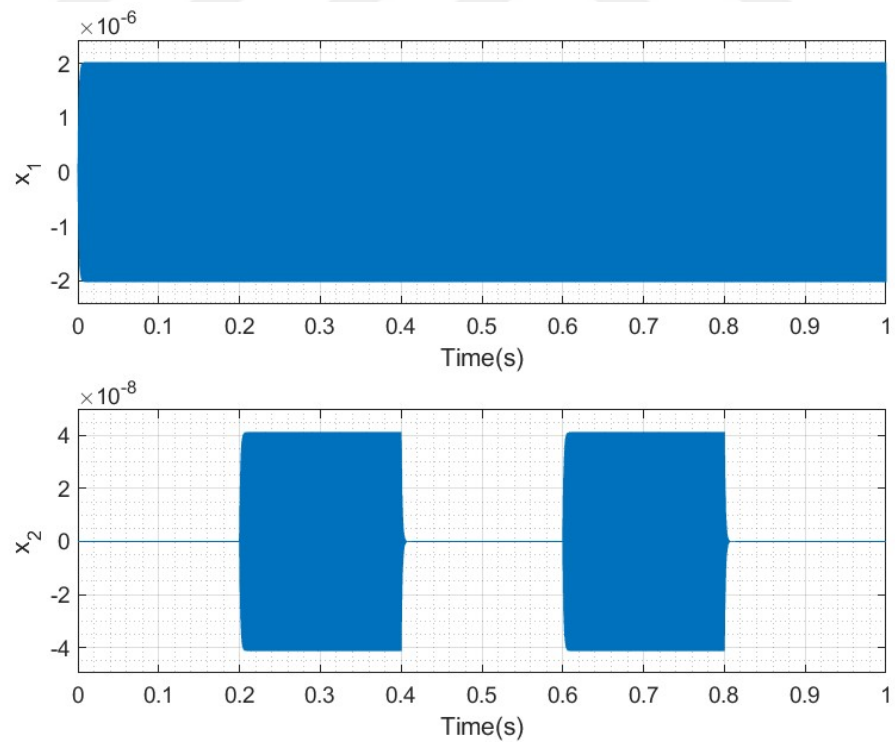


Figure 3.6. Primary and Secondary Motion Value Under 20 rad/s Angular Velocity

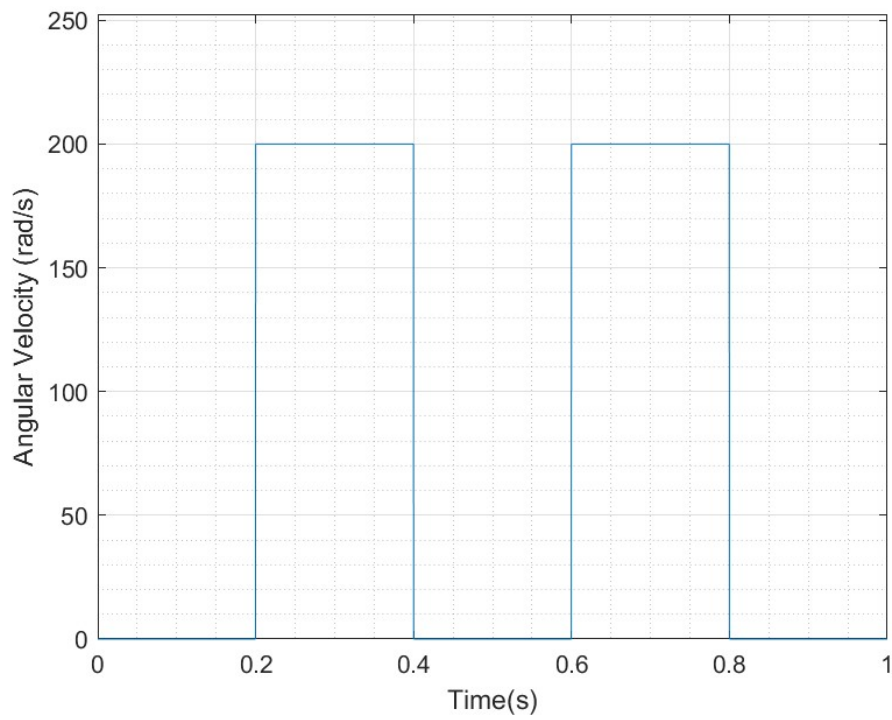


Figure 3.7. Input Angular Velocity for 200 rad/s Angular Velocity Scenario

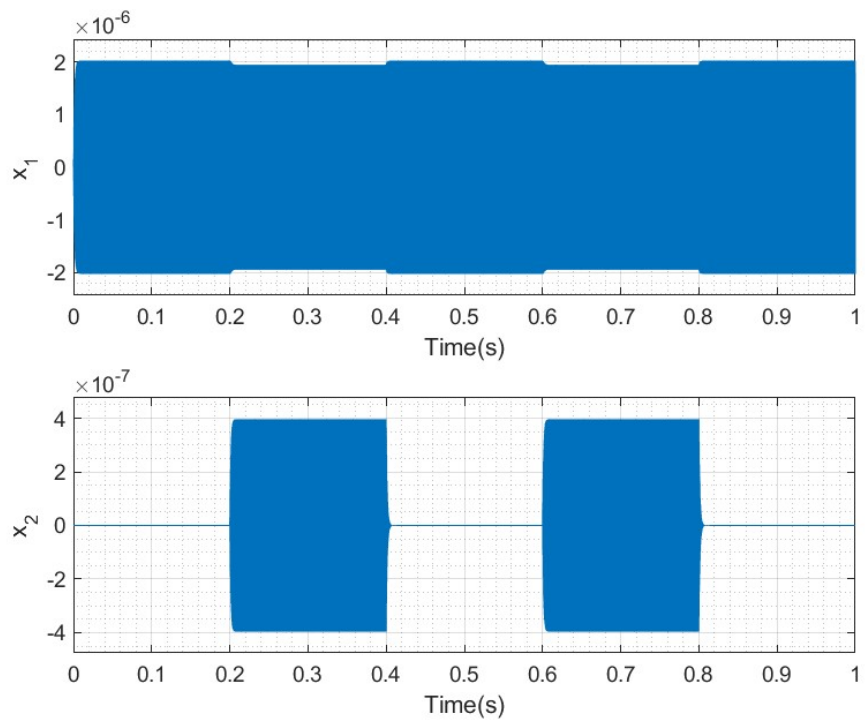


Figure 3.8. Primary and Secondary Motion Value Under 200 rad/s Angular Velocity

The effect of the angular velocity in the primary and secondary motions is shown above. The amplitude of the primary and secondary motion changes with the angular velocity. When the angular velocity increases, the amplitude increases accordingly. Therefore, results verify previous analytic solutions.

The mathematical model can simulate different scenarios easily. However, the solution of secondary motion is an oscillation. Therefore, extra derivations need to be done to find the angular velocity.

3.3 Demodulation of the Solution

In the previous part, a solution for the main equation set has been found. However, the solution of primary and secondary motions are oscillations, which is not an angular velocity value. Therefore, further processes have been made to obtain the angular velocity. The first process is the demodulation of the solution.

Demodulation is the process of extracting the original information from a signal. In this case, demodulation is used to extract angular velocity information from the secondary motion, which is an oscillation. A block diagram of the demodulation process has been given in Figure 3.9.

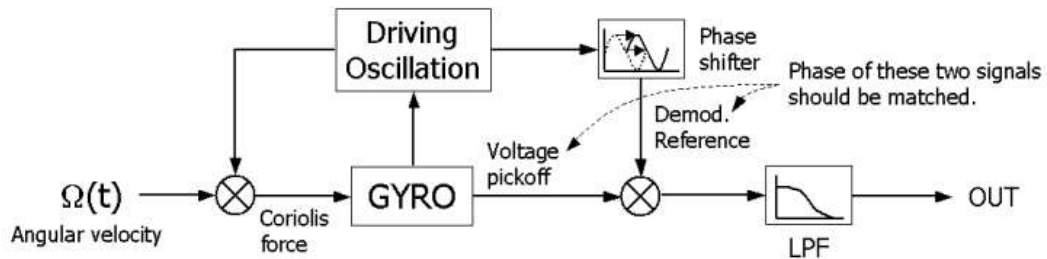


Figure 3.9. Block Diagram of Demodulation Process [49]

In CVG, the excitation oscillation can demodulate the secondary motion. However, the phase of the secondary motion must be equal to the demodulation reference signal. Therefore, the phase difference between the secondary motion and excitation oscillation is examined.

In the previous part, the phases of the primary and secondary motions were found, which are given below.

$$\varphi_1 = \text{atan}\left(\frac{2\delta\omega\zeta_1}{1-\delta\omega}\right) \quad (3.36)$$

$$\varphi_2 = \text{atan}\left(\frac{\delta k^2 - (1 + 4\zeta_1\zeta_2\delta k + \delta k^2)\delta\omega^2 + \delta\omega^4}{2\delta k\delta\omega(\zeta_2 + \zeta_1\delta k) - 2\delta\omega^3(\zeta_1 + \zeta_2\delta k)}\right) \quad (3.37)$$

From these equations, one can see that if the frequency of the excitation force is the same as the natural frequency of the primary and secondary motion, the phase of the primary motion is $\pi/2$. Furthermore, in the same conditions, the phase of the secondary motion is equal to the $-\pi/2$. Depending on these calculations, the phase of the demodulation reference signal should be found and used in the phase shifter. Therefore, a demodulation reference signal with phase shift is given below [42].

$$q_{10} \sin(\omega t + \varphi_2) \quad (3.38)$$

Secondary motion oscillation could be written as below.

$$A_{20} \sin(\omega t + \varphi_2) \quad (3.39)$$

For demodulation processes, Equation (3.38) and (3.39) multiplied.

$$-\frac{A_{20}q_{10}}{2} [\cos(2\omega t + 2\varphi_2) - \cos(2\varphi_2)] \quad (3.40)$$

Solution of the demodulation process filtered with a low-pass filter to eliminate the high-frequency signals. Additionally, a low-pass eight-order Butterworth filter was used in this mathematical model [7].

$$LPF \left\{ -\frac{A_{20}q_{10}}{2} [\cos(2\omega t + 2\varphi_2) - \cos(2\varphi_2)] \right\} = -\frac{A_{20}q_{10}}{2} \cos(2\varphi_2) \quad (3.41)$$

One should see that the cosine signal with phase shift frequency isn't filtered. As an example, the previous example with 20 rad/s is continued. In Figure 3.10, the result of the demodulation has been given. Furthermore, the filtered signal can be seen in Figure 3.11.

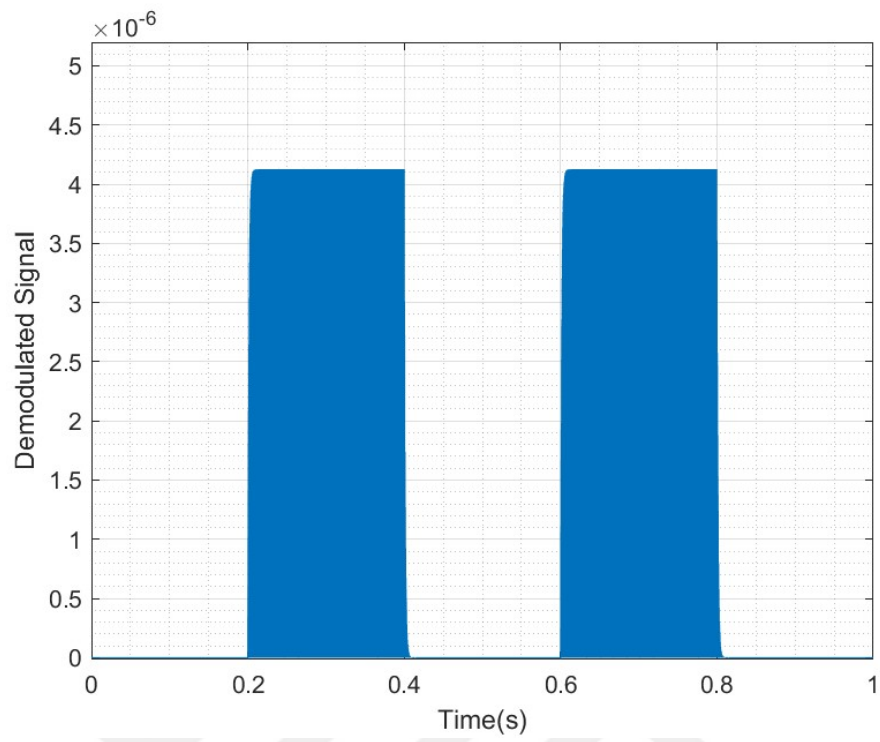


Figure 3.10. Demodulation Signal

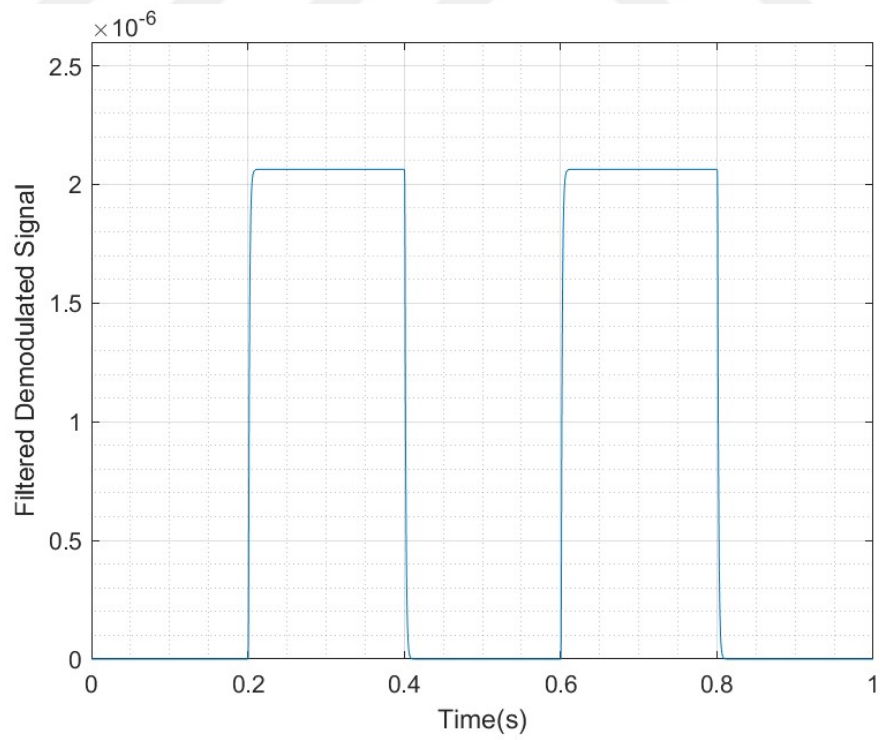


Figure 3.11. Filtered Demodulation Signal

3.4 Scale Factor

In the previous part, the filtered value of the signal was found. However, the scale factor is also needed to get the angular velocity. The scale factor is not dependent on the angular velocity, which can be expressed as below.

$$A_{20}^* = S_{\Omega} \cdot \Omega \quad (3.42)$$

A_{20}^* is the ideal output for the CVG and S_{Ω} symbolizes the scale factor. To find the scale factor, the below equation can be used [7].

$$S_{\Omega} = \left. \frac{\partial A_{20}^*}{\partial \Omega} \right|_{\Omega \rightarrow 0} = \frac{2cq_{10}\delta\omega}{k^3 \sqrt{[(1-\delta\omega^2)^2 + 4\zeta_1^2\delta\omega^2][(\delta k^2 - \delta\omega^2)^2 + 4\zeta_2^2\delta\omega^2\delta k^2]}} \quad (3.43)$$

At the end of the demodulation process, the amplitude is different from the ideal output given in Equation (3.42). Since the scale factor is found for the ideal output, variables from the demodulation process have been multiplied. The final process to get the angular velocity is shown below.

$$\begin{aligned} \left| \frac{A_{20}q_{10}}{2} \right| &= S_{\Omega} \cdot \Omega \\ \left| \frac{A_{20}q_{10}}{2} \right| \cdot \left(\frac{2}{q_{10}S_{\Omega}} \right) &= \Omega \end{aligned} \quad (3.44)$$

The mathematical model was constructed with derived equations in previous sections. That can be seen in Figure 3.12. Additionally, one can see the final process after the sensitive element dynamics is given in Figure 3.13.

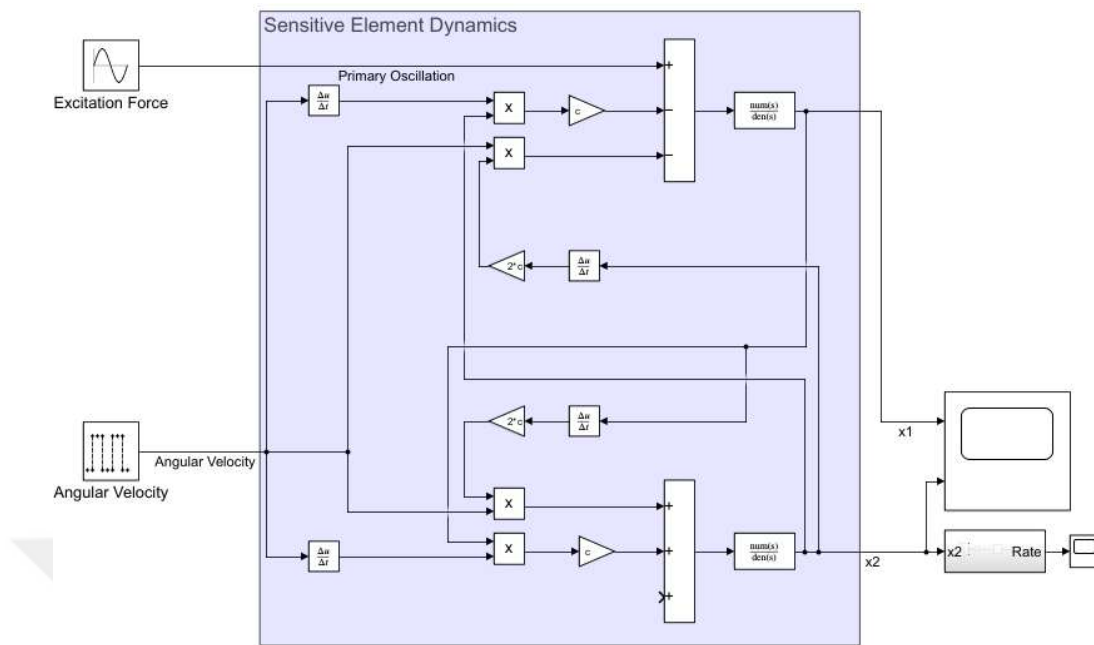


Figure 3.12. The Mathematical Model of the CVG

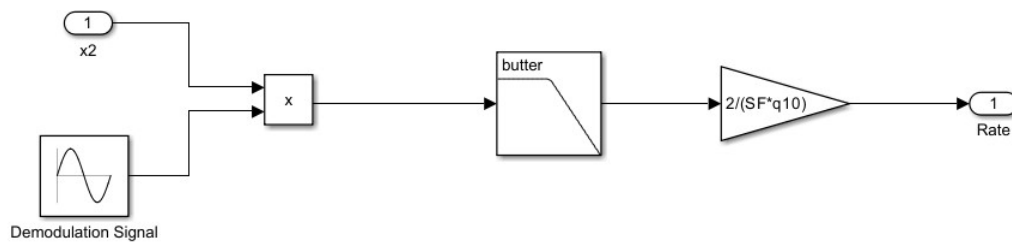


Figure 3.13. Result Processes

3.5 Example of the Result of the Mathematical Model

Previous examples of 20 rad/s and 200 rad/s angular velocities are continued in this section. Results are shown in Figure 3.14 and Figure 3.15.

In both examples, one can easily see that there is an initial large error in the settling time with the rate of the angular velocity in the input. Therefore, this error doesn't occur in the steady-state region of the measurement.

Additionally, in the first example, one can see that the mathematical model perfectly measures the 20 rad/s. However, the second example has an error in measured

angular velocity. The reason for the error is that the measured angular velocity is much closer to the excitation natural frequency. Therefore, validation of the assumption made in previous sections is done.

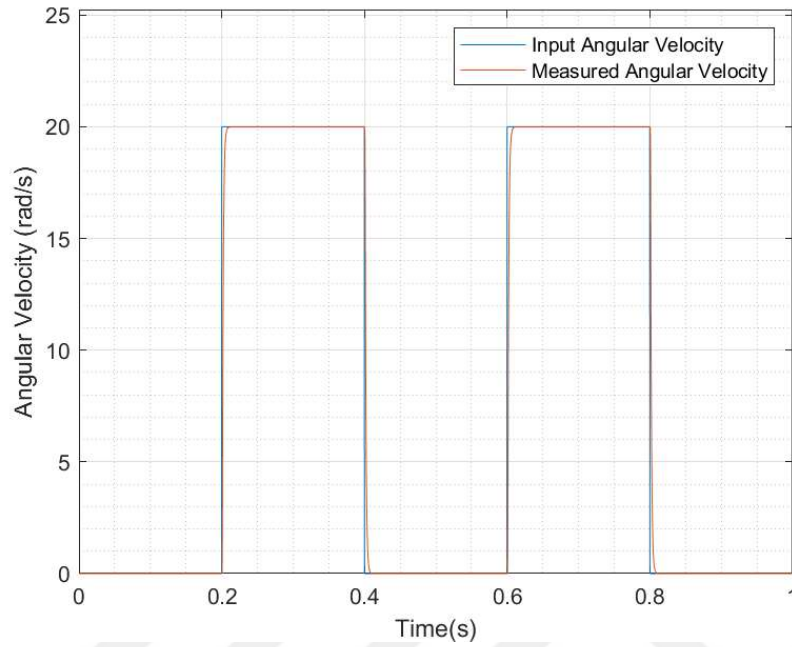


Figure 3.14. Measured Angular Velocity Under 20 rad/s Angular Velocity

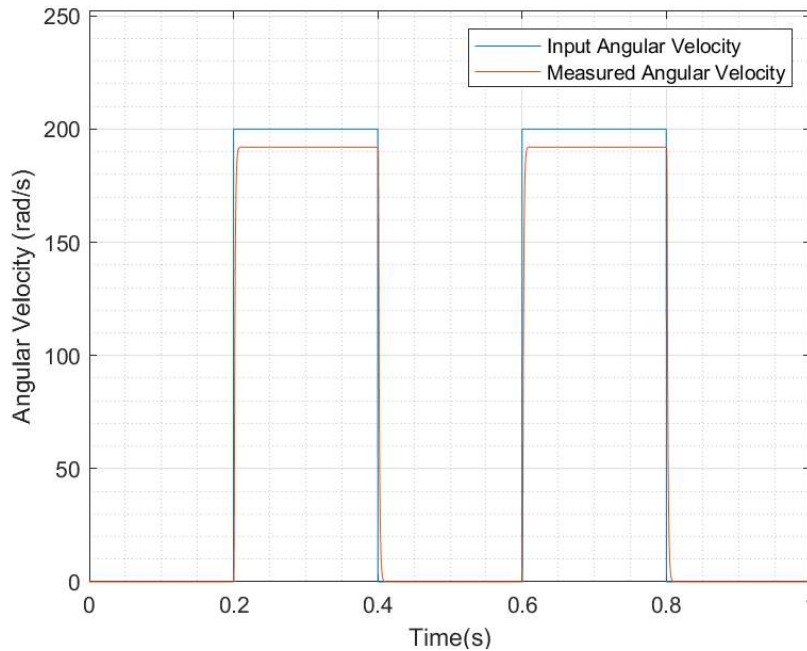


Figure 3.15. Measured Angular Velocity Under 200 rad/s Angular Velocity

3.6 Model Order Reduction with State-Space

The mathematical model of the CVG has been found in the previous section. However, the mathematical model doesn't have a geometrical dependency until now. A model order reduction method, state space equations, is included in the system to have geometrical dependency. State space equations use the natural frequency and mode shapes, which are the modal analysis results. State space equations can be seen below.

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\tag{3.45}$$

In these equations, x is the state vector, u is the input vector, and y is the output vector. Equation (3.35) can be expressed in matrices as below.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & 2c\Omega \\ -2c\Omega & 2\zeta_2\omega_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & c\dot{\Omega} \\ -c\dot{\Omega} & \omega_2^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix}\tag{3.46}$$

In the modal domain, the forced output can be rewritten as below. [27]

$$Q_i(t) = \phi_i^T \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}\tag{3.47}$$

Combining the Equation (3.46) and (3.47), this equation can be written as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & 2c\Omega \\ -2c\Omega & 2\zeta_2\omega_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & c\dot{\Omega} \\ -c\dot{\Omega} & \omega_2^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [\phi]^T \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}\tag{3.48}$$

$[\phi]$ is the displacement normalized mode shape matrix of the system. Additionally, Ω is the second input. Therefore, the equation is rewritten as only the left side of the equation has the state vector.

$$\begin{aligned}& \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & 0 \\ 0 & 2\zeta_2\omega_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ &= [\phi]^T \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} + \begin{bmatrix} 0 & -2c\Omega \\ 2c\Omega & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & -c\dot{\Omega} \\ c\dot{\Omega} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}\end{aligned}\tag{3.49}$$

The right side of the equation, the forced output of the system, is included in the mathematical model to reduce the complexity in the input “u” matrix. However, the left side of the equation will be written as state space matrices and the system matrices, which can be seen below.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \quad (3.50)$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & -2\zeta_1\omega_1 & 0 \\ 0 & -\omega_2^2 & 0 & -2\zeta_2\omega_2 \end{bmatrix} \quad (3.51)$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.52)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (3.53)$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (3.54)$$

$$u = [\phi]^T \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad (3.55)$$

This equation and matrices are implemented in the mathematical model.

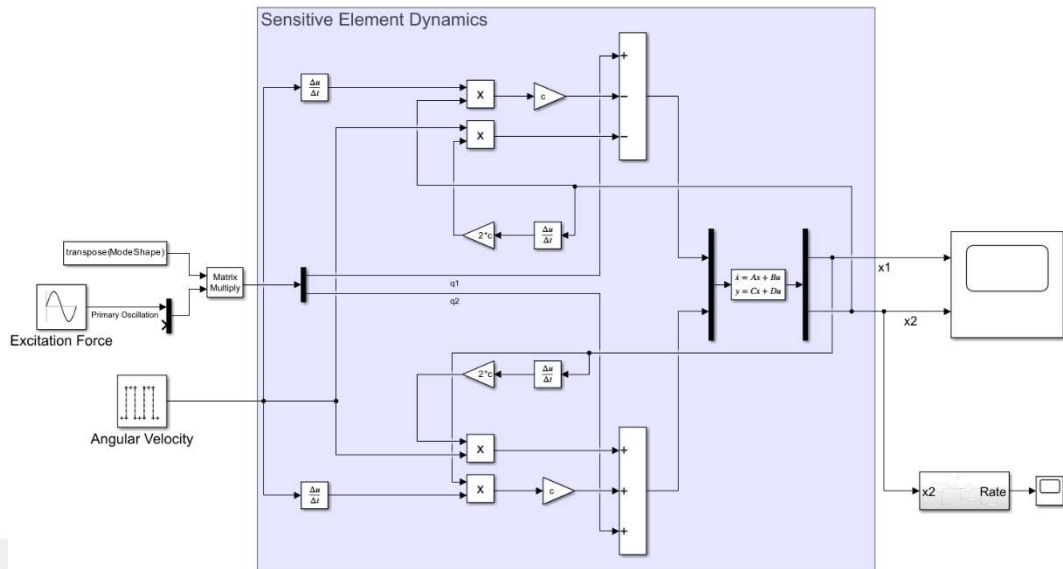


Figure 3.16. The Mathematical Model of the CVG with Model Order Reduction

The results are found when the same examples throughout this section are continued.

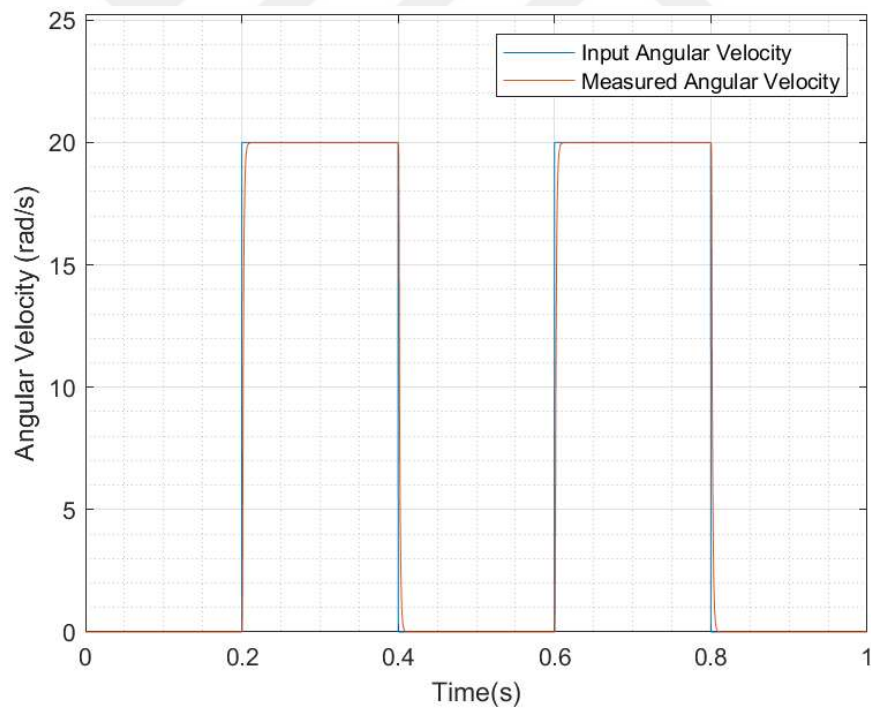


Figure 3.17. Measured Angular Velocity Under 20 rad/s Angular Velocity with Identity Mode Shape Matrix

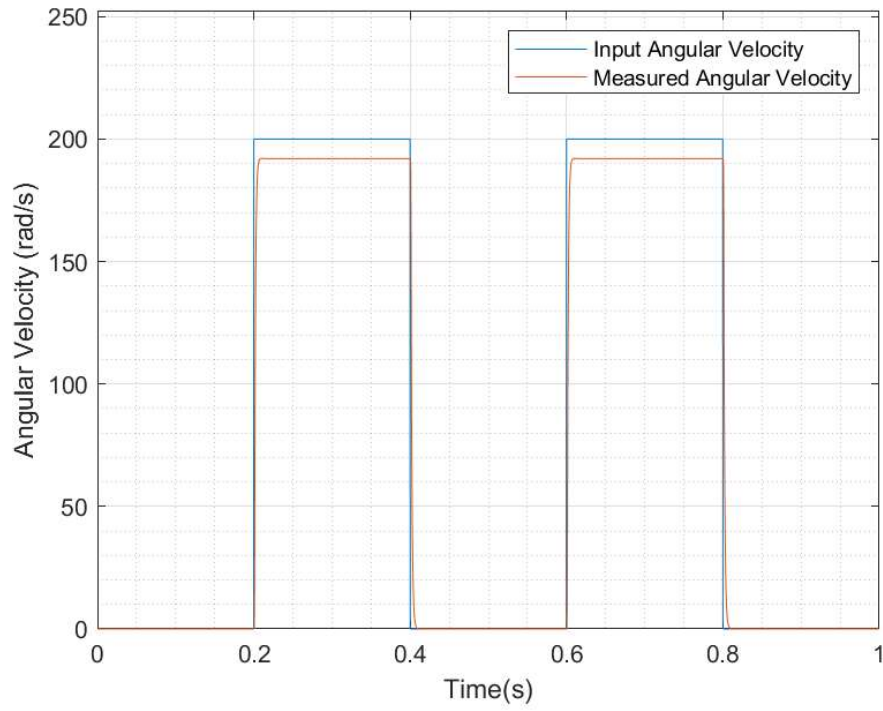


Figure 3.18. Measured Angular Velocity Under 200 rad/s Angular Velocity with Identity Mode Shape Matrix

The mode shape matrix used in these simulations is given below.

$$[\phi] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.56)$$

Different examples are given below.

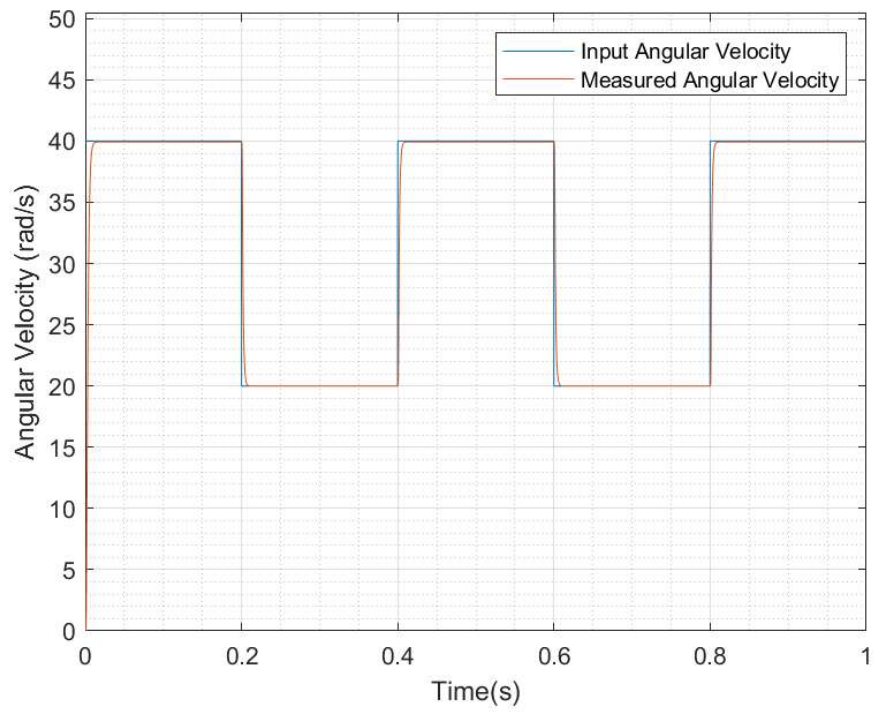


Figure 3.19. Input Scenario Example One

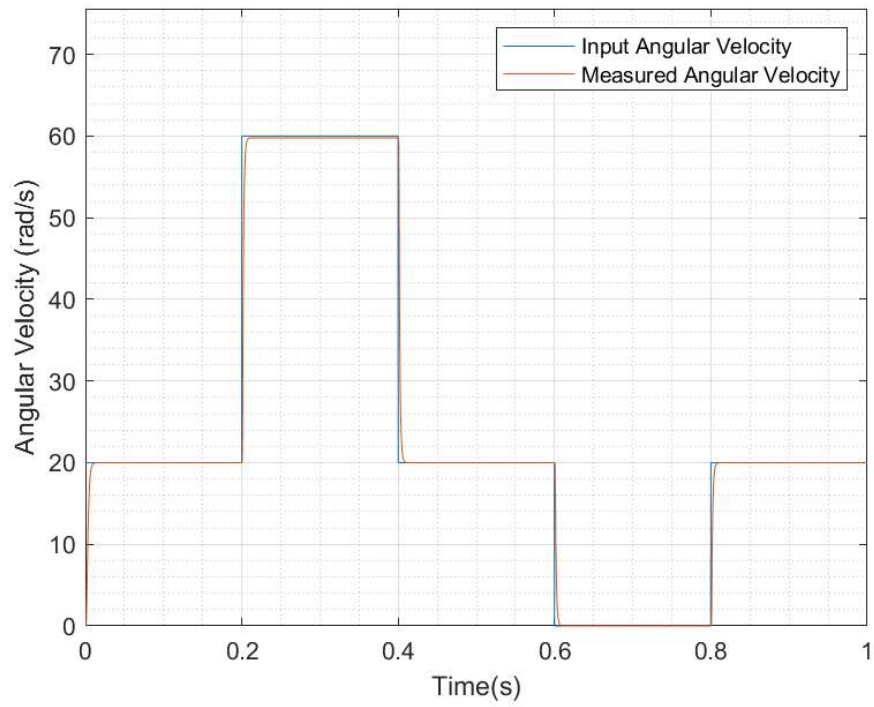


Figure 3.20. Input Scenario Example Two

Therefore, the results are the same without the model order reduction method. However, the results are different when non-diagonal mode shape matrices are used. For example, when the below mode shape matrix is used, the result is given in Figure 3.21 for 20 rad/s angular velocity.

$$[\phi] = \begin{bmatrix} 1 & 0.01 \\ 0.01 & 1 \end{bmatrix} \quad (3.57)$$

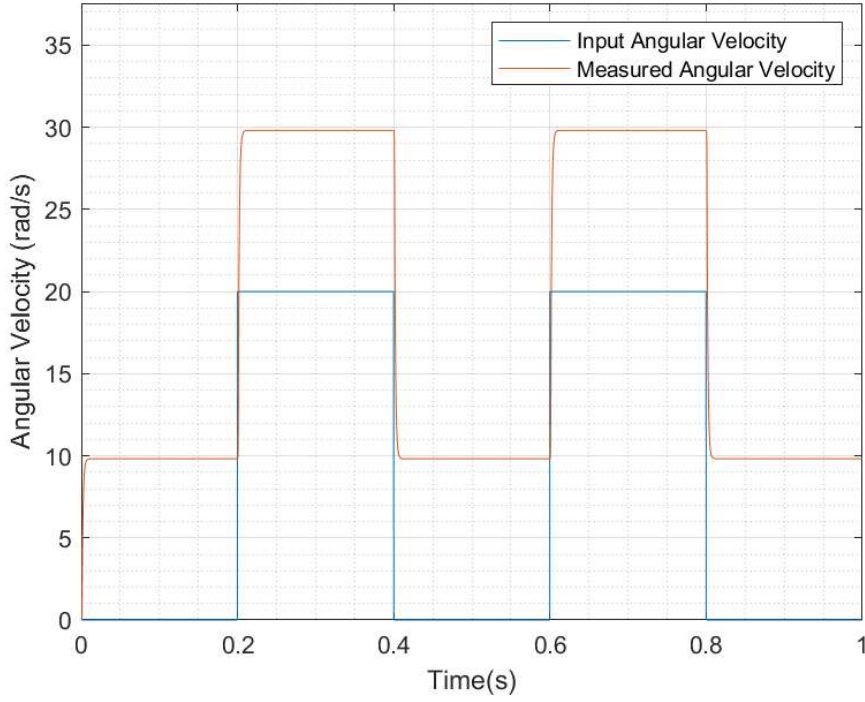


Figure 3.21. Measured Angular Velocity Under 20 rad/s Angular Velocity with Non-Identity Mode Shape Matrix

The results show a bias error in the sensor. The excitation force component along the secondary axis causes this error. Therefore, the mode shape of the resonator is important. This mode shape can differ from the mistake in producing the resonator. Additionally, material properties can vary. Therefore, a split in the natural frequencies leads to a non-zero mode shape matrix. These errors are examined in the later sections.

