HOMOGENEOUS SPACETIME SOLUTIONS OF MINIMAL MASSIVE GRAVITY

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ABSTRACT

HOMOGENEOUS SPACETIME SOLUTIONS OF MINIMAL MASSIVE GRAVITY

In this thesis we first study general mathematical properties of homogeneous spaces. We then review classification of three-dimensional Lie algebras, which was done by Bianchi. There are nine classes. In the last part we investigate which of these nine classes exist as solutions in three-dimensional Minimal Massive Gravity model.

ÖZET

MİNİMAL KÜTLELİ YERÇEKİM KURAMININ HOMOJEN UZAYZAMAN ÇÖZÜMLERİ

Bu tezde ilk olarak homojen uzayların genel matematiksel özelliklerini çalışacağız. Ardından Bianchi tarafından yapılmış üç boyutlu Lie cebirlerinin sınıflandırılmasını gözden geçireceğiz. Böyle dokuz sınıf bulunmaktadır. Son kısımda bu dokuz sınıftan hangilerinin Minimal Kütleli Yerçekim modelinde çözüm olarak yer aldığı saptanmaktadır.

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LIST OF SYMBOLS

A^T	Transpose of a matrix A
$ad(a)$	The adjoint representation
$\mathrm{Aut}(V)$	The group of automorphisms of a linear space V
C^k	k times continuously differentiable
df_x	The differential of a map f at the point x
$d\theta$	Exterior derivative of a differential form θ
e	The unit element of a group
$E(n)$	The Euclidean group of affine transformations in $\mathbb{R}^n$
exp	Exponential map on a manifold
Exp	Exponential map of a Lie group
G_0	The identity component of a Lie group G
$G_{k,n}(\mathbb{R})$	Grassmannian manifolds
$GL(n, \mathbb{K})$	The group of the $n \times n$ invertible matrices over the field $\mathbb{K}$
$GL(V)$	The group of invertible linear transformations of a linear space
$\mathbb{H}$	The upper-half plane
Id	The identity map
Im	Imaginary part
L_g	Left translation on a Lie group
$O(n)$	The group of the n-dimensional orthogonal real matrices
$\mathbb{PR}^n$	n-dimensional real projective space
$\mathbb{S}^n$	n-dimensional unit sphere
$SL(n, \mathbb{R})$	The special linear group of the $n \times n$ real matrices
$SO(n)$	The special orthogonal group of the $n \times n$ real matrices
$Sp(n)$	n-dimensional Spin group
$SU(n)$	The special unitary group of the $n \times n$ complex matrices
T_xM	The tangent space of a manifold M at the point x
$U(n)$	The unitary group of the n-dimensional complex matrices
V^*	Dual of a linear space V

$\delta^a{}_b$	Kronecker delta
$\phi^*\theta$	The pullback of a form θ by a map ϕ
ϕ_*X	The pushforward of a vector X by a map ϕ
$\wedge$	Wedge product
∇	Levi-Civita connection
$\langle\cdot,\cdot\rangle$	An inner product on a Euclidean space
$(\cdot,\cdot)$	An inner product on a linear space
$[\cdot,\cdot]$	Lie bracket
$\lvert\cdot\rvert$	Norm of a vector
$\oplus$	Direct sum
$\otimes$	Tensor product

LIST OF ACRONYMS/ABBREVIATIONS

AdS	Anti-de Sitter
BCH	Baker-Campbell-Hausdorff
MMG	Minimal Massive Gravity
TMG	Topological Massive Gravity

1. INTRODUCTION

Minimal Massive Gravity (MMG) is a three-dimensional gravity model proposed in [1], which attracted much interest during the last two years, is an extension of another gravity model known as Topological Massive Gravity (TMG) [2].

MMG is a higher derivative and highly non-linear gravity model. Therefore, in general the solutions of the field equations of MMG for special types of spacetime metrics or at the special points in the parameter space of the model are studied. In this thesis we construct homogeneous spacetime solutions of MMG.

Homogeneity is essential, since contemporary cosmological models are based on the idea that the universe looks almost the same everywhere, an idea known as the cosmological principle. This principle is assumed to apply only on very large scales, where local variations in density are averaged over. In mathematical formulation the cosmological principle is related to two properties, which a manifold might possess, namely, isotropy and homogeneity. Isotropy is the property that the space looks the same in all directions i.e., given any two vectors at the tangent space of that point there is an isometry the pushforward of which takes one to another. Homogeneity is the property that the metric is the same throughout the manifold i.e., given any two points on a manifold, there is an isometry that takes one to another.

Homogeneous manifolds or homogeneous spaces are also important in differential geometry. As we will see in the next chapter, studying them reduces to studying the Lie groups and their actions. As we know, Lie groups possess many interesting properties from having an analytic group structure to being geodesically complete with left-invariant metric. Therefore, we expect for homogeneous spaces have similar nice features.

The thesis is organized as follows. In the second chapter we introduce homogeneous spaces, provide examples of them, discuss invariant metrics and Riemannian homogeneous manifolds. In the third chapter, after presenting the structure of a Lie group with Maurer-Cartan forms we discuss left-invariant metrics and then give the classification of three-dimensional Lie algebras. Finally, we introduce MMG model and construct solutions on the homogeneous spaces.

We assume that a reader is familiar with the basic concepts in differential and Riemannian geometries such as Lie groups, vector fields, flows, exponential mapping, normal neighborhood, Riemannian metric, covariant derivative, geodesics and so on. Some good sources are [3–7].

Throughout the thesis we use the Einstein summation convention: when an upper index and lower index are the same, then it is understood that there is a summation over that index.

2. HOMOGENEOUS SPACES

It is interesting to study the transitive group actions on a manifold, since this leaves invariant some properties of it (such as distances in some metric, or a class of curves such as straight lines in the plane). Therefore, the manifold "looks the same" everywhere from the point of view of this property. Often, such transitive group actions on a manifold are models for various kinds of geometric structures, and as such they have an important role in many areas of differential geometry. They are interesting in physics as well. Observations show that our spatial universe is a three-dimensional homogeneous space.

In the first part of this chapter we provide some basic examples of homogeneous spaces and describe a very general construction that can be used to generate a great number of homogeneous spaces. In the second part we study homogeneous spaces from algebraic point of view. Finally, we discuss homogeneous Riemannian manifolds, in which isometries play a central role.

2.1. Homogeneous Spaces in General

We start with the basic definitions on group actions.

Definition 2.1. *Let G be a Lie group and M be a manifold. The action of G on M is a differentiable map $\sigma : G \times M \mapsto M$ which satisfies the conditions*

$$(i)\ \sigma(e,p) = p, \qquad (ii)\ \sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$$

for any $p \in M$ and $g_1, g_2 \in G$ (the notation $g \cdot p$ is often used instead of $\sigma(g,p)$).

Definition 2.2. *Let G be a Lie group acting on a manifold M by $\sigma : G \times M \mapsto M$. The action σ is said to be*

(a) ***transitive*** *if, for any $p_1, p_2 \in M$, there exists an element $g \in G$ such that $g \cdot p_1 = p_2$;*

(b) ***free*** *if every non-trivial element $g \neq e$ of G has no fixed points in M, that is if there exists an element $p \in M$ such that $g \cdot p = p$, then g must be the unit element e;*

(c) ***effective*** *if the unit element $e \in G$ is the unique element that defines the trivial action on M, i.e., if $g \cdot p = p$ for all $p \in M$, then g must be the unit element.*

(d) ***proper*** *if the map $G \times M \mapsto M \times M$ given by $(g, p) \mapsto (g \cdot p, p)$ is a proper map (i.e., the preimage of any compact set is compact).*

Definition 2.3. *A smooth manifold endowed with a transitive smooth action by a Lie group G is called a* ***homogeneous G-space*** *(or a homogeneous space or homogeneous manifold if it is not important to specify the group).*

Definition 2.4. *Let G be a Lie group acting on a manifold M. The* ***isotropy group*** *of $p \in M$ is a subgroup of G defined by $H(p) = \{g \in G : g \cdot p = p\}$. $H(p)$ is also called the stabilizer of p or a stationary group.*

Now let us have a look at some fundamental examples of homogeneous spaces.

(i) The natural action of $O(n)$ on $\mathbb{S}^{n-1}$ is transitive, since $O(n)$ is the set of rotations in $\mathbb{R}^n$.

(ii) Let $E(n)$ denote the subgroup of $GL(n+1, \mathbb{R})$ consisting of matrices of the form

$$\{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} : A \in O(n),\ b \in \mathbb{R}^n\} \tag{2.1}$$

where b is considered as an $n \times 1$ column matrix. It is straightforward to check that $E(n)$ is an embedded Lie subgroup. If $S \subset \mathbb{R}^{n+1}$ denotes the affine subspace defined by $x^{n+1} = 1$, then a simple computation shows that $E(n)$ takes S to itself. If we identify S with $\mathbb{R}^n$ in the obvious way, this induces a smooth action of $E(n)$ on $\mathbb{R}^n$, in which the matrix as above sends x to $Ax + b$. It is well known that these are precisely the diffeomorphisms of $\mathbb{R}^n$ that preserve the Euclidean distance function i.e., they are isometries of $\mathbb{R}^n$. For this reason, $E(n)$ is called the Euclidean group. Because any point in $\mathbb{R}^n$ can be taken to any other by a translation, $E(n)$ acts transitively on $\mathbb{R}^n$, so $\mathbb{R}^n$ is a homogeneous $E(n)$-space.

(iii) $SL(2,\mathbb{R})$ acts smoothly and transitively on the upper half-plane $\mathbb{H}$=$\{z \in \mathbb{C}: \text{Im}z > 0\}$ by the Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \tag{2.2}$$

where $ad - bc = 1$.

(iv) The natural action of $GL(n,\mathbb{C})$ on $\mathbb{C}^n$ restricts to natural actions of both $U(n)$ and $SU(n)$ on $\mathbb{S}^{2n-1}$, thought of as the set of unit vectors in $\mathbb{C}^n$.

Let G be a Lie group and let $H \subset G$ be a Lie subgroup. For each $g \in G$, the left coset of g modulo H is the set $gH = \{gh : h \in H\}$. The set of left cosets modulo H is denoted by G/H; with the quotient topology determined by the natural map $\pi : G \to G/H$ sending each element $g \in G$ to its coset, namely, the left coset space of G modulo H. Two elements $g_1, g_2 \in G$ are in the same coset modulo H if and only if ${g_1}^{-1}g_2 \in H$; in this case we write $g_1 \equiv g_2 (modH)$.

Next we describe a very general construction that can be used to generate a great number of homogeneous spaces, as quotients of Lie groups by closed Lie subgroups.

Theorem 2.5. *[3] Let G be a Lie group and let H be a closed Lie subgroup of G. The left coset space G/H has a unique smooth manifold structure such that the quotient* *map* $\pi : G \to G/H$ *is a smooth submersion. The left coset of* G *on* G/H *given by* $g_1 \cdot (g_2 H) = (g_1 g_2)H$ *turns* G/H *into a homogeneous* G*-space.*

The homogeneous spaces constructed in the above theorem turn out to be of central importance because, as the next theorem shows, "every" homogeneous space is equivalent to one of this type.

Theorem 2.6. *[4] Suppose* M *is a homogeneous* G*-space. Let* p *be any point of* M *and let* G_p *denote the subgroup of* G *that leaves* p *fixed. Then*
(a) G_p *is closed;*
(b) The map $F : G/G_p \to M$ *defined by* $F(gG_p) = g \cdot p$ *is a diffeomorphism such that*

$$F(h \cdot gG_p) = h \cdot F(gG_p)$$

holds for any $h \in G$*;*
(c) If M *is connected, then* G_0*, the identity component of* G*, acts transitively on* M*.*

In some sense the above theorem says that the study of homogeneous spaces can "almost" be reduced to the algebraic problem of understanding closed Lie subgroups of Lie groups. Thus, most books define homogeneous space to be a quotient manifold of the form G/H, where G is a Lie group and H is a closed Lie subgroup of G.

