TRANSFINITE DIAMETERS AND POLYA INEQUALITY

by
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Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Sabancı University
August 2015

Transfinite diameters and Polya inequality

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To my beloved family

Transfinite diameters and Polya inequality

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Mathematics, Doctorate Thesis, 2015

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Keywords: Transfinite Diameter, internal transfinite diameter, Chebyshev Constant, Polya Inequality

Abstract

This dissertation deals with two main problems concerning Polya's inequality, mostly, in several variables. We investigate the problems about obtaining the new version of Polya inequality for domains in terms of internal transfinite diameter, due to V. Zakharyuta, and the sharpness of Polya inequality in one and multivariable case.

First part is devoted to the sharpness of Polya's inequality. We make a classification of sharpness properties of a Polya's inequality related to a compact set in multivariate case and examine the stability of these properties by using the considerations obtained from the stability of transfinite diameter with respect to the approximations from inside and outside by compact sets. For real compact sets in $\mathbb{C}^n$, we prove that they have the strong sharpness property. The main ingredient we exploit in proving this is the Bloom-Levenberg integral representation of Vandermondians.

In the second part of thesis, we study internal characteristics of domains in $\mathbb{C}^n$. As a consequence of classical Polya's inequality, we give first the new version of Polya inequality including the internal transfinite diameter in one variable. For multivariable case, given a linearly convex domain with an approximation of sufficiently good sets from inside, it is proved that the internal transfinite diameter of boundary viewed from a point is equal to the transfinite diameter of the compact conjugate set to the aforementioned domain. This will enable us to establish the domain analogue of Polya inequality involving internal transfinite diameter for domains called linearly convex by using the duality due to Aizenberg-Martineau.

Sonluötesi çaplar ve Polya eşitsizliği

Ozan Günyüz

Matematik, Doktora Tezi, 2015

Tez Danışmanı: Prof. Dr. Vyacheslav Zakharyuta

Anahtar Kelimeler: Sonluötesi çap, iç sonluötesi çap, Chebyshev sabiti ve Polya eşitsizliği

Özet

Bu tez çalışması Polya eşitsizliği ile ilgili çoğunlukla çok değişkende olmak üzere iki ana problemle uğraşmaktadır. Bölgeler için Polya eşitsizliğinin, V. Zakharyuta tarafından tanımlanan iç sonlu ötesi çapa göre yeni versiyonunu elde etme ile tek ve çok değişkende Polya eşitsizliğinin eşitlik durumunun sağlanması ile ilgili problemler araştırılmıştır.

Birinci kısım Polya eşitsizliğinin eşitlik durumunun elde edilmesine ayrılmıştır. Çok değişkenli durumda Polya eşitsizliğinin eşitlik durumu özellikleri ile ilgili bir sınıflandırma yapıyoruz ve verilen bir kompakt kümenin sonlu ötesi çapının içeriden ve dışarıdan yapılan yaklaştırımlara göre kararlılığından elde edilen sonuçlar yardımıyla bu özelliklerin kararlılığını araştırıyoruz. Bu eşitlik durumlarının kanıtlarında kullanılan esas içerik Vandermond determinantlar için Bloom-Levenberg integral gösterimidir.

İkinci bölümde $\mathbb{C}^n$ deki bölgelerin iç karakteristiklerini inceliyoruz. Tek değişkende bilinen Polya eşitsizliğinin bir sonucu olarak, ilk olarak iç sonluötesi çapı içeren Polya eşitsizliğinin yeni versiyonunu veriyoruz. Çok değişkenli durumda, içeriden yeterince düzgün kümelerin yaklaştırımına sahip verilen

doğrusal konveks bir bölge için, bölge içinde bir noktadan görülen sınır kümesinin iç sonluötesi çapının bu bölgenin kompakt eşleniği olan kümenin sonluötesi çapına eşit olduğu ispatlanmıştır. Bu bize doğrusal konveks bölgeler için Aizenberg-Martineau dualitesi kullanılarak, iç sonluötesi çapın içerildiği Polya eşitsizliğinin bölge benzerinin oluşturulmasını sağlamıştır.

ACKNOWLEDGEMENTS

First and foremost I want to thank my advisor Prof. Vyacheslav Zakharyuta. It has been an honor to be his last Ph.D. student. He has taught me, both consciously and unconsciously, how good theoretical maths and thinking are done. I appreciate all his contributions of time, ideas, and huge knowledge to make my Ph.D. experience productive and stimulating. I never felt myself so lucky as I feel when I'm a Ph.D. student of Prof. Zakharyuta. The joy and enthusiasm he has for his research was quite motivational for me, even during difficult times in the Ph.D. pursuit. I am also thankful for the excellent example he has provided as a successful, intelligent mathematician and professor.

I want to thank sincerely Prof. Aydın Aytuna and Assoc. Prof. Cem Güneri for their support, guidance and understanding since I came to Sabancı University.

For this dissertation I would like to thank my reading committee members Prof. Erhan Çalışkan, Assoc. Prof. Mert Çağlar and Assist. Prof. Kamer Kaya.

Time at Sabancı University was made so enjoyable in large part due to the friends and groups that became a part of my life. I am grateful for the time spent with roommates and friends in Sabancı University.

Lastly, I would like to thank my beloved parents Seher Günyüz, Veli Günyüz, and my brother and sisters Orkun Günyüz, Ayzen Günyüz and Feyzan Günyüz for all their endless love, patience, support and encouragement, and being with me in tough times during the preparation of this work.

Contents

Introduction

Polya's inequality in one variable ([28]) states that transfinite diameter $d(K)$ of a polynomially convex compact set $K \subseteq \mathbb{C}$ is an upper bound for the Hankel determinants H_s formed by the coefficients of Taylor series expansion of the analytic function f defined on the complement of K around ∞ in the extended complex plane. That is, the following holds

$$D(f) := \limsup_{s \to \infty} |H_s(f)|^{1/s^2} \leq d(K).$$

Generalization of this inequality to the multidimensional case for a compact set $K \subseteq \mathbb{C}^n$ was done by V. Zakharyuta in [44] where, instead of using an analytic function in the complement, so-called analytic functionals on compact sets are used. The initial attempts regarding the sharpness of Polya's inequality were made by G. M. Goluzin ([15]). For a compact set $K \subseteq \mathbb{C}$ whose boundary consists of a finite number of closed Jordan curves, Goluzin has achieved to prove that ([15], and also [16]) Polya's inequality is sharp.
In the second chapter of this thesis, we will be interested in the sharpness of Polya's inequality in several variables. We will define two sharpness properties as "strong sharpness property" and "sharpness property" on compact subsets of $\mathbb{C}^n$. Stability of transfinite diameter of a compact set from outside with compact sets is known ([51], [23]). In this thesis, under certain conditions, the stability of transfinite diameter of a compact set with regard to the approximation from inside is proven. Unweighted energy version of Rumely's

formula plays an important role in proving the stability of this kind. Subject to these stability features, we investigate the stability of these sharpness properties. We obtain that the sharpness property is preserved under the approximation from inside by compact sets. We show at the end of second chapter that any real compact subset of $\mathbb{C}^n$ has the strong sharpness property by using the Bloom-Levenberg representation of Vandermondians ([9]).
The third chapter deals with the internal characteristics of domains. The internal versions of Chebyshev constant and transfinite diameter, defined by V. Zakharyuta ([48]), are investigated. We concentrate on linearly convex domains with an approximation of good enough sets from inside, considering the shifted domains $D_a := D-a$ and conjugate sets $\widetilde{D_a} := \widetilde{D - a}$ to these sets, we will show that the Aizenberg-Martineau duality for the linearly convex domains with this sort of approximation remains to hold. Having given the one-dimensional internal version of Polya inequality, we disprove Sheinov's claims regarding the internal analogue of Polya's inequality in Theorem 2 of [37] and Theorem A of [38] by giving a counterexample in the one dimensional case. Finally, for linearly convex domains in $\mathbb{C}^n$ with good approximation from inside, we obtain the internal version of Polya's inequality involving the internal transfinite diameter.

Chapter 1

Preliminaries

This chapter is devoted to the preliminary information and results which we use throughout the thesis.

1.1 Transfinite Diameter, Chebyshev Constant and Capacity

In this section, first we will give the one variable versions of transfinite diameter, Chebyshev constant and capacity, and secondly as several variable versions, transfinite diameter and Chebyshev constant will be taken into consideration.

For a compact set K in $\mathbb{C}$, the *transfinite diameter of* K is defined as:

$$d(K) := \limsup_{s\to\infty} d_s(K), \tag{1.1.1}$$

where $d_s(K) := \max\{|\det(z_\nu^{\mu-1})_{\mu,\nu=1}^s|^{\frac{2}{s(s+1)}} : z_\nu \in K,\ \nu = 1,\dots,s\}$.

This notion was introduced by Fekete [14] for $n = 1$. It was also proved there that there is usual limit in (1.1.1). Transfinite diameter can be expressed as a geometric mean of extremal pairwise distances among s points on K (if $s \geq 1$):

$$d_s(K) := \max\left\{\left(\prod_{\nu<\mu\leq s}|z_\mu - z_\nu|\right)^{2/s(s+1)} : z_\nu \in K\right\}, \tag{1.1.2}$$

The Chebyshev constant of a compact set K is defined via:

$$\tau(K) := \lim_{s\to\infty}\left(\inf\left\{\max_{z\in K}\left|z^s + \sum_{j=0}^{s-1} c_j z^j\right| : c_j \in \mathbb{C},\ j = 0, 1, \ldots s-1\right\}\right)^{1/s}.$$

The *capacity of a compact set* K is the number defined by

$$c(K) = \exp(-\rho_K), \tag{1.1.3}$$

where

$$\rho_K := \lim_{z\to\infty}(g_K(z) - \ln|z|) \tag{1.1.4}$$

is the *Robin constant of* K. Here $g_K(z)$ is the Green function of K with logarithmic singularity at ∞ defined, by Peron approach, as follows

$$g_K(z) := \limsup_{\zeta\to z}\sup\{v(\zeta) : v \in S(K)\}, \tag{1.1.5}$$

where $S(K)$ is the set of all subharmonic functions in $\mathbb{C}$ such that v is non-positive on K and $v(\zeta) - \ln|\zeta|$ is bounded in a neighborhood of ∞. Capacity is one of the crucial set characteristics in potential theory in the complex plane. For a thorough investigation of capacity, [43] and [30] might be quite useful.

There is a cornerstone result in geometric function theory pertaining to above three set characteristics for a compact set $K \subseteq \mathbb{C}$ called *Fekete-Szegö relation* which expresses that $d(K) = \tau(K) = c(K)$ ([14], [16], [42]).

We are going to use the notation, for $D \subseteq \mathbb{C}^n$, $|f|_D := \sup\{|f(z)| : z \in D\}$ for a function $f : D \to \mathbb{C}$. Let $\mathbb{Z}_+^n$ be the collection of all n-dimensional vectors with non-negative integer coordinates. For $k = (k_1, \dots, k_\nu, \dots, k_n) \in \mathbb{Z}_+^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, let $z^k = z_1^{k_1} \dots z_n^{k_n}$ and $|k| := k_1 + \dots + k_n$ be the degree of the monomial z^k. We consider the enumeration $\{k(i)\}_{i\in\mathbb{N}}$ of the set $\mathbb{Z}_+^n$ such that $|k(i)| \le |k(i+1)|$ and on each set $\{|k(i)| = s\}$ the enumeration coincides with the lexicographic order relative to $k_1, \dots, k_n$. We will write $s(i) := |k(i)|$. The number of multiindices of degree at most s is $m_s := C_{s+n}^s$ and the number of those of degree exactly s is $N_s = m_s - m_{s-1} = C_{n+s-1}^s$, $s \ge 1$; $N_0 = 1$. Let $l_s := \sum_{q=0}^s qN_q$ for $s = 0, 1, 2, \dots$.