CHAPTER 4

MODAL ANALYSIS OF CYLINDRICAL VIBRATORY GYROSCOPE

The resonator of the CVG is the heart of the sensor. For given dynamical equations, it is essential to perform a modal analysis of the resonator. Eigenvalues and eigenmodes can describe the vibration characteristic of the CVG in terms of both space and time. The modal characteristics of the system are mostly influenced by the design of the sensor. Also, both modal characteristics and the sensor's design influence the sensor's sensitivity. This section will explain the modeling of the resonator and the properties of the modal analysis of the resonator. Additionally, a sample modal analysis is given at the end of the chapter.

4.1 Analytical Solution of the Mode Shape of 2D Ring Shape

Firstly, the analytical solution of the eigenmodes of 2D ring shape has been given in this section. The analytical solution is based on the 2D ring shapes due to the shape of the main operating modes. The coordinate system that will be used in the analytical solution is given in Figure 4.1.

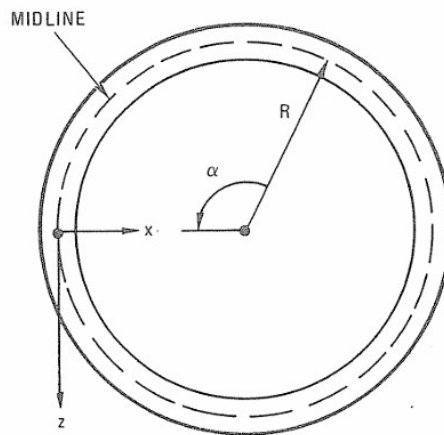


Figure 4.1. Coordinate System for 2D Ring Shape [50]

One should define the mode type from deformation to write the analytical solution. In-plane flexural modes of a 2D ring have been given in Figure 4.2.

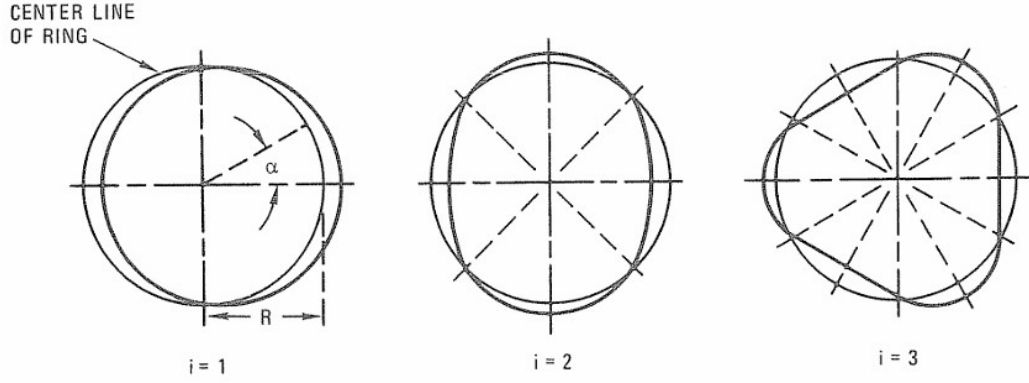


Figure 4.2. In-plane Flexural Modes of a Ring [50]

Based on the Figure 4.2, one can see that $i = 2$ for the operating modes for the CVG. Therefore, the mode shape equation has been given below:

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 2 \cos(2\alpha) \\ \sin(2\alpha) \end{bmatrix} \quad (4.1)$$

However, this analytical solution will not be enough for the resonator's modal analysis due to the geometric anisotropy of the resonator. Therefore, the modal analysis was done using the ABAQUS software.

4.2 Hypotheses

The resonator of the CVG can be modeled as a thin-wall structure. The theory of thin plates is rooted in the Kirchhoff hypothesis, and the theory of thin shells is developed from the Love hypothesis [27]. These are used to study the vibration characteristics of the resonator. The bottom of the resonator can be treated as a thin plate model. Additionally, the wall of the resonator can be modeled as a shell model. The simple resonator of the CVG can be seen in Figure 4.3.

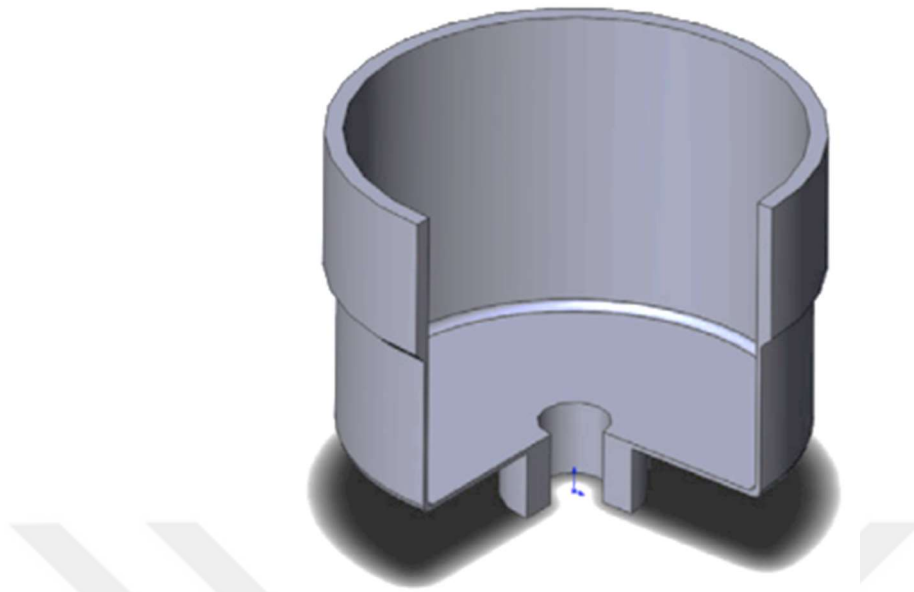


Figure 4.3. Resonator of the CVG [31]

4.2.1 The Theory of Thin Plates

The Kirchhoff hypothesis has been used in the theory of thin plates. The Kirchhoff hypothesis states the following:

- The thickness of the plate is small in comparison to other dimensions. The maximum deflection is much smaller than its thickness. Therefore, the strain and rotation angles are minimal.
- Stresses perpendicular to the plate can be neglected due to how small the thickness is.
- Shear deformation is ignored.
- Resistance, which happens due to the moment of inertia, to the bending is neglected. However, inertia caused by the rotation and lateral displacement are still included.

4.2.2 The Theory of Shells

The love hypothesis has been used in the theory of shells. The Love hypothesis states the following:

- The thickness of the shell is thinnest among the dimensions.
- Strains and displacements are small enough to ignore second or higher terms.
- The radial normal stress is neglected.
- The length of the centerline does not change since the normal length of the centerline is a straight line before and after the deformation.

4.3 Mechanical Modeling of the Resonators

The cylindrical shell of the CVG resonator is the main inertial component. Additionally, it is used to determine the main vibration characteristics. Therefore, it is essential to model and analyze the resonator. Mechanical models are based on the plate and shell theories [27]. This section shows mechanical parameter models in their working modes. These models will help construct the modal analysis.

4.3.1 Concentration Stiffness Model

A concentration stiffness model has been constructed to examine the stiffness characteristics of the CVG resonator. This model helps to investigate the relationship between the excitation force and displacements. The model in the x-y plane can be seen in Figure 4.4.

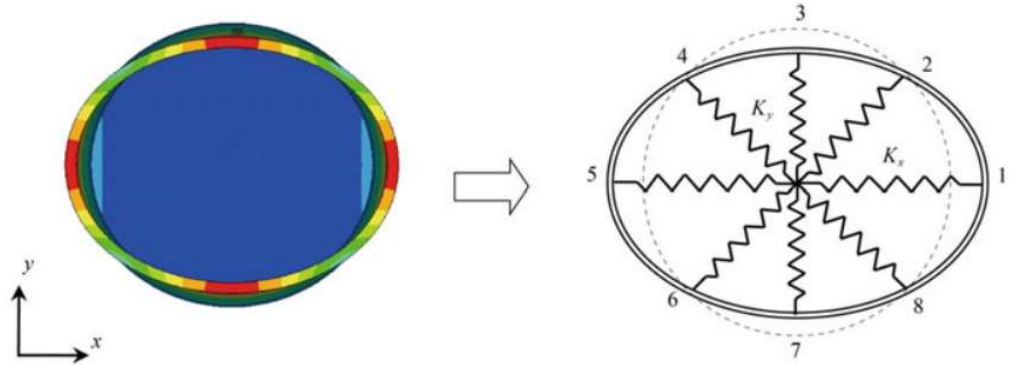


Figure 4.4. Concentration Stiffness Model in x-y plane [27]

Stiffnesses are distributed evenly in eight different positions. These positions are selected according to the location of the piezoelectrics, which are the nodal and anti-nodal points of the operating mode shapes. These stiffnesses are equal to each other when the resonator is undeformed due to the axial symmetry of the resonator. The model in the x-z plane can be seen in Figure 4.5.

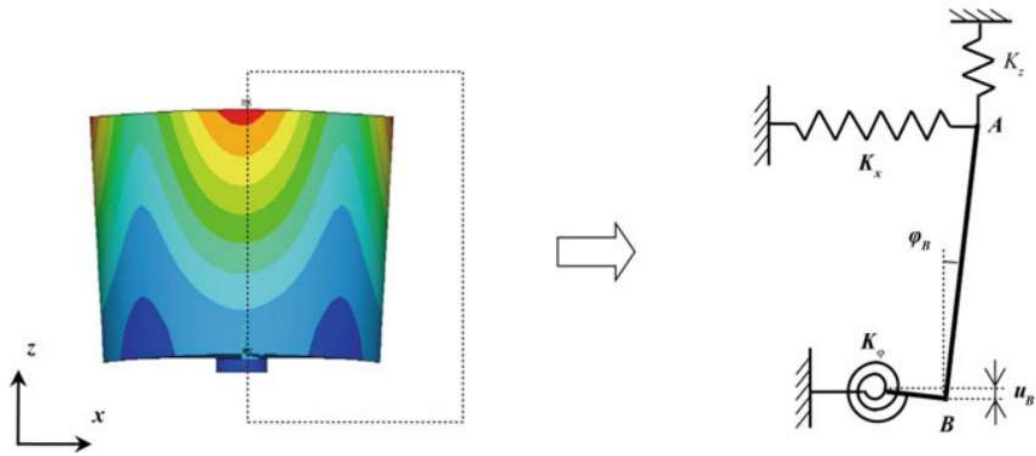


Figure 4.5. Concentration Stiffness Model in x-z plane [27]

Stiffness values K_z and K_x at point A are determined by the displacement in the shell wall of point A. The displacement at the bottom thin plate determines the torsional spring K_ϕ at the bottom. Deformation at the shell wall is ϕ_B , and deformation at the bottom thin plate is u_B .

4.3.2 Lumped Mass Model

When investigating the main vibration characteristics of the resonator, it is essential to analyze the effective inertial masses. A lumped mass model has been constructed to analyze the inertial masses. This model can be seen in Figure 4.6.

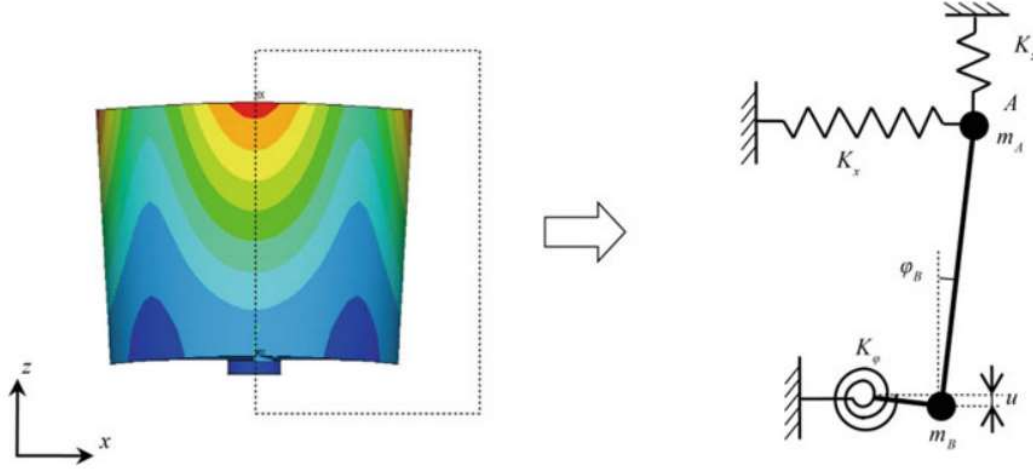


Figure 4.6. Lumped Mass Model of CVG Resonator [27]

m_A is the effective mass value in the shell wall of point A. The effective mass at the bottom thin plate of point B is m_B . Piezoelectric materials deflect the resonator from the bottom and create u_B displacement. Due to Love the hypothesis, the relationship between the deflection values φ_B and u_B can be taken as a linear [27]. Additionally, the excitation moment and the displacement at point A are added to the model, as shown in Figure 4.7. The excitation moment is created by the piezoelectrics. An excitation moment is used as an excitation force in dynamic analysis to reduce the complexity of the equations. Excitation force is used with the displacement at point A in the mathematical model.

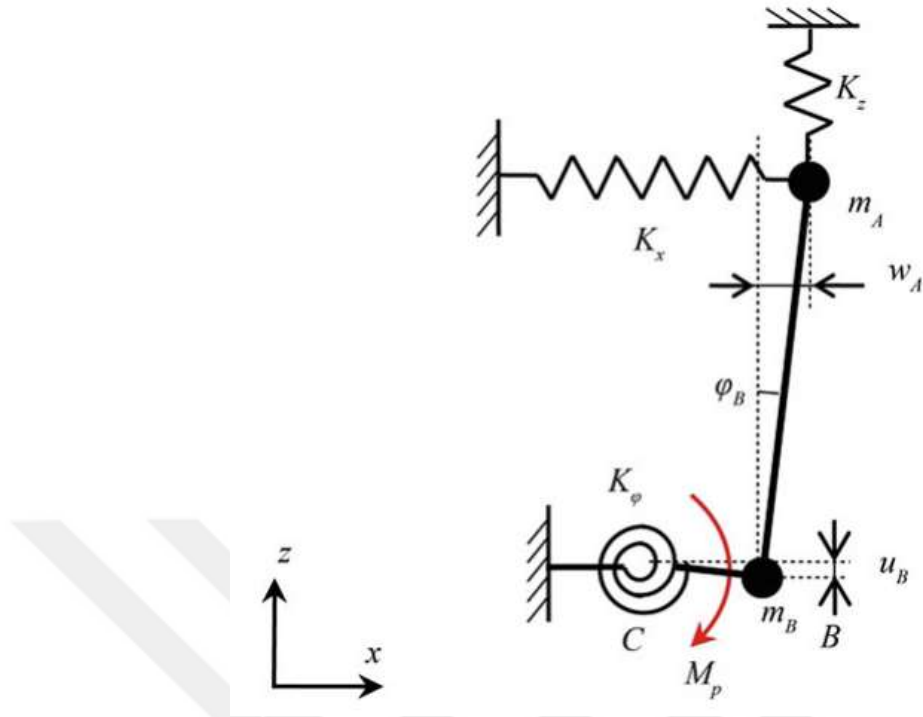


Figure 4.7. Lumped Mass Model in Dynamical Analyses [27]

4.4 Parameters and Outputs of the Modal Analysis

Due to the axial symmetry of the resonators, meshes are very important in the modal analysis. Meshes must be symmetrical and evenly distributed to get the two operating modes to have the same natural frequency. Modal analysis was performed in the ABAQUS program, and C3D20 quadratic elements were used. No boundary condition is given to the system. Therefore, studies were done on free body condition. Resonators are divided into sixteen different parts, which are perpendicular to the sensitive axis. The outputs of the analyses are the mode shapes in two operating modes and natural frequencies at the two operating modes.

4.4.1 Results of the Mode Shapes

Mode shapes are found in modal analysis at ABAQUS software. Generally, other than the first six rigid body mode shapes, the first two modes are the operating modes which can be seen in Figure 4.8.

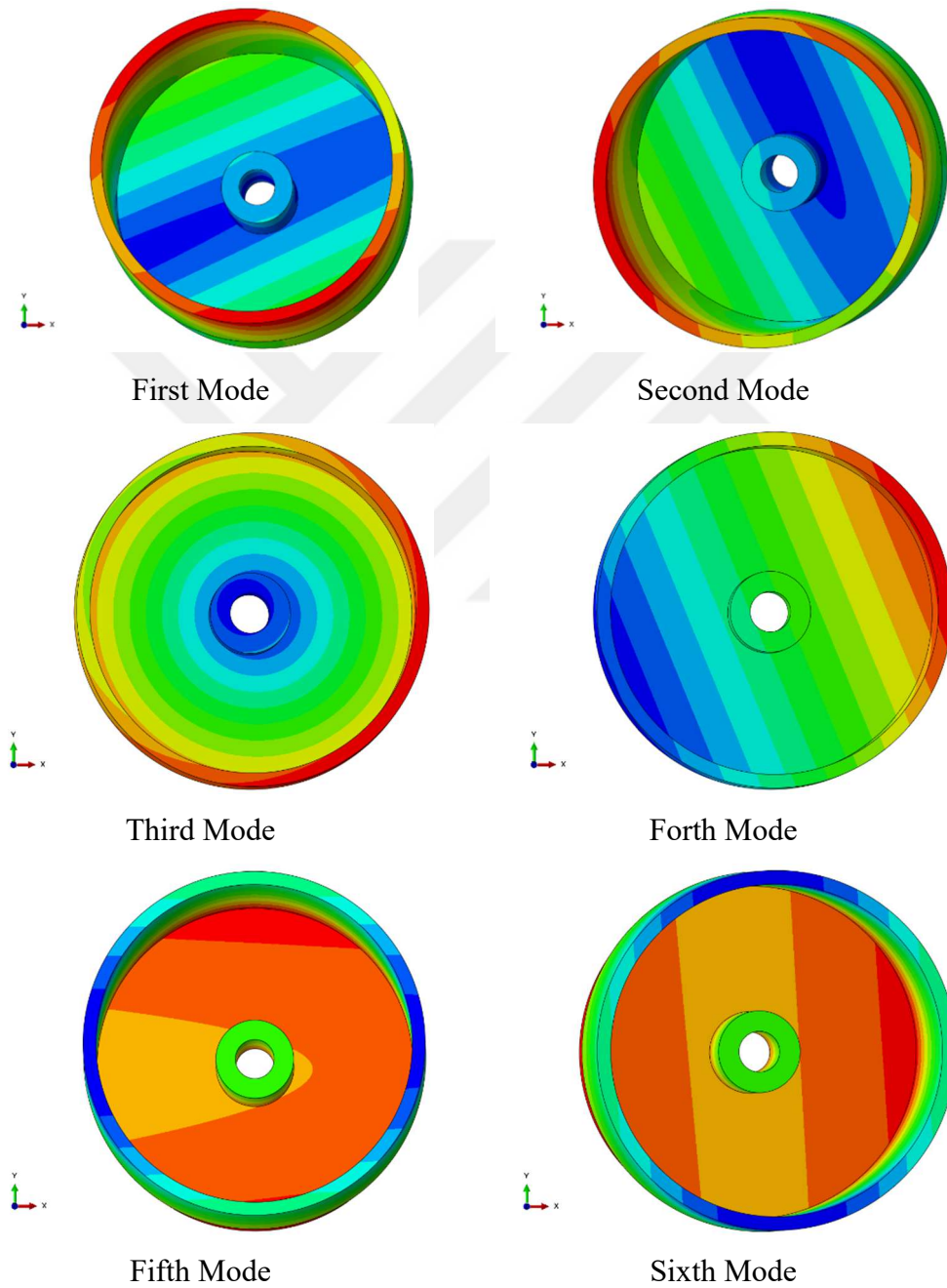


Figure 4.8. Mode Shapes of the CVG Resonator

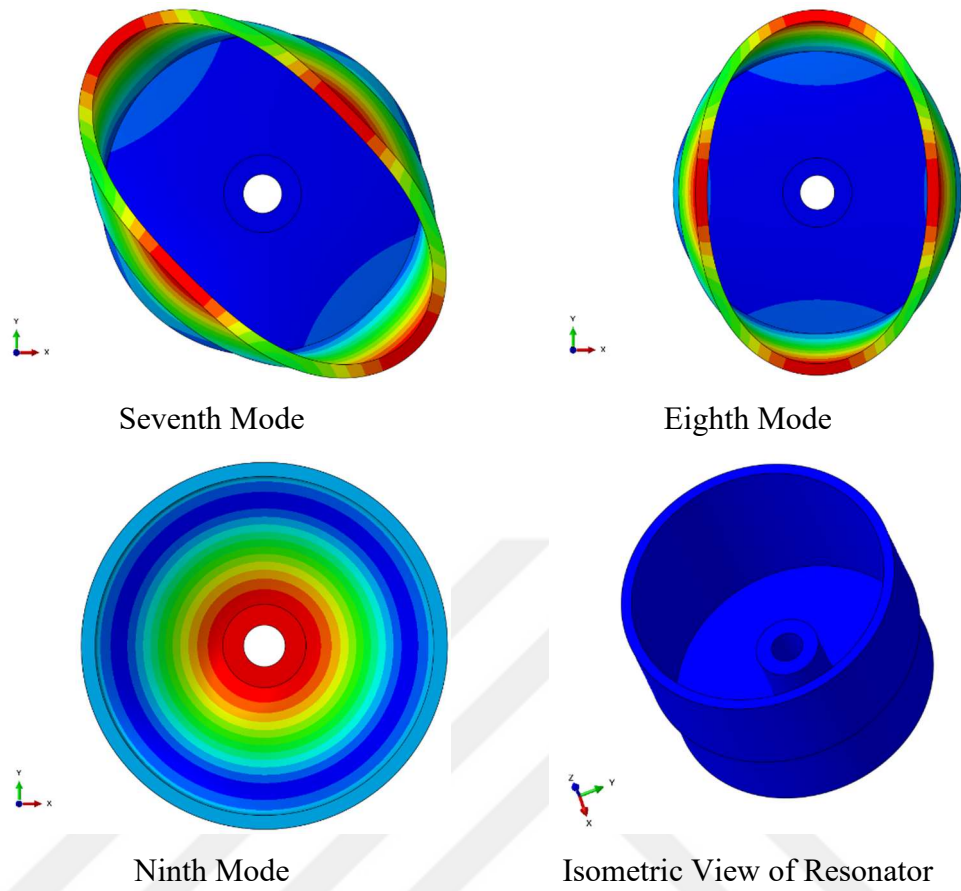


Figure 4.8. (continued) Mode Shapes of the CVG Resonator

Mode shapes are founded on the relative displacement of the points on the cylindrical coordinate system in the operating modes. The cylindrical coordinates of the system can be seen in Figure 4.9.

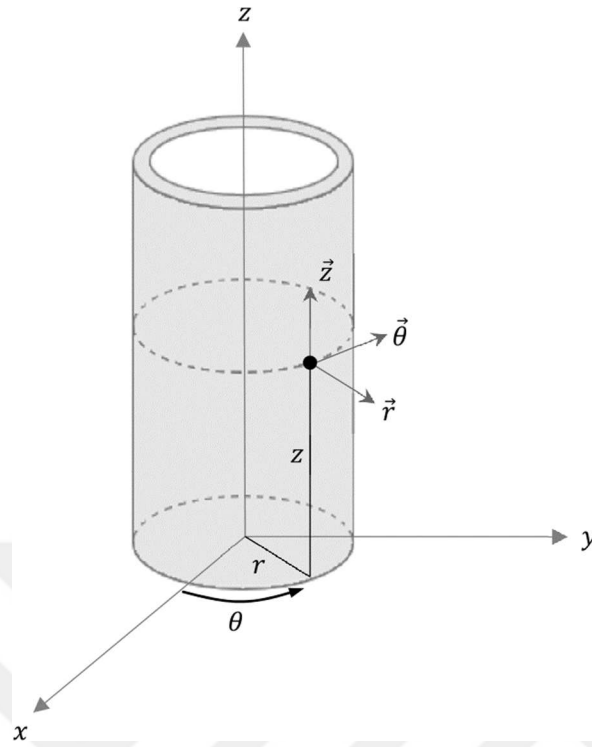


Figure 4.9. Cylindrical Coordinates in Hollow Cylinder [51]

Displacements of the points that are placed in the middle plane of the top of the shell wall are used in the mathematical model. Due to the hypotheses, the top plane of the shell wall has the same deformation along the $\vec{r}$ coordinate of the cylindrical coordinate system. Displacements of the points are fitted to the cosine equation [52]. When fitting has been done to the two operating modes, the difference in the cosine angle is always 45° . The equation can be seen below.

$$x_w = A \cos(2x - 2\theta_w) \quad (4.2)$$

A is the amplitude of the equation. θ_w is the angle of the nodal point. x_w is the total deformation at an angle.

An example can be seen in Figure 4.10 and Figure 4.11.

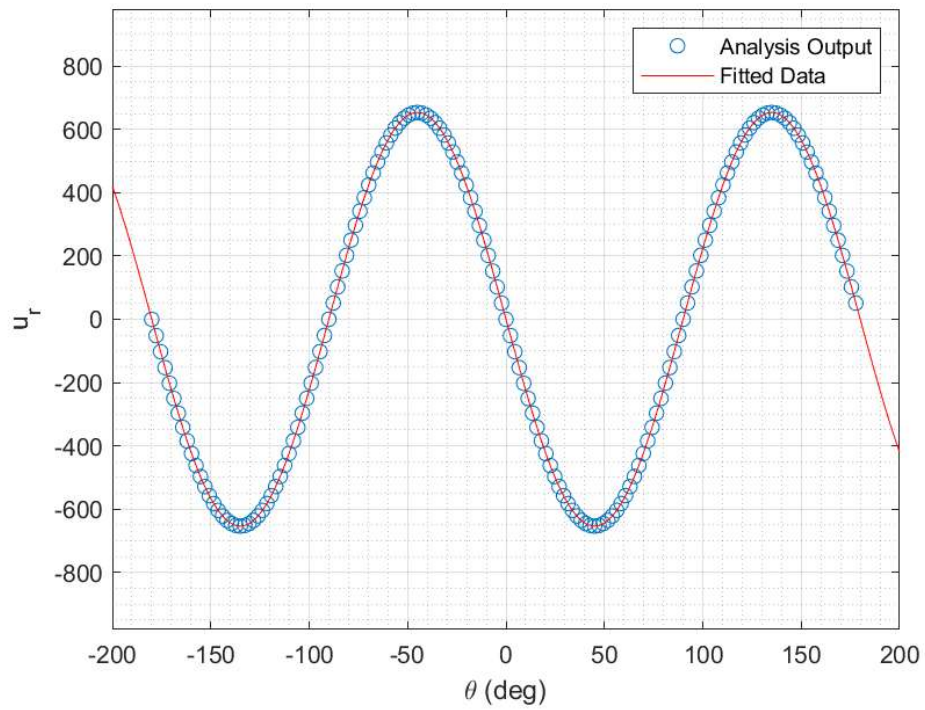


Figure 4.10. Cos Fitting Figure for First Operating Mode

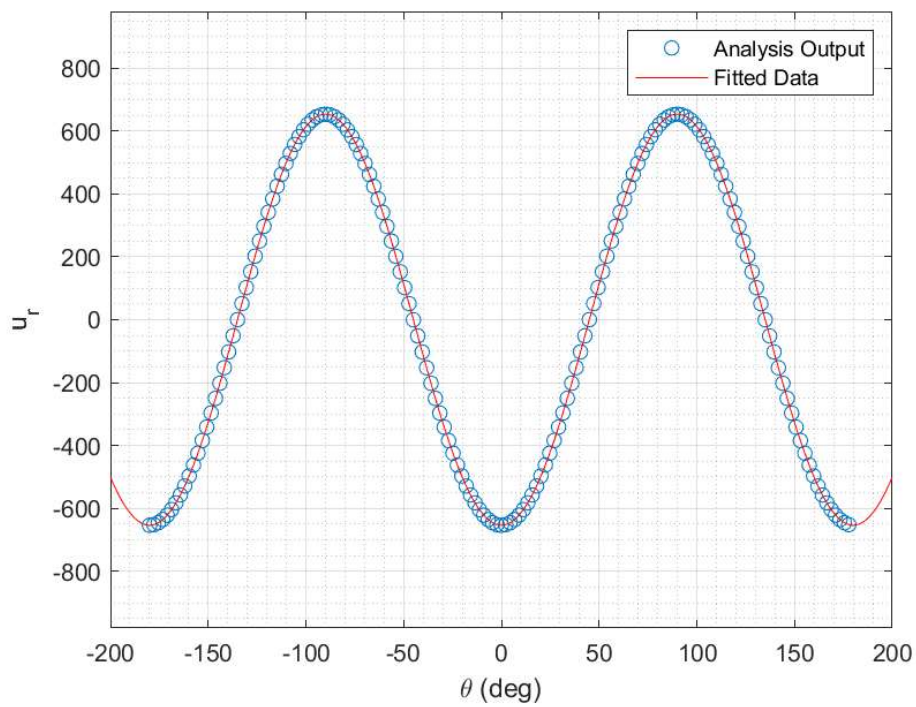


Figure 4.11. Cos Fitting Figure for Second Operating Mode

One can find the nodal and anti-nodal points from the cos-fitting equation. Two points are nodal points of the two-mode shapes used in the modal matrices. These points can be seen in Figure 4.12.

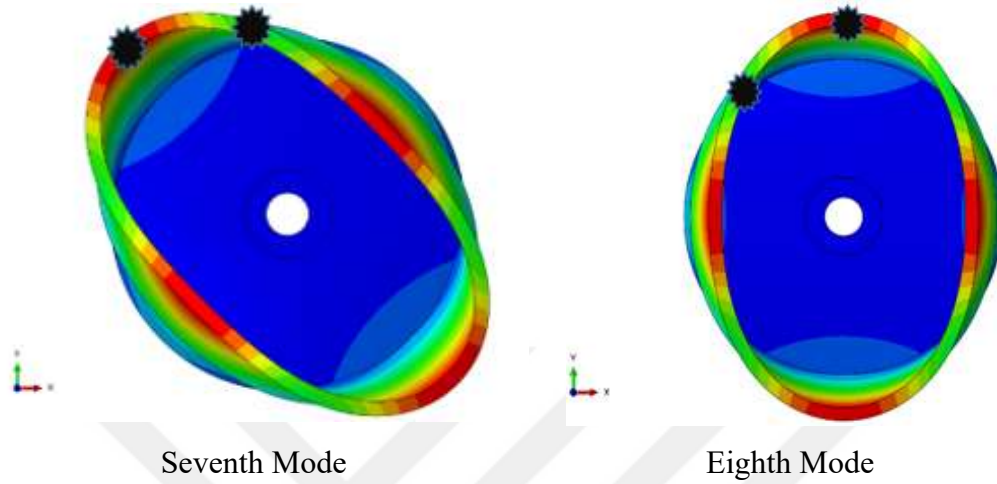


Figure 4.12. Nodal and Anti-Nodal Points Examples for Both Modes

ABAQUS program uses the mass normalized displacement for the relative displacement. However, displacement-normalized mode shapes will be used in the dynamic analysis. Thus, the mode shape matrix always has the biggest value as one. The mode shape matrix can be seen below.

$$[\phi] = \begin{bmatrix} x_{1,1} & x_{2,1} \\ x_{1,2} & x_{2,2} \end{bmatrix} \quad (4.3)$$

The first subscript of the x describes the direction, and the second subscript of the x represents the mode number.

4.4.2 Results of Natural Frequency

Natural frequencies are found in modal analysis at ABAQUS software. If the material is homogenous along the resonator and the resonator is axisymmetric, then two operating modes must have the same natural frequency. Therefore, mesh quality is an important factor in modal analysis. Mesh quality analyses have been done in all

modal analyses to have the same natural frequency in two modes, up to seven significant figures.

4.5 Sample Analysis and the Results of the Analysis

A sample analysis has been given in this section. Firstly, the resonator is given in the Figure 4.13.

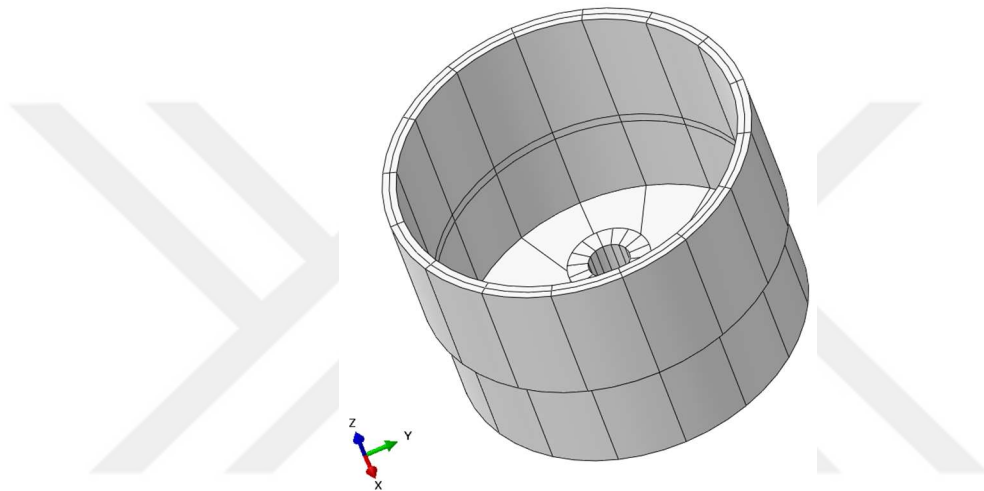


Figure 4.13. Resonator of the CVG in Sample Analysis

The resonator has the free boundary condition and the 3J53 nickel-cobalt alloy, which will be explained later. Axial-symmetrical and C3D20 quadratic elements have been applied to the resonator, as shown in Figure 4.14.