Applying the characterization in Theorem 2.5 to the examples of transitive group actions we provided after Definition 2.4 it can be observed that some of them are indeed homogeneous.

(i) Consider the natural action of $O(n)$ on $\mathbb{S}^{n-1}$. Let the "north pole" $N = (0, ..., 0, 1)$ serve as a base point in $\mathbb{S}^{n-1}$. Then it is easy to verify that the isotropy group is $O(n-1)$, thought of as orthogonal transformations of $\mathbb{R}^n$ that fix the last variable. Thus $\mathbb{S}^{n-1}$ is diffeomorphic to the quotient manifold $O(n)/O(n-1)$. For the action of $SO(n)$ on $\mathbb{S}^{n-1}$, the isotropy group is $SO(n-1)$, so $\mathbb{S}^{n-1}$ is also diffeomorphic to $SO(n)/SO(n-1)$.

(ii) Since the Euclidean group $E(n)$ in the equation (2.1) acts smoothly and transitively on $\mathbb{R}^n$, and the isotropy group of the origin is the subgroup $O(n) \subset E(n)$ (identified with the $(n+1)\times(n+1)$ matrices of the form $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ with $A \in O(n)$), so $\mathbb{R}^n$ is diffeomorphic to $E(n)/O(n)$.

(iii) Next consider the transitive action of $SL(2,\mathbb{R})$ on the upper half-plane by Möbius transformations in the equation (2.2). Direct computation shows that the isotropy group of the point $i \in \mathbb{H}$ consists of matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2+b^2 = 1$. This subgroup is exactly $SO(2) \subset SL(2,\mathbb{R})$, so this gives rise to a diffeomorphism $\mathbb{H} \approx SL(2,\mathbb{R})/SO(2)$.

(iv) We also have $\mathbb{S}^{2n-1} \approx U(n)/U(n-1) \approx SU(n)/SU(n-1)$.

(v) The group $O(n+1)$ acts transitively on $\mathbb{PR}^n$ from the left. Note first that $O(n+1)$ acts on $\mathbb{R}^{n+1}$ in the usual manner and preserves the equivalence relation employed to define $\mathbb{PR}^n$. Accordingly this action of $O(n+1)$ on $\mathbb{R}^{n+1}$ induces the natural action of $O(n+1)$ on $\mathbb{PR}^n$. Clearly this action is transitive on $\mathbb{PR}^n$. If we take a point $p \in \mathbb{PR}^n$, which corresponds to a point $(1, 0, ..., 0) \in \mathbb{R}^{n+1}$, the isotropy group G_p is $\begin{pmatrix} \pm 1 & 0 \\ 0 & O(n) \end{pmatrix}$$=O(1) \times O(n)$, where $O(1)$ is the set $\{-1, +1\} = \mathbb{Z}_2$. Now we find that $O(n+1)/[O(1) \times O(n)] \approx \mathbb{S}^n/\mathbb{Z}_2 \approx \mathbb{PR}^n$.

(vi) The above result can be easily generalized to the Grassmannian manifolds (for details see [5]): $G_{k,n}(\mathbb{R}) = O(n)/[O(k) \times O(n-k)]$. This fact also shows that the Grassmannian manifold is compact.

(vii) Any Lie group G is itself a homogeneous G-space, when it acts by left translation on itself. In this case the isotropy group of any point in G consists only of the unit element.

2.2. Algebra on Homogeneous Spaces

Every group automorphism ϕ of a Lie group G induces an automorphism ϕ_* of its Lie algebra $\mathfrak{g}$; $\phi_*[U,V] = [\phi_* U, \phi_* V]$ for every $U, V \in \mathfrak{g}$. Particularly, for every $g \in G$, ad_g mapping x to gxg^{-1} induces an automorphism of $\mathfrak{g}$, also denoted by ad_g. The representation $g \mapsto ad_g$ of G is called the **adjoint representation** of G in $\mathfrak{g}$.

Definition 2.7. *Let $M = G/H$ be a homogeneous space on which a connected Lie group G acts transitively and effectively. We say that a space G/H is* ***reductive*** *if the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an ad_H-invariant subspace $\mathfrak{m}$, that is, if*

$$(i)\ \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad (ii)\ \mathfrak{h} \cap \mathfrak{m} = 0, \qquad (iii)\ ad_H \mathfrak{m} \subset \mathfrak{m}.$$

The last condition implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and, conversely, if H is connected, then they are actually equivalent (see [6]).

Lemma 2.8. *[6] In any of the following cases, a homogeneous space G/H is reductive:*

(a) H is compact;

(b) H is connected and $\mathfrak{h}$ is reductive in $\mathfrak{g}$ in the sense that ad_H in $\mathfrak{g}$ is completely reducible. This is the case if H is connected and semisimple;

(c) H is a discrete subgroup of G.

Proof. To prove that G/H is reductive in the compact case, let $(\cdot,\cdot)'$ be an arbitrary inner product on $\mathfrak{g}$. Define a new inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ by

$$(X,Y) = \int_H (ad_h(X), ad_h(Y))' dh \tag{2.3}$$

where dh denotes the Haar measure on H. The new inner product is ad_H-invariant. If we denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ with respect to inner product $(\cdot,\cdot)$, then we have an ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. □

Let M be a homogeneous space G/H, and let H be an isotropy group of $x \in M$, say. We shall often identify the tangent space T_xM at the "origin" x with the quotient space $\mathfrak{g}/\mathfrak{h}$ in a natural manner. If G/H is reductive with ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then both T_xM and $\mathfrak{g}/\mathfrak{h}$ will be identified with $\mathfrak{m}$ also in a natural manner. Since, for each $h \in H$, $ad_h : \mathfrak{g} \to \mathfrak{g}$ maps the subalgebra $\mathfrak{h}$ into itself, it induces a linear transformation of $\mathfrak{g}/\mathfrak{h}$ which will be also denoted by ad_h. We shall identify each element $X \in \mathfrak{g}$ with the vector field on M as follows: we first identify X with the left-invariant vector field $\tilde{X}$ on G induced by the left translations and then identify $\tilde{X}$ with the vector field $\phi_*\tilde{X}$, where ϕ_* is the pushforward of the map $\phi : G \to M$ given by $\phi(g) = g \cdot x$.

Let $M = G/H$ be a homogeneous space with H being an isotropy group of $x \in M$. Then a Riemannian metric g on M is called ***G*-invariant** (or invariant, if it is not important to specify the group) metric, if for any $y \in G$ and for $a \in G$ with $a \cdot x = y$, we have $g_x(X, Y) = g_y((\phi_a)_*X, (\phi_a)_*Y)$ for $X, Y \in T_xM$, where ϕ_a is a map from M to M defined as $\phi_a(y) = a \cdot y$. It is easy to verify that g_x is independent of the choice of $a \in G$ with $a \cdot x = y$ (see [7]).

For example, let G be any (connected) Lie group. Then G acts transitively on itself by the left translations. Hence, any inner product on the Lie algebra $\mathfrak{g}$ of G gives rise by the left translations to G-invariant Riemannian metric on G. Such a metric is called a **left-invariant Riemannian metric**. Here $H = \{e\}$, so $\mathfrak{h} = 0$ and $\mathfrak{m} = \mathfrak{g}$.

Let us express some of the basic properties of an invariant metric on G/H in the Lie algebraic terms. We first prove

Proposition 2.9. *[6] There is a natural one-to-one correspondence between the G-invariant indefinite Riemannian metrics g on $M = G/H$ and the ad_H-invariant non-*

degenerate symmetric bilinear forms B on $\mathfrak{g}/\mathfrak{h}$. The correspondence is given by

$$B(\bar{X},\bar{Y}) = g_x(X,Y) \qquad\qquad X,Y \in \mathfrak{g} \tag{2.4}$$

where $\bar{X}$ and $\bar{Y}$ are the elements of $\mathfrak{g}/\mathfrak{h}$ represented by X and Y respectively. A form B is positive definite if and only if the corresponding metric g is positive definite.

Proof. Any point of M is of the form $f(x)$ for some $f \in G$ and any vector at $f(x) \in M$ is of the form $df_x(X)$ for some $X \in \mathfrak{g}$. It is straightforward to verify that the following equality defines a G-invariant metric g on M:

$$g(df_x(X), df_x(Y)) = B(\bar{X},\bar{Y}) \tag{2.5}$$

for $X,Y \in \mathfrak{g}$. The rest follows easily. □

Corollary 2.10. *[6] If $M = G/H$ is reductive with ad_H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, then there is a natural one-to-one correspondence between the G-invariant metrics g on $M = G/H$ and the ad_H-invariant non-degenerate symmetric bilinear forms B on $\mathfrak{m}$. The correspondence is given by*

$$B(X,Y) = g_x(X,Y) \tag{2.6}$$

for $X,Y \in \mathfrak{m}$.

The invariance of B by ad_H implies

$$B([Z,X],Y) + B(X,[Z,Y]) = 0 \tag{2.7}$$

for $X,Y \in \mathfrak{m}$ and $Z \in \mathfrak{h}$, and the converse holds true if H is connected (see [6]).

Let us have a close look to the sphere example. We already know that $SO(n+1)/SO(n) = S^n$. Let $\{e_0, e_1, ..., e_n\}$ be the natural basis of $\mathbb{R}^{n+1}$ and let S^n be the unit sphere in $\mathbb{R}^{n+1}$. The rotation group $SO(n+1)$ acts on S^n transitively, and the isotropy group at $e_0 \in S^n$ consists of all $A \in SO(n+1)$ such that $Ae_0 = e_0$, namely, such that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}, \qquad B \in SO(n). \tag{2.8}$$

The canonical decomposition of the Lie algebra is given by

$$\mathfrak{o}(n+1) = \mathfrak{o}(n) + \mathfrak{m} \tag{2.9}$$

where $\mathfrak{o}(n+1)$ is the Lie algebra of all $(n+1)\times(n+1)$ skew-symmetric matrices, $\mathfrak{o}(n)$ the subalgebra of $\mathfrak{o}(n+1)$ consisting of all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \tag{2.10}$$

where C is skew-symmetric of degree n, and $\mathfrak{m}$ the subspace of all matrices of the form

$$\begin{pmatrix} 0 & -\xi^T \\ \xi & C \end{pmatrix} \tag{2.11}$$

where ξ is a column vector in $\mathbb{R}^n$. The adjoint representation of $SO(n)$ in $\mathfrak{m}$ is given by

$$ad\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & -\xi^T \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(B\xi)^T \\ B\xi & 0 \end{pmatrix} \tag{2.12}$$

in other words, it is essentially the action of $SO(n)$ on $\mathbb{R}^n$. Similarly, for $C \in \mathfrak{o}(n)$

$$ad\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} 0 & -\xi^T \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(C\xi)^T \\ C\xi & 0 \end{pmatrix} \tag{2.13}$$

that is, the adjoint representation of $\mathfrak{o}(n)$ in $\mathfrak{m}$ is essentially the action of $\mathfrak{o}(n)$ on $\mathbb{R}^n$.

