ISTANBUL TECHNICAL UNIVERSITY ★ INSTITUTE OF SCIENCE AND TECHNOLOGY

A SEMICLASSICAL KINETIC THEORY OF THE DIRAC PARTICLES

M.Sc. THESIS

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ FEN BİLİMLERİ ENSTİTÜSÜ

DİRAC PARÇACIKLARININ YARI KLASİK KİNETİK KURAMI

YÜKSEK LİSANS TEZİ

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Eda KILINÇARSLAN, a M.Sc. student of ITU Institute of Science and Technology 509131102 successfully defended the thesis entitled **"A SEMICLASSICAL KINETIC THEORY OF THE DIRAC PARTICLES"**, which he/she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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to Berkin Elvan,

FOREWORD

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ABBREVIATIONS

GBM	**:** Gosselin-Berard-Mohrbach
BMT	**:** Bargmann-Michel-Teledgi

A SEMICLASSICAL KINETIC THEORY OF THE DIRAC PARTICLES

SUMMARY

The semiclassical kinetic theory of massive spin-$\frac{1}{2}$ particles interacting with the external electromagnetic fields is formulated in terms of differential forms which are matrix valued in spin space. Semiclassical approximation is performed by employing the wave packet constructed as superposition of positive energy plane wave solutions of the free Dirac equation. A symplectic two-form is derived using the wave packet. It is a matrix in "spin indices" and possesses a term related to the Berry curvature obtained from a non-Abelian Berry gauge field. Time evolution of phase space variables in terms of phase space themselves are attained by making use of the volume form which is also a matrix. Continuity equation for particle number density and the particle current density are obtained by introducing a change of basis in order to define distribution functions in the helicity basis. The massless limit is derived by constructing the helicity states explicitly.

When one deals with a non-relativistic formulation of massive particles the equations of motion should be corrected with a relativistic kinematic factor known as Thomas precession. Its origin lies in the fact that when one would like to write two successive Lorentz boost as one Lorentz boost it should be accompanied with a rotation whose angle depends on the related velocities. It is shown that Thomas precession can be included straightforwardly into the semiclassical formulation adopted in the thesis. It alters the equations of motion and cancels the anomalous velocity terms appearing due to the Berry curvature.

Initially, I will derive the semiclassical block diagonal Hamiltonian for a Dirac particle in the electromagnetic field including all terms at the first order in Planck constant using the Gosselin-Berard-Mohrbach method. In this method the unitary transformation which block diagonalizes the Hamiltonian possesses terms related to the Berry gauge fields. In general curvature of the Berry gauge fields appear as the phase factor of a quantum state transported adiabatically. When the block diagonalization is carried out by unitary transformation, the dynamical operators should also be transformed and they become non-commutative. I will use these non-commutative phase space operators to derive the time evolution of spin matrices which will be introduced in the course of finding the semiclassical formulation.

The one-form corresponding to first order Lagrangian is defined by making use of the wave packet built with the positive energy solutions of the Dirac equation. This one-form can be written as a matrix whose indices correspond to the positive energy solutions which are called spin indices. It has a term depending on the non-Abelian Berry gauge fields given by the degenerate positive energy solutions. Then the symplectic two-form derived from this one-form includes a term which depends on the Berry curvature. I use the differential form formalism to obtain the equations of motion of phase space variables. A straightforward method is applied to find solutions of the equations of motion for the

phase space velocities in terms of the phase space variables employing Liouville equation and the differential form formalism.

To get the kinetic theory of Dirac particles I need the distribution function which can be used to define the particle number density. However, it is a matrix whose elements should be interpreted appropriately. The mostly adopted procedure is to choose a specific configuration where the third component of spin is a conserved quantity. Then one can set the off-diagonal terms to zero. In general spin is not a conserved quantity but helicity operator gives a vanishing commutator with the free Dirac Hamiltonian. Moreover when I discuss the massless limit it would be essential to split the right-handed and the left-handed contributions. Therefore, the appropriate basis is the one where the helicity operator is diagonal. I define this new basis and obtain the continuity equation for the Dirac particle using the distribution function which is diagonal. Then I derive the continuity equation for the particle number density and the particle number current density. Obviously, because of possessing the solutions of the equations of motion for the velocities in terms of phase space variables one can directly obtain the particle current.

Obtaining the massless limit in the helicity basis is straightforward. It yields the continuity equation which has an anomaly term. The particle current possesses an anomalous velocity term and a term leading to the chiral magnetic effect.

Thomas precession which shows up as the relativistic correction in the equations of motion are obtained. I briefly discuss what is the source of the Thomas precession. Then I present how one should introduce this correction into the wave packet formalism. It gives a contribution to the initial one-form on the same footing with the Berry gauge field. In fact up to higher order terms in momentum it gives the opposite contribution of the Berry gauge field and cancels the anomalous velocity terms given by the Berry curvature. This result coincides with the ones obtained within the relativistic formulations of the Dirac particles.

Originally the Thomas precession is used to obtain the corrections to the non-relativistic formulation of the time evolution of spin matrices. However, the formalism which I adopted is not aware of the time evolution of spin. For completeness I show that it can be integrated into the formalism by making use of the non-commutative charter of the dynamical variables obtained in the Gosselin-Berard-Mohrbach method. Time evolution of spin matrices are shown to be the same with the Bargmann-Michel-Telegdi equation.

Lastly, the results obtained in the thesis and the possible extensions are discussed.

DİRAC PARÇACIKLARININ YARI KLASİK KİNETİK KURAMI

ÖZET

Kütleli spin-$1/2$ parçacıkların hareketini ifade etmek için Dirac denklemi kullanılır. İkisi pozitif enerji, ikisi de negatif enerji çözümlerine ait olmak üzere Dirac denkleminin dört tane çözümü vardır. Kuantum mekaniksel olarak parçacıkların hareketi dalga paketi kurularak ifade edilmektedir. Fakat negatif enerji çözümlerinin de olması parçacık yorumunu zorlaştırır. Bu nedenle relativistik olmayan yarı klasik dinamiği, Dirac denkleminin pozitif enerji çözümlerini içeren dalga paketi oluşturarak elde edeceğim.

Yarı klasik limit bazı kuantum mekaniksel etkileri daha iyi anlamak için yararlı olabilecek bir yöntemdir. Dirac denkleminde kütlesiz limite gidildiğinde kiral ya da Weyl parçacığı adı verilen parçacıkların hareketini ifade eden denklem elde edilmiş olur. Son zamanlardaki çalışmalarda, $3+1$ boyutta, dış elektromanyetik alan nedeniyle oluşan anomali terimlerinin kiral parçacıklarının yarı klasik kinetik kuramında nasıl yerleştirilebileceği gösterilmiştir. Yüksek boyutta genelleştirilmiş hareket denklemlerinin, faz uzayı değişkenlerine bağlı çözümlerini de tam olarak veren yöntem matris değerlidir. Bu yöntemde diğerlerinden farklı olarak faz uzayı değişkenleri konum ve momentumdur, spine karşı gelen klasik bir nicelik yoktur. Klasik faz uzayı değişkenleri matris olmadıkları halde hareket denklemleri ile bulunan hız değişkenleri matris değerlidir. Matrislerin “spin indisleri” farklı pozitif çözümlere karşılık gelir.

Kiral kinetik kuramının yarı klasik formalizminin en önemli bileşenlerinden biri kuantum mekaniksel bir faz faktörü olan Berry fazını veren Berry ayar alanlarıdır. Kuantum mekaniksel bir sistemde, Hamilton yoğunluğunun bağlı olduğu dış parametreler, bunlar elektrik ve manyetik alan olarak düşünülebilir, çok yavaş değiştirildiğinde adiyabatik kurama göre, sistemin kuantum durumu değişmez. Burada bahsedilen yavaş değişim, parametrelerin çevrimsel bir eğri üzerinde hareket etmesi olarak ifade edilebilir. Bu kapalı eğri tamamlandığında sistemdeki durum vektörü bir faz kazanır. Bu faz dinamik ve geometrik iki kısımdan oluşmaktadır. Adiyabatik değişim altındayken, durum vektörünün kazandığı geometrik faz çarpanına Berry fazı denir. Berry ayar alanlarının eğriliği kuantum mekaniksel bir faz çarpanı olan Berry fazını verir.

Bu yöntemle birinci derece Lagrange yoğunluğu ile yarıklasik Hamilton dinamiğini elde etmek için ilk olarak yarı klasik blok köşegen Hamilton yoğunluğu verilmelidir. Yarı klasik Hamilton yoğunluğunu köşegenleştirecek uniter matris dönüşümünün Planck sabiti bölü 2π, $\hbar$, ye göre birinci mertebe tüm terimleri verecek şekilde yazılması gerekir.

Dalga paketi ve diferansiyel formlar kullanılarak Dirac parçacıklarının yarı klasik kinetik kuramı elde edilebilir. Bu yöntemin bazı avantajları bulunmaktadır. Öncelikle, formalizmde spin özgürlük derecesine karşılık gelen klasik bir nicelik bulunmadığı için hareket denklemlerinin faz uzayı hızlarının faz değişkenleri cinsinden veren çözümleri açıkça bulunabilir. Böylece parçacık akısı rahatlıkla yazılabilir.

Tez kapsamında, öncelikle elektromanyetik alan içindeki kütleli spin-$1/2$ parçacıklar için Dirac Hamilton yoğunluğunu blok köşegen hale getireceğim. Dalga paketi ve

diferansiyel form yöntemiyle hareket denklemlerini ve çözümlerini bulacağım. Burada faz uzayı değişkenlerinin hız denklemi çözümleriyle hesaplanan, hız denklemlerinin faz uzayı değişkenlerine bağlı çözümleridir. Bu denklemlerin elde edilişi sırasında denklemlerimizin içine Berry ayar alanlarının girdiğini göreceğiz. Daha sonra yarı klasik Dirac parçacıkları için dağılım fonksiyonunu uygun bazda yazıp süreklilik denklemini bulacağım. Kütleli fermiyonlar için elde ettiğim hareket denklemlerinin çözümlerinde kütlesiz limite giderek ve baz değiştirerek elektromanyetik alandaki Weyl parçacıklarının hareket denklemlerinin çözümlerine ulaşacağım. Thomas presesyonun ne olduğunu anlatıp incelediğim yarı klasik formalizmde Thomas presesyonunun elde ettiğim hız denklemlerine nasıl bir katkı yapacağını inceleyeceğim. Son olarak ise elektromanyetik alan altındaki yarı klasik Dirac parçacığının spinin zaman içindeki değişimini veren denklemi bulacağım.

