NUMERICAL STUDY OF ORTHOGONAL POLYNOMIALS FOR FRACTAL MEASURES

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By
Ahmet Nihat Şimşek
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Alexander Goncharov(Advisor)

Hakkı Turgay Kaptanoğlu

Oktay Duman

Approved for the Graduate School of Engineering and Science:

Levent Onural
Director of the Graduate School

ABSTRACT

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Ahmet Nihat Şimşek
M.S. in Mathematics
Advisor: Alexander Goncharov
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In recent years, potential theory has an essential effect on approximation theory and orthogonal polynomials. Basic concepts of the modern theory of general orthogonal polynomials are described in terms of Potential Theory. One of these concepts is the Widom factors which are the ratios of norms of extremal polynomials to a certain degree of capacity of a set. While there is a theory of Widom factors for finite gap case, very little is known for fractal sets, particularly for supports of continuous singular measures. The motivation of our numerical experiments is to get some ideas about how Widom factors behave on Cantor type sets.

We consider weakly equilibrium Cantor sets, introduced by A.P. Goncharov in [16], which are constructed by iteration of quadratic polynomials that change from step to step depending on a sequence of parameters. Changes in these parameters provide a Cantor set with several desired properties. We give an algorithm to calculate recurrence coefficients of orthogonal polynomials for the equilibrium measure of such sets. Our numerical experiments point out stability of this algorithm.

Asymptotic behaviour of the recurrence coefficients and the zeros of orthogonal polynomials for the equilibrium measure of four model Cantor sets are studied via this algorithm. Then, several conjectures about asymptotic behaviour of the recurrence coefficients, Widom factors, and zero spacings are proposed based on these numerical experiments. These results are accepted for publication [1] (jointly with G. Alpan and A.P. Goncharov).

Keywords: Cantor Sets, Parreau-Widom sets, Orthogonal Polynomials, Zero spacing, Potential Theory, Widom Factors.

ÖZET

FRAKTAL ÖLÇÜMLERİN ORTOGONAL POLİNOMLARININ NUMERİK ÇALIŞMASI

Ahmet Nihat Şimşek
Matematik, Yüksek Lisans
Tez Danışmanı: Alexander Goncharov
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Son yıllarda potansiyel teorisinin yaklaşım teorisi ve interpolasyon üzerinde temel etkileri olmuştur. Genel ortogonal polinomların modern teorisinin temel kavramları potansiyel teorisi açısından tanımlanmıştır. Bu kavramlardan biri de ekstremal polinomların normlarının bir kümenin belli bir derecesine oranı olan Widom faktörleridir. Widom faktörlerinin sonlu boşluk durumu için teorisi varken, fraktal kümeler için çok az şey bilinmektedir, özellikle de dayanağı sürekli tekil ölçümler için. Numerik deneyimizin motivasyonu Cantor tipindeki kümelerde Widom faktörlerinin nasıl davrandığı hakkında bilgi edinmektir.

A.P. Goncharov tarafından [16]'te tanıtılan zayıf dengeli Cantor kümelerini inceliyoruz. Bu kümeler, adım adım bir dizi parametreye bağlı olarak değişen ikinci dereceden polinomların yinelenmesiyle elde edilmektedir. Bu parametrelerdeki değişimler çeşitli istenilen özelliklere sahip Cantor kümeleri sağlamaktadır. Bu tür kümelerin denge ölçümleriyle ilgili ortogonal polinomlarının rekürens katsayılarını hesaplamak için bir algoritma veriyoruz. Numerik deneylerimiz bu algoritmaya itimat edilebileceğini göstermektedir.

Dört model Cantor kümenin denge ölçümleriyle ilgili ortogonal polinomların rekürens katsayıların ve sıfırlarının asimptotik davranışları bu algoritma ile incelenmiştir. Daha sonra rekürens katsayıların asimptotik davranışları, Widom faktörleri ve sıfırlar arasındaki aralıklar hakkında çeşitli sanılarda bulunulmuştur. Bu sonuçlar yayın için kabul edilmiştir [1] (G. Alpan ve A. Goncharov ile ortak çalışmadır).

Anahtar sözcükler: Cantor Kümeleri, Parreau-Widom Kümeleri, Ortogonal Polinomlar, Sfr Aralıkları, Potansiyel Teorisi, Widom Faktörleri.

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Chapter 1

Introduction

We present our numerical experiments and give the necessary preliminaries for them. Motivation of our numerical experiments is to get a few ideas about how Widom factors behave on Cantor type sets. For this purpose we consider a Cantor set $K(\gamma)$ introduced by A. Goncharov in [16]. Construction and some properties of $K(\gamma)$ are given. However, to that end we need to give some preliminary information. Thus, we first start with elements of Potential Theory. Then, we give some basic concepts of orthogonal polynomials on the real line. And finally we introduce Widom factors and some of their properties.

In Chapter 2, we define some concepts from potential theory that we will use. The term $'potential'$ arise from the idea that forces in nature can be modelled using potentials satisfing Laplace's equation. Potential theory focuses on the properties of harmonic functions. One of the main reasons, why potential theory is useful, is that there is a direct connection between monic polynomials and logarithmic potentials, that is, for any monic polynomial $p(z) = (z-z_1)\cdots(z-z_n)$ we have $\log\left(\frac{1}{|p(z)|}\right) = \int \log\frac{1}{|z-\omega|}d\mu(\omega) = U^{\mu}(z)$, where μ is the counting measure on the zeros of polynomial p. And thanks to this, potential theory has a huge impact on approximation theory and the theory of orthogonal polynomials.

We give necessary concepts from potential theory in three sections: Potential and Energy, Equilibrium Measure and Capacity, Green's Functions and Parreau-Widom Sets. In the first section we introduce the core concepts of potential theory, the logarithmic potential and logarithmic energy. In the following section we talk about logarithmic capacity and equilibrium measure which arise from minimal energy the idea that a charge placed on a conductor will be distributed to minimize its total energy. Then, in the last section we give a relation between Green's functions and capacity, which help us to calculate capacity. Also, we briefly talk about regularity with respect to Dirichlet Problem and introduce Parreau-Widom sets. Note that some Cantor sets are Parreau-Widom (for examples see [6] and [19]).

Chapter 3 is divided into two subsections: Orthogonal Polynomials on the Real Line and Widom Factors. In the first section, we begin with the definition of the orthogonal polynomials for a measure μ. Then, their fundamental properties, recurrence relation and the Jacobi matrix H_μ rise from the sequences of the recurrence coefficients. Then, we give a relation between eigenvectors of H_μ and the zeros of associated orthogonal polynomials via Gauss-Jacobi quadrature and we introduce Christoffel numbers which are used to determine our algorithm's reliability.

For the next section of Chapter 3, we introduce a relatively new concept, called Widom factors, due to the fundamental paper by H. Widom in 1969 [35], where he considered the ratios $\frac{||T_n||_{L^\infty(K)}}{(Cap(K))^n}$ and $\frac{||Q_n(x;\mu)||_{L^2(K)}}{(Cap(K))^n}$ for finite unions of smooth Jordan curves and arcs. Also, we discuss some properties of Widom factors.

The last chapter begins with the construction of $K(\gamma)$. In the construction we use a sequence $\gamma = (\gamma_s)_{s=1}^\infty$. Note that with different γ one obtains different $K(\gamma)$. Then, for the next section we talk about orthogonal polynomials on $K(\gamma)$ and we provide the algorithm, we have used in our experiments, to calculate the recurrence coefficients of $H_{\mu_{K(\gamma)}}$. Followed by a section for some properties of $K(\gamma)$, we note that, due to its construction $K(\gamma)$ is somewhat flexible, that it presents many properties but we cover only the ones that suits our purposes. We introduce there four models, i.e., different γ's, used for our experiments and give the properties of each model.

The rest of the last chapter sections are for our numerical results obtained in [1], joint work of G. Alpan and A. Goncharov. In calculations of these types, one must ensure whether the algorithm used is stable or not. We show that our experiments point out numerical stability. Then, for the next section we propose conjectures linking geometric properties of sets and on asymptotic behaviour of recurrence coefficients. In the next section, we introduce a notion of almost periodicity and analyze the Jacobi matrix of $K(\gamma)$ in this respect. In the following section, we examine the Widom factors of $K(\gamma)$. Here, one of our conjectures gives a possible relation between Parreau-Widom sets and Widom factors. And we finish with spacing properties of orthogonal polynomials on $K(\gamma)$. This is related with the recent paper by G. Alpan [2]. There is one more section for figures done for easy access.

Chapter 2

Elements of Potential Theory

2.1 Potential and Energy

Edward B. Saff describes Potential Theory as an elegant blend of real and complex analysis. It is important to state that Potential Theory had and still has major impacts on Approximation Theory in recent years. As we will explain partly, logarithmic potentials have a direct relation with polynomial and rational functions. Some problems that Potential Theory resolved can be listed in short tiles as: rate of polynomial approximation, asymptotic behaviour of zeros of polynomials, fast decreasing polynomials, recurrence coefficients of orthogonal polynomials, generalized Weierstrass problem, optimal point arrangements on the sphere and rational approximation.

In this Chapter we followed [21], [25], [24] interchangeably to stay in our scope of interest.

Let $\mathcal{M}$ be the collection of all finite Borel measures with compact support, and given a compact set K, let $\mathcal{M}(K)$ be the collection of all finite Borel measures on K. Also, let us denote the collection of all unit Borel measures on K with

$\mathcal{M}_u(K)$. Note that the support of a Borel measure μ is defined by

$$\text{supp}(\mu) = \{z \in \mathbb{C} : \forall \varepsilon > 0 \text{ we have } \mu(B(z, \varepsilon)) > 0\}.$$

Definition 2.1.1. Let $\mu \in \mathcal{M}$. Then, the *logarithmic potential* of μ defined as the function

$$U^{\mu}(z) := \int \log \frac{1}{|z - \omega|} d\mu(\omega), \tag{2.1}$$

where $U^{\mu} : \mathbb{C} \to (-\infty, \infty]$.

Let us give an example:

Example 2.1.2. Let $K = \{z_1, z_2, \ldots, z_N\}$ and $\mu(\cdot) = \sum_{i=1}^{N} \delta_{z_i}$ where $N \leq \infty$ and for any $i = 1, 2, \ldots, N$, $\delta_i(E) = 1$ if $z_i \in E$ and it is 0 otherwise. Then,

$$U^{\mu}(z) = \int \log \frac{1}{|z - \omega|} d\mu(\omega) = \sum_{i=1}^{N} \log \frac{1}{|z - z_i|} =$$

$$= -\sum_{i=1}^{N} (\log |z - z_i|) = -\log |\prod_{i=1}^{N} (z - z_i)|.$$

Definition 2.1.3. The *logarithmic energy* $I(\mu)$ of a measure μ from the collection $\mathcal{M}(K)$ is defined by

$$I(\mu) := \int U^{\mu}(z) d\mu(z) = \int \int \log \frac{1}{|z - \omega|} d\mu(\omega) d\mu(z). \tag{2.2}$$

Note that, $I(\mu)$ takes values from $(-\infty, \infty]$. Then, see a trivial example:

Example 2.1.4. Let $K = \{z_k\}_{k=1}^{N}$ and $\mu(E) = \sum_{i=1}^{N} \alpha_k \delta_{z_i}$ where $N \leq \infty$ and, for any $i = 1, 2, \ldots, N$, let $\delta_i(E) = 1$ if $z_i \in E$ and 0 otherwise. From Example 2.1.2 we know that

$$U^{\mu}(z) = \sum_{i=1}^{N} \log \frac{1}{|z - z_i|}.$$

Then, logarithmic energy of the measure μ is

$$I(\mu) = \int U^{\mu}(z) d\mu(z) = \int \sum_{i=1}^{N} \log \frac{1}{|z - z_i|} \mu(z) = \sum_{j=1}^{N} \sum_{i=1}^{N} \log \frac{1}{|z_j - z_i|}.$$

To finalize, observe that, setting $i = j$ we see that $\log \frac{1}{|z_i - z_j|} = \infty$. Therefore, logarithmic energy of μ is ∞.