We transfer the Euclidean inner product $\langle \xi, \eta \rangle$ on $\mathbb{R}^n$ to $\mathfrak{m}$ under the identification of the column vector ξ with in the matrix with the usual vector in $\mathbb{R}^n$, getting an inner product on $\mathfrak{m}$ which is invariant by $ad(SO(n))$. We note that if $\xi, \eta \in \mathbb{R}^n$ correspond to $X, Y \in \mathfrak{m}$, respectively, then $\langle \xi, \eta \rangle = -\frac{1}{2} trXY$; thus our inner product on $\mathfrak{m}$ is the restriction of the inner product in $\mathfrak{o}(n+1)$ given by $(A, B) = -\frac{1}{2} trAB$ for $A, B \in \mathfrak{o}(n+1)$, which is invariant by $ad(SO(n+1))$ on $\mathfrak{o}(n+1)$.

2.3. Homogeneous Riemannian Manifolds

In this section we discuss an important class of homogeneous spaces, namely, Homogeneous Riemannian Manifolds, in which the group of isometries act transitively.

Definition 2.11. *(a) A* ***homogeneous Riemannian manifold*** (M, g) *is a Riemannian manifold on which the group of its isometries,* $I(M, g)$*, is acting transitively (i.e., for any two points* $x, y \in M$*, there exists an isometry transforming* x *into* y*).*
(b) A Riemannian manifold (M, g) *is said to be* ***G-homogeneous Riemannian manifold*** *if* G *is a closed subgroup of* $I(M, g)$ *acting on* M *transitively.*

Several groups can act transitively on a given manifold. For example, the isometry group of Euclidean space consists of all motions, and its proper subgroup of parallel translations act transitively. Note also that, since an isometry preserves the metric g, it preserves the Levi-Civita connection, geodesics, volume elements, and curvatures.

The condition that G is closed is used only to simplify the exposition. Indeed, if G is any Lie group acting transitively and effectively on M, and if G is leaving invariant some Riemannian metric on M, then there exists a unique subgroup $\bar{G}$ of $D(M)$, the group of diffeomorphisms of M, such that, for any G-invariant Riemannian metric g on M, the group $\bar{G}$ is the closure of G in $I(M, g)$.

Now we state a crucial result regarding the group of isometries of Riemannian manifolds, which enables connects with the results of the previous section.

Theorem 2.12. *[8] Let (M,g) be a (pseudo)Riemannian manifold. Then the following assertions hold:*
(a) The group I(M,g) of all isometries of M is a Lie group smoothly acting on M.
(b) For any point $x \in M$, the isotropy group

$$I_x(M, g) = \{f \in I(M, g) : f(x) = x\}$$

is closed in I(M,g). Moreover, the isotropy representation

$$\rho : I_x(M, g) \to GL(T_xM), \qquad f \mapsto \rho(f) = T_xf$$

defines an isomorphism of the group $I_x(M, g)$ onto a closed subgroup of the orthogonal group $O(T_xM, g_x) \subset GL(T_xM)$.

Corollary 2.13. *[8] If (M,g) is a Riemannian manifold, then the following hold:*
(a) $I_x(M, g)$ is a compact subgroup of the group $I(M, g)$. Moreover, if M is compact, then I(M,g) is also compact;
(b) dim $I(M, g) \leq \frac{1}{2}n(n+1)$, and, moreover, the equality is possible here if and only if (M,g) has constant sectional curvature.

We note that $I(M, g)$ acts properly on any Riemannian manifold and that $I(M, g)$ may be compact (e.g., trivial), even if (M, g) is non-compact or pseudo-Riemannian (see [8]).

In applying the theory of Lie group actions to differential geometry, it is important to show that a given group of differentiable action of a manifold can be made into a Lie group action by introducing a suitable differentiable structure in it.

Below we provide a generalized version of Theorem 2.12.

Theorem 2.14. *[7] Let G be a locally compact group acting effectively on a connected manifold M of class C^k, $1 \leq k \leq \infty$, and let each action be of class C^1. Then G is a Lie group and the mapping $G \times M \to M$ is of class C^k.*

Theorem 2.15. *[7] The group G of isometries of a connected, locally compact metric space M is locally compact with respect to the compact-open topology.*

We recall that the compact-open topology of G is defined as follows. For any finite number of pairs (K_i, U_i) of compact subsets K_i and open subsets U_i of M, let $W = W(K_1, ..., K_s, U_1, ..., U_s) = \{\phi \in G : \phi(K_i) \subset U_i \text{ for } i = 1, ..., s\}$. Then the sets W of this form are taken as a base for the open sets of G. The conclusion of Theorem 2.15 is still true if M has finitely many connected components. Moreover, if M is compact, then G is also compact. The detailed proof of the above theorems and their subsequent results can be found in [7].

Let $G \subset I(M, g)$ be closed. By Corollary 2.13, the isotropy group of an arbitrary point $x \in M$ is the compact subgroup $H = \{f \in G : f(x) = x\}$ of the group $I(M, g)$. The compactness of M is equivalent to the compactness of G. Since an isometry f is uniquely defined by the image $f(x)$ of a single point x and by the corresponding tangent mapping $T_x f$, it follows that the linear isotropy representation $\rho(f) = T_x f$ of the isotropy subgroup H in $GL(T_x M)$ is exact (i.e. injective), hence the action of G is effective.

The following spaces are examples of the simplest non-compact homogeneous spaces (see [8]):

(i) the hyperbolic space $H^n = SO_0(1, n)/SO(n)$, where $SO_0(1, n)$ is the connected component of the identity of the group $O(1, n)$, which is given by a quadratic form of signature $(1, n)$ in $\mathbb{R}^{n+1}$;

(ii) the complex and quaternion analogs of the hyperbolic space:
$\mathbb{CH}^n = SU(1, n)/S(U(1) \times U(n))$, $\mathbb{HH}^n = Sp(1, n)/[Sp(1) \times Sp(n)]$.

Riemannian symmetric spaces are important examples of homogeneous spaces. A Riemannian manifold (M, g) is said to be **symmetric**, if for any point $x \in M$, there exists an isometry s_x of the manifold (M, g) such that $s_x(x) = x$ and $d(s_x)_x = -Id_{T_xM}$. The isometry s_x (it is uniquely defined if M is connected) is called the (central) symmetry centered at the point x. A Riemannian symmetric space (M, g) is complete, since any geodesic can be extended by using the symmetries centered at its endpoints. Furthermore, for any points $x, y \in M$, the symmetry with respect to the middle point of any geodesic segment connecting x and y (which exists because of completeness) interchanges x and y. Therefore, the isometry group acts transitively. For more detailed study of symmetric spaces refer to chapter XI of [6].

The following theorem regarding the completeness of homogeneous Riemannian manifold is easily obtained from the transitivity of the action of the Lie group G on the homogeneous space $M = G/H$ and the existence of a normal neighborhood for any point of a Riemannian manifold.

Theorem 2.16. *[7] A homogeneous Riemannian manifold is complete.*

Proof. Let x be a point of a homogeneous Riemannian manifold (M, g). There exists $r > 0$ such that, for every unit vector X at x, the geodesic $exp(sX)$ is defined for $|s| \leq r$. Let $\tau = x(s)$, $0 \leq s \leq a$, be any geodesic with canonical parameter s in M ($|\dot{x}(s)| = 1$). We shall show that $\tau = x(s)$ can be extended to a geodesic defined for $0 \leq s \leq a + r$. Let f be an isometry of M which maps x into $x(a)$. Then f^{-1} maps the unit vector $\dot{x}(a)$ at $x(a)$ into a unit vector X at x: $X = f^{-1}(\dot{x}(s))$. Since $exp(sX)$ is a geodesic through x, $f(exp(sX))$ is a geodesic through $x(a)$. We set

$$x(a + s) = f(exp(sX)), \qquad 0 \leq s \leq r.$$

Then $\tau = x(s)$, $0 \leq s \leq a + r$, is a geodesic. □

Remember that a vector field X on M is called a **Killing vector field** if the local one-parameter group of diffeomorphisms generated by this field consists of local isometries. Moreover, the bracket of two Killing vector fields is a Killing field. Therefore, the space of all Killing vector fields is a Lie subalgebra of the Lie algebra of all vector fields. Now we have the theorem relating the Killing vector fields to the group of isometries of M.

Theorem 2.17. *[7] On a complete manifold (M,g), any Killing field is complete, i.e., it generates a one-parameter group of isometries, and the Lie algebra of Killing vector fields is the Lie algebra of the Lie group I(M,g).*

To see the connection between the Lie algebra of Killing vector fields and the Lie algebra of the Lie group $I(M,g)$ fix a G-homogeneous Riemannian manifold (M,g) (always connected) and a point x in M. Then $H = G \cap I_x(M,g)$ is a compact subgroup of G by Corollary 2.13 and M is diffeomorphic to the quotient manifold G/H. We note that G acts (nearly) effectively on M, that is, H contains no (non-discrete) invariant subgroup of G. We denote by $\mathfrak{g}$ the Lie algebra of G as usual. For each X in $\mathfrak{g}$ we denote by $Exp(tX)$ the one-parameter subgroup of G generated by X. The action of $Exp(tX)$ on M turns it into a one-parameter group ϕ_t of diffeomorphisms of M, defined by

$$\phi_t(y) = Exp(tX)y.$$

We identify X in $\mathfrak{g}$ with the vector field on M generated by ϕ_t. In doing so, we identify $\mathfrak{g}$ with the set of those Killing vector fields of (M,g) which generate one-parameter subgroups of G (we recall that G is not necessarily the whole isometry group). There is one subtle point in this identification. Let $[\ ,\ ]$ be the Lie bracket of vector fields in M and $[\ ,\ ]_{\mathfrak{g}}$ the Lie algebra bracket of $\mathfrak{g}$. Then, using the identification given above, we have for X and Y in $\mathfrak{g}$

$$[X,Y]_{\mathfrak{g}} = -[X,Y].$$

The Lie algebra $\mathfrak{h}$ of $\mathfrak{g}$ generated by H is then identified with the subalgebra of those Killing vector fields in $\mathfrak{g}$ which vanish at x. Since H is compact, ad_H is a compact subgroup of the linear group of $\mathfrak{g}$. By already mentioned averaging procedure in the discussion of the equation (2.3), there exists an ad_H-invariant subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is the direct sum $\mathfrak{h} \oplus \mathfrak{m}$.

We choose once and for all such an $\mathfrak{m}$ (which is not necessarily unique). Then we identify $\mathfrak{m}$ with T_xM by taking the value of the corresponding Killing vector field at x. In this way, the isotropy representation ρ of H in T_xM is identified with the restriction of the adjoint representation ad of H to $\mathfrak{m}$.

Recall that the Corollary 2.10 states that a G-homogeneous Riemannian manifold (M, g) is completely determined by an ad_H-invariant inner product on $\mathfrak{m}$. A consequence of this fact is that the curvature of (M, g) may be computed in terms of the inner product on $\mathfrak{m}$ and the Lie algebra structure $\mathfrak{g}$. Notice that, because of the invariance of the curvature under isometries, we need only to know the curvature at the point x. Moreover, the curvature tensor at x is identified with a tensor on the vector space $\mathfrak{m}$. Since H acts by isometries, the resulting tensor on $\mathfrak{m}$ is in particular ad_H-invariant. In order to see how the computations of the curvature reduce to the computations in terms of the inner product on $\mathfrak{m}$ we look at some results. We first recall a useful result regarding the Levi-Civita connection and Killing vector fields.