İlk kısımda, Gosselin-Berard-Mohrbach yöntemini kullanarak elektromanyetik alanda hareket eden kütleli parçacıklar için yarı klasik blok köşegen Hamilton yoğunluğunu hesaplayacağım. Bu yöntem birkaç adımdan oluşmaktadır. Hesaplanacak olan Hamilton yoğunluğunun $\hbar$ ye göre birinci dereceden olan tüm terimleri içermesini istiyorum. Elektromanyetik alan altındaki Hamilton yoğunluğunun bağlı olduğu değişkenleri, $\boldsymbol{x}$ ve $\boldsymbol{p}$ yi, birbirleriyle komute edecek şekilde aldığınızda Hamilton yoğunluğu klasik bir büyüklüktür olur. Klasik Hamiltonyeni köşegenleştirmek için uniter Foldy-Wouthuysen dönüşümleri, U_{FW}, kullanılır. Yarı klasik blok köşegen Hamilton yoğunluğunu hesaplamak için, Hamilton yoğunluğunun klasik faz uzayı değişkenleri $\boldsymbol{x}$ ve $\boldsymbol{p}$ yerine birbirleriyle komute etmeyen kuantum mekaniksel faz uzayı operatörleri olan $(\boldsymbol{P}, \boldsymbol{R})$ ye bağlı olduğunu düşünelim. Bu durumda Foldy-Wouthuysen dönüşümleri uniter olmaktan çıkarlar ve uniterliği sağlamak için birinci mertebe $\hbar$ içeren bir terim eklemek gerekir. Bu durumu şöyle ifade edebiliriz:

$$U_{FW} \to U_{FW} + X\, U_{FW}.$$

Buradaki X Berry ayar alanlarına bağlıdır. Sonraki aşamada ise yeni dönüşüm kullanılarak birinci derece $\hbar$ mertebesinde olan tüm terimleri içeren yarı klasik blok köşegen Hamilton yoğunluğu tam olarak hesaplanmış olur. Fakat blok köşengenleştirme işlemi sırasında Hamilton yoğunluğunun bağlı olduğu faz uzayı değişkenlerinin cebri non-komutatif olur. Hesaplanmış olan Hamilton yoğunluğu tezimin bir sonraki kısımlarındaki hareket denklemlerinin çözümünde ve spinin zaman içindeki değişiminin hesabında kullanılacaktır.

Tezin ikinci kısmında, yarıklasik yöntemde tek parçacık durumunu ifade edebilmek için Dirac denkleminin pozitif enerji çözümlerinden oluşan bir dalga paketi kuracağım. Kurulan dalga paketi ile yarıklasik birinci derece Lagrange yoğunluğuna karşılık gelen η bir-form elde edilir.

Sonraki bölümde ise bir-form η aracılığıyla simplektik iki-form $\tilde{\omega}$ oluşturulur. Bu spin indisleri ile yazılan bir matristir. Berry ayar alanı içeren $\tilde{\omega}$ Hamilton formalizmini elde etmek için kulanılır.

Simplektik iki-form üzerinde diferansiyel yöntem olan matris değerli vektör alanının iç çarpım işlemi yapılarak hareket denklemleri elde edilmiş olur. Dirac parçacıklarının hızları için hareket denklemi çözümlerine ise yine bir diferansiyel form yöntemi olan Lie türevi işlemiyle ulaşılır. Bu bölümde, hareket denklemi çözümlerini iki ayrı işlemin sonuçlarını karşılaştırarak hesapladım. Bunların ilki, $3+1$ uzayzaman boyutu için tanımlanan hacim formun Pfaffian matrise bağlı olarak yazılmasıdır. Pfaffian matris bir kare matris için determinantının karekökü olarak tanımlanır. Diğer yol ise, yine $3 + 1$ uzayzaman boyutu için, hacim formu simpletik iki-forma bağlı olarak tanımlamaktır. Hareket denklemi çözümlerine, iki farklı terimle belirlenmiş hacim formun Lie türevlerinin hesaplanması ve elde edilen denklemlerin karşılaştırılmasıyla ulaşılır.

Kütleli, spin-$1/2$ parçacıklar ile çalışmama rağmen kullandığım yarıklasik yöntemde spine karşılık gelen klasik serbestlik derecesi yoktur. Sistemdeki hız ifadeleri matris değerli olduklarından bulduğum parçacık akım yoğunluğu da matris değerlidir. Buradaki önemli nokta, serbest Dirac parçacığı için spin korunumlu bir büyüklük olmamasına rağmen helisite operatörü korunumlu bir büyüklüktür. Bu nedenle helisite operatörünü kullanarak bir spin akısı türetilebilir. Dağılım fonksiyonu ve süreklilik denklemi başlığındaki kısımda, parçacıkları sağ elli ve sol elli olmak üzere iki kısma ayırarak dağılım fonksiyonunu elde etmek istiyorum. Bu nedenle dağılım fonksiyonunu köşegen olarak yazmak için sistemimdeki bazı değiştirerek helisite bazına geçeceğim.

Daha sonra ise helisitenin köşegen olduğu bazda kurduğum dağılım fonksiyonunu ters dönüşüm ile ilk bazda ifade edeceğim. Kütleli fermiyonlar için sağ elli ve sol elli parçacıklar dengede olduğundan dağılım fonksiyonunun ilk bazda da köşegen olarak ifade edilebileceğini göstereceğim. Bu bölümde son olarak ise süreklilik denklemini türeterek kütleli fermiyonların süreklilik denklemini sağladığını göstereceğim.

Önceki bölümlerde Dirac parçacıklarının hızları için hareket denklemlerinin çözümlerini, kütleli fermiyonların dağılım fonksiyonunu ve süreklilik denklemini elde ettim. Bu işlemlerin ardından ise Dirac parçacıklarının hızları için bulunan hareket denklemlerindeki tüm ifadeleri helisite bazında yazarak kütlesiz limitini hesaplayacağım. Böylece kütlesiz fermiyonlar için hareket denklemlerinin çözümünü elde edeceğim. Ayrıca, kütlesiz fermiyonların parçacık akısını ve süreklilik denklemlerini bulacağım. Dirac parçacıklarının aksine, süreklilik denklemini sağlamadıklarını ve anomaliye sahip olduklarını göstereceğim.

Dalga paketi yöntemiyle Dirac parçacıkları için elde edilen hareket denklemi çözümlerindeki hız ifadeleri Berry eğriliği terimlerini içeren "anormal hız" terimlerine sahiptir. Oysa, Dirac parçacıklarının kovaryant formalizmi ile elde edilen hareket denklemlerinde Thomas presesyonu nedeniyle anormal hız terimleri yoktur. Thomas presesyonu spin matrisinin relativistik olmayan hareket denklemlerinde bir kinematik düzeltme terimi olarak bulunmuştur. Bununla birlikte Thomas presesyonu faz uzayı değişkenlerinin hareket denklemlerine katkı sağlamalıdır. Diferansiyel form ve dalga paketi formalizmi ile kurduğum relativistik olmayan sistemin, Thomas dönmesi katkısını içermemesi beklenen bir durumdur. Bu durumu düzeltmek için yarıklasik formalizmine Thomas dönmesi yerleştirilmelidir.

Thomas presesyonu bir Lorentz ötelemesinin ard arda uygulanan iki Lorentz ötelemesi ve dönme ifadesi cinsinden yazılmasından kaynaklanır. Buradaki dönme ifadesine aynı zamanda Thomas dönmesi de denilmektedir. Bu sayede Dirac denklemini kullanmadan elektronun spinin zamana göre değişiminin ifadesi doğru bir şekilde hesaplanabilmektedir.

Kullandığım yarı klasik yöntemin Thomas presesyonun katkısını hesaplamak için çok uygun olduğunu göreceğim. Thomas dönmesini ve sistemime nasıl bir katkı verdiğini bularak, momentumun yüksek mertebe katkısını ihmal ettiğimde anormal hız terimlerinin kaybolduğunu göstereceğim. Bu sonuç ilk defa bu tezde bulunmuştur.

Thomas presesyonu katkısının incelenmesinin ardından spin matrislerinin zaman içindeki değişimi hesaplanacaktır. Kullandığım yöntemde spin matrisleri Pauli spin matrisleri ile ifade edilmektedirler ve faz uzayı yöntemi spinin hareketlerini belirlemez. Bu nedenle spin hareket denklemleri başka yöntemlerle bulunmalıdır. Bunun için blok köşegen Hamilton yoğunluğunu bulmakta kullandığım Gosselin-Berard-Mohrbach yöntemini kullanacağım. O formalizmde uniter dönüşüm sonrası faz uzayı işlemcileri non-komütatif olurlar. Dolayısıyla Pauli matrisleri ile de komüte etmezler. Bu katkılar göz önüne alındığında Gosselin-Berard-Mohrbach yöntemiyle elde ettiğim sonucun, elektromanyetik alanda hareket eden elektronun spininin zaman içindeki değişimini veren Bargmann-Michel-Teledgi denklemiyle aynı sonucu verdiğini göstereceğim.

Tezin son bölümünde ise elde ettiğim sonuçlar ve bazı uygulamaları tartışılmıştır.

1. Introduction

Dirac equation which describes massive spin-$1/2$ particles possesses either positive or negative energy solutions described by spinors. However, to get a well defined one particle interpretation a wave packet build of only positive energy plane wave solutions should be preferred. Employing this wave packet one can obtain a non-relativistic semiclassical dynamics which may be useful to have a better understanding of some quantum mechanical phenomena. In the massless limit Dirac equation yields chiral particles called Weyl particles. Recently it has been shown that chiral semiclassical kinetic theory can be formulated embracing the anomalies due to the external electromagnetic fields in $3+1$ dimensions [1, 2]. This remarkable result was extended to any even dimensional space-time by making use of differential forms in [3] by introducing some classical variables corresponding to spin. Although in [3] non-Abelian anomalies have been incorporated into the particle currents successfully, the solutions of phase space velocities in terms of phase space variables were missing. In [4] a complete description of the chiral semiclassical kinetic theory in the presence of the external electromagnetic fields, in any even space-time dimension was established by introducing a matrix valued symplectic two-form, without introducing any classical variable corresponding to spin degrees of freedom. In this formalism, although the classical phase space variables are the ordinary ones, the velocities arising from the equations of motion are matrix valued. It has been shown in [5] that this matrix valued symplectic two form can be derived within the semiclassical wave packet formalism [6, 7].