2.2 Equilibrium Measure and Capacity

A charge placed on a conductor will be distributed to minimize its total energy, which suggests:

Definition 2.2.1. Assume that there is a measure $\mu_0 \in \mathcal{M}_u(K)$ such that $I(\mu_0) < \infty$. Then, there is a measure $\mu_K \in \mathcal{M}_u(K)$ that satisfies

$$I(\mu_K) = \inf_{\mu \in \mathcal{M}_u(K)} I(\mu) = V_K. \tag{2.3}$$

This measure is called *equilibrium measure* for K and V_K is called the *minimal energy* of the set K.

Let us demonstrate the equilibrium measures for the interval $[-1, 1]$ and the unit disc.

Example 2.2.2. Let $K = [-1, 1]$, then $d\mu_K = \dfrac{dx}{\pi\sqrt{1-x^2}}$(the arcsine measure) is the equilibrium measure for K.

$$\begin{aligned} U^{\mu_K}(z) &= \int_{-1}^{1} \log \frac{1}{|z-x|\sqrt{1-x^2}} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{|z-cos\theta|} d\theta \end{aligned}$$

which equals $\log 2$ for $z \in [-1, 1]$ and $\log 2 - \log|z + \sqrt{z^2-1}|$ otherwise. Also,

$$I(\mu_K) = \int U^{\mu_K}(z) d\mu_K = \frac{1}{\pi} \int_{-1}^{1} \log 2 \frac{1}{\sqrt{1-x^2}} dx = \log 2.$$

Example 2.2.3. For $\mathbb{D}$ we have $\mu_{\mathbb{D}} = d\lambda_{arc}/2\pi$ (the normalized arclength measure) as the equilibrium measure.

$$\begin{aligned} U^{\mathbb{D}}(z) &= \int \log \frac{1}{|z-\omega|} d\mu(\omega) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{|z-e^{i\theta}|} d\theta, \end{aligned}$$

which equals 0 for $|z| \leq 1$ and $\log \frac{1}{|z|}$ otherwise. Moreover,

$$I(\mu_{\mathbb{D})} = 0$$

Definition 2.2.4. (p. 25 in [25]) For a compact set $K \subset \mathbb{C}$, the *logarithmic capacity* of K is defined as

$$\mathrm{Cap}(K) := e^{-V_K}. \tag{2.4}$$

Moreover, the capacity of an arbitrary Borel set E defined as

$$\mathrm{Cap}(E) := \sup\{\mathrm{Cap}(K) : K \subset E, \;\; K \;\; \text{compact}\}. \tag{2.5}$$

Note that, we may use just capacity to refer to logarithmic capacity. The following theorem is for basic properties of $\mathrm{Cap}(E)$ as a set function.

Theorem 2.2.5. *For A and B, Borel subsets of $\mathbb{C}$, we have that*

i) *if* $A \subset B$, *then* $\mathrm{Cap}(A) \leq \mathrm{Cap}(B)$,

ii) $\forall \alpha, \beta \in \mathbb{C}$ *we have* $\mathrm{Cap}(\alpha A + \beta) = |\alpha|\mathrm{Cap}(A)$,

iii) $\mathrm{Cap}(A) = \sup\{\mathrm{Cap}(K) : \textit{compact } K \subset A\}$,

iv) *if A is compact, then* $\mathrm{Cap}(A) = \mathrm{Cap}(\partial_e A)$ *(the exterior boundary of set A).*

It is easy to see the following proposition by 2.4 and the definition of the equilibrium measure.

Proposition 2.2.6. *Let $K \subset \mathbb{C}$ be compact and has non-zero capacity. Then, we have*

$$I(\mu_K) = \log \frac{1}{\mathrm{Cap}(K)}$$

where μ_K is the equilibrium measure for K.

Let us give capacities of some basic cases. Remark that by Examples 2.2.2 and 2.2.3 we have $I(\mu_K) = \log 2$ for $K = [-1, 1]$ and $I(\mu_K) = 0$ for $K = \overline{\mathbb{D}}$.

Example 2.2.7. Let K be a line segment and set its length as t. Then we have

$$Cap(K) = Cap([-\frac{t}{2}, \frac{t}{2}]) = \frac{t}{2}Cap([-1, 1]) = \frac{t}{2}e^{-log2} = \frac{t}{4}.$$

Example 2.2.8. Take a closed ball $\overline{B(r, z_0)} \subset \mathbb{C}$ with radius r and center z_0. Then,

$$Cap(\overline{B(r, z_0)}) = r.Cap(\overline{\mathbb{D}}) = r.$$

Observe that Cap(·) is a monotone function. One might wonder the continuity of Cap(·) for nested family of sets. We have the following theorem:

Theorem 2.2.9. *(Theorem 5.1.3 in [21]) Let $\mathbb{C} \supset A_1 \supset A_2 \supset A_3 \supset \ldots$ for compact A_n. Then, we have*

$$\mathrm{Cap}(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathrm{Cap}(A_n).$$

On the other hand, if we have Borel sets $B_1 \subset B_2 \subset B_3 \subset \ldots \subset \mathbb{C}$ then,

$$\mathrm{Cap}(\cup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mathrm{Cap}(B_n).$$

Now, we will give an important notion and some remarks about it.

Definition 2.2.10. In Potential Theory, a set is called *polar* if its capacity is zero.

Remark that polarity gives a concept of negligible sets in Potential Theory.

Remark 2.2.11. Any set that is a subset of a polar Borel set is polar. Then, it is easy to see that the countable union of polar sets is polar.

Remark 2.2.12. By the definition of capacity, for a Borel set to be non-polar it has to be the support of a positive measure that has finite minimal energy. And a Borel set A is polar if and only if for each $\mu \in \mathcal{M}(K)$ for every compact subset K of A, logarithmic energy $I(\mu)$ is infinite.

Now, we will introduce a notion called *quasi-everywere* to exclude the negligible (polar) parts of a set.

Definition 2.2.13. We use *quasi-everywere* (*q.e.*) for a property to state that this particular property holds everywhere on a set except on a set of zero capacity.

The following theorem gives conditions for existence and uniqueness of equilibrium measures.

Theorem 2.2.14. *(Theorem 3.3.2 and 3.7.6 in [21]) For every compact set* $K \subset \mathbb{C}$ *there exists an equilibrium measure* $\mu_K \in \mathcal{M}(K)$*. If, in addition,* $Cap(K) > 0$*, then the equilibrium measure for* K *is unique and support of* μ_K *is a subset of exterior boundary of* K*.*

Now, with the definition of polarity and the previous theorem in mind we will give a lemma about the relation of capacity and equilibrium measure.

Lemma 2.2.15. *(Lemma 1.2.7 in [23]) If we have a compact non-polar set* $K \subset \mathbb{C}$*, then*

$$\mathrm{Cap}(supp(\mu_K)) = \mathrm{Cap}(K).$$

The following theorem is called Frostman Theorem and it is considered as the fundamental theorem of potential theory due to its importance for determining equilibrium measures via potential and energy.

Theorem 2.2.16. *(Frostman Theorem)([24]) For a non-polar compact set* $K \subset \mathbb{C}$ *and its equilibrium measure* μ_K*, we have*

- $U^{\mu_K}(z) \le I(\mu_K)$ *for all* $z \in \mathbb{C}$.
- $U^{\mu_K}(z) = I(\mu_K)$ *q.e. on* K.

In general, it is difficult to show the equilibrium measure for a given set. However, sometimes the Frostman Theorem can be used for this purpose. It is easy to see by Frostman Theorem that the measures used in Examples 2.2.2 and 2.2.3 are the equilibrium measures of respected sets.

2.3 Green's Function and Parreau-Widom Set

As it can be seen from the examples, although Definition 2.2.4 is conceptually useful, it is hard to calculate exactly the capacity of a set even in simple cases.

However, thanks to a relation between capacity and Green functions, we can compute the capacities of compact sets.

Definition 2.3.1. (p. 53 in [25]) Suppose that Ω_K is the component of $\overline{\mathbb{C}}\backslash K$ that contains ∞ where K is non-polar and compact. Then, remark that K and $\partial\Omega_K$ have the same equilibrium measure, also we have $\text{Cap}(K) = \text{Cap}(\partial\Omega_K)$ (see Corallary 4.5 in [25]). Then, *Green's function* $g_{\Omega_K}(z)$ *of* Ω_K *with pole at* ∞ is defined uniquely with the following properties:

i) $g_{\Omega_K}(z)$ is nonnegative and harmonic on $\Omega_K\backslash\{\infty\}$,

ii) $g_{\Omega_K}(z) = \log|z| + \log\frac{1}{\text{Cap}(K)}$ as $|z| \to \infty$,

iii) $\lim\limits_{z\to t,\, t\in\Omega_K} g_{\Omega_K}(z) = 0$ for q.e. $t \in \partial\Omega_K$.

Note that since K is chosen of positive capacity the existence follows if we set

$$g_{\Omega_K}(z) = \log\frac{1}{\text{Cap}(K)} - U^{\mu_K}(z). \tag{2.6}$$

Now, we will introduce another notion called regularity with respect to the Dirichlet problem. Dirichlet Problem, basically, is to find a harmonic function on a domain with given initial boundary values.

Definition 2.3.2. A point $z_0 \in \partial_e K$ is called a *regular point* of the unbounded component Ω_K of $\overline{\mathbb{C}}\backslash K$, if $g_{\Omega_K}(z)$ is continuous at z_0. Otherwise, it is called *irregular*. This implies that $z \in \partial\Omega_K$ is a regular point if and only if

$$g_{\Omega_K}(z) = 0$$

which is equivalent to

$$U^{\mu_K}(z) = \log\frac{1}{Cap(K)}$$

by Equation 2.6. And if every point of $\partial\Omega_K$ is regular, then Ω_K is said to be regular with respect to the Dirichlet problem.

Remark 2.3.3. ([25], p. 54) The set of all irregular points has capacity zero.

For further discussion on Green's functions see [25] section I.4.

Now, we will combine compactness, regularity and non-polarity to obtain another notion called Parreau-Widom sets.

Definition 2.3.4. A compact, regular set $K \subset \mathbb{R}$ with positive logarithmic capacity is called a $Parreau - Widom\, set$ if $\sum_i g_{\Omega_K}(c_i) < \infty$ where $\{c_i\}$ are the critical points of $g_{\Omega_K}(z)$.

Remark 2.3.5. (see [36]) If a compact non-polar regular set $K \subset \mathbb{R}$ is a finite union of closed disjoint intervals, then K is Parreau-Widom. Moreover, each gap in between intervals contains one critical point of g_{Ω_K} and g_{Ω_K} does not have any other critical points.

And also note that a Parreau-Widom set has positive Lebesgue measure (see [28]).

2.3.1 Smoothness of Green's Functions

Definition 2.3.6. Let f be real or complex function on the Euclidean space. If there exists $\alpha \in (0, 1]$ and $\beta \in (0, \infty)$ such that

$$|f(x) - f(y)| \leq \beta |x - y|^{\alpha}$$

for all x, y in the domain of f, then we say that f is Hölder *continuous of order* α.

Then, note that, a Green's function is said to be optimally smooth if $K \subset \mathbb{R}$ it is Hölder continuous of order 1/2. Now, let us show some basic examples.

Example 2.3.7. Let $K = [-1, 1]$, then by 2.6 we have

$$g_{\Omega_K}(z) = \log |z + \sqrt{z^2 - 1}|.$$

Then, for $z = 1 + x \;\; x > 0$ we have

$$g_{\Omega_K}(1 + x) = \log |1 + x + \sqrt{x^2 + 2x}| \leq \log |1 + \sqrt{3x} + 3x/2| \leq \sqrt{3}x^{1/2}.$$

It is possible to show that $g_{\Omega_K}(z) \leq \sqrt{3}(\text{dist}(z, [-1,1]))^{1/2}$ for every z. Thus, g_{Ω_K} is Hölder continuous of order 1/2.