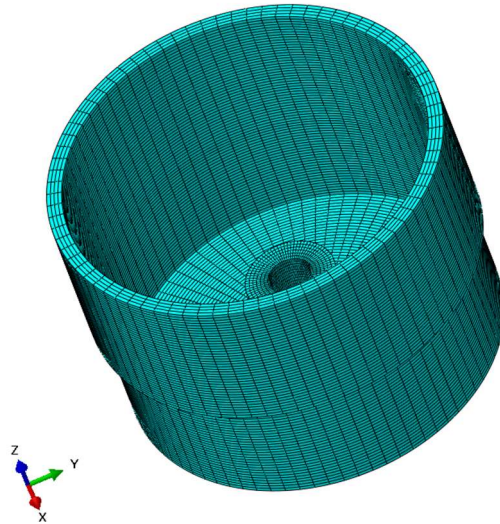


Figure 4.14. Mesh of the Resonator in Sample Analysis

Result nodes are selected as shown below in Figure 4.15. Results of the modal analysis in the cartesian coordinate system are shown in Figure 4.16 and Figure 4.17. Only operating modes are provided.

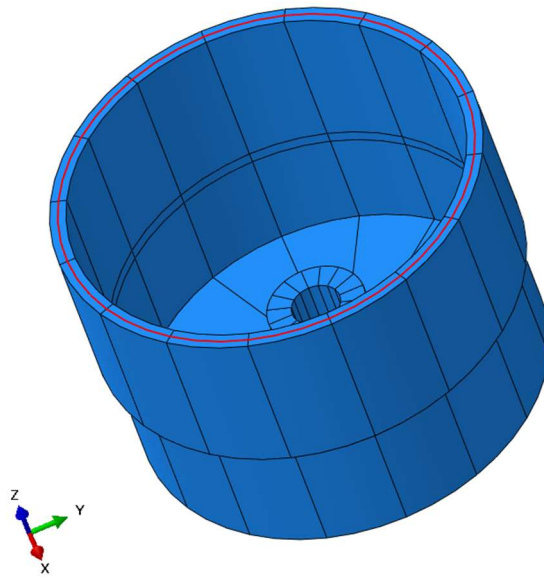
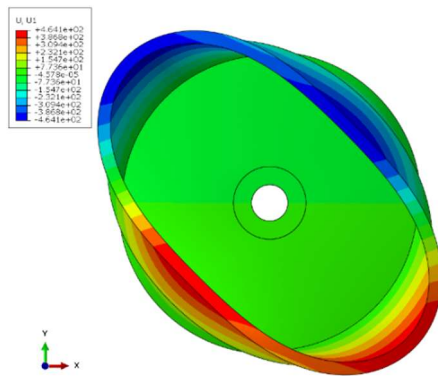
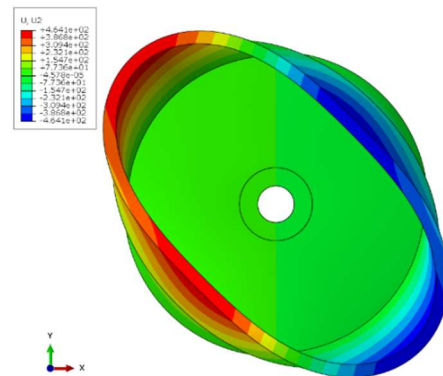


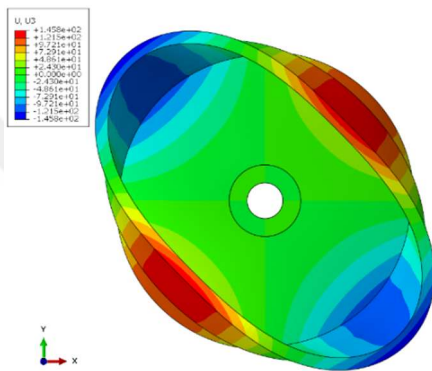
Figure 4.15. Result Set of the Resonator in Sample Analysis



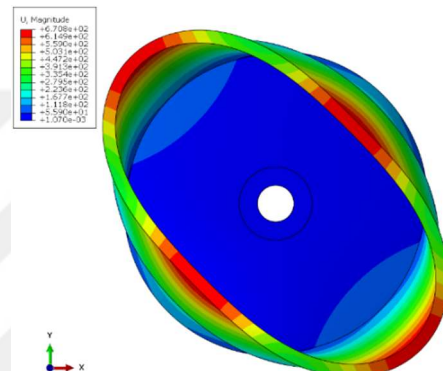
Deformation in X Direction



Deformation in Y Direction



Deformation in Z Direction



Total Deformation

Figure 4.16. Deformation at the First Mode in the Cartesian Coordinate System

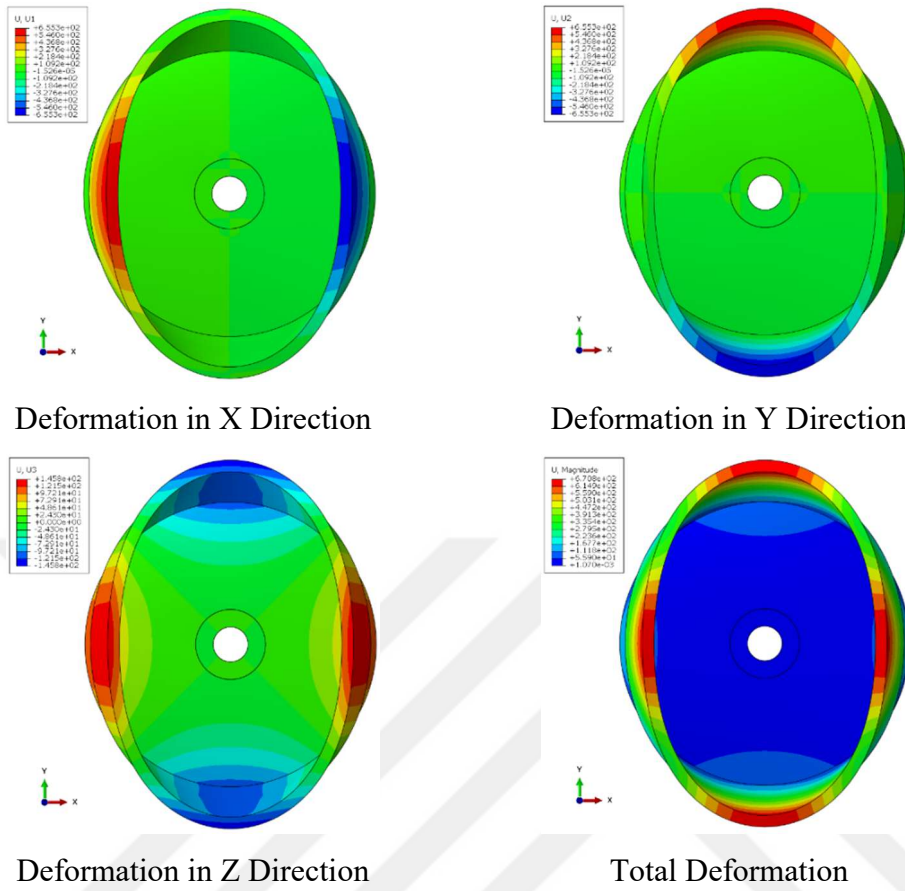
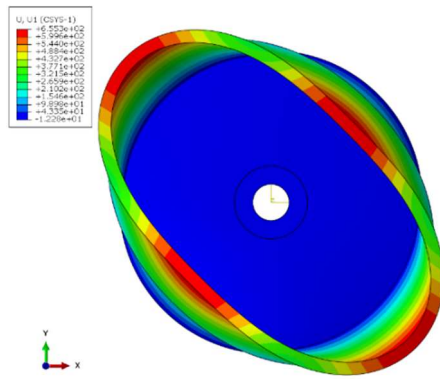
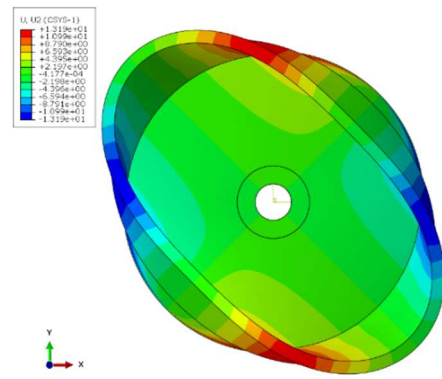


Figure 4.17. Deformation at the Second Mode in the Cartesian Coordinate System

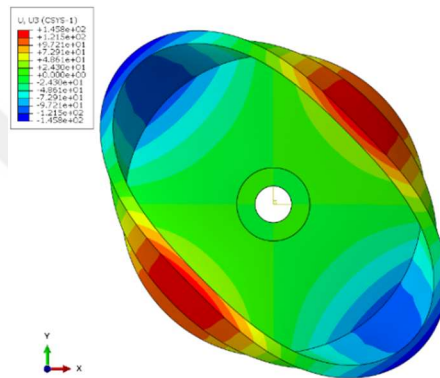
The results of the modal analysis in the cylindrical coordinate system are shown in Figure 4.18 and Figure 4.19.



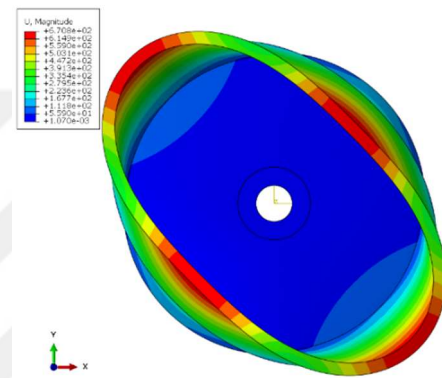
Deformation in R Direction



Deformation in θ Direction



Deformation in Z Direction



Total Deformation

Figure 4.18. Deformation at the First Mode in the Cylindrical Coordinate System

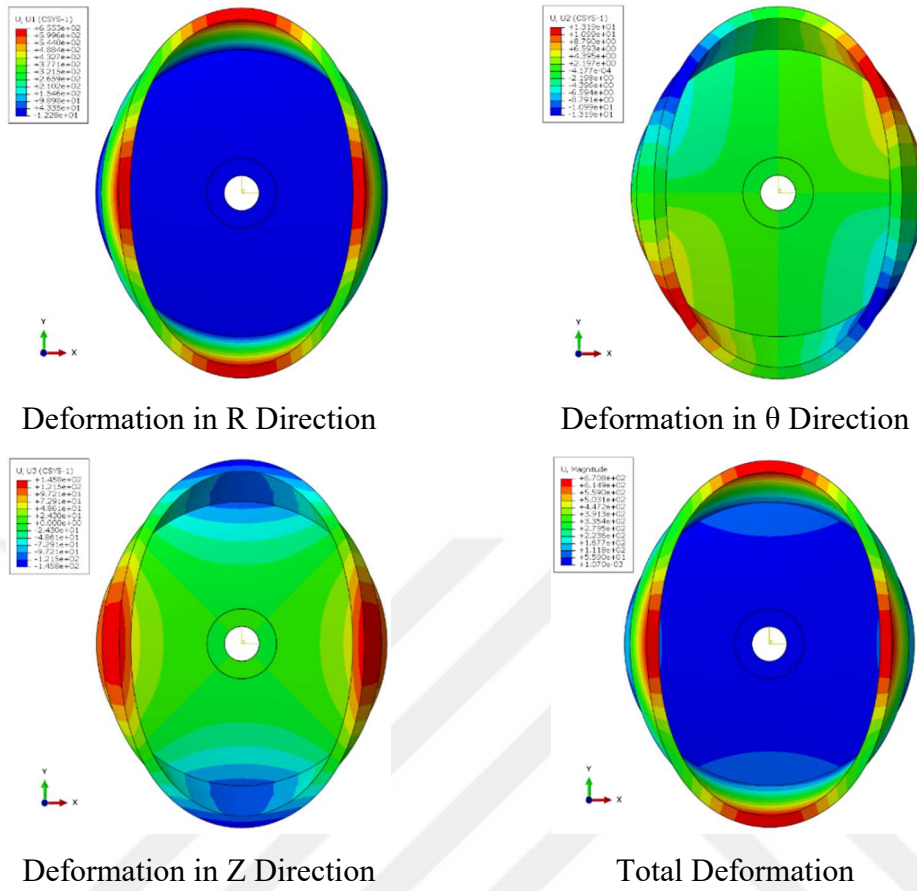


Figure 4.19. Deformation at the Second Mode in the Cylindrical Coordinate System

Additionally, one can see the nodal and anti-nodal points with the cylindrical coordinate system. In Figure 4.20 and Figure 4.21, deformations at the r direction are given, and the legend is restricted to between 1 and -1 to make the nodal points visible.

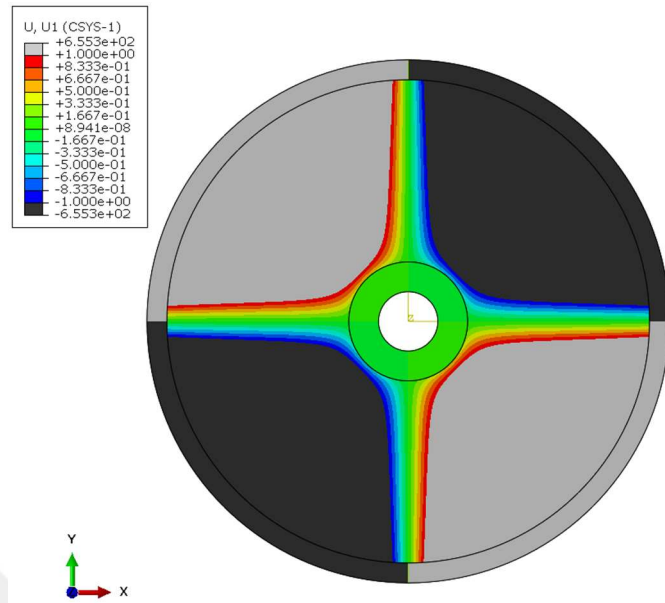


Figure 4.20. Nodal Points in First Mode

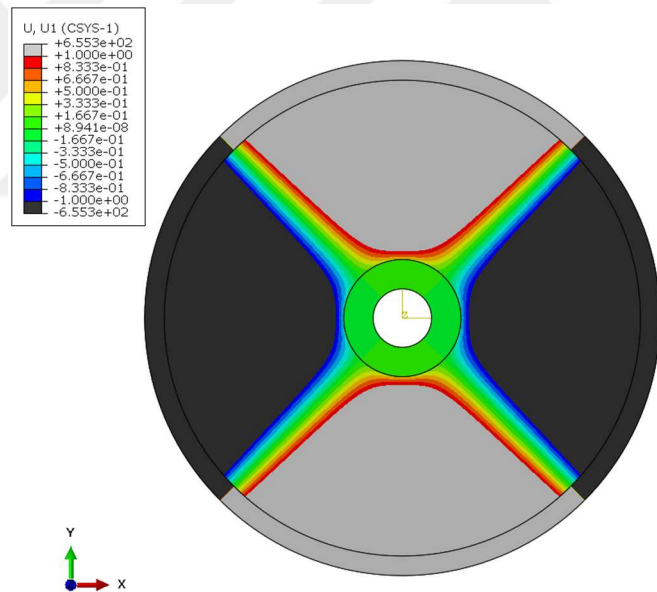


Figure 4.21. Nodal Points in Second Mode

The two modes have the same natural frequency, which is 4310.4 Hz. The results from the nodes are fitted to the cosine equation. The equation plots can be seen in Figure 4.22 and Figure 4.23, and equations can be seen in Equations (4.4) and (4.5).

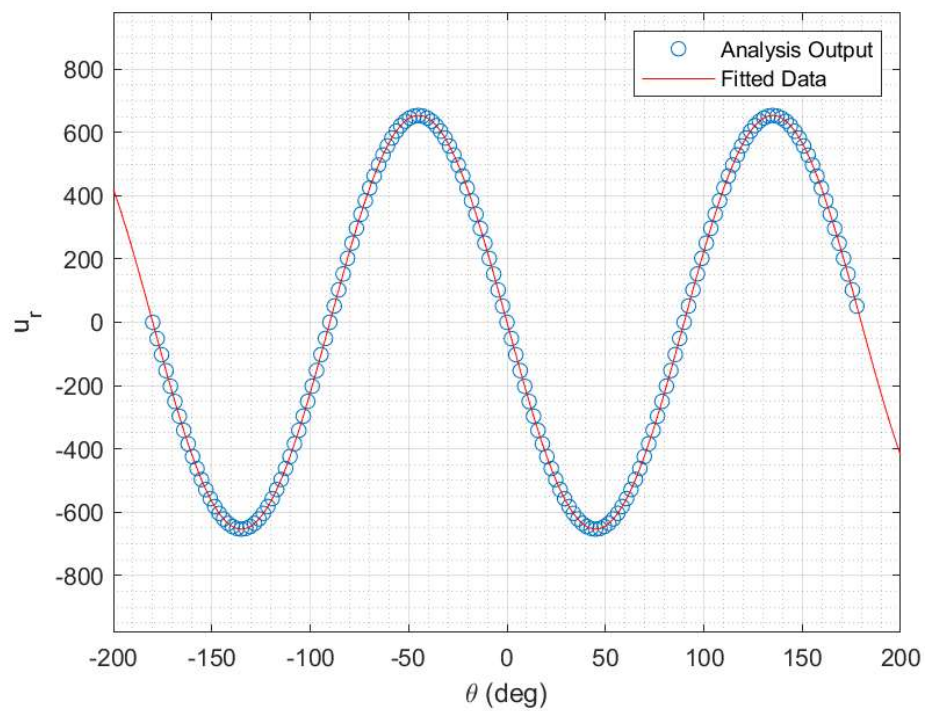


Figure 4.22. Cos Fitting Figure for First Operating Mode

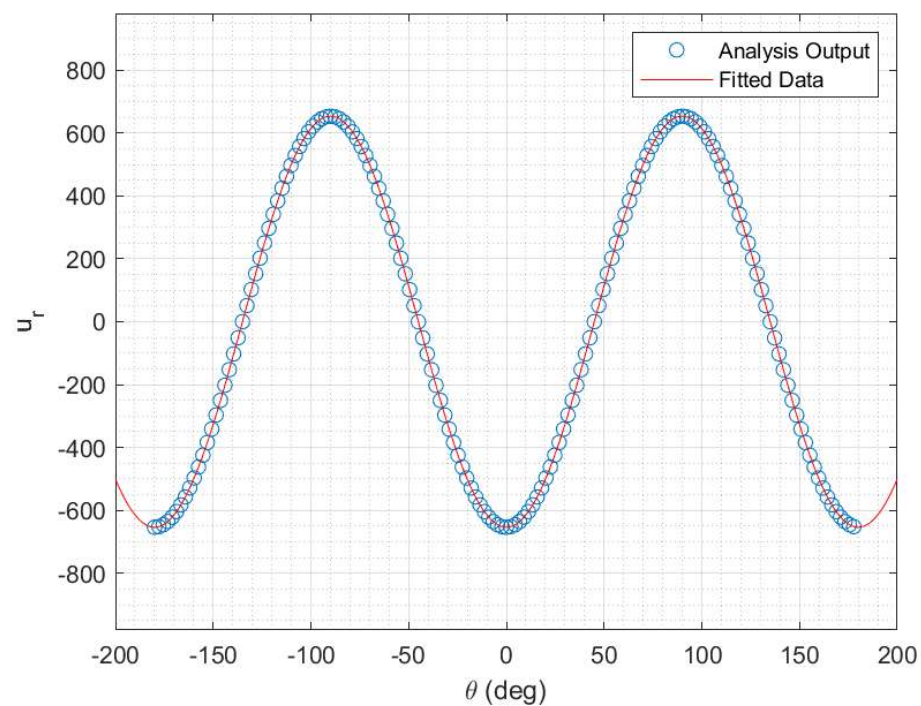


Figure 4.23. Cos Fitting Figure for Second Operating Mode

$$653.8 \cos(2x + 90.02) \quad (4.4)$$

$$-653.8 \cos(2x + 0.02) \quad (4.5)$$

Additionally, a mode shape matrix is constructed from the solution, which can be seen below.

$$[\phi] = \begin{bmatrix} 1 & -5.5372e-09 \\ -5.5372e-09 & 1 \end{bmatrix} \quad (4.6)$$

State-space matrices are found in the results of the analysis. Damping ratios are taken as 0.025 [7].

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7.3281e+08 & 0 & -1.3535e+03 & 0 \\ 0 & -7.3281e+08 & 0 & -1.3535e+03 \end{bmatrix} \quad (4.7)$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.8)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (4.9)$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (4.10)$$

Moreover, this sample analysis was simulated using the above information and mathematical model. The results can be seen in Figure 4.24.

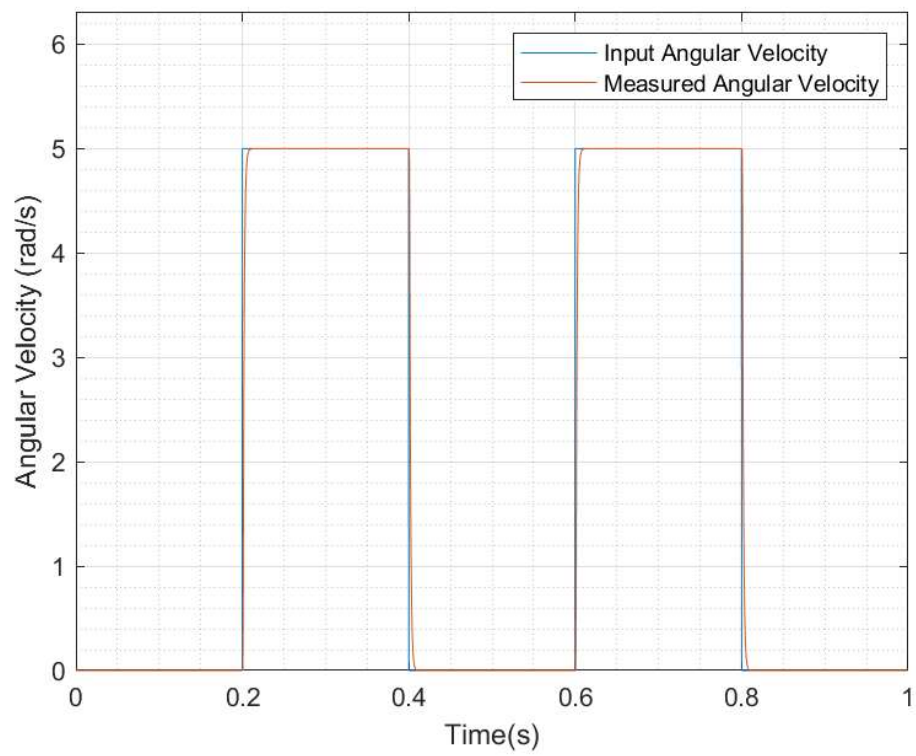


Figure 4.24. Result of the Sample Analysis

CHAPTER 5

MOST COMMON CVG RESONATORS and INVESTIGATION OF THE FACTORS THAT AFFECT ANGULAR VELOCITY ESTIMATION

In the previous chapters, the mathematical model and modal analysis are explained. Therefore, a complete mathematical vibrational model has been constructed. In this chapter, the most common CVG resonators in the industry are introduced. Moreover, CVG resonators are analyzed by using the mathematical vibrational model in different scenarios. In these scenarios, an error has been added to the resonator with different error magnitudes for density or Young's modulus in error analysis. Finally, these resonators are compared in the error analysis in different scenarios and the results are given.

5.1 Most Common Resonators in the Industry

There are many different resonators designed in the industry from the start of the invention of the CVG. After the performed literature research, three different resonators have been found in the industry to be the most popular. These common resonators are listed below.

5.1.1 Cylindrical Resonator Gyroscope

A cylindrical resonator gyroscope is one of the types of CVG. The cylindrical resonator gyroscope has many advantages such as high stability, high reliability, long life, and low power consumption as stated in the paper [17]. This resonator can be seen in Figure 5.1.

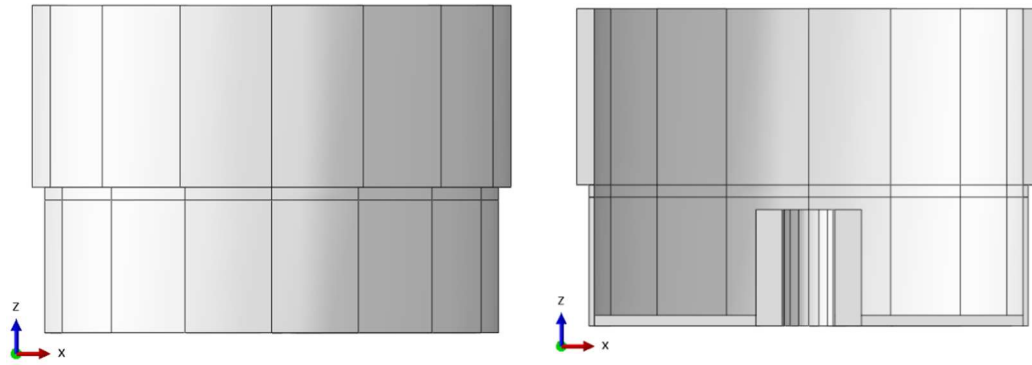


Figure 5.1. Cylindrical Resonator Gyroscope

In the work of Luo et al. [17], it is stated that the used material is brass. Brass has 0.33 Poisson's ratio, 110 *GPa* Young's Modulus, and 8430 *kg/m*³. The resonant natural frequency is 2981.9 Hz as stated in the paperwork of Luo et al. [17]. The resonator has been designed and analyzed in ABAQUS software. Four different mesh densities are selected and analyzed to make sure sufficient mesh density is achieved. As stated before, mesh quality is compared with the natural frequencies. Table 5.1 shows mesh density, the seed number at the one edge of the cell, the total number of elements in the analyses, natural frequencies, and the run time.

Table 5.1. Mesh Convergence Analysis of the Cylindrical Resonator Gyroscope

Mesh Density	Total Number of Elements	First Natural Freq.	Second Natural Freq.	Run Time
5 Seeds	33760	3174.9	3174.9	4 Minutes
10 Seeds	67520	3174.6	3174.6	10 Minutes
15 Seeds	101280	3174.6	3174.6	30 Minutes
20 Seeds	135040	3174.6	3174.6	2 Hours

The difference between the natural frequencies can be explained with uncertain dimensions in the work. Therefore, the resonator design is derived from the paper. According to the above table, ten seeds are selected. Ten seeds in one edge of the

cell are shown in Figure 5.2. In Figure 5.3 solution of the analyzed resonator has been given. Additionally, in Figure 5.4 nodal points of the resonator have been shown.

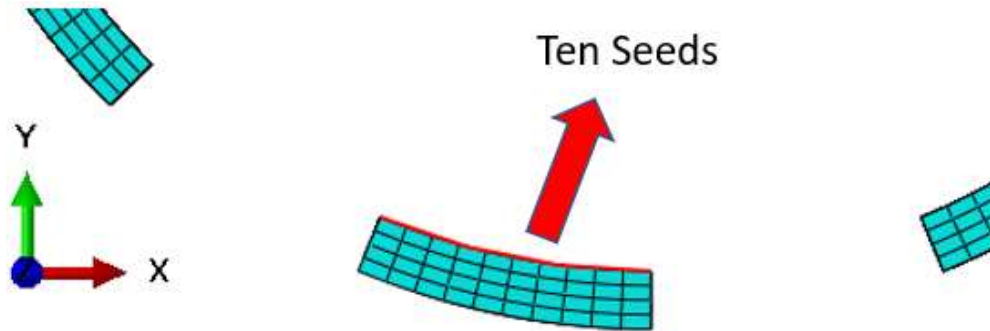


Figure 5.2. Seed Number of the Analysis of the Cylindrical Resonator Gyroscope

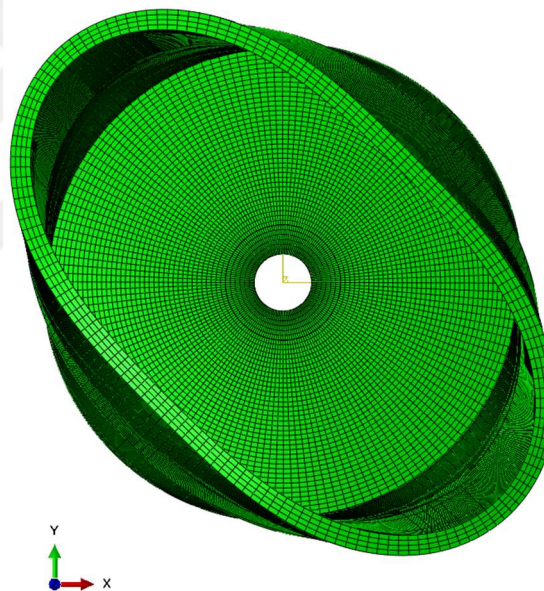


Figure 5.3. Solution of the Analysis of the Cylindrical Resonator Gyroscope

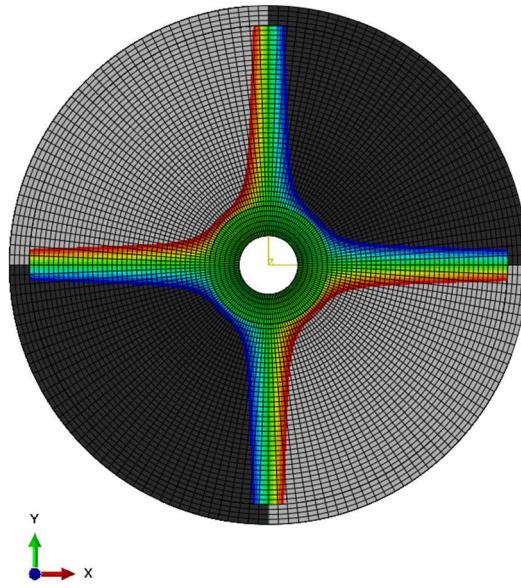


Figure 5.4. Nodal Points of the Analysis of the Cylindrical Resonator Gyroscope

5.1.2 Bell-Shaped Resonator Gyroscope

A bell-shaped resonator gyroscope is another type of CVG. The bell-shaped resonator gyroscope has a vibration stability and high impact resistance advantage over other CVG types [14]. This resonator can be seen in Figure 5.5.

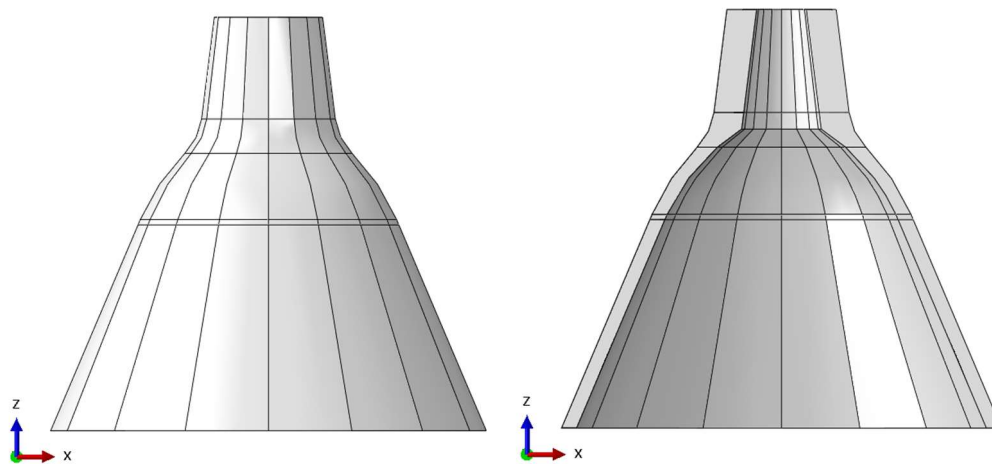


Figure 5.5. Bell-Shaped Resonator Gyroscope

In the work of Su et al. [15], it is stated that the used material is a 3J53 which is a nickel-cobalt alloy. 3J53 has 0.3 Poisson's ratio, 196.76 *GPa* Young's Modulus, and 8170 *kg/m*³. The resonant natural frequency is 5909.6 Hz as stated in the work of Su et al. [15]. To reach sufficient mesh convergence, four different mesh densities are selected and analyzed. Table 5.2 shows the mesh density, which is the seed number at the one edge of the cell, the total number of elements in the analysis, natural frequencies, and the run time.

Table 5.2. Mesh Convergence Analysis of the Bell-shaped Resonator Gyroscope

Mesh Density	Total Number of Elements	First Natural Freq.	Second Natural Freq.	Run Time
5 Seeds	8560	6469.7	6469.7	1 Minute
10 Seeds	17120	6469.3	6469.3	1.5 Minute
15 Seeds	25680	6469.2	6469.2	2 Minutes
20 Seeds	34240	6469.2	6469.2	3 Minutes

The difference between the natural frequencies can be explained as the paper does not include exact dimensions. In the work of Su et al. [15], only main dimensions (height, radius, and thickness) are given. Therefore, the resonator design is derived from the paper. According to the above table, fifteen seeds are selected. Fifteen seeds in one edge of the cell are shown in Figure 5.6. In Figure 5.7, the solution of the analyzed resonator has been given. Additionally, the nodal points of the resonator are shown in Figure 5.8.

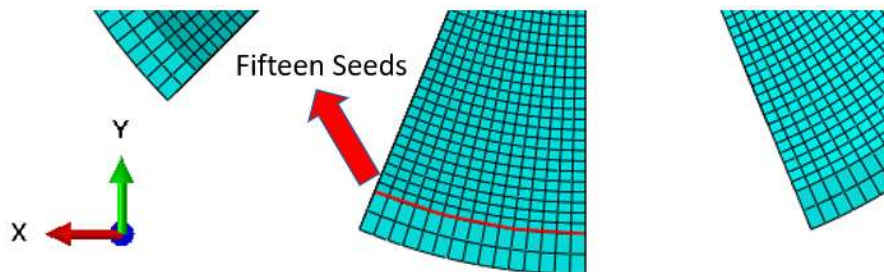


Figure 5.6. Seed Number of the Analysis of the Bell-Shaped Resonator Gyroscope

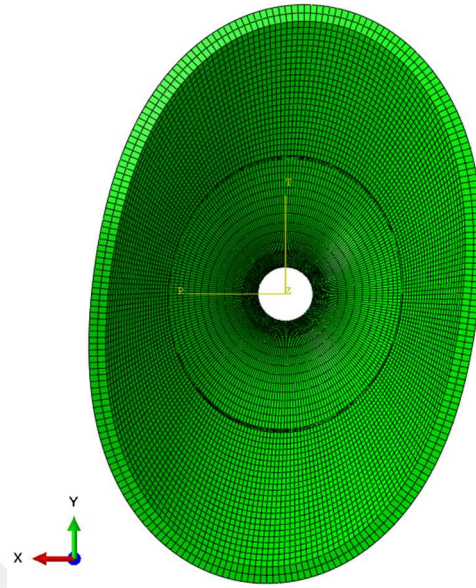


Figure 5.7. Solution of the Analysis of the Bell-Shaped Resonator Gyroscope

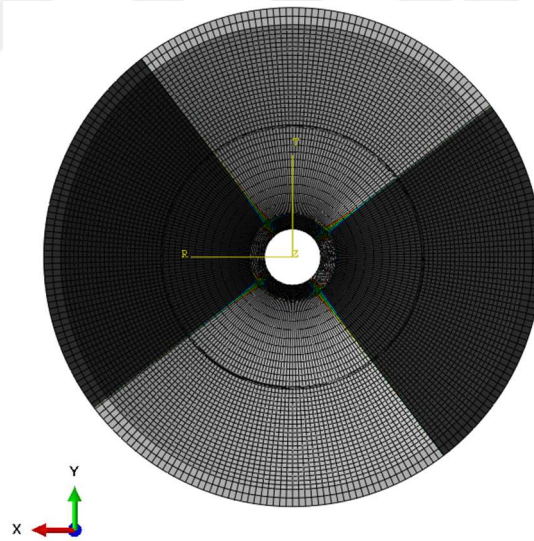


Figure 5.8. Nodal Points of the Analysis of the Bell-Shaped Resonator Gyroscope

5.1.3 Ring Resonator Gyroscope

Ring resonator gyroscope is one of the types of CVG. The ring resonator gyroscope has many advantages like high operation accuracy, long life, and good shock resistance [16]. This resonator can be seen in Figure 5.9.