Lemma 2.18. *[8] Let X, Y, and Z be Killing vector fields on a Riemannian manifold (M, g). Then*

$$2g(\nabla_X Y, Z) = g([X, Y], Z) + g([X, Z], Y) + g(X, [Y, Z]). \tag{2.14}$$

Proof. We have that if X is a Killing vector field, then for arbitrary vector fields V and W the following holds:

$$g(\nabla_V X, W) + g(\nabla_W X, V) = 0. \tag{2.15}$$

If X and Z are Killing vector fields, then

$$g([X,Z],X) = g(\nabla_X Z, X) - g(\nabla_Z X, X) = g(\nabla_X X, Z). \tag{2.16}$$

By permuting the Killing vector fields in the above identity, we get

$$g(\nabla_X Y, Z) + g(\nabla_Y X, Z) = g([X,Z],Y) + g(X,[Y,Z]). \tag{2.17}$$

Then the result follows from $\nabla_X Y - \nabla_Y X = [X,Y]$. □

Proposition 2.19. *[8] Let (M,g) be G-homogeneous and* $\mathfrak{m}$ *be defined as earlier. Let X, Y be Killing vector fields in* $\mathfrak{m}$*. Then, at the point x, we have*

$$(\nabla_X Y)_{\mathfrak{m}} = -\frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y) \tag{2.18}$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ *is defined by*

$$2(U(X,Y),Z) = ([Z,X]_{\mathfrak{m}}, Y) + (X,[Z,Y]_{\mathfrak{m}}) \tag{2.19}$$

for all Z in $\mathfrak{m}$*.*

The result of the above proposition is obtained directly from Lemma 2.18. Here and in the following, $[\ ,\ ]_{\mathfrak{m}}$ and $[\ ,\ ]_{\mathfrak{h}}$ are the components on $\mathfrak{m}$ and $\mathfrak{h}$ of $[\ ,\ ]_{\mathfrak{g}}$ respectively.

Theorem 2.20. *[8] Given two vectors X and Y at a point x of a homogeneous Riemannian manifold M, we have the following formula for the curvature*

$$\begin{aligned} g_x(R(X,Y)X,Y) &= -\frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 - \frac{1}{2}([X,[X,Y]_{\mathfrak{g}}]_{\mathfrak{m}}, Y) \\ &- \frac{1}{2}([Y,[Y,X]_{\mathfrak{g}}]_{\mathfrak{m}}, X) + |U(X,Y)|^2 - (U(X,X),U(Y,Y)) \end{aligned} \tag{2.20}$$

In particular, if $|X| = |Y| = 1$ *and (X,Y)=0, then this formula gives the sectional curvature of the 2-plane spanned by X and Y.*

Proof.

$$
\begin{aligned}
g_x(R(X,Y)X,Y) &= (\nabla_{[X,Y]}X,Y) - (\nabla_X\nabla_Y X,Y) + (\nabla_Y\nabla_X X,Y) \\
&= -(\nabla_Y X,[X,Y]) - X(\nabla_Y X,Y) + (\nabla_Y X,\nabla_X Y) \\
&+ Y(\nabla_X X,Y) - (\nabla_X X,\nabla_Y Y) \\
&= |\nabla_Y X|^2 - (\nabla_X X,\nabla_Y Y) + Y([X,Y],X) \\
&= \frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 + ([X,Y]_{\mathfrak{m}},U(X,Y)) + |U(X,Y)|^2 \\
&- (U(X,X),U(Y,Y)) + ([Y,[X,Y]],X) + ([X,Y],[Y,X]) \\
&= |U(X,Y)|^2 - (U(X,X),U(Y,Y)) + \frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 \\
&+ \frac{1}{2}([[X,Y]_{\mathfrak{m}},X]_{\mathfrak{m}},Y) + \frac{1}{2}([[X,Y]_{\mathfrak{m}},Y]_{\mathfrak{m}},X) \\
&+ ([Y,[X,Y]_{\mathfrak{h}}]_{\mathfrak{m}},X) + ([Y,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}},X) - |[X,Y]_{\mathfrak{m}}|^2. \qquad (2.21)
\end{aligned}
$$

Using the ad_H-invariance of (,) we have $([Y,[X,Y]_{\mathfrak{h}}]_{\mathfrak{m}},X) = (Y,[[X,Y]_{\mathfrak{h}},X]_{\mathfrak{m}})$ and the result follows. □

3. GEOMETRY ON LIE GROUPS

In the first section we discuss the left-invariant vector fields on Lie groups and their structure coefficients, which determine the geometry of the group. Then we define Lie-algebra valued Maurer-Cartan forms, which carry the information about left-invariant one-forms. In the third part, the left-invariant metrics are studied and the curvature tensors are determined in terms of metric components and structure coefficients. Finally, we give the Bianchi classification of the three-dimensional Lie algebras.

3.1. Frames and Structure Equation

Consider the set of n vectors $\{V_1, V_2, ..., V_n\}$ which is a basis of T_eG where G is an n-dimensional Lie group with the algebra $\mathfrak{g}$. Then the basis can be extended to the set of n linearly independent left-invariant vector fields $\{X_1, X_2, ..., X_n\}$ by setting $X_a(g) = (L_g)_* V_a$ at each point $g \in G$. The set $\{X_a\}$ is a frame of basis defined throughout G. Since $[X_a, X_b](g)$ is again an element of $\mathfrak{g}$ at g, it can be expressed in terms of $\{X_a\}$ as

$$[X_a, X_b] = f_{ab}{}^c X_c \tag{3.1}$$

where $f_{ab}{}^c$ are called the structure coefficients of the Lie group G. If G is a matrix group, at $g = e$, the unit element of the group G, the equation (3.1) turns out to be the commutator of matrices V_a and V_b. In fact the coefficients $f_{ab}{}^c$ are independent of g. Let $f_{ab}{}^c(e)$ be the structure coefficients at the unit element. When $(L_g)_*$, the pushforward of the left action diffeomorphism, is applied to the Lie bracket $[X_a, X_b](e) = f_{ab}{}^c(e) X_c(e)$, we get

$$[X_a, X_b](g) = f_{ab}{}^c(e) X_c(g) \tag{3.2}$$

which shows that the structure coefficients are g-independent i.e., the structure coefficients determine a Lie group completely (Lie's Theorem). Now consider a dual basis to $\{X_a\}$ and denote it by $\{\theta^a\}$; $\theta^a(X_b) = \delta^a{}_b$. $\{\theta^a\}$ becomes a basis for the left-invariant one-forms. We state the properties of structure coefficients, left-invariant vector fields, and left-invariant one-forms, proofs of which follow from direct computations on this basis (see [5] p. 176):

(a) Skew-symmetry

$$f_{ab}{}^c = -f_{ba}{}^c \tag{3.3}$$

(b) Jacobi identity

$$f_{ab}{}^c f_{cd}{}^e + f_{da}{}^c f_{cb}{}^e + f_{bd}{}^c f_{ca}{}^e = 0 \tag{3.4}$$

(c) First Maurer-Cartan structure equation

$$d\theta^a + \frac{1}{2} f_{bc}{}^a \theta^b \wedge \theta^c = 0 \tag{3.5}$$

(d) Lie derivatives

$$\mathcal{L}_{X_a} X_b = f_{ab}{}^c X_c \tag{3.6}$$

$$\mathcal{L}_{X_b} \theta^a = -f_{bc}{}^a \theta^c \tag{3.7}$$

3.2. Maurer-Cartan Forms

Define a Lie-algebra valued one-form $\theta : T_g G \to T_e G$ by

$$\theta : X \mapsto (L_{g^{-1}})_* X = (L_g)_*^{-1} X \qquad X \in T_g G. \tag{3.8}$$

θ is called the canonical one-form or Maurer-Cartan form on G.

Theorem 3.1. *[5] (a) The canonical one-form θ is expanded as*

$$\theta = V_a \otimes \theta^a \tag{3.9}$$

where $\{V_a\}$ is the basis of T_eG and $\{\theta^a\}$ is the dual basis of T_g^*G.

(b) The canonical one-form θ satisfies

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0 \tag{3.10}$$

where $d\theta = V_a \otimes d\theta^a$ and

$$[\theta \wedge \theta] \equiv [V_a, V_b] \otimes \theta^a \wedge \theta^b \tag{3.11}$$

Proof. (a) Let $Y = Y^a X_a \in T_gG$, where $\{X_a\}$ is the set of frame vectors generated by $\{V_a\}$. From the equation (3.8) we find

$$\theta(Y) = Y^a \theta(X_a) = Y^a (L_g)_*^{-1}[(L_g)_* V_a] = Y^a V_a. \tag{3.12}$$

On the other hand

$$(V_a \otimes \theta^a)(Y) = Y^b V_a \theta^a(X_b) = Y^b V_a \delta^a{}_b = Y^a V_a. \tag{3.13}$$

(b) We use the first Maurer-Cartan structure equation (3.5):

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = -\frac{1}{2} V_a \otimes f_{bc}{}^a \theta^b \wedge \theta^c + \frac{1}{2} f_{bc}{}^a V_a \otimes \theta^b \wedge \theta^c = 0. \tag{3.14}$$

□

In some sense, θ is an identity automorphism. It is common to write (3.8) as

$$\theta_g = g^{-1}dg. \tag{3.15}$$

This is not a very proper notation, but correctly indicates that tangent vectors at g are brought back to vectors at e by left translation along g^{-1}. If G is a group of matrices, then $g^{-1}dg$ has a precise meaning as written and it is straightforward to verify that θ_g in (3.15) is left-invariant:

$$(hg)^{-1}d(hg) = g^{-1}h^{-1}hdg = g^{-1}dg \tag{3.16}$$

and satisfies the first Maurer-Cartan equation (3.10):

$$d(g^{-1}dg) = dg^{-1}\wedge dg = -g^{-1}dgg^{-1}\wedge dg = -g^{-1}dg\wedge g^{-1}dg = -\frac{1}{2}[g^{-1}dg, g^{-1}dg]. \tag{3.17}$$

For a more detailed discussion refer to [4].

One can determine Maurer-Cartan forms θ^a explicitly in terms of the structure coefficients ${f_{ab}}^c$. To see this consider N_0, an open star-shaped neighborhood of 0 in $\mathfrak{g}$. N_0 is mapped diffeomorphically by Exp, the exponential mapping of a Lie group, onto an open neighborhood N_e of e in G. Then take $(x^1, x^2, ..., x^n)$ as the canonical coordinates of $x = ExpV$ $(V \in N_0)$, where $V = x^a V_a$, with respect to the basis $\{V_a\}$. Exp is a smooth map from $\mathfrak{g}$ into G, thus the forms $Exp^*\theta^a$, the pullbacks of the one-forms θ^a, can be expressed in terms of the canonical coordinates $(x^1, x^2, ..., x^n)$ of $\mathfrak{g}$ with respect to the basis $\{V_a\}$,

$$(Exp^*\theta^a)(V_b) = {A^a}_b(x^1, x^2, ..., x^n), \qquad {A^a}_b \in C^\infty(\mathbb{R}^n). \tag{3.18}$$

Then, if $f \in C^\infty(G)$,

$$(Exp_*V_b)f = V_b(f \circ Exp) = (\frac{d}{dt}f(Exp(V + tV_b)))|_{t=0} \tag{3.19}$$

whence

$$Exp_* V_b = \frac{\partial}{\partial x^b}. \tag{3.20}$$

Consequently,

$$\theta^a(\frac{\partial}{\partial x^b}) = \theta^a(Exp_* V_b) = (Exp^* \theta^a)(V_b) \tag{3.21}$$

so

$$\theta^a = A^a{}_b(x^1, x^2, ..., x^n)dx^b. \tag{3.22}$$

Using the formula for the differential of the exponential map (see [4] page 105)

$$Exp_*|_V = ((L_{ExpV})_*)|_e \circ \frac{1 - e^{-ad_V}}{ad_V} \qquad (V \in \mathfrak{g}) \tag{3.23}$$

and the left-invariance of θ^a we have

$$A^a{}_b(x^1, x^2, ..., x^n) = \theta^a(Exp_* V_b) = \theta^a|_e(\frac{1 - e^{-ad_V}}{ad_V}(V_b)) \tag{3.24}$$

where ad_V is a linear transformation of the Lie algebra $\mathfrak{g}$ given by $ad_V(W) = [V, W]$ and where the fraction is given by the power series

$$\frac{1 - e^{-ad_V}}{ad_V} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}(ad_V)^k. \tag{3.25}$$

For the details of the above derivation refer to [4].