The standard $(n-1)$-simplex will be taken into consideration

$$\Sigma := \{\theta = (\theta_\nu) \in \mathbb{R}^n : \theta_\nu \ge 0,\ \nu = 1, \dots, n;\ \sum_{\nu=1}^n \theta_\nu = 1\}, \tag{1.1.6}$$

and its interior (with respect to the relative topology on the hyperplane containing Σ)

$$\Sigma^\circ := \{\theta = (\theta_\nu) \in \Sigma : \theta_\nu > 0,\ \nu = 1, \dots, n\}.$$

For $\theta \in \Sigma$ we denote by $\mathcal{L}_\theta$ the set of all infinite sequences $L \subset \mathbb{N}$ such that $\frac{k(i)}{s(i)} \xrightarrow{L} \theta$. We use also the notation $k! := k_1! \cdots k_\nu! \cdots k_n!$, $k = (k_\nu) \in \mathbb{Z}_+^n$.

Now we are ready to give the definitions of multivariate chararacteristics of a compact set $K \subseteq \mathbb{C}^n$. Let $\{\zeta_1, \dots, \zeta_i\} \subseteq K$. Consider Vandermondians:

$$V(\zeta_1, \dots, \zeta_i) := \det (e_\alpha(\zeta_\beta))_{\alpha,\beta=1}^i,\ i \in \mathbb{N},$$

where $e_\alpha(z) := z^{k(\alpha)}$, $\alpha \in \mathbb{N}$ and $(\zeta_\beta) \in K^i$. Define "*maximal Vandermondians*":

$$V_i := \sup\{|V(\zeta_1, \dots, \zeta_i)| : (\zeta_j) \in K^i\}.$$

Set $d_s(K) := (V_{m_s})^{1/l_s}$. The *transfinite diameter of* K defined by Leja in [22] is the number:

$$d(K) := \limsup_{s\to\infty} d_s(K). \tag{1.1.7}$$

Leja raised the problem as to whether there is usual limit in (1.1.7) ([22]). This problem was solved by Schiffer and Siciak for a special case when K is the topological product of plane compact sets in [35]. V.Zakharyuta in [44] solved the problem affirmatively for an arbitrary compact set $K \subseteq \mathbb{C}^n$ by introducing the following what is called as *directional Chebyshev constant*

$$\tau(K,\theta) \ : \ = \limsup_{\substack{i\to\infty \\ \frac{k(i)}{|k(i)|}\to\theta}} \tau_i := \sup_{L\in\mathcal{L}_\theta} \limsup_{i\in L} \tau_i, \ \ \theta \in \Sigma, \tag{1.1.8}$$

$$\tau_i = \tau_i(K) := (M_i)^{1/s(i)}, \ \ i \in \mathbb{N}, \tag{1.1.9}$$

where

$$M_i := \inf\left\{ |p|_K : p = e_i + \sum_{j=1}^{i-1} c_j \ e_j \right\}, \ \ i \in \mathbb{N} \tag{1.1.10}$$

The constants M_i are called as the *least uniform deviation of monic polynomials* from the identical zero on compact set K. A polynomial which attains its infimum in (1.1.10) is called a *Chebyshev polynomial*. In regard to the theory of best approximation in Banach spaces ([2], section 8), this sort of polynomials always exist, but uniqueness is not guaranteed. The *principal Chebyshev constant* is defined as the continual geometric mean of directional Chebyshev constants

$$\tau(K) := \exp \int_\Sigma \ln \tau(K,\theta) d\sigma(\theta), \tag{1.1.11}$$

where σ is the normalized Lebesgue measure on Σ. Next theorem can be thought of the several complex variable version of Fekete-Szegö relation which was proved in [44].

Theorem 1.1.1. *The usual limit exists in (1.1.7) and*

$$d(K) = \tau(K) = \exp \int_\Sigma \ln \tau(K,\theta) \ \ d\sigma(\theta).$$

As we specified at the beginning of the section, we won't touch the multidimensional capacity and related capacitative notions. Defining the multivariate analogues of capacity, contrary to the one-dimensional case, depends on the choice of the norm on $\mathbb{C}^n$. A noteworthy aspect of the pluripotential theory is that there are numerous capacities. Many capacities were defined using different considerations by various authors. A comprehensive information concerning multidimensional capacity notions might be found in [44], [45], [33], [39], [40], [3], [4], [5], [8], [7] and [20].

The set $\hat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq |p|_K \ \textit{for all polynomials } p\}$ is called as *polynomially convex hull* of a compact set K.

Definition 1.1.2. A compact set $K \subseteq \mathbb{C}^n$ is said to be *polynomially convex* if $K = \hat{K}$.

Lastly, we mention *pluripotential Green function* of a compact set $K \subseteq \mathbb{C}^n$. It is defined as follows

$$g_K(z) = \limsup_{\zeta \to z} \sup\{u(z) : u|_K \leq 0, \ u \in \mathcal{L}(\mathbb{C}^n)\},$$

where $\mathcal{L}(\mathbb{C}^n)$ represents the Lelong class consisting of all functions $u \in Psh(\mathbb{C}^n)$ such that $u(\zeta) - \ln|\zeta|$ is bounded from above near infinity, where $Psh(\mathbb{C}^n)$ is the collection of all plurisubharmonic functions defined on $\mathbb{C}^n$. We will also consider the class of functions $\mathcal{L}^+(\mathbb{C}^n) := \{u \in \mathcal{L}(\mathbb{C}^n) : u(z) \geq log^+|z| + C\}$. The function $g_K(z)$ is either plurisubharmonic in $\mathbb{C}^n$ or identically equal to $+\infty$. For more detail about the pluripotential Green function, [20], [44], [33] and [45] may be useful.

Definition 1.1.3. An open set $\Omega \subseteq \mathbb{C}^n$ is said to be *pseudoconvex* if there exists a continuous plurisubharmonic function f in Ω such that

$$\Omega_c = \{z \in \Omega : f(z) < c\} \Subset \Omega,$$

for every $c \in \mathbb{R}$.

Definition 1.1.4. A subset E of a domain Ω in $\mathbb{C}^n$ is called *pluripolar* in Ω if for each point $z_0 \in E$ there exists a connected neighbourhood U of z_0 and a nontrivial plurisubharmonic function $u(z)$ defined in U such that $E \cap U \subseteq u^{-1}(-\infty)$.

Definition 1.1.5. An open set $D \subseteq \mathbb{C}^n$ is called a *domain of holomorphy* if there are no open sets D_1 and D_2 in $\mathbb{C}^n$ satisfying the following properties:

(i) $\emptyset \neq D_1 \subseteq D_2 \cap D$,

(ii) D_2 is connected and is not contained in D,

(iii) For each $f \in A(D)$, there is a function $g \in A(D_2)$ such that $f = g$ on D_1.

1.2 Spaces of Analytic Functions and Differential Forms

The *space of all analytic functions* on a complex manifold Ω, with the topology of uniform convergence on compact subsets of Ω, will be denoted by $A(\Omega)$. The topology on $A(\Omega)$ is the locally convex topology generated by seminorms,

$$|f|_K = \max_{z \in K} |f(z)| \tag{1.2.1}$$

where K is any compact set in Ω.

Definition 1.2.1. An n-dimensional complex analytic manifold Ω is called a *Stein manifold* if

(i) Ω is countable at infinity, i.e., if there exists a countable number of compact subsets $\{K_i : i \in \mathbb{N}\}$ such that every compact subset of Ω is contained in some K_i.

(ii) The holomorphically convex hull

$$\widehat{K}_h := \{z \in \Omega : |f(z)| \leq \sup_{w \in K} |f(w)| \text{ } for \text{ } all \text{ } f \in A(\Omega)\}$$

is a compact subset of Ω for every compact subset K of Ω.

(iii) For any different points z_1 and z_2 in Ω, there exists a function $f \in A(\Omega)$ such that $f(z_1) \neq f(z_2)$.

(iv) For any $z \in \Omega$, there exist n functions $f_1, \ldots, f_n \in A(\Omega)$ forming a coordinate system at z.

As an example, every domain of holomorphy in $\mathbb{C}^n$ is a Stein manifold since an open set Ω in $\mathbb{C}^n$ is a domain of holomorphy if and only if K is relatively compact in Ω implies $\widehat{K}_h$ is relatively compact in Ω by the characterization of domain of holomorphy ([19], Theorem 2.5.5). Any submanifold of a Stein manifold is also Stein manifold itself ([19], Theorem 5.1.5).

If Ω is countable at infinity, e.g., Ω is a Stein manifold, then the topology on $A(\Omega)$ can be defined by some countable sequence of seminorms of the form (1.2.1). We have also completeness of this space by using the fact that, for any sequence in $A(\Omega)$ converging locally uniformly to a function $f : \Omega \to \mathbb{C}$, we obtain $f \in A(\Omega)$. Hence, $A(\Omega)$ becomes a *Fréchet* space when Ω is a Stein manifold.

For an arbitrary subset U of a complex manifold Ω, let $\mathcal{N}(U)$ be the collection of all open neighbourhoods of U in Ω. We define an equivalence relation by expressing that two functions $f \in A(D_f)$ and $g \in A(D_g)$, where $D_f, D_g \in \mathcal{N}(U)$, are equivalent if there exists $D \in \mathcal{N}(U)$ such that $D \subseteq D_f \cap D_g$ and $f(z) = g(z)$ for every $z \in D$. An equivalence class of an element with respect to this equivalence relation is said to be a *germ of analytic functions*, or briefly, a *germ*.

If U is a non-empty open set in Ω, then if any two functions $f, g \in A(U)$ are in the same germ, we then have $f \equiv g$ on U. Since U is a non-empty open set in Ω and $f - g \equiv 0$ on U, we should have $f - g \equiv 0$ on the whole Ω. So, $f \equiv g$ on any neighborhood of U, which implies that any germ on U consists of a unique analytic function on U.

Let us denote by $A(U)$ the space of all analytic germs on U equipped with the inductive limit topology

$$A(U) = \liminf_{D \in \mathcal{N}(U)} A(D) \tag{1.2.2}$$

i.e., the finest topology on $A(U)$ for which the natural restriction mappings from $A(D)$ to $A(U)$, where $D \in \mathcal{N}(U)$, are continuous. Then, $A(U)$ is also a locally convex space.

If K is a compact set in Ω, then we can express $A(K)$ as the countable inductive limit

$$A(K) = \liminf_{n \to \infty} A(D_n) \tag{1.2.3}$$

where $\{D_n\}$ is an arbitrary countable basis of $\mathcal{N}(K)$. Without loss of generality, we can select the sets D_n such that D_{n+1} is relatively compact in D_n for every n, and no D_n includes a connected component disjoint from K. So, in this setting, $x_n \to x$ in $A(K)$ if there exists a neighborhood $D \in \mathcal{N}(K)$ such that $x_n \in A(D)$ for every n, $x \in A(D)$ and (x_n) converges uniformly to x on any compact subset of D.

Let K be a compact set in $\mathbb{C}^n$, and $J : A(K) \to C(K)$ the natural restriction homomorphism. $AC(K)$ is the Banach space obtained as the completion of the set $J(A(K))$ in the space $C(K)$ with respect to the uniform norm. Let $K = \bigcap_{n=1}^{\infty} D_n$, $D_{n+1} \Subset D_n$. We will also consider the following countable inductive limit of Banach spaces taken with respect to the set inclusion $AC(D_n)$ as the completion of the spaces $A(\overline{D_n})$ with regard to the uniform norm $|f|_{D_n}$

$$A(K) = \liminf_{n \to \infty} AC(D_n). \tag{1.2.4}$$

A differential form ω defined on $\Omega \subseteq \mathbb{C}^n$ is said to be *of type* (p, q) if it can be written as

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \omega_{IJ} dz^I d\overline{z}^J,$$

where $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$, $1 \leq i_1 < \ldots < i_p \leq n$, $1 \leq j_1 < \ldots < j_q \leq n$ and $dz^I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, $d\overline{z}^J = d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$.