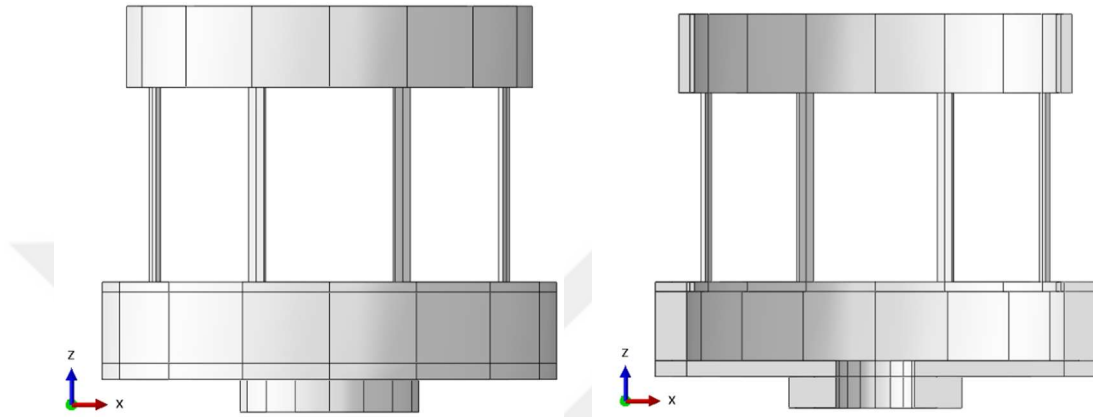


Figure 5.9. Ring Resonator Gyroscope

In the work of Tao et al. [16], it is stated that the used material is a nickel-cobalt alloy. 3J53 has 0.3 Poisson's ratio, 106 GPa Young's Modulus, and 8600 kg/m³. The resonant natural frequency is 2975 Hz as stated in the work of Tao et al. [16]. Four different mesh densities are selected and analyzed to ensure mesh convergence. In Table 5.3, one can see mesh density, which is the seed number at the one edge of the cell, the total number of elements, natural frequencies, and the run time.

Table 5.3. Mesh Convergence Analysis of the Ring Resonator Gyroscope

Mesh Density	Total Number of Elements	First Natural Freq.	Second Natural Freq.	Run Time
9 Seeds	83040	3019.3	3019.3	8 Minutes
18 Seeds	166080	3018.8	3018.8	20 Minutes
27 Seeds	249120	3018.6	3018.6	36 Minutes
36 Seeds	332160	3018.6	3018.6	120 Minutes

According to the above table, twenty-seven seeds are selected. The twenty-seven seeds in one edge can be seen in Figure 5.10. Figure 5.11 shows, the solution of the analyzed resonator. Moreover, the nodal points of the ring resonator gyroscope can be seen in Figure 5.12.

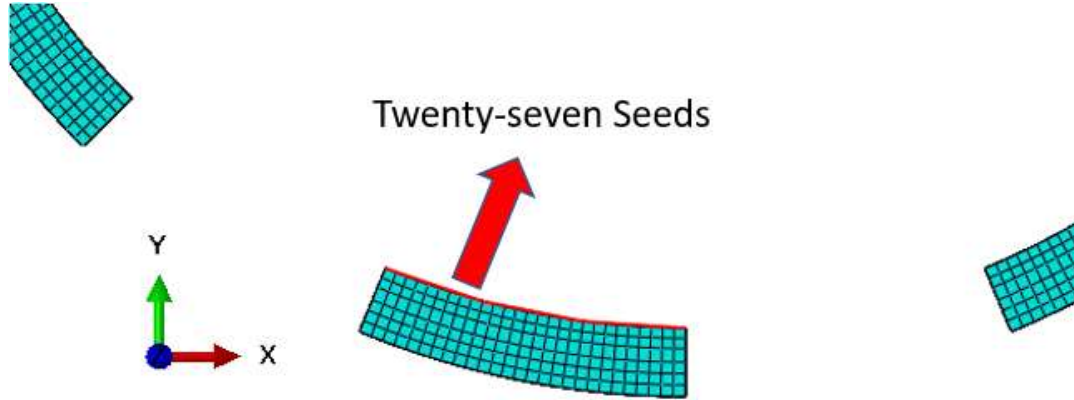


Figure 5.10. Seed Number of the Analysis of the Ring Resonator Gyroscope

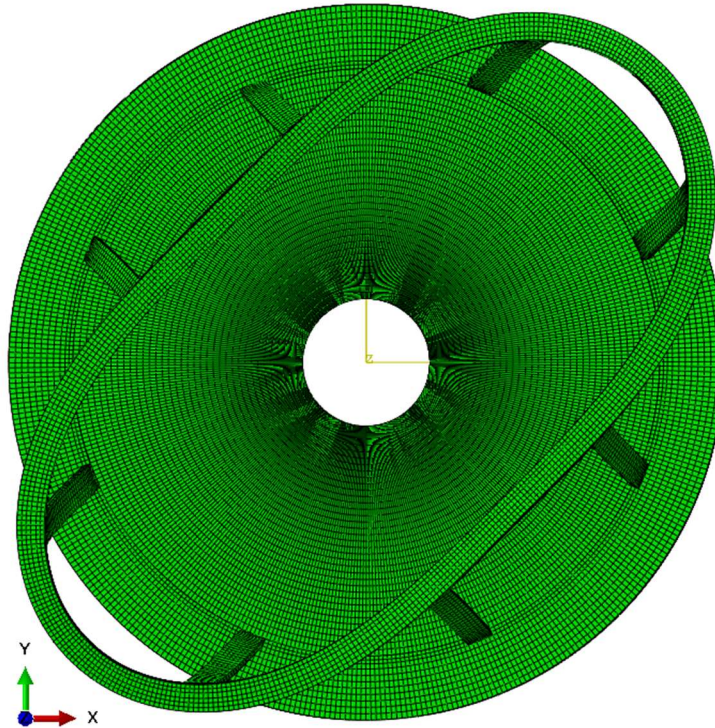


Figure 5.11. Solution of the Analysis of the Ring Resonator Gyroscope

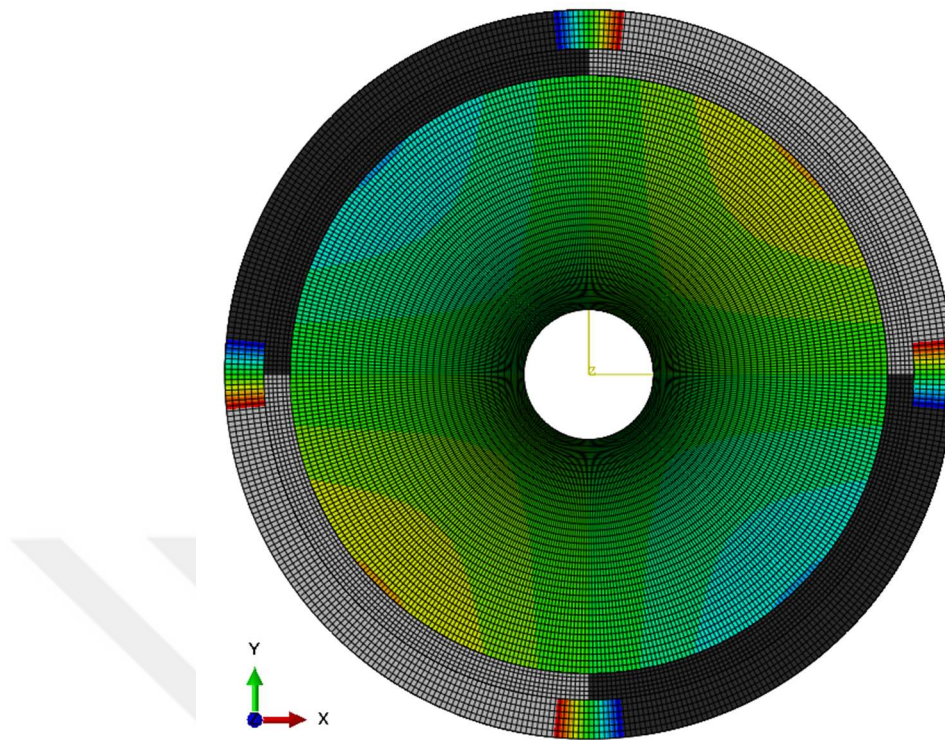


Figure 5.12. Nodal Points of the Analysis of the Ring Resonator Gyroscope

5.2 Error Analysis

In CVG resonators, it is essential to have a perfectly axisymmetric body to have zero errors in measured angular velocity. However, when there is any imperfection, it causes a natural frequency split and determines the mode shape. Since there is an asymmetric feature in the resonator when there is an imperfection, the main direction of the exciting mode will be in the same direction as the imperfection [33]. An example with added mass can be seen in Figure 5.13. In Figure 5.14, two added masses can be seen in the resonator.

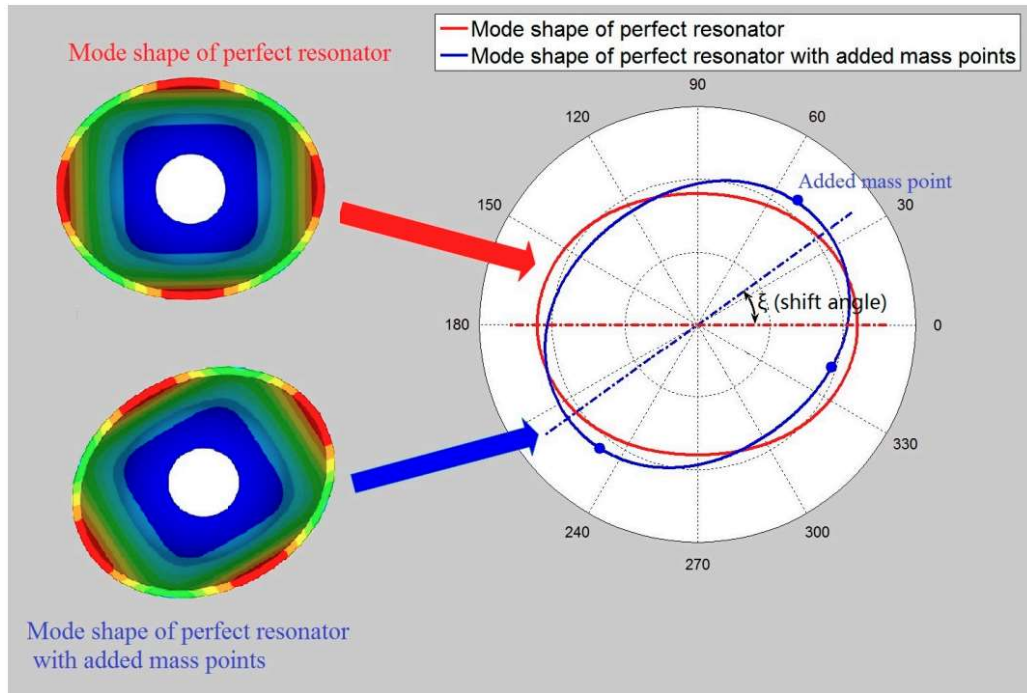


Figure 5.13. Mode Shape of the Imperfect Resonator [33]

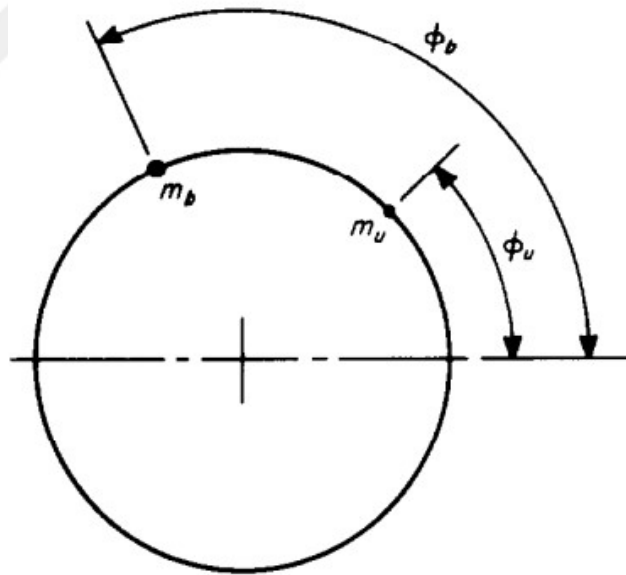


Figure 5.14. Imperfect Resonator with Two Added Mass [28]

Imperfections are not limited only to the mass; spring constants can be different too which is seen back in the concentration stiffness model. These imperfections fix the vibration axis of the resonator and differ the natural frequencies from each other. To

include new masses or stiffnesses, density or Young's Modulus are changed in some cells as imperfections to axisymmetric. Adding new masses and stiffnesses is modeling the manufacturing errors or casting errors that can happen in the resonator. In the next subsections, an error comparison of the common CVG resonators due to manufacturing errors is done.

5.3 Scenarios

Different scenarios are constructed to analyze the resonators. Three different scenarios have been used. These scenarios are selected based on errors from various perspectives. Scenarios are based on several parameter testing methods [27]. These parameter test methods are a simple method to measure the resonator parameters based on acoustic waves and vibration measurements [53]. Parameter test methods are given in Figure 5.15 and Figure 5.16.

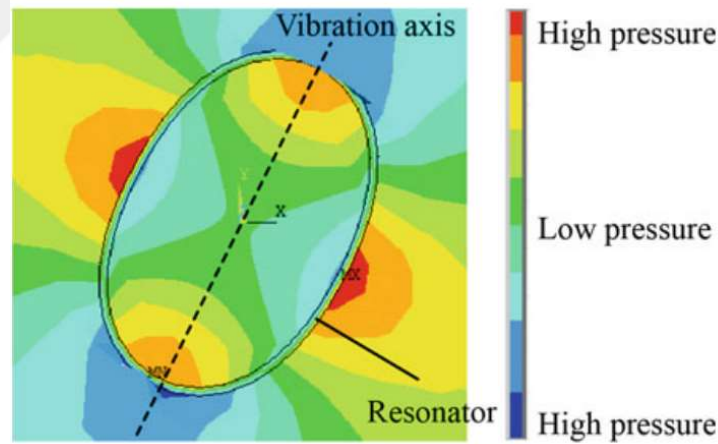


Figure 5.15. Acoustic Parameter Testing [53]

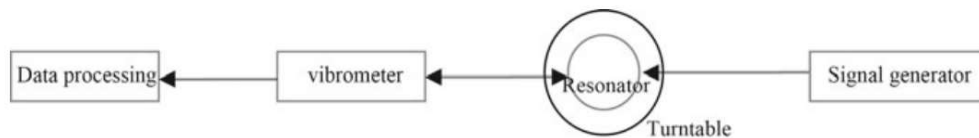


Figure 5.16. Vibration Parameter Testing [27]

These methods measure the mode shape, phase angles, and natural frequency. The setup for the acoustic measurement can be seen in Figure 5.17.

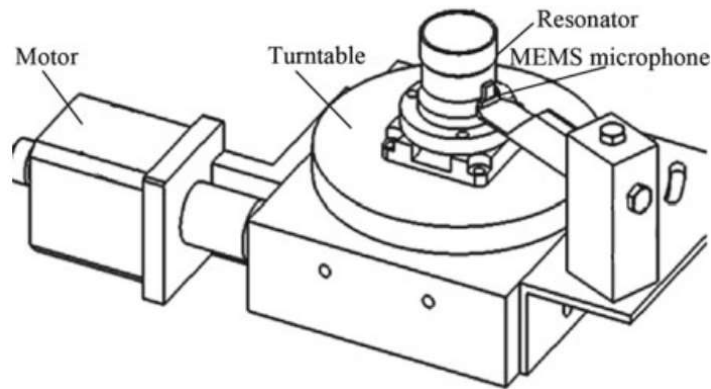


Figure 5.17. Setup for Acoustic Measurement [27]

Vibrometer measurement is nearly the same as the acoustic method. However, there is a laser vibrometer instead of a MEMS microphone. Based on these methods three different scenarios are constructed. All scenarios cover different situations. In scenario 1, everything is perfectly measured. In the second scenario, the phase angle of the second mode is not measured. Lastly, nothing is measured in scenario 3. These scenarios are explained below in subsections.

5.3.1 Scenario 1

In the first scenario, all parameters are perfectly measured, which are the phase angles, natural frequencies, and mode shapes of the two operating modes. Resonators have imperfections but performance depends on the mathematical model too. This scenario is assumed to be the baseline case for the axially perfect CVG.

5.3.2 Scenario 2

In the second scenario, parameters other than the phase angle of the second mode are perfectly measured. The calculation of the phase angle of the second mode compared to the excitation force can be seen below.

$$\varphi_2 = \text{atan} \left(\frac{\delta k^2 - (1 + 4\zeta_1\zeta_2\delta k + \delta k^2)\delta\omega^2 + \delta\omega^4}{2\delta k\delta\omega(\zeta_2 + \zeta_1\delta k) - 2\delta\omega^3(\zeta_1 + \zeta_2\delta k)} \right) \quad (5.1)$$

When the natural frequencies are the same, the phase angle of the second mode is shown below.

$$\varphi_2 = -\frac{\pi}{2} \quad (5.2)$$

Since the phase angle of the second mode is not measured in this scenario, it is assumed as in Equation (5.2). This scenario will limit the compensation capabilities of the mathematical model, and it will create a steady-state bias error in the measured angular velocities.

5.3.3 Scenario 3

In the third scenario, natural frequencies and the phase angle of the second mode are not measured. Therefore, the frequency of the excitation oscillation is not given precisely, equal to the natural frequency of the first mode, to the system. This will increase the bias error in the measured angular velocity. In this scenario, the frequency of the excitation oscillation will be given as the resonator is perfect, not have any imperfections.

5.4 Analysis Parameters in Each Resonator

Analysis parameters of each resonator are investigated and standardized to compare the designs of the resonators. Therefore, one will see the effect of the imperfections

in comparison to the design of the resonator. These analysis parameters are the material of the resonator, imperfection area, imperfection values

5.4.1 Material of the Resonator

Each resonator has different materials in corresponding references [15], [16], [17]. However, to compare the design of the resonators these materials are fixed in 3J53. In the below table, materials of the resonator in reference and analyses are given.

Table 5.4. Materials of the Resonators

Resonator	Material in Reference	Material in Analyses
Cylindrical Resonator Gyroscope	Brass	3J53
Bell-Shaped Resonator Gyroscope	3J53	3J53
Ring Resonator Gyroscope	Nickel-Cobalt Alloy	3J53

The material properties of the 3J53 material can be seen below.

Table 5.5. Material Properties of the 3J53 Material [15]

Material	Density	Young's Modulus
3J53	8170 kg/m^3	196.50 GPa

5.4.2 Q Factor of the Resonators

The Q factor (damping ratio) of the sensors is taken as a constant value (0.025 for damping ratio) for all resonators [7]. Despite the Q factor can change with the design and unbalance, the objective is to decrease the complexity of the analysis.

Additionally, the Q factor has an impact on the settling time which can be seen below [27].

$$t_s = \frac{4}{\omega\zeta} \quad (5.3)$$

However, steady-state performances will be compared in the scenarios. Therefore, the settling time effect of the Q factor can be neglected.

5.4.3 Imperfection Areas

The imperfections are given to the resonator in certain areas. The mesh should be carefully structured at the imperfection areas for accurate analysis. Circular regions of all resonators where measurements are performed are divided into sixteen equal cells parallel along the sensitive axis. These imperfection areas will be in the shell wall of the resonator due to the modeling of the resonator. One needs to select several parts from the sixteen equal cells to have imperfections. Each resonator is analyzed, and the most sensitive part of the cells is found. The imperfection areas are decided accordingly.

5.4.3.1 Imperfection in the Parallel Along to the Sensitive Axis

There are sixteen equal cells in each resonator. The mode-shape and natural frequency split vary with the selected imperfect cells. Therefore, different analyses have been done and several cells have been selected. In below Figure 5.18, one can see the sixteen equal-numbered cells.

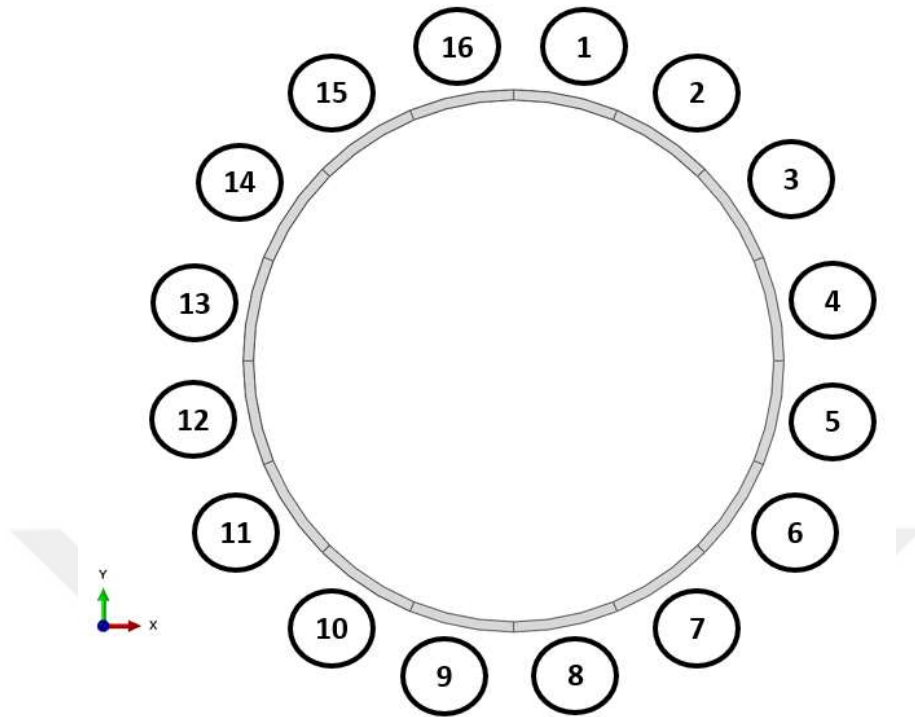


Figure 5.18. Sixteen Cells of the Resonators

One, two, three, and four imperfect cells are analyzed. All cases are investigated separately.

5.4.3.1.1 One Imperfect Cell

In this part, one imperfect cell is investigated. In Figure 5.19, one imperfect cell has been shown

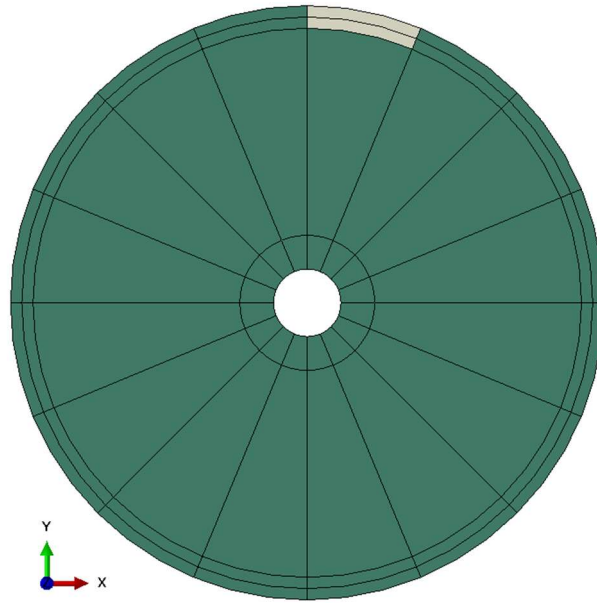


Figure 5.19. One Imperfect Cell

As a result of this analysis, a 1.2 Hz frequency split was found. In this analysis, density changed from 8170 kg/m^3 to 8100 kg/m^3 in the imperfect cell. The nodal axis of the mode shape is the middle of the imperfect cell. The location of the imperfect cell is not important. Only the nodal axis of the mode shape differs with the location of the imperfect cell which will be used as a variable in the mathematical model part. The split of the natural frequency is not dependent on the location of the imperfect cell.

5.4.3.1.2 Two Imperfect Cells

In this part, two imperfect cells are investigated. In Figure 5.20, two imperfect cells have been shown

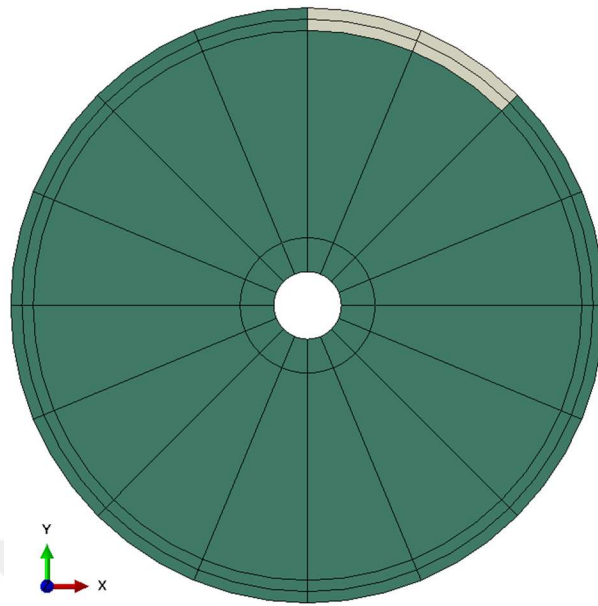


Figure 5.20. Two Imperfect Cells

As a result of this analysis, a 1.7 Hz frequency split was found. As an example, density changed from 8170 kg/m^3 to 8100 kg/m^3 in the imperfect cells. The nodal axis of the mode shape is the middle of the two imperfect cells. The nodal axis of the mode shape differs from the location of the imperfect cells. The split of the natural frequency is not dependent on the location of the imperfect cells, and it is higher than the one imperfect cell.

5.4.3.1.3 Three Imperfect Cells

In this part, three imperfect cells are investigated. In Figure 5.21, three imperfect cells have been shown

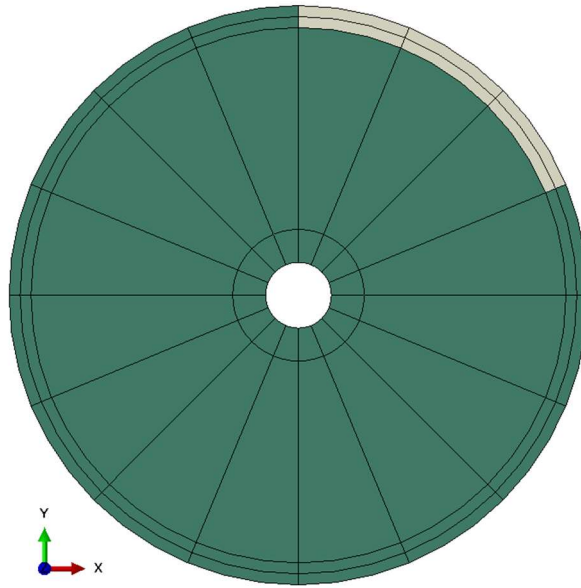


Figure 5.21. Three Imperfect Cells

As a result of this analysis, a 1.3 Hz frequency split was found. For example, density changed from 8170 kg/m^3 to 8100 kg/m^3 in the imperfect cells. The split of the natural frequency is dependent on the location of the imperfect cells. It can be lower or higher than the two imperfect cells. Imperfect cells at every 45° degrees cancel each other which comes from the mode shapes. The nodal axis of the mode shape differs with the location of the imperfect cells. The nodal axis of the mode shape is the middle of the three imperfect cells.

5.4.3.1.4 Four Imperfect Cells

In this part, four imperfect cells are investigated. In Figure 5.22, four imperfect cells have been shown

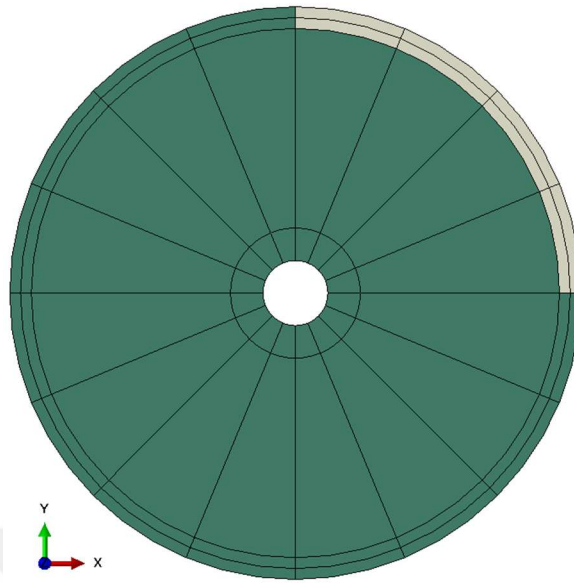


Figure 5.22. Four Imperfect Cells

As a result of this analysis, a 0.0 Hz frequency split was found. Density changed from 8170 kg/m^3 to 8100 kg/m^3 in the imperfect cells. The nodal axis of the mode shape is the middle of the four imperfect cells. The split of the natural frequency depends on the imperfect cells' location. It can be lower or higher than the two imperfect cells. Imperfect cells at every 45-degree that is inside every 90-degree section cancel each other and decrease frequency split due to the 45-degree differences in the mode shapes. Additionally, if an imperfect cell is repeating every 45-degree in other 90-degree sections, it increases the value of the frequency split. Several examples are given in Figure 5.23, Figure 5.24, and Figure 5.25.

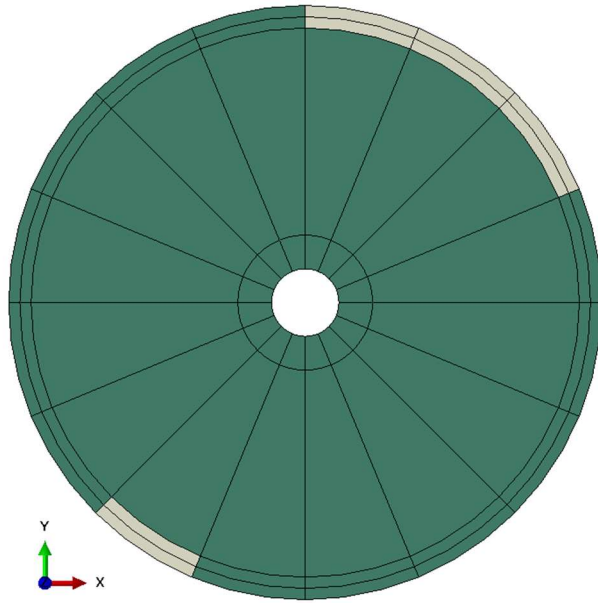


Figure 5.23. Four Imperfect Cells in 1, 2, 3, and 10 Cell

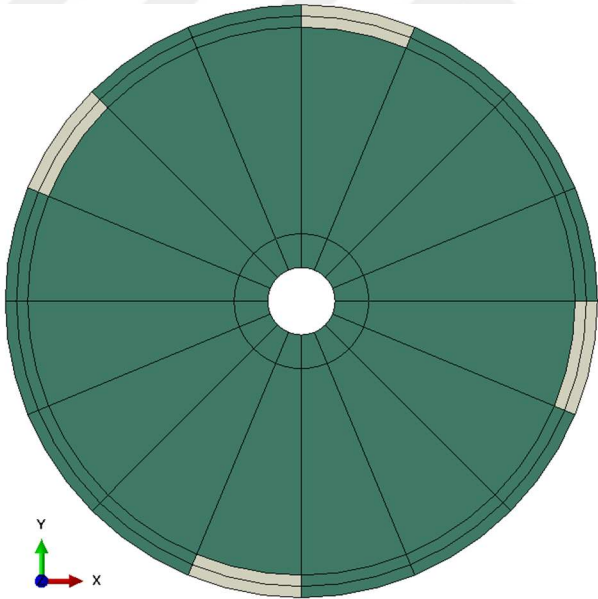


Figure 5.24. Four Imperfect Cells in 1, 5, 9, and 14 Cell

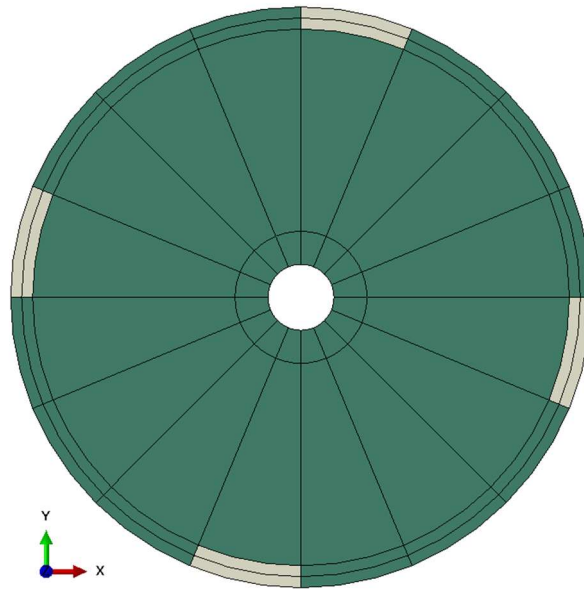


Figure 5.25. Four Imperfect Cells in 1, 5, 9, and 13 Cell

Therefore, one should look for the effective imperfect cells in an analysis. The result table can be seen below.

Table 5.6. Value of the Frequency Split According to Effective Imperfect Cells

Imperfect Cells	Frequency Split	Effective Imperfect Cells
1-2-3-4	0 Hz	Zero
1-3-5-7	0 Hz	Zero
1-2-3-7	1.7 Hz	Two (2-3)
1-2-3-10	2.5 Hz	Two (2-2)
1-2-9-10	3.5 Hz	Four (1-2-3-4)
1-5-9-14	3.9 Hz	Four (1-1-1-2)
1-5-9-13	4.9 Hz	Four (1-1-1-1)

According to the above understandings, two imperfect cells are selected. Due to, more stability in the error values and high error values respect to other number of imperfect cells.

5.4.3.2 Imperfection in the Perpendicular Plane to the Sensitive Axis

For every resonator, the volume of the imperfection cells depends. The volume of the imperfection cells is decided according to the sensitive part of the resonator. Therefore, the walls of the resonators are analyzed and investigated for the most sensitive parts.

5.4.3.2.1 Cylindrical Resonator Gyroscope

The wall of the cylindrical resonator gyroscope is divided into two parts: The upper cell which is in white, and the lower cell which is in green which can be seen in Figure 5.26.

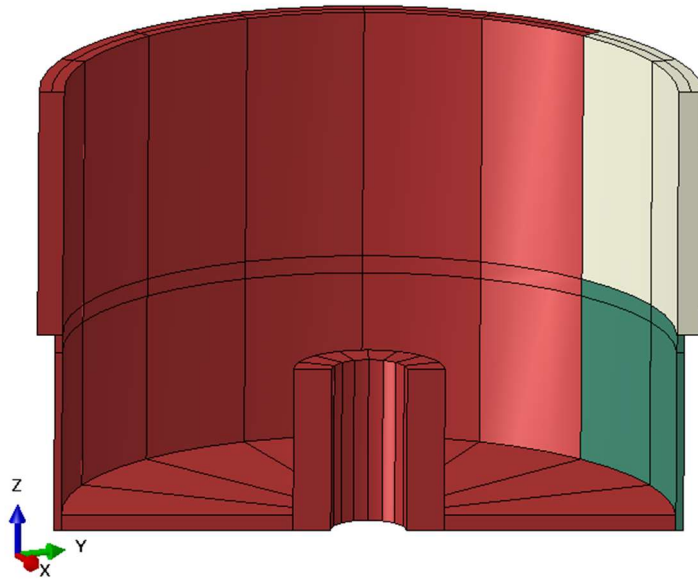


Figure 5.26. Parts of the Cylindrical Resonator Gyroscope

For example, density changed to 8100 kg/m^3 separately in the imperfect cells. The results are given in Table 5.7.