Example 2.3.8. Take $K = \overline{\mathbb{D}}$. By (2.6) we have

$$g_{\Omega_K}(z) = \log|z|$$

for $z \in \overline{\mathbb{C}} \setminus K$ and 0 otherwise. Then observe that,

$$g_{\Omega_K}(z) = \log|z| \leq \log(1+r) \leq r$$

for all $z \in \overline{B(1+r, 0)}$. Therefore, g_{Ω_K} is Hölder continuous of order 1.

Chapter 3

Orthogonal Polynomials on the Real Line and Widom Factors

3.1 Orthogonal Polynomials On The Real Line

P. L. Chebyshev developed Orthogonal polynomials in 19th century from a study of fractions. Since then the field has been pursued by many great mathematicians, and as mentioned before, potential theory lead to major developments in this field recently.

We followed [29] and [33] interchangeably in this section.

Let us begin with definition of *orthonormal relation*.

Definition 3.1.1. A set of functions $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ from $L^2(\mu)$ is called orthonormal if the relation

$$(\phi_i(x), \phi_j(x)) = \int \phi_i(x)\phi_j(x)d\mu(x) = \delta_{ij}$$

holds for $i, j = 0, 1, \ldots, n$.

Using this relation one can orthogonalize a set of linearly independent functions. It is well known that for a set of real-valued and linearly independent functions $f_0(x), f_1(x), f_2(x), \ldots$ of the class $L^2(\alpha)$ defined on (a, b) there exists an orthonormal set $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ such that

$$\phi_n(x) = \sum_{i=0}^{n} c_{ni} f_i(x).$$

The process to obtain this orthonormal set from the set of linearly independent functions is called Gram-Schmidt orthogonalization and the orthonormal set generated by this process is uniquely determined. Now, we apply this orthogonalization process to $\{1, x, x^2, \ldots\}$ to obtain orthogonal polynomials:

Definition 3.1.2. Let the moments

$$m_n = \int x^n d\mu(x)$$

exist and be finite for $n = 0, 1, 2, \ldots$(They always exist and are finite for a measure $\mu \in \mathcal{M}(K)$ where $K \subset \mathbb{R}$). Then, apply Gram-Schmidt process to the set $\{1, x, x^2, \ldots\}$ to get the polynomials $q_0(x;\mu), q_1(x;\mu), q_2(x;\mu), \ldots$. Note that these polynomials satisfy

$$\int q_n(x;\mu) q_m(x;\mu) d\mu(x) = \delta_{nm}$$

where $n, m = 0, 1, 2, \ldots$, the degree of q_n is n and $\kappa_n > 0$, the coefficient of x^n in q_n. Then, we call $Q_n(x;\mu) := \dfrac{q_n(x;\mu)}{\kappa_n}$ the n-th monic orthogonal polynomial for μ (and q_n is the n-th orthonormal polynomial for μ).

Note that, $||x^{n+1} - Q_{n+1}(x)||_{L^2(\mu)}$ is the projection of x^{n+1} on the set $\{1, x, x^2, \ldots, x^n\}$.

Now, we give some elementary properties of the zeros of orthogonal polynomials.

Theorem 3.1.3. *([33], p. 7)The zeros of the orthogonal polynomials $q_n(x;\mu)$ are real, distinct and in (a, b) where $a = \inf \operatorname{supp}(\mu)$ and $b = \sup \operatorname{supp}(\mu)$.*

Theorem 3.1.4. *(Theorem 3.3.2 in [29]) Let the set $\{x_1, x_2 \ldots x_n\}$ be the zeros of the orthogonal polynomial $q_n(x)$ such that they are enumerated in ascending order. For $m = 1, \ldots n-1$ every interval $[x_m, x_{m+1}]$ there is exactly one zero of q_{n+1}.*

Theorem 3.1.5. *(Theorem 3.3.3 in [29]) At least one zero of $q_i(x)$ lies in between two zeros of $q_j(x)$ for $i > j$.*

Remark that zeros of orthogonal polynomials generate the Gauss-Jacobi quadrature:

Definition 3.1.6. (Lemma 0.2 in [33]) Let $x_1 < x_2 < \ldots < x_n$ be the zeros of the polynomial $q_n(x;\mu)$. Then, for any polynomial $p(x)$ of degree at most $2n-1$ there are positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ called *Christoffel numbers* such that

$$\int p(x)d\mu(x) = \sum_{i=1}^{n} \lambda_i p(x_i). \tag{3.1}$$

Note that $d\mu(x)$ and n determine Christoffel numbers λ_i uniquely. In fact, we have

$$\lambda_n = \frac{-\kappa_{n+1}}{\kappa_n q_{n+1}(x_i) q_n'(x_i)} = \frac{\kappa_n}{\kappa_{n-1} q_{n-1}(x_i) q_n'(x_i)}. \tag{3.2}$$

Now, we will give a recurrence relation which is a significant property of orthogonal polynomials on the real line.

Theorem 3.1.7. *(Lemma 0.3 in [33]) Assume that $q_{-1}(x;\mu) := 0$ and $q_0(x;\mu) := 1$. For every three consecutive orthogonal polynomials we have the following recurrence formula:*

$$xq_n(x;\mu) = a_{n+1}q_{n+1}(x;\mu) + b_{n+1}q_n(x;\mu) + a_n q_{n-1}(x;\mu) \;\; n \in \mathbb{N}_0, \tag{3.3}$$

where a_n, b_n are real constants such that $a_n > 0$ and

$$a_n = \frac{\kappa_n}{\kappa_{n-1}}, \quad b_n = \int x q_n^2(x) d\mu(x).$$

Here, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are called the *recurrence* (or *Jacobi*) *coefficients*. Then, observe that the relation evolves for monic orthogonal polynomials $Q_n(x;\mu)$ defined in Definition 3.1.2 with distribution $d\mu(x)$ such that

$$Q_{n+1}(x;\mu) = (x - b_{n+1})Q_n(x;\mu) - a_n^2 Q_{n-1}(x;\mu), \;\; n \in \mathbb{N}_0, \tag{3.4}$$

where $a_n \in \mathbb{R}^+$ and $b_n \in \mathbb{R}$. Moreover, we have $||Q_n(\cdot;\mu)||_{L^2(\mu)} = a_1 \cdots a_n$ since $||Q_n(x;\mu)||_{L^2(\mu)} = \kappa_n^{-1}$ and $a_n = \frac{\kappa_n}{\kappa_{n-1}}$.

Now, given this recurrence relation (3.4), we can introduce a Jacobi matrix of order n. If we are given two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ where a_n is positive and b_n is real for all $n \in \mathbb{N}$ and both are bounded, then we can define the corresponding Jacobi Matrix

$$H_{\mu} = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.5}$$

Remark that if μ is the scalar valued spectral measure of H_{μ} for the cyclic vector $e = (1, 0, \dots, 0)^T$ (i.e, $\ell^2(\mathbb{N})$ can be spanned by $\{e, H_{\mu}e, (H_{\mu})^2 e, \dots\}$), then it has $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ as recurrence coefficients. Here, $H_{\mu} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is a self-adjoint bounded operator. For more on spectral theory of orthogonal polynomials, see [27, 33].

Moreover, if we set

$$H_{\mu^n_{K(\gamma)}} = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & a_{n-1} & b_n \end{pmatrix}, \tag{3.6}$$

then, by expanding $\det\left(H^n_{\mu} - xI\right)$ along the n-th row we have the following result:

Lemma 3.1.8. *(p. 9 of [33]) The eigenvalues of the matrix H^n_{μ} are equal to the zeros of the associated orthogonal polynomial $q_n(x;\mu)$. And a normalized eigenvector ν for the eigenvalue $v = x_m$, the m-th zero of $q_n(x;\mu)$, is given by*

$$\sqrt{\lambda_m}(q_0(x_m), q_1(x_m), \dots, q_{n-1}(x_m))$$

where $(\lambda_m)_{m=1}^{n}$ are the Christoffel numbers.

Therefore, the monic orthogonal polynomials with respect to measure μ can be written as

$$Q_n(x;\mu) = \det\left(xI - H^n_{\mu}\right).$$

3.2 Widom Factors

Definition 3.2.1. The polynomial $T_n(x) = x^n + \ldots$ is called the $n-th$ Chebyshev polynomial of the first kind on K if

$$||T_n||_{L^\infty(K)} = \min\{||Q_n||_{L^\infty(K)} : Q_n \text{ monic polynomial of degree } n\}$$

where $K \subset \mathbb{R}$ is an infinite compact set and $||.||_{L^\infty(K)}$ denotes the supremum norm on K.

Remark 3.2.2. (Corollary 5.5.5 in [21]) Between the Chebyshev polynomial $T_n(x)$ on the set K and the logarithmic capacity of that set there is a relation of the form

$$\lim_{n\to\infty} ||T_n||_{L^\infty(K)}^{1/n} = Cap(K).$$

Definition 3.2.3. Let T_n be the n-th Chebyshev polynomial on a non-polar compact $K \subset \mathbb{C}$. The *n-th Widom Factor* for the supremum norm on K is defined as

$$W_n(K) = \frac{||T_n||_{L^\infty(K)}}{(Cap(K))^n}.$$

Let us give a couple of examples.

Example 3.2.4. For $K = [-1, 1]$ we have $||T_n||_{L^\infty(K)} = 2^{1-n}$ (see [22]), and from Chapter 2 we know that $Cap(K) = 1/2$. So,

$$W_n(K) = \frac{2^{1-n}}{(1/2)^n} = 2.$$

Example 3.2.5. For $K = \overline{\mathbb{D}}$ we have $||T_n||_{L^\infty(K)} = 1$(see [22]), and from Chapter 2 we know that $Cap(K) = 1$. Hence,

$$W_n(K) = 1.$$

Let us give an important remark

Proposition 3.2.6. *Widom factor is invariant under dilation and translation for any compact non-polar $K \subset \mathbb{C}$, i.e.,*

$$W_n(\alpha K + \beta) = W_n(K)$$

where $\alpha > 0$, $\beta \in \mathbb{C}$.

Proof. It is easy to see that $||T_n||_{L^\infty(\alpha K+\beta)} = \alpha^n ||T_n||_{L^\infty(K)}$, and recall part (ii) of Theorem 2.2.5; thus we have the desired equality. $\square$

Recall that one of the main points of this research is to analyse the asymptotic behaviour of Widom factors. By previous remark, Examples 3.2.4 and 3.2.5, we have established that Widom factors of any disc or interval is a constant sequence. But except for very few cases the limit of $(W_n(K))_{n=1}^\infty$ does not exist. And the behaviour of the sequence is quite irregular for even simple cases. Thus, we consider lower and upper estimates. We have the following theorem by Schiefermayr for sets on the real line.

Theorem 3.2.7. *[26]For a compact non-polar set $K \subset \mathbb{R}$ we have*

$$W_n(K) \geq 2$$

for all $n \in \mathbb{N}$.

Note that, we have $\lim_{n\to\infty} W_n(K) = 1$ for any disc or circle (by Example 3.2.5). However, V. Totik showed that this is not true for the case when the unbounded connected component Ω_K of $\overline{\mathbb{C}}\backslash K$ is not simply connected (see [35]).

Theorem 3.2.8. *(Theorem 2 in [32]) For a compact set $K \subset \mathbb{C}$ let Ω_K denote the unbounded connected component of $\overline{\mathbb{C}}\backslash K$. If Ω_K is not simply connected, then there is a $\varepsilon > 0$ such that*

$$W_{n_i}(K) \geq \varepsilon + 1$$

for some subsequence of $(W_n(K))_{n=1}^\infty$.

Now, observe that, Theorem 3.2.7 and Remark 3.2.2 imply the following lemma:

Lemma 3.2.9.

$$\lim_{n\to\infty} (W_n(K))^{1/n} = 1$$

Hence, we have

$$\frac{1}{n} \log W_n(K) \to 0$$

as $n \to \infty$. That is, $(W_n)_{n=1}^\infty$ has subexponential growth.