One of the main ingredients of these semiclassical formalisms of chiral kinetic theory is the Berry gauge field whose field strength yields the quantum mechanical phase factor known as the Berry phase summarized in Appendix A. To study dynamics starting with a first order Lagrangian, the related Hamiltonian should be provided. In the development of the chiral kinetic theory the Hamiltonian was taken as the positive relativistic energy of the free Weyl Hamiltonian, i.e. the magnitude of the momentum vector. However, later it was shown that the adequate Hamiltonian should contain all the first order terms in Planck constant [8] which can be attained by employing the method introduced in

[9]. Independently, the same Hamiltonian was conjectured in [10] to restore the Lorentz invariance of the semiclassical chiral theory. To obtain this semiclassical Hamiltonian one first has to derive the Hamiltonian of the massive spin-$1/2$ fermion and then take the massless limit.

Massive fermions also appear in condensed matter systems which were studied in terms of wave packets in [11, 12]. The semiclassical kinetic theory of Dirac particles was also discussed in [13], where the Berry gauge fields described in a different basis and some classical degrees of freedom have been assigned to spin. In the thesis the formalism given in [4] is applied to attain the semiclassical kinetic theory of the Dirac particles. There are some advantages of employing this method. First of all because of not attributing any classical variable to spin but considering matrix valued quantities in spin space the calculations can be done explicitly. The differential forms method of [4] provides us the solutions of the equations of motion for the phase space velocities in terms of the phase space variables straightforwardly. Thus the particle currents can readily be derived except the difficulty interpreting the matrix elements of the distribution function. It is a matrix in "spin indices" but spin components are not conserved. In contrary to spin, helicity operator is a conserved quantity for the free Dirac particle. To consider the massless limit and chiral currents when there is an imbalance of chiral particles, one has to split the particles as right-handed and left-handed. Therefore introducing a change of basis to the helicity basis would be appropriate. Moreover, I will show that within this formalism one can study the relativistic corrections known as Thomas precession [14] in a comprehensible manner.

Thomas precession stems from the fact that a Lorentz boost can be written as two successive Lorentz boosts accompanied by a rotation which is called Thomas rotation. This purely kinematic phenomenon is essential to obtain time evolution of electron's spin correctly without referring to the Dirac equation. Thomas precession should also contribute to the equations of motion of phase space variables. In fact, due to the Thomas precession the covariant formalism of the Dirac particles yield equations of motion where anomalous velocity terms do not emerge [15, 16]. However, as we will see the equations of motion derived within the wave packet formalism possess anomalous velocity terms proportional to the Berry curvature. This would have been expected because, our non-relativistic formalism is not aware of the Thomas rotation. Correction due to the Thomas rotation should be installed in the formalism. I will show that to take this correction into account the adopted formalism is very suitable: It contributes to one-form obtained by semiclassical wave packet on the same footing as the Berry gauge field. It yields the cancellation of the anomalous velocity terms up to higher order terms in momentum.

For completeness, the semiclassical formalism which is adopted in this thesis should be supported by an equation governing time evolution of spin. This will be attained employing Gosselin-Berard-Mohrbach (GBM) method [9] which is also needed to derive the semiclassical Hamiltonian. Thus, I briefly review this method which is a generalization of the Foldy-Wouthuysen transformation [17] in Section 2. In Section 3 the one-form corresponding to the first order Lagrangian is obtained by the wave packet composed of the positive energy plane wave solutions of the Dirac equation. Then the related symplectic form is constructed and solutions of the equations of motion for the velocities are established in Section 4. I clarified in Section 5 how to introduce a change of basis where the helicity operator is diagonal. Employing distribution function in the adequate basis I then can write the particle number density and the related current by the velocities obtained in terms of the phase space variables in Section 4. In Section 6 the massless case is established by constructing the helicity eigenstates explicitly. In Section 7 first a brief review of the Thomas rotation is presented . Then, I show how it contributes to the one-form built with the semiclassical wave packet. We will see that up to higher order terms in momentum it contributes as the the Berry gauge field but with an opposite sign. I will show that GBM method can be used to derive time evolution of spin matrices which coincides with the Bargmann-Michel-Teledgi equation [18] in Section 8. In the last section the results obtained and some possible applications are discussed.

2. Semiclassical Diagonalization of The Dirac Hamiltonian

In this section, I present the semiclassical block diagonal Hamiltonian for the massive fermions in the electromagnetic field using Gosselin-Berard-Mohrbach method [9]. This method consists of several steps. To begin with, since I would like to write semiclassical block diagonal Hamiltonian, the diagonalization matrix must be established up to the first order terms in the Planck constant $\hbar$. I will see that the terms which are linear in $\hbar$ are related to the Berry gauge fields. Then, I apply the block diagonalization process. After this, canonical variables become non-commutative.

Let us start with the Hamiltonian given by

$$H = H_0 + e\,A_0(\boldsymbol{x}). \qquad \textbf{(2.1)}$$

where H_0 is

$$H_0(\boldsymbol{p} - e\,\boldsymbol{a}(\boldsymbol{x})) = \beta m + \boldsymbol{\alpha}\cdot(\boldsymbol{p} - e\,\boldsymbol{a}(\boldsymbol{x})). \qquad \textbf{(2.2)}$$

This Hamiltonian describes the Dirac particle interacting with the external electromagnetic fields $\mathscr{E}$, $\boldsymbol{B}$ whose vector and scalar potentials are $\boldsymbol{a}(\boldsymbol{x})$ and $A_0(\boldsymbol{x})$. H is classical because I deal with $\boldsymbol{x}$ and $\boldsymbol{p}$ which are commuting phase space variables.

I choose the representation of the α_i and β matrices as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where σ_i are the Pauli spin matrices. I set the speed of light $c = 1$ and charge $e < 0$ for electron.

The Hamiltonian which has commuting space and momentum variables is diagonalized by the unitary Foldy-Wouthuysen transformation as

$$U(\boldsymbol{p} - e\boldsymbol{a}) = \frac{\beta H_0 + E}{\sqrt{2E(E+m)}}, \qquad \textbf{(2.3)}$$

where $E = \sqrt{(\boldsymbol{p} - e\,\boldsymbol{a})^2 + m^2}$ is the relativistic energy.

I would like to obtain a block diagonal Hamiltonian considering the quantum mechanical phase space operators $(\boldsymbol{P},\ \boldsymbol{R})$, which satisfy the canonical commutation relations

$$[P_i, R_j] = i\hbar\delta_{ij}.$$

When one maps $\boldsymbol{p} \to \boldsymbol{P}$ and $\boldsymbol{x} \to \boldsymbol{R}$, the Foldy-Wouthuysen transformation matrix (2.3) ceases to be unitary. To restore unitary at the first order in $\hbar$, one replaces the Foldy-Wouthuysen transformation adding a term:

$$U(\boldsymbol{p}-e\boldsymbol{a}) \to U(\boldsymbol{P}-e\boldsymbol{a}(R)) + X\,U(\boldsymbol{P}-e\boldsymbol{a}(R)).$$

Here, X is of order $\hbar$ and it is not uniquely defined. One can write it as

$$X = \frac{i}{4\hbar}[A_{P_i}, A_{R_i}],$$

where A_{P_i} and A_{R_i} are defined by

$$A_{R_i} = i\hbar\, U \nabla_{P_i} U^\dagger, \qquad \textbf{(2.4)}$$

$$A_{P_i} = i\hbar\, U \nabla_{R_i} U^\dagger. \qquad \textbf{(2.5)}$$

I would like to emphasize that $\boldsymbol{R}$ dependence of $U(\boldsymbol{P}-e\boldsymbol{a}(R))$ is due to the electromagnetic vector potential $\boldsymbol{a}(\boldsymbol{R})$.

The semiclassical block diagonalization transformation

$$H_D = (U(\boldsymbol{P}-e\boldsymbol{a}(R)) + X\,U)H_0(\boldsymbol{P}-e\boldsymbol{a}(\boldsymbol{R}))(U^\dagger(\boldsymbol{P}-e\boldsymbol{a}(R)) + U^\dagger\,X^\dagger).$$

is exact up to order $\hbar$ terms. I compute this transformation and obtain the block diagonal Hamiltonian

$$H_D(\tilde{\boldsymbol{\pi}}, \tilde{\boldsymbol{r}}) = \beta\, E(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{\pi}}) + \frac{i}{2\hbar}\, P[\beta\, E(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{\pi}}), A_{R_i}]\, A_{P_i} - [\beta\, E(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{\pi}}), A_{P_i}]\, A_{R_i}, \qquad \textbf{(2.6)}$$

where P denotes the projection on to the block diagonal terms and $\tilde{\boldsymbol{\pi}} = \tilde{\boldsymbol{p}} - e\boldsymbol{a}(\tilde{\boldsymbol{r}})$. The transformed phase space variables are

$$\tilde{\boldsymbol{r}} = P[U(\boldsymbol{P},\boldsymbol{R})\boldsymbol{R}\,U^\dagger(\boldsymbol{P},\boldsymbol{R})] = \boldsymbol{R} + P(A_R), \qquad \textbf{(2.7)}$$

$$\tilde{\boldsymbol{p}} = P[U(\boldsymbol{P},\boldsymbol{R})\boldsymbol{P}\,U(\boldsymbol{P},\boldsymbol{R})] = \boldsymbol{P} + P(A_P). \qquad \textbf{(2.8)}$$

As one can easily observe these dynamical variables are non-commutative.

Then I explicitly get

$$H_D(\tilde{\boldsymbol{\pi}}, \tilde{\boldsymbol{r}}) = \beta\, E - \hbar e\left(m\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{B}}{2E^2} + \frac{(\boldsymbol{B}\cdot\tilde{\boldsymbol{\pi}})(\boldsymbol{\Sigma}\cdot\tilde{\boldsymbol{\pi}})}{2E^2(E+m)}\right), \qquad \textbf{(2.9)}$$

where $\boldsymbol{\Sigma}$ matrices are given by

$$\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}. \qquad \textbf{(2.10)}$$

It is the semiclassical block diagonal Hamiltonian for the massive fermions in the external electromagnetic fields.