Table 5.7. Analysis of the Cells at the Wall in Cylindrical Resonator Gyroscope

Cell of the Resonator	First Natural Frequency	Second Natural Frequency	Frequency Split
Bottom Cell	4308.0	4308.1	0.1
Top Cell	4309.3	4311.1	1.8
All Cells	4309.3	4311.2	1.9

Therefore, only the top cell of the resonator has been selected.

5.4.3.2.2 Bell-Shaped Resonator Gyroscope

The bell-shaped resonator gyroscope is divided into three cells: the upper cell which is white, the middle cell which is blue, and the lower cell which is green which can be seen in Figure 5.27.

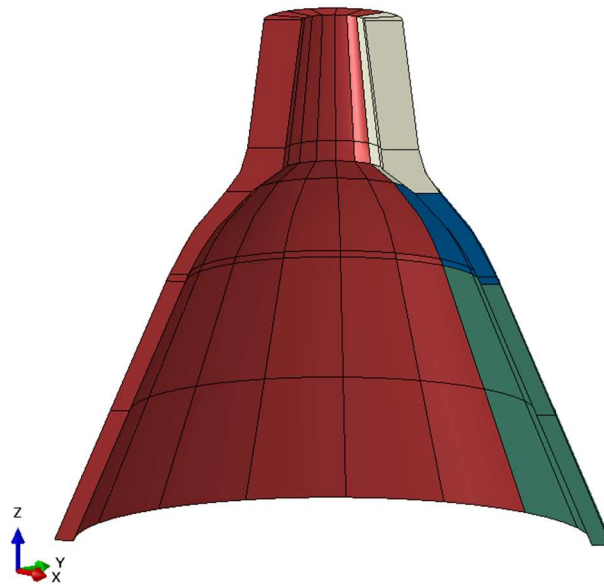


Figure 5.27. Cells of the Bell-Shaped Resonator Gyroscope

For example, density changed to 8100 kg/m^3 in the imperfect cells. The results are given in Table 5.8.

Table 5.8. Analysis of the Cells at the Wall in Bell-Shaped Resonator Gyroscope

Cell of the Resonator	First Natural Frequency	Second Natural Frequency	Frequency Split
Upper Cell	6464.2	6464.2	0.0
Lower Cell	6466.1	6469.1	3.0
Middle Cell	6464.2	6464.3	0.1
Lower + Middle Cells	6466.2	6469.2	3.0
Middle + Upper Cells	6464.2	6464.3	0.1
All Cells	6466.2	6469.2	3.0

Therefore, the lower bottom cell is selected.

5.4.3.2.3 Ring Resonator Gyroscope

The ring resonator gyroscope is divided into three cells: the top cell which is blue, the beam which is green, and the bottom cell which is white. These cells are shown in Figure 5.28.

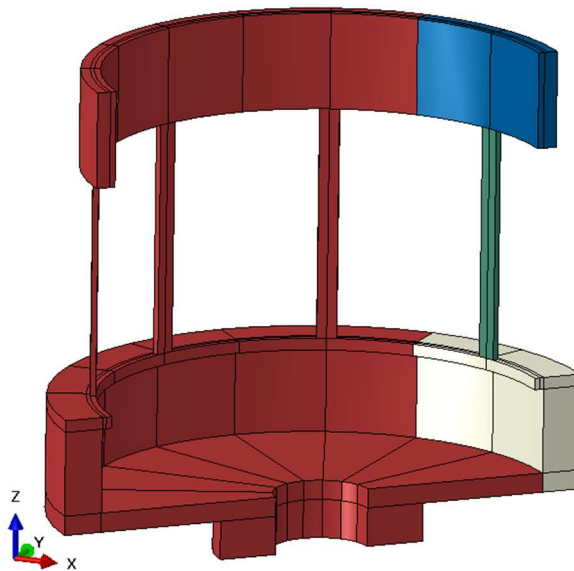


Figure 5.28. Cells of the Ring Resonator Gyroscope

For example, density changed to 8100 kg/m^3 in the imperfect cells. The results are given in Table 5.9.

Table 5.9. Analysis of the Cells at the Wall in Ring Resonator Gyroscope

Cell of the Resonator	First Natural Frequency	Second Natural Frequency	Frequency Split
Top Cell	4218.1	4219.8	1.7
Beam Cell	4216.8	4216.9	0.1
Bottom Cell	4216.8	4216.8	0.0
Bottom + Top Cells	4218.1	4219.9	1.8
Bottom + Beam Cells	4216.8	4216.9	0.1
Beam + Top Cells	4218.1	4219.9	1.8
All Cells	4218.1	4219.9	1.8

Therefore, only the top cell is selected.

5.4.4 Imperfection Values

The imperfections are given to the resonator in certain areas with certain values. To compare effectively, density and Young's Modulus values are equal in all imperfect analyses. These values can be seen in Table 5.10.

Table 5.10. Imperfection Values for the Density and the Young's Modulus

Material Properties	Values			
Density (kg/m^3)	7900	7950	8000	8030
	8065	8100	8135	8170
	8205	8240	8275	8310
	8345	8380	8415	8450
Young's Modulus (GPa)	190	191	192	193
	194	195	196	196.50
	197	198	199	200
	201	202	203	

Analyses have been done according to these density and Young's modulus values.

5.5 The Result of the Resonator Analysis

Modal analyses have been done on resonators, as explained in previous parts. The results of the three resonators are given below. The results of the density change are presented in Figure 5.29, Figure 5.30, and Figure 5.31.

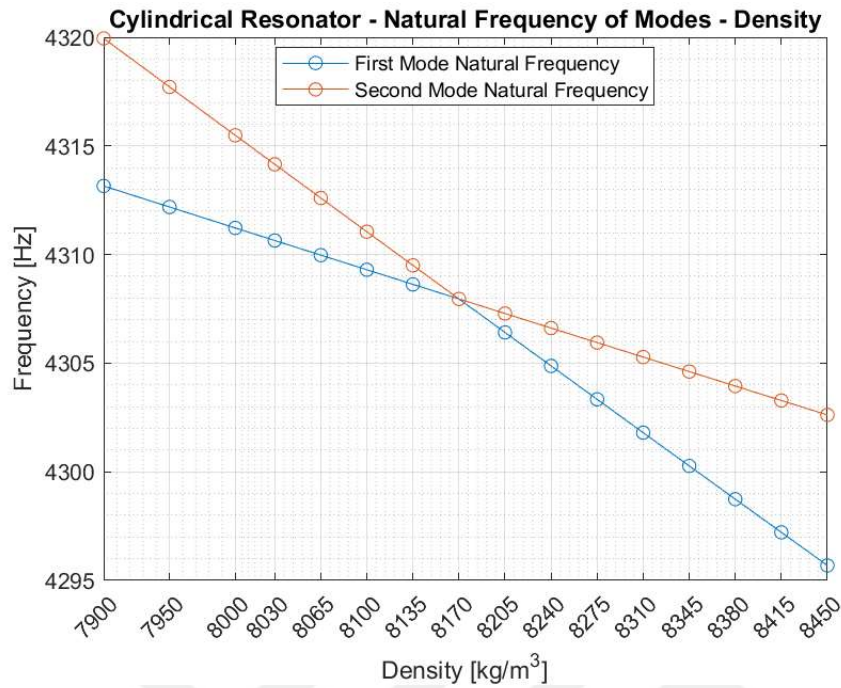


Figure 5.29. Density Imperfection Natural Frequency Results of the Cylindrical Resonator

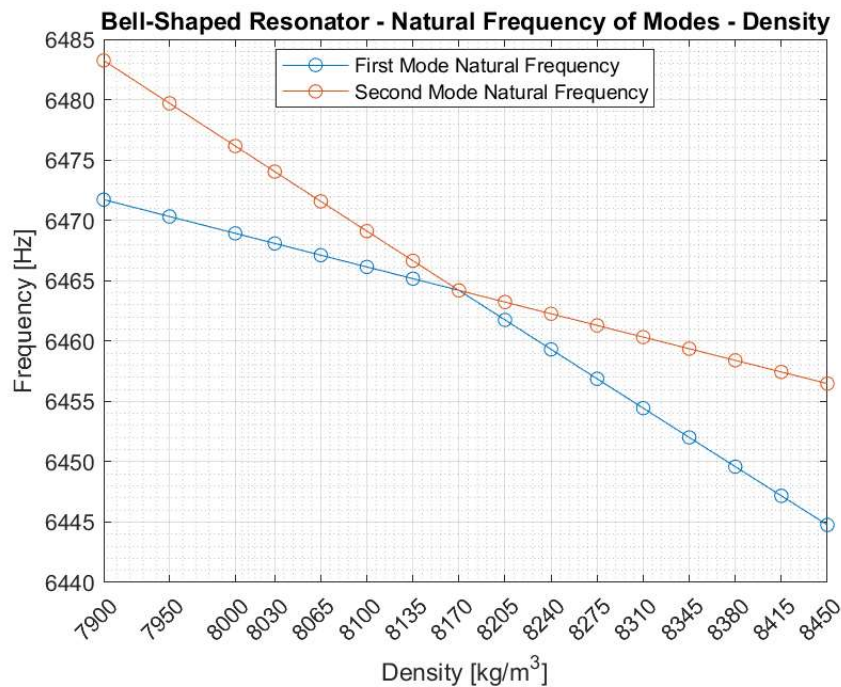


Figure 5.30. Density Imperfection Natural Frequency Results of the Bell-Shaped Resonator

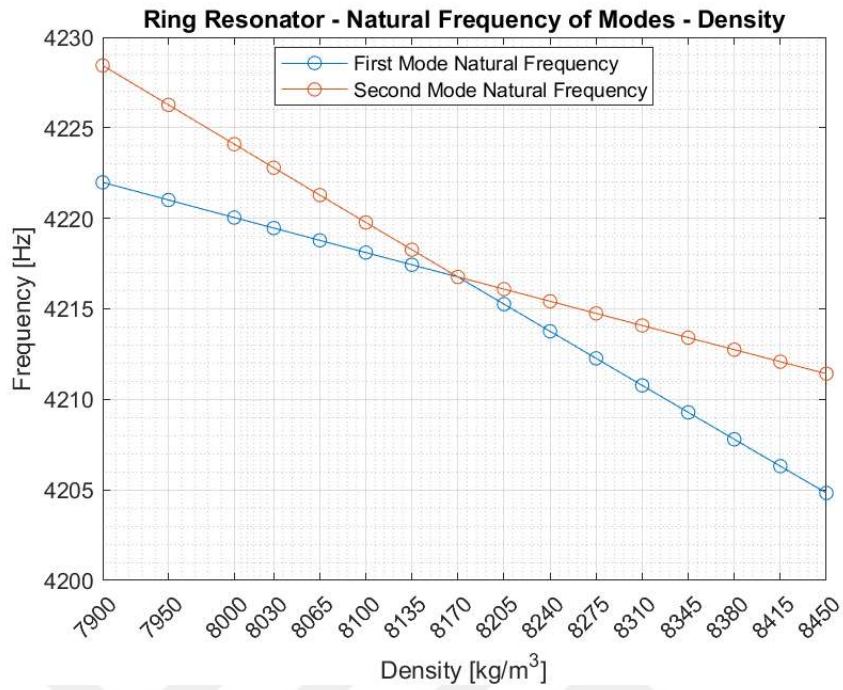


Figure 5.31. Density Imperfection Natural Frequency Results of the Ring Resonator

In Figure 5.32, Figure 5.33, and Figure 5.34, the results of the changed Young's modulus are given.

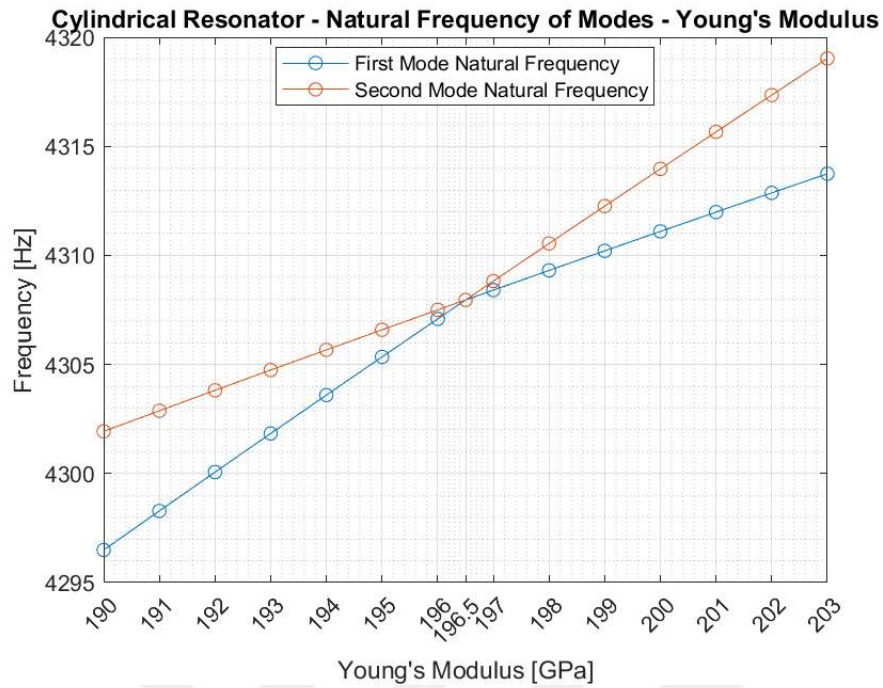


Figure 5.32. Young's Modulus Imperfection Natural Frequency Results of the Cylindrical Resonator

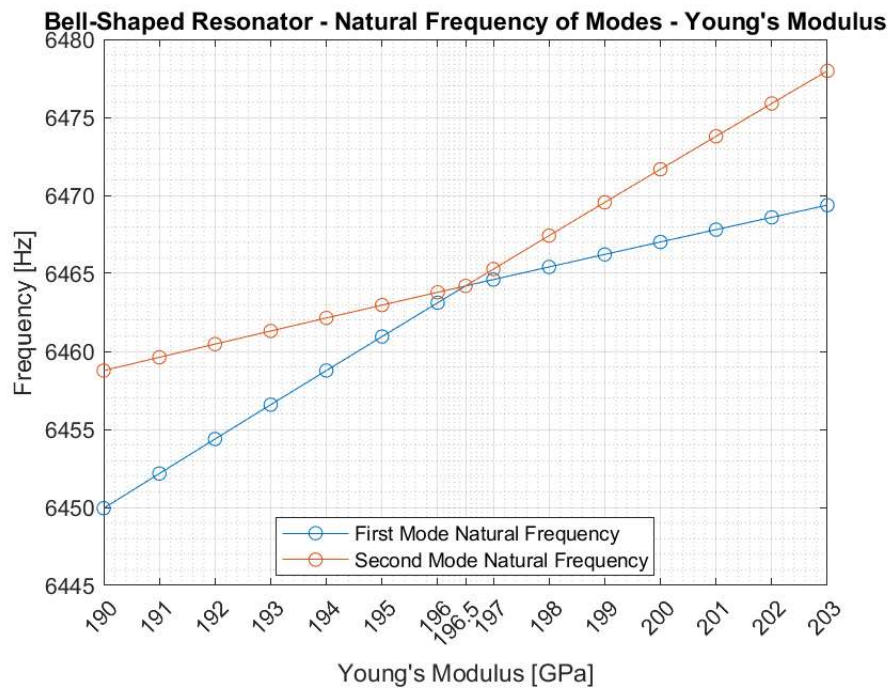


Figure 5.33. Young's Modulus Imperfection Natural Frequency Results of the Bell-Shaped Resonator

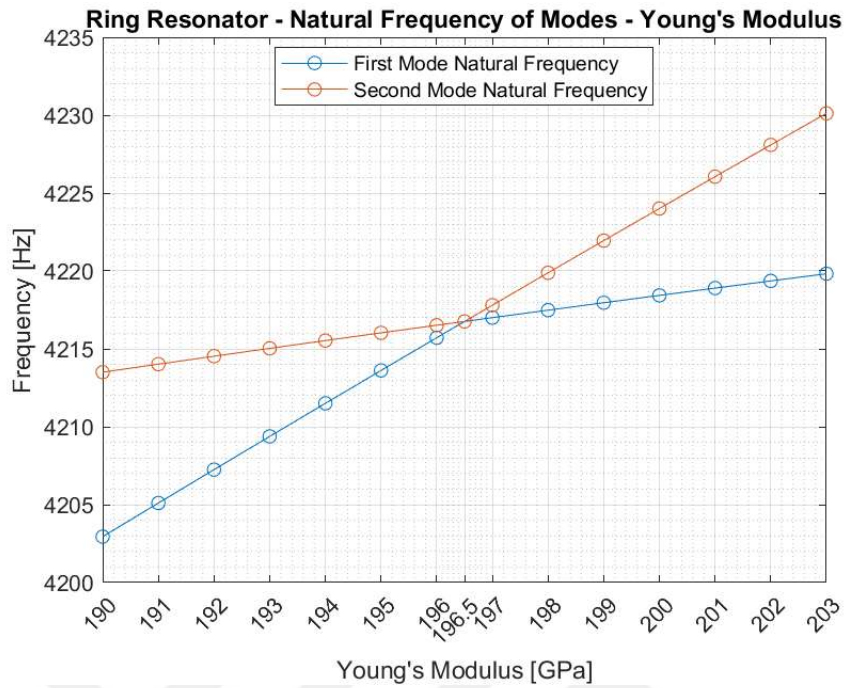


Figure 5.34. Young's Modulus Imperfection Natural Frequency Results of the Ring Resonator

In Figure 5.35, Figure 5.36, and Figure 5.37, the frequency splits of the imperfect resonators depending on the density are given.

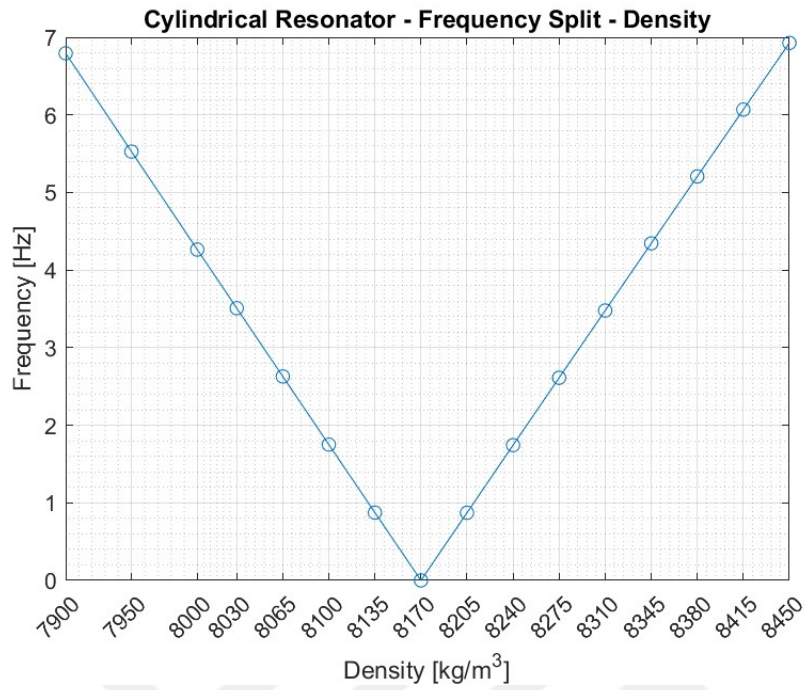


Figure 5.35. Density Imperfection Natural Frequency Split Results of the Cylindrical Resonator

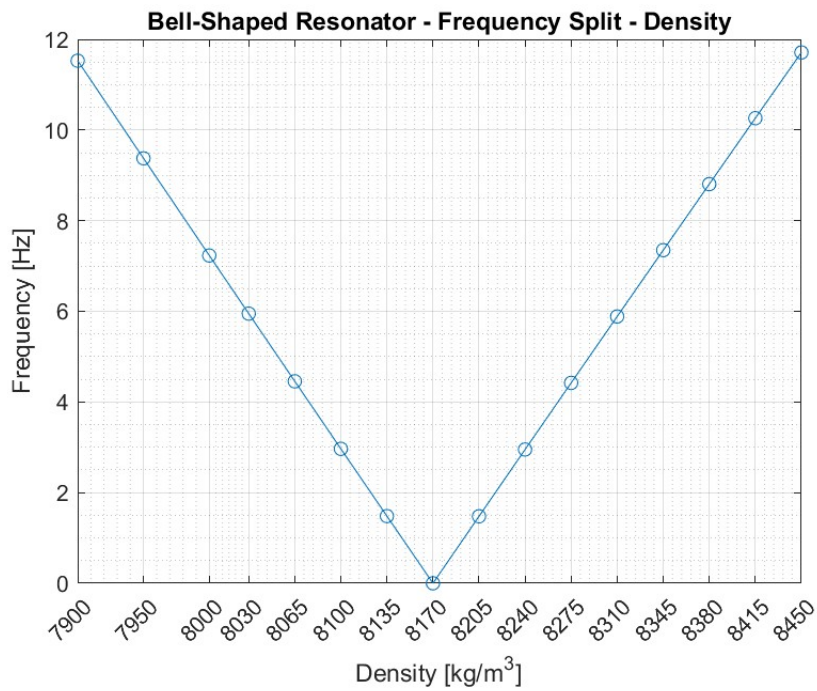


Figure 5.36. Density Imperfection Natural Frequency Split Results of the Bell-Shaped Resonator

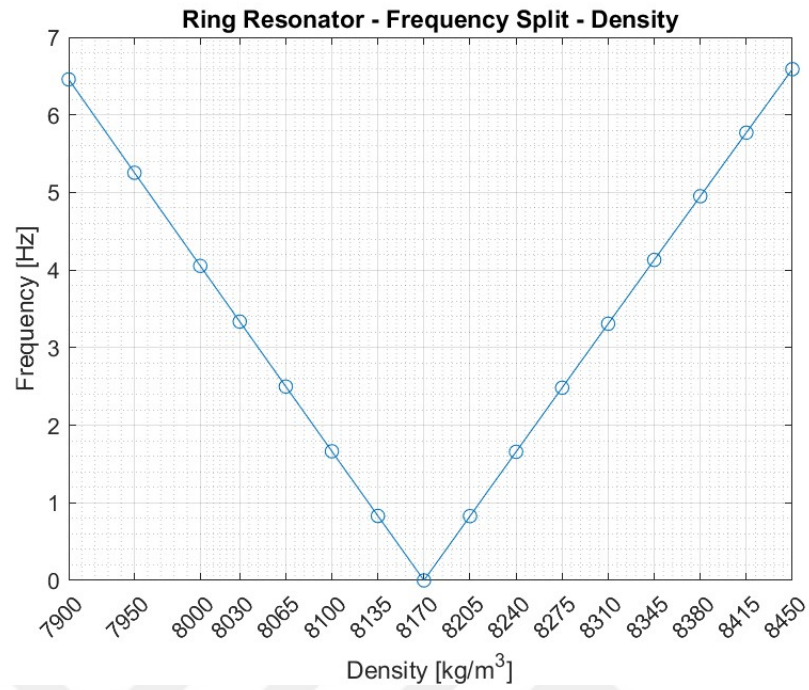


Figure 5.37. Density Imperfection Natural Frequency Split Results of the Ring Resonator

In Figure 5.38, Figure 5.39, and Figure 5.40, the frequency splits of the imperfect resonators depending on Young's Modulus are given.

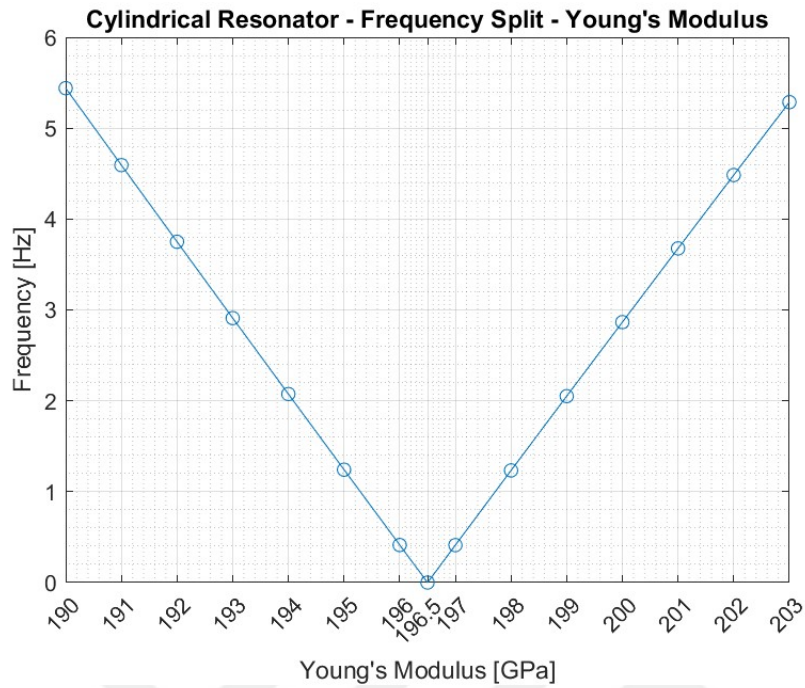


Figure 5.38. Young's Modulus Imperfection Natural Frequency Split Results of the Cylindrical Resonator

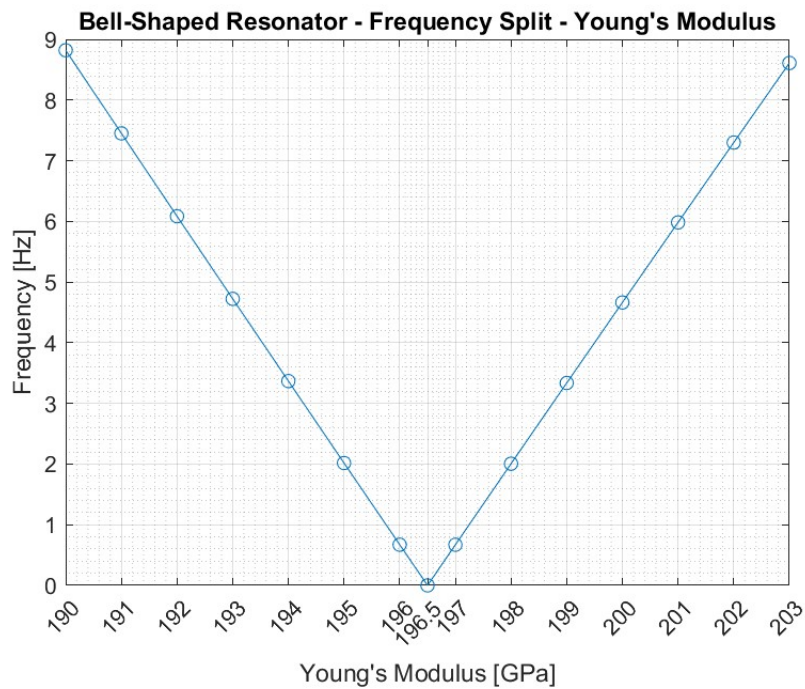


Figure 5.39. Young's Modulus Imperfection Natural Frequency Split Results of the Bell-Shaped Resonator

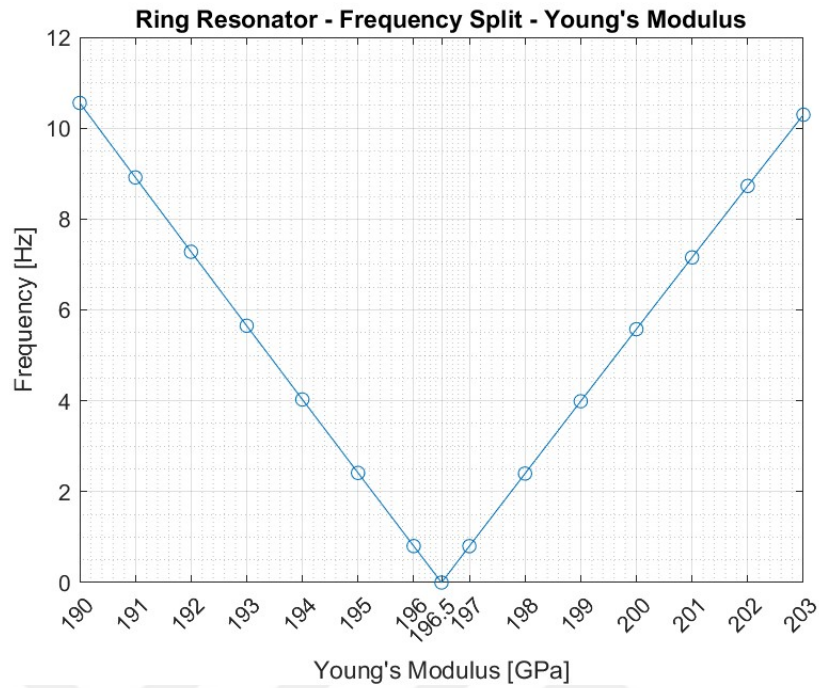


Figure 5.40. Young's Modulus Imperfection Natural Frequency Split Results of the Ring Resonator

In Figure 5.41, Figure 5.42, and Figure 5.43, the measured angular velocities of the imperfect resonators depending on the density in all scenarios are given.

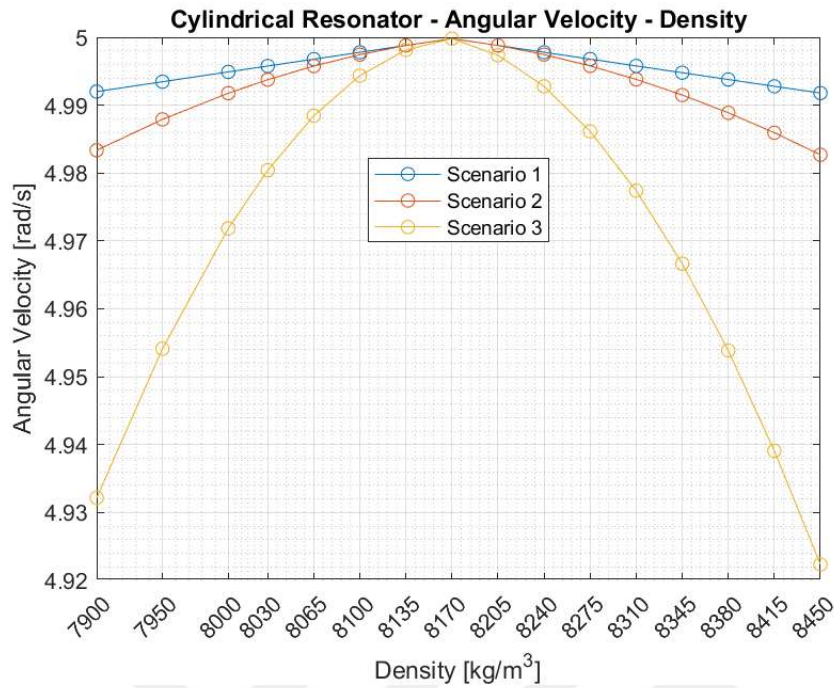


Figure 5.41. Density Imperfection Measured Angular Velocity Results of the Cylindrical Resonator

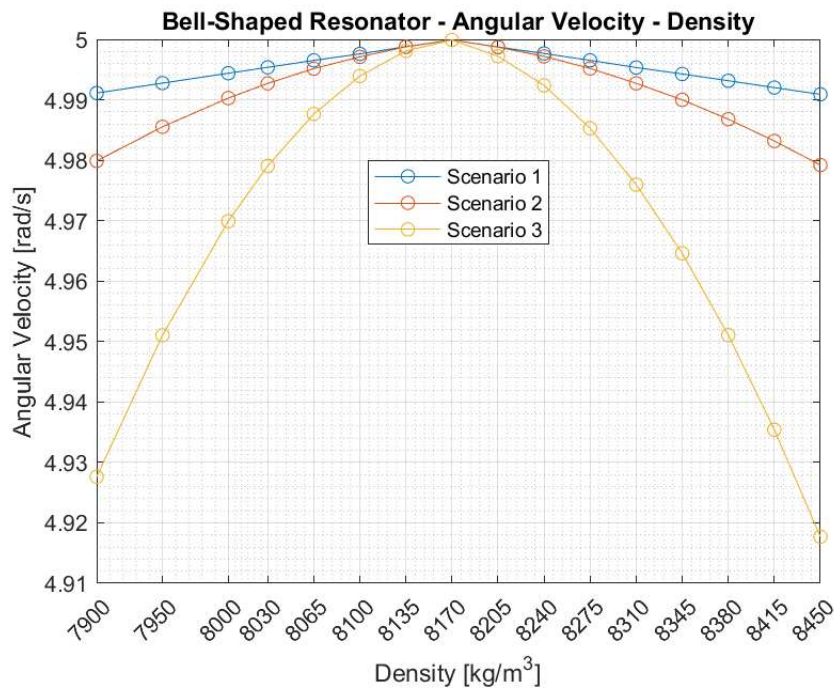


Figure 5.42. Density Imperfection Measured Angular Velocity Results of the Bell-Shaped Resonator

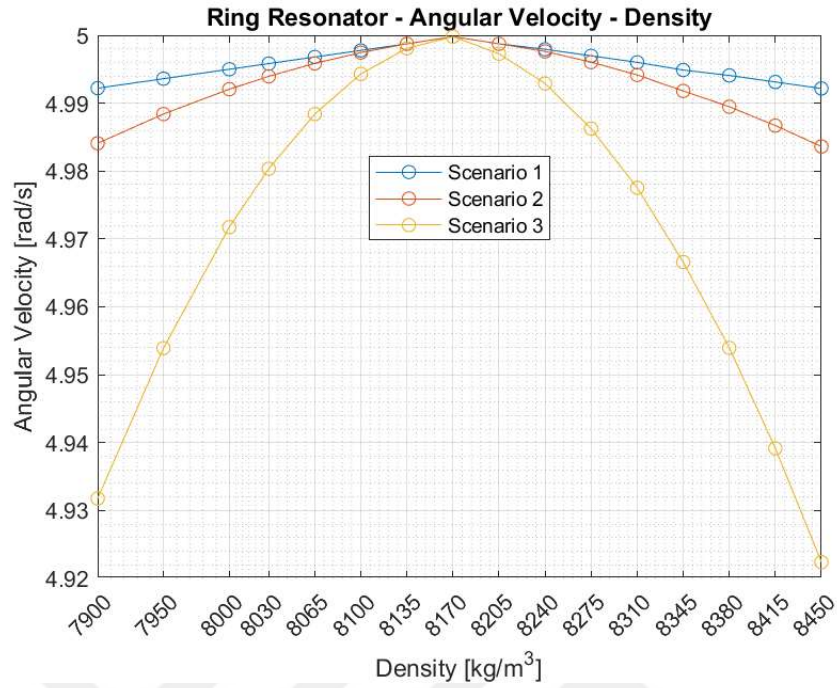


Figure 5.43. Density Imperfection Measured Angular Velocity Results of the Ring Resonator

In Figure 5.44, Figure 5.45, and Figure 5.46, the measured angular velocities of the imperfect resonators depending on Young's Modulus in all scenarios are given.