Obviously, $p+q$ is the degree of ω. Every differential form of degree s can be written in a unique way as a sum of differential forms of types (p, q) with $p+q = s$. If ω is a form of type (p, q), then the differential of ω is defined as

$$d\omega = \sum_{I,\,J} d\omega_{I,\,J} \wedge dz^I \wedge d\overline{z}^J$$

$$\partial\omega = \sum_{I,\,J} \partial\omega_{I,\,J} \wedge dz^I \wedge d\overline{z}^J$$

$$\overline{\partial}\omega = \sum_{I,\,J} \overline{\partial}\omega_{I,\,J} \wedge dz^I \wedge d\overline{z}^J$$

As seen above, $d = \partial + \overline{\partial}$, and the forms $\partial\omega$ and $\overline{\partial}\omega$ are of type $(p+1, q)$ and $(p, q+1)$, respectively. d is called *exterior differentiation operator.*

Another differential operator d^c is defined by

$$d^c := i(\overline{\partial} - \partial).$$

From the definitions of d and d^c, we obtain

$$dd^c = 2i\partial\overline{\partial}.$$

If $f \in C^2(\Omega)$, then

$$dd^c f = 2i \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

The *complex Monge-Ampére operator* in $\mathbb{C}^n$ is defined as the $n-$th exterior power of dd^c, that is

$$(dd^c)^n = \underbrace{dd^c \wedge \ldots \wedge dd^c}_{n-times}.$$

If $f \in C^2(\Omega)$, we have

$$(dd^c f)^n = 4^n n! \det\left[\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}\right] dV,$$

where

$$dV = \left(\frac{i}{2}\right) dz_1 \wedge d\overline{z_1} \wedge dz_2 \wedge d\overline{z_2} \wedge \ldots \wedge dz_n \wedge d\overline{z_n},$$

is the usual volume form in $\mathbb{C}^n$. More detailed treatment of differential forms and related concepts can be found in [19], [20] and [26].

The *Monge-Ampere energy $\mathcal{E}(u,v)$ of u relative to v* for $u, v \in \mathcal{L}^+(\mathbb{C}^n)$ is defined as follows ([11], Section 5):

$$\mathcal{E}(u,v) := \int_{\mathbb{C}^n} (u-v) \sum_{j=0}^{n} (dd^c u)^j \wedge (dd^c v)^{n-j}.$$

The next lemma whose proof can, for instance, be found in [11] will be of use.

Lemma 1.2.2. *Let $\{w_j\}, \{v_j\}$ be two sequences in $\mathcal{L}^+(\mathbb{C}^n)$ with $w_j \downarrow w \in \mathcal{L}^+(\mathbb{C}^n)$ and $v_j \downarrow v \in \mathcal{L}^+(\mathbb{C}^n)$. Then*

$$\mathcal{E}(w_j, v) \to \mathcal{E}(w,v) \;\; \textit{and} \;\; \mathcal{E}(w_j, v_j) \to \mathcal{E}(w,v). \tag{1.2.5}$$

1.3 Duality

The following theorem is what is known as *Gröthendieck-Köthe-Silva duality*, or briefly *GKS-duality*. It provides a representation of linear continuous functionals with analytic functions.

Theorem 1.3.1. *Let $D \subseteq \overline{\mathbb{C}}$ be any set. Let $A_0(\overline{\mathbb{C}} \backslash D) := \{f \in A(\overline{\mathbb{C}} \backslash D) : f(\infty) = 0\}$. Then one has the isomorphism*

$$T : A(D)^* \to A_0(\overline{\mathbb{C}} \backslash D).$$

such that

$$f^*(f) = \int_\gamma f'(z) f(z) dz, \quad f \in A(D) \tag{1.3.1}$$

where $T(f^*) = f'$ and $\gamma = \gamma(f, f')$ is a union of finite number of closed smooth Jordan curves that separates the singularities of the germs f and f'.

There is no general duality for sets in several complex variables. Nonetheless, if we restrict ourselves to polydiscs, we have the following well-known duality:

Lemma 1.3.2. *Let U_R be a polydisc in $\mathbb{C}^n$ around zero with polyradius $R = (R_1, \ldots, R_n)$ and set U_R^* as*

$$U_R^* := \{z = (z_i) \in \overline{\mathbb{C}^n} : |z_i| \geq R_i,\ i = 1, 2, \ldots, n\} \tag{1.3.2}$$

Then there exists a natural isomorphism $T : A(U_R)^* \to A(U_R^*)$ such that for $f' = T(f^*)$, we obtain

$$f^*(f) = (\frac{1}{2\pi i})^n \int_{S_\lambda} f(\zeta) f'(\zeta)\, d\zeta_1 d\zeta_2 \ldots d\zeta_n\,,\ f \in A(U_R), \tag{1.3.3}$$

where

$$S_\lambda = S_\lambda(f^*) = \{\zeta = (\zeta_i) \in \overline{\mathbb{C}^n} : |\zeta_i| = \lambda\, R_i\,,\ i = 1, 2, \ldots, n\}\,, \lambda = \lambda(f^*) < 1.$$

By Lemma 1.3.2, we can identify the dual space of entire functions $A(\mathbb{C}^n)^*$ with the space $A_0(\{\infty^n\})$ of all germs of analytic functions f at the point

$\{\infty^n := \infty \times \cdots \times \infty\} \subset \overline{\mathbb{C}^n}$, having an expansion

$$f(z) = \sum_{k \in \mathbb{Z}_+^n} \frac{a_k(f)}{z^{k+I}}, \tag{1.3.4}$$

converging uniformly in a neighborhood of ∞^n:

$$\{z = (z_\nu) \in \overline{\mathbb{C}^n} : |z_\nu| \geq r\},$$

with $r = r(f)$. That is to say:

Lemma 1.3.3. *There is an isomorphism,*

$$T : A(\mathbb{C}^n)^* \to A_0(\{\infty^n\}), \tag{1.3.5}$$

such that, for each f^* and $f' = Tf^*$, we have

$$f^*(f) := [f, f'] := \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}_R^n} f(\zeta) f'(\zeta)\ d\zeta,\ f \in A(\mathbb{C}^n),$$

where $R = R(f^*)$ and

$$\mathbb{T}_R^n := \{z = (z_\nu) \in \mathbb{C}^n : |z_\nu| = R,\ \nu = 1, \ldots, n\}. \tag{1.3.6}$$

Following Hörmander, ([19], Section 4) we call an element $f^* \in A(\mathbb{C}^n)^*$ *analytic functional.* The expansion (1.3.4) can be considered as its *Taylor expansion at ∞^n.*

Next lemma will be used actively in subsequent chapters. For the sake of completeness, we include this lemma along with its proof.

Lemma 1.3.4. *Let X, Y be locally convex spaces and $J : X \to Y$ be an injective continuous linear operator such that $J(X)$ is dense in Y. Then the adjoint operator $J^* : Y^* \to X^*$ defined by $J^*(y^*) = y^* \circ J$ is also linear, injective and continuous operator between dual spaces. Furthermore if X is* *reflexive, then the image $J^*(Y^*)$ is dense in X^*.*

Proof. First of all, the linearity and continuity of J^* follows immediately from its definition. In order to prove that J^* is injective, let $J^*(y_1^*) = J^*(y_2^*)$, where $y_1^*, y_2^* \in Y^*$. Then by definition of J^*, we have $y_1^* \circ J = y_2^* \circ J$, i.e., $y_1^*(J(x)) = y_2^*(J(x))$ for all $x \in X$. This means that $y_1^* = y_2^*$ on the dense image set $J(X) \subseteq Y$. Since continuous functions which are equal to each other on a dense subset $J(X)$ are equal on the whole domain Y, one concludes that $y_1^* = y_2^*$ on Y. Hence J^* is injective.

For the second part of lemma, assume now that X is reflexive. Then there exists an isometric isomorphism such that

$$\alpha : X \to X^{**} \tag{1.3.7}$$

$$\alpha(x)(v) = v(x), \tag{1.3.8}$$

where $v \in X^*$. Now pick $v' \in X^{**}$ with $v'(x^*) = 0$ for every $x^* \in J^*(Y^*)$. By reflexivity of X, for the element $v' \in X^{**}$, there exists $x \in X$ such that $\alpha(x) = v'$. So for all $x^* \in J^*(Y^*)$, we have

$$v'(x^*) = \alpha(x)(x^*) = x^*(x) = 0. \tag{1.3.9}$$

Since $x^* \in J^*(Y^*)$, $x^* = J^*(y^*)$ for some (unique) $y^* \in Y^*$. From the relation (1.3.9) and the definition of J^*, one gets

$$x^*(x) = J^*(y^*(x)) = y^*(J(x)) = 0$$

for every $y^* \in Y^*$. Now because Y^* separates the points of Y, one must have $J(x) = 0$, but J is injective, thus $x = 0$. This gives that $v' = \alpha(0) = 0$.

Therefore, for each $v' \in X^{**}$ such that $v'(x^*) = 0$ for all $x^* \in J^*(Y^*)$, we get $v' \equiv 0$. By using a corollary of Hahn-Banach Theorem, for example ([32], Theorem 3.5), $J^*(Y^*)$ is dense in X^*, which was to be shown. $\square$

Hereinafter, given a linear continuous injection $J : X \to Y$, we are going to write shortly $X \hookrightarrow Y$, and under the conditions of Lemma 1.3.4, for dual spaces, we will denote the linear continuous injection $J^* : Y^* \to X^*$ as $Y^* \hookrightarrow X^*$.

Chapter 2

Polya Inequality

2.1 Polya Inequality in One and Multivariable Cases

The following result is due to Polya, which can be found in [28].

Theorem 2.1.1. *Let K be a polynomially convex compact set in $\mathbb{C}$ and $f \in A\left(\overline{\mathbb{C}} \setminus K\right)$ have the following expansion in a neighbourhood of ∞:*

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}. \tag{2.1.1}$$

Let $A_s(f) := \det(a_{k+m})_{k,m=0}^{s-1}$, $s \in \mathbb{N}$, be a sequence of Hankel determinants composed from the coefficients of the expansion (2.1.1). Then,

$$D(f) := \limsup_{s\to\infty} |A_s(f)|^{1/s^2} \leq d(K). \tag{2.1.2}$$

An attempt to obtain a direct multivariate analog of the inequality (2.1.2) gives not much because if we consider functions analytic on the complement of $K \subseteq \mathbb{C}^n$, then the space $A(\overline{\mathbb{C}^n} \backslash K)$ consists only of constant functions. Schiffer and Siciak ([35]) have obtained, as a special case, some analog for the product of plane compact sets $K = K_1 \times K_2 \times \ldots \times K_n \subset \mathbb{C}^n$ and functions $f \in A((\overline{\mathbb{C}} \backslash K_1) \times \ldots \times (\overline{\mathbb{C}} \backslash K_n))$. Sheinov ([37], [38]) studied another analogue of Polya's inequality for a *linearly convex* compact set K, considering the Taylor expansion at the origin for functions analytic in the domain $D = K^*$, where K^* is linearly convex adjoint (conjugate) to K (projective complement of K by Martineau [27]).

In [44], Zakharyuta investigated the case of an arbitrary polynomially convex compact set $K \subset \mathbb{C}^n$ and obtained the generalizon of Theorem 2.1.1 for several variables. What was done there is to consider the analytic functionals in $A(\mathbb{C}^n)$ that are extendible continuously onto the space $A(\widehat{K})$, instead of analytic functions on some complement set of K. We will now give this generalized form of Polya's theorem.