Observe that, this lemma imposes a theoretical constraint on the growth rate of $W_n(K)$, i.e. $\liminf W_n(K) \geq 1$. Moreover, if we have a infinite and compact set K which is union of disjoint closed intervals, there are several results (see [30, 31, 32, 35]) saying that $(W_n(K))_{n=1}^{\infty}$ is bounded. Now, recall that one goal of this research to analyze Widom factors on Cantor sets, particularly, their bounds. It is recently proven in [10] that there are some Cantor sets K such that the Widom factor $W_n(K)$ is bounded.

Chapter 4

Weakly Equilibrium Cantor Sets and Numerical Experiments

In this chapter we discuss our numerical experiments from [1], joint work of G. Alpan, A. Goncharov, A. N. Şimşek, as mentioned before.

4.1 Construction of Weakly Equilibrium Cantor Sets

In this section, we give the construction of the Cantor set $K(\gamma)$ we used in our numerical experiments that is introduced by A. Goncharov in [16]. Let us begin by taking a sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$, where γ_s is in the interval $(0, 1/4)$ for all s. Then, define $r = (r_s)_{s=0}^{\infty}$ with $r_0 = 1$ and $r_s := \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Now, let

$$P_1(x) := x - 1 \text{ and } P_{2^{s+1}}(x) := P_{2^s}(P_{2^s}(x) + r_s) \tag{4.1}$$

for $s \in \mathbb{N}_0$. For any choice of $\gamma = (\gamma_s)_{s=1}^{\infty}$ this recursive relation yields

$$P_2(x) = x(x - 1).$$

However, for $s \geq 2$ the polynomial P_{2^s} heavily depends on the sequence γ, hence, a different set of polynomials for different choices of γ. Then, for $s = 0, 1, 2, \ldots$

consider the sets

$$E_s = \{x \in \mathbb{R} | P_{2^{s+1}}(x) \leq 0\}.$$

Now, we define our set as

$$K(\gamma) := \bigcap_{s=0}^{\infty} E_s.$$

Note that, E_s is equivalent to

$$\left[\frac{2}{r_s} P_{2^s} + 1\right]^{-1} ([-1, 1]) = \bigcup_{j=1}^{2^s} I_{j,s} \ \forall s.$$

Here, $I_{j,s}$ are closed basic intervals of the s-th level which are necessarily disjoint. Then, setting $l_{j,s}$ as the length of $I_{j,s}$, we see that by Lemma 2 in [5], $\max_{1 \leq j \leq 2^s} l_{j,s} \to 0$ as $s \to \infty$. Hence, $K(\gamma)$ is a Cantor set.

4.2 Orthogonal Polynomials On Weakly Equilibrium Cantor Sets

Now that we have constructed our set, we can begin discussing orthogonal polynomials on $K(\gamma)$. To that end in [5], G. Alpan and A. Goncharov gave some important Theorems about orthogonal polynomials on $K(\gamma)$ and also an algorithm to calculate the elements of the Jacobi matrix $H_{\mu_{K(\gamma)}}$.

Theorem 4.2.1. *(Prop. 1 in [16] and Thm. 2.1 in [5]) The polynomial $P_{2^s} + \frac{r_s}{2}$ is the Chebyshev polynomial for $K(\gamma)$ for all $s = 0, 1, 2, \ldots$.*

Theorem 4.2.2. *(Theorem 2.4 [16]) For a non-polar compact set K let μ_K be its equilibrium measure. And let the normalized counting measures on the zeros $(x_i)_{i=1}^{2^s}$ of the Chebyshev polynomial $P_{2^s} + \frac{r_s}{2}$ be $\sigma_s := 2^{-s} \sum_{i=1}^{2^s} \delta_{x_i}$. Then, $\sigma_n \to \mu_K$ in weak star topology.*

Lemma 4.2.3. *(Lemma 2.5 in [5]) If $s > n$ with $s \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we have*

$$\int \left(P_{2^n} + \frac{r_n}{2}\right) d\sigma_s = 0.$$

Lemma 4.2.4. *(Lemma 2.5 in [5]) Take indices (k_i) such that $0 \leq k_1 < k_2 < \ldots < k_n < s$. Then,*

i) $$\int P_{2^{k_1}} P_{2^{k_2}} \ldots P_{2^{k_n}} d\sigma_s = \int P_{2^{k_1}} d\sigma_s \ldots \int P_{2^{k_n} d\sigma_s} = (-1)^n \prod_{i=1}^{n} \frac{r_{k_i}}{2}.$$

ii) $$\int \left(P_{2^{k_1}} + \frac{r_{k_1}}{2}\right) \ldots \left(P_{2^{k_n}} + \frac{r_{k_n}}{2}\right) d\sigma_s = 0.$$

Observe that, by Theorem 4.2.2, $\mu_{K(\gamma)}$ can be used instead of σ_n in previous two lemmas. Now, we have the following important theorem:

Theorem 4.2.5. *(Theorem 2.8 in [5]) For all $s \in \mathbb{N}_0$, the 2^s-th monic orthogonal polynomial $Q_{2^s}(\cdot; \mu_{K(\gamma)})$ for the equilibrium measure of $K(\gamma)$ equals $P_{2^s} + \dfrac{r_S}{2}$. Then, by (4.1) we have (Corollary 2.9 in [5])*

$$Q_{2^{s+1}}(\cdot; \mu_{K(\gamma)}) = Q_{2^s}^2(\cdot; \mu_{K(\gamma)}) - (1 - 2\gamma_{s+1})\frac{r_s^2}{4}. \tag{4.2}$$

Then, by (4.2) for all $s \in \mathbb{N}_0$ we have

$$||Q_{2^s}\left(\cdot; \mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)} = \sqrt{(1 - 2\,\gamma_{s+1})\, r_s^2/4}. \tag{4.3}$$

We already know that the diagonal elements, the b_n's of $H_{\mu_{K(\gamma)}}$, are equal to 1/2 by Section 4 in [5]. For the outdiagonal elements, a_n, by Theorem 4.3 in [5] we can calculate $(a_n)_{n=1}^{\infty}$ recursively; here is the algorithm:

$$a_1 = \|Q_1\left(\cdot; \mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}, \tag{4.4}$$

$$a_2 = \frac{\|Q_2\left(\cdot; \mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}}{\|Q_1\left(\cdot; \mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}}. \tag{4.5}$$

If $n + 1 = 2^s > 2$ then

$$a_{n+1} = \frac{||Q_{2^s}\left(\cdot; \mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)}}{||Q_{2^{s-1}}\left(\cdot; \mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)} \cdot a_{2^{s-1}+1} \cdot a_{2^{s-1}+2} \cdots a_{2^s-1}}. \tag{4.6}$$

If $n+1 = 2^s(2k+1)$ for some $s \in \mathbb{N}$ and $k \in \mathbb{N}$, then

$$a_{n+1} = \sqrt{\frac{\|Q_{2^s}\left(\cdot;\mu_{K(\gamma)}\right)\|^2_{L^2\left(\mu_{K(\gamma)}\right)} - a^2_{2^{s+1}k}\cdots a^2_{2^{s+1}k-2^s+1}}{a^2_{2^s(2k+1)-1}\cdots a^2_{2^{s+1}k+1}}}, \tag{4.7}$$

If $n+1 = (2k+1)$ for $k \in \mathbb{N}$ then

$$a_{n+1} = \sqrt{\|Q_1\left(\cdot;\mu_{K(\gamma)}\right)\|^2_{L^2\left(\mu_{K(\gamma)}\right)} - a^2_{2k}}. \tag{4.8}$$

To see how we used this algorithm in Matlab see Codes, Appendix A.

4.3 Properties of Weakly Equilibrium Cantor Sets

We will now give some properties of $K(\gamma)$.

Theorem 4.3.1. *(see [5]) If $\gamma_s \leq 1/6$ $\forall s \in \mathbb{N}$, then $K(\gamma)$ has zero Lebesgue measure, and $\mu_{K(\gamma)}$ is purely singular continuous and* $\liminf a_n = 0$ *for $\mu_{K(\gamma)}$.*

We use the following theorem to determine whether the corresponding Green's function is optimally smooth or not:

Theorem 4.3.2. *(see [6]) $g_{\Omega_{K(\gamma)}}$ is optimally smooth (Hölder continuous with exponent 1/2) if and only if $\sum_{s=1}^{\infty}(1-4\gamma_s) < \infty$.*

Parreau-Widom characterization for $K(\gamma)$:

Theorem 4.3.3. *(see [6]) $K(\gamma)$ is a Parreau-Widom set if and only if $\sum_{s=1}^{\infty}\sqrt{1-4\gamma_s} < \infty$.*

Upper bound characterization for Widom-Hilbert factor ($W_n^2(\mu_{K(\gamma)})$ which is a special case of Widom factors) for $K(\gamma)$:

Theorem 4.3.4. *(see [7]) If $\sum_{s=1}^{\infty}(1-4\gamma_s) < \infty$, then there is $C_n > 0$ such that for all $n \in \mathbb{N}$ we have*

$$C_n \geq W_n^2(\mu_{K(\gamma)}) := \frac{\|Q_n\left(\cdot;\mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}}{(\mathrm{Cap}(K(\gamma)))^n} = \frac{a_1\cdots a_n}{(\mathrm{Cap}(K(\gamma)))^n}.$$

Capacity of $K(\gamma)$:

Theorem 4.3.5. *(see [16]) $Cap(K(\gamma)) = \exp\left(\sum_{k=1}^{\infty} 2^{-k}\log\gamma_k\right)$, which implies that $K(\gamma)$ is non-polar if and only if*

$$\sum_{n=1}^{\infty} 2^{-n}\log\left(1/\gamma_n\right) < \infty.$$

How to obtain the zeros of the 2^s-th monic orthogonal polynomial for $\mu_{K(\gamma)}$:

Theorem 4.3.6. *(see [2]) Let $v_{1,1}(t) = 1/2 - (1/2)\sqrt{1-2\gamma_1+2\gamma_1 t}$ and $v_{2,1}(t) = 1 - v_{1,1}(t)$. For each $n > 1$, let $v_{1,n}(t) = \sqrt{1-2\gamma_n+2\gamma_n t}$ and $v_{2,n}(t) = -v_{1,n}(t)$. Then the zero set of $Q_{2^s}\left(\cdot;\mu_{K(\gamma)}\right)$ is $\{v_{i_1,1}\circ\ldots\circ v_{i_s,s}(0)\}_{i_s\in\{1,2\}}$ for all $s \in \mathbb{N}$.*

Theorem 4.3.7. *(see [2]) We have $\mathrm{supp}(\mu_{K(\gamma)}) = \mathrm{ess\,supp}(\mu_{K(\gamma)}) = K(\gamma)$. If*

$$K(\gamma) = [0,1] \setminus \bigcup_{k=1}^{\infty}(c_i, d_i)$$

where $c_i \neq d_j$ for all $i, j \in \mathbb{N}$, then $\mu_{K(\gamma)}([0,e_i]) \subset \{m2^{-n}\}_{m,n\in\mathbb{N}}$, where $e_i \in (c_i, d_i)$. Moreover for each $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $m2^{-n} < 1$, there is an $i \in \mathbb{N}$ such that $\mu_{K(\gamma)}([0,e_i]) = m2^{-n}$.

4.4 Models

Now, having some idea about $K(\gamma)$ we will give the models we used for numerical experiments and the properties of those models briefly.

We use the following four models for our numerical experiments:

1: $(\gamma_s) = 1/4 - (50+s)^{-4}$

2: $(\gamma_s) = 1/4 - (50+s)^{-2}$

3: $(\gamma_s) = 1/4 - (50+s)^{-5/4}$

4: $(\gamma_s) = 1/4 - 1/50$

And each of them represent different properties:

i) Model 1 represents as an example where $K(\gamma)$ is Parreau-Widom.

ii) Model 2 gives a non-Parreau-Widom set with a fast growth of γ and $g_{\Omega_{K(\gamma)}}$ optimally smooth.

iii) Model 3 gives a non-Parreau-Widom set with a relatively slow growth of γ but still with $g_{\Omega_{K(\gamma)}}$ optimally smooth.

iv) Model 4 produces a set which is neither Parreau-Widom nor with Green's function for its complement optimally smooth.