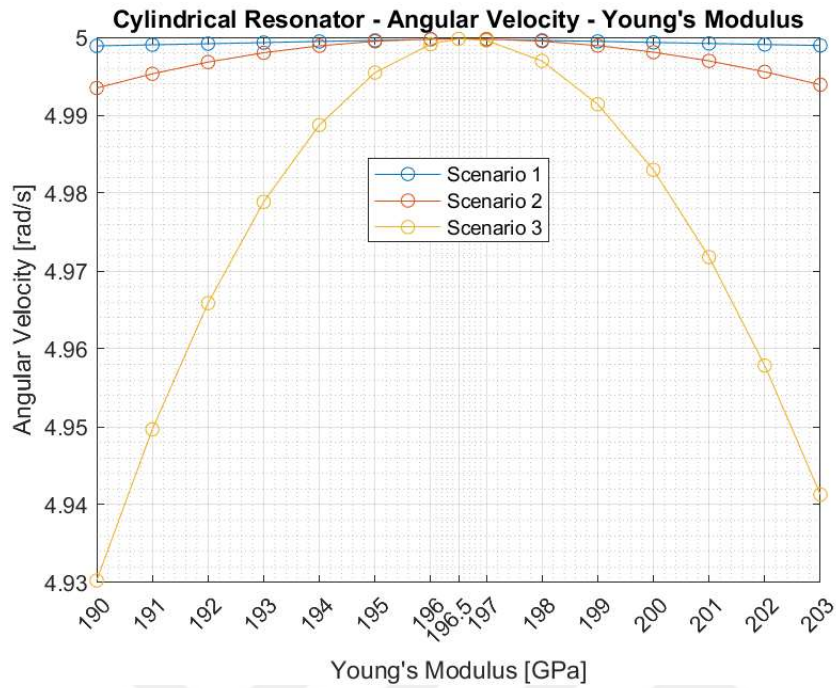


Figure 5.44. Young's Modulus Imperfection Measured Angular Velocity Results of the Cylindrical Resonator

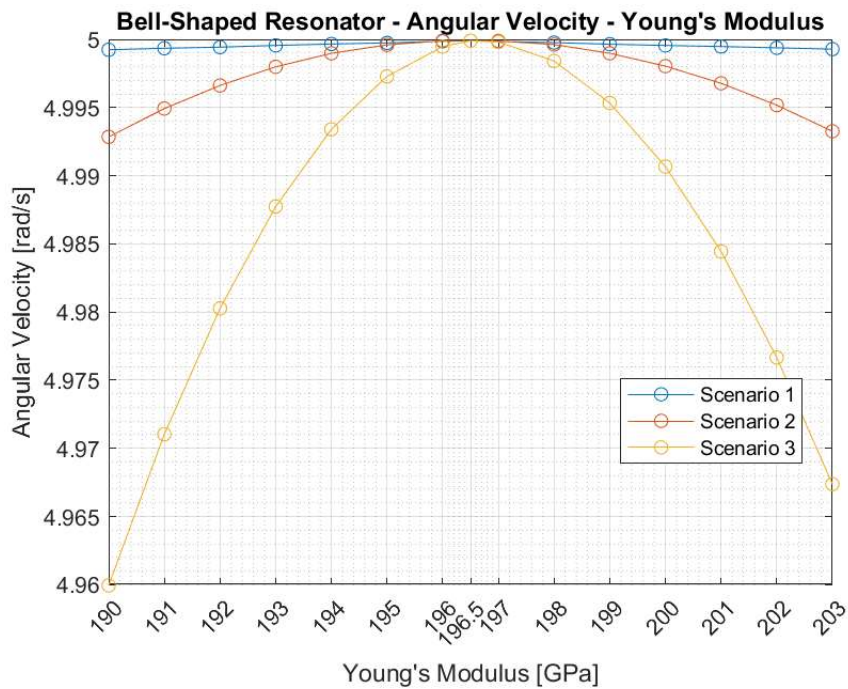


Figure 5.45. Young's Modulus Imperfection Measured Angular Velocity Results of the Bell-Shaped Resonator

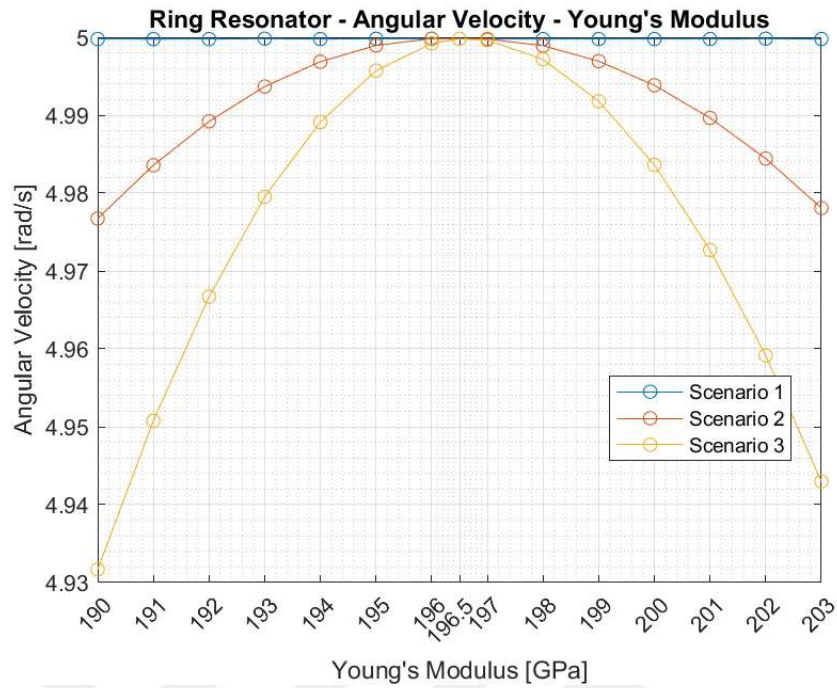


Figure 5.46. Young's Modulus Imperfection Measured Angular Velocity Results of the Ring Resonator

The measured angular velocities for all scenario 3 error analyses in the time domain by each resonator are given. In Figure 5.47, results for the density imperfection for the cylindrical resonator are given with a closer look. In other figures, results are given with only a close look to increase visibility. These results can be seen in Figure 5.48, Figure 5.49, Figure 5.50, Figure 5.51, and Figure 5.52.

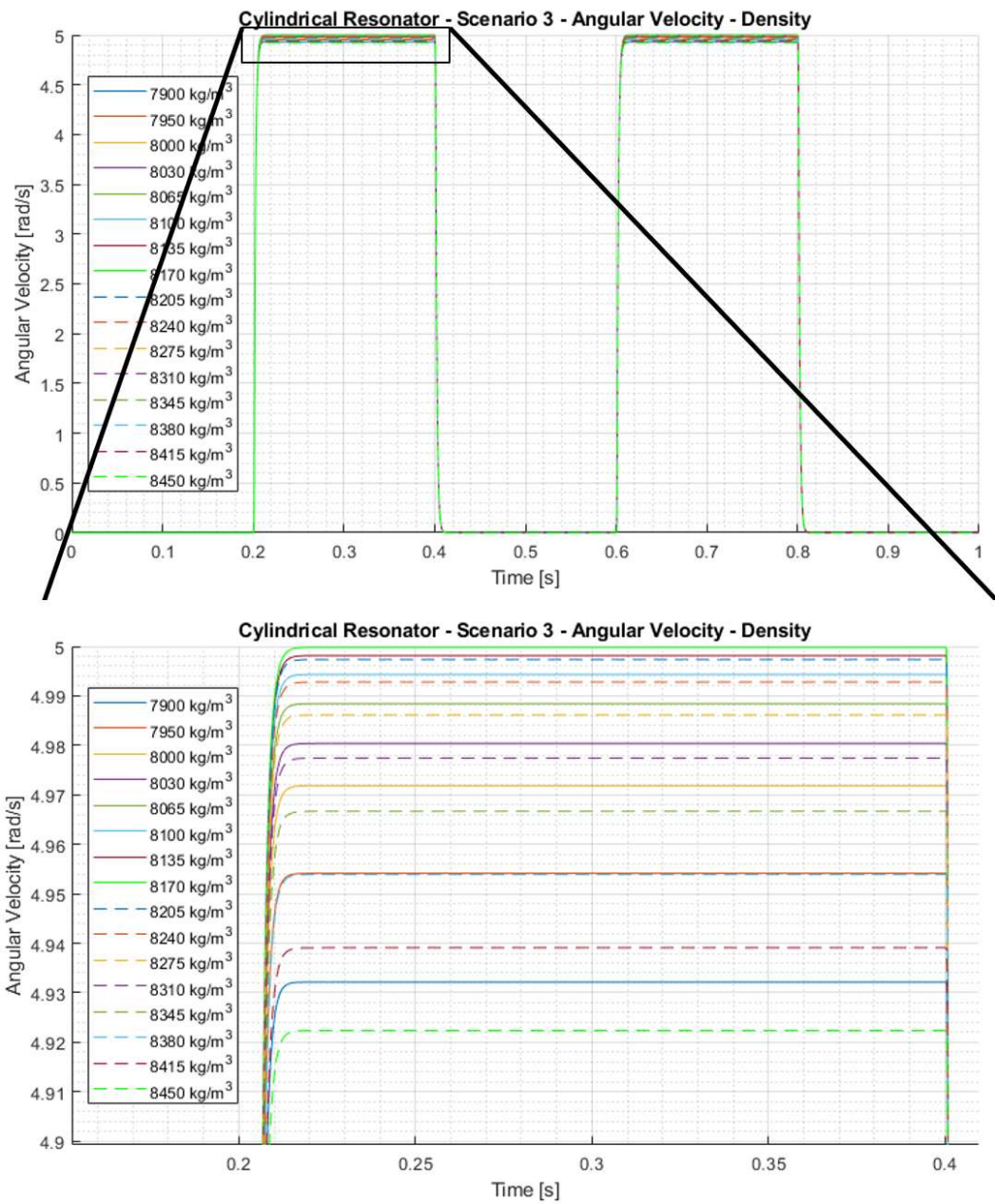


Figure 5.47. Density Imperfection Measured Angular Velocity Results versus Time in Scenario 3 for the Cylindrical Resonator

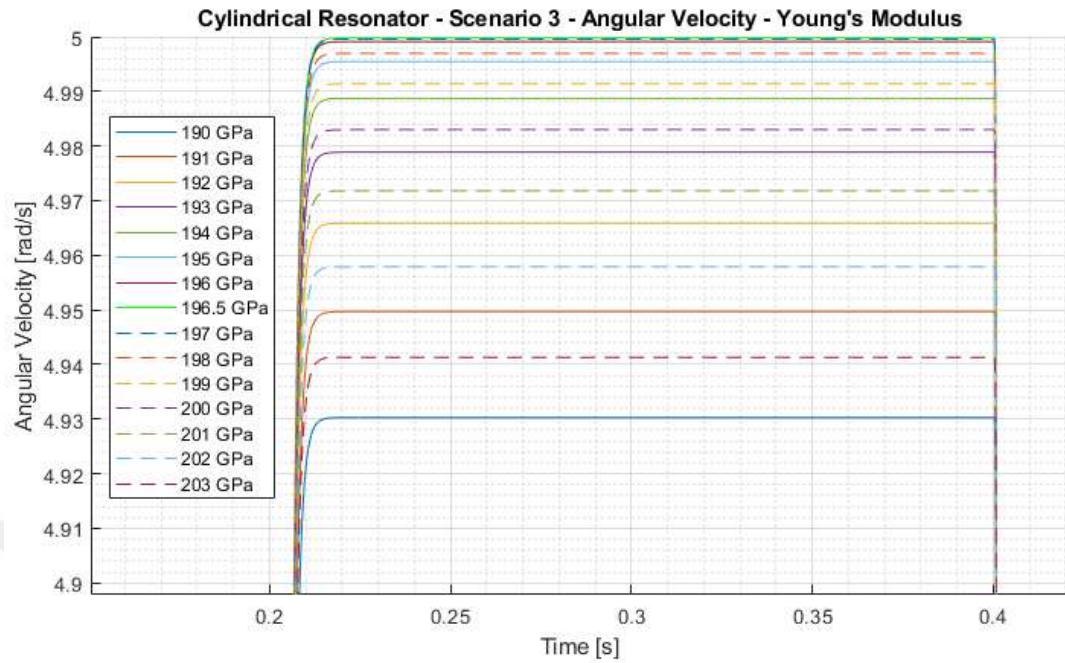


Figure 5.48. Young's Modulus Imperfection Measured Angular Velocity Results versus Time in Scenario 3 for the Cylindrical Resonator

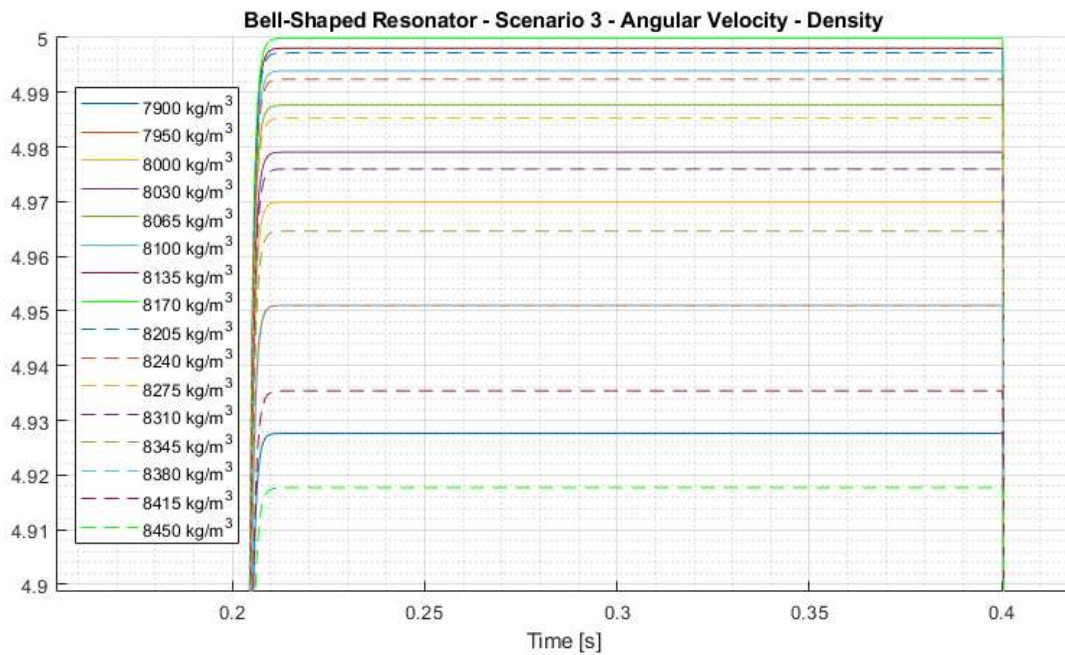


Figure 5.49. Density Imperfection Measured Angular Velocity Results versus Time in Scenario 3 for the Bell-Shaped Resonator

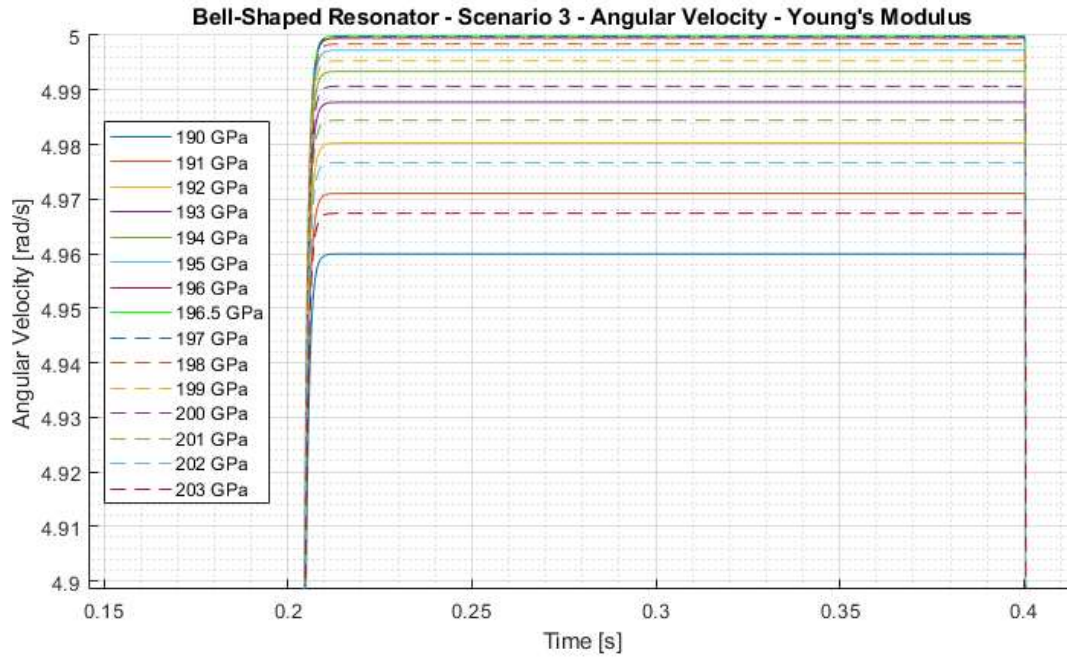


Figure 5.50. Young's Modulus Imperfection Measured Angular Velocity Results versus Time in Scenario 3 for the Bell-Shaped Resonator

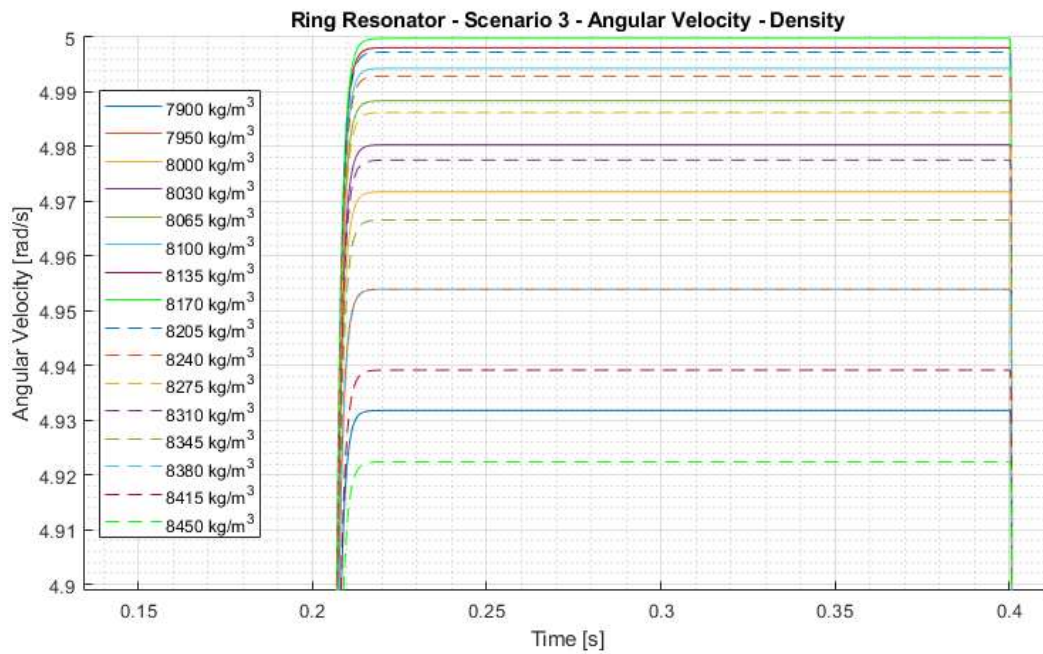


Figure 5.51. Density Imperfection Measured Angular Velocity Results versus Time in Scenario 3 for the Ring Resonator

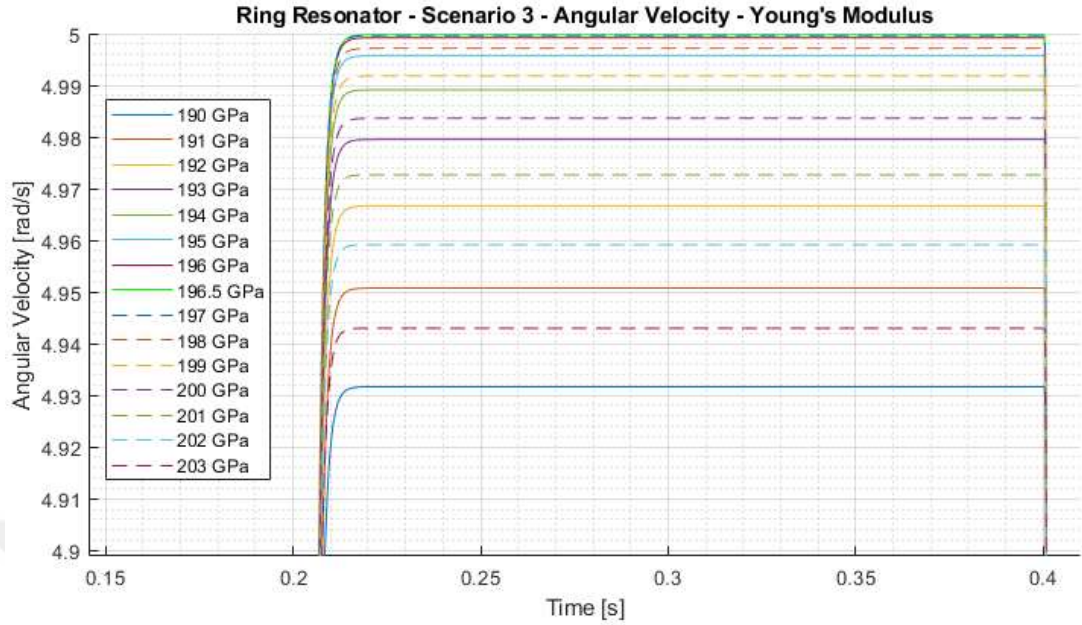


Figure 5.52. Young's Modulus Imperfection Measured Angular Velocity Results versus Time in Scenario 3 for the Ring Resonator

A comparison of the measured angular velocities for scenario 1 and the density change is given in Figure 5.53. A comparison of the measured angular velocities for scenario 1 and Young's Modulus change is given in Figure 5.54. Scenario 2 and the density change comparison of the measured angular velocities is given in Figure 5.55. Scenario 2 and Young's Modulus change comparison of the measured angular velocities is given in Figure 5.56. The measured angular velocities for scenario 3 and the density change in each resonator are given in Figure 5.57. The measured angular velocities for scenario 3 and Young's Modulus change in each resonator are given in Figure 5.58. Additionally, a comparison frequency split when density imperfection occurs at all resonators is given in Figure 5.59. Moreover, a comparison of frequency split when Young's Modulus imperfection occurs at all resonators is given in Figure 5.60.

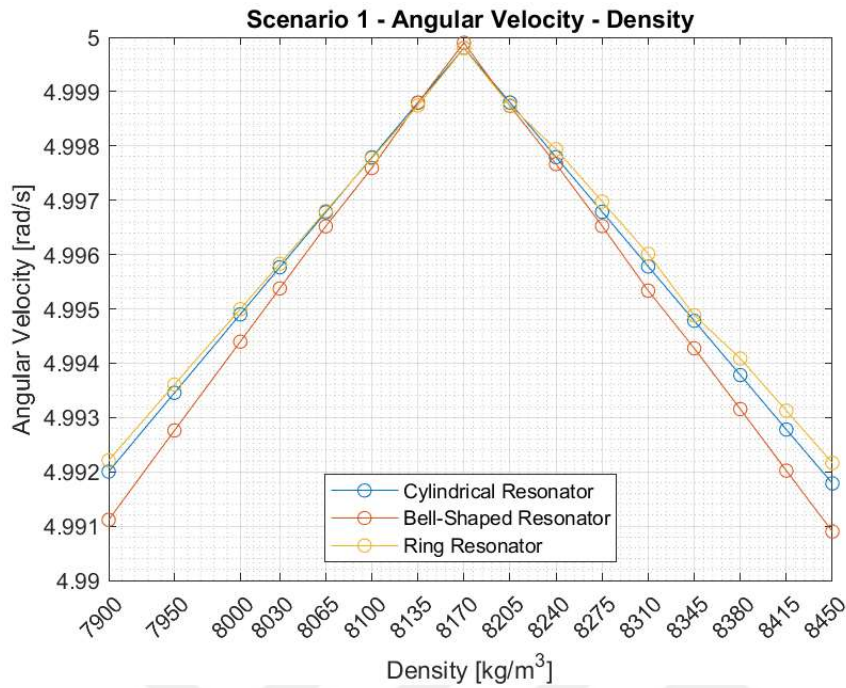


Figure 5.53. Comparison of the Resonators in Scenario 1 Depending on the Density as Measured Angular Velocity

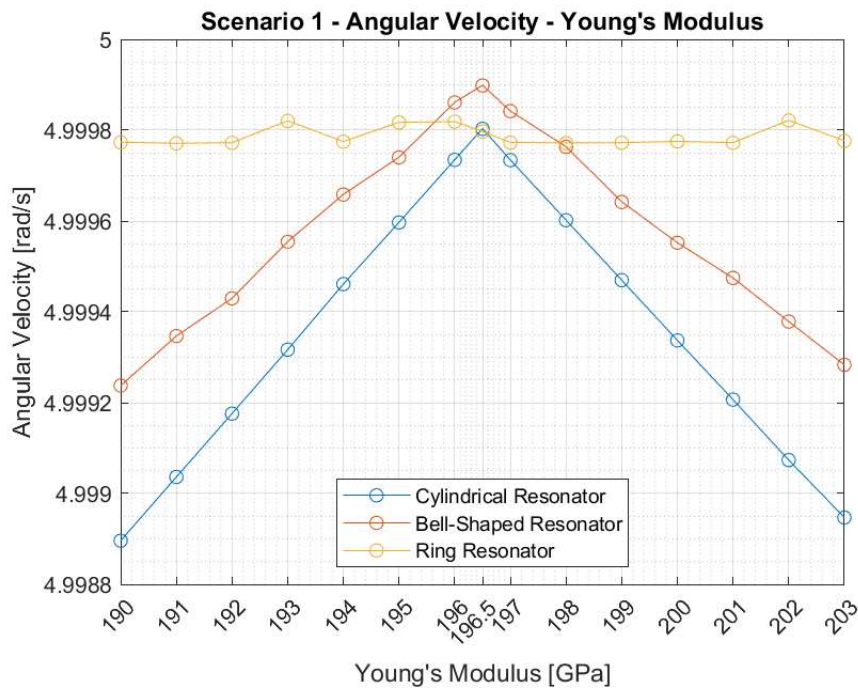


Figure 5.54. Comparison of the Resonators in Scenario 1 Depending on the Young's Modulus as Measured Angular Velocity

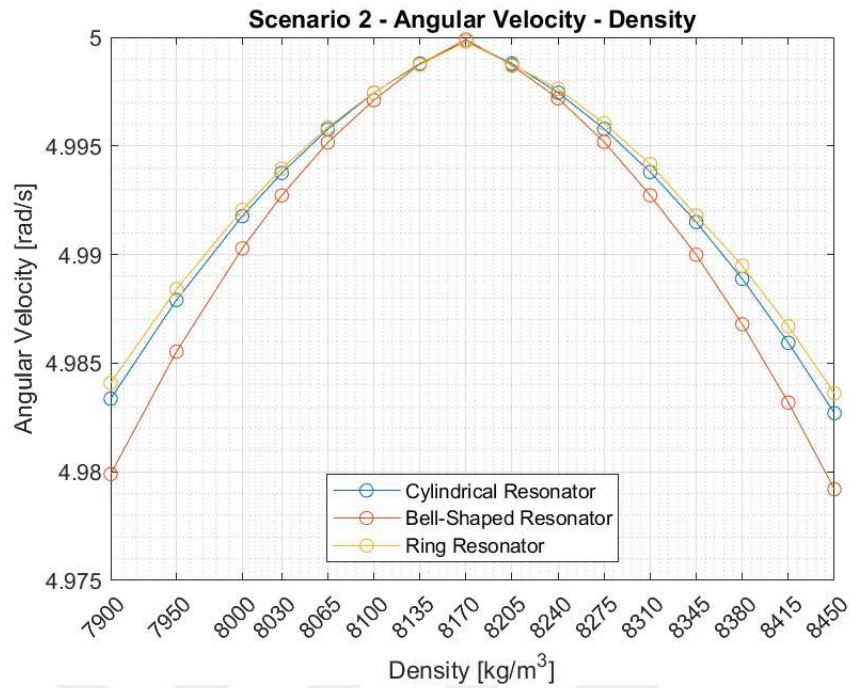


Figure 5.55. Comparison of the Resonators in Scenario 2 Depending on the Density as Measured Angular Velocity

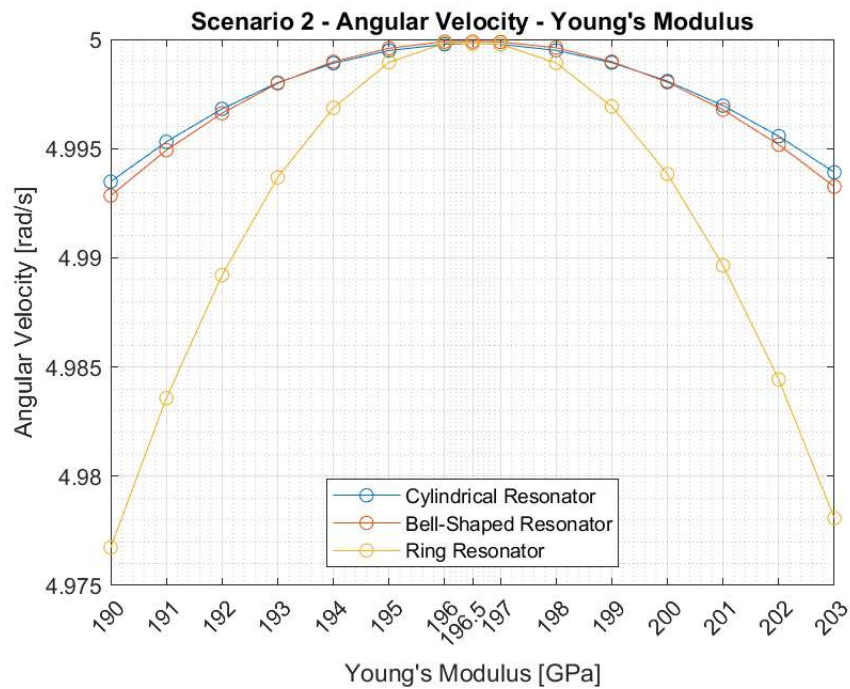


Figure 5.56. Comparison of the Resonators in Scenario 2 Depending on the Young's Modulus as Measured Angular Velocity

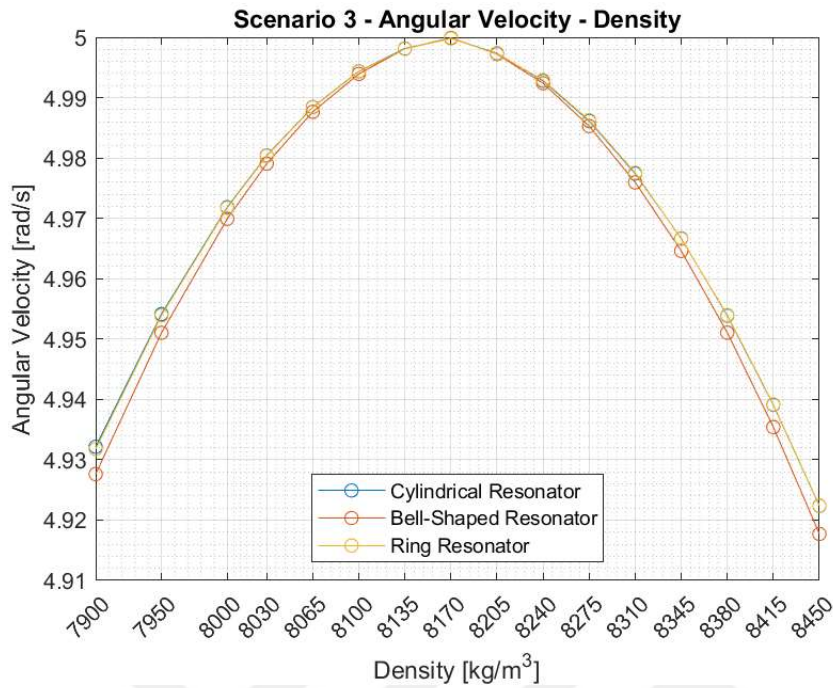


Figure 5.57. Comparison of the Resonators in Scenario 3 Depending on the Density as Measured Angular Velocity

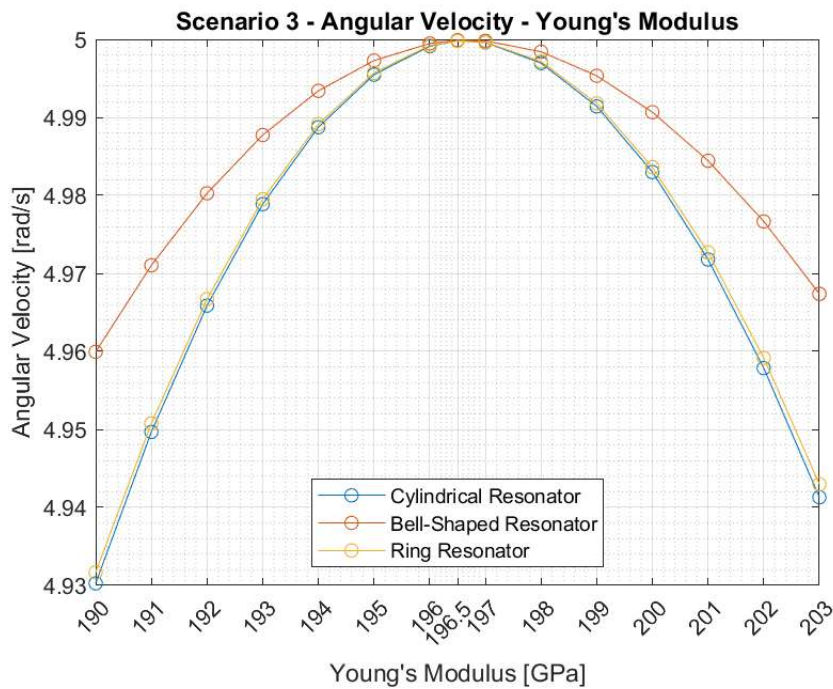


Figure 5.58. Comparison of the Resonators in Scenario 3 Depending on the Young's Modulus as Measured Angular Velocity

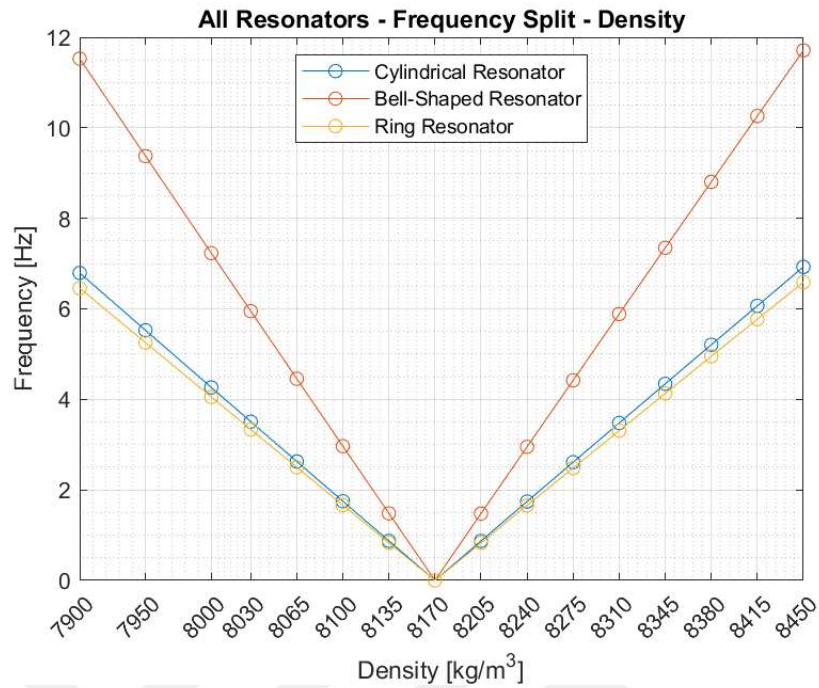


Figure 5.59. Comparison of the Resonators Depending on the Density as Frequency Split

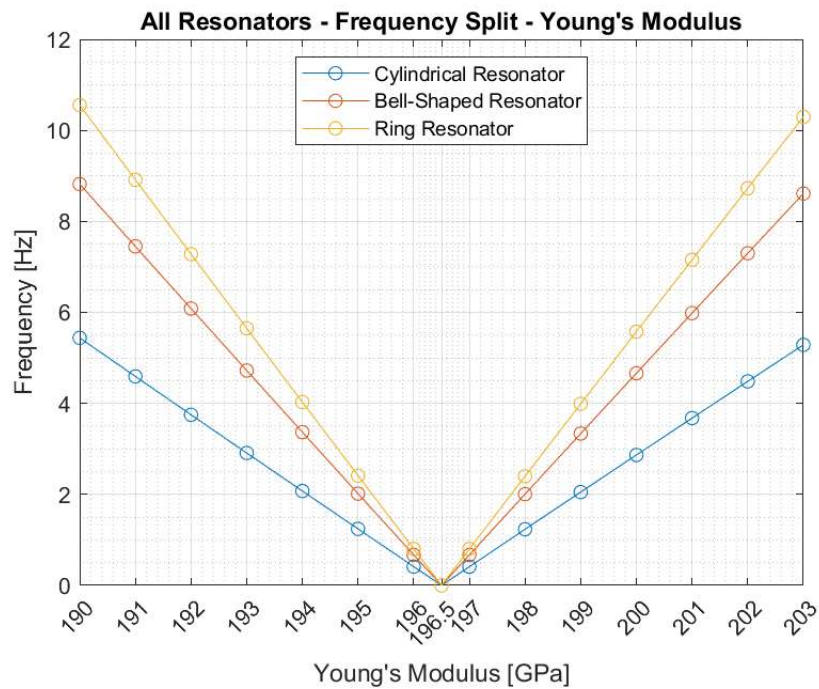


Figure 5.60. Comparison of the Resonators Depending on the Young's Modulus as Frequency Split

5.6 Discussion of the Analysis Results

The results of the analysis have been given above. According to these results, several outcomes can be found.