By Lemma 1.3.3, let us define, for every functional f^*, a related sequence of multivariate Hankel-like determinants constructed from the coefficients of the expansion (1.3.4):

$$H_i = H_i\left(f^*\right) := \det \left(a_{k(\alpha)+k(\beta)}\right)_{\alpha,\beta=1}^{i}, \ i \in \mathbb{N} \tag{2.1.3}$$

with

$$a_{k(\alpha)} := f^*\left(e_\alpha\right) := [e_\alpha, f'], \ \alpha \in \mathbb{N}, \ f' = Tf^*. \tag{2.1.4}$$

Now we are ready to formulate the general form of multivariate Polya's inequality. We will give the proof of it as well for the sake of completeness.

Theorem 2.1.2. *(V. Zakharyuta, 1975) Suppose that K is a polynomially convex compact set in $\mathbb{C}^n$, f^* is an analytic functional which has a continuous extension onto $A(K)$ and $f' = Tf^*$ is the corresponding analytic germ at ∞^n. Then for the determinants (2.1.3), the following inequality holds:*

$$D(f') := \limsup_{i \to \infty} \left|H_i\left(f^*\right)\right|^{\frac{1}{2l_{s(i)}}} \leq d(K). \tag{2.1.5}$$

Proof. Let us first choose a large enough $R > 0$ such that $K \subseteq U_R$, where U_R is an equilateral polydisc with radius R. We then have the dense and continuous embedding $A(\overline{U_R}) \hookrightarrow A(K)$. By Lemma 1.3.4, one has $A(K)^* \hookrightarrow A(\overline{U_R})^*$. Therefore by Lemma 1.3.2, for every functional $f^* \in A(K)^*$, there is the germ of analytic function f' such that $f'(z) = \sum_{k\in\mathbb{Z}_+^n} \frac{a_k}{z^{k+I}},\ I = (1,\ldots,1),\ a_k = f^*(z^k)$, in the neighborhood of ∞.

We select a sequence of open sets D_m with $D_{m+1} \Subset D_m$ and $K = \bigcap_{m=1}^{\infty} D_m$. We have $\widehat{d}(K) = \lim_{m\to\infty} d(K_m)$, where $K_m = \overline{D_m}$ and $\widehat{d}(K)$ is the *outer transfinite diameter*. We will take $A(K)$ into consideration with the countable inductive limit topology (see 1.2.4). By definition of Hankel-like determinants 2.1.3, we first have

$$H_i = \begin{vmatrix} f^*\big((z^{(1)})^{k(1)+k(1)}\big) & \ldots & f^*\big((z^{(1)})^{k(1)+k(i)}\big) \\ f^*\big((z^{(2)})^{k(2)+k(1)}\big) & \ldots & f^*\big((z^{(2)})^{k(1)+k(i)}\big) \\ \vdots & \ldots & \vdots \\ f^*\big((z^{(i)})^{k(i)+k(1)}\big) & \ldots & f^*\big((z^{(i)})^{k(i)+k(i)}\big) \end{vmatrix} \tag{2.1.6}$$

Since determinant is linear in each row separately, by using the notation $f^*_{z^{(m)}},\ m = 1,\ldots,i$; meaning that the linear functional f^* is applied sequentially to a function of the variable $z^{(m)}$ by keeping the other variables fixed, we have

$$H_i = f^*_{z^{(m)}}(\ldots(f^*_{z^{(1)}}((z^{(1)})^{k(1)}\ldots(z^{(i)})^{k(i)} \begin{vmatrix} (z^{(1)})^{k(1)} & \ldots & (z^{(1)})^{k(i)} \\ (z^{(2)})^{k(1)} & \ldots & (z^{(2)})^{k(i)} \\ \vdots & \ldots & \vdots \\ (z^{(i)})^{k(1)} & \ldots & (z^{(i)})^{k(i)} \end{vmatrix})) \ldots) \tag{2.1.7}$$

The determinant appearing very inside is the Vandermondian determinant $V(z^{(1)},\ldots,z^{(i)})$ for the variables $z^{(1)},\ldots,z^{(i)}$, therefore we have

$$H_i = f^*_{z^{(m)}}(\ldots(f^*_{z^{(1)}}(V(z^{(1)},\ldots,z^{(i)}).(z^{(1)})^{k(1)}\ldots(z^{(i)})^{k(i)}))\ldots). \tag{2.1.8}$$

If we consider all $i!$ permutations of the variables $z^{(1)},\ldots,z^{(i)}$, applications of f^* in each of these orders not changing the value of (2.1.8) and sum all together using the linearity of f^*, we obtain, passing to the absolute value, the following

$$i! \mid H_{\ i}(f^*)| = \mid f^*_{z^{(i)}}(\ldots f^*_{z^{(j)}}(\ldots(f^*_{z^{(1)}}([V(z^{(1)}, z^{(2)}, \ldots, z^{(i)})]^2)\ldots)\ldots)|. \tag{2.1.9}$$

Consider maximal Vandermondians for the set K_m; that is,

$$V_i = \sup\{|V(z^{(1)}, \ldots, z^{(i)})| : z^{(1)}, \ldots, z^{(i)} \in K_n\}$$

By Theorem 1.1.1, for the compact set K_n, given any $\epsilon > 0$, there exists $N_0(\epsilon) \in \mathbb{N}$ such that, whenever $i \geq N_0$, we have

$$\sup\{|V(z^{(1)}, \ldots, z^{(i)})| : z^{(1)}, \ldots, z^{(i)} \in K_m\}^{l_{s(i)}} < d(K_m) + \epsilon \tag{2.1.10}$$

or

$$\sup\{|V(z^{(1)}, \ldots, z^{(i)})| : z^{(1)}, \ldots, z^{(i)} \in K_m\} < (d(K_m) + \epsilon)^{l_{s(i)}}. \tag{2.1.11}$$

On the other hand, since the linear functional $f^* \in A(K)^* \hookrightarrow A(\overline{U_R})^*$ is, by definition of inductive topology on $A(K)$ (1.2.4), continuous on each space $AC(D_m)$, therefore there are constants $C_m < \infty,\ \ n = 1, 2, \ldots$ such that

$$|f^*(f)| \leq C_m |f|_{K_m},\ \ f \in AC(K_m). \tag{2.1.12}$$

Iterating (2.1.12) to (2.1.9) i times, one gets

$$i!|H_i| \leq |C_n|^{2i} |V(z^{(1)}, \ldots, z^{(i)})|^2. \tag{2.1.13}$$

(2.1.11) yields

$$i!|H_i| \leq (C_n)^{2i}(d(K_m) + \epsilon)^{2l_{s(i)}}. \tag{2.1.14}$$

Because $(i!)^{\frac{1}{2l_{s(i)}}} \to 1$ and $\frac{i}{l_{s(i)}} \to 0$ as $i \to \infty$, we obtain, passing to the upper limit,

$$\limsup_{i\to\infty} |H_i(f^*)|^{\frac{1}{2l_{s(i)}}} \le d(K_m) + \epsilon, \quad m \ge m_0(\epsilon), \tag{2.1.15}$$

Since ϵ is arbitrary, we have

$$D(f') \le \widehat{d}(K) \tag{2.1.16}$$

To conclude the proof, we use the Proposition 2.2.1 which will be given later and obtain (2.1.5). □

Remark 2.1.3. A bit weaker inequality (2.1.16) is given in [44].

The relation (2.1.9) which is very important in the above proof will be used essentially in the section 2.3.

Remark 2.1.4. The classical Polya's Theorem (Theorem 2.1.1) is a particular case of Theorem 2.1.2 since, due to Gröthendieck-Köthe-Silva duality (Theorem 1.3.1), every $f \in A(\overline{\mathbb{C}}\backslash K)$ satisfying (2.1.1) in a neighborhood of ∞ represents a linear continuous functional $f^* \in A(K)^* \hookrightarrow A(\mathbb{C})^*$.

Definition 2.1.5. Let $K \subset \mathbb{C}^n$ be a compact set, and μ be a bounded positive Borel measure on K. The pair (K, μ) is said to satisfy *Bernstein-Markov inequality* for holomorphic polynomials in $\mathbb{C}^n$ if, given $\varepsilon > 0$, there exists a constant $M = M(\varepsilon)$ such that for all polynomials p_s of degree at most s

$$|\, p_s|_K \le M(1+\varepsilon)^s \|\, p_s\|_{L^2(\mu)}.$$

Theorem 2.1.6. *(Bloom-Levenberg) Let $K \subset \mathbb{C}^n$ be a compact set, μ be a bounded positive Borel measure on K and let (K, μ) satisfy Bernstein-Markov inequality. Then,*

$$\lim_{s\to\infty} Z_s(K,\mu)^{\frac{1}{2l_s(n)}} = d(K),$$

where

$$Z_s(K,\mu) = \int_{K^{m_s(n)}} | V(\zeta^{(1)}, \ldots, \zeta^{(m_s(n))})|^2 d\mu(\zeta^{(1)}) \ldots d\mu(\zeta^{(m_s(n))}). \quad (2.1.17)$$

Remark 2.1.7. In [10] (Proposition 3.4 and Corollary 3.5), the same authors proved that for any compact set $K \subseteq \mathbb{C}^n$, there exists a measure $\mu \in \mathcal{M}(K)$ such that (K, μ) satisfies Bernstein-Markov property.

2.2 Stability of Transfinite Diameter

The following proposition provides the stability of transfinite diameter of a compact set in $\mathbb{C}^n$ approximated from outside.

Proposition 2.2.1. *(V.A. Znamenskii [51, 52], Levenberg [23]) Let K be a compact set in $\mathbb{C}^n$ and $\{K_j\}$ a sequence of compact sets such that $K_{j+1} \subseteq K_j$ for all $j \in \mathbb{N}$ and $K = \bigcap_{j=1}^{\infty} K_j$. Then,*

$$\widehat{d}(K) := \lim_{j\to\infty} d(K_j) = d(K).$$

In this section, we prove a stability property of transfinite diameter relative to the approximation from inside. The following is an easy consequence of Lemma 6.5 in [7]:

Lemma 2.2.1. *Suppose that K is a non-pluripolar compact set in $\mathbb{C}^n$, and $\{K_j\}$ is a sequence of non-pluripolar compact sets such that $K_j \subseteq K_{j+1} \subseteq K$, $j \in \mathbb{N}$ and for $L := \bigcup_{j=1}^{\infty} K_j$, we have*

$$\int_{K\setminus L} (dd^c g_K)^n = 0. \quad (2.2.1)$$

Then

$$\lim_{j\to\infty} g_{K_j}(z) = g_K(z), \quad z \in \mathbb{C}^n.$$

Theorem 2.2.2. *Under the conditions of Lemma 2.2.1, we have,*

$$\lim_{j\to\infty} d(K_j) = d(K).$$

Proof. We will use the unweighted energy version of Rumely's formula (See e.g., Theorem 5.1 of [25], or Section 9.1 of [11]). Since, by Lemma 2.2.1, $g_{K_j} \downarrow g_K$, applying Lemma 1.2.2, one obtains the following

$$-\ln d(K_j) = \frac{1}{n(2\pi)^n}\mathcal{E}(g_{K_j}, g_T) \uparrow \frac{1}{n(2\pi)^n}\mathcal{E}(g_K, g_T) = -\ln d(K), \ \ as \ \ j \to \infty,$$

where T is the unit torus in $\mathbb{C}^n$. □

2.3 Sharpness of Polya's Inequality

Goluzin obtained the following theorem ([15]) which gives a relation about how transfinite diameter $d(K)$ of a compact set $K \subseteq \mathbb{C}$ changes under polynomial change of variables.