3.3. Left-invariant Metrics

As we discussed earlier in section 2.2, a left-invariant metric g on the group G is given by a non-degenerate metric B on $\mathfrak{g}$,

$$g = B(\theta, \theta) = B_{ab}\theta^a\theta^b. \tag{3.26}$$

The metric g is uniquely determined up to the action of the automorphism group on B. By using the action of the automorphism group we fix the metric B into classes where in each class the metric B depends on a small set of continuous parameters $\{u, v, w, ...\}$. To see this, fix a basis $\{V_a\}$ of $\mathfrak{g}$ and its dual basis $\{\theta^a\}$. Then the symmetric non-degenerate bilinear form $B(\theta, \theta) = B_{ab}\theta^a\theta^b$ on $\mathfrak{g}$ corresponds to a left-invariant metric on G. Consider a new basis $\{V_{\bar{a}}\}$ with its dual basis $\{\theta^{\bar{a}}\}$ on $\mathfrak{g}$ obtained from an automorphism of the Lie algebra, which preserves the Lie brackets. Suppose $V_{\bar{b}} = f^b{}_{\bar{b}}V_b$, so $V_b = f^{\bar{b}}{}_bV_{\bar{b}}$, where $f^a{}_{\bar{b}}f^{\bar{b}}{}_b = \delta^a{}_b$. Now, if $\theta^a = \xi^a{}_{\bar{a}}\theta^{\bar{a}}$, then we get

$$\delta^a{}_b = \theta^a(V_b) = \xi^a{}_{\bar{a}}\theta^{\bar{a}}(f^{\bar{b}}{}_bV_{\bar{b}}) = \xi^a{}_{\bar{a}}f^{\bar{b}}{}_b\delta^{\bar{a}}{}_{\bar{b}} = \xi^a{}_{\bar{a}}f^{\bar{a}}{}_b. \tag{3.27}$$

Thus, it follows that $\xi^a{}_{\bar{a}} = f^a{}_{\bar{a}}$. Hence, the left-invariant metrics are, indeed, uniquely determined up to the action $B \mapsto S^TBS$ with $S \in$Aut$(\mathfrak{g})$.

For example, consider a three dimensional Lie group $SU(2)$. One can fix a basis $\{V_1, V_2, V_3\}$ for the Lie algebra $\mathfrak{su}_2$ with

$$[V_1, V_2] = V_3 \qquad [V_2, V_3] = V_1 \qquad [V_3, V_1] = V_2. \tag{3.28}$$

Consider the following Lie algebra automorphism:

$$\begin{pmatrix} \tilde{V_1} \\ \tilde{V_2} \\ \tilde{V_3} \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \tag{3.29}$$

For simplicity denote the rows of the 3×3 matrix as u, v, and w, respectively. From being a Lie algebra automorphism it directly follows that $u \times v = w$, $v \times w = u$, and $w \times u = v$, when u, v, and w are considered as vectors in $\mathbb{R}^3$. From this we can deduce that the set $\{u, v, w\}$ is an orthonormal basis of $\mathbb{R}^3$. Hence, the automorphism group of $\mathfrak{su}_2$ is precisely $O(3)$. Since the bilinear form is symmetric, it can be diagonalized by an $SO(3)$ action.

When we use the basis of the left-invariant one-forms θ^a, all curvature calculations become algebraic. For the Levi-Civita connection with $\nabla_a \theta^b = \Gamma_{ac}{}^b \theta^c$, we obtain the following results:

$$\nabla_a X_b = -\Gamma_{ab}{}^c X_c \tag{3.30}$$

$$B_{bd}\Gamma_{ac}{}^d + B_{dc}\Gamma_{ab}{}^d = 0 \tag{3.31}$$

$$\Gamma_{ba}{}^c - \Gamma_{ab}{}^c = f_{ab}{}^c \tag{3.32}$$

from direct computation, from the metric compatibility, namely, from $0 = \nabla_a(B_{bc}\theta^b\theta^c)$, and from Torsion-free condition, $\nabla_a X_b - \nabla_b X_a = [X_a, X_b]$, respectively.

By permuting the indices a, b, and c in the equation (3.31) we get three new equations. Using repeatedly obtained equations and the equality (3.32) for the Levi-Civita coefficients we obtain the following identity:

$$\Gamma_{ab}{}^c = \frac{1}{2}(B^{ce}B_{da}f_{be}{}^d - B^{ce}B_{bd}f_{ea}{}^d - f_{ab}{}^c). \tag{3.33}$$

The Riemann curvature

$$R^a{}_{bcd}X_a = \nabla_c\nabla_d X_b - \nabla_d\nabla_c X_b - \nabla_{[X_c, X_d]}X_b \tag{3.34}$$

is given by

$$R^a{}_{bcd} = \Gamma_{db}{}^k\Gamma_{ck}{}^a - \Gamma_{cb}{}^k\Gamma_{dk}{}^a + f_{cd}{}^k\Gamma_{kb}{}^a. \tag{3.35}$$

3.4. Three-dimensional Lie Algebras

Now we present the classification of three-dimensional Lie algebras. Linear transformation of the basis $\{V_a\}$ of $\mathfrak{g}$, the Lie algebra of a Lie group G, transform the structure coefficients tensorially. To identify distinct algebras one must determine the sets of constants ${f_{ab}}^c$ which cannot be related by linear transformations. For example, in a two-dimensional Lie algebra there are only two types, the first being abelian i.e., with the vanishing bracket and the second with the bracket $[V_1, V_2] = V_1$.

The three-dimensional Lie algebras were originally classified by Bianchi (1898) into eleven classes (see [9]), nine of which are single groups and two of which have a continuum of isomorphism classes. Sometimes two of the groups are included in the infinite families, giving nine, called Bianchi I-IX, instead of eleven classes. Bianchi classified them by considering the dimension of the derived algebra, which is rarely presented using the original method of him. Another way of identifying the three-dimensional Lie algebras is given in [10]. The structure coefficients of the Lie algebra are allocated into two classes. The unimodular Lie algebras, where ${f_{ab}}^b = 0$ and the non-unimodular ones with $\frac{1}{2}{f_{ab}}^b = h_a \neq 0$. The structure coefficients of the three-dimensional real Lie algebras can be parametrized by the vector components h_a and a symmetric tensor density n^{ab} (see [11] and [12] and references therein),

$$ {f_{bc}}^a = \varepsilon_{bcd} n^{da} + h_b {\delta_c}^a - h_c {\delta_b}^a \tag{3.36} $$

where ε_{bcd} is totally antisymmetric tensor. The Jacobi identity reduces to $h_a n^{ab} = 0$. Thus, the classification of the structure coefficients reduces to the classification of symmetric tensor densities that annihilate vector, a problem solved in [10].

In the following discussion and sections we use convention of [13]. We first give an alternative classification, which will be convenient for us later and at the end present all eleven classifications of Bianchi.

Besides the abelian $\mathbb{R}^3$ and the two simple algebras $\mathfrak{sl}_2$ and $\mathfrak{su}_2$, we also have the Lie algebras $\mathfrak{a}_0$ and $\mathfrak{a}_\infty$, and two continuous families of Lie algebras: $\mathfrak{iso}(1,1;\theta)$ and $\mathfrak{iso}(2;\theta)$. The parameter θ varies in $[0, \frac{\pi}{2}]$ and $\mathfrak{iso}(1,1;0)$=$\mathfrak{iso}(2;0)$. For convenience we start by introducing continuous families.

The Lie algebra $\mathfrak{iso}(2;\theta)$ is spanned by l, m_1, and m_2, with the only non-vanishing brackets

$$[l, m_1] = 2\cos\theta m_1 + 2\sin\theta m_2, \qquad [l, m_2] = 2\cos\theta m_2 - 2\sin\theta m_1. \tag{3.37}$$

That is, l acts as a euclidean rotation twisted by a simultaneous dilation on the vector $m_i \in \mathbb{R}^2$. The second family $\mathfrak{iso}(1,1;\theta)$ is spanned by l', m'_1, and m'_2 with non-vanishing brackets

$$[l', m'_1] = 2\cos\theta m'_1 + 2\sin\theta m'_2, \qquad [l, m_2] = 2\cos\theta m'_2 + 2\sin\theta m'_1. \tag{3.38}$$

That is, l' acts as a Lorentzian rotation twisted by a dilation on the vector $m'_i \in \mathbb{R}^{1,1}$. Both Lie algebras limit at $\theta = 0$ to the same Lie algebra $\mathfrak{iso}(1,1;0) = \mathfrak{iso}(2;0)$, in which the action of l on m_1 and m_2 is a rescaling by the same factor. For simplicity we drop primes in (3.38).

It is convenient to introduce a continuous family of Lie algebras $\mathfrak{a}_\lambda$, $\lambda \in \mathbb{R}$ to define $\mathfrak{a}_0$ and $\mathfrak{a}_\infty$. Let it be spanned by r, x, and y, and with non-vanishing brackets

$$[r, x] = x - \lambda y, \qquad [r, y] = x + y. \tag{3.39}$$

The Lie algebra $\mathfrak{a}_0$ is precisely given by $\mathfrak{a}_\lambda$ at the value of $\lambda = 0$. The Lie algebra $\mathfrak{a}_\infty$ can be thought of as a limit of the family: first rescale the basis by $(r, x, y) \mapsto \frac{1}{\lambda}(r, x, y)$ and then send λ to infinity. The only non-vanishing Lie bracket of the limit $\mathfrak{a}_\infty$ is $[r, x] = -y$.