When imperfection has a higher density than the remaining resonator, the natural frequencies are decreased.

When imperfection has a higher Young's Modulus than the resonator, the natural frequencies are decreased. These results are correct according to engineering intuition.

Additionally, in both cases, one can see that the relationship between density or Young's Modulus and natural frequencies is linear. For example, natural frequencies in Figure 5.29 still, decrease or increase linearly. However, in the default value, for density is 8170 kg/m^3 and for Young's Modulus is 196.50 GPa , first and second natural frequencies switch the order.

The frequency split increases linearly with the higher imperfection both for density and Young's Modulus.

In three scenarios, one can see that Scenario 3 has the most errors in other scenarios which is the toughest among other scenarios. In default values, all resonators have the same 5 rad/s measured angular velocity which implies the scenarios are working perfectly. Additionally, errors are increasing with the increasing frequency split.

However, all resonators have different error values in the scenarios which implies that design is important for the measured angular velocity error. As a distinction in Figure 5.54, the ring resonator has a different behavior in the imperfections. However, one can see that the difference from the actual angular velocity value is very minimal. Therefore, it can be acceptable as a round-off error. Other than this round-off error, all resonators nearly have the same behavior in the scenarios.

As the comparison of scenarios

- The bell-shaped resonator has the worst error behavior in all scenarios related to density error. However, has the nearly best behavior in all scenarios depending on Young's Modulus
- The ring resonator and cylindrical resonator have similar behavior in the scenarios that depend on the density. However, in scenarios that depend on Young's Modulus, in scenario 1 the cylindrical resonator is worse but in scenario 2 the ring resonator has the worst errors. Scenario 2 is more important than scenario 1 since in general the errors are minimal in scenario 1.
- When one looks at the frequency splits of all resonators, one can see that the cylindrical resonator has the nearly best frequency splits in all imperfections.
- Simplicity of the design is important to have minimal errors in manufacturing. From this perspective, the cylindrical resonator is the best among the other resonators.

Since the comparison of the results has been done. The cylindrical resonator can be selected as the best resonator among the others for design purposes with the reasonings of the above comments.

CHAPTER 6

DESIGN PARAMETERS AND IMPROVED DESIGN RESONATOR

This chapter mainly discusses the design criteria and design parameters of an improved cylindrical resonator and a new proposed resonator. Throughout the thesis, several design criteria have been found. Additionally, the best performing resonator under different errors is found to be the cylindrical resonator. According to this knowledge, an improved cylindrical resonator has been designed.

6.1 Design Criteria

In this thesis, the sensor's working principles, the resonator's model, and different resonators have been discussed. Based on this research several design requirements have been found.

- The most important criterion is the operating modes have the same frequency.
- Secondly, the assumption made in Section 0 which is the natural frequency is much bigger than the measured angular velocity. Therefore, natural frequency should be as high as possible.
- Thirdly, the natural frequency of the operating modes must have a large enough difference from other modes [8]. For example, in the research of Su et al. [15] this difference was taken as 1,500 Hz.
- Fourthly and lastly, there is a natural frequency limitation due to the electronics in the sensor. Liu et al. [8] limited the natural frequency between 5-10 kHz. Rozelle [9] limited the natural frequency between 4-8 kHz due to the high-frequency AC pickoff component in the electrical circuit.

Based on the above requirements, the new improved design's requirements are determined as below:

- Operating modes must have the same frequency.
- The natural frequency of the operating modes must be as high as possible. At a minimum, there is a three-order-of-magnitude difference between the measured angular velocity and the natural frequency of the operating modes.
- The natural frequency of the operating modes must be in the range of 5-8 kHz due to the electronics in the sensor.
- The difference in the natural frequency of the operating modes with other modes must be a minimum of 1,500 Hz. This criterion is based on the research of Su et al. [15].
- The main diameter of the resonator will be the same due to industry needs.

6.2 Design Parameters

Several design parameters are selected one can see the parameters on the resonator in Figure 6.1. Some of the parameters are important for the operating modes and others are important for the closest mode. Important criteria for the operating mode:

- Height of the Lower Part which is $h, lower$
- Outer Radius of the Upper Part which is $R, o, upper$

The important criteria for the closest mode:

- Thickness of the Base which is $t, base$
- Outer Radius of the Lower Part which is $R, o, lower$

There are some criteria that are important for both modes:

- Height of the Upper Part which is $h, upper$
- Inner Radius of the Resonator which is R, i

The dimensions of these sensitive parts are analyzed using different dimensions. The default values of the parameters are given in Table 6.1. Results can be seen in the following section.

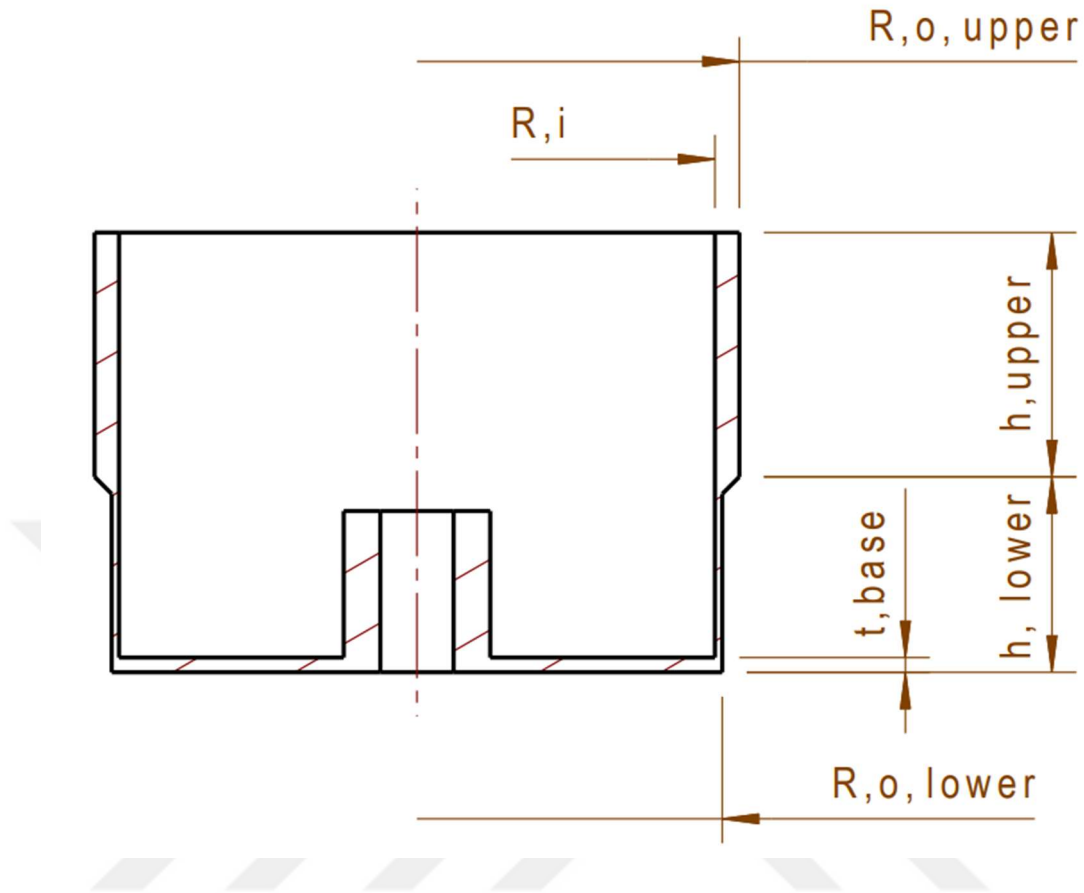


Figure 6.1. Design Parameters of the Cylindrical Resonator

Table 6.1. Default Values of the Design Parameters

	$h, lower$	$R, o, upper$	$t, base$	$R, o, lower$	$h, upper$	R, i
Default Values	8 mm	13.2 mm	0.6 mm	12.5 mm	10 mm	12.2 mm

6.2.1 Height of the Lower Part

The height of the lower part of the resonator has been analyzed using seven different values. In Figure 6.2, it is shown that the natural frequency of operating modes decreases with a longer height. The natural frequency of the closest mode decreases as well with a longer height. However, the decrease in operating modes is higher than the closest mode.

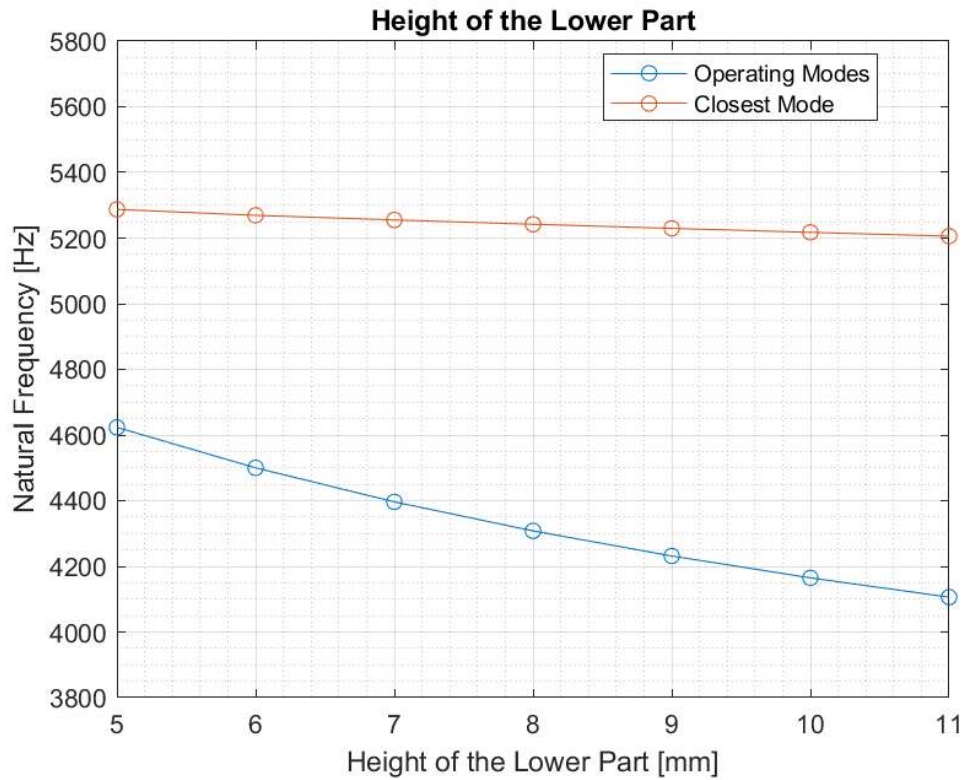


Figure 6.2. Natural Frequency of the Operating Modes and Closest Mode by the Height of the Lower Part

6.2.2 Outer Radius of the Upper Part

The outer radius of the upper part of the resonator has been analyzed using seven different values. In Figure 6.3, the natural frequency of operating modes increases with a longer radius. On the other hand, the natural frequency of the closest mode decreases with a longer radius. The increase in the natural frequency of operating modes is significantly higher than the decrease in the natural frequency of the closest mode.

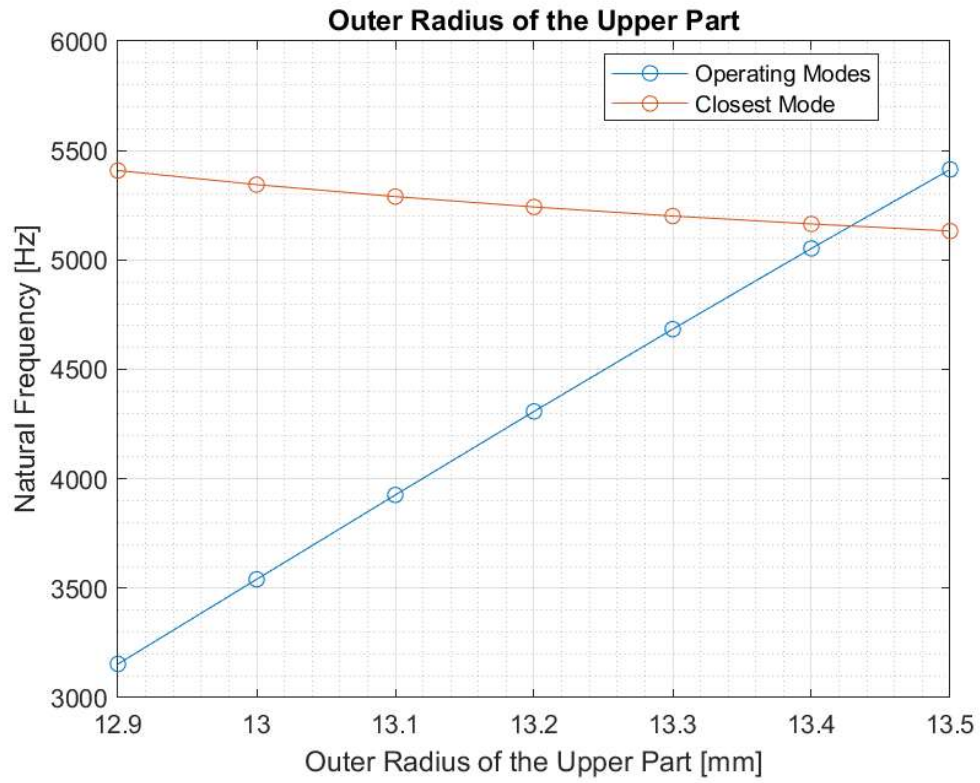


Figure 6.3. Natural Frequency of the Operating Modes and Closest Mode by the Outer Radius of the Upper Part

6.2.3 Thickness of Basement

The thickness of the basement of the resonator has been analyzed using nine different values. In Figure 6.4, the natural frequency of operating modes is nearly the same in all thicknesses. However, the natural frequency of the closest mode is higher with the thicker thicknesses.

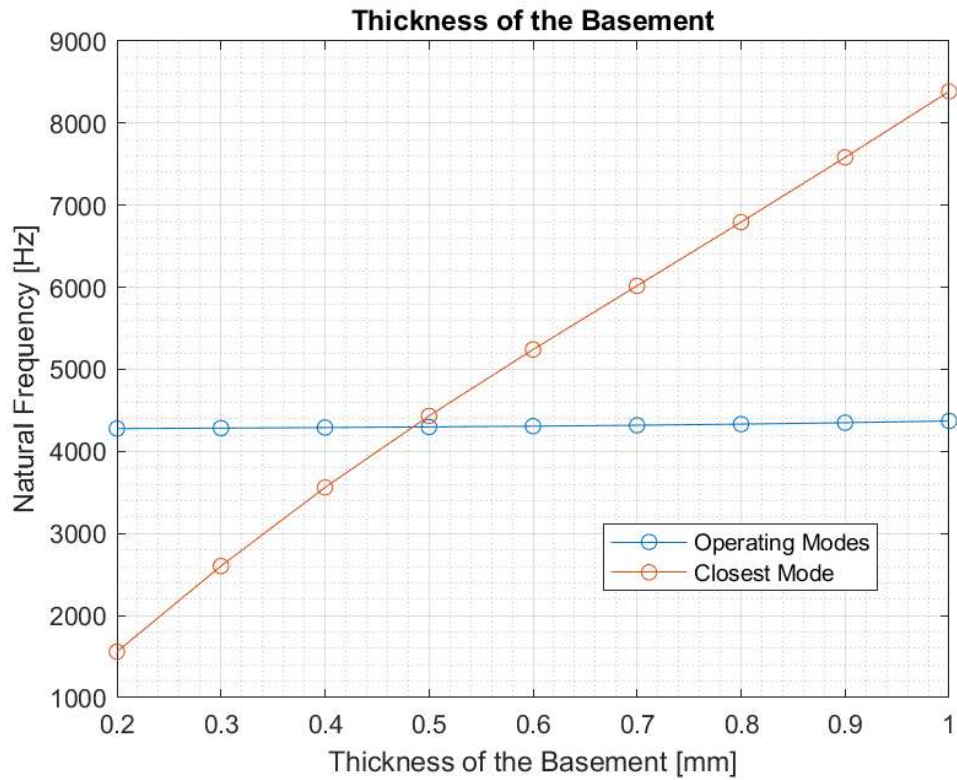


Figure 6.4. Natural Frequency of the Operating Modes and Closest Mode by the Thickness of the Basement

6.2.4 Outer Radius of the Lower Part

The outer radius of the lower part of the resonator has been analyzed using seven different values. In Figure 6.5, the natural frequency of operating modes increases mildly with a longer radius, and an appreciable increase begins after the 12.6 mm radius. The natural frequency of the closest mode increases with a longer radius.

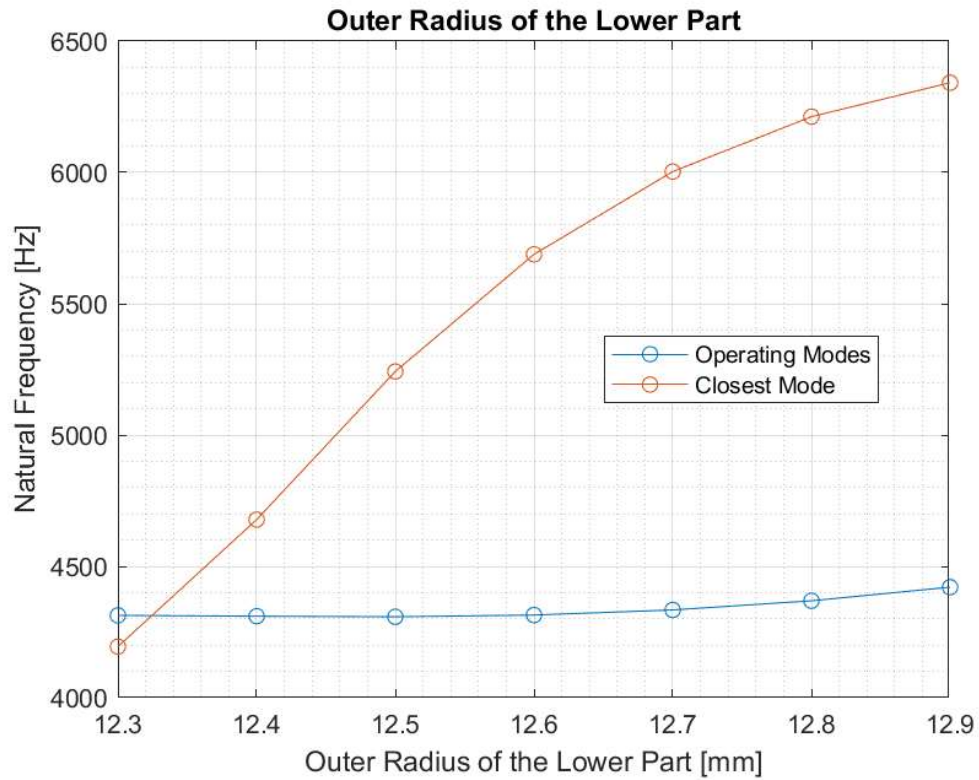


Figure 6.5. Natural Frequency of the Operating Modes and Closest Mode by the Outer Radius of the Lower Part

6.2.5 Height of the Upper Part

The height of the upper part of the resonator has been analyzed using seven different values. In Figure 6.6, the natural frequency of the operating modes continues to decrease with a longer height in the upper part. Also, the natural frequency of the closest mode decreases with a longer height. The decrease in the natural frequency of the closest mode is larger than the natural frequency of the operating modes.

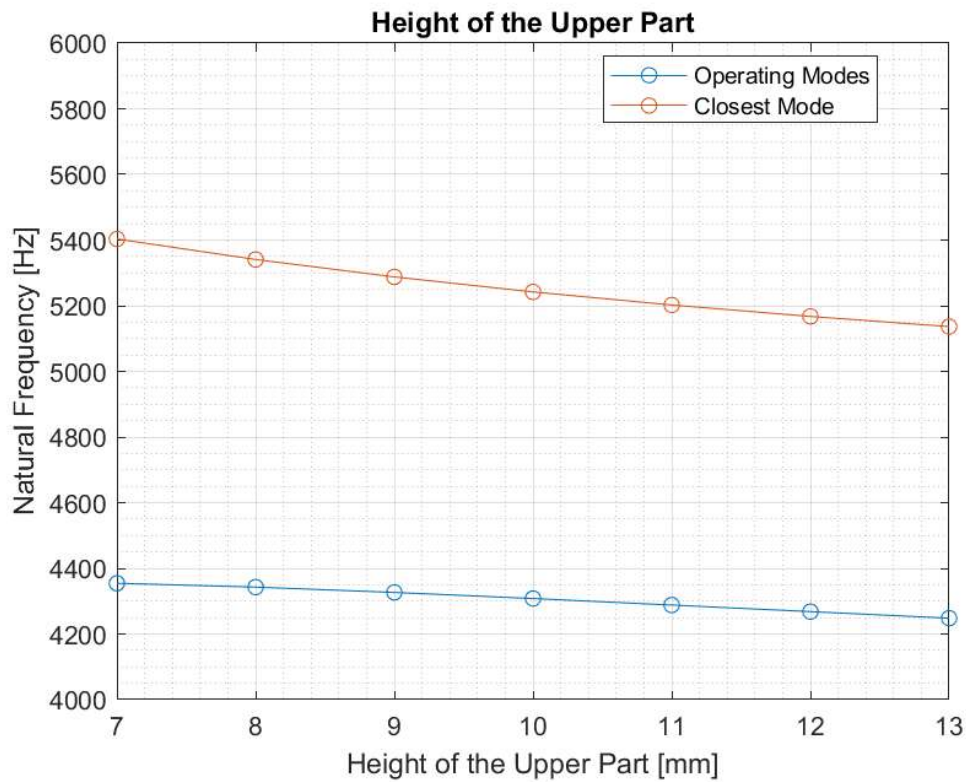


Figure 6.6. Natural Frequency of the Operating Modes and Closest Mode by the Height of the Upper Part

6.2.6 Inner Radius of the Resonator

The inner radius of the resonator has been analyzed using six different values. In Figure 6.7, the natural frequency of the operating modes decreases with a longer radius. Also, the natural frequency of the closest mode decreases with a longer radius. The decrease in the natural frequency of the closest mode is nearly the same as the natural frequency of the operating modes.

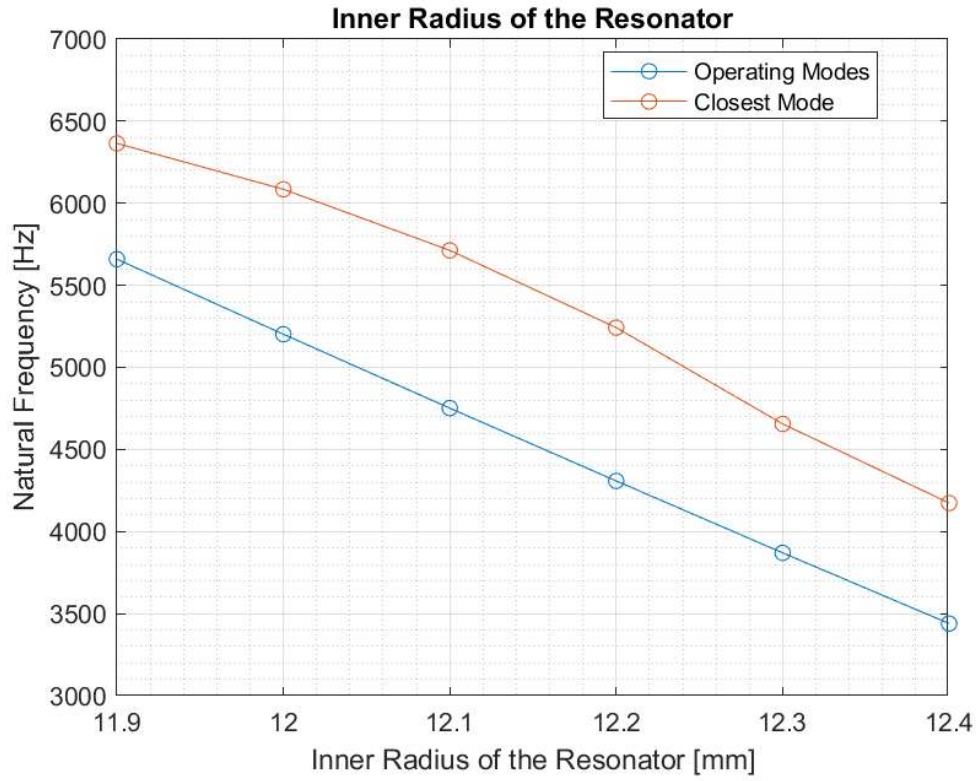


Figure 6.7. Natural Frequency of the Operating Modes and Closest Mode by the Inner Radius of the Resonator

6.3 Finding Improved Resonator Design

To improve the performance of the cylindrical resonator different design options are generated using the information from the isolated parameter sweeps presented in the previous section with the regression analysis. Regression analysis was done with the Minitab software. The regression equation for the natural frequency of the operation mode can be seen below.

$$\omega_{1,2} = 7262.0 - 18.1 * h_{upper} - 4441.5 * R_{i} + 3772.1 * R_{o,upper} + 179.5 * R_{o,lower} + 111.7 * t_{base} - 85.2 * h_{lower} \quad (6.1)$$

The r-sq of this regression equation is 99.84%. The same regression analysis has been done for the closest mode too. The regression equation for the natural frequency of the closest mode can be seen below.

$$\omega_{closest} = 16729.0 - 44.0 * h_{upper} - 4431.0 * R_i - 456.0 * R_{o,upper} + 3524.0 * R_{o,lower} + 8384.0 * t_{base} - 13.4 * h_{lower} \quad (6.2)$$

The r-sq of this regression equation is 99.06%. Different design options are found with this method. Additionally, modal analyses have been done for these design options. The comparison of the results of the modal analysis for different design options can be seen in Table 6.2.

Table 6.2. The Regression and Modal Analysis Results of the New Designs

Design Number	Natural Frequency Result of the Operating Modes [Hz]		Natural Frequency Result of the Closest Mode [Hz]	
	The Regression Analysis	The Modal Analysis	The Regression Analysis	The Modal Analysis
Design 1	6614.77	6643.50	7169.70	7302.80
Design 2	6637.12	6663.10	8846.50	9098.20
Design 3	6648.30	6673.60	9684.80	9909.90
Design 4	5143.56	5084.80	4788.40	4773.40
Design 5	5058.32	5028.10	4775.00	4791.80
Design 6	6817.48	7400.20	10520.30	10580.00
Design 7	6763.65	7160.80	9463.00	9903.50
Design 8	6319.49	6674.40	9019.90	9448.20
Design 9	6283.60	6575.00	8315.00	8436.80
Design 10	6727.75	7055.40	8758.20	9049.90
Design 11	5958.01	5925.10	5980.20	6026.20
Design 12	6539.02	6566.80	8995.80	9230.90
Design 13	6709.51	6946.30	9022.50	9339.70
Design 14	6763.34	7076.40	10079.80	10252.00
Design 15	7261.63	7898.90	10963.40	10808.00

The used design parameters in the designs are given in Table 6.3.

Table 6.3. Used Design Parameters of the New Designs

	$h, lower$	$R, o, upper$	$t, base$	$R, o, lower$	$h, upper$	R, i
Design 1	5 mm	13.4 mm	0.7 mm	12.5 mm	13 mm	11.9 mm
Design 2	5 mm	13.4 mm	0.9 mm	12.5 mm	13 mm	11.9 mm
Design 3	5 mm	13.4 mm	1.0 mm	12.5 mm	13 mm	11.9 mm
Design 4	8 mm	13.3 mm	0.5 mm	12.5 mm	9 mm	12.1 mm
Design 5	9 mm	13.3 mm	0.5 mm	12.5 mm	9 mm	12.1 mm
Design 6	5 mm	13.4 mm	0.9 mm	12.9 mm	7 mm	11.9 mm
Design 7	5 mm	13.4 mm	0.9 mm	12.6 mm	7 mm	11.9 mm
Design 8	5 mm	13.4 mm	0.9 mm	12.6 mm	7 mm	12.0 mm
Design 9	5 mm	13.4 mm	0.9 mm	12.4 mm	7 mm	12.0 mm
Design 10	5 mm	13.4 mm	0.9 mm	12.4 mm	7 mm	11.9 mm
Design 11	8 mm	13.4 mm	0.6 mm	12.5 mm	10 mm	12.0 mm
Design 12	7 mm	13.4 mm	0.9 mm	12.5 mm	9 mm	11.9 mm
Design 13	5 mm	13.4 mm	0.9 mm	12.5 mm	9 mm	11.9 mm
Design 14	5 mm	13.4 mm	0.9 mm	12.8 mm	9 mm	11.9 mm
Design 15	5 mm	13.4 mm	0.9 mm	12.9 mm	7 mm	11.8 mm

One can see that the regression analysis can predict the modal analysis result. Moreover, the modal analysis results of the new design options are added to the regression analysis to increase the accuracy of the regression analysis. The new regression equation for the natural frequency of the operation mode can be seen below.

$$\omega_{1,2} = 5920.0 - 61.2 * h, upper - 4653.0 * R, i + 3799.0 * R, o, upper + 513.6 * R, o, lower + 224.9 * t, base - 116.9 * h, lower \quad (6.3)$$

The r-sq of this regression equation is 99.67%. The same regression analysis has been done for the closest mode too. The new regression equation for the natural frequency of the closest mode can be seen below.

$$\omega_{closest} = 19717.0 - 49.5 * h_{upper} - 4584.0 * R_{i-290.0 * R_{o,upper}} \quad (6.4) \\ + 3273.0 * R_{o,lower} + 8542.0 * t_{base} - 38.1 * h_{lower}$$

The r-sq of this regression equation is 99.65%. The results can be seen below.

Table 6.4. The Improved Regression and Modal Analysis Results of the New Designs

Design Number	Natural Frequency Result of the Operating Modes [Hz]		Natural Frequency Result of the Closest Mode [Hz]	
	The Improved Regression Analysis	The Modal Analysis	The Improved Regression Analysis	The Modal Analysis
Design 1	6650.40	6643.50	7332.80	7302.80
Design 2	6695.38	6663.10	9041.20	9098.20
Design 3	6717.87	6673.60	9895.40	9909.90
Design 4	5190.91	5084.80	4820.20	4773.40
Design 5	5074.01	5028.10	4782.10	4791.80
Design 6	7270.90	7400.20	10647.10	10580.00
Design 7	7116.83	7160.80	9665.40	9903.50
Design 8	6651.53	6674.40	9206.90	9448.20
Design 9	6548.82	6575.00	8552.40	8436.80
Design 10	7014.12	7055.40	9010.90	9049.90
Design 11	5996.95	5925.10	6054.30	6026.20
Design 12	6708.31	6566.80	9162.90	9230.90
Design 13	6942.11	6946.30	9239.10	9339.70
Design 14	7096.18	7076.40	10220.90	10252.00
Design 15	7736.20	7898.90	11105.50	10808.00

Designs 6, 7, 14, and 15 are selected as better designs since their natural frequency of operating mode is higher than 7,000 Hz. Additionally, their closest mode is far enough to meet the design criteria. These four designs are compared in scenario 3 and with the “7900 kg/m^3 ” imperfection. The results are given below.

Table 6.5. The Comparison of the Best Designs

Design Name	First Nat. Freq.	Second Nat. Freq.	Freq. Split	Measured Ang. Vel.	Nat. Freq. when Perfect
Design 6	7408.40	7419.80	11.40	4.9391	7400.2
Design 7	7169.00	7180.40	11.40	4.9356	7160.8
Design 14	7084.70	7096.00	11.30	4.9338	7076.4
Design 15	7907.70	7919.90	12.20	4.9393	7898.9

Design 6 and Design 15 are the best designs among the others. Therefore, the comparison of the designs continued for selected designs. Design 6 and Design 15 are compared in Scenario 3 for all imperfections. In Figure 6.8 and Figure 6.9, Design 6 and Design 15 are compared in terms of frequency split and measured angular velocity in scenario 3. Also, the default cylindrical resonator has been added to the comparisons.

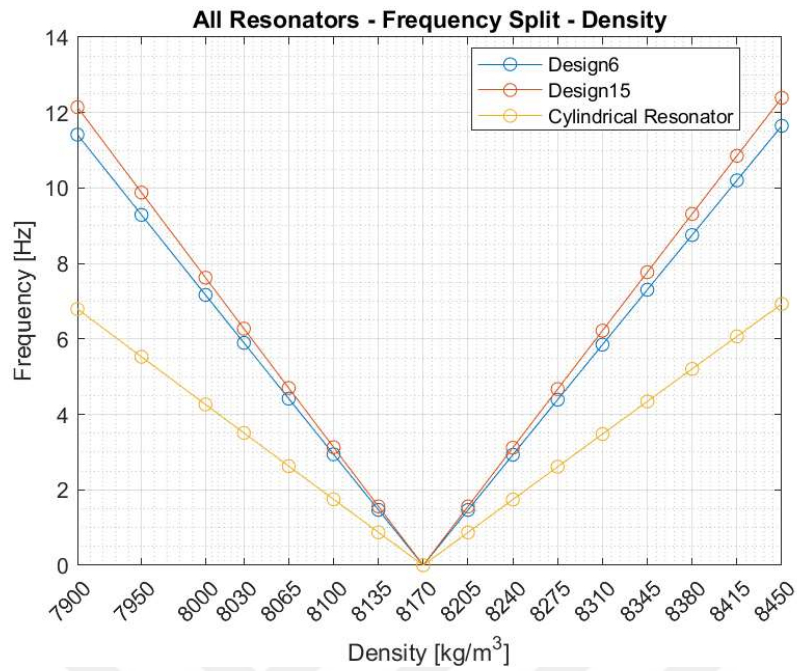


Figure 6.8. Design 6, Design 15, and Default Cylindrical Resonator Comparison in terms of Natural Frequency Split in Density Imperfection

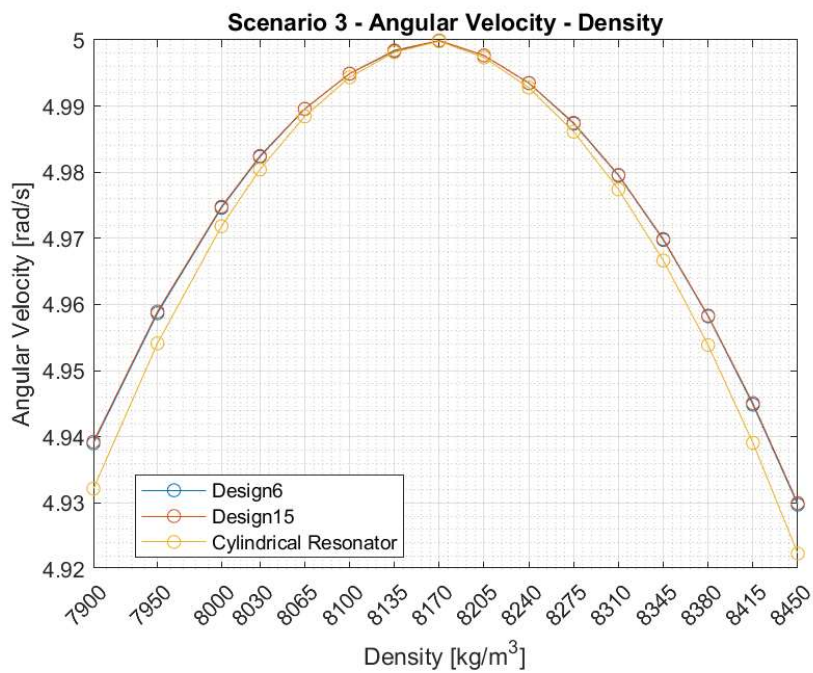


Figure 6.9. Design 6, Design 15, and Default Cylindrical Resonator Comparison in terms of Measured Angular Velocity in Scenario 3 and Density Imperfection

In terms of frequency split, the design 6 is better. In terms of performance, measured angular velocities are nearly the same. Therefore, design 6 was selected as the new better design.