Theorem 2.3.1. *Let $f(z)$ be an analytic function for large z and have the expansion*

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k},$$

in a neighbourhood of $z = \infty$. Let $p(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n$, where $\alpha_0 \neq 0$, $n \geq 1$. Assume that the function $f_*(z) := f(p(z))$ possesses the expansion around $z = \infty$

$$f_*(z) = \sum_{k=1}^{\infty} \frac{a_k^*}{z^k}.$$

If we let

$$A_m := \begin{vmatrix} a_1 & a_2 & \dots & a_m \\ a_2 & a_3 & \dots & a_{m+1} \\ \vdots & \vdots & \vdots & \\ a_m & a_{m+1} & \dots & a_{2m-1} \end{vmatrix},$$

$$A_m^* := \begin{vmatrix} a_1^* & a_2^* & \dots & a_m^* \\ a_2^* & a_3^* & \dots & a_{m+1}^* \\ \vdots & \vdots & \vdots & \\ a_m^* & a_{m+1}^* & \dots & a_{2m-1}^* \end{vmatrix},$$

then the A_m^**, for* $m = 1, 2, \dots,$ *are independent of the values* $\alpha_1, \alpha_2, \dots,$ *and are expressed in terms of the* A_m *according to the formulae*

$$A_m^* = \begin{cases} \mp\alpha_0^{-np^2} A_p^n & for\ m = pn, \\ 0 & for\ other\ m. \end{cases}$$

Moreover, if we write

$$D := \limsup_{m\to\infty} |A_m|^{\frac{1}{m^2}}, \ D^* := \limsup_{m\to\infty} |A_m^*|^{\frac{1}{m^2}}, \tag{2.3.1}$$

then we have

$$D^* = \left(\frac{D}{|\alpha_0|}\right)^{\frac{1}{n}}.$$

As an application of the above theorem, Goluzin proved the following ([15], see also [16], Section 11):

Theorem 2.3.2. *For functions which are analytic in an infinite domain* B *with boundary* K *consisting of a finite number of closed Jordan curves and having the expansion*

$$f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k}, \tag{2.3.2}$$

in a neighborhood of $z = \infty$, the inequality $D(f) \leq d(K)$ given by Theorem 2.1.1 is sharp. Here $d(K)$ is the transfinite diameter of K.

Another way of expressing Theorem 2.3.2 is, for a compact set $K \subseteq \mathbb{C}$

$$d(K) = \sup\left\{D(f) : f \in A\left(\overline{\mathbb{C}} \setminus K\right)\right\}, \tag{2.3.3}$$

if the boundary ∂K consists of a finite number of closed Jordan curves.

Definition 2.3.3. Let K be a compact set in $\mathbb{C}^n$. K is said to satisfy the *sharpness property* in Polya inequality, shortly denoted as $K \in (SP)$, if

$$d(K) = \sup\{D(f') : f' = T(f^*), f^* \in A(K)^*\}.$$

We say that K has a *strong sharpness property* in Polya inequality, denoted by $K \in (SSP)$, if there exists a $f^* \in A(K)^*$ such that

$$D(f') = d(K)$$

for $f' = T(f^*)$, where T is defined as in Lemma 1.3.3.

For an approximation from inside, we have the stability of the property (SP), that is:

Proposition 2.3.1. *Let the conditions of Lemma 2.2.1 be given. Suppose further that $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then $K \in (SP)$.*

Proof. By Definition 2.3.3, for each $i \in \mathbb{N}$, there exists $f_i^* \in A(K_i)^*$ with $f_i' = T(f_i^*)$ such that $d(K_i) \leq D(f_i') + \frac{1}{i}$. Theorem 2.1.2 gives $D(f_i') \leq d(K_i)$. By Theorem 2.2.2, we have

$$d(K) \leq \lim_{i\to\infty} D(f_i') = \sup\{D(f_i') : i \in \mathbb{N}\} \leq d(K),$$

which concludes that $d(K) = \sup\{D(f_i') : i \in \mathbb{N}\}$ and so $K \in (SP)$ by Definition 2.3.3. $\square$

For an approximation from outside, (SP) is not preserved but we obtain the proposition below, in a sense weaker than (SP) :

Proposition 2.3.2. *Let K be a compact set in $\mathbb{C}^n$, $\{K_i\}$ a sequence of compact sets with $K = \bigcap_{i=1}^{\infty} K_i$. Assume $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then there exists a sequence of analytic functionals $\{f_i^*\}$ such that $f_i^* \in A(K_i)^*$ for each $i \in \mathbb{N}$ and*

$$\lim_{i\to\infty} D(f_i') = d(K). \tag{2.3.4}$$

Proof. Proof is almost the same as the proof of Proposition 2.3.1 except we only use Proposition 2.2.1 instead of Theorem 2.2.2 in the end, hence we have the following

$$d(K) = \lim_{i\to\infty} d(K_i) \leq \lim_{i\to\infty} D(f_i') \leq \lim_{i\to\infty} d(K_i) = d(K),$$

which gives the limit (2.3.4). □

For an arbitrary compact set in $\mathbb{C}$, the following sharpness statement, which is weaker than (SP) and is analogous to Proposition 2.3.2, is derived easily from Goluzin's result above.

Proposition 2.3.3. *Let K be a compact set in $\mathbb{C}$, $\{K_i\}$ a sequence of compact sets with the properties $K_{i+1} \Subset K_i$ for all $i \in \mathbb{N}$, $K = \bigcap_{i=1}^{\infty} K_i$. Then there exists a sequence of functions $f_i \in A(\overline{\mathbb{C}} \backslash K_i)$ such that*

$$\lim_{i\to\infty} D(f_i) = d(K). \tag{2.3.5}$$

Proof. For each $i \in \mathbb{N}$, we can find a compact set L_i whose boundary consists of a finite number of closed Jordan curves so that $K_{i+1} \Subset L_i \Subset K_i$ holds. By the result of Goluzin, there exists $f_i \in A(\overline{\mathbb{C}} \setminus L_i)$ such that, $d(L_i) < D(f_i) + \frac{1}{i}$. Since $f_i \in A(\overline{\mathbb{C}} \setminus K_i)$ holds, we get by Theorem 2.1.1, $D(f_i) \leq d(K_i)$. Hence by using Proposition 2.2.1 we obtain the following

$$d(K) = \lim_{i\to\infty} d(L_i) \leq \lim_{i\to\infty} D(f_i) \leq \lim_{i\to\infty} d(K_i) = d(K),$$

which gives the desired limit (2.3.5). $\square$

Let now K be a pluripolar compact set in $\mathbb{C}^n$. Then $K \in (SSP)$ by the result of Levenberg-Taylor ([24]) which says that $d(K) = 0$ if and only if K is pluripolar.

From now on, we only consider non-pluripolar compact sets.

Let K be a compact set in $\mathbb{C}^n$, and $J : A(K) \to C(K)$ the natural restriction homomorphism. $AC(K)$ is the Banach space obtained as the completion of the set $J(A(K))$ in the space $C(K)$ with respect to the uniform norm.

In what follows, we prove that any real compact subset of $\mathbb{C}^n$ satisfies the property (SSP). We will need the following lemma.

Lemma 2.3.4. *Let K be an infinite polynomially convex compact set in $\mathbb{C}^n$. Then, for each bounded Borel measure $\mu \in \mathcal{M}(K)$, there exists an analytic functional $f^* \in A(K)^* \hookrightarrow A(\mathbb{C}^n)^*$ and a corresponding analytic germ $f' = Tf^*$ such that*

$$f^*(f) = \int_K f(\zeta) d\mu(\zeta), \tag{2.3.6}$$

for every $f \in A(\mathbb{C}^n)$.

Proof. By Lemma 1.3.4, the dense embedding $A(K) \hookrightarrow AC(K)$ implies, for the dual spaces, the following embedding: $AC(K)^* \hookrightarrow A(K)^*$. Since $AC(K)$ is a closed subspace of $C(K)$, every bounded Borel measure $\mu \in \mathcal{M}(K)$ defines a linear continuous functional $F^* \in AC(K)^*$ such that

$$F^*(f) = \int_K f(\zeta) d\mu(\zeta)$$

for every $f \in AC(K)$. Then, the restriction $f^* = F^*|_{A(K)}$ belongs to $A(K)^*$.

By lemma 1.3.3, since $A(K)^* \hookrightarrow A(\mathbb{C}^n)^*$, there is $f' \in A_0(\{\infty^n\})$ such that

$$f^*(f) = [f, f'] = \left(\frac{1}{2\pi i}\right)^n \int_{\mathbb{T}_R^n} f(\zeta) f'(\zeta)\ d\zeta,\ f \in A(\mathbb{C}^n),$$

where $\mathbb{T}_R^n$ is defined as in (1.3.6), and R is sufficiently large. $\square$

Now we show that, for any real compact set in $\mathbb{C}^n$, the equality in the estimate (2.1.5) is attained at some $f^* \in A(K)^*$.

Theorem 2.3.5. *Let $K \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n$ be a compact set. Then $K \in (SSP)$.*

Proof. By Theorem 2.1.6 and Remark 2.1.7, there exists a measure $\mu \in \mathcal{M}(K)$ such that (K, μ) satisfies the Bernstein-Markov inequality. Let f^* be an analytic functional corresponding to μ by Lemma 2.3.4. Initially, we show that $Z_s(K,\mu) = m_s(n)!\,|\,H_{m_s(n)}(f^*)|$, where $Z_s(K,\mu)$ and $H_{m_s(n)}(f^*)$ are defined in Section 2. Indeed, considering the relation (2.1.9) gives :

$$m_s(n)!\,|H_{\,m_s(n)}(f^*)| = |\ f^*_{\zeta^{(m_s(n))}}(\ldots(f^*_{\zeta^{(1)}}([V(\zeta^{(1)},\ldots,\zeta^{(m_s(n))})]^2)\ldots)|, \tag{2.3.7}$$

Since K is a real subset and so $[V(\zeta^{(1)},\zeta^{(2)},\ldots,\zeta^{(m_s(n))})]^2$ is nonnegative, by iterating (2.3.6) $m_s(n)$ times, the righthand side of (2.3.7) becomes :

$$\int_K \cdots \int_K |\ V(\zeta^{(1)},\zeta^{(2)},\ldots,\zeta^{(m_s(n))})|^2 d\mu(\zeta^{(1)})\ldots d\mu(\zeta^{(m_s(n)}),$$

which is equal to $Z_s(K,\mu)$. Since $(m_s(n)!)^{\frac{1}{2l_s(n)}} \to 1$ as $s \to \infty$, we have, by Theorem 2.1.6,

$$d(K) = \lim_{s\to\infty} Z_s(K,\mu)^{\frac{1}{2l_s(n)}} = \lim_{s\to\infty} |\ H_{\,m_s(n)}(f^*)|^{\frac{1}{2l_s(n)}} = D(f').$$

$\square$

It is an open question which type of compact sets in $\mathbb{C}^n$ satisfy either the property (SP) or (SSP).

Chapter 3

Internal Set Characteristics in $\mathbb{C}^n$

In [48], notions of transfinite diameter and Chebyshev constant are studied in a new setting by considering domains. In this chapter we mention these concepts along with some results.

3.1 Internal Versions of Transfinite Diameter and Chebyshev Constant

We shall denote by $H^\infty(D)$ the space of all bounded functions $f \in A(D)$ with the uniform norm $\|f\|_{H^\infty(D)} := |f|_D$. Let $D \Subset \mathbb{C}^n$ be a domain and $a \in D$. The following two systems are taken into consideration there:

1. System of monomials:

$$e_{a,i}(z) := (z-a)^{k(i)},\ i \in \mathbb{N}, \tag{3.1.1}$$

2. System of analytic functionals biorthogonal to the system (3.1.1):

$\{e'_{a,i}\}_{i\in\mathbb{N}}$ defined by

$$e'_{a,i}(f) := \frac{f^{(k(i))}(a)}{k(i)!},\ i \in \mathbb{N},\ f \in A(\{a\}), \tag{3.1.2}$$

where $A(\{a\})$ is the space of analytic germs at the point a. If there is no confusion about the point a, our notation for (3.1.1) and (3.1.2) will be shortly as $e_i(z)$ and $e'_i(f)$, respectively.