3. Wave Packet Formalism of the First Order Lagrangian

In this section I would like to construct the semiclassical first order Lagrangian in terms of the positive energy plane wave solutions of the Dirac equation [5]. I would like to use differential forms, so that I start with deriving the one form η which will be used to write the symplectic two-form needed for obtaining Hamiltonian dynamics.

A semiclassical wave packet formulation may provide us a well defined one particle interpretation. Therefore, I define the wave packet consisting of positive energy solutions of the Dirac equations. The position of the wave packet center in the coordinate space is $\boldsymbol{x}_c$ and the corresponding momentum is $\boldsymbol{p}_c$. The wave packet can be defined by using the positive energy solutions, $u^\alpha(\boldsymbol{x},\boldsymbol{p})$; $\alpha = 1,2$, as

$$\psi_{\boldsymbol{x}}(\boldsymbol{p}_c,\boldsymbol{x}_c) = \sum_\alpha \xi_\alpha\, u^\alpha\, (\boldsymbol{p}_c,\boldsymbol{x}_c) e^{\frac{-i\boldsymbol{p}_c\cdot\boldsymbol{x}}{\hbar}}.$$

For simplicity we deal with the constant coefficients ξ_α.

As I would like to attain the one-form η, it is defined through dS as

$$dS \equiv \int [dx]\delta(\boldsymbol{x}_c - \boldsymbol{x})\Psi^\dagger_{\boldsymbol{x}}(-i\hbar d - H_D dt)\Psi_{\boldsymbol{x}} = \sum_{\alpha\beta}\xi^*_\alpha \eta^{\alpha\beta}\xi_\beta, \tag{3.1}$$

where H_D is the block diagonal Hamiltonian found in (2.9). One can show that

$$\begin{aligned} d\psi_x &= \frac{\partial \psi_x}{\partial \boldsymbol{x}_c}\cdot d\boldsymbol{x}_c + \frac{\partial \psi_x}{\partial \boldsymbol{p}_c}\cdot d\boldsymbol{p}_c \\ &= [\sum_\beta \xi_\beta \frac{\partial u^\beta(\boldsymbol{p}_c,\boldsymbol{x}_c)}{\partial \boldsymbol{x}_c} e^{\frac{-i\boldsymbol{p}_c\cdot\boldsymbol{x}}{\hbar}}]\cdot d\boldsymbol{x}_c + [\sum_\beta \xi_\beta (\frac{\partial u^\beta(\boldsymbol{p}_c,\boldsymbol{x}_c)}{\partial \boldsymbol{p}_c} - i\hbar \boldsymbol{x} u^\beta) e^{\frac{-i\boldsymbol{p}_c\cdot\boldsymbol{x}}{\hbar}}]\cdot d\boldsymbol{p}_c. \end{aligned}$$

Plugging this into (3.1) I obtain

$$\begin{aligned} dS &= -\left(\sum_{\alpha,\beta}\xi_\alpha{}^* i\hbar u^{\alpha\dagger}\frac{\partial u^\beta}{\partial \boldsymbol{x}_c}\xi_\beta\right)\cdot d\boldsymbol{x}_c - \left(\sum_{\alpha,\beta}\xi_\alpha{}^* i\hbar u^{\alpha\dagger}\frac{\partial u^\beta}{\partial \boldsymbol{p}_c}\xi_\beta\right)\cdot d\boldsymbol{p}_c \\ &- \sum_{\alpha,\beta}\xi_\alpha{}^* E_\alpha \delta^{\alpha\beta}\xi_\beta dt - \boldsymbol{x}_c\cdot d\boldsymbol{p}_c. \end{aligned}$$

Let us introduce the matrix valued Berry gauge fields

$$\begin{aligned} \boldsymbol{A}^{\alpha\beta} &= i\hbar u^{\dagger(\alpha)}(\boldsymbol{p}_c,\boldsymbol{x}_c)\frac{\partial}{\partial \boldsymbol{p}_c}u^{(\beta)}(\boldsymbol{p}_c,\boldsymbol{x}_c). \\ \boldsymbol{a}^{\alpha\beta} &= i\hbar u^{\dagger(\alpha)}(\boldsymbol{p}_c,\boldsymbol{x}_c)\frac{\partial}{\partial \boldsymbol{x}_c}u^{(\beta)}(\boldsymbol{p}_c,\boldsymbol{x}_c), \end{aligned} \tag{3.2}$$

which can be used to write dS as

$$dS = \int [dx] \Psi_{\boldsymbol{x}}^{\dagger} (-i\hbar d - H_D dt) \Psi_{\boldsymbol{x}} = \sum_{\alpha\beta} \xi_{\alpha}^{*} \Big(-\boldsymbol{x}_c \cdot d\boldsymbol{p}_c \delta^{\alpha\beta} - \boldsymbol{a}^{\alpha\beta} \cdot d\boldsymbol{x}_c - \boldsymbol{A}^{\alpha\beta} \cdot d\boldsymbol{p}_c - {H_D}^{\alpha\beta} dt \Big) \xi_{\beta}.$$

Hence I calculated (3.1) and attained the one-form η in the general form as

$$\eta^{\alpha\beta} = -\delta^{\alpha\beta} \boldsymbol{x}_c \cdot d\boldsymbol{p}_c - \boldsymbol{a}^{\alpha\beta} \cdot d\boldsymbol{x}_c - \boldsymbol{A}^{\alpha\beta} \cdot d\boldsymbol{p}_c - {H_D}^{\alpha\beta} dt.$$

Although we deal with the $(3+1)$ dimensional space time, the derivation of the η is independent of dimension.

4. Semiclassical Dynamics of the Dirac Particles

In this section instead of solving the Dirac equation in the presence of the electromagnetic gauge potential $\boldsymbol{a}(\boldsymbol{x})$, I would like to consider the free solutions whose Hamiltonian is given by (2.2) with the replacement $\boldsymbol{p} \to \boldsymbol{p} + e\boldsymbol{a}(\boldsymbol{x})$. Then the positive energy solutions will not have $\boldsymbol{x}$ dependence.

Therefore by renaming $(\boldsymbol{x}_c, \boldsymbol{p}_c) \to (\boldsymbol{x}, \boldsymbol{p})$ and setting $\boldsymbol{a}^{\alpha\beta} = 0$, I obtain the following one form

$$\eta = p_i dx_i + e a_i dx_i - A_i dp_i - H dt \tag{4.1}$$

where the repeated indices $i = 1, 2, 3$, are summed over. Here

$$H = H_D(\boldsymbol{p}) + eA_0(\boldsymbol{x}),$$

where $H_D(\boldsymbol{p})$ is found by using (2.9) which is became 2×2 matrices by the projection operator onto the positive energy subspace I_+.

$$H_D(\boldsymbol{p}) = E - \hbar e \left(m \frac{\boldsymbol{\sigma} \cdot \boldsymbol{B}}{2E^2} + \frac{(\boldsymbol{B} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \boldsymbol{p})}{2E^2(E+m)} \right). \tag{4.2}$$

I would like to derive the symplectic two-form $\tilde{\omega}$ to obtain the Hamiltonian formalism. I adopt the definition of the symplectic two-form $\tilde{\omega}$ to be

$$\tilde{\omega} = d\eta - i\eta \wedge \eta.$$

Employing the one-form given in (4.1) it yields

$$\begin{aligned} \tilde{\omega} &= dp_i \wedge dx_i + \mathscr{E}_i dx_i \wedge dt + f_i dp_i \wedge dt - \frac{1}{2} G_{ij} dp_i \wedge dp_j + \frac{e}{2} F_{ij} dx_i \wedge dx_j \\ &= dp_i \wedge dx_i + \mathscr{E}_i dx_i \wedge dt + f_i dp_i \wedge dt - G + eF, \end{aligned}$$

where $\mathscr{E}_i$ and f_i are given by

$$\mathscr{E}_i = -\left(e \frac{\partial a_i}{\partial t} + \frac{\partial H}{\partial x_i} \right),$$

$$f_i = \left(-\frac{\partial H}{\partial p_i} - \frac{i}{\hbar} [A_i, H] \right).$$

The two-forms F and G are written in terms of the Berry field strength

$$G_{ij} = \left(\frac{\partial A_j}{\partial p_i} - \frac{\partial A_i}{\partial p_j} - i[A_i, A_j] \right),$$

and the electromagnetic field strength tensor

$$F_{ij} = \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right)$$

as $G = \frac{1}{2} G_{ij} dp_i dp_j$ and $F = \frac{1}{2} F_{ij} dx_i dx_j$.

The equations of motion can be found calculating the following:

$$i_{\tilde{v}} \tilde{\omega} = 0. \quad \textbf{(4.3)}$$

where $i_{\tilde{v}}$ is the interior product of the vector field

$$\tilde{v} = \frac{\partial}{\partial t} + \dot{\tilde{x}}_i \frac{\partial}{\partial x_i} + \dot{\tilde{p}}_i \frac{\partial}{\partial p_i}. \quad \textbf{(4.4)}$$

$(\dot{\tilde{x}}_i, \dot{\tilde{p}}_i)$ are the matrix-valued time evolutions of the phase space variables (x_i, p_i). The equation of motion is given by making use of (4.3) and (4.4):

$$\dot{\tilde{x}}_i = f_i \; + \; \dot{\tilde{p}}_c \, G_{ci},$$

$$\dot{\tilde{p}}_i = \mathscr{E}_i \; - \; e \, \dot{\tilde{x}}_c \, F_{ci}.$$

The solutions of the equation of motions for the velocities of the Dirac particle are attained thanks to Lie derivative of the volume form as I will derive in the following. The volume form for $(3+1)$ spacetime dimensions is given by

$$\tilde{\Omega} = \frac{1}{3!} \tilde{\omega} \wedge \tilde{\omega} \wedge \tilde{\omega} \wedge dt.$$

It can be written in terms of the canonical volume element of the phase space, $dV = dx_i \wedge dx_j \wedge dx_k \wedge dp_i \wedge dp_j \wedge dp_k$,

$$\tilde{\Omega} = \tilde{\omega}_{1/2} \, dV \wedge dt$$

where $\tilde{\omega}_{1/2}$ is the Pfaffian of the following matrix,

$$\begin{pmatrix} F_{ij} & -\delta_{ij} \\ \delta_{ij} & G_{ij} \end{pmatrix}$$

Obviously, $\tilde{\omega}_{1/2}$ is still a matrix in the (α, β) indices. For an $n \times n$ matrix, the Pfaffian matrix is described as the square root of its determinant. It is basically given by completely antisymmetric tensor. However, I do not need its explicit definition as we will see.