In fact, the family $\mathfrak{a}_\lambda$ for $\lambda < 0$ is isomorphic to the family $\mathfrak{iso}(1,1;\theta)$ with $\lambda = -\tan^2\theta$ and the family $\mathfrak{a}_\lambda$ with $\lambda > 0$ is isomorphic to $\mathfrak{iso}(2;\theta)$ with $\lambda = \tan^2\theta$. However $\mathfrak{a}_0$ is not isomorphic to $\mathfrak{iso}(1,1;0) = \mathfrak{iso}(2;0)$. None of the Lie algebras $\mathfrak{a}_\infty$, $\mathfrak{iso}(2;\frac{\pi}{2})$, and $\mathfrak{iso}(1,1;\frac{\pi}{2})$ are isomorphic to each other. Hence, any three-dimensional Lie algebra is isomorphic to one of

$$\mathbb{R}^3,\quad \mathfrak{sl}_2,\quad \mathfrak{su}_2,\quad \mathfrak{a}_0,\quad \mathfrak{a}_\infty,\quad \mathfrak{iso}(1,1;\theta),\quad \mathfrak{iso}(2;\theta).$$

An alternative classification of three dimensional Lie algebras is thus given by

$$\mathbb{R}^3,\quad \mathfrak{sl}_2,\quad \mathfrak{su}_2,\quad \mathfrak{a}_\lambda,\quad \mathfrak{a}_\infty,\quad \mathfrak{iso}(1,1;\frac{\pi}{2}),\quad \mathfrak{iso}(2;\frac{\pi}{2}),\quad \mathfrak{iso}(2;0).$$

Finally, let us present all eleven classes of Bianchi:

Bianchi	**Our**	$f_{ab}{}^b$	**Lie brackets**
I	$\mathbb{R}^3$	0	trivial
II	$\mathfrak{a}_\infty$	0	$[r,x] = -y$
III	$\mathfrak{iso}(1,1;\frac{\pi}{4})$	h_a	$[l_0,m_1] = m_1 + m_2,\ [l_0,m_2] = m_1 + m_2$
IV	$\mathfrak{a}_0$	h_a	$[r,x] = x,\ [r,y] = x + y$
V	$\mathfrak{iso}(2;0)$	h_a	$[l_0,m_1] = m_1,\ [l_0,m_2] = m_2$
VI	$\mathfrak{iso}(1,1;\theta)$	h_a	$[r,x] = x + \tan^2\theta\ y,\ [r,y] = x + y$
VI_0	$\mathfrak{iso}(1,1;\frac{\pi}{2})$	0	$[l_0,m_1] = m_2,\ [l_0,m_2] = m_1$
VII	$\mathfrak{iso}(2;\theta)$	h_a	$[r,x] = x - \tan^2\theta\ y,\ [r,y] = x + y$
VII_0	$\mathfrak{iso}(2;\frac{\pi}{2})$	0	$[l_0,m_1] = m_2,\ [l_0,m_2] = -m_1$
VIII	$\mathfrak{sl}_2$	0	$[\tau_0,\tau_1] = \tau_2,\ [\tau_2,\tau_1] = \tau_0,\ [\tau_2,\tau_0] = \tau_1$
IX	$\mathfrak{su}_2$	0	$[\tau_1,\tau_2] = \tau_3,\ [\tau_2,\tau_3] = \tau_1,\ [\tau_3,\tau_1] = \tau_2$

Table 3.1. Classification of three-dimensional Lie algebras. In the third row h_a stands for a nonzero vector. Here $\theta \in (0,\frac{\pi}{2})$ and some brackets are presented in more suitable basis.

4. SOLUTIONS OF MINIMAL MASSIVE GRAVITY

We begin this chapter by introducing the Minimal Massive Gravity (MMG) and in the subsequent sections construct the homogeneous spacetime solutions of it.

4.1. Minimal Massive Gravity

MMG is a recently constructed three-dimensional higher derivative gravity model, which attracted great attention due to its interesting physical properties [1]. The theory is defined by the field equation

$$G_{\mu\nu} + a g_{\mu\nu} + b C_{\mu\nu} + c J_{\mu\nu} = 0 \tag{4.1}$$

where $G_{\mu\nu}$ is the Einstein tensor and where the symmetric, traceless, parity-odd and covariantly conserved Cotton tensor $C_{\mu\nu}$ is defined in terms of Schouten tensor $S_{\sigma\nu}$ as

$$C^{\mu}{}_{\nu} \equiv \epsilon^{\mu\rho\sigma} \nabla_{\rho} S_{\sigma\nu}, \qquad S_{\sigma\nu} = R_{\sigma\nu} - \frac{1}{4} R g_{\sigma\nu}. \tag{4.2}$$

The Levi-Civita pseudo tensor is defined as $\epsilon_{\mu\rho\sigma} = \sqrt{-g}\varepsilon_{\mu\rho\sigma}$ in terms of the weight ± 1 tensor density $\varepsilon_{\mu\rho\sigma}$, where we use the convention $\varepsilon_{012} = +1$. The symmetric, curvature-squared tensor $J_{\mu\nu}$ is

$$\begin{aligned} J^{\mu\nu} &\equiv -\frac{1}{2}\epsilon^{\mu\rho\sigma}\epsilon^{\nu\tau\eta} S_{\rho\tau} S_{\sigma\eta} \\ &= S^{\mu\rho} S^{\nu}{}_{\rho} - S^{\mu\nu} S - \frac{1}{2} g^{\mu\nu} (S^{\rho\sigma} S_{\rho\sigma} - S^2) \\ &= R^{\mu\rho} R^{\nu}{}_{\rho} - \frac{3}{4} R^{\mu\nu} R - \frac{1}{2} g^{\mu\nu} (R^{\rho\sigma} R_{\rho\sigma} - \frac{5}{8} R^2) \end{aligned} \tag{4.3}$$

with $S \equiv g^{\mu\nu} S_{\mu\nu}$. J-tensor is not covariantly conserved on its own, i.e. it does not satisfy Bianchi identity so that MMG field equation (4.1) can not be derived from an action that only involves the metric and its curvature tensors (see [1]).

Finally, the coefficients a, b and c in terms of physical parameters are

$$a = \frac{\bar{\Lambda}_0}{\bar{\sigma}}, \qquad b = \frac{1}{\mu\bar{\sigma}}, \qquad c = \frac{\gamma}{\mu^2\bar{\sigma}}. \tag{4.4}$$

When $\gamma = 0$ ($c = 0$) the model reduces to the Topological Massive Gravity (TMG) model, where such solutions were studied before (see [12–15]).

There are two special points in the parameter space of the MMG theory (see [1]). The first one is called the 'chiral point' for which the central charges vanish and is given by

$$\bar{\sigma} + \frac{\gamma}{2\mu^2 l_{ch}^2} \pm \sqrt{\bar{\sigma}^2 - \frac{\gamma\bar{\Lambda}_0}{\mu^2}} = 0 \qquad \text{or} \qquad 1 + \frac{c}{2b^2}(1 - ac) \pm \sqrt{1 - ac} = 0. \tag{4.5}$$

The second one is called the 'merger point' where:

$$\bar{\Lambda}_0 = \frac{\mu^2\bar{\sigma}^2}{\gamma} \qquad \text{or} \qquad ac = 1. \tag{4.6}$$

The strategy of solving the field equations of MMG on three-dimensional Lie groups that posses a left-invariant metric is based on the results and discussions of section 3.3. A Lie algebra basis τ_a is fixed up to the automorphism group for each Lie algebra $\mathfrak{g}$ of the three-dimensional Lie algebras. The metric g_{ab} of the group is identified with a metric B_{ab} on the Lie algebra. Then the metric B_{ab} on $\mathfrak{g}$ is fixed in the space of orbits of metrics under the action of the algebra's automorphisms. The field equations then reduce to algebraic equations on B_{ab}.

Instead of solving algebraic equations for the metric parameters $\{u, v, w, ...\}$ in terms of the parameters $\{a, b, c\}$ of the MMG theory it is more convenient and illuminating to display the parameters in the theory in terms of the parameters of the metric, since solving for the parameters of the metric in terms the parameters of the MMG theory is very complicated in general, except for some special cases. This reduces to solving a system of linear equations

$$A \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = V \tag{4.7}$$

for a,b, and c, where A is a matrix is of the dimension $k \times 3$ and V is a vector $k \times 1$ vector with $k = 3, 4$ or 6. The rank of the matrix A can be at most three. When the rank of A is three, the linear equation (4.7) has a unique solution, provided that the solution exists. If the solution exists, the cases when the rank of A is less than three should be considered separately as in such cases the solutions may totally differ from the general solutions. If A is of the dimension 3×3, then computing the determinant of A is enough to determine when the rank of A is less than three. For the cases when A is not a square matrix and for the cases when (4.7) does not have a general solution more careful analysis is required. For example, it may be the case that for a particular relation among the parameters of the metric the system becomes consistent. In the following section we provide the general solutions and the special solutions for simpler cases, some of which are also homogeneous solutions of TMG theory.

In the following sections we assume the metrics to be Lorentzian with mostly plus signature $(-, +, +)$.

4.2. Solutions on $SL(2, \mathbb{R})$

For the Lie algebra $\mathfrak{sl}_2$ of $SL(2, \mathbb{R})$ a basis $\{\tau_0, \tau_1, \tau_2\}$ can be fixed with

$$[\tau_0, \tau_1] = \tau_2, \qquad [\tau_2, \tau_1] = \tau_0, \qquad [\tau_2, \tau_0] = \tau_1. \tag{4.8}$$

Let θ^a be a dual basis of τ_a. Elements of $SL(2, \mathbb{R})$ can be parametrized by a group representative as (see [16])

$$\mathcal{V}(x) = e^{t(\tau_0+\tau_2)} e^{\sigma\tau_1} e^{\zeta\tau_2}. \tag{4.9}$$

As we discussed earlier after the equation (3.15), it follows that the Maurer-Cartan one-forms are

$$\mathcal{V}^{-1}d\mathcal{V} = (e^{\sigma}\cosh\zeta dt - \sinh\zeta d\sigma)\tau_0 + (\cosh\zeta d\sigma - e^{\sigma}\sinh\zeta dt)\tau_1 + (d\zeta + e^{\sigma}dt)\tau_2. \quad (4.10)$$

A left-invariant metric g on $SL(2,\mathbb{R})$ is given by a non-degenerate metric B on $\mathfrak{sl}_2$, $g = B_{ab}\theta^a\theta^b$, and is unique up to the action of $B \mapsto S^T BS$ with $S \in SO(1,2)$. Under the action of the automorphism group B there are four classes (see e.g. [13]).

4.2.1. 111-type metric

111-type metric is of the form

$$g = u\theta^0\theta^0 + v\theta^1\theta^1 + w\theta^2\theta^2. \quad (4.11)$$

The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$\begin{aligned} a &= \frac{1}{Q}\cdot\frac{(u^2+(v-w)^2+2u(v+w))^3}{8uvw} \\ b &= \frac{-1}{Q}\cdot 8\sqrt{-uvw}(u+v-w)(u-v+w)(u+v+w) \\ c &= \frac{1}{Q}\cdot 8uvw(u^2+(v-w)^2+2u(v+w)) \\ R &= -\frac{u^2+(v-w)^2+2u(v+w)}{2uvw} \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} Q &= 9u^4+4u^3(v+w)+4u(v-w)^2(v+w) \\ &- 2u^2(5v^2-2vw+5w^2)+(v-w)^2(9v^2+14vw+9w^2). \end{aligned} \quad (4.13)$$

Moreover A in the equation (4.7) is a 3×3 matrix with the determinant

$$detA = -Q \cdot \frac{(u+v)(v-w)(u+w)}{8(-uvw)^{5/2}}. \tag{4.14}$$

Thus the cases $Q = 0$, $u = -v$ and $v = w$ should be considered separately. We skip the case $Q = 0$ due to its complexity.

$\boldsymbol{u = -v}$: This type of the metric is called as spacelike warped AdS, in which case $w > 0$ and the coefficients a, b, and c in terms of u, v, and w are equal to

$$a = \frac{16v^2(-4v+w) + c(4v-7w)(4v-3w)}{192v^4}, \qquad b = \frac{8v^2 + c(4v-3w)}{12\sqrt{v^2w}}. \tag{4.15}$$

The spacelike warped AdS metric in [17] is given by

$$ds^2 = \frac{l^2}{(\nu^2+3)}[-dt^2(1+r^2) + \frac{dr^2}{1+r^2} + \frac{4\nu^2}{\nu^2+3}(d\zeta + rdt)^2]. \tag{4.16}$$

The metric parameters v and w in the equation (4.16) correspond to

$$v = \frac{l^2}{(\nu^2+3)}, \qquad w = \frac{4l^2\nu^2}{(\nu^2+3)^2}. \tag{4.17}$$

The limit $\nu \to 1$ in (4.16) corresponds to the AdS metric, where $-u = v = w$. It is worthwhile to note that in all the above cases when $b = 0$ we get the equality (4.6).