Define a sequence

$$\delta_i = \delta_i(a, D) := \inf\{|f|_D : f \in \mathcal{N}_i\}, \tag{3.1.3}$$

where

$$\mathcal{N}_i = \mathcal{N}_i(a, D) := \{f \in H^\infty(D) : e'_{j,a}(f) = 0,\ j < i\,;\ e'_{i,a}(f) = 1\}. \tag{3.1.4}$$

We shall stick to the convention $\inf\emptyset = +\infty$. (This might happen when $H^\infty(D)$ consists only of constants).

Definition 3.1.1. ([48]) The *directional Chebyshev constant of D relative to a point a in a direction $\theta \in \Sigma$* is the number

$$\tau(a, D; \theta) := \limsup_{\frac{k(i)}{|k(i)|} \xrightarrow{L} \theta} (\delta_i)^{\frac{1}{s(i)}} := \sup_{L\in\mathcal{L}_\theta} \limsup_{i\in L} (\delta_i)^{\frac{1}{s(i)}}, \tag{3.1.5}$$

where δ_i is defined as in (3.1.3) above.

Lemma 3.1.2. *([48]) The set $\Sigma(a, D) := \{\theta \in \Sigma : \tau(a, D; \theta) < \infty\}$ is convex and the function $\ln\tau(a, D; \theta)$ is convex on $\Sigma(a, D)$.*

Corollary 3.1.1. ([48]) The function $\ln\tau(a, D; \theta)$ is continuous on the interior of the set $\Sigma(a, D)$.

Lemma 3.1.3. *([48]) If r is the radius of an inscribed equilateral polydisc for D centered at a, then $\tau(a, D; \theta) \geq r$ for all $\theta \in \Sigma$. If D is bounded and R*

is the radius of a circumscribed equilateral polydisc for D centered at a, then $\tau(a, D; \theta)$ is uniformly bounded from above by R from above.

With the aid of above three considerations, the following definition is meaningful:

Definition 3.1.4. ([48]) The *principal Chebyshev constant of D relative to $a \in D$* is the number:

$$\tau(a, D) := \exp\Big(\int_{\Sigma} \ln \tau(a, D; \theta) d\sigma(\theta) \Big), \tag{3.1.6}$$

where σ is the normalized Lebesgue measure on Σ.

The following proposition provides a relation between internal analogue of principal Chebyshev constant and principal Chebyshev constant of a compact set.

Proposition 3.1.1. *([48]) Let D be a bounded complete logarithmically convex n-circular domain in $\mathbb{C}^n$ with $0 \in D$ and*

$$h(\theta) = h_D(\theta) := \sup \Big\{ \sum_{i=1}^{n} \theta_i \ln |z_i| : z = (z_i) \in D \Big\}, \ \theta = (\theta_i) \in \Sigma, \tag{3.1.7}$$

be its characteristic function. Then $\tau(0, D; \theta) = \tau(\overline{D}, \theta) = \exp h(\theta)\,, \ \theta \in \Sigma$, and

$$\tau(0, D) = \tau(\overline{D}) = \exp \int_{\Sigma} h(\theta) d\sigma(\theta), \tag{3.1.8}$$

where σ is the normalized Lebesgue measure on Σ, $\tau(\overline{D}, \theta)$ and $\tau(\overline{D})$ are, respectively, the directional and the principal Chebyshev constants of compact set K .

For the unbounded case, we have have the following:

Theorem 3.1.5. *Let D be an n-circular, unbounded, complete and logarith-*

mically convex domain in $\mathbb{C}^n$ *with* $0 \in D$*. Then*

$$\tau(0, D) = \exp \int_{\Sigma} h(\theta) d\sigma(\theta), \tag{3.1.9}$$

where σ *is the normalized Lebesgue measure on* Σ*. Furthermore,* $\tau(0, D) < \infty$ *if and only if* h_D *is integrable on* Σ*.*

Proof. Let D be as in the theorem. Since any n-circular domain is the union of polydiscs, which can be chosen as an increasing union, we can write

$$D = \bigcup_{i=1}^{\infty} D_i \ ; \ D_i \subseteq D_{i+1}, \ i = 1, 2, \ldots.$$

It is well-known that any polydisc is logarithmically convex, n-circular and complete. Now, on these domains we define the pointwise limit:

$$h_D(\theta) := \lim_{k \to \infty} h_{D_k}(\theta) \ ; \ h_{D_k}(\theta) \uparrow h_D(\theta) \ \ as \ \ k \to, \infty$$

where h_{D_k} and h_D are the characteristic functions of D_k and D, respectively. By ([48], Definition 33), we write

$$\tau(0, D) := \widetilde{\tau}(0, D) = \lim_{k \to \infty} \tau(0, D_k).$$

We then have, by Proposition 3.1.1 and continuity of exponential function

$$\tau(0, D) = \lim_{k \to \infty} \tau(0, D_k) = \exp\big(\lim_{k \to \infty} \int_{\Sigma} h_{D_k}(\theta) d\sigma(\theta)\big). \tag{3.1.10}$$

From the equality (3.1.10), by invoking monotone convergence theorem we get

$$\tau(0, D) = \exp\big(\int_{\Sigma} h_D(\theta) d\sigma(\theta)\big).$$

For the second assertion in the theorem, by using the first part proved above, we readily have $\tau(0, D) < \infty$ if and only if $\int_{\Sigma} h_D(\theta) d\sigma(\theta) < \infty$, concluding that h_D is integrable on Σ. $\square$

Definition 3.1.6. A Stein manifold X is called *strictly pluriregular* if there is a pseudoconvex domain E with $D \Subset E \subseteq X$ and a continuous function $f \in Psh(E)$ such that $D = \{z \in E : f(z) < 0\}$.

For an arbitrary stein manifold D, one can define the directional and principal Chebyshev constant given local coordinates h at the point $a \in D$. Let us consider a continuous plurisubharmonic function f in D such that $\{z \in D : f(z) < s\}$ for every $s \in \mathbb{N}$ and $u(a) < 1$. Take the increasing sequence of sets A_s that are connected components of $\{z \in D : f(z) < s\}$ containing the point a and $A_s \Uparrow D$. Then we give the following definition:

Definition 3.1.7. ([48]) The *directional Chebyshev constant* of a Stein manifold D with respect to the point a and with local coordinates h is defined as follows

$$\tilde{\tau}_h(a, D; \theta) := \lim_{s\to\infty} \tau_h(a, A_s; \theta) = \sup_{s\in\mathbb{N}} \tau_h(a, A_s; \theta),$$

and the *principal Chebyshev constant* is defined by

$$\tilde{\tau}_h(a, D) := \lim_{s\to\infty} \tau_h(a, A_s).$$

In the case $D \subseteq \mathbb{C}^n$ and $h(z) = z - a$, we write $\tilde{\tau}(a, D; \theta)$ and $\tilde{\tau}(a, D)$.

For a strictly pluriregular domain D in $\mathbb{C}^n$, the quantity $\tau(a, \partial D; \theta)$ and $\tau(a, \partial D)$ make sense (See [48], section 5). By [48], Theorem 12, we have the following important relation

$$\tau(a, D; \theta) = \tau(a, \partial D; \theta)^{-1}, \ \ \theta \in \Sigma^\circ,$$

where $\tau(a, \partial D; \theta)$ is the directional Chebyshev constant of ∂D viewed from the point a in the direction $\theta \in \Sigma$.

Under the above considerations, for an arbitrary domain D in $\mathbb{C}^n$, one can define the *directional Chebyshev constant of ∂D viewed from the point a*

as follows:

$$\tilde{\tau}(a, \partial D) := \tilde{\tau}(a, \partial D') = \lim_{s\to\infty} \tilde{\tau}(a, \partial V_s),$$

where $D' \supseteq D$ is the envelope of holomorphy, identified as an unbranched Riemann domain over $\mathbb{C}^n$ with a projection $\pi : D' \to \mathbb{C}^n$ such that $\pi(z) = z$ on D and $\{V_s\}_{s=1}^{\infty}$ is an increasing sequence of strictly pluriregular domains with $V_s \Uparrow D'$.

Definition 3.1.8. ([48]) Let $D \subseteq \mathbb{C}^n$ and $a \in D$ be given. The *transfinite diameter of the boundary ∂D viewed from the point $a \in D$* is the number

$$d(a, \partial D) := \limsup_{i\to\infty} (\widetilde{\mathcal{V}}_i)^{\frac{1}{l_{s(i)}}}, \tag{3.1.11}$$

where

$$\widetilde{\mathcal{V}}_i = \sup\left\{\left|\det\left(e'_\alpha(f_\beta)\right)_{\alpha,\beta=1}^{i}\right| : f_\beta \in \mathbb{B}_{H^\infty(D)}, \beta = 1, \ldots, i\right\} \tag{3.1.12}$$

is the sequence of extremal Vandermondians, $\mathbb{B}_{H^\infty(D)}$ is the closed unit ball in $H^\infty(D)$.

We end this section by quoting the following important theorem which provides an internal analogue of Fekete-Szegö relation (Theorem 1.1.1) for strictly pluriregular domains :

Theorem 3.1.9. *([48]) Let D be a strictly pluriregular domain in $\mathbb{C}^n$ and $a \in D$. Then the following holds*

$$\tau(a, \partial D) = d(a, \partial D) = (\exp \sum_{\nu=1}^{n+1} \frac{1}{\nu}) \lim_{i\to\infty} \frac{\mathcal{W}_{i,a}^{\frac{1}{\lambda_{s(i)}}}}{s(i)},$$

where $\lambda_{s(i)} := \frac{s^{n+1}}{(n-1)!(n+1)}$ and $\lambda_s \sim l_s$ as $s \to \infty$,

$$\mathcal{W}_{i,a} := \sup\{|\mathcal{W}_a((f_\nu)_{\nu=1}^{i})| : |f_\nu| \le 1,\ f_\nu \in H^\infty(D), \nu = 1, \ldots, i\},$$

and

$$\mathcal{W}_a((f_\nu)_{\nu=1}^i) = \det\left(f_\nu^{(k(\mu))}(a)\right)_{\mu,\nu=1}^i$$

is the multivariate Wronskian of the system $\{f_\nu\}_{\nu=1}^i$, evaluated at the point a.

3.2 Linear Convexity in $\mathbb{C}^n$

We recall some terminology and results from [1]. A domain $D \subseteq \mathbb{C}^n$ is called *linearly convex* if for any point $\zeta \in \partial D$, there exists a complex $(n-1)$-dimensional analytic plane passing through ζ and not intersecting D. A domain $D \subseteq \mathbb{C}^n$ is called *linearly convex in the sense of Martineau* if, through each point of the complement of D, there passes an $(n-1)$-dimensional analytic plane not touching D. Obviously every Martineau linearly convex domain is linearly convex.

The following exterior differential form will be in consideration:

$$\omega(u,z) = \frac{(n-1)!}{(2\pi i)^n (u,z)^n} \sum_{k=1}^{n} (-1)^{k-1} u_k du_1 \wedge \ldots \wedge du_{[k]} \wedge \ldots \wedge dz_1 \wedge \ldots \wedge dz_n,$$

where $(u,z) := u_1 z_1 + \ldots + u_n z_n$ and $[k]$ means that k^{th} term is omitted.