4.5 Numerical Stability of Algorithm

We need to show that our algorithm points out numerical stability. To that end, we need to compare zeros and Christoffel numbers obtained by our algorithm and their theoretical values. By using the following remark we have compared the eigenvalues of $H^{2^n}_{\mu_{K(\gamma)}}$ to zeros obtained by Theorem 4.3.6. Recall Lemma 3.1.8; the eigenvalues of $H^{2^n}_{\mu_{K(\gamma)}}$ are equal to the zeros of $Q_{2^n}(.;\mu_{K(\gamma)})$.

Let $\{v_i^n\}_{i=1}^{2^n}$ be the set of eigenvalues of $H^{2^n}_{\mu_{K(\gamma)}}$ and $\{x_i^n\}_{i=1}^{2^n}$ be the set of zeros of $Q_{2^n}(.;\mu_{K(\gamma)})$ (where zeros are enumerated in ascending order). Then, setting $E_n^1 := 2^{-n}(\sum_{k=1}^{2^n} |v_k^n - x_k^n|)$ we have Figure 4.1.

Before we can draw Figure 4.2 we need two remarks.

Remark 4.5.1. (see [15]) The squares of the eigenvectors' first components give the Christoffel numbers corresponding to $Q_{2^n}(.;\mu_{K(\gamma)})$.

Remark 4.5.2. (see Theorem 1.3.5 in [27]) The Christoffel numbers corresponding to $Q_{2^n}(.;\mu_{K(\gamma)})$ are exactly equal to 2^{-n}.

Now, let $\{\nu_i^n\}_{i=1}^{2^n}$ be the set of squared first components of normalized eigenvectors of $H^{2^n}_{\mu_{K(\gamma)}}$. Then, setting $E_n^2 := 2^{-n}(\sum_{k=1}^{2^n} |2^{-n} - \nu_k^n|)$ we have the Figure 4.2. Thus, as it can be seen from Figure 4.1 and 4.2, our Algorithm is reliable with small errors. These values can be compared with Figure 4.2 in [18].

4.6 First Observations

Now that we have established that we can rely on our algorithm up to a small error, we can begin our analysis. Our numerical experiments (we found the minimum via the code in Appendix A) suggests that $\min_{i\in 1,\ldots,2^n} a_i = a_{2^n}$ for $n \leq 14$. Therefore, we make the following conjecture:

Conjecture 4.6.1. *For $\mu_{K(\gamma)}$ we have $\min\limits_{i\in\{1,\ldots,2^n\}} a_i = a_{2^n}$ and, in particular, $\liminf\limits_{s\to\infty} a_{2^s} = \liminf\limits_{n\to\infty} a_n$.*

Also, remark that by (4.4) and (4.8) we have $\max\limits_{n\in\mathbb{N}} a_n = a_1$.

Before we continue let us give a remark about Parreau-Widom sets.

Remark 4.6.2. For Parreau-Widom K we have $\liminf a_n > 0$ where a_n's are outdiagonal elements of H_{μ_K} (see Remark 4.8 in [5]).

Now, consider Theorem 4.3.3 and the previous remark. Then, we have $\liminf a_n > 0$ for $\mu_{K(\gamma)}$ if $\sum_{s=1}^{\infty} \sqrt{1-4\gamma_s} < \infty$. In addition, by previous remark and [13], if $\liminf a_n = 0$, where a_n's are outdiagonal elements of $H_{\mu_{K(\gamma)}}$, then $K(\gamma)$ has zero Lebesgue measure. In this respect, we have computed the ratio

$\rho_n := \frac{a_{2n}}{a_{2n+1}}$ for $n = 1, \ldots, 13$ to find for which of our models we have $\liminf a_n = 0$ (see Figures 4.3 and 4.4). And, note that, we assume that Conjecture 4.6.1 is correct.

For the first model, ρ_n is very close to 1, however, this is expected since for this model $\liminf a_n > 0$ since it is Parreau-Widom. For the rest of the models, it seems that $(\rho_n)_{n=1}^{13}$ behaves like a constant. Thus, this experiment can be read as: $\liminf a_n = 0$ unless $\sum_{s=1}^{\infty} \sqrt{1 - 4\gamma_s} < \infty$. Hence, the following conjecture:

Conjecture 4.6.3. *$K(\gamma)$ is of positive Lebesgue measure if and only if* $\sum_{s=1}^{\infty} \sqrt{1 - 4\gamma_s} < \infty$ *if and only if* $\liminf a_n > 0$.

4.7 Almost Periodicity

Definition 4.7.1. A sequence $\alpha = (\alpha_n)_{n=-\infty}^{\infty}$ with $\alpha_n \in \mathbb{C}$ for all n is called *almost periodic* if the set $\{\alpha^m = (\alpha_{n+m})_{n=-\infty}^{\infty} : m \in \mathbb{Z}\}$ is precompact in $\ell^\infty(\mathbb{Z})$. And a one-sided sequence is called almost periodic if it is the restriction of a two-sided almost periodic sequence to natural numbers.

However, they are essentially the same objects since every one-sided almost periodic sequence has a unique extension to a two-sided almost periodic sequence(see 5.13 in [27]).

Now, we extend this notion to the Jacobi matrix H_μ.

Definition 4.7.2. A Jacobi matrix H_μ is called almost periodic if the recurrence coefficients $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ for the measure μ are almost periodic.

Definition 4.7.3. We call a sequence $\beta = (\beta_n)_{n=1}^{\infty}$ *asymptotically almost periodic* if there exists an almost periodic sequence $\alpha = (\alpha_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty}(\alpha_n - \beta_n) = 0$. Note that if it exists, α is unique and called the *almost periodic limit*.

In this section, we shift our focus to a more interesting problem: Is $H_{\mu_{K(\gamma)}}$ almost periodic or, at least, asymptotically almost periodic? Before we begin our analysis, first note that, $K(\gamma)$ is a generalized Julia set for $\inf_s \gamma_s > 0$ ([8]). And Jacobi matrix of an equilibrium measure for Julia sets is almost periodic (see [8, 36]), so we suspect that it is almost periodic or at least asymptotically almost periodic. Here, a Julia set (named after Gaston Julia(1893-1978)) can be defined as:

Definition 4.7.4. ([33], p. 35 or see [9]) Take the monic polynomial $Q(z) = z^n + \cdots$ and let $Q_m(z) = Q_{m-1}(Q(z))$ to be its m-th iterate with $Q_0(z) = z$. Then, the *Julia* set for the polynomial Q is

$$J := \{z \in \mathbb{C} : Q_m(z) = z \text{ and } |Q_n'(z)| > 1\}.$$

Lemma 4.7.5. *(see [9]) We have the following properties for a Julia set J for a monic polynomial Q:*

i) J is compact,

ii) $J \neq \emptyset$,

iii) J is completely invariant ($Q(J) = Q = Q^{-1}(J)$),

iv) $Cap(J) = 1$,

v) $supp(\mu_J) = J$.

We refer the reader to [9] for more about Julia sets. Now, recall that $b_n = 1/2$ for all $n \in \mathbb{N}$, hence, it is periodic. So, we need to analyse $(a_n)_{n=1}^{\infty}$ for periodicity. We need a few definitions for this.

Definition 4.7.6. Suppose that μ is a measure with infinite compact support and also suppose that ω_n be the normalized counting measure on the zeros of $Q_n(\cdot;\mu)$. Then, if there exists a measure ω such that $\omega_n \to \omega$ (here the convergence is weak star convergence: $\int f d\omega_n \to \int f d\omega$ for a continuous function f), we call ω *density of states* (DOS) measure for H_μ. Moreover, *integrated density of states* (IDS) is defined as the integral $\int_{-\infty}^{x} d\omega$.

By Theorem 1.7 and 1.12 in [27] and by [34] we have the following for $H_{\mu_{K(\gamma)}}$:

Remark 4.7.7. Density of states of $H_{\mu_{K(\gamma)}}$ is $\mu_{K(\gamma)}$. This implies that if one chooses $x \in (c_i, d_i)$ (see Theorem 4.3.7) (choosing x from a gap of $\text{supp}(\mu_{K(\gamma)})$), then IDS is $\int_{-\infty}^{x} d\mu_{K(\gamma)} = m2^{-n}$ and $m2^{-n} \leq 1$. Moreover, for any $m, n \in \mathbb{N}$ if $m2^{-n} < 1$, then there exists a gap (c_k, d_k) where IDS is $m2^{-n}$.

Note that, in the previous remark, a bounded component of $\mathbb{R}\backslash K$ is what we mean by a gap of a compact set $K \subset \mathbb{R}$. Now, we need one more definition.

Definition 4.7.8. We define the *frequency module* $\mathfrak{M}(\alpha)$ *of* an almost periodic sequence $\alpha = (\alpha_n)_{n=1}^{\infty}$ as the $\mathbb{Z}$-module of the real numbers modulo 1 generated by

$$\{\theta : \lim_{n\to\infty} \frac{1}{N} \alpha_n e^{2i\pi n\theta} \neq 0\}.$$

Remark 4.7.9. We have several results/properties for frequency module:

i) $\mathfrak{M}(\alpha)$ is countable.

ii) α can be written as a uniform limit of Fourier series, where frequencies chosen from $\mathfrak{M}(\alpha)$.

iii) Frequency module $\mathfrak{M}(H)$ of a Jacobi matrix H is generated by $\mathfrak{M}(a)$ and $\mathfrak{M}(b)$ where $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$ coefficients of H.

iv) For an almost periodic Jacobi matrix H the values of IDS in gaps are from $\mathfrak{M}(H)$ (see Theorme III.1 in [12]).

v) DOS measure of an asymptotically almost periodic Jacobi matrix is the same as its almost periodic limit (see Theorem 2.4 in [14]).

Definition 4.7.10. For $N \in \mathbb{N}$ the discrete Fourier transform $\widehat{\alpha} = (\widehat{\alpha_n})_{n=1}^{N}$ of $(\alpha_n)_{n=1}^{N}$ is defined by

$$\widehat{\alpha_k} := \sum_{n=1}^{N} \alpha_n e^{-2(k-1)i\pi(n-1)/N}$$

where $k = 1, 2, \ldots, N$.

We computed the discrete Fourier transform $(\widehat{a_n})_{n=1}^{2^{14}}$ for the first 2^{14} recurrence coefficients a_n. Note that, for every model the frequencies run from 0 to 1. Also, we normalized $|\widehat{a}|^2$ by dividing it by $\sum_{n=1}^{2^{14}} |\widehat{a_n}|^2$. We plotted this normalized power spectrum (see Figure 4.5) without the big peak at 0.

For all models, the spectrum yield a small number of peaks compared to 2^{14} frequencies which points out almost periodicity of a_n's. In here, we consider only model 1 but we have similar pictures for other models. The highest peaks are at $0.5, 0.25, 0.75, 0.375, 0.625, 0.4375, 0.5625, 0.125, 0.875, 0.3125$ which are of the form $m2^{-n}$ where $n \leq 4$. Note that these frequencies are exactly the values of IDS for $H_{\mu_{K(\gamma)}}$ in the gaps. Note that they appear earlier in the construction of the weakly equilibrium Cantor set. Thus, we have the following conjecture naturally:

Conjecture 4.7.11. *For any γ, $(a_n)_{n=1}^{\infty}$ for $H_{\mu_{K(\gamma)}}$ is asymptotically almost periodic where the almost periodic limit has frequency module equal to $\{m2^{-n}\}_{m,n\in\{\mathbb{N}_0\}}$ modulo 1.*

4.8 Widom Factors

We examine Widom Factors for $K(\gamma)$ in this section. We have the following relation between sequences with subexponetial growth and Widom factors:

Remark 4.8.1. (Theorem 4.4 in [17]) For each sequence $(c_n)_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} \frac{1}{n}\log c_n = 0$, there is a Cantor set $K(\gamma)$ such that $W_n(K(\gamma)) > c_n$ for all $n \in \mathbb{N}$.