$\boldsymbol{v = w}$: This type of metric is called as timelike warped AdS, in which case $u < 0$ and the coefficients a, b, and c in terms of u, v, and w are equal to

$$a = \frac{-16w^2(u+4w) + c(3u+4w)(7u+4w)}{192w^4}, \qquad b = \frac{8w^2 + c(3u+4w)}{12w\sqrt{-u}}. \tag{4.18}$$

4.2.2. 12-type metric

12-type metric is of the form

$$g = v(-\theta^0\theta^0 + \theta^1\theta^1) + w\theta^2\theta^2 + z(\theta^0 + \theta^1)^2 \tag{4.19}$$

with $vwz \neq 0$, and it is always Lorentzian. It is mostly plus for $w > 0$. Here, z can be rescaled freely and the metric depends only on its sign. It is a z-deformation of type 111 with $u = -v$.

The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$\begin{aligned} a &= \frac{1}{Q} \cdot \frac{(w-4v)^3}{8v^2} \\ b &= \frac{1}{Q} \cdot 8(2v-w)\sqrt{v^2 w} \\ c &= \frac{1}{Q} \cdot 8v^2(w-4v) \\ R &= \frac{w-4v}{2v^2} \end{aligned} \tag{4.20}$$

where $Q = 16v^2 + 8vw - 9w^2$. A in the equation (4.7) is a 4×3 matrix. We find out that the cases $Q = 0$ and $v = w$ should be considered separately. The case $16v^2 + 8vw - 9w^2 = 0$ does not provide interesting results, however in the case $v = w$ the metric is known to be as null warped AdS. We also note that, when $w = 4v$ the solution is Kundt type (see [18]).

$\boldsymbol{v = w}$: The coefficients a, b, and c in terms of u, v, and w are equal to

$$a = -\frac{16w + c}{64w^2}, \qquad b = \frac{8w + c}{12\sqrt{w}}. \tag{4.21}$$

Note that in all the above cases when $b = 0$ we get the equality (4.6).

4.2.3. 3-type metric

3-type metric is of the form

$$g = 2u(-\theta^0\theta^0 + \theta^1\theta^1 + \theta^2\theta^2) + 2z(\theta^0\theta^2 + \theta^1\theta^2) \tag{4.22}$$

with $uz \neq 0$, which always has a Lorentzian signature and it is mostly plus for $u > 0$. It is a z-deformation of type 111 with $-u = v = w$ (anti de-Sitter).

The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$a = -\frac{9}{80u}, \qquad b = \frac{8\sqrt{2}}{15}\sqrt{u}, \qquad c = -\frac{16u}{5}, \qquad R = -\frac{3}{4u}. \tag{4.23}$$

We also note that the solution is attained only at the chiral point i.e. the coefficients satisfy the equality (4.5) with the plus sign.

4.2.4. $1z\bar{z}$-type metric

$1z\bar{z}$-type metric is of the form

$$g = u(-\theta^0\theta^0 + \theta^1\theta^1) + 2z\theta^0\theta^1 + w\theta^2\theta^2 \tag{4.24}$$

with $uzw \neq 0$. The metric is always Lorentzian and mostly plus for $w > 0$. It is a z-deformation of type 111 with $u = -v$.

The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$\begin{aligned} a &= \frac{1}{Q}\cdot\frac{(4uw-w^2+4z^2)^3}{8w(u^2+z^2)} \\ b &= \frac{1}{Q}\cdot 8(w-2u)(w^2+4z^2)\sqrt{w(u^2+z^2)} \\ c &= \frac{1}{Q}\cdot 8w(4uw-w^2+4z^2)(u^2+z^2) \\ R &= -\frac{4uw-w^2+4z^2}{2w(u^2+z^2)} \end{aligned} \tag{4.25}$$

where $Q=(3w^2+4z^2)^2-16u^2(w^2+8z^2)-8uw(w^2-4z^2)$. Here A of the equation (4.7) is also a 4×3 matrix. The only special case that should be considered separately is when $Q=0$. We also note that when $b=0$ the equality (4.6) holds.

4.3. Solutions on $SU(2)$

We can fix a basis $\{\tau_1,\tau_2,\tau_3\}$ and its dual basis θ^a for the Lie algebra $\mathfrak{su}_2$ with

$$[\tau_1,\tau_2]=\tau_3 \qquad [\tau_2,\tau_3]=\tau_1 \qquad [\tau_3,\tau_1]=\tau_2. \tag{4.26}$$

An element of $SU(2)$ can be parametrized by (see [16])

$$\mathcal{V}=e^{\phi\tau_3}e^{\xi\tau_2}e^{\psi\tau_3}. \tag{4.27}$$

The Maurer-Cartan one-forms are

$$\begin{aligned} \mathcal{V}^{-1}d\mathcal{V} &= (\sin\psi d\xi-\cos\psi\sin\xi d\phi)\tau_1 \\ &+ (\cos\psi d\xi+\sin\psi\sin\xi d\phi)\tau_2+(d\psi+\cos\xi d\phi)\tau_3. \end{aligned} \tag{4.28}$$

A left-invariant metric g on $SU(2)$ is given by a non-degenerate metric B on $\mathfrak{su}_2$, $g=B_{ab}\theta^a\theta^b$, and is unique up to the action of the automorphism group $SO(3)$. Since symmetric matrices are diagonalizable by $SO(3)$, we take the metric to be

$$g = u\theta^1\theta^1 + v\theta^2\theta^2 + w\theta^3\theta^3. \tag{4.29}$$

The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$\begin{aligned} a &= \frac{1}{Q}\cdot\frac{(u^2+(v-w)^2-2u(v+w))^3}{8uvw} \\ b &= \frac{1}{Q}\cdot 8\sqrt{-uvw}(u-v-w)(u+v-w)(u-v+w) \\ c &= \frac{1}{Q}\cdot 8uvw(u^2+(v-w)^2-2u(v+w)) \\ R &= -\frac{u^2+(v-w)^2-2u(v+w)}{2uvw} \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} Q &= 9u^4-4u^3(v+w)-4u(v-w)^2(v+w) \\ &- 2u^2(5v^2-2vw+5w^2)+(v-w)^2(9v^2+14vw+9w^2). \end{aligned} \tag{4.31}$$

Moreover A in the equation (4.7) is a 3×3 matrix with the determinant

$$detA = \frac{Q}{8(uvw)^3}\cdot\sqrt{-uvw}(u-v)(u-w)(v-w). \tag{4.32}$$

Thus the cases $Q=0$, $u=v$ (is enough due to the symmetry) should be considered separately. We again skip the case $Q=0$ due to its complexity.

$\boldsymbol{u=v}$: This is the so-called stretched/squashed sphere deformed over its Hopf fibration. For the coefficients a, b, and c in terms of u, v, and w we have

$$a = \frac{16v^2(4v-w)+c(4v-3w)(4v-7w)}{192v^4}, \qquad b = -\frac{8v^2+c(3w-4v)}{12\sqrt{-v^2w}}. \tag{4.33}$$

Notice that in all the above cases when $b=0$ we get the equality (4.6).

4.4. Solutions on A_∞

The Lie algebra $\mathfrak{a}_\infty$ of A_∞ is spanned by r, x, and y and has non-vanishing bracket $[r, x] = -y$. Define a dual basis $\{\tilde{r}, \tilde{x}, \tilde{y}\}$. The Baker-Campbell-Hausdorff (BCH) formula allows us to write a representative as (see [16])

$$\mathcal{V} = e^{sr} e^{tx} e^{\rho y}. \tag{4.34}$$

The Maurer-Cartan one-forms are

$$\mathcal{V}^{-1} d\mathcal{V} = (ds)r + (dt)x + (d\rho - tds)y. \tag{4.35}$$

By the automorphism group, which is given by $\mathrm{Aut}(\mathfrak{a}_\infty) = (\mathbb{Z}_2 \times SL(2, \mathbb{R}) \times \mathbb{R}^+) \ltimes \mathbb{R}^2$, a left-invariant metric can be fixed as (see [13])

$$g = u\tilde{r}\tilde{r} + 2w\tilde{r}\tilde{x} + v\tilde{x}\tilde{x} \pm \tilde{y}\tilde{y}. \tag{4.36}$$

The $SO(2) \subset SL(2, \mathbb{R})$ automorphisms can be used to diagonalize the metric, $w = 0$, and we can furthermore rescale u or v freely. The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$\begin{aligned} a &= \frac{16|w^2 - uv| + 21c}{192(w^2 - uv)^2} \\ b &= \frac{-8(w^2 - uv) \pm 3c}{12\sqrt{|w^2 - uv|}} \\ R &= \frac{1}{2|w^2 - uv|}. \end{aligned} \tag{4.37}$$

Notice that when $b = 0$ we have the equality (4.6).

4.5. Solutions on A_0

The Lie algebra $\mathfrak{a}_0$ of A_0, spanned by r, x, and y, has non-vanishing brackets

$$[r,x]=x, \qquad [r,y]=x+y. \tag{4.38}$$

Define a dual basis $\{\tilde{r},\tilde{x},\tilde{y}\}$. Again by BCH formula we can choose the representative

$$\mathcal{V}=e^{\xi x+\rho y}e^{sr}. \tag{4.39}$$

Then the Maurer-Cartan one-forms are

$$\mathcal{V}^{-1}d\mathcal{V}=(e^{-s}d\xi - se^{-s}d\rho)x+(e^{-s}d\rho)y+(ds)r. \tag{4.40}$$

The automorphism group is described by the transformation

$$(r,x,y)\mapsto(r+px+qy,hx,kx+hy). \tag{4.41}$$

Under the action of the automorphism group there are four types of metric (see [13]).

4.5.1. B_1-type metric

Metric is given by

$$B_1=z\tilde{r}^2\pm\tilde{x}^2+v\tilde{y}^2. \tag{4.42}$$

There is no solution for a, b, and c.

4.5.2. B_2-type metric

Metric is given by

$$B_2 = z\tilde{r}^2 \pm 2\tilde{x}\tilde{y}. \tag{4.43}$$

It is always Lorentzian and mostly plus for $z > 0$. The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$a = -\frac{4z + c}{4z^2}, \qquad b = \mp\frac{2z + c}{2\sqrt{z}}, \qquad R = -\frac{6}{z}. \tag{4.44}$$

Note that the solution is attained at the chiral point (4.5). Moreover, when $b = 0$ we have the merger point (4.6).

Under the coordinate transformation

$$s \to log(\bar{z}), \qquad \rho \to lx^+, \qquad \xi \to lx^- \tag{4.45}$$

the solution (4.44) corresponds to pp-wave solution in [19], where in the equation (3.2) the metric ansatz is given as

$$ds^2 = \frac{l^2}{\bar{z}^2}(-F(x^+, \bar{z})(dx^+)^2 - 2dx^+dx^- + d\bar{z}^2) \tag{4.46}$$

for which

$$z = l^2, \qquad F(x^+, \bar{z}) = 2log(\bar{z}). \tag{4.47}$$

Moreover, the condition $\mu^2 l^2\sigma + \mu l + \gamma/2 = 0$ in our formulations is equivalent to $z + b\sqrt{z} + c/2 = 0$, which is precisely the equation (4.44).

4.5.3. B_3-type metric

Metric is given by

$$B_3 = z\tilde{r}^2 + \tilde{r}\tilde{x} + v\tilde{y}^2. \tag{4.48}$$

In this case $a = 0$ is the necessary and the sufficient condition to solve the equation. The scalar curvature is identically zero.