Time evolution of the volume form $\tilde{\Omega}$ can be found by calculating the Lie derivative associated with $\tilde{v}$:

$$\mathscr{L}_{\tilde{v}}\tilde{\Omega} = (i_{\tilde{v}}d + di_{\tilde{v}})\tilde{\omega}_{1/2}.$$

It can be calculated into the two different ways. Firstly, the Lie derivative of the volume form can be written in terms of the Pfaffian,

$$\begin{aligned} \mathscr{L}_{\tilde{v}}\tilde{\Omega} &= (i_{\tilde{v}}d + di_{\tilde{v}})(\tilde{\omega}_{1/2}dV \wedge dt) \\ &= \left(\frac{\partial \tilde{\omega}_{1/2}}{\partial t} + \frac{\partial(\tilde{\omega}_{1/2}\dot{\tilde{x}}_i)}{\partial x_i} + \frac{\partial(\tilde{\omega}_{1/2}\dot{\tilde{p}}_i)}{\partial p_i}\right) dV \wedge dt \end{aligned} \quad \textbf{(4.5)}$$

The second way is

$$\begin{aligned} \mathscr{L}_{\tilde{v}}\tilde{\Omega} &= (i_{\tilde{v}}d + di_{\tilde{v}})(\frac{1}{3!}\tilde{\omega}^3 \wedge dt) \\ &= \frac{1}{3!}d\tilde{\omega}^3 \end{aligned} \quad \textbf{(4.6)}$$

To calculate it let me write explicitly

$$d\tilde{\omega}^3 = \frac{\partial \tilde{\omega}^3}{\partial t}dt + \frac{\partial \tilde{\omega}^3}{\partial x_i}dx_i + \frac{\partial \tilde{\omega}^3}{\partial p_i}dp_i.$$

where $\tilde{\omega}^3$ is,

$$\begin{aligned} \tilde{\omega}^3 &= dp_i \wedge dx_i \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k + 3\mathscr{E}_i dx_i \wedge dp_j \wedge dx_j \wedge dp_k \wedge dx_k \wedge dt \\ &- 6F \wedge G dp_j \wedge dx_j + 6f_i F \wedge dp_i \wedge dx_j \wedge dp_j \wedge dt - 6\mathscr{E}_i G \wedge dp_j \wedge dx_j \wedge dx_i \wedge dt \\ &- 6\mathscr{E}_i F \wedge G \wedge dx_i \wedge dt - 6f_i F \wedge G dp_i \wedge dt. \end{aligned}$$

$d\tilde{\omega}^3$ can be written

$$\begin{aligned} d\tilde{\omega}^3 &= [\frac{\partial}{\partial t}(3! + \frac{6}{4}e(F_{mn}G_{mn} - F_{nm}G_{mn})) \\ &+ \frac{\partial}{\partial x_i}(-3.2!f_i + 6\mathscr{E}_n G_{an} - 6ef_l F_{kl} G_{ak} - \frac{6}{2}ef_i F_{kl} G_{kl}) \\ &+ \frac{\partial}{\partial p_i}(2!3\mathscr{E}_i + 6ef_n B_{an} - 6e\mathscr{E}_l G_{kl} F_{ak} - \frac{6}{2}e\mathscr{E}_i G_{kl} F_{kl})]dV \wedge dt, \end{aligned} \quad \textbf{(4.7)}$$

where F_{ij} is written by

$$F_{ij} = \varepsilon_{ijk}B_k, \quad \textbf{(4.8)}$$

and G_{ij} is given as

$$G_{ij} = \varepsilon_{ijk}G_k. \quad \textbf{(4.9)}$$

When (4.7) is substituted in (4.6) and compared with (4.5), we find the solutions of the equations of motion in general form [4]:

$$\tilde{\omega}_{1/2} = 1 + e\,\boldsymbol{B}\cdot\boldsymbol{G} \tag{4.10}$$

$$\tilde{\omega}_{1/2}\cdot\dot{\tilde{x}}_i = -f_i + (\mathscr{E}\times\boldsymbol{G})_i - eB_i(\boldsymbol{f}\cdot\boldsymbol{G}) \tag{4.11}$$

$$\tilde{\omega}_{1/2}\cdot\dot{\tilde{p}}_i = \mathscr{E}_i - (\boldsymbol{f}\times e\boldsymbol{B})_i + G_i(\mathscr{E}\cdot e\boldsymbol{B}) \tag{4.12}$$

In order to write the solution of the equations of motion for Dirac particle explicitly, let me first give the positive solutions for $H_0(\boldsymbol{p}) = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$. They can be obtained as

$$U^{\alpha}(\boldsymbol{p}) = U(\boldsymbol{p}){u_0}^{\alpha}.$$

where the rest frame solutions are

$${u_0}^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad {u_0}^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}.$$

$U(\boldsymbol{p})$ is the Foldy-Wouthuysen transformation:

$$\begin{aligned} U(\boldsymbol{p}) &= \frac{\beta H_0(\boldsymbol{p}) + E}{\sqrt{2E(E+m)}} \\ &= \begin{pmatrix} \frac{m+E}{(\sqrt{2E(E+m)}}I & \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2E(E+m)}} \\ -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2E(E+m)}} & \frac{m+E}{\sqrt{2E(E+m)}}I \end{pmatrix}. \end{aligned} \tag{4.13}$$

The Berry gauge field can be found by using Foldy-Wouthuysen transformation matrix.

$$A_i = i\hbar\, I_+\, U(\boldsymbol{p})\frac{\partial U^{\dagger}(\boldsymbol{p})}{\partial p_i}\, I_+, \tag{4.14}$$

Then I acquire,

$$\boldsymbol{A} = -\hbar\frac{\boldsymbol{\sigma}\times\boldsymbol{p}}{2E(E+m)}. \tag{4.15}$$

Hence, by substituting (4.15) in the definition of the Berry curvature (4.9), I get

$$\boldsymbol{G} = -\frac{\hbar}{2E^3}m\left(\boldsymbol{\sigma} + \frac{\boldsymbol{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{m(m+E)}\right). \tag{4.16}$$

Observe that the covariant derivative of the Berry curvature vanishes,

$$D_iG_i = \frac{\partial G_i}{\partial p_i} - \frac{i}{\hbar}[A_i, G_i] = 0. \tag{4.17}$$

The first term can be calculated as

$$\frac{\partial G_i}{\partial p_i} = \frac{\hbar m}{E^4(E+m)}\boldsymbol{\sigma}\cdot\boldsymbol{p}.$$

The latter term is obtained by using (4.15) and (4.16):

$$\begin{aligned}
\frac{i}{\hbar}[A_i, G_i] &= \frac{-im}{2E^3}[A_i, \sigma_j](\delta_{ij} + \frac{p_i p_j}{m(m+E)}) \\
&= \frac{\hbar m}{2E^4(E+m)}(\boldsymbol{\sigma}\cdot\boldsymbol{p}\delta_{ij} - \sigma_i p_j)(\delta_{ij} + \frac{p_i p_j}{m(m+E)}) \\
&= \frac{\hbar m}{E^4(E+m)}\boldsymbol{\sigma}\cdot\boldsymbol{p}.
\end{aligned}$$

Hence I established the result reported in (4.17).

I explicitly obtain the velocities (4.11), (4.12), weighted by (4.10) as

$$\tilde{\omega}_{1/2} = 1 - e\frac{\hbar}{2E^3}m\left(\boldsymbol{\sigma}\cdot\boldsymbol{B} + \frac{\boldsymbol{p}\cdot\boldsymbol{B}(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{m(m+E)}\right), \quad \textbf{(4.18)}$$

$$\tilde{\omega}_{1/2}\dot{\tilde{x}}_i = -f_i - \varepsilon_{ijk}\frac{\hbar}{2E^3}m\left(\mathscr{E}_i\sigma_k + \frac{\mathscr{E}_i p_k(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{m(m+E)}\right) - e\,B_i(\boldsymbol{f}\cdot\boldsymbol{G}), \quad \textbf{(4.19)}$$

$$\tilde{\omega}_{1/2}\dot{\tilde{p}}_i = \mathscr{E}_i - e(\boldsymbol{f}\times\boldsymbol{B})_i - e\frac{\hbar}{2E^3}m\left(\sigma_i + \frac{p_i(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{m(m+E)}\right)(\mathscr{E}\cdot\boldsymbol{B}). \quad \textbf{(4.20)}$$

where f_i is written by

$$f_i = -\frac{p_i}{E} + \hbar e\left(-\frac{p_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{B})}{E^4} + \frac{B_i(\boldsymbol{\sigma}\cdot\boldsymbol{p})(E-m)}{2E^3(E+m)} + m\,\sigma_i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{E^3(E+m)} - p_i\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})((\boldsymbol{B}\cdot\boldsymbol{p}))}{2E^4(E+m)}\right).$$

5. Distribution Function and Continuity Equation

I study spin-$\frac{1}{2}$ massive particles where the velocities are matrices in "spin spaces". So that one should consider a matrix valued distribution function. For a Dirac particle spin is given by the Σ matrices defined in (2.10). For the semiclassical wave packet composed of the positive energy solutions it is projected on to

$$u^{\alpha\dagger}\,\boldsymbol{\Sigma}\,u^{\beta}=\boldsymbol{\sigma}^{\alpha\beta}.$$

They do not commute with the free Dirac Hamiltonian so that they are not conserved in time. However the helicity operator

$$\lambda^{\alpha\beta}=u^{\alpha\dagger}\,(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p})\,u^{\beta}=\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}$$

commutes with the free Dirac Hamiltonian.