6.4 The Improved Design Resonator

After a couple of optimization processes, a new resonator has been designed. This improved design resonator's design parameters can be seen in Figure 6.10 and Table 6.6.

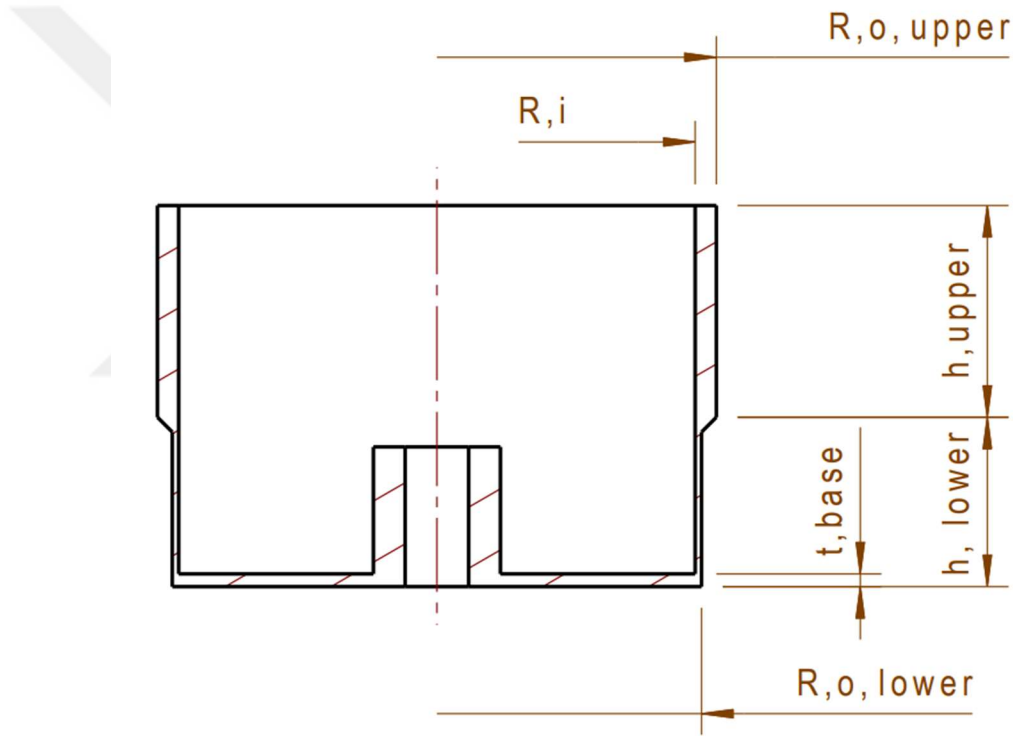


Figure 6.10. Design Parameters of the Cylindrical Resonator

Table 6.6. The Design Parameters of the Improved Design

	$h, lower$	$R, o, upper$	$t, base$	$R, o, lower$	$h, upper$	R, i
Default Values	5 mm	13.4 mm	0.9 mm	12.9 mm	7 mm	11.9 mm

6.5 The Performance of the Improved Designed Resonator

The same error analysis was performed on the improved design resonator. The frequency split is higher. However, measured angular velocity is more accurate in this new design. The results of the new design are given in Figure 6.11, Figure 6.12, Figure 6.13, Figure 6.14, Figure 6.15, and Figure 6.16.

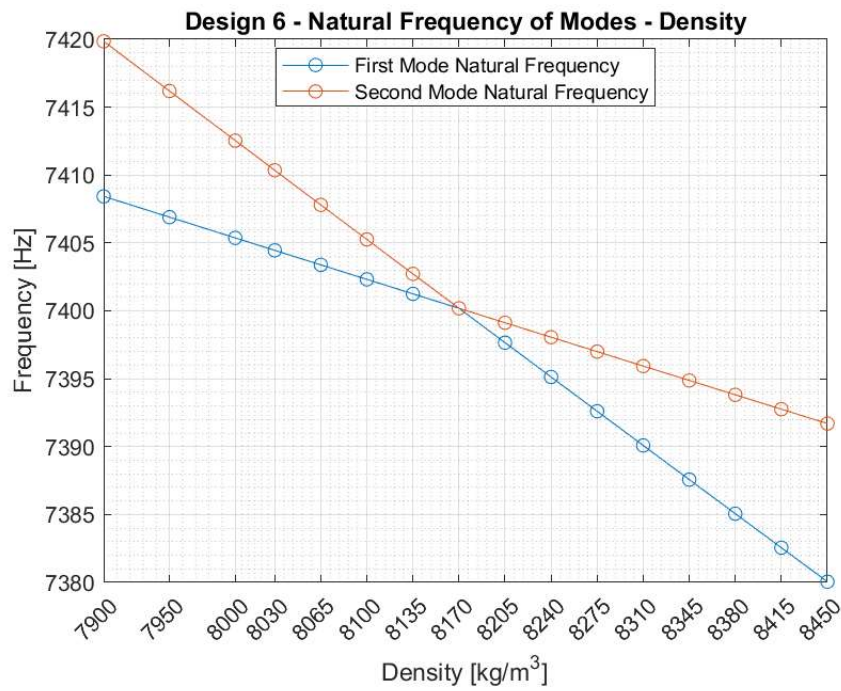


Figure 6.11. Density Imperfection Natural Frequency Results of the Improved Designed Resonator

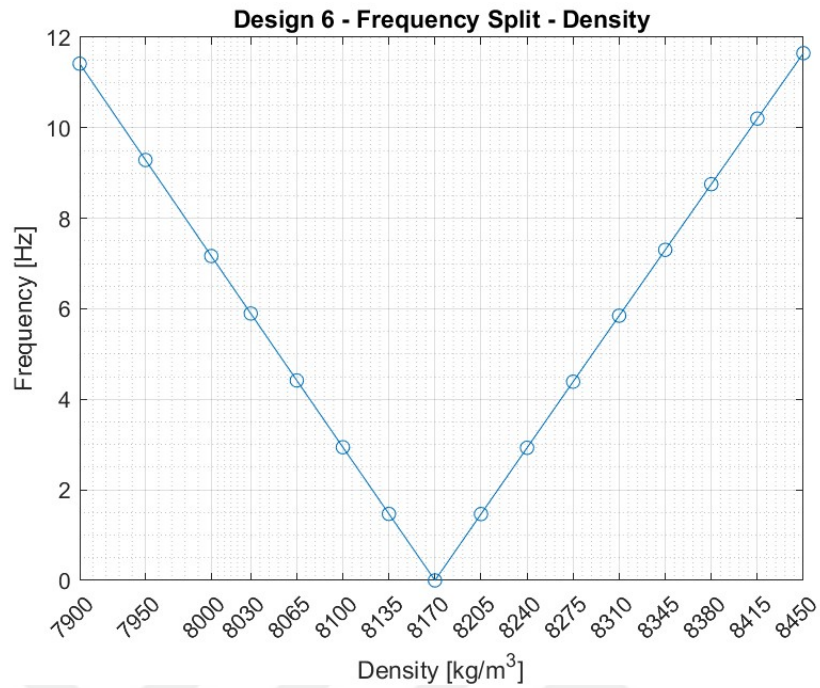


Figure 6.12. Density Imperfection Natural Frequency Split Results of the Improved Design Resonator

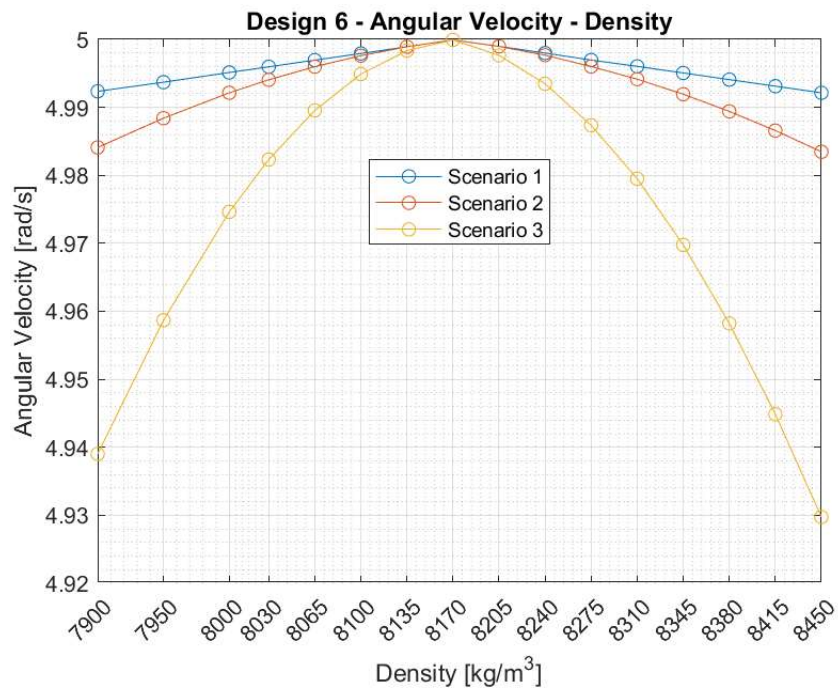


Figure 6.13. Density Imperfection Measured Angular Velocity Results of the Improved Design Resonator

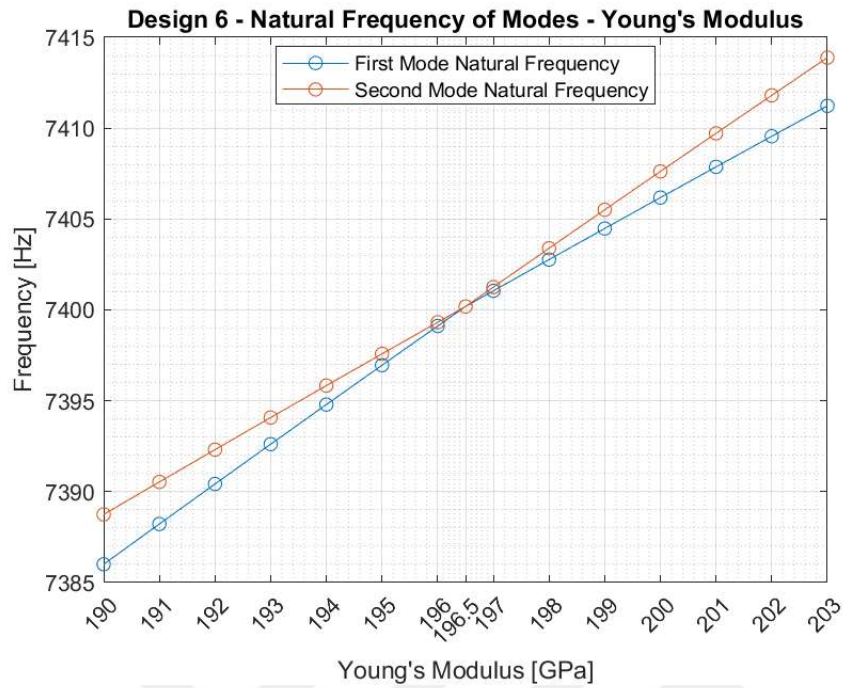


Figure 6.14. Young's Modulus Imperfection Natural Frequency Results of the Improved Design Resonator

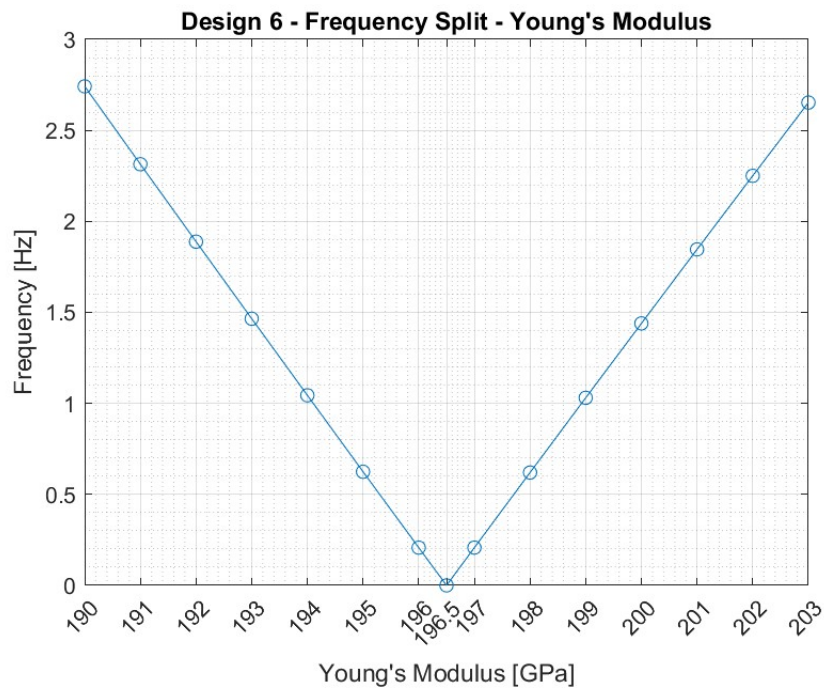


Figure 6.15. Young's Modulus Imperfection Natural Frequency Split Results of the Improved Design Resonator

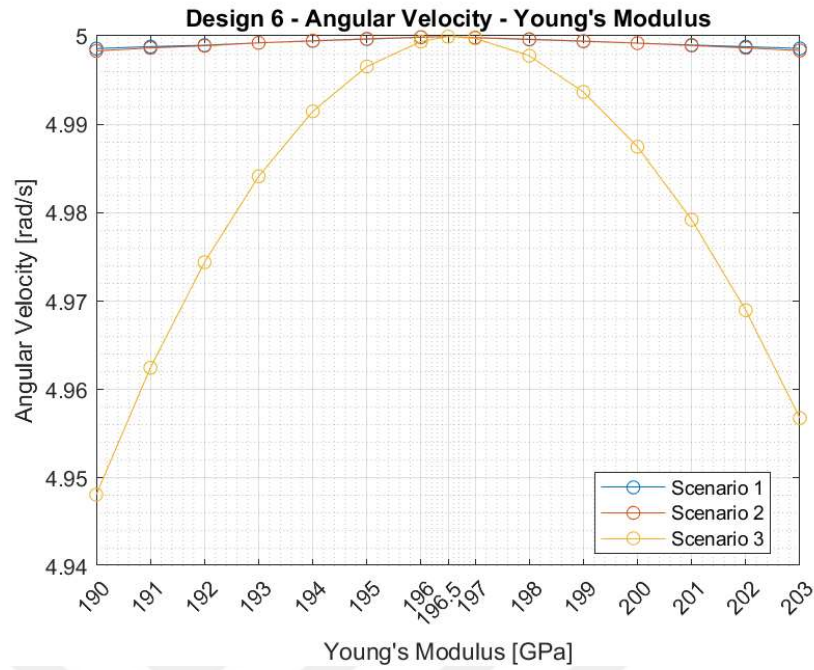


Figure 6.16. Young's Modulus Imperfection Measured Angular Velocity Results of the Improved Design Resonator

The improved design is compared with the most common resonators in scenarios 2 and 3 and with the “7900 kg/m^3 ” and “190 GPa” imperfection. The results can be seen in Table 6.7 and Table 6.8.

Table 6.7 The Scenario 2 and 3 Density Imperfection Performance Comparison of the Improved Design with Most Common CVG Resonators

Design Name	First Nat. Freq.	Second Nat. Freq.	Freq. Split	Scenario 2 Measured Ang. Vel.	Scenario 3 Measured Ang. Vel.
The Improved Design	7408.40	7419.80	11.40	4.9841	4.9391
Cylindrical	4313.16	4319.95	6.79	4.9834	4.9322
Ring	4247.10	4253.55	6.45	4.9839	4.9315
Bell-Shaped	6471.74	6483.27	11.53	4.9798	4.9275

Table 6.8 The Scenario 2 and 3 Performance Young's Modulus Imperfection Comparison of the Improved Design with Most Common CVG Resonators

Design Name	First Nat. Freq.	Second Nat. Freq.	Freq. Split	Scenario 2 Measured Ang. Vel.	Scenario 3 Measured Ang. Vel.
The Improved Design	7386.01	7388.75	2.74	4.9983	4.9480
Cylindrical	4296.94	4302.39	5.44	4.9935	4.9303
Ring	4228.21	4238.65	10.43	4.9767	4.9317
Bell-Shaped	6449.97	6458.80	8.82	4.9928	4.9599

CHAPTER 7

CONCLUSION

In conclusion, this thesis provides an in-depth exploration of the CVG. Throughout the thesis, both theoretical and practical aspects of its innovation, design, and analysis have been examined. This research systematically examines the existing knowledge, develops robust models, and proposes improvements in the design.

A literature review is explained by discussing the history, key advancements, and challenges of the CVG technology in Section 2. The literature review explains the importance of this research within the broader landscape of gyroscopic sensors. This review is divided into subsequent sections to understand the place of CVG performance in the industry and sets the ground for the understanding of prior work. Additionally, the novelty of this thesis is given briefly.

The mathematical model presented in Section 3 provides a simulation for the angular velocity measurements of the CVG. This mathematical model serves as one of the critical tools for analyzing and optimizing the performance of the resonator. From this model, one can see the natural frequency splits and mode shape differences have resulted as an error in the measurements. By using theoretical principles and simulation data, the model leads the way for a deeper investigation of CVG dynamics.

Section 4, continues the investigation of CVG dynamics by conducting a modal analysis of the CVG resonator using FEM. This analysis examines the structural and dynamic characteristics of the resonator. Additionally, it helps to gain insights into the natural frequency and mode shapes of the resonator. The hypotheses, axisymmetric meshes, and modeling are documented in the thesis to ensure the reliability of the results. Furthermore, the integration of modal analysis results into the mathematical model, constructing a complete mathematical vibrational model,

enhances the predictive capabilities, establishing a strong link between the computer simulations and real-world applications.

Section 5 introduces the industry-level perspective of this study. This section provides an overview of the most common CVG resonators in the industry which are cylindrical, ring, and bell-shaped resonators. These most common designs are compared under density and Young's modulus imperfections. Young's modulus imperfection causes more errors in the measurements than the density imperfections. Therefore, the research highlights their relative strength and weaknesses in various scenarios. Based on these scenarios, the best-performing resonators were selected as the cylindrical and bell-shaped resonators. However, the cylindrical resonator is selected as the best-performing resonator due to manufacturing practically. Therefore, the analyses identify optimal configurations for specific applications and offer practical insights for engineers and designers in this field.

Section 6 introduces contribution to the field with the introduction of an improved design resonator. This new improved design was developed from the gained insights from the previous sections. A set of design criteria is constructed with the literature research about the electronics and gained insights about the importance of the natural frequency of the resonator. The design parameters are carefully selected, well-described, and used in the new innovative design. When selecting the new innovative design regression analysis and modal analysis results have been used. This new innovative design was better than the most common CVG designs.

In future works, several topics that can be used are given below:

- Piezoelectric actuators in the resonator can be modeled and added to the mathematical model. Furthermore, piezoelectrical element errors can be used in the resonator error analysis.
- Balancing and tuning processes can be studied. Moreover, these processes can be modeled in finite element programs, and the mathematical model can show the performance of the resonators.

- The control loop of the sensor can be studied. Moreover, the electronics of the sensor can be studied and modeled.
- The compared resonators and the new resonator can be produced, and the sensor performance and modal analysis can be validated.



REFERENCES

- [1] A. Noureldin, T. B. Karamat, and J. Georgy, *Fundamentals of Inertial Navigation, Satellite-based Positioning and their Integration*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013. doi: 10.1007/978-3-642-30466-8.
- [2] V. M. N. Passaro, A. Cuccovillo, L. Vaiani, M. De Carlo, and C. E. Campanella, “Gyroscope Technology and Applications: A Review in the Industrial Perspective,” *Sensors*, vol. 17, no. 10, p. 2284, Oct. 2017, doi: 10.3390/s17102284.
- [3] A. Bhadra, S. Ghose, and B. Raychaudhuri, “A quest for the origin of the Sagnac effect,” *Eur. Phys. J. C*, vol. 82, no. 7, p. 649, Jul. 2022, doi: 10.1140/epjc/s10052-022-10620-6.
- [4] E. J. Post, “Sagnac Effect,” *Rev. Mod. Phys.*, vol. 39, no. 2, pp. 475–493, Apr. 1967, doi: 10.1103/RevModPhys.39.475.
- [5] H. C. Lefèvre, “The fiber-optic gyroscope, a century after Sagnac’s experiment: The ultimate rotation-sensing technology?,” *Comptes Rendus Phys.*, vol. 15, no. 10, pp. 851–858, Nov. 2014, doi: 10.1016/j.crhy.2014.10.007.
- [6] R. Antonello and R. Oboe, “MEMS Gyroscopes for Consumers and Industrial Applications,” in *Microsensors*, O. Minin, Ed., InTech, 2011. doi: 10.5772/17689.
- [7] V. Apostolyuk, *Coriolis Vibratory Gyroscopes: Theory and Design*. Cham: Springer International Publishing, 2016. doi: 10.1007/978-3-319-22198-4.
- [8] N. Liu, Z. Su, and Q. Li, “Design and experiment of a novel bell-shaped vibratory gyro,” *Sens. Actuators Phys.*, vol. 238, pp. 37–50, Feb. 2016, doi: 10.1016/j.sna.2015.10.036.
- [9] D. Rozelle, “The Hemispherical Resonator Gyro: From Wineglass to the Planets,” *Adv. Astronaut. Sci.*, vol. 134, pp. 1157–1178, Jan. 2009.

- [10] “VI. The waves on a rotating liquid spheroid of finite ellipticity”, doi: <https://royalsocietypublishing.org/doi/epdf/10.1098/rsta.1889.0006>.
- [11] A. M. Shkel, “Type I and Type II Micromachined Vibratory Gyroscopes,” in *2006 IEEE/ION Position, Location, And Navigation Symposium*, Coronado, CA: IEEE, 2006, pp. 586–593. doi: 10.1109/PLANS.2006.1650648.
- [12] V. V. Chikovani, I. M. Okon, A. S. Barabashov, and P. Tewksbury, “A set of high accuracy low cost metallic resonator CVG,” in *2008 IEEE/ION Position, Location and Navigation Symposium*, Monterey, CA, USA: IEEE, 2008, pp. 238–243. doi: 10.1109/PLANS.2008.4569975.
- [13] X. Xi, Y. L. Wu, D. B. Xiao, Y. M. Zhang, and X. Z. Wu, “A new metallic coriolis vibratory gyroscope (CVG) with multiple-shell structure,” in *2016 DGON Inertial Sensors and Systems (ISS)*, Karlsruhe, Germany: IEEE, Sep. 2016, pp. 1–10. doi: 10.1109/InertialSensors.2016.7745668.
- [14] N. Liu, Z. Su, Q. Li, M. Fu, H. Liu, and J. Fan, “Characterization of the Bell-Shaped Vibratory Angular Rate Gyro,” *Sensors*, vol. 13, no. 8, pp. 10123–10150, Aug. 2013, doi: 10.3390/s130810123.
- [15] Z. Su, M. Fu, Q. Li, N. Liu, and H. Liu, “Research on Bell-Shaped Vibratory Angular Rate Gyro’s Character of Resonator,” *Sensors*, vol. 13, no. 4, pp. 4724–4741, Apr. 2013, doi: 10.3390/s130404724.
- [16] Y. Tao *et al.*, “Design, analysis and experiment of a novel ring vibratory gyroscope,” *Sens. Actuators Phys.*, vol. 168, no. 2, pp. 286–299, Aug. 2011, doi: 10.1016/j.sna.2011.04.039.
- [17] Y. Luo, T. Qu, B. Zhang, Y. Pan, and P. Xiao, “Simulations and Experiments on the Vibrational Characteristics of Cylindrical Shell Resonator Actuated by Piezoelectric Electrodes with Different Thicknesses,” *Shock Vib.*, vol. 2017, pp. 1–6, 2017, doi: 10.1155/2017/2314858.

- [18] *IEEE Standard Specification Format Guide and Test Procedure for Coriolis Vibratory Gyros*. doi: 10.1109/IEEESTD.2004.95744.
- [19] L. C. Ng, “On the application of Allan variance method for Ring Laser Gyro performance characterization,” Oct. 1993, doi: 10.2172/10196087.
- [20] J. Beitia and S. Kowaltschek, “Rate Measurement Unit for Attitude Determination and Control Subsystem,” in *41st Annual AAS Guidance, Navigation and Control Conference*, Breckenridge, Colorado, USA, 2018.
- [21] V. V. Chikovani, “Trends of Ukrainian all digital coriolis vibratory gyroscopes developments,” in *2014 IEEE 3rd International Conference on Methods and Systems of Navigation and Motion Control (MSNMC)*, Kiev, Ukraine: IEEE, Oct. 2014, pp. 25–28. doi: 10.1109/MSNMC.2014.6979720.
- [22] V. V. Chikovani, “Performance parameters comparison of ring laser, coriolis vibratory and fiber-optic gyros based on Allan variance analysis,” in *2013 IEEE 2nd International Conference Actual Problems of Unmanned Air Vehicles Developments Proceedings (APUAVD)*, Kiev, Ukraine: IEEE, Oct. 2013, pp. 153–156. doi: 10.1109/APUAVD.2013.6705312.
- [23] G. K. M. Mittapally, D. Dantala, I. M. Chhabra, N. V. N. R. Pasalapudi, and K. C. Das, “Influence of vacuum and its degradation on the sensing element of the Coriolis Vibratory Gyroscope,” *J. Vibroengineering*, vol. 23, no. 8, pp. 1716–1726, Dec. 2021, doi: 10.21595/jve.2021.22103.
- [24] G. Remillieux and F. Delhaye, “Sagem Coriolis Vibrating Gyros: A vision realized,” in *2014 DGON Inertial Sensors and Systems (ISS)*, Karlsruhe, Germany: IEEE, Sep. 2014, pp. 1–13. doi: 10.1109/InertialSensors.2014.7049409.
- [25] Y. Pan *et al.*, “Monolithic Cylindrical Fused Silica Resonators with High Q Factors,” *Sensors*, vol. 16, no. 8, p. 1185, Jul. 2016, doi: 10.3390/s16081185.

- [26] J. Sun, Y. Wu, X. Xi, Y. Zhang, and X. Wu, "Analysis of the Damping Characteristics of Cylindrical Resonators Influenced by Piezoelectric Electrodes," *Sensors*, vol. 17, no. 5, p. 1017, May 2017, doi: 10.3390/s17051017.
- [27] X. Wu, X. Xi, Y. Wu, and D. Xiao, *Cylindrical Vibratory Gyroscope*. in Springer Tracts in Mechanical Engineering. Singapore: Springer Singapore, 2021. doi: 10.1007/978-981-16-2726-2.
- [28] C. H. J. Fox, "A simple theory for the analysis and correction of frequency splitting in slightly imperfect rings," *J. Sound Vib.*, vol. 142, no. 2, pp. 227–243, Oct. 1990, doi: 10.1016/0022-460X(90)90554-D.
- [29] S.-Y. Choi and J.-H. Kim, "Natural frequency split estimation for inextensional vibration of imperfect hemispherical shell," *J. Sound Vib.*, vol. 330, no. 9, pp. 2094–2106, Apr. 2011, doi: 10.1016/j.jsv.2010.11.014.
- [30] X. Xi, Y. Zhang, J. Sun, and X. Wu, "Frequency Split Analysis of a Ring Vibratory Gyroscope Based on Harmonic Transformation of Parametric Errors," Aug. 26, 2020, *In Review*. doi: 10.21203/rs.3.rs-63047/v1.
- [31] D. Vakhlyarsky, F. Sorokin, A. Gouskov, M. Basarab, and B. Lunin, "Approximation method for frequency split calculation of coriolis vibrating gyroscope resonator," *J. Sound Vib.*, vol. 526, p. 116733, May 2022, doi: 10.1016/j.jsv.2021.116733.
- [32] A. K. Rourke, S. Mewilliam, and C. H. J. Fox, "MULTI-MODE TRIMMING OF IMPERFECT RINGS," *J. Sound Vib.*, vol. 248, no. 4, pp. 695–724, Dec. 2001, doi: 10.1006/jsvi.2001.3811.
- [33] Z. Lin, M. Fu, Z. Deng, N. Liu, and H. Liu, "Frequency Split Elimination Method for a Solid-State Vibratory Angular Rate Gyro with an Imperfect Axisymmetric-Shell Resonator," *Sensors*, vol. 15, no. 2, pp. 3204–3223, Feb. 2015, doi: 10.3390/s150203204.

- [34] Z. X. Hu, B. J. Gallacher, J. S. Burdess, S. R. Bowles, and H. T. D. Grigg, "A systematic approach for precision electrostatic mode tuning of a MEMS gyroscope," *J. Micromechanics Microengineering*, vol. 24, no. 12, p. 125003, Dec. 2014, doi: 10.1088/0960-1317/24/12/125003.
- [35] Y. Hu, H. Zhong, Y. Lu, J. Duan, and X. Sun, "Laser Trimming Method for Reducing Frequency Split of Cylindrical Vibrating Gyroscope," *IOP Conf. Ser. Mater. Sci. Eng.*, vol. 793, no. 1, p. 012027, Mar. 2020, doi: 10.1088/1757-899X/793/1/012027.
- [36] P. Xiao *et al.*, "Influence of Temperature Variation on the Vibrational Characteristics of Fused Silica Cylindrical Resonators for Coriolis Vibratory Gyroscopes," *Sensors*, vol. 20, no. 4, p. 1032, Feb. 2020, doi: 10.3390/s20041032.
- [37] Y. Lin, X. Zheng, S. Liu, W. Ma, and Z. Jin, "Temperature-dependence improvement for a MEMS gyroscope using triangular-electrode based capacitive detection method," *Micro Nano Lett.*, vol. 12, no. 11, pp. 828–833, Nov. 2017, doi: 10.1049/mnl.2017.0431.
- [38] S. A. Zotov, I. P. Prikhodko, A. A. Trusov, and A. M. Shkel, "Frequency modulation based angular rate sensor," in *2011 IEEE 24th International Conference on Micro Electro Mechanical Systems*, Cancun, Mexico: IEEE, Jan. 2011, pp. 577–580. doi: 10.1109/MEMSYS.2011.5734490.
- [39] X. Ren, X. Zhou, S. Yu, X. Wu, and D. Xiao, "Frequency-Modulated MEMS Gyroscopes: A Review," *IEEE Sens. J.*, vol. 21, no. 23, pp. 26426–26446, Dec. 2021, doi: 10.1109/JSEN.2021.3117939.
- [40] V. V. Chikovani, V. M. Azarskov, and S. V. Golovach, "Whole Angle Mode for Low Q-factor Vibratory Gyroscopes," in *The 9-th World Congress "Aviation in the XXI-st century,"* Kyiv, Ukraine, Sep. 2020.
- [41] X. Wang, W. Wu, B. Luo, Z. Fang, Y. Li, and Q. Jiang, "Force to Rebalance Control of HRG and Suppression of Its Errors on the Basis of FPGA," *Sensors*, vol. 11, no. 12, pp. 11761–11773, Dec. 2011, doi: 10.3390/s111211761.

- [42] N. Liu, Z. Su, and H. Liu, *Research on The Control Loop for Solid Vibratory Gyroscope*. 2013, p. 4968.
- [43] C. H. He *et al.*, “A novel narrow-band force rebalance control method for the sense mode of MEMS vibratory gyroscopes,” *Measurement*, vol. 62, pp. 197–204, Feb. 2015, doi: 10.1016/j.measurement.2014.11.009.
- [44] M. H. Kline *et al.*, “Quadrature FM gyroscope,” in *2013 IEEE 26th International Conference on Micro Electro Mechanical Systems (MEMS)*, Taipei, Taiwan: IEEE, Jan. 2013, pp. 604–608. doi: 10.1109/MEMSYS.2013.6474314.
- [45] Y. Zhang, T. Miao, K. Guo, J. Sun, X. Wu, and D. Xiao, “Electronic Gain Error Compensation For Whole-Angle Mode Coriolis Vibrating Gyroscopes With High Q Factor,” in *2020 IEEE International Symposium on Inertial Sensors and Systems (INERTIAL)*, Hiroshima, Japan: IEEE, Mar. 2020, pp. 1–4. doi: 10.1109/INERTIAL48129.2020.9090062.
- [46] S. V. Joubert, M. Y. Shatalov, and T. H. Fay, “Rotating structures and Bryan’s effect,” *Am. J. Phys.*, vol. 77, no. 6, pp. 520–525, Jun. 2009, doi: 10.1119/1.3088877.
- [47] S. V. Joubert, M. Y. Shatalov, T. H. Fay, and A. V. Manzhurov, “Bryan’s effect and anisotropic nonlinear damping,” *J. Sound Vib.*, vol. 416, pp. 125–135, Mar. 2018, doi: 10.1016/j.jsv.2017.11.043.
- [48] J. Seok, H. F. Tiersten, and H. A. Scarton, “An analysis of a vibratory angular-rate gyroscope using polarized piezoceramic bimorph plates. Part 2: solution procedure for the gyroscope with superposed angular velocity,” *J. Sound Vib.*, vol. 280, no. 1–2, pp. 289–310, Feb. 2005, doi: 10.1016/j.jsv.2003.12.031.
- [49] W.-T. Sung, J. G. Lee, and T. Kang, “A novel demodulation method in MEMS gyroscope,” presented at the ICMIT 2005: Mechatronics, MEMS, and Smart Materials, Y. Wei, K. T. Chong, T. Takahashi, S. Liu, Z. Li, Z. Jiang, and J. Y. Choi, Eds., Chongqing, China, Dec. 2005, p. 604019. doi: 10.1117/12.664215.

- [50] R. D. Blevins, *Formulas for Natural Frequency and Mode Shape*. Van Nostrand Reinhold Company, 1979. [Online]. Available: <https://books.google.com.tr/books?id=ekaPQgAACAAJ>
- [51] A. Anssari-Benam, A. Bucchini, and G. Saccomandi, “Modelling the Inflation and Elastic Instabilities of Rubber-Like Spherical and Cylindrical Shells Using a New Generalised Neo-Hookean Strain Energy Function,” *J. Elast.*, vol. 151, no. 1, pp. 15–45, Sep. 2022, doi: 10.1007/s10659-021-09823-x.
- [52] Z. Ruan, X. Ding, Y. Pu, Y. Gao, and H. Li, “In-Run Automatic Mode-Matching of Whole-Angle Micro-Hemispherical Resonator Gyroscope Based on Standing Wave Self-Precession,” *IEEE Sens. J.*, vol. 22, no. 14, pp. 13945–13957, Jul. 2022, doi: 10.1109/JSEN.2022.3181302.
- [53] Xiang Xi, Yulie Wu, Yongmeng Zhang, Xiaomei Wu, Yu Zheng, and Xuezhong Wu, “A Simple Acoustic Method for Modal Parameter Measurement of the Resonator for Vibratory Shell Gyroscope,” *IEEE Sens. J.*, vol. 14, no. 11, pp. 4069–4077, Nov. 2014, doi: 10.1109/JSEN.2014.2332879.