Note that, for a unit Borel measure with infinite compact support on $\mathbb{R}$ we have

$$||Q_n(.;\mu_K)||_{L^2(\mu_K)} \leq ||T_n||_{L^2(\mu_K)} \leq ||T_n||_{L^\infty(K)}, \tag{4.9}$$

where Q_n is the n-th monic polynomial for μ_K. Also, by [21, 27], if a non-polar compact $K \subset \mathbb{R}$ is regular, then $\text{supp}(\mu_K) = K$ (recall Lemma 2.2.15; if we lift the regularity condition for K, instead of the last equality we have $\text{Cap}(\text{supp}(\mu_K)) = \text{Cap}(K)$). Now, recall the definition of Widom-Hilbert factors from Theorem 4.3.4: the n-th Widom-Hilbert factor for μ is defined as

$$W_n^2(\mu) := \frac{||Q_n(.;\mu)||_{L^2(\mu)}}{(\mathrm{Cap}(supp(\mu)))^n}. \tag{4.10}$$

Then, observe that by Equation 4.9 we have

$$W_n^2(\mu_K) \le W_n(K). \tag{4.11}$$

Also, by [10] when $K \subset \mathbb{R}$ is Parreau-Widom, we have $\liminf a_n > 0$. It would be interesting to find (if any exists) a non-Parreau-Widom set on the real line such that it is regular and sequence of Widom factors is bounded. To see this problem in a different way observe that Equation (4.11) lead us to a weaker problem: Is there a non-Parreau-Widom but regular set $K \subset \mathbb{R}$ such that $(W_n^2(\mu_K))_{n=1}^\infty$ is bounded? Therefore, we will examine the behaviour of $(W_n^2(\mu_{K(\gamma)}))_{n=1}^\infty$ for non-Parreau-Widom $K(\gamma)$. For this, first consider that for $\gamma_k \le 1/6$ for all $k \in \mathbb{N}$ $(W_n^2(\mu_{K(\gamma)}))_{n=1}^\infty$ is unbounded (see [5]). Recall that none of our models satisfy this, so let us continue with another remark (by [5]) to begin our analysis.

Remark 4.8.2. For any γ we have $W_n^2(\mu_{K(\gamma)}) \ge \sqrt{2}$ for all $n \in \mathbb{N}_0$.

Thus, we have

$$W_{2^n-1}^2(\mu_{K(\gamma)}) = W_{2^n}^2(\mu_{K(\gamma)})\frac{Cap(K(\gamma))}{a_{2^n}} \ge \sqrt{2}\frac{Cap(K(\gamma))}{a_{2^n}}. \tag{4.12}$$

Assuming Conjectures 4.6.1 and 4.6.3 are true; for non-Parreau-Widom $K(\gamma)$ we have $\liminf_{n\to\infty} a_{2^n} = 0$. This implies that by (4.12), we have $\limsup_{n\to\infty} W_{2^n-1}^2(\mu_{K(\gamma)}) = \infty$ if $\liminf_{n\to\infty} a_{2^n} = 0$. Hence, we conjecture:

Conjecture 4.8.3. *$K(\gamma)$ is a Parreau-Widom set if and only if $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^\infty$ is bounded if and only if $(W_n(K(\gamma)))_{n=1}^\infty$ is bounded.*

Now, let K be a union of finitely many compact non-degenerate intervals on $\mathbb{R}$ and ω be the Radon-Nikodym derivative of μ_K with respect to the Lebesgue meeasure on the line. Then μ_K satisfies the Szegő condition: $\int_K \omega(x)\log\omega(x)\,dx > -\infty$. This implies by Corollary 6.7 in [11] that $(W_n^2(\mu_K))_{n=1}^\infty$ is asymptotically almost periodic. Note that, μ_K satisfies the Szegő condition if K is a Parreau-Widom set(see [20]).

In Figure 4.7, we plotted the Widom-Hilbert factors for Model 1 until $n = 2^{20}$ and apparently

$$\limsup(W_n^2(\mu_{K\gamma})) \neq \sup_n(W_n^2(\mu_{K\gamma})).$$

Then, we plotted (Figure 4.6) the power spectrum for $(W_n^2(\mu_{K\gamma}))_{n=1}^{2^{14}}$ where we normalized $|\hat{W}^2|^2$ by dividing it by $\sum_{n=1}^{2^{14}} |\hat{W}_n^2(\mu_{K\gamma})|$. Again, frequencies run from 0 to 1 and we omitted the big peak at 0. Like the previous power spectrum there are only a few peaks which is an important indicator of almost periodicity as mentioned before. The highest ten peaks are at $0.5, 0.00006103515625, 0.25, 0.75, 0.125, 0.875, 0.375, 0.625, 0.0625, 0.9375$, however, they are quite different from the power spectrum for a_n; which may indicate that almost periodic limit has a different frequency module.

If Conjecture 4.8.3 is correct, then the sequence of Widom-Hilbert factors $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^{\infty}$ is unbounded and cannot be asymptotically almost periodic if $K(\gamma)$ is not Parreau-Widom. Therefore, we conjecture:

Conjecture 4.8.4. $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^{\infty}$ *is asymptotically almost periodic if and only if $K(\gamma)$ is Parreau-Widom. If $K(\gamma)$ is Parreau-Widom then the frequency module of the almost periodic limit includes the module generated by $\{m2^{-n}\}_{m,n\in\{N_0\}}$ modulo 1.*

4.9 Spacing Propeties of Orthogonal Polynomials

In this section we give some spacing properties of orthogonal polynomials. For a given γ, define $Z_n(\mu) := \{x : q_n(x;\mu) = 0\}$ for all $n \in \mathbb{N}$ and enumerate its elements x_i^n in ascending order for $i = 1, \ldots, n$. And also define

$$M_n(\mu) := \inf\{|x-y| : x, y \in Z_n(\mu) \text{ and } x \neq y\}.$$

In [2], G. Alpan studied the behaviour of $(M_n(\mu_{K(\gamma)}))_{n=1}^{\infty}$, that is, the global behaviour of the spacing of zeros. We will give our numerical study (from [1]) on the local behaviour of the zeros.

Here, we will only discuss Model 1 since for other cases the results are similar. Let us define

$$S_n^N := |x_{2n}^N - x_{2n-1}^N|$$

for $N = 2^3, 2^4, \ldots, 2^{14}$, where $n = 1, \ldots, N$. Then, we computed

$$R_N := \max\{\frac{S_n^N}{S_m^N} : n, m = 1, \ldots, N/2\}$$

for $N = 2^3, 2^4, \ldots, 2^{14}$. As also can be seen from Figure 4.8 these ratios $(R_{2^k})_{k=3}^{14}$ increase fast which indicates that $(R_{2^k})_{k=1}^{\infty}$ is unbounded.

We also plotted $\dfrac{S_t^N}{S_1^N}$ (see Figure 4.9) where $N = 2^14$ and $t = 2$, $t = 2^6$. And these ratios seem to converge fast.

Now, we will give our last conjecture but it won't contain the case when $\gamma_k < \frac{1}{32}$ for all $k \in \mathbb{N}$, i.e., the case when γ is small. The reason is that for a γ with $\gamma_k < \frac{1}{32}$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \gamma_k = M < \infty$ we have, by Lemma 6 in [16],

$$S_i^{2^k} \leq \exp(16M)\gamma_1 \cdots \gamma_{k-1}$$

for all $k > 1$. Also, by Lemmas 4 and 6 in [16], we have

$$S_i^{2^k} \geq \frac{7}{8}\gamma_1 \cdots \gamma_{k-1},$$

hence, we get $R_{2^k} \leq \frac{8}{7}\exp(16M)$, i.e., $(R_{2^n})_{n=2}^{\infty}$ is bounded.

Conjecture 4.9.1. *For each $\gamma = (\gamma_k)_{k=1}^{\infty}$ with $\inf_k \gamma_k > 0$, $(R_{2^k})_{k=1}^{\infty}$ is an unbounded sequence. If $t = 2^k$ for some $k \in \mathbb{N}$, there is a $c_0 \in \mathbb{R}$ depending on k such that*

$$\lim_{n\to\infty} \frac{S_t^{2^n}}{S_1^{2^n}} = c_0.$$

4.10 Figures

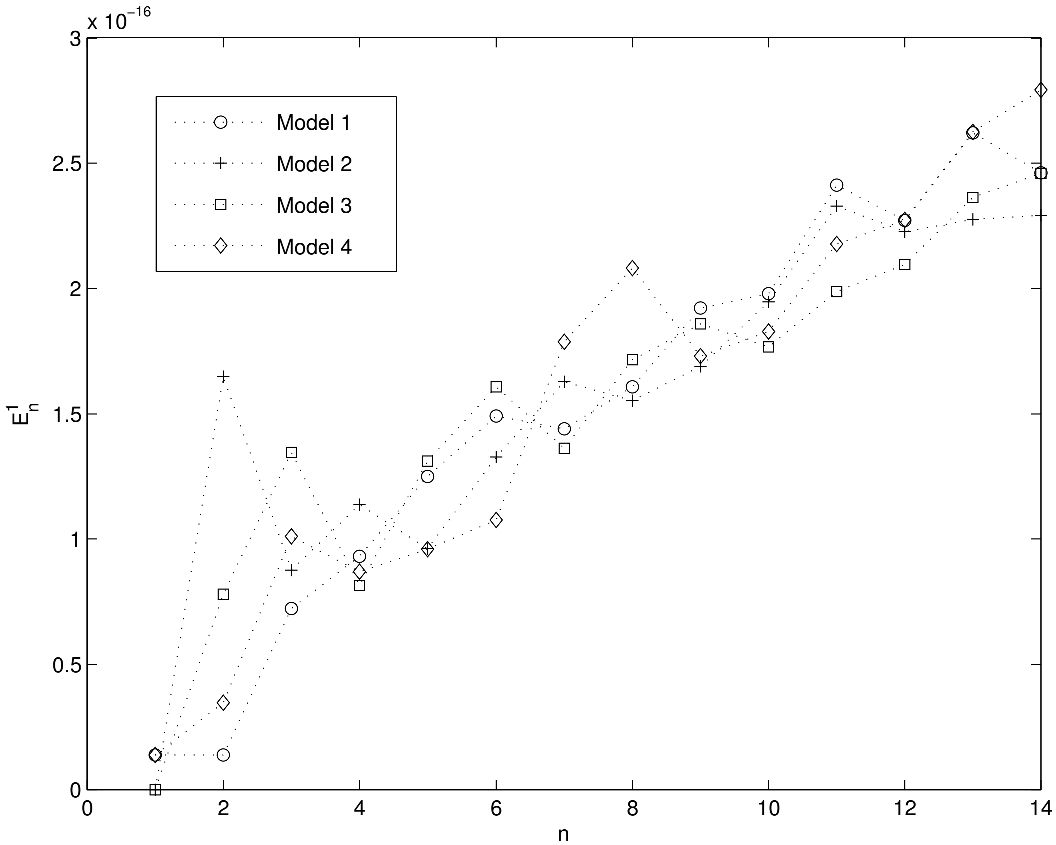

Figure 4.1: Errors associated with eigenvalues.