We use the map

$$\tau(\rho) = (\tau_1(\rho), \ldots, \tau_n(\rho)),$$

where

$$\tau_k(\rho) = \rho'_{z_k} \frac{1}{(\nabla\rho(z), z)} = \frac{\partial\rho}{\partial z_k} \frac{1}{(\nabla\rho(z), z)},$$

here ∇ is the holomorphic gradient operator defined by

$$\nabla\rho(z) := (\frac{\partial\rho}{\partial z_1}, \ldots, \frac{\partial\rho}{\partial z_n}).$$

A domain $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ in $\mathbb{C}^n$ is called *regular linearly* *convex* if it is linearly convex and the real function ρ is defined and twice continuously differentiable in a neighbourhood of $\overline{D}$ with $\nabla\rho \neq 0$ on ∂D. For such a domain, by Cauchy-Fantappiè formula ([1], section 8)

$$f(z) = \int_{\partial D} f(\zeta)\omega(\nabla\rho, \zeta - z) \tag{3.2.1}$$

for $f \in A_C(D)$, where $A_C(D) := A(D) \cap C(\overline{D})$, and

$$\omega(\nabla\rho, \zeta - z) = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{k=1}^n \delta_k d\overline{\zeta}_{[k]} \wedge d\zeta}{[\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n)]^n}$$

with

$$\delta_k = \begin{vmatrix} \rho'_{\zeta_1} & \ldots & \rho'_{\zeta_n} \\ \rho''_{\zeta_1\bar{\zeta}_1} & \ldots & \rho''_{\zeta_n\bar{\zeta}_1} \\ \ldots & [k] & \ldots \\ \psi''_{\zeta_1\bar{\zeta}_n} & \ldots & \psi''_{\zeta_n\bar{\zeta}_n} \end{vmatrix}, \quad k = 1, 2, \ldots, n; k^{th} \textit{ row omitted.}$$

From (3.2.1), we have, for $k = (k_1, \ldots, k_n)$

$$f^{(k)}(z) = \frac{(n+|k|-1)!}{(n-1)!} \int_{\partial D} f(\zeta) w^k \omega(\nabla\rho(\zeta), \zeta - z), \quad w = (w_1, \ldots, w_n), \tag{3.2.2}$$

where

$$w_i = \frac{\rho'_{\zeta_i}}{[\rho'_{\zeta_1}(\zeta_1 - z_1) + \ldots + \rho'_{\zeta_n}(\zeta_n - z_n)]}, \quad i = 1, \ldots, n. \tag{3.2.3}$$

Since $f \in A(D) \cap C(\overline{D})$, we have $f(z) = \sum_{k \in \mathbb{Z}_+^n} a_k z^k$ and

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{(n+|k|-1)!}{(n-1)!\, k!} \int_{\partial D} f(\zeta) w^k \omega(\nabla\rho(\zeta), \zeta), \tag{3.2.4}$$

where $k! = k_1! \ldots k_n!$.

Let D be a set in $\mathbb{C}^n$ with $0 \in D$. Then the set $\widetilde{D} = \{w \in \mathbb{C}^n : (w, z) =$

$w_1z_1 + \ldots + w_nz_n \neq 1 \; for \;\; any \;\; z \in D\}$ is called *conjugate set to* D. If D is open, then $\tilde{D}$ is compact, and conversely, if D is compact, then $\tilde{D}$ is open. One can look at, for instance, [1] for more detail about conjugate sets.

Now, assume that $D_m = \{z \in \mathbb{C}^n : \rho_m(z) < 0\}$, $m = 1, 2, , \ldots,$ are the approximation of the linearly convex domain D from inside with regular linearly convex domains. Then we have

Theorem 3.2.1. *(Martineau and Aizenberg) There is an isomorphism*

$$S : A(D)^* \to A(\widetilde{D}) \tag{3.2.5}$$

such that for each $f^* \in A(D)^*$ *and* $\psi = S(f^*)$ *we have*

$$f^*(f) =< f, \psi >:= \int_{\partial D_m} f(z)\psi(\tau(\rho_m))\omega(\nabla\rho_m(z), z), \tag{3.2.6}$$

m *depends only on* ψ.

Remark 3.2.2. By reflexivity of the space $A(D)$, one can consider the dual mapping

$$L := S^* : A(\widetilde{D})^* \to A(D)^{**} = A(D).$$

We will consider shifted domains

$$D_a := D - a = \{z - a : z \in D\}, \tag{3.2.7}$$

and the conjugate set of D_a denoted as

$$\widetilde{D_a} := (\widetilde{D - a}) = \{w \in \mathbb{C}^n : (w, z-a) = w_1(z_1-a_1)+\ldots+w_n(z_n-a_n) \neq 1 \;\; for \; all \; z \in D\}. \tag{3.2.8}$$

These amount to saying that we make a change of variable $\zeta = z - a$, where $z \in D$. We have, by letting $D_{m,a} := D_m - a$ and $\partial D_{m,a} := \partial(D_m - a)$,

$\widetilde{\rho_m}(\zeta) := \rho_m(z-a)$

$$D_a = \bigcup_{m=1}^{\infty} D_{m,a} \text{ and } \overline{D}_{m,a} \subseteq D_{m+1,a}, \tag{3.2.9}$$

where $D_{m,a} = \{\zeta \in \mathbb{C}^n : \widetilde{\rho_m}(\zeta) < 0\}$, $\widetilde{\rho_m} \in C^2$, and $\nabla\widetilde{\rho_m}(\zeta) \neq 0$ at any $\zeta \in \partial D_{m,a}$.

In (3.2.6), writing $\zeta = z - a$, we get

$$f^*(f) = \int_{\partial D_{m,a}} f(\zeta)\, \psi(\tau(\widetilde{\rho_m}(\zeta)))\, \omega(\nabla\widetilde{\rho_m}(\zeta), \zeta). \tag{3.2.10}$$

As a result of the above considerations, we have

Lemma 3.2.3. *If one takes the translated counterparts (3.2.7) and (3.2.8) into consideration in Theorem 3.2.1, then it continues to be true. There is the following isomorphism between the topological vector spaces $A(D_a)^*$ and $A(\widetilde{D_a})$*

$$U : A(D_a)^* \to A(\widetilde{D_a}) \tag{3.2.11}$$

such that for every $f^* \in A(D_a)^*$ and $\psi = U(f^*) \in A(\widetilde{D_a})$, one gets

$$f^*(f) = \int_{\partial D_{m,a}} f(\zeta)\, \psi(\tau(\widetilde{\rho_m}(\zeta)))\, \omega(\nabla\widetilde{\rho_m}(\zeta), \zeta), \tag{3.2.12}$$

where m depends only on ψ.

3.3 Internal Polya Inequality

For the time being, we will be concentrating on one variable case. Let D be a simply connected domain in $\overline{\mathbb{C}}$ and $a \in D$. The *conformal radius of D with respect to the point a* is defined as follows ([13], [29], [43])

$$r(a, D) := \frac{1}{|\omega'(a)|}, \tag{3.3.1}$$

where $\omega : D \to \mathbb{U}$ is a biholomorphic mapping such that $\omega(a) = 0$ and $\mathbb{U}$ is open unit disk in $\mathbb{C}$.

The *capacity of D relative to a point $a \in D$* is defined as $c(a, D) := \exp(-\rho(a, D))$ where $\rho(a, D) := \lim_{z\to a}(g_D(a, z) - \ln|z - a|)$ is the *Robin constant of D relative to $a \in D$* and $g_D(a, z)$ is the generalized Green function of D with the normalized (negative) logarithmic singularity at a.

If D is a simply connected domain in $\overline{\mathbb{C}}$ and $a \in D$, then $r(a, D) = c(a, D)$. Another relevant capacity is the *radius of ∂D viewed from a point $a \in D$* defined via:

$$c(a, \partial D) := \exp \rho(a, D) = \frac{1}{c(a, D)}. \tag{3.3.2}$$

If we use an appropriate biholomorphic mapping, the above capacities can be turned into the logarithmic capacity of a compact set obtained as the image of biholomorphic mapping. Define $K_a := \{\frac{1}{z-a} : z \in \overline{\mathbb{C}}\backslash D\}$. Then

$$c(a, D) = \frac{1}{c(K_a)}\,,\ \ c(a, \partial D) = c(K_a), \tag{3.3.3}$$

where $c(K_a)$ is the logarithmic capacity of the compact set K_a in $\mathbb{C}$. Due to Fekete-Szegö result, we have $c(K_a) = d(K_a)$.

Let D be a domain with $D \neq \overline{\mathbb{C}}$ and $a \in D$. The *transfinite diameter of ∂D viewed from the point a* as ([48]):

$$d(a, \partial D) := \lim_{s\to\infty} \left(\sup\{\left|\det(e_{i,a}(z_\nu))_{i,\nu=1}^{s}\right| : (z_\nu) \in (\mathbb{C}\backslash D)^s\}\right)^{\frac{2}{s(s+1)}}, \tag{3.3.4}$$

where $e_{j,a}(z) := \frac{1}{(z-a)^j}$. Change of variable with $z = a + \frac{1}{w}$ gives that

$$d(K_a) = d(a, \partial D) = c(K_a) = c(a, \partial D). \tag{3.3.5}$$

Let $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ and $a \in \mathbb{D}_R$. We have the following automorphism of $\mathbb{D}_R$ onto itself taking a to 0 :

$$\varphi_a(z) = \frac{z-a}{1-\frac{z\overline{a}}{R^2}} = \gamma, \tag{3.3.6}$$

Consider the Möbius transformation $\alpha(\gamma) = \frac{\gamma}{R}$ from $\mathbb{D}_R$ to $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Composition of φ_a with α gives an analytic bijection of $\mathbb{D}_R$ onto $\mathbb{D}_1$ mapping a to 0, that is to say, $\beta(z) := (\alpha \circ \varphi_a)(z) : \mathbb{D}_R \to \mathbb{D}_1$ by $z \to \frac{z-a}{R-\frac{z\overline{a}}{R}}$.

Since $\mathbb{D}_R$ is simply connected, $r(a, \mathbb{D}_R)$ and $c(a, \mathbb{D}_R)$ will be equal and so, by (3.3.1) and (3.3.3), we have

$$c(a, \partial\mathbb{D}_R) = \frac{1}{c(a, D)} = \frac{1}{r(a, \mathbb{D}_R)} = \frac{R}{R^2 - |a|^2}. \tag{3.3.7}$$

As we see, $c(a, \partial\mathbb{D}_R)$ depends obviously on the choice of the point $a \in \mathbb{D}_R$.

Let D be a domain in $\mathbb{C}$ and $a \in D$. Then we have

$$\begin{aligned} \widetilde{D}_a &= \{w \in \mathbb{C} : w.(z-a) \neq 1 \ for \ \ all \ \ z \in D\}, \\ &= \{w = \frac{1}{z-a} : z \in \overline{\mathbb{C}} \backslash D\}. \end{aligned} \tag{3.3.8}$$

Now let us take $D = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $a \in D$. Then

$$\begin{aligned} \widetilde{D}_a &= \{w = \frac{1}{z-a} : z \in \overline{\mathbb{C}} \backslash D\}, \\ &= \{w = \frac{1}{z-a} : |z| \geq 1\}. \end{aligned}$$

In particular if $a = 0$, we have

$$\widetilde{D} := \widetilde{D}_0 = \{w \in \mathbb{C} : |w| \leq 1\} = \overline{\mathbb{D}}. \tag{3.3.9}$$

Lastly, from the definition of the set K_a, one easily has

$$\widetilde{D}_a = K_a. \tag{3.3.10}$$

In the following theorem, we will obtain another version of Theorem 2.1.1 using internal transfinite diameter.

Theorem 3.3.1. *Let D be a domain, $h \in D$, $D \neq \overline{\mathbb{C}}$ and $f \in A(D)$ be given. Then*

$$D(f,h) := \limsup_{s\to\infty} |A_s(f;h)|^{\frac{1}{s^2}} \leq d(h, \partial D),$$

where $d(h,\partial D)$ is the internal transfinite diameter of the boundary viewed from the point h, and $A_s(f,h) = \det(c_{i+j})_{i,j=0}^{s}$ is Hankel determinant with entries extracted from the Taylor series expansion of f around the point h.