It is appropriate to split up the particles as right-handed and left-handed. This can be performed in the basis where the helicity λ is diagonal. In order to establish the diagonal basis, I use the spherical coordinates given by,

$$p_x=p\sin\theta\cos\phi$$

$$p_y=p\sin\theta\sin\phi$$

$$p_z=p\cos\theta.$$

λ is diagonalized by the unitary matrix:

$$R=\begin{pmatrix}\cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2})e^{-i\varphi}\\ \sin(\frac{\theta}{2})e^{i\varphi} & \cos(\frac{\theta}{2})\end{pmatrix}.$$

Thus the helicity basis is defined by

$$\phi=R\,u.$$

Now, the distribution function can be written in the diagonal basis as

$$f_\phi=\begin{pmatrix}f_R & 0\\ 0 & f_L\end{pmatrix}.$$

The distribution function in the initial basis can be obtained by using transformation as

$$
\begin{aligned}
f_u = R f_\phi R^\dagger &= \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2})e^{-i\phi} \\ -\sin(\frac{\theta}{2})e^{i\phi} & \cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} f_R & 0 \\ 0 & f_L \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2})e^{-i\phi} \\ \sin(\frac{\theta}{2})e^{i\phi} & \cos(\frac{\theta}{2}) \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(\frac{\theta}{2})f_R + \sin^2(\frac{\theta}{2})f_L & \frac{\sin(\theta)}{2}e^{-i\phi}(f_R - f_L) \\ \frac{\sin(\theta)}{2}e^{i\phi}(f_R - f_L) & \cos^2(\frac{\theta}{2})f_R + \sin^2(\frac{\theta}{2})f_L \end{pmatrix}.
\end{aligned}
$$

Since the number of the right-handed and left-handed particles are equal for massive spin-$\frac{1}{2}$ particles , I set $f_R = f_L = f$. So that the distribution function becomes

$$f_u = \begin{pmatrix} f_R & 0 \\ 0 & f_L \end{pmatrix} = f\mathbb{I}. \tag{5.1}$$

Let the distribution function,f, satisfy the collisionless Boltzmann equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\dot{\tilde{x}}_i + \frac{\partial f}{\partial p_i}\dot{\tilde{p}}_i = 0. \tag{5.2}$$

In order to write the continuity equation, one should identify the particle number density $n(x,p,t)$, and the particle current density $j(x,p,t)$. As all of these equations which I found are matrix valued, I have to introduce an appropriate definition of the classical limit. This can be done by taking their trace and define the classical velocities as

$$\sqrt{W} \equiv \mathrm{Tr}[\tilde{\omega}_{1/2}],\ \sqrt{W}\dot{x}_i \equiv \mathrm{Tr}[\tilde{\omega}_{1/2}\dot{\tilde{x}}_i],\ \sqrt{W}\dot{p}_i \equiv \mathrm{Tr}[\tilde{\omega}_{1/2}\dot{\tilde{p}}_i].$$

I can write the probability density function as $\rho(x,p,t) = \sqrt{W}f$. Hence, the particle number density and the particle current density are given by:

$$
\begin{aligned}
n(x,t) &= \int \frac{d^3p}{(2\pi)^3}\rho(x,p,t) = \int \frac{d^3p}{(2\pi)^3}\mathrm{Tr}[\tilde{\omega}_{1/2}]f, \\
\boldsymbol{j}(x,t) &= \int \frac{d^3p}{(2\pi)^3}\rho(x,p,t)\dot{\boldsymbol{x}} = \int \frac{d^3p}{(2\pi)^3}\mathrm{Tr}[\tilde{\omega}_{1/2}\dot{\tilde{\boldsymbol{x}}}]f,
\end{aligned} \tag{5.3}
$$

In order to attain the continuity equation, we need the Liouville equation. Equation (4.6) can be written as

$$
\begin{aligned}
\mathscr{L}_{\tilde{v}}\tilde{\Omega} &= (i_{\tilde{v}}d + di_{\tilde{v}})(\frac{1}{3!}\tilde{\omega}^3 \wedge dt) \\
&= \frac{1}{2}d\tilde{\omega} \wedge \tilde{\omega}^2
\end{aligned} \tag{5.4}
$$

where $d\tilde{\omega}$ and $\tilde{\omega}^2$ are given by,

$$
\begin{aligned}
d\tilde{\omega} &= -\frac{1}{2}\frac{\partial G_{ij}}{\partial p_k}dp_k \wedge dp_i \wedge dp_j \\
&= -\frac{\hbar\, m\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E^4(E+m)}d^3p,
\end{aligned} \tag{5.5}
$$

$$\begin{aligned}\tilde{\omega}^2 &= \mathscr{E}_i F_{jk} dx_i \wedge dx_k \wedge dx_k \wedge dt \\ &= 2\mathscr{E} \cdot \boldsymbol{B}\, d^3x \wedge dt \end{aligned} \quad \textbf{(5.6)}$$

Hence,by making use of (5.5) and (5.6), (5.4) is rewritten as

$$\mathscr{L}_v \Omega = \frac{1}{2} d\omega \wedge \omega^2 = \left(\hbar \frac{m\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E^4(E+m)} \mathscr{E} \cdot \boldsymbol{B} \right) dV \wedge dt. \quad \textbf{(5.7)}$$

Comparing (4.5) with (5.7), I can easily write

$$\frac{\partial \tilde{\omega}_{1/2}}{\partial t} + \frac{\partial (\tilde{\omega}_{1/2} \tilde{\dot{x}}_i)}{\partial x_i} + \frac{\partial (\tilde{\omega}_{1/2} \tilde{\dot{p}}_i)}{\partial p_i} = \hbar \frac{m\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E^4(E+m)} \mathscr{E} \cdot \boldsymbol{B} \quad \textbf{(5.8)}$$

Here $\int \frac{d^3p}{(2\pi)^3} \frac{\partial (\tilde{\omega}_{1/2} \tilde{\dot{p}}_i)}{\partial p_i} = 0$ because I suppose that there is no contribution from the boundary of the momentum space.

In conclusion I reach the continuity equation using (5.3) and (5.8):

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}[\hbar \frac{m\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E^4(E+m)} \mathscr{E} \cdot \boldsymbol{B}\, f] = 0.$$

One can also obtain the particle current

$$j_i = \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}[\tilde{\omega}_{1/2} \dot{\tilde{x}}_i f] = \int \frac{d^3p}{(2\pi)^3} \left(\frac{p_i\, f}{E} \right).$$

The Dirac particles satisfy the continuity equation, which has no anomaly.

6. Massless Fermions

I found the solution of the equations of motion for the phase space velocities in terms of phase space variables in general form in section 4. In this part, I would like to study the massless case. The massless Dirac equation can be written as two Weyl equations for the right-handed and left-handed fermions. Therefore, the helicity basis is suitable to discuss the massless case.

First of all in the Berry curvature (4.16) yields

$$\boldsymbol{G} = -\hbar\, \boldsymbol{b}\, \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p}.$$

Here $\boldsymbol{b} = \frac{\boldsymbol{p}}{2\, p^3}$ is the monopole field situated at $p = 0$: $\boldsymbol{\nabla} \cdot \boldsymbol{b} = 2\pi \delta^3(p)$

Now, the Berry curvature is singular and instead of (4.17) it satisfies

$$\frac{\partial G_{\phi i}}{\partial p_i} = -2\pi\hbar\sigma_z \delta^3(p).$$

When I change the bases from u to ϕ, the Berry curvature becomes

$$\boldsymbol{G}_\phi = R^\dagger\, \boldsymbol{G}\, R = -\hbar\boldsymbol{b}\, R^\dagger\, \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{p}\, R = -\hbar\boldsymbol{b} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Obviously, one can deal with the right-handed and left-handed fermions independently. They produce the similar results. Then, let us deal only with the projection onto the right-handed massless fermions. The projection matrix is,

$$P_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$P_R\, \tilde{\omega} P_R$ yields the scalar value

$$\sqrt{\omega} \equiv 1 - e\, \hbar\boldsymbol{b} \cdot \boldsymbol{B}. \tag{6.1}$$

The other solutions (4.19) and (4.20) obtain the massless limit:

$$\sqrt{\omega}\dot{x}_i = -f_i^\phi - \hbar\, \varepsilon_{ijk}\, \mathscr{E}_j\, b_k + e\, \hbar\, B_i\, \boldsymbol{f} \cdot \boldsymbol{b}, \tag{6.2}$$

$$\sqrt{\omega}\dot{p}_i = \mathscr{E}_i - e\, \varepsilon_{ijk}\, f_j^\phi\, B_k + e\, \hbar\, b_i\, (\mathscr{E} \cdot \boldsymbol{B}). \tag{6.3}$$

where f_i denotes the massless limit of f_i.

In order to obtain the particle current density, one defines the probability density function $\rho(x,p,t)$ as $\sqrt{\omega}\, f$. f is distribution function for the right-handed fermions and satisfies the collisionless Boltzmann equation (5.2). Thus the particle current density $\boldsymbol{j}$ is

$$j_i = \int \frac{d^3p}{(2\pi)^3}\sqrt{\omega}\dot{x}_i f = \int \frac{d^3p}{(2\pi)^3}(-f_i^{\phi} - \hbar\, \varepsilon_{ijk}\, \mathscr{E}_j\, b_k + e\, \hbar\, B_i\, \boldsymbol{f}\cdot\boldsymbol{b})f.$$

where the last term where the current is parallel to the magnetic field is the chiral magnetic effect term.

The continuity equation for massless particle can be calculated by using the Liouville equation and making use of equation (4.5) and equation (5.4), the Liouville equation possesses anomalies:

$$\left(\frac{\partial}{\partial t}\sqrt{\tilde{\omega}} + \frac{\partial}{\partial x_i}(\sqrt{\tilde{\omega}}\dot{x}_i) + \frac{\partial}{\partial p_i}(\sqrt{\tilde{\omega}}\dot{p}_i)\right) = 2\pi\delta^3(p)\mathscr{E}\cdot\boldsymbol{B}.$$

Thus I can write by using the definition of the probability density

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\dot{x}_i)}{\partial x_i} + \frac{\partial(\rho\dot{p}_i)}{\partial p_i} = 2\pi e f\delta^3(p)\mathscr{E}\cdot\boldsymbol{B}.$$

The chiral anomaly is explained as non-conservation of the classically conserved chiral current at the quantum level in quantum field theory. I show this phenomena by using the probability function f, which satisfies the collisionless Boltzmann equation.

In conclusion I derive the anomalous continuity equation for massless fermions by using the Boltzmann equation and (5.3) as

$$\frac{\partial n(x,t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = \frac{e}{4\pi^2}f(x,p=0,t)\mathscr{E}\cdot\boldsymbol{B}.$$

The Berry monopole situated on the boundary $|\boldsymbol{p}| = 0$ is responsible for the non-conservation of the chiral particle current.

7. Thomas Precession

I would like to make clear how to take into account the Thomas precession [14] within the wave packet formalism. To this end let us first briefly recall Thomas precession following [19].