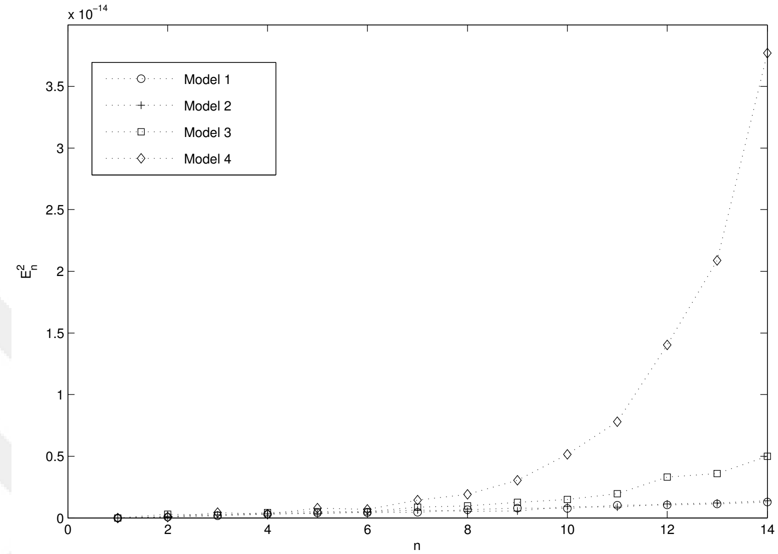

Figure 4.2: Errors associated with eigenvectors.

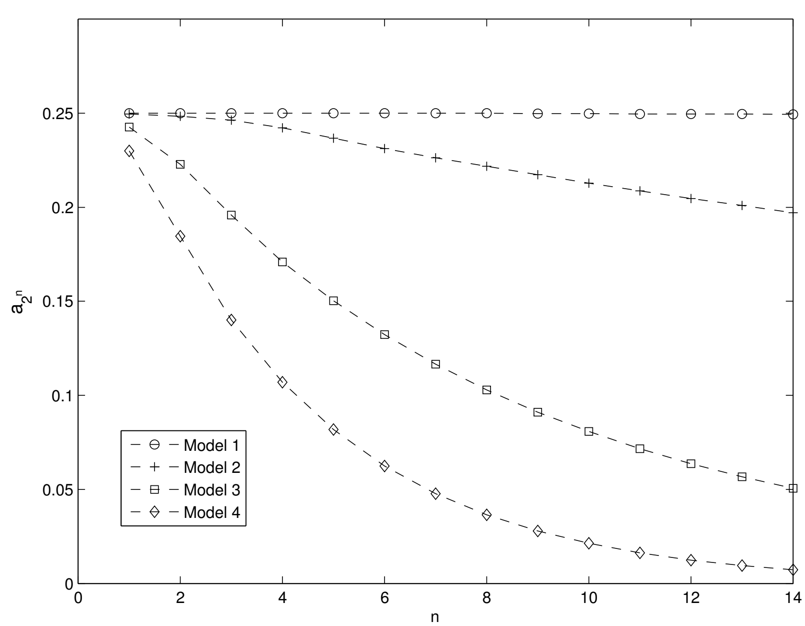

Figure 4.3: The values of outdiagonal elements of Jacobi matrices at the indices of the form 2^s.

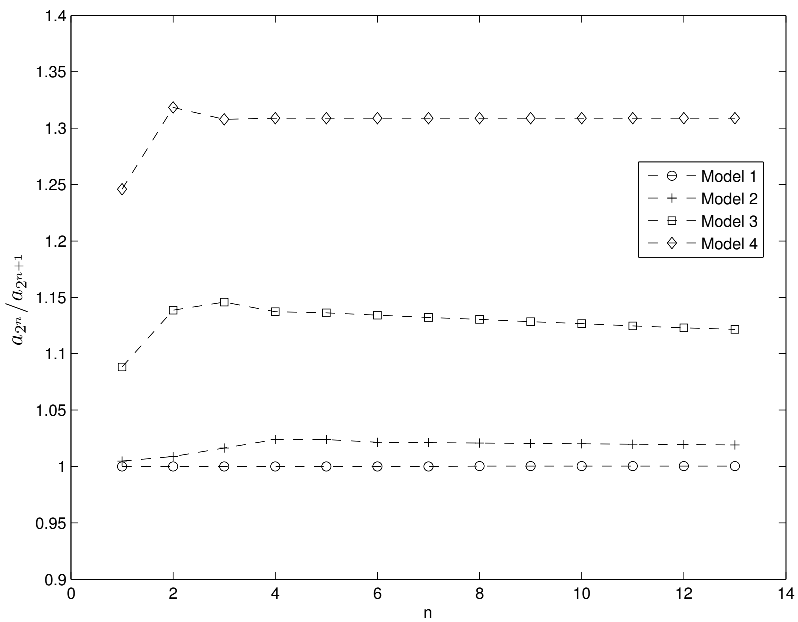

Figure 4.4: The ratios of outdiagonal elements of Jacobi matrices at the indices of the form 2^s.

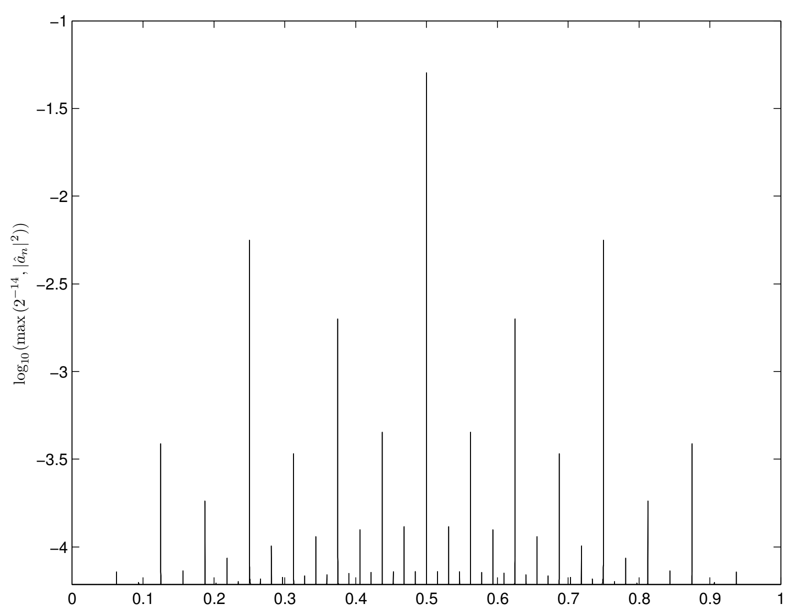

Figure 4.5: Normalized power spectrum of the a_n's for Model 1.

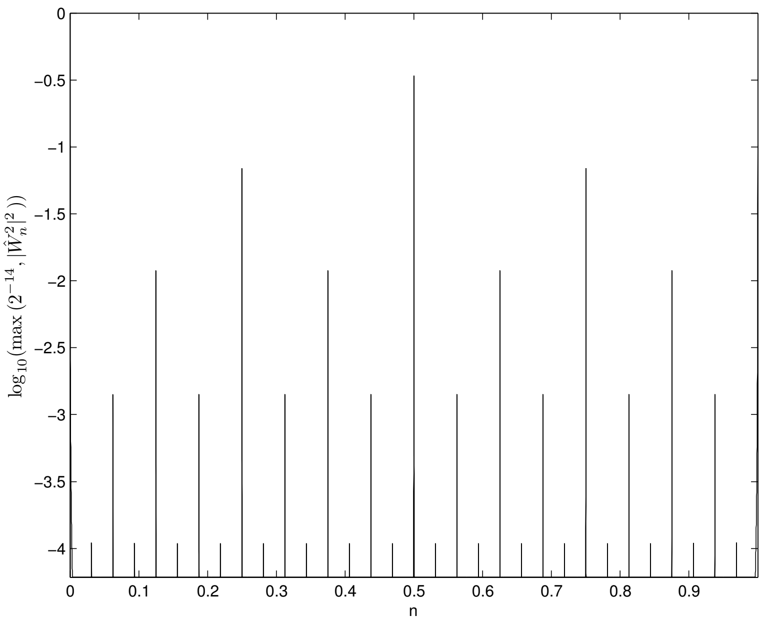

Figure 4.6: Normalized power spectrum of the $W_n^2\left(\mu_{K(\gamma)}\right)$'s for Model 1.

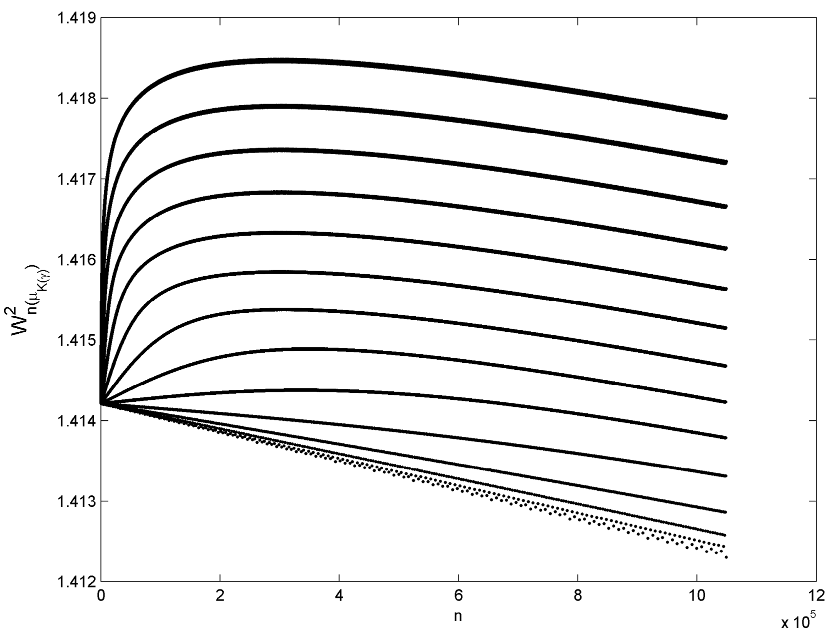

Figure 4.7: Widom-Hilbert factors for Model 1

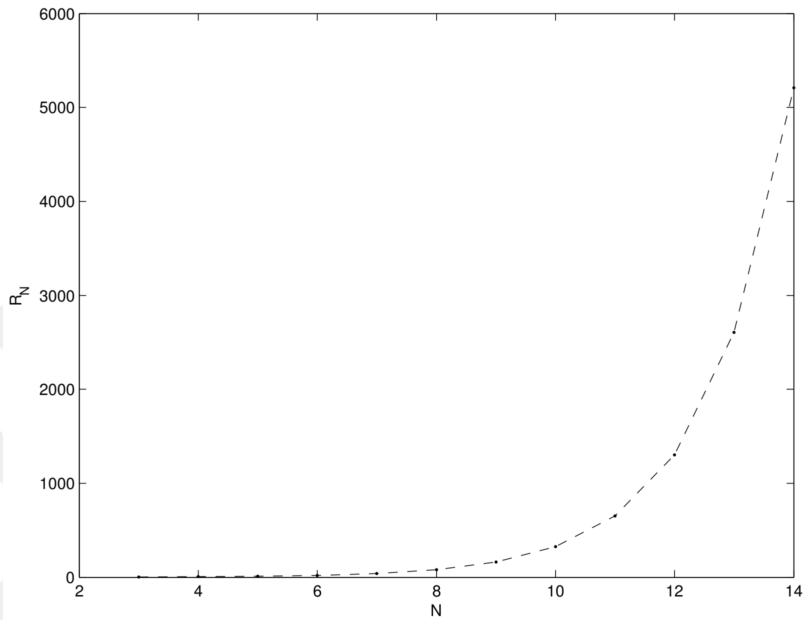

Figure 4.8: Maximal ratios of the distances between adjacent zeros

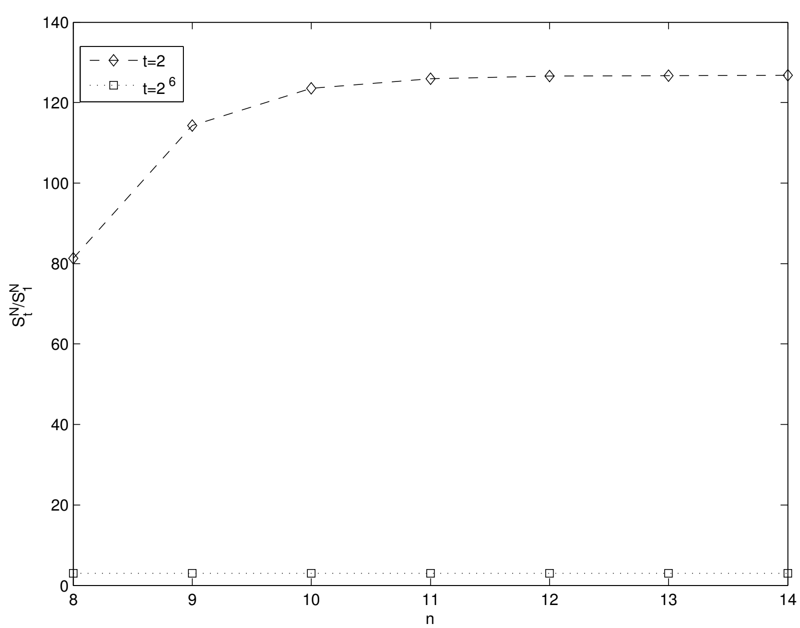

Figure 4.9: Ratios of the distances between prescribed adjacent zeros

Bibliography

[1] Alpan, G., Goncharov A. and Şimşek, A. N.: "Asymptotic properties of Jacobi matrices for a family of fractal measures," accepted for publication in *Experimental Mathematics*, arXiv:1603.02312v1.