4.5.4. B_4-type metric

Metric is given by

$$B_4 = z\tilde{r}^2 + \tilde{r}\tilde{y} + u\tilde{x}^2. \tag{4.49}$$

It is always Lorentzian and mostly plus for $u > 0$. The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$a = -\frac{u}{18}, \qquad b = -\frac{4}{9\sqrt{u}}, \qquad c = -\frac{2}{9u}, \qquad R = 2u. \tag{4.50}$$

4.6. Solutions on $ISO(2;\theta)$

We define the dual basis $\{\tilde{l}, \tilde{m_1}, \tilde{m_2}\}$ to the Lie algebra basis $\{l, m_1, m_2\}$ of $\mathfrak{iso}(2;\theta)$, where the only non-vanishing brackets are

$$[l, m_1] = 2\cos\theta m_1 + 2\sin\theta m_2, \qquad [l, m_2] = 2\cos\theta m_2 - 2\sin\theta m_1. \tag{4.51}$$

We choose the group representative

$$\mathcal{V} = e^{xm_1 + ym_2} e^{\rho l}. \tag{4.52}$$

Then the Maurer-Cartan one-forms are

$$\begin{aligned} \mathcal{V}^{-1}d\mathcal{V} = (d\rho)l \quad &+ \quad e^{-2\rho\cos\theta}[\cos(2\rho\sin\theta)dx + \sin(2\rho\sin\theta)dy]m_1 \\ &+ \quad e^{-2\rho\cos\theta}[-\sin(2\rho\sin\theta)dx + \cos(2\rho\sin\theta)dy]m_2. \end{aligned} \tag{4.53}$$

The automorphism group is given by $\mathrm{Aut}(\mathfrak{iso}(2;\theta)) = (SO(2)\times\mathbb{R}^+)\ltimes\mathbb{R}^2$ if $\theta \neq 0$ and $\mathrm{Aut}(\mathfrak{iso}(2;0)) = GL(2,\mathbb{R})\ltimes\mathbb{R}^{1,1}$ if $\theta = 0$. One can fix two types of metrics (see [13]).

4.6.1. B_1-type metric

Metric is given by

$$B_1 = u\tilde{l}\tilde{l} + v\tilde{m}_1\tilde{m}_1 + w\tilde{m}_2\tilde{m}_2 \tag{4.54}$$

where we can furthermore rescale v or w freely by using the $\mathbb{R}^+$ automorphisms. Here we assume Lorentzian mostly plus signature.

There is no general solution for a, b, and c. The scalar curvature is given by

$$R = -\frac{2[12vw\cos^2\theta + (v-w)^2\sin^2\theta]}{uvw}. \tag{4.55}$$

A in the equation (4.7) is a 4×3 matrix and the cases $\theta = 0$, $\theta = \frac{\pi}{2}$, and $v = w$ should be considered separately.

$\boldsymbol{\theta = 0}$: In this case Cotton tensor vanishes identically, the metric becomes Einstein, and one can fix the metric B_1 to be

$$B_1 = |z|(\pm\tilde{l}\tilde{l} \pm \tilde{m}_1\tilde{m}_1 \pm \tilde{m}_2\tilde{m}_2). \tag{4.56}$$

Then we have the relation

$$a = -\frac{4(\pm|z| - c)}{z^2}. \tag{4.57}$$

$\boldsymbol{\theta = \frac{\pi}{2}}$: The coefficients a, b, and c in terms of u, v, and w are equal to

$$a = \frac{1}{Q} \cdot \frac{(v-w)^4}{2uvw}, \qquad b = \frac{1}{Q} \cdot 4\sqrt{-uvw}(v+w), \qquad c = \frac{1}{Q} \cdot 2uvw \tag{4.58}$$

where $Q = 9v^2 + 14vw + 9w^2$. Note that when $b = 0$ we get the equality (4.6).

$\boldsymbol{v = w}$: In this case Cotton tensor (4.2) vanishes identically, the metric becomes Einstein, and

$$a = -\frac{4\cos^2\theta(u + c\cdot\cos^2\theta)}{u^2}. \tag{4.59}$$

4.6.2. B_2-type metric

Metric is given by

$$B_2 = u\tilde{l}\tilde{l} + \tilde{l}\tilde{m}_1 + w\tilde{m}_2\tilde{m}_2 \tag{4.60}$$

which is Lorentzian, mostly plus for $w > 0$.

For θ nonzero the coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$a = -\frac{2}{9}w\sin^2\theta, \qquad b = \frac{2}{9\sqrt{w}\sin\theta}, \qquad c = -\frac{1}{18w\sin^2\theta}, \qquad R = 8w\sin^2\theta. \tag{4.61}$$

When $\theta = 0$ the metric is flat and the equation is solved if and only if $a = 0$.

4.7. Solutions on $ISO(1,1;\theta)$

The basis $\{l, m_1, m_2\}$ of $\mathfrak{iso}(1,1;\theta)$ has the brackets

$$[l, m_1] = 2\cos\theta m_1 + 2\sin\theta m_2, \qquad [l, m_2] = 2\cos\theta m_2 + 2\sin\theta m_1 \tag{4.62}$$

and the dual basis $\{\tilde{l}, \tilde{m_1}, \tilde{m_2}\}$. We choose the group representative

$$\mathcal{V} = e^{xm_1+ym_2}e^{\rho l} \tag{4.63}$$

and the Maurer-Cartan one-forms are

$$\begin{aligned}\mathcal{V}^{-1}d\mathcal{V} = (d\rho)l \quad &+ \quad e^{-2\rho\cos\theta}[\cosh(2\rho\sin\theta)dx - \sinh(2\rho\sin\theta)dy]m_1 \\ &+ \quad e^{-2\rho\cos\theta}[\cosh(2\rho\sin\theta)dy - \sinh(2\rho\sin\theta)dx]m_2.\end{aligned} \tag{4.64}$$

The automorphism group is given by $\mathrm{Aut}(\mathfrak{iso}(1,1;\theta)) = (\mathbb{Z}_2 \times SO(1,1) \times \mathbb{R}^+) \ltimes \mathbb{R}^{1,1}$, from which two types of metrics can be fixed (see [13]).

4.7.1. B_1-type metric

The B_1-type metric is given by

$$B_1 = z\tilde{l}\tilde{l} + u(\tilde{m_1} + \tilde{m_2})^2 + v(\tilde{m_1} - \tilde{m_2})^2 + 2w(\tilde{m_1}\tilde{m_1} - \tilde{m_2}\tilde{m_2}) \tag{4.65}$$

with $w^2 \neq uv$. We may fix at least two parameters of (u, v, w) that appear in the metric, if non-zero up to their sign to be ± 1. If $w^2 - uv > 0$, then the metric is Lorentzian and mostly plus for for $z > 0$.

There is no general solution for a, b, and c. The scalar curvature is given by

$$R = -\frac{8[3(uv - w^2)\cos^2\theta + uv\sin^2\theta]}{z(uv - w^2)}. \tag{4.66}$$

However, the cases $\theta = 0$, $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$, and $uv = 0$ should be considered separately. When $\theta = 0$ the Lie algebras of $ISO(1,1;0)$ and $ISO(2;0)$ coincide. So the case $\theta = 0$ was studied in section 4.6.

$\boldsymbol{\theta = \frac{\pi}{4}}$: The coefficients a, b, and c in terms of u, v, and w are equal to

$$\begin{aligned} a &= \frac{2z(w^2 - uv)(4uv - 3w^2) + c(4uv - w^2)(4uv + 3w^2)}{3z^2(w^2 - uv)^2} \\ b &= \frac{-z(w^2 - uv) + c(4uv - w^2)}{3w\sqrt{2z(w^2 - uv)}}. \end{aligned} \tag{4.67}$$

$\boldsymbol{\theta = \frac{\pi}{2}}$: The coefficients a, b, and c in terms of u, v, and w are equal to

$$a = \frac{2u^2v^2}{z(uv + 8w^2)(uv - w^2)}, \quad b = -\frac{2w\sqrt{z(w^2 - uv)}}{uv + 8w^2}, \quad c = \frac{z(uv - w^2)}{2(uv + 8w^2)}. \tag{4.68}$$

$\boldsymbol{u = 0}$: The coefficients a, b, and c in terms of u, v, and w are equal to

$$a = -\frac{4\cos^2\theta(z + c\cdot\cos^2\theta)}{z^2}, \qquad b = \frac{w(z + 2c\cdot\cos^2\theta)}{2\sqrt{w^2 z}(\cos\theta - 2\sin\theta)}. \tag{4.69}$$

$\boldsymbol{v = 0}$: The coefficients a, b, and c in terms of u, v, and w are equal to

$$a = -\frac{4\cos^2\theta(z + c\cdot\cos^2\theta)}{z^2}, \qquad b = -\frac{w(z + 2c\cdot\cos^2\theta)}{2\sqrt{w^2 z}(\cos\theta + 2\sin\theta)}. \tag{4.70}$$

We note that in all the above cases when $b = 0$ we have the equality (4.6).

4.7.2. B_2-type metric

The B_2-type metric is given by

$$B_2 = z\tilde{l}\tilde{l} + \tilde{l}\tilde{m}_1 + u(\tilde{m}_1 + \tilde{m}_2)^2 + v(\tilde{m}_1 - \tilde{m}_2)^2 + 2w(\tilde{m}_1\tilde{m}_1 - \tilde{m}_2\tilde{m}_2) \tag{4.71}$$

with $w^2 = uv$ and $u + v \neq 2w$. We can fix a or b up to a sign to be ± 1. The metric is always Lorentzian and mostly plus.

The coefficients a, b, and c and the scalar curvature R in terms of u, v, and w are equal to

$$a = -\frac{32vw^2 \sin^2\theta}{9(v-w)^2}, \qquad b = -\frac{|v-w|}{18w\sqrt{v}\sin\theta}$$

$$c = -\frac{(v-w)^2}{288vw^2\sin^2\theta}, \qquad R = \frac{128vw^2\sin^2\theta}{(v-w)^2}. \tag{4.72}$$

The only nontrivial simple case that should be considered separately is $\theta = \frac{\pi}{4}$.

$\boldsymbol{\theta = \frac{\pi}{4}}$: The coefficients a, b, and c in terms of u, v, and w are equal to

$$a = \frac{32vw^2[(v-w)^2 + 168c \cdot vw^2]}{3(v-w)^4}, \qquad b = -\frac{(v-w)^2 - 48c \cdot vw^2}{12w|v-w|\sqrt{2v}}. \tag{4.73}$$

Note that for $b = 0$ we have the equality (4.6).

5. CONCLUSION

In this thesis after reviewing general mathematical properties of homogeneous spaces, geometry of Lie groups, and Bianchi classification of three-dimensional Lie algebras we constructed homogeneous spacetime solutions of MMG, more precisely we looked for the solutions among the three-dimensional Lie groups with the left-invariant metric. From the obtained solutions the following conclusions can be drawn.

(i) There are no solutions for B_1-type metric neither in $ISO(2;\theta)$ nor in $ISO(1,1;\theta)$, except in some special cases. There are no solutions at all for A_0 with B_1-type and B_3-type metrics.

(ii) The solutions (4.23) for $SL(2,\mathbb{R})$ with 3-type metric and (4.44) for A_0 with B_2-type metric are attained only at the chiral point i.e., the equality (4.5) holds.

(iii) When it is allowed to set the parameter c to zero, then the solutions reduce to the solutions of TMG (see [12–15]). If not, for example as in the solution (4.23), then such a solution does not exist in TMG.

(iv) If we set $b = 0$ in the solutions, then we get the merger point (4.6).

(v) Interestingly, in the solutions for A_0 with B_4-type metric (4.50), $ISO(2;\theta)$ with B_2-type metric (4.61), and $ISO(1,1;\theta)$ with B_2-type metric (4.72) we have the relation $ac = \frac{1}{81}$, which does not have any apparent physical significance in the model.

Among the solutions of $SL(2,\mathbb{R})$ we identified warped AdS-type metrics (refer to the discussion of the equation (4.16)). The next task is to compare our solutions with the already known solutions of MMG and determine those which are new.

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