The source of this phenomenon lies in the fact that if one would like to write a Lorentz boost as two successive Lorentz boosts, one also should rotate the coordinates with an angle depending on the velocities. This rotation yields an angular velocity known as the Thomas precession.

Suppose that the Dirac particle is moving with the velocity $\boldsymbol{v}$ with respect to laboratory frame at time t. Hence the particle's co-moving frame denoted by the inertial spacetime coordinates x', is connected to the spacetime coordinates of the laboratory frame by the Lorentz boost $\lambda_{boost}(\boldsymbol{v})$ at time t:

$$x' = \lambda_{boost}(\boldsymbol{v})\, x. \tag{7.1}$$

Let the particle accelerate, so that it moves with the velocity $\boldsymbol{v}+d\boldsymbol{v}$ with respect to the laboratory frame at time $t+dt$. Then at time $t+dt$ the co-moving coordinate frame coordinates x'' will be connected to the laboratory frame by the Lorentz transformation

$$x'' = \lambda_{boost}(\boldsymbol{v}+d\boldsymbol{v})\, x. \tag{7.2}$$

Let me write the connection between the two co-moving frame coordinates x' and x'' as

$$x'' = \lambda_T x'.$$

Inspecting (7.1) and (7.2) the transformation λ_T can be written as

$$\lambda_T = \lambda_{boost}(\boldsymbol{v}+d\boldsymbol{v})\, \lambda_{boost}(-\,\boldsymbol{v}). \tag{7.3}$$

The Lorentz boost $\lambda_{boost}(\boldsymbol{v}+d\boldsymbol{v})$ can be separated into two successive Lorentz boosts accompanied by the rotation $R(d\boldsymbol{\theta})$

$$\lambda_{boost}(\boldsymbol{v}+d\boldsymbol{v}) = R(d\boldsymbol{\theta})\, \lambda_{boost}(d\boldsymbol{v})\, \lambda_{boost}(\boldsymbol{v}), \tag{7.4}$$

where the angle of the rotation is

$$d\boldsymbol{\theta} = \frac{\gamma^2}{\gamma+1}\boldsymbol{v}\times d\boldsymbol{v}. \tag{7.5}$$

Here $\gamma = \frac{1}{\sqrt{1-v^2}}$ is the relativistic factor

Therefore by plugging (7.4) into (7.3)

$$\lambda_T = R(d\boldsymbol{\theta})\ \lambda_{boost}(d\boldsymbol{v}). \tag{7.6}$$

In the nonrelativistic systems successive co-moving frames of the Dirac particles are connected by only boosts without any rotations. Therefore, x''' coordinates of the frame moving with the velocity $\boldsymbol{v}+d\boldsymbol{v}$ at time $t+dt$ will be obtained from the system moving with the velocity $\boldsymbol{v}$ at time t only with the boost $\lambda_{boost}(d\boldsymbol{v})$ without any rotation:

$$x''' = \lambda_{boost}(d\boldsymbol{v})\ x'.$$

Then by making use of (7.1), (7.2) and (7.3) the coordinates of the co-moving frame, x''', are written in terms of the laboratory frame coordinate as

$$x''' = R(-d\boldsymbol{\theta})\ \lambda_{boost}(\boldsymbol{v}+d\boldsymbol{v})\ x. \tag{7.7}$$

In the wave packet formalism one deals with the group velocity

$$\boldsymbol{v} \equiv \frac{\partial E}{\partial \boldsymbol{p}} = \frac{\boldsymbol{p}}{E}. \tag{7.8}$$

Since $\gamma = \frac{E}{m}$, by plugging (7.8) into (7.5) one gets

$$d\boldsymbol{\theta} = \frac{\boldsymbol{p}\times d\boldsymbol{p}}{m(E+m)}.$$

Let the laboratory frame and co-moving reference frames coincide at the time $t=0$ when the particle is at rest. Hence the solution of the Dirac equation in laboratory frame is $u(0)$. Now in the nonrelativistic system, one deals with,

$$\begin{aligned} du(\boldsymbol{p})_{NR} &= u'''(\boldsymbol{p}+d\boldsymbol{p}) - u'(\boldsymbol{p}) = R(-d\boldsymbol{\theta})\ \lambda_{boost}(\boldsymbol{v}+d\boldsymbol{v})u(0) - \lambda_{boost}(\boldsymbol{v})u(0) \\ &= R(-d\boldsymbol{\theta})u(\boldsymbol{p}+d\boldsymbol{p}) - u(p) \end{aligned}$$

Hence when one considers the Thomas precession, the one-form η of the nonrelativistic wave packet formalism will lead to

$$i\hbar u^\dagger(\boldsymbol{p})du(\boldsymbol{p})_{NR} = i\hbar u^\dagger(\boldsymbol{p})\left[R(d\boldsymbol{\theta})u(\boldsymbol{p}+d\boldsymbol{p}) - u(\boldsymbol{p})\right] \tag{7.9}$$

Rotation of the spinors is given as

$$R(d\boldsymbol{\theta}) = 1 - \frac{i}{4}\,\sigma_{ij}\,d\omega^{ij}$$

Here σ_{ij} and $d\omega^{ij}$ are written as

$$\sigma_{ij} = \frac{i}{2}\,[\gamma_i\,,\,\gamma_j] = \varepsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix},$$

$$d\omega^{ij} = \varepsilon^{ijm}\,d\theta_m = -\varepsilon^{ijm}\,d\theta^m.$$

I can write the rotation as

$$R(d\boldsymbol{\theta}) = 1 + \mathscr{D}(d\boldsymbol{\theta}).$$

where $\mathscr{D}(d\boldsymbol{\theta})$ is

$$\mathscr{D}(d\boldsymbol{\theta}) = \begin{pmatrix}\boldsymbol{\sigma}\cdot d\boldsymbol{\theta} & 0\\ 0 & \boldsymbol{\sigma}\cdot d\boldsymbol{\theta}\end{pmatrix}.$$

(7.9) yields

$$i\hbar u^\dagger(\boldsymbol{p})du(\boldsymbol{p})_{NR} = i\hbar u^\dagger(\boldsymbol{p})\frac{\partial u(\boldsymbol{p})}{\partial \boldsymbol{p}}\cdot d\boldsymbol{p} + i\hbar u^\dagger(\boldsymbol{p})\mathscr{D}(-d\boldsymbol{\theta})u(\boldsymbol{p}),$$

keeping only the first order terms in $d\boldsymbol{p}$ and $d\boldsymbol{v} = \frac{d\boldsymbol{p}}{m}$ The first term is the Berry gauge field calculated in (4.15). The Thomas precession term can be shown to be

$$i\hbar u^\dagger(\boldsymbol{p})\mathscr{D}(-d\boldsymbol{\theta})u(\boldsymbol{p}) = \frac{(\hbar\boldsymbol{\sigma}\times\boldsymbol{p})}{4m^2}\cdot d\boldsymbol{p}. \tag{7.10}$$

Hence when the Thomas correction is considered the velocities can be read from (4.18), (4.19) and (4.20) by the replacement of the Berry gauge fields (4.15) as

$$\boldsymbol{A}_T = -\frac{(\hbar\boldsymbol{\sigma}\times\boldsymbol{p})}{4m^2}\cdot d\boldsymbol{p} \;+\; \frac{(\hbar\boldsymbol{\sigma}\times\boldsymbol{p})}{4m^2}\cdot d\boldsymbol{p} \;=\; 0$$

up to p^2 terms, which were ignored in the calculation of the Thomas corrections. Therefore, I conclude that the anomalous velocity terms disappear when the Thomas rotation is considered.

8. Time Evolution of Spin

I would like to write the semiclassical equation governing time evolution of the spin for the semiclassical Dirac particle in the electromagnetic field. However, in my formalism, there is no classical degrees of freedom corresponding to the spin of the particle. The spin for the wave packet composed of the positive energy solutions is $\boldsymbol{\sigma}$. Therefore, I use the $\boldsymbol{\sigma}$ matrices which corresponds to spin and use the semiclassical Hamiltonian (4.2).

However as it has been discussed in sec. 2, the new dynamical variables $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{p}}$ are non-commutative and depend on spin matrices. Thus I get

$$\frac{d\sigma_i}{dt} = \frac{1}{i\hbar}[\sigma_i, H(\tilde{\boldsymbol{r}}_+)] = \frac{1}{i\hbar}\left([\sigma_i, H] + [\sigma_i, \tilde{\boldsymbol{r}}_+]\frac{\partial H}{\partial \tilde{\boldsymbol{r}}_+}\right), \tag{8.1}$$

where $\frac{\partial H}{\partial \tilde{\boldsymbol{r}}_+} = -e\boldsymbol{E}$ and $\tilde{\boldsymbol{r}}_+$ is defined by (2.7) projecting on positive energy solutions:

$$\tilde{\boldsymbol{r}}_+ = \boldsymbol{R} + (I_+\, \boldsymbol{A}\, I_+) = \boldsymbol{R} - \hbar\frac{\boldsymbol{\sigma}\times\boldsymbol{p}}{2E(E+m)}.$$

Thus the last term in (8.1) can be written as

$$\frac{d\sigma_i}{dt} = \frac{1}{i\hbar}[\sigma_i, H] - \frac{e\,E_j}{i\hbar}\,\Theta_{ij}^{\sigma\tilde{r}}. \tag{8.2}$$

where $\Theta_{ij}^{\sigma\tilde{r}_+}$ is

$$\Theta_{ij}^{\sigma\tilde{r}} = [\sigma_i, \tilde{r}_{+j}] = [\sigma_i, A_j] = \frac{-i\hbar}{E(E+m)}[(\boldsymbol{\sigma}\cdot\boldsymbol{p})\,\delta_{ij} - \sigma_j\, p_i].$$

In the Hamiltonian (4.2) I ignore higher order terms in p^2:

$$H = E - \frac{e\,\hbar\,\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2E}. \tag{8.3}$$

By plugging (8.3) into 8.2 I get

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{e}{E}\boldsymbol{\sigma}\times\boldsymbol{B} + \frac{e}{E(E+m)}[E_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \;-\; p_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{E})],$$

where the last term can be written as

$$E_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \;-\; p_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{E}) = \varepsilon_{ijm}\,\varepsilon_{mkl}\,\sigma_j\,E_k\,p_l. \tag{8.4}$$