[2] Alpan, G: "Spacing properties of the zeros of orthogonal polynomials on Cantor sets via a sequence of polynomial mappings," *Acta Mathematica Hungarica*, vol. 149, no. 2, pp. 509–522, 2016.

[3] Alpan, G. and Goncharov, A.: "Two measures on Cantor sets," *Journal of Approximation Theory*, vol. 186, pp. 28–32, 2014.

[4] Alpan, G. and Goncharov, A.: "Widom factors for the Hilbert norm," *Banach Center Publications*, vol. 107, pp. 11–18, 2015.

[5] Alpan, G. and Goncharov, A.: "Orthogonal polynomials for the weakly equilibrium Cantor sets," *Proceedings of the American Mathematical Society*, vol. 144, no. 9, pp. 3781–3795 , 2016.

[6] Alpan, G. and Goncharov, A.: "Orthogonal polynomials on generalized Julia sets," Preprint, arXiv:1503.07098v3, 2015.

[7] Alpan, G., Goncharov, A. and Hatinoğlu, B.: "Some asymptotics for extremal polynomials," *Computational Analysis: Contributions from AMAT 2015*, Springer-New York, pp. 87–101, 2016.

[8] Barnsley, M. F., Geronimo, J. S. and Harrington, A. N.: "Almost periodic Jacobi matrices associated with Julia sets for polynomials," *Communications in Mathematical Physics*, vol. 99, no. 3, pp. 303–317, 1985.

[9] Brolin, H.: "Invariant sets under iteration of rational functions," *Arkiv fr Matematik*, vol. 6, no. 2, pp. 103–144, 1965.

[10] Christiansen, J.S., Simon, B. and Zinchenko, M., "Asymptotics of Chebyshev Polynomials, I. Subsets of ℝ," Preprint, arXiv:1505.02604v1, 2015.

[11] Christiansen, J.S., Simon, B. and Zinchenko, M.: "Finite Gap Jacobi Matrices, II. The Szegö Class," *Constructive Approximation*, vol. 33, no. 3,pp. 365–403, 2011.

[12] Delyon, F. and Souillard, B.: "The rotation number for finite difference operators and its properties," *Communications in Mathematical Physics*, vol. 89, no. 3, pp. 415–426, 1983.

[13] Dombrowski, J.: "Quasitriangular matrices," *Proceedings of the American Mathematical Society*, vol. 69, no. 1, pp. 95–96, 1978.

[14] Geronimo, J. S., Harrell E. M. II and Van Assche, W.:. "On the asymptotic distribution of eigenvalues of banded matrices," *Constructive Approximation*, vol. 4, no. 1, pp. 403–417, 1988.

[15] Golub, G. H., Welsch, J. H.: "Calculation of Gauss Quadrature Rules," *Mathematics of Computation*, vol. 23, no. 106, pp. 221–230, 1969.

[16] Goncharov, A.: "Weakly equilibrium Cantor type sets," *Potential Analysis*, vol. 40, no. 2, pp. 143–161, 2014.

[17] Goncharov A. and Hatinoğlu, B.: "Widom Factors," *Potential Analysis*, vol. 42, no. 3, pp. 671–680, 2015.

[18] Mantica, G.: "Orthogonal polynomials of equilibrium measures supported on Cantor sets," *Journal of Computational and Applied Mathematics*, vol. 290, pp. 239–258, 2015.

[19] Peherstorfer, F. and Yuditskii, P.: "Asymptotic behavior of polynomials orthonormal on a homogeneous set," *Journal dAnalyse Mathmatique*, vol. 89, no. 1, pp. 113–154,2003.

[20] Pommerenke, Ch.: "On the Green's function of Fuchsian groups," *Annales Academim Scientiarum Fennicre*, vol. 2, pp. 409-427, 1976.

[21] Ransford, T.: "Potential theory in the complex plane," Cambridge University Press, 1995.

[22] Rivlin, T.J.: "Chebyshev Polynomials," John Wiley and Sons, USA, 1990.

[23] Stahl, H. and Totik, V.: "General Orthogonal Polynomials," Cambridge University Press, 1992.

[24] Saff, E. B.: "Logarithmic Potential Theory with Applications to Approximation Theory," *Surveys in Approximation Theory*, vol. 5, pp. 165–200, 2010.

[25] Saff, E. B. and Totik, V.: "Logarithmic potentials with external fields," Springer-Verlag, New York, 1997.

[26] Schiefermayr, K.: "A lower bound for the minimum deviation of the Chebyshev polynomial on a compact real set," *East Journal on Approximations*, vol. 14, no. 2, pp. 223–233, 2008.

[27] Simon, B.: "Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials," Princeton University Press, Princeton, New York, 2011.

[28] Sodin, M. and Yuditskii, P.: "Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions," *Journal of Geometric Analysis*, vol. 7, no. 3, pp. 387–435, 1997.

[29] Szegő, G.: "Orthogonal Polynomials," AMS, USA, 1975.

[30] Totik, V.: "Chebyshev constants and the inheritance problem," *Journal of Approximation Theory*, vol. 160, no.1–2, pp. 187–201, 2009.

[31] Totik, V. and Yuditskii, P.: "On a conjecture of Widom," *Journal of Approximation Theory*, vol. 190, pp. 50–61, 2015.

[32] Totik, V.: "Chebyshev Polynomials on Compact Sets," *Potential Analysis*, vol. 40, no. 4, pp. 511–524, 2014.

[33] Van Assche, W.: "Asymptotics for orthogonal polynomials," *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1987.

[34] Widom, H.: "Polynomials associated with measures in the complex plane," *Journal of Applied Mathematics and Mechanics*, vol. 16, pp. 997–1013, 1967.

[35] Widom, H: "Extremal polynomials associated with a system of curves in the complex plane," *Advances in Mathematics*, vol. 3, no. 2, pp. 127–232 (1969)

[36] Yudistkii, P: "On the Direct Cauchy Theorem in Widom Domains: Positive and Negative Examples," *Computational Methods and Function Theory*, vol. 11, no. 2, pp. 395–414, 2012.

Appendix A

Codes

We used following code to do our numerical experiments via MATLAB.

```
format longEng

%Constants
N = 2^14; %The N
GL = 100; %length of the gamma(G) vector(sequence)
RL = 30; %%length of the recurrence(R) vector(sequence)
Z=2^14;

%Gamma
G = zeros(0,GL);
for i=1:GL
    G(i)= 1/4 - 1/50;
end

%Recurrence Relation
R = zeros(0,RL);
R(1) = 1;
for i=2:RL
    R(i) = G(i-1)*(R(i-1))^2;
end
B = zeros(0,RL);
```

```
for i=1:RL
    B(i) = (1-2*G(i))*((R(i))^2)/4;
end

%Capacity
CSum = (log(G(1)))/2;
for i=2:30
    CSum = CSum+(log(G(i)))/(2^i);
end
C = exp(CSum);

A = zeros(0,N); %a_n
A(1) = sqrt(B(1));
A(2) = sqrt(B(2)/B(1));
Io2 = zeros(0,24); %Index keeper vector for 2^n's
Io2(1) = 2;

for i=3:N
    s=0;
    while mod(i/(2^s),2)==0
        s = s + 1;
    end
    A(i)=1;
    %exp 2 case
    if i/(2^s)==1
        Io2(s)= i;
        for j=1:s
            for k=1:(2^(j-1))
                A(i) = ((A(i))/A(i-2^(j-1)+1-k))*((G(s-j+1))^2)/A(i-2^(j-1)+1-k);
            end
        end
        A(i) = sqrt((A(i))*(1-2*G(s+1))/4);
    %odd number case
    elseif mod(i,2)==1
        A(i) = sqrt(B(1)-(A(i-1))^2);
    %otherwise
    else
        %first part
        O1 = 1;
        for j=1:s
```

```
            for k=1:(2^(j-1))
                t = 1;
                t = t*(A(i-2^(j-1)+1-k));
                O1 = O1*((G(s-j+1))^2)/(t^2);
            end
        end
        O1 = O1*(1-2*G(s+1))/4;
        %second part
        O2 = (A(i-2^s));
        for j=1:(2^s-1)
            O2 = O2*A(i-(2^s)-j)/A(i-j);
        end
        %Finishing touches
        A(i) = sqrt(O1-(O2)^2);
    end
end

%GAUSS.m Gauss-Jacobi quadrature
J=zeros(Z);
for n=1:Z
    J(n,n)=0;
end
for n=2:Z
  J(n,n-1)=A(n-1);
  J(n-1,n)=J(n,n-1);
end
[V,D]=eig(J); %Columns of V is the right eigenvectors
%D is the eigenvalues(diagonal matrix)
[D,I]=sort(diag(D));
V=V(:,I);
xw=[D A(1,2)*V(1,:)'.^2];

Q=0;
for i=1:Z
    Q = Q+xw(i,2);
end
Q=1/Q;
E=xw*Q;
%Error calculations
Y=zeros(Z,1);
```

```
for i=1:Z
    Y(i)=1/Z;
end
ERROR1=0;
for i=1:Z
    ERROR1 = ERROR1+abs(Y(i)-E(i,2));
end
ERROR1=ERROR1*1/Z;

o6=[sqrt(1-2*G(log2(Z))),1-sqrt(1-2*G(log2(Z)))];
o5=zeros(1,4);
for i=1:log2(Z)-2;
    o5=[sqrt(1-2*G(log2(Z)-i)+2*G(log2(Z)-1)*o6),1-sqrt(1-2*G(log2(Z)-i)
        +2*G(log2(Z)-1)*o6)];
    o6=o5;
    o5=zeros(1,2^(i+2));
    o5=o5-0.5;
end
o5=sort([1/2-((1/2)*sqrt(1-2*G(1)+2*G(1)*o6)) ,1/2+((1/2)*sqrt(1-2*G(1)
    +2*G(1)*o6))]);
ERROR2=0;
for i=1:Z
    ERROR2=ERROR2+abs(o5(i)-D(i));
end
ERROR2=ERROR2*(1/Z);

%WIDOM
W = zeros(0,N*2);
W(1) = A(1)/C;
for i=2:N
    W(i) = W(i-1)*(A(i)/C);
end
W2 = zeros(0,20);
for i=1:13
    W2(i) = W((2^i)-1);
end

%Finding places of Mins&Maxs of Widom factors
[k1,Wn_max] = find(W==max(W(:)));
[k2,Wn_min] = find(W==min(W(:)));
```

```
%Finding places of Mins&Maxs of a_n's
[l1,n_max] = find(A==max(A(:)));
[l2,n_min] = find(A==min(A(:)));

%Ratios of a_n for n of power 2
T = zeros(0,23);
for i=1:12
    T(i) = A(Io2(i+1))/A(Io2(i));
end
```