Proof. Consider the transform

$$T_h(z) = \frac{1}{z-h}, \quad z \in \overline{\mathbb{C}} \backslash D.$$

Then $T_h(\overline{\mathbb{C}}\backslash D) = K_h = \tilde{D}_h$ by (3.3.8) and (3.3.10), which is a compact set in $\mathbb{C}$. Since $f \in A(D)$, there is a Taylor series expansion around the point h

$$f(z) = \sum_{k=0}^{\infty} c_k (z-h)^k.$$

Now, by the change of variables that $z = h + \frac{1}{\zeta}$, we have $F(\zeta) = \sum_{k=0}^{\infty} c_k \frac{1}{\zeta^k}$, from which we can easily obtain the coefficients:

$$c_n = \frac{1}{2\pi i} \int_\Gamma F(\zeta)\zeta^{n-1} d\zeta, \;\; n = 0, 1, 2, ...$$

Now, for the compact set K_h, Theorem 2.1.1 gives:

$$\limsup_{s\to\infty} |A_s|^{\frac{1}{s^2}} \leq d(K_h), \;\; A_s = \det(c_{i+j})_{i,j=0}^{s}.$$

By Fekete-Szegö relation, we get $d(K_h) = c(K_h)$. By (3.3.5), $c(K_h) = c(h,\partial D) = d(h, \partial D)$, which completes the proof. □

We will now record some information from [37]. For $\Lambda = (\lambda_1, \ldots, \lambda_n) \in$

$\mathbb{C}^n$, $h = (h_1, \ldots, h_n) \in \mathbb{C}^n$, we denote by $\Lambda_h K$ the image of the compact set K under the mapping $z_i \to \lambda_i(z_i - h_i)$, $(i = 1, 2, \ldots, n)$. In particular, for $\Lambda = (\lambda, \ldots, \lambda)$, we get a homothety $\lambda_h K$ with the center at h and a homothety coefficient λ.

In [37], Sheinov came up with the theorem below, and it appears also in [38]:

Theorem 3.3.2. *Let $f(z)$ be a function that is holomorphic in a regular linearly convex region $D \subseteq \mathbb{C}^n$ which contains the origin of coordinates and such that for at least one point $h \in D$, we have $\lambda_h(\overline{D}) \subseteq D$ for any $\lambda \in (0, 1)$. Then*

$$\limsup_{s\to\infty} |G_s|^{\frac{1}{2^s\binom{s+n}{n+1}}} \le d(\widetilde{D}), \tag{3.3.11}$$

where G_s is the determinant obtained by multiplying by $k![(n-1+|k|)!]^{-1}$ each Taylor coefficient c_k in the expansion of $f(z)$ in a neighbourhood of the point h, $d(\widetilde{D})$ is the transfinite diameter of the conjugate compact set $\widetilde{D}$ to D.

It is also noted there as a remark after the above theorem that, for $n = 1$, one obtains Theorem 2.1.1 from Theorem 3.3.2. We show that Sheinov's claim is wrong by exhibiting a counterexample on the open unit disk $\mathbb{D}$ in the case $n = 1$. Let $\mathbb{D}$ be the open unit disk in $\overline{\mathbb{C}}$, $h \in \mathbb{D}$. Since $c(h, \partial\mathbb{D}) = \frac{1}{1-|h|^2}$, by (3.3.5) and (3.3.7), $d(h, \partial\mathbb{D}) = d(K_h) = \frac{1}{1-|h|^2}$. Note that, for $h \neq 0$, $d(K_h) > d(K_0)$. By (3.3.10), $K_0 = \widetilde{D}$. Therefore Sheinov's claim says that

$$\limsup_{s\to\infty} |A_s(f;h)|^{1/s^2} \le d(K_0) = d(\widetilde{D}). \tag{3.3.12}$$

Let $0 \neq h \in \mathbb{D}$. By using the sharpness result of Goluzin, i.e. the relation (2.3.3), in Theorem 3.3.1, given any $\epsilon > 0$, there exists a function $F_{\epsilon,h} \in A(\overline{\mathbb{C}} \backslash \widetilde{D_h})$ such that

$$D(F_{\epsilon,h}) > d(\widetilde{D_h}) - \epsilon. \tag{3.3.13}$$

The last remaining part to reach a contradiction with (3.3.12) is to pick a suitable $\epsilon > 0$ to conclude that $D(F_{\epsilon,h}) > d(K_0) = d(\widetilde{D})$. As we see from what we have proved above, Sheinov's claim is true only in the case $h = 0$.

In the sequel, we will obtain the internal analogue of Polya's inequality for linearly convex domains with an approximation sequence $\{D_m\}_{m=1}^{\infty}$ of regular linearly convex domains from inside. From now on, D shall be such a domain. We begin with a lemma.

Lemma 3.3.3. *If $a \in D$, then $d(a, \partial D) = d(\widetilde{D_a})$.*

Proof. By Lemma 3.2.3, it is sufficient to prove the assertion for $a = 0$. Let us see how the operator $S : A(D)^* \to A(\widetilde{D})$ acts on the system (3.1.1). By (3.2.4), S is diagonal with $\lambda_k := \frac{(n+|k|-1)!}{(n-1)!\,k!}$. We write the following

$$e_{\alpha}^{'} \to S(e_{\alpha}^{'}) := h_{\alpha},\ \alpha \in \mathbb{N}, \tag{3.3.14}$$

and by (3.2.6) we get

$$e_{\alpha}^{'}(e_{\beta}) =< e_{\beta}, h_{\alpha} > . \tag{3.3.15}$$

By Remark 3.2.2, for $e_{\beta} \in A(D)$, we have a linear continuous functional $h_{\beta}^{*} \in A(\widetilde{D})^{*}$ with $L(h_{\beta}^{*}) := e_{\beta}$ such that

$$h_{\beta}^{*}(h_{\alpha}) =< e_{\beta}, h_{\alpha} > \tag{3.3.16}$$

and also since, due to Lemma 1.3.4, $A(\widetilde{D})^{*} \hookrightarrow A(\mathbb{C}^n)^{*}$, we get, by Lemma 1.3.3, there is a corresponding germ $T(h_{\beta}^{*}) := h_{\beta}'$ such that

$$h_{\beta}^{*}(h_{\alpha}) = [h_{\alpha}, h_{\beta}'] \tag{3.3.17}$$

and from (3.3.16) and (3.3.17), we have

$$< e_{\beta}, h_{\alpha} >= [h_{\alpha}, h_{\beta}^{'}], \tag{3.3.18}$$

and thus, combining this with (3.3.15), one gets

$$e'_\alpha(e_\beta) = [h_\alpha, h'_\beta]. \tag{3.3.19}$$

Now let $f \in A(D)$ with a Taylor expansion $\sum_{\beta\in\mathbb{Z}^n_+} c_\beta e_\beta(z)$. Then Remark 3.2.2 gives that there is $f^* \in A(\widetilde{D})^*$ such that $L(f^*) = f$ and $f^*(h_\alpha) =< h_\alpha, f >$, also by Lemma 1.3.4, $f^* \in A(\widetilde{D})^* \hookrightarrow A(\mathbb{C}^n)^*$ and by Lemma 1.3.3 we have the corresponding germ $T(f^*) = f' \in A_0(\{\infty^n\})$ such that $f^*(h_\alpha) = [h_\alpha, f']$. Therefore we obtain the following

$$f^*(h_\alpha) = [h_\alpha, f'] =< h_\alpha, f >=< h_\alpha, \sum_{\beta\in\mathbb{Z}^n_+} c_\beta e_\beta > \tag{3.3.20}$$

and using (3.3.15), we get

$$f^*(h_\alpha) = \sum_{\beta\in\mathbb{Z}^n_+} c_\beta e'_\alpha(e_\beta) = e'_\alpha(\sum_{\beta\in\mathbb{Z}^n_+} c_\beta e_\beta(z)) = e'_\alpha(f). \tag{3.3.21}$$

Let us now consider the restricted functionals $g^*_\beta := f^*_\beta|_{AC(\widetilde{D})}$ and write the following Vandermondians defined in [45], section 5

$$V'_i = \sup\{\det\,(g^*_\beta(h_\alpha))^i_{\alpha,\,\beta=1} : \|g^*_\beta\|_{AC(\widetilde{D})^*} \leq 1,\ \beta = 1, \ldots, i\}. \tag{3.3.22}$$

By combining (3.1.12) and (3.3.21) with (3.3.22) above, we obtain that extremal Vandermondians in $d(0, \partial D)$ and $d(\widetilde{D})$ are equal to each other, which gives that, considering the formula (5.5) in [47] and after passing to the limit in i, $d(0, \partial D) = d(\widetilde{D})$. $\square$

Theorem 3.3.4. *Let $a \in D$, $f \in A(D)$. Then*

$$\limsup_{i\to\infty} |H_i(f)|^{\frac{1}{2l_{s(i)}}} \leq d(a, \partial D). \tag{3.3.23}$$

Here $H_i(f) = \det\,(b_{k+l})^i_{k,l=1}$ are generalized Hankel determinants, where

$b_{k+l} = \frac{1}{\lambda_{k+l}} a_{k+l}$; a_k are the coefficients of Taylor expansion of f around a and $\lambda_k = \frac{(n+|k|-1)!}{(n-1)!\,k!}$.

Proof. In the light of Lemma 3.2.3, it is enough to consider the case $a = 0$. We have $f(z) = \sum_{k \in \mathbb{Z}_+^n} a_k z^k$. By the relation (3.2.4), for $k = (k_1, \ldots, k_n)$

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{(n+|k|-1)!}{(n-1)!\,k!} \int_{\partial D_m} f\, w^k \omega(\nabla\rho(\zeta), \zeta), \tag{3.3.24}$$

where

$$w^k := w_1^{k_1} \ldots w_n^{k_n}, \tag{3.3.25}$$

with

$$w_i = \frac{\rho'_{\zeta_i}}{[\rho'_{\zeta_1}\zeta_1 + \ldots + \rho'_{\zeta_n}\zeta_n]}, \quad i = 1, \ldots, n. \tag{3.3.26}$$

It follows that, by writing $\lambda_k := \frac{(n+|k|-1)!}{(n-1)!\,k!}$

$$a_k = \frac{f^{(k)}(0)}{k!} = \lambda_k \int_{\partial D_m} f\, w^k \omega(\nabla\rho(\zeta), \zeta) = \lambda_k < f, w^k > . \tag{3.3.27}$$

By Remark 3.2.2, for $f \in A(D)$, there is $f^* \in A(\widetilde{D})^* \hookrightarrow A(\mathbb{C}^n)^*$ such that $L(f^*) = f$ and for $\varphi_k(w) := w^k$, we have $f^*(\varphi_k) =< f, \varphi_k >$. By Lemma 1.3.4, there is a corresponding germ $f' \in A_0(\{\infty^n\})$ with $f' = T(f^*)$ such that

$$f^*(\varphi_k) = [f', \varphi_k] =< f, \varphi_k >$$

By using (3.3.27), it follows that

$$a_k = \lambda_k\, [f', \varphi_k]. \tag{3.3.28}$$

Now if we write

$$b_{k(i)} := f^*(\varphi_{k(i)}) = \frac{1}{\lambda_{k(i)}} a_{k(i)} \tag{3.3.29}$$

and form the generalized Hankel determinants

$$H_i = H_i(f) := \det(b_{k(\alpha)+k(\beta)})_{\alpha,\beta=1}^{i}, \;\; i \in \mathbb{N}, \tag{3.3.30}$$

then, by Theorem 2.1.2 with $K := \widetilde{D}$, we obtain

$$D(f) := \limsup_{i\to\infty} |H_i\,(f)\,|^{\frac{1}{2l_{s(i)}}} \leq d(\widetilde{D}). \tag{3.3.31}$$

Finally, using Lemma 3.3.3 finishes the proof.

$\square$

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