Therefore by using (8.4) and setting $\gamma = \frac{E}{m}$, one reaches the time evolution of the spin

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{e}{m}\boldsymbol{\sigma}\times\left[\frac{1}{\gamma}\boldsymbol{B} + \frac{1}{\gamma+1}\boldsymbol{E}\times\boldsymbol{v}\right]. \tag{8.5}$$

Observe that (8.5) is the BMT equation as composed in Jackson . [18, 19]

9. Results and Discussion

Semiclassical kinetic theory of massive spin-$1/2$ particles are studied within the method introduced in [4]. The main ingredients are the symplectic-two form which is a matrix in the spin indices related to the positive energy solutions of the Dirac equation. These solutions constitute the wave packet which leads to the semiclassical approximation. The block diagonal Hamiltonian including all terms which are at the first order in Planck constant obtained from the Dirac Hamiltonian in the presence of the external electromagnetic fields is presented. By projecting it on the positive energy subspace I obtain the Hamiltonian which is used to define the semiclassical symplectic two-form which is the starting point of the formulation. Differential forms are used to define the semiclassical Hamiltonian dynamics of Dirac particles. The solutions of the equations of motion for the phase space velocities in terms of the phase space variables are derived. Then I used them to define the continuity equation of the particle number density and particle number current. To achieve it one has to define distribution function adequately. This is possible in the basis where the helicity operator is diagonal. Therefore, I performed a change of basis such that the helicity operator becomes diagonal. This is also needed to obtain the massless limit. I showed that the massless limit yields the continuity equation with an anomaly term and also to the particle current yielding the chiral magnetic effect as expected.

Thomas precession correction needed in the non-relativistic formulation is studied within the wave packet formalism. I showed that up to higher order terms in momentum it sweeps out the contribution coming from the Berry gauge field. This coincides with the results obtained in relativistic formulation of Dirac particles. This is a result which is reported for the first time in physics literature. The method of introducing the Thomas precession correction which I presented is valid in general. It can be applied to other semiclassical approaches of Dirac like systems where the underlying Hamiltonian of the theory is given by Dirac like Hamiltonian as in some condensed matter systems.

The semiclassical kinetic theory formulation of Dirac particles and obtaining the massless limit by explicitly constructing the suitable basis which are developed in this thesis can be generalized to systems which have some other interaction terms in the Hamiltonian. Obviously in the development of the semiclassical kinetic theory the next important step is to switch-on interactions in the Boltzmann equation which are ignored in my study.

REFERENCES

[1] **Son, D.T. and Yamamoto, N.** (2012). Berry Curvature, Triangle Anomalies, and the Chiral Magnetic Effect in Fermi Liquids, *Physical Review Letters*, ***109***(18), 181602, 1203.2697.

[2] **Stephanov, M.A. and Yin, Y.** (2012). Chiral Kinetic Theory, *Physical Review Letters*, ***109***(16), 162001, 1207.0747.

[3] **Dwivedi, V. and Stone, M.** (2014). Classical chiral kinetic theory and anomalies in even space-time dimensions, *Journal of Physics A Mathematical General*, ***47***(2), 025401, 1308.4576.

[4] **Dayi, O.F. and Elbistan, M.** (2014). A Semiclassical Formulation of the Chiral Magnetic Effect and Chiral Anomaly in Even d+1 Dimensions, *ArXiv e-prints*, 1402.4727.

[5] **Dayi, Ö.F. and Yunt, E.** (2014). Relation Between the Spin Hall Conductivity and the Spin Chern Number for Dirac-like Systems, *ArXiv e-prints*, 1408.1596.

[6] **Sundaram, G. and Niu, Q.** (1999). Wave-packet dynamics in slowly perturbed crystals: Gradient corrections and Berry-phase effects, *Phys. Rev. B*, ***59***, 14915–14925, cond-mat/9908003.

[7] **Culcer, D.**, **Yao, Y. and Niu, Q.** (2005). Coherent wave-packet evolution in coupled bands, *Phys. Rev. B*, ***72***(8), 085110, cond-mat/0411285.

[8] **Manuel, C. and Torres-Rincon, J.M.** (2014). Chiral transport equation from the quantum Dirac Hamiltonian and the on-shell effective field theory, *Physical Review D*, ***90***(7), 076007, 1404.6409.

[9] **Gosselin, P.**, **Bérard, A. and Mohrbach, H.** (2007). Semiclassical diagonalization of quantum Hamiltonian and equations of motion with Berry phase corrections, *European Physical Journal B*, ***58***, 137–148, hep-th/0603192.

[10] **Chen, J.Y.**, **Son, D.T.**, **Stephanov, M.A.**, **Yee, H.U. and Yin, Y.** (2014). Lorentz Invariance in Chiral Kinetic Theory, *Physical Review Letters*, ***113***(18), 182302, 1404.5963.

[11] **Chang, M.C. and Niu, Q.** (2008). Berry curvature, orbital moment, and effective quantum theory of electrons in electromagnetic fields, *Journal of physics. Condensed matter*, ***20***, 193202, 0953-8984.

[12] **Chuu, C.P.**, **Chang, M.C. and Niu, Q.** (2010). Semiclassical dynamics and transport of the Dirac spin, *Solid State Communications*, ***150***, 533–537, 0911.4760.

[13] **Chen, J.W.**, **Pang, J.y.**, **Pu, S. and Wang, Q.** (2014). Kinetic equations for massive Dirac fermions in electromagnetic field with non-Abelian Berry phase, *Physical Review D*, ***89***(9), 094003, 1312.2032.

[14] **Thomas, L.H.** (1926). The Motion of the Spining Electron, *Nature*, ***117***, 514.

[15] **Stone, M.**, **Dwivedi, V. and Zhou, T.** (2015). Berry Phase, Lorentz Covariance, and Anomalous Velocity for Dirac and Weyl Particles, *Physical Review D*, ***91***(2), 025004, 1406.0354.

[16] **Duval, C. and Horváthy, P.A.** (2004). Anyons with anomalous gyromagnetic ratio and the Hall effect, *Physics Letters B*, ***594***, 402–409, hep-th/0402191.

[17] **Foldy, L.L. and Wouthuysen, S.A.** (1950). On the Dirac Theory of Spin 1/2 Particles and Its Non-Relativistic Limit, *Phys. Rev.*, ***78***, 29.

[18] **Bargmann, V.**, **Michel, L. and Telegdi, V.L.** (1954). Precession of the Polarization of Particles Moving in a Homogeneous Electromagnetic Field, *Physical Review Letters*, ***2***, 435.

[19] **Jackson, J.D.** (1975). *Classical Electrodynamics*, John Wiley, second Edition.

[20] **Sakurai, J.J.** (1967). *Advanced Quantum Mechanics*, Addison-Wesley.

[21] **Bohm, A.**, **Mostafazadeh, A.**, **Koizumi, H.**, **Nui, Q. and Zwanziger, J.** (2003). *The Geometric Phase in Quantum System*, Springer-Verlag.

APPENDICES

APPENDIX A :Berry Gauge Fields

Berry Gauge Fields

Consider the Hamiltonian depending on the external variables such as $\boldsymbol{R}(R_1, R_2, ..., t)$ which rely on time dependence. The physical meaning can be thought as electric field, magnetic field etc. When the parameters change slowly in a quantum mechanical system, the quantum mechanical state returns to the initial condition only up to phase factor after the cyclic evolution completed. The phase factor consist of the dynamical phase in addition to a purely geometrical part. The geometrical phase term is called the Berry phase. Derivation of the Berry phase can obtained by the quantum adiabatic approximation which is only related to slowly altering Hamiltonians. The Berry gauge fields are defined by using the Berry phase factor [20, 21]. To begin with, the eigenvalue equation is given by

$$H(R(t))|n(\boldsymbol{R}(t)) >= E_n(\boldsymbol{R}(t))|n(\boldsymbol{R}(t)) > .$$

where $R(T) = R(0)$. This means that the cyclic evolution is closed.

In the adiabatic approximation the evolving state vector $\psi(t)$ can be written in the basis $|n(\boldsymbol{R}(t)) >$ as

$$\psi(t) = c_n(t)|n(\boldsymbol{R}(t)) > .$$

The Schrodinger equation,

$$\begin{aligned} H(R(t))c_n(t)|n(\boldsymbol{R}(t)) > &= E_n(t)|n(\boldsymbol{R}(t)) > \\ &= i\hbar\frac{\partial}{\partial t}[c_n(t)|n(\boldsymbol{R}(t)) >] \end{aligned} \quad \textbf{(A.1)}$$

One can obtain

$$i\hbar\frac{\partial c_n(t)}{\partial t} = c_n(t)\left[E_n(t) - i\hbar < n(\boldsymbol{R})|\frac{\partial|n(\boldsymbol{R}) >}{\partial t}\right].$$

Then it becomes

$$\int_{c_n(0)}^{c_n(T)} \frac{dc_n(t)}{c_n} = \int_0^T \frac{-i}{\hbar}E_n(t')dt' - \int_0^T < n(\boldsymbol{R})|\frac{\partial|n(\boldsymbol{R}) >}{\partial t'}dt'. \quad \textbf{(A.2)}$$

where $c_n(t)$ is the phase factor including the dynamical and geometrical phase terms. It can be written as

$$c_n(t) = e^{\frac{-i}{\hbar}\int_0^T E_n(t')}e^{\frac{i}{\hbar}\gamma_n(t)}$$

$\gamma_n(t)$ is the Berry phase which is real valued

$$\gamma_n(t) = \int_0^T i\hbar < n(\boldsymbol{R})|\frac{\partial|n(\boldsymbol{R}) >}{\partial t'}dt' = \int_0^T i\hbar < n(\boldsymbol{R})|[\nabla_{\boldsymbol{R}}|n(\boldsymbol{R}) >] \cdot d\boldsymbol{R}$$

As $R(T) = R(0)$ in the adiabatic approximation,

$$\gamma_n(t) = \oint i\hbar < n(\boldsymbol{R})|[\nabla_{\boldsymbol{R}}|n(\boldsymbol{R}) >] \cdot d\boldsymbol{R}.$$

In the conclusion one reach the Berry gauge fields,

$$\boldsymbol{A}_n(\boldsymbol{R}) \equiv i\hbar < n(\boldsymbol{R})|[\nabla_{\boldsymbol{R}}|n(\boldsymbol{R}) >].$$

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