DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

REPRESENTATIONS OF QUIVERS

by
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İZMİR

REPRESENTATIONS OF QUIVERS

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We have read the thesis entitled "**REPRESENTATIONS OF QUIVERS**" completed by **ÇİĞDEM YIRTICI** under supervision of **ASST. PROF. DR. MURAT ALTUNBULAK** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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REPRESENTATIONS OF QUIVERS

ABSTRACT

The goal of representation theory of quivers is to classify all representations of a given quiver Q and all morphisms between them up to isomorphism. Krull-Remak-Schmidt theorem states that every representation can be expressed in a unique way as a direct sum of indecomposable representations. By using this theorem, we obtain that it is enough to classify all indecomposable representations and morphisms between them and the obtained result provides convenience to achieve the goal of this theory. On the other hand, a well-known theorem states that the category of (finite-dimensional) representations of the quiver Q is equivalent to the category of (finitely generated) kQ-modules. This theorem is very important in representation theory since by using it, we can switch between two descriptions and use which one is convenient for our needs. Moreover, Auslander-Reiten quiver plays an essential role in this theory since it provides information about the representation theory of quivers, and also algebras. To understand Auslander-Reiten quiver more detailed, it is necessary to be familiar with Auslander-Reiten theory. Additionally, on the purpose of classifying all representations by using indecomposable representations and developing the representation theory, some special results were obtained. Brauer-Thrall conjectures, which include two conjectures and help to construct Auslander-Reiten quiver as well, are among these results. Another important result is Gabriel's theorem which is the first classification result in the representation theory of finite-dimensional algebras. Last one is Kac's theorem. In this thesis, we research the theory of representations of quivers and present the main results of this theory in consideration of above information.

Keywords: Representations of quivers, indecomposable representations, the category of representations, the category of kQ-modules, Auslander-Reiten quiver, Auslander-Reiten theory.

KUİVERLERİN TEMSİLLERİ

ÖZ

Kuiverlerin temsil teorisinin amacı, verilen bir Q kuiverinin bütün temsillerini ve onların aralarındaki bütün morfizmaları izomorfizma anlamında sınıflandırmaktır. Krull-Remak-Schmidt teoremi, her temsilin parçalanamaz temsillerin bir direkt toplamı olarak tek bir yolla ifade edildiğini belirtir. Bu teoremi kullanarak, tüm parçalanamaz temsilleri ve onların aralarındaki morfizmaları sınıflandırmanın yeterli olduğunu elde ederiz ve elde edilen sonuç, bu teorinin amacına ulaşmak için kolaylık sağlar. Bunun yanı sıra, tanınmış bir teorem, Q kuiverinin (sonlu boyutlu) temsillerinin kategorisinin (sonlu üreteçli) kQ-modüllerinin kategorisine denk olduğunu belirtir. Bu teoremi kullanarak, iki tanım arasında yer değişikliği yapabildiğimizden ve ihtiyaçlarımız için hangi tanım uygunsa onu kullabildiğimizden dolayı, bu teorem temsil teorisinde çok önemlidir. Dahası, Auslander-Reiten kuiveri, kuiverlerin temsil teorisi ve aynı zamanda cebirler hakkında bilgi sağladığından dolayı bu teoride temel rol oynar. Auslander-Reiten kuiverini daha detaylı anlamak için, Auslander-Reiten teorisine aşina olmak gereklidir. Ek olarak, parçalanamaz temsilleri kullanarak bütün temsilleri sınıflandırmak ve temsil teorisini geliştirmek amacıyla, bazı özel sonuçlar elde edilmiştir. İki hipotez içeren ve Auslander-Reiten kuiverini de inşa etmeye yarayan Brauer-Thrall hipotezleri bu sonuçlar arasındadır. Diğer önemli sonuç ise sonlu boyutlu cebirlerin temsil teorisinde ilk sınıflandırma sonucu olan Gabriel'in teoremidir. Sonuncusu ise Kac'ın teoremidir. Bu tezde, yukarıdaki bilgiler ışığında, kuiverlerin temsillerinin teorisini araştırırız ve bu teorinin ana sonuçlarını sunarız.

Anahtar kelimeler: Kuiverlerin temsilleri, parçalanamaz temsiller, temsillerin kategorisi, kQ-modüllerinin kategorisi, Auslander-Reiten kuiveri, Auslander-Reiten teorisi.

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CHAPTER ONE
INTRODUCTION

The theory of representations of quivers was started in the early seventies of the past century. It was introduced by Gabriel and has its origin in Gabriel (1972). Representations of quivers were originally introduced to treat problems of linear algebra, for example, the classification of tuples of subspaces of a prescribed vector space or the study of the examples of matrix problems with a new language. This approach leads naturally to a more sophisticated language known as categories and functors. Additionally, soon it began to play an important role in representation theory of finite-dimensional algebras. More clearly, the use of quivers in the representation theory of finite-dimensional algebras gives the possibility to visualize the modules of a given algebra as a collection of matrices, each of which is associated to an arrow in a certain diagram which is called *quiver*. Note that a representation of a quiver assigns a vector space to each vertex, and a linear map to each arrow in the quiver. The principal results of representations of quivers were obtained by Gabriel, Kac, Auslander, Reiten, Ringel, Schofield and Crawley-Boevey.

Representations of quivers have an important place in mathematics, and also in physics. By now, a number of remarkable connections to other algebraic topics have been discovered, in particular to Lie algebras, Hall algebras, quantum groups, Coxeter groups, geometric invariant theory and more recently to cluster theory which is an advanced topic in representation theory of quivers and has applications in geometry. Moreover, the theory of quiver representations is an open field of mathematics to improvement and new applications of this field on different areas arise day by day.

The goal of representation theory of quivers is to classify all representations of a given quiver Q and all morphisms between them up to isomorphism. Main motivation of this thesis comes from this aim. At this point, *Krull-Remak-Schmidt theorem* plays an important role in this thesis. Because, Krull-Remak-Schmidt theorem states that every representation can be expressed in a unique way as a direct sum of indecomposable representations. Therefore, to achieve this goal, for simplicity, it is enough to classify all indecomposable representations and morphisms between them. Hence, in this thesis, we generally study with indecomposable representations.

In this thesis, we study with two categories: the category of (finite-dimensional) quiver representations $\operatorname{rep} Q$ of a given quiver Q and the category of (finitely generated) kQ-module $\operatorname{mod} kQ$; and the second interest of this thesis is the equivalence of these two categories. To be more precise, for every quiver Q, one can associate the path algebra kQ, whose elements are finite sums of paths in the quiver and whose multiplication is given by concatenation of paths. The modules of the path algebra correspond precisely to the representations of the quiver. Thus the quiver does not give only an example of an algebra but also a very concrete model for the representation theory of the algebra. The beauty of the theory is that the quiver approach can be used to study the representation theory of arbitrary finite-dimensional algebras.

In the light of the goal of representation theory, another important notion called Auslander-Reiten quiver arises. The main tool for describing the category of finite-dimensional representations of quivers (or the category of finitely generated modules) is the Auslander-Reiten quiver which gives explicit information about the representations (or modules) as well as the morphisms between them in a most convenient way. Furthermore, the Auslander-Reiten quiver is a very good approximation of the module category. More clearly, if the number of isoclasses of the indecomposable representations of a given quiver Q is finite, then the Auslander-Reiten quiver gives the complete picture of the module category. To understand the Auslander-Reiten quiver in details, a well-known theory called Auslander-Reiten theory, which studies the representation theory of finite-dimensional algebras using techniques such as irreducible morphisms and almost split sequences, is necessary to know. The Auslander-Reiten theory was developed by Auslander & Reiten (1975, 1977a). Auslander-Reiten theory, that is the theory of irreducible morphisms and almost split sequences, has been playing a fundamental role in the modern representation theory of artin algebras. It has also an important impact to many other areas of mathematics such as algebraic geometry and algebraic topology.

By the above considerations, in this thesis, we first think the goal of representation theory, that is, we think to classify all representations of a given quiver Q and all morphisms between them up to isomorphism. Krull-Remak-Schmidt theorem helps to achieve this goal since it states that every representation can be expressed in a unique way as a direct sum of indecomposable representations. Here, we notice that studying with indecomposable representations and morphisms between them is enough for this aim. On the other hand, we focus the equivalence of the category of (finite-dimensional) representations of the quiver Q and the category of (finitely generated) kQ-modules and we see this equivalence in a theorem. This theorem is very important in representation theory since by using it, we can switch between two descriptions and use which one is convenient for our needs. For example, if we compute Auslander-Reiten quiver, it is more convenient to use the graphical notation coming from the representations of quivers, and if we use algebraic arguments, using modules is generally easier. At this point, we again emphasize the importance of the Auslander-Reiten quiver to say that why we are interested in this notion. Auslander-Reiten quiver provides information about the representation theory of quivers, and also algebras. We stated that our aim is to classify all representations of a given quiver and all morphism between them up to isomorphism. By Krull-Remark-Schmidt theorem, it is sufficient to classify all indecomposable algebras and irreducible morphisms between them. It leads to construct the Auslander-Reiten quiver whose vertices consist of indecomposable modules and arrows represent the irreducible morphisms between them.

On the purpose of classifying all representations by using indecomposable representations and developing the representation theory, many important results were obtained. For example, first one brought into life by Brauer around 1940 as exercise for his students and which remained unproved for a long time. At the origin of many recent developments of representation theory are these statements which are known as the Brauer-Thrall conjectures. Brauer-Thrall conjectures include two conjectures and help to construct Auslander-Reiten quiver as well. Another important result is

Gabriel's theorem, which is the first classification result in the representation theory of finite-dimensional algebras. Gabriel's theorem, proved by Pierre Gabriel, classifies the quivers, which has only finitely many isomorphism classes of indecomposable representations, in terms of Dynkin diagrams. It was stated that the main problem in the representation theory of quivers is to classify the indecomposable representations up to isomorphism. But, in some cases, it is too difficult since it includes the classification of all finite dimensional indecomposable representations over the complex quivers. However, Kac proved that the set of dimension vectors of indecomposable representations can be classified.

In consideration of above information, we researched the theory of representations of quivers and collected many important results. In this thesis, all of the information given above are explained in more details as follows:

In Chapter 2, necessary definitions and results from module theory, ring theory, homological algebra and category theory are given to prepare a substructure to understand this thesis more deeply. We first introduce the definitions and results on algebras and modules, and we briefly recall some of the basic facts from module theory. After that we introduce some of the basic notions related with categories and functors. Lastly, we collect some of the basic concepts from homological algebra.

In Chapter 3, we introduce the representations of quivers. More clearly, it starts with the definition of quivers and their representations and then develops the basic tools from homological algebra such as morphisms, direct sum, exact sequences, etc. After that, it introduces the concepts of simple, projective and injective representations to present indecomposable representations.

In Chapter 4, path algebras and bound quiver algebras are introduced. Also, a proof of the equivalence of the notions of modules over the bound quiver algebra and representations of the bound quiver are given. More clearly, we first define path algebras which are an important class of algebras and later we establish bound quiver algebras. Additionaly, we explain an important result which states that the category rep Q of finite-dimensional representations of the quiver Q is equivalent to the category mod kQ of finitely generated modules over the path algebra kQ.

In Chapter 5, our main objective is to introduce the Auslander-Reiten quiver. To understand the theoretical background of the Auslander-Reiten quiver, we need some definitions and properties. Therefore, step by step we give the necessary information and reach the Auslander-Reiten quiver. Thus we begin with introducing a series of notions and results named after Auslander and Reiten such that all of them create the Auslander-Reiten theory. More clearly, we first present irreducible morphisms and almost split sequences. Later, we explain the Auslander-Reiten translation. Well-known Auslander-Reiten formulas are given; and lastly, we reach the Auslander-Reiten quiver which is the main goal in this chapter.

Lastly, in Chapter 6, some deepest results in the representation theory of quivers, and also algebras are introduced. More clearly, we first present two important conjectures called the Brauer-Thrall conjectures. Later, we introduce a well-known theorem which is Gabriel's theorem and we give some necessary definitions and results for this theorem. Another important theorem called Kac's theorem is presented as well.

CHAPTER TWO
PRELIMINARIES

This chapter is devoted to introduce necessary definitions and results from module theory, ring theory, homological algebra and category theory to understand this thesis more deeply. In this chapter, all of the information to prepare a substructure for this thesis are explained. More information on ring theory can be found in (Anderson & Fuller, 1992; Dummit & Foote, 2004; Lam, 1999, 2001) and on algebras in (Assem et al., 2006; Auslander et al., 1997).

2.1 Algebras and Modules

Definition 2.1.1. A field k is called **algebraically closed** if any nonconstant polynomial $f(t)$ in one indeterminate t with coefficients in k has a root in k.

Throughout this thesis, we assume that k is algebraically closed, unless otherwise stated. Now, we give some definitions and notions from ring theory.

Definition 2.1.2. Let R be a ring with unity. A **right ideal** (respectively **left ideal**) I is a subgroup of the additive group of R such that $ar \in I$ (respectively $ra \in I$) for all $a \in I$ and $r \in R$. A **two-sided ideal** is a right ideal that is also a left ideal.

Definition 2.1.3. A proper (left, right or two-sided) ideal I in R is called **maximal** if for any (left, right or two-sided) ideal J such that $I \subset J \subset R$, we have $I = J$ or $J = R$.

In this thesis, an ideal plays an important role. Thus we now define this ideal.

Definition 2.1.4. The (Jacobson) **radical** $\operatorname{rad} R$ is the intersection of all maximal right ideals in R.

In Chapter 5, we use the following lemma to define the radical of a category. Thus it is necessary to know.

Lemma 2.1.5. *(Schiffler, 2014, Lemma 4.1) Let R be a ring and let $a \in R$. Then the following are equivalent:*

(1) $a \in \operatorname{rad} R$.

(2) For all $b \in R$*, the element* $1 - ab$ *has a right inverse.*

(3) For all $b \in R$*, the element* $1 - ab$ *has a two-sided inverse.*

(4) a lies in the intersection of all maximal left ideals in R.

(5) For all $b \in R$*, the element* $1 - ba$ *has a left inverse.*

(6) For all $b \in R$*, the element* $1 - ba$ *has a two-sided inverse.*

For more information, see Schiffler (2014).

2.1.1 Algebras

Definition 2.1.6. Let k be a field. A k-**algebra** is a ring A with unity 1 such that A has a k-vector space structure compatible with the multiplication of the ring, that is, such that

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for all $\lambda \in k$ and all $a, b \in A$.

A k-algebra A is called **finite-dimensional** if the dimension $\dim_k A$ of the k-vector space A is finite.

Definition 2.1.7. Let A and B be k-algebras. A k-linear map $f : A \to B$ is called a k-**algebra homomorphism** if $f(1) = 1$ and, for all $a, a' \in A$,

$$f(aa') = f(a)f(a').$$

Definition 2.1.8. Let A be a k-algebra. The **opposite algebra** A^{op} of A is defined to be the k-algebra whose underlying set and vector space structure are just those of A, but the multiplication $*$ in A^{op} is defined by formula $a * b = ba$.

Definition 2.1.9. An algebra A is called **local** if A has a unique maximal right ideal, or equivalently, if A has a unique maximal left ideal.

The following results are used in Chapter 4.

Lemma 2.1.10. *Let A be a finite dimensional k-algebra. Then A is a local algebra if and only if A has only two idempotents, 0 and 1.*

Proof. See Assem et al. (2006),(Lemma I.4.6). ◻

Corollary 2.1.11. *Let A be an arbitrary k-algebra and M be a right A-module. If the algebra* End *M is local, then M is indecomposable.*

Proof. See Assem et al. (2006),(Corollary I.4.8). ◻

2.1.2 Modules

Definition 2.1.12. Let R be a ring with unity. A (right) R-**module** M is an abelian group together with a binary operation $M \times R \to M$, $(m,r) \mapsto mr$, satisfying the following conditions:

(1) $(m_1 + m_2)r = m_1 r + m_2 r$,

(2) $m(r_1 + r_2) = mr_1 + mr_2$,

(3) $m(r_1 r_2) = (mr_1)r_2$,

(4) $m1 = m$,

where $m, m_1, m_2 \in M$ and $r, r_1, r_2 \in R$.

The *left* R-modules can be defined simply by multiplying the elements of the ring from the left and writing the analogous axioms.

In this thesis, an R-module M always means right R-module M, unless otherwise stated.

Definition 2.1.13. An R-module M is said to be *generated by the elements* $m_1, m_2, \cdots, m_s$ if for every $m \in M$, there exists $a_1, \cdots a_s \in R$ such that $m = m_1 a_1 + m_2 a_2 + \cdots + m_s a_s$. The module M is called **finitely generated** if it is generated by a finite subset of M.

Note that if M is generated by $m_1, m_2, \cdots, m_s$, then $M = m_1R + m_2R + \cdots + m_sR$.

Definition 2.1.14. Let M and N be two R-modules. A map $f : M \to N$ is called a **morphism** of R-modules if for all $m, m' \in M$ and all $r \in R$, the following is satisfied:

$$f(m + m') = f(m) + f(m'),$$
$$f(mr) = f(m)r.$$

Remark 2.1.15. If A is an k-algebra, then a morphism of A-modules is also a homomorphism of the underlying k-vector spaces, so a linear map.

The **kernel** of f is the set $\mathrm{Ker}\, f = \{m \in M \mid f(m) = 0\}$, the **image** of f is the set $\mathrm{Im}\, f = \{f(m) \mid m \in M\}$, and the **cokernel** of f is the set $\mathrm{Coker}\, f = N/\,\mathrm{Im}\, f$.

2.1.3 Direct Sum Decompositions

From now on, A denotes the finite-dimensional k-algebra.

Definition 2.1.16. Let $M_1, M_2, \cdots, M_s$ be A-modules. Then the **direct sum** $M_1 \oplus M_2 \oplus \cdots \oplus M_s$ is the A-module whose vector space is the direct sum of the vector spaces of the M_i and whose module structure is given by $(m_1, m_2, \cdots, m_s)a = (m_1a, m_2a \cdots m_sa)$.

Definition 2.1.17. A module is called **indecomposable** if it cannot be written as the direct sum of two proper submodules.

In the study of indecomposable modules over a k-algebra A, an important role is played by idempotent elements of A defined as follows.

Definition 2.1.18. An element $e \in A$ is called an **idempotent** if $e^2 = e$. The idempotent e is called **central** if $ae = ea$ for all $a \in A$. The idempotents $e_1, e_2 \in A$ are called **orthogonal** if $e_1e_2 = e_2e_1 = 0$. The idempotent e is called **primitive** if e cannot be written as a sum $e = e_1 + e_2$, where e_1 and e_2 are nonzero orthogonal idempotents of A.

By Assem et al. (2006, I.4), it is written that since the algebra A is finite-dimensional, the module A_A admits a direct sum decomposition

$A_A = P_1 \oplus \cdots \oplus P_n$, where $P_1, \cdots, P_n$ are indecomposable right ideals of A. Moreover, $P_1 = e_1A, \cdots, P_n = e_nA$, where $e_1, \cdots, e_n$ are primitive pairwise orthogonal idempotents of A such that $1 = e_1 + \cdots + e_n$. Conversely, every set of idempotents with the preceding properties induces a decomposition $A_A = P_1 \oplus \cdots \oplus P_n$ with indecomposable right ideals $P_1 = e_1A, \cdots, P_n = e_nA$. This decomposition is called an **indecomposable decomposition** of A and such a set $\{e_1, \cdots, e_n\}$ is called a **complete set of primitive orthogonal idempotents** of A.

We end this section with the following definitions which are necessary to know.

Definition 2.1.19. An algebra A is called **connected** if A is not a direct product of two algebras, or equivalently, if 0 and 1 are the only central idempotents of A.

Definition 2.1.20. A finite-dimensional k-algebra A is called **representation-finite** if the number of the isomorphism classes of indecomposable A-modules is finite. A k-algebra A is called **representation-infinite** if A is not representation-finite.

2.2 Concepts from Category Theory

This section is devoted to giving some definitions and notions from category theory which plays a central role in this thesis. For more detailed information, see (Assem et al., 2006; Bucur et al., 1968; Osborne, 2000).

2.2.1 Categories

Definition 2.2.1. A **category** is a triple $\mathcal{C} = (\mathrm{Ob}\,\mathcal{C}, \mathrm{Hom}\,\mathcal{C}, \circ)$, where $\mathrm{Ob}\,\mathcal{C}$ is called the **class of objects** of $\mathcal{C}$, $\mathrm{Hom}\,\mathcal{C}$ is called the **class of morphisms** of $\mathcal{C}$, and $\circ$ is binary operation called the **composition of morphisms** of $\mathcal{C}$ satisfying the following conditions:

(a) to each pair of objects X, Y of $\mathcal{C}$, we associate a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, called the **set of morphisms** from X to Y, such that if $(X,Y) \neq (Z,U)$ then the intersection of the sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(Z,U)$ is empty; and

(b) for each triple of objects X, Y, Z of C, the operation

$$\circ : \mathrm{Hom}_C(Y,Z) \times \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_C(X,Z),$$
$$(g,f) \mapsto g \circ f$$

is defined and has the following two properties:

(i) $h \circ (g \circ f) = (h \circ g) \circ f$ for every triple $f \in \mathrm{Hom}_C(X,Y)$, $g \in \mathrm{Hom}_C(Y,Z)$, $h \in \mathrm{Hom}_C(Z,U)$ of morphisms; and

(ii) for each object X of C, there exists an element $1_X \in \mathrm{Hom}_C(X,X)$ called the identity morphism on X, such that if $f \in \mathrm{Hom}_C(X,Y)$ and $g \in \mathrm{Hom}_C(Y,Z)$ then $f \circ 1_X = f$ and $1_X \circ g = g$.

Remark 2.2.2. We can write $f : X \to Y$ instead of $f \in \mathrm{Hom}_C(X,Y)$ and we say that f is a morphism from X to Y. Also, we write $X \in \mathrm{Ob}C$ to mean that X is an object of C.

Definition 2.2.3. A category C is called an **additive category** if the following conditions are satisfied:

(a) for any finite set of objects $X_1, \cdots, X_n$ of C there exists a direct sum $X_1 \oplus \cdots \oplus X_n$ in C,

(b) for each pair $X, Y \in \mathrm{Ob}C$, the set $\mathrm{Hom}_C(X,Y)$ of all morphisms from X to Y in C is equipped with an abelian group structure;

(c) for each triple of objects $X, Y, Z \in C$, the composition of morphisms in C

$$\circ : \mathrm{Hom}_C(Y,Z) \times \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_C(X,Z)$$

is *bilinear*, that is $(f + f') \circ g = f \circ g + f' \circ g$ and $f \circ (g + g') = f \circ g + f \circ g'$ for all morphisms $f, f' \in \mathrm{Hom}_C(Y,Z)$ and all morphisms $g, g' \in \mathrm{Hom}_C(X,Y)$; and

(d) there exists an object $0 \in \mathrm{Ob}C$ (called the *zero object* of C) such that the identity morphism 1_0 is the element zero of the abelian group $\mathrm{Hom}_C(0,0)$.

Definition 2.2.4. Let C be an additive category. The **opposite category** C^{op} of C is defined to be the additive category such that whose objects are the objects of C, and

$\mathrm{Hom}_{C^{op}}(X,Y) = \mathrm{Hom}_C(Y,X)$ for all objects X and Y in $\mathrm{Ob}C$; the addition in $\mathrm{Hom}_{C^{op}}(X,Y)$ is the addition in $\mathrm{Hom}_C(Y,X)$; and the composition $\circ'$ in $\mathrm{Hom}C^{op}$ is given by the formula $g \circ' f = f \circ g$, where $\circ$ is the composition in $\mathrm{Hom}C$. Clearly, $(C^{op})^{op} = C$.

Definition 2.2.5. A category C is called an **abelian category** if

(a) C is additive; and

(b) each morphism $f : X \to Y$ in C admits a kernel $u : \mathrm{Ker}\, f \to X$ of f and a cokernel $p : Y \to \mathrm{Coker}\, f$ of f and the induced morphism $\overline{f} : \mathrm{Coker}\, u \to \mathrm{Ker}\, p$ is an isomorphism.

Definition 2.2.6. Let C be an abelian category. A sequence

$$\begin{tikzcd}\cdots \arrow[r] & X_{n+1} \arrow[r, "f_n"] & X_n \arrow[r, "f_{n-1}"] & X_{n-1} \arrow[r] & \cdots\end{tikzcd}$$

in C is said to be **exact** if $\mathrm{Ker}\, f_{n-1} = \mathrm{Im}\, f_n$ for all n. Any exact sequence of the form

$$\begin{tikzcd}0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0\end{tikzcd}$$

in C is called a **short exact sequence**.

Definition 2.2.7. An abelian category C is called **abelian k-category** if C is k-category, that is, $\mathrm{Hom}C$ is a k-vector space, and the composition of morphisms is bilinear.

In this thesis, we are mainly interested in the following two classes of abelian k-categories:

(1) the category $\mathrm{rep}Q$ of finite dimensional representations of a quiver Q which is defined in the next chapter,

(2) the category $\mathrm{mod}A$ of finitely generated A-modules.

Abelian properties of $\mathrm{rep}Q$ and $\mathrm{mod}A$ can be found in (Assem et al., 2006; Schiffler, 2014).

In Chapter 5, we reformulate the definition of irreducible morphisms by using a notion from category theory. Thus we introduce the following definition for this reformulation.

Definition 2.2.8. Let C be an additive k-category. A class I of morphisms of C is called an **ideal** in C if I has the following properties:

(i) For any object X in C, the zero morphism $0_X \in \mathrm{Hom}(X,X)$ belongs to I,

(ii) If $f,g \in \mathrm{Hom}(X,Y)$ are morphisms in I and $\lambda,\mu \in k$, then $\lambda f + \mu g \in I$,

(iii) If $f,g \in \mathrm{Hom}(X,Y)$ is a morphism in I and $g \in \mathrm{Hom}(Y,Z)$ is a morphism in C, then $g \circ f \in I$, and

(iv) If $g \in \mathrm{Hom}(Y,Z)$ is a morphism in I and $f \in \mathrm{Hom}(X,Y)$ is a morphism in C, then $g \circ f \in I$.

2.2.2 Functors

Definition 2.2.9. Let C, C' be two k-categories. A **covariant functor** $\mathcal{F} : C \to C'$ is a mapping that associates

(1) to each object $X \in C$ and object $\mathcal{F}(X) \in C'$ and

(2) to each morphism $f : X \to Y$ in C a morphism $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ in C'

such that $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ and $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$, for all objects X and all morphisms f and g in C.

A **contravariant functor** $\mathcal{F} : C \to C'$ is a mapping that associates

(1) to each object $X \in C$ and object $\mathcal{F}(X) \in C'$ and

(2) to each morphism $f : X \to Y$ in C a morphism $\mathcal{F}(f) : \mathcal{F}(Y) \to \mathcal{F}(X)$ in C'

such that $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ and $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for all objects X and all morphisms f and g in C.

We often see the duality functor D in this thesis. Thus it is necessary to define this functor as follows.

Definition 2.2.10. The **duality functor**

$$D = \mathrm{Hom}_k(-,k) : \mathrm{mod}A \to \mathrm{mod}A^{op}$$

is the contravariant functor which takes a morphism $f : M \to N$ in $\mathrm{mod}A$ to the morphism $Df : DN \to DM$ in $\mathrm{mod}A^{op}$.

In this thesis, we are mainly interested in two very important functors $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$, where X is an arbitrary fixed object in the category $\mathcal{C}$. For more details, see (Osborne, 2000; Schiffler, 2014).

In order to state and prove the Auslander-Reiten formulas in Chapter 5, it is necessary to give the following two definitions:

Definition 2.2.11. Let $F,G : \mathcal{C} \to \mathcal{D}$ be two covariant functors between two categories. A **functorial morphism** $\omega : F \to G$ is a family of morphisms $\omega_L : F(L) \to G(L)$ where L runs over all objects in the category $\mathcal{C}$, such that, for any morphism $f : L_1 \to L_2$ in $\mathcal{C}$, the following diagram is commutative:

$$\begin{tikzcd}
F(L_1) \arrow[r, "F(f)"] \arrow[d, "\omega_{L_1}"'] & F(L_2) \arrow[d, "\omega_{L_2}"] \\
G(L_1) \arrow[r, "G(f)"] & G(L_2).
\end{tikzcd}$$

Dually, if $F,G : \mathcal{C} \to \mathcal{D}$ are two contravariant functors, then a **functorial morphism** $\omega : F \to G$ is a family of morphisms $\omega^L : F(L) \to G(L)$ where L runs over all objects in the category $\mathcal{C}$, such that, for any morphism $f : L_1 \to L_2$ in $\mathcal{C}$, the following diagram is commutative:

$$\begin{tikzcd}
F(L_2) \arrow[r, "F(f)"] \arrow[d, "\omega^{L_2}"'] & F(L_1) \arrow[d, "\omega^{L_1}"] \\
G(L_2) \arrow[r, "G(f)"] & G(L_1).
\end{tikzcd}$$

Definition 2.2.12. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Two functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are called **functorially isomorphic** if for every object $M \in \mathcal{C}$, there exists an isomorphism $\xi_M : F_1(M) \to F_2(M) \in \mathcal{D}$ such that for every morphism $f : M \to N$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
F_1(M) \arrow[r, "F_1(f)"] \arrow[d, "\xi_M"'] & F_1(N) \arrow[d, "\xi_N"] \\
F_2(M) \arrow[r, "F_2(f)"] & F_2(N).
\end{tikzcd}$$

And we denote it by $F_1 \cong F_2$.

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong 1_{\mathcal{C}}$ and $F \circ G \cong 1_{\mathcal{D}}$. The functor G is called a quasi-inverse functor for F.

Definition 2.2.13. Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories. A covariant additive functor $T : \mathcal{C} \to \mathcal{C}'$ is said to be **right exact** (respectively **left exact**) if, for any exact sequence

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

(respectively $\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$) in $\mathcal{C}$, the induced sequence

$$\begin{tikzcd}
T(X) \arrow[r, "T(f)"] & T(Y) \arrow[r, "T(g)"] & T(Z) \arrow[r] & 0
\end{tikzcd}$$

(respectively $\begin{tikzcd} 0 \arrow[r] & T(X) \arrow[r, "T(f)"] & T(Y) \arrow[r, "T(g)"] & T(Z) \end{tikzcd}$) in $\mathcal{C}'$ is exact. The functor T is **exact** if it is both left and right exact.

Similarly, a contravariant additive functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be **left exact** (respectively **right exact**) if, for any exact sequence

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

(respectively $\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$) in $\mathcal{C}$, the induced sequence

$$\begin{tikzcd}
0 \arrow[r] & F(Z) \arrow[r, "F(g)"] & F(Y) \arrow[r, "F(f)"] & F(X)
\end{tikzcd}$$

(respectively $\begin{tikzcd} F(Z) \arrow[r, "F(g)"] & F(Y) \arrow[r, "F(f)"] & F(X) \arrow[r] & 0 \end{tikzcd}$) in $\mathcal{C}'$ is exact.

2.3 Concepts from Homological Algebra

Throughout this section, k is a field and A is a k-algebra (not necessarily finite dimensional). Also, $\mathrm{mod}A$ denotes the category of finitely generated A-modules.

In this section, we collect basic notions and elementary facts from homological algebra needed in this thesis. In particular, we define Ext_A^n. For more information on homological algebra, see (Assem et al., 2006; Gabriel & Roiter, 1997; Osborne, 2000; Roiter, 1968).

Let $M, N, L \in \mathrm{mod}A$ and let $f \in \mathrm{Hom}_A(M,N)$ and $g \in \mathrm{Hom}_A(N,L)$. By Definition 2.2.6, the sequence

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, "g"] & L \end{tikzcd}$$

is called **exact** if $\mathrm{Ker}\, g = \mathrm{Im}\, f$. It is called a **complex** if $gf = 0$. The **homology** of the complex is defined to be the quotient $\mathrm{Ker}\, g / \mathrm{Im}\, f$. The homology measures how much the sequence differs from being exact, that is, how complex it is. Now, the following definitions are necessary to know.

Definition 2.3.1. Let $P \in \mathrm{mod}A$. P is called **projective** if for every surjective module homomorphism $f : N \twoheadrightarrow M$ and every module homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $fh = g$.

Equivalently, P is called **projective** if $\mathrm{Hom}_A(P, -)$ is an exact functor.

Definition 2.3.2. Let $M \in \mathrm{mod}A$. A **projective resolution** of M is an exact sequence of A-modules

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "f"] & M \arrow[r] & 0, \end{tikzcd}$$

where all P_i are projective.

Remark 2.3.3. Any module in $\mathrm{mod}A$ has a projective resolution, which can be assembled inductively as follows using the fact that there are enough projectives:

Choose P_0, and $f : P_0 \to M$ onto.

Choose P_1, and $d_1 : P_1 \to \operatorname{Ker} f$ onto.

Choose P_2, and $d_2 : P_2 \to \operatorname{Ker} d_1$ onto.

Etc.

Definition 2.3.4. Let A be an algebra. If each submodule of a projective module is projective, then the algebra A is called **hereditary.**

Here we define the projective and injective dimensions of a module as well as the global dimension of an algebra. Roughly speaking, the projective dimension (respectively injective dimension) measures how far a module is from being projective (respectively injective), and the global dimension measures how far an algebra is from being hereditary.

Definition 2.3.5. Let M be an A-module. The **projective dimension** pd$M$ of M is the smallest integer d such that there exists a projective resolution of the form

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

If no such resolution exists, then we say that M has infinite projective dimension.

Dually, the **injective dimension** id$M$ of M is the smallest integer d such that there exists an injective resolution of the form

$$0 \to M \to I_0 \to I_1 \to \cdot \to I_{d-1} \to I_d \to 0.$$

If no such resolution exists, then we say that M has infinite injective dimension.

Definition 2.3.6. The **global dimension** gldim$A$ of the algebra A is defined as the supremum of the projective dimensions of all A-modules, that is,

$$\operatorname{gldim} A = \sup\{\operatorname{pd} M \mid M \in \operatorname{mod} A\}.$$

Remark 2.3.7. The global dimension of A can equivalently be defined as the supremum of the injective dimensions of all A-modules.

Remark 2.3.8. (1) A module M is projective if and only if pd$M$=0.

(2) An algebra A is hereditary if and only if gldim$A \leq 1$.

Now we are able to define the functor Ext_A^n. Let $N \in \mathrm{mod}\,A$. Applying $\mathrm{Hom}_A(-,N)$ to the projective resolution of M gives

$$\begin{tikzcd}
\cdots & \mathrm{Hom}(P_1,N) \arrow[l] & \mathrm{Hom}(P_0,N) \arrow[l, "f_1"'] & 0 \arrow[l, "f_0"'] \\
\cdots & \mathrm{Hom}(P_{n+1},N) \arrow[l] & \mathrm{Hom}(P_n,N) \arrow[l, "f_{n+1}"'] & \cdots \arrow[l, "f_n"']
\end{tikzcd}$$

with $\mathrm{Hom}(M,N)$ deleted as before. Here $f_i = \mathrm{Hom}(d_i,N)$ for any i. Then the nth homology of this complex, $\mathrm{Ker}\, f_n / \mathrm{Im}\, f_{n+1}$ is $\mathrm{Ext}_A^n(M,N)$.

CHAPTER THREE

REPRESENTATIONS OF QUIVERS

In this chapter, we introduce the representations of quivers. More clearly, it starts with the definition of quivers and their representations and then develops the basic tools from homological algebra such as morphisms, direct sum, exact sequences, etc. After that, it introduces the concepts of simple, projective and injective representations.

3.1 Quivers

This first section is devoted to defining the quivers. The *quiver* means literally a box for holding arrows. This word was chosen by Gabriel (1972). Before Gabriel, the word *diagram schemes* were used instead of quivers by Grothendieck.

Definition 3.1.1. A **quiver** $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets such that Q_0 is the set of vertices, Q_1 is the set of arrows and $s, t : Q_1 \to Q_0$ are two maps which assign the starting vertex and the terminal vertex for each arrow.

An arrow is usually denoted by $i \xrightarrow{\alpha} j$ or $\alpha : i \to j$ where $i = s(\alpha)$ is its starting point and $j = t(\alpha)$ is its terminal point. Note that a quiver is a directed graph.

A quiver Q is called **finite** if Q_0 and Q_1 are finite sets. In this thesis, we always assume that Q is finite quiver, unless otherwise stated. A **subquiver** of a quiver $Q = (Q_0, Q_1, s, t)$ is a quiver $Q' = (Q'_0, Q'_1, s', t')$ such that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$ and the restrictions $s|_{Q'_1} = s'$, $t|_{Q'_1} = t'$.

Definition 3.1.2. The **underlying graph** $\overline{Q}$ of a quiver Q is obtained from Q by forgetting the orientation of the arrows. Moreover, if $\overline{Q}$ is a connected graph, the quiver Q is called **connected**.

We end this section with an example as follows.

Example 3.1.3. The following diagram is a quiver given by $Q_0 = \{1, 2, 3, 4\}$, $Q_1 =$

$\{\alpha,\beta,\gamma,\delta\}$, $s(\alpha) = 1$, $s(\beta) = 2$, $s(\gamma) = 3$, $s(\delta) = 1$ and $t(\alpha) = 2$, $t(\beta) = 3$, $t(\gamma) = 4$, $t(\delta) = 3$.

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\delta"'] & 2 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] & 4
\end{tikzcd}$$

3.2 Representations

Our objectives in this section is to define representations of quivers and to develop the basic morphisms, direct sum, exact sequences, etc. Note that representation theory of quivers aims to classify all representations of a given quiver Q and all morphisms between them up to isomorphism. To classify these, it is sufficient to look at indecomposable representations and morphisms between them. Why is it sufficient to look at indecomposable representations? *Krull-Remak-Schmidt theorem* which is given in this section answers this question.

We start with the definition of quiver representations. For this, we need a field k. In this thesis, for simplicity, we assume that k is an algebraically closed field.

Definition 3.2.1. Let Q be a quiver. A **representation** $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ of Q is a pair of two collections: the first one is the collection of k-vector spaces M_i for each vertex $i \in Q_0$ and the second one is the collection of k-linear maps $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ for each arrow $\alpha \in Q_1$.

We write a representation M "graphically" by replacing each vertex $i \in Q_0$ by the vector space M_i and each arrow $\alpha : i \to j$ by the linear map $M_\alpha : M_i \to M_j$.

A representation $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ is called **finite-dimensional** if each vector space M_i is finite-dimensional. The **dimension vector** of a representation M is the vector $(\dim M_i)_{i\in Q_0}$ of the dimensions of the vector spaces and denoted by $\underline{\dim} M$.

Example 3.2.2. Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"]
\end{tikzcd}$$

Then the following is a representation of Q,

$$\begin{tikzcd} M & k^2 \arrow[r, "{[1\ 0]}"] & k \arrow[loop right, "1"] \end{tikzcd}$$

and the dimension vector is $\underline{\dim} M = (2,1)$.

Definition 3.2.3. Let $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ and $M' = (M'_i, M'_\alpha)_{i\in Q_0,\alpha\in Q_1}$ be two representations of a quiver Q. A **morphism** (of representations) $f : M \to M'$ is a family of k-linear maps $f = (f_i : M_i \to M'_i)_{i\in Q_0}$ such that for each arrow $\alpha : i \to j$, we have $M'_\alpha f_i = f_j M_\alpha$ or, equivalently, the following diagram commutes:

$$\begin{tikzcd} M_i \arrow[r, "M_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M'_i \arrow[r, "M'_\alpha"] & M'_j \end{tikzcd}$$

A morphism $f = (f_i) : M \to M'$ is an **isomorphism** if each f_i is bijective. Also, M and M' are called **isomorphic** if there exists an isomorphism between M and M'. The class of all representations that are isomorphic to a given representation M is called the **isoclass** of M.

Example 3.2.4. Let Q be the quiver $1 \longrightarrow 2$, and let

$$\begin{tikzcd} M & k \arrow[r, "1"] & k, \end{tikzcd}$$

$$\begin{tikzcd} M' & k \arrow[r, "0"] & 0 \end{tikzcd}$$

be two representations of Q. Then the map $f = (f_1, f_2)$, where f_1 is the multiplication by $c \in k$ and f_2 is the zero map, is a morphism from M to M', that is, $M \xrightarrow{f} M'$ such that

$$\begin{tikzcd} k \arrow[r, "1"] \arrow[d, "c"'] & k \arrow[d, "0"] \\ k \arrow[r, "0"] & 0 \end{tikzcd}$$

Thus we have $\mathrm{Hom}(M, M') \cong \{(c, 0) \mid c \in k\} \cong k$.

Now, let $g = (g_1, g_2)$ be a morphism from M' to M, that is, $M' \xrightarrow{g} M$. Suppose that the diagram

$$\begin{tikzcd}
k \arrow[r, "0"] \arrow[d, "g_1"'] & 0 \arrow[d, "g_2=0"] \\
k \arrow[r, "1"] & k
\end{tikzcd}$$

commutes. Then $g_1 = 1 \circ g_1 = 0$. Thus the only morphism from M' to M is the zero morphism, that is, $g = (0,0)$. Hence, $\mathrm{Hom}(M', M) \cong 0$.

Definition 3.2.5. A **subrepresentation** of a representation $(M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is given by a tuple $(U_i)_i$ of subspaces U_i of M_i such that $M_\alpha(U_{s(\alpha)}) \subseteq U_{t(\alpha)}$ for all $\alpha \in Q_1$. Hence, we obtain a representation $(U_i, U_\alpha)_{i \in Q_0, \alpha \in Q_1}$ where $U_\alpha : U_{s(\alpha)} \to U_{t(\alpha)}$ is defined by $U_\alpha(u) = M_\alpha(u)$ for all $u \in U_{s(\alpha)}$. Alternatively, U is called a subrepresentation of M if there is an injective morphism $i : U \hookrightarrow M$.

We work with categories throughout this thesis. Thus category theory is very important for us. Necessary definitions of category theory are given in Section 2.2. More information can be found in (Assem et al., 2006; Osborne, 2000). From now on, we use the categorical language for our study.

Let Q be a quiver. Then the finite-dimensional representations of Q together with the morphisms form a category. We denote this category by $\mathrm{rep}\, Q$. So, if M is an object in $\mathrm{rep}\, Q$, then M is a finite-dimensional representation of the quiver Q.

The following remark summarize one of the fundamental points of this thesis.

Remark 3.2.6. The goal of representation theory of quivers is to classify all representations of a given quiver Q and all morphisms between them up to isomorphism. So to achieve this goal, for simplicity, it is enough to classify all indecomposable representations and morphisms between them.

Before explaining this remark, it is necessary to give the following two definitions and a well-known theorem called *Krull-Remak-Schmidt theorem*.

Definition 3.2.7. Let $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $M' = (M'_i, M'_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be representations of a given quiver Q. Then

$$M \oplus M' = \left(M_i \oplus M'_i, \begin{bmatrix} M_\alpha & 0 \\ 0 & M'_\alpha \end{bmatrix} \right)_{i \in Q_0, \alpha \in Q_1}$$

is a representation of Q. This representation is called **direct sum** of M and M'.

Example 3.2.8. Consider the representations M and M' given in the Example 3.2.4. Then the direct sum $M \oplus M'$ is the representation

$$k \oplus k \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]} k \oplus 0$$

which is isomorphic to

$$k^2 \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} k.$$

Definition 3.2.9. A representation $M \in \operatorname{rep} Q$ is called **indecomposable** if M is nonzero and M has no direct sum decomposition $M \cong L \oplus N$ where L and N are nonzero representations in $\operatorname{rep} Q$.

Example 3.2.10. The representations M and M' in Example 3.2.4 are indecomposable. But the representation

$$k^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} k^2$$

of the quiver given in Example 3.2.4 is not indecomposable since it is isomorphic to

$$(k \xrightarrow{1} k) \oplus (k \xrightarrow{1} k).$$

Theorem 3.2.11. ***Krull-Remak-Schmidt Theorem.*** *Let Q be a quiver and let $M \in$ $\operatorname{rep} Q$. Then M has a decomposition*

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t,$$

where the $M_i \in \operatorname{rep} Q$ are indecomposable for each $i = 1, \cdots, t$ and unique up to order.

Proof. Existence. Let M be indecomposable. Then, it is clear. So, suppose that M is not indecomposable. Then $M \cong M' \oplus M''$, where M' and M'' are strictly smaller dimension than M. Continuing this process, since $M \in \operatorname{rep} Q$, and the dimensions of M' and M'' are smaller than M, this process stops and we obtain $M' \cong M'_{1'} \oplus M'_{2'} \oplus \ldots \oplus M'_{t'}$, $M'' \cong M''_{1''} \oplus M''_{2''} \oplus \ldots \oplus M''_{t''}$, where all $M'_{i'}, M''_{i''}$ are indecomposable. This shows the existence of the decomposition.

Uniqueness. See Assem et al. (2006),(Theorem I.4.10). □

This theorem asserts that any representation M of a given quiver Q is written as a direct sum of indecomposable representations. So it is easier to study with indecomposable representations to classify them. Note that the category of indecomposable representations are denoted by ind$Q$.

Notice that we only mentioned on objects in rep Q. So it is natural to ask that it is really possible to recover the category rep Q by the category ind$Q$ not only with the objects, but also with the morphisms. By Ringel (2014), it is explained that it is possible because of the following:

Let $M, M' \in \operatorname{rep} Q$. Decompose them as $M = \bigoplus_{i=1}^{n} M_i$ and $M' = \bigoplus_{j=1}^{n'} M'_j$, where all M_i, M'_j are indecomposable representations. We denote the set of morphisms from M to M' by $\operatorname{Hom}(M, M')$. Then we can write $\operatorname{Hom}(M, M')$ as a kind of matrix

$$\operatorname{Hom}(M, M') = (\operatorname{Hom}(M_i, M'_j))_{ij},$$

and the composition of morphisms (this is what the real essence of the category rep Q) is given by the usual rules of matrix multiplication.

Remark 3.2.12. Representations of quivers are just the higher form of linear algebra.

We now introduce some necessary definitions and results for rep Q and end this section.

Definition 3.2.13. A sequence of morphisms

$$\begin{tikzcd} \cdots \arrow[r] & M_{n-1} \arrow[r, "f_{n-1}"] & M_n \arrow[r, "f_n"] & M_{n+1} \arrow[r, "f_{n+1}"] & M_{n+2} \arrow[r] & \cdots \end{tikzcd}$$

is called **exact** if $\operatorname{Ker} f_n = \operatorname{Im} f_{n-1}$ for any n.

In particular,

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

is called a **short exact sequence** if f is a monomorphism, g is an epimorphism and $\operatorname{Ker} g = \operatorname{Im} f$.

Example 3.2.14. Let Q be the quiver and M be the representation given in Example 3.2.4, and let $S(1)$ and $S(2)$ be the following representations:

$$\begin{tikzcd} S(1) & (1 \arrow[r] & 0), \end{tikzcd}$$

$$\begin{tikzcd} S(2) & (0 \arrow[r] & 1). \end{tikzcd}$$

Then

$$\begin{tikzcd} 0 \arrow[r] & S(2) \arrow[r, "f"] & M \arrow[r, "g"] & S(1) \arrow[r] & 0 \end{tikzcd}$$

where $f = (f_1, f_2) = (0,1)$, $g = (g_1, g_2) = (1,0)$; and

$$\begin{tikzcd} 0 \arrow[r] & S(2) \arrow[r, "f'"] & S(1) \oplus S(2) \arrow[r, "g'"] & S(1) \arrow[r] & 0 \end{tikzcd}$$

where $f' = (f'_1, f'_2) = (0,1)$, $g' = (g'_1, g'_2) = (1,0)$ are short exact sequences.

Definition 3.2.15. A morphism $f : L \to M$ is called a **section** if there exists a morphism $h : M \to L$ such that $h \circ f = 1_L$. A morphism $g : M \to N$ is called a **retraction** if there exists a morphism $h : N \to M$ such that $g \circ h = 1_N$.

Definition 3.2.16. A short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

is called **split** if f is a section.

Example 3.2.17. The first short exact sequence in Example 3.2.14 does not split since there is no nonzero morphism from M to $S(2)$ and so f cannot be a section; and the second one splits since the morphism $h' = (0,1) : S(1) \oplus S(2) \longrightarrow S(2)$ satisfies $h' \circ f' = 1_{S(2)}$.

Proposition 3.2.18. *(Schiffler, 2014, Proposition 1.8) Let*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence in $\operatorname{rep} Q$*. Then*

(i) f is a section if and only if g is a retraction.

(ii) If f is a section, then $\operatorname{Im} f (= \operatorname{Ker} g)$ *is a direct summand of M.*

Corollary 3.2.19. *(Schiffler, 2014, Corollary 1.9) If the sequence*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

is split exact, then $M \cong L \oplus N$.

3.3 Simple, Projective and Injective Representations

The aim of this section is to present an explicit computation of the simple, projective and injective representations which are the key concepts, not only in this thesis, but also in representation theory. Also, some important results are established.

Now, before giving definitions and properties of simple, projective and injective representations, we need to introduce another definition of a special notion. This one is called *paths* in a quiver. Throughout this thesis, the paths are one of the essential notions. More information are given in Chapter 4. However, for this section, it is necessary to give the definition of path.

Definition 3.3.1. Let Q be a quiver and $i, j \in Q_0$. A **path** of length $l \geq 1$ from i to j is a sequence

$$(i \mid \alpha_1, \alpha_2, \cdots \alpha_l \mid j),$$

where $\alpha_k \in Q_1$ for all $1 \leq k \leq l$, and we have $s(\alpha_1) = i$, $t(\alpha_k) = s(\alpha_{k+1})$ and $t(\alpha_l) = j$ for each $1 \leq k \leq l$. This path is shortly denoted by $\alpha_1\alpha_2 \cdots \alpha_l$ and also visualized by

$$i = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \longrightarrow \cdots \xrightarrow{\alpha_l} a_l = j.$$

The following structures are often seen in this thesis, so these are required to know.

- The path of length $l = 0$ is called **constant path** (or **trivial path**) at the vertex i and denoted by $e_i = (i \parallel i)$.
- A path given by $(i \mid \alpha_1, \alpha_2, \cdots, \alpha_l \mid i)$ where $l \geq 1$ is called an **oriented cycle**. An oriented cycle of length 1 is also called **loop**. If a quiver does not contain any oriented cycle, then it is called **acyclic.**
- An arrow $i \xrightarrow{\alpha} j$ is also a path of length 1 and denoted by $(i \mid \alpha \mid j)$.

Example 3.3.2. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"] \end{tikzcd}$$

Then $(1 \mid \alpha \mid 2)$, $(2 \mid \beta \mid 2)$, $(1 \mid \alpha, \beta, \beta \mid 2)$ are paths, but $(2 \mid \beta, \alpha \mid 1)$ is not.

Here, it is necessary to give the following definition which is necessary to know for Section 5.4.

Definition 3.3.3. Let Q be a quiver. If there exists a path from i to j, then i is said to be **predecessor** of j and j is said to be a **successor** of i. Also, if there exists an arrow $i \rightarrow j$, then i is said to be a **direct predecessor** of j, and j is said to be **direct successor** of i. For $i \in Q_0$, we denote by i^- (or i^+) the set of all direct predecessor (successor, respectively) of i. The elements of $i^+ \cup i^-$ are called the **neighbours** of i.

Now, we can start to give main parts of this section. Throughout this section, we assume that Q is an acyclic quiver.

3.3.1 Simple Representations

Definition 3.3.4. Let i be a vertex of Q and let $S(i) = (S(i)_j, S(i)_\alpha)_{j\in Q_0,\alpha\in Q_1}$ be the representation such that $S(i)$ is one-dimensional k-vector space in the vertex i and zero elsewhere, and $(S_i)_\alpha = 0$ for each arrow α of the quiver Q, that is,

$$S(i)_j = \begin{cases} k & \text{if} \quad i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S(i)_\alpha = 0 \ \text{ for all arrow } \ \alpha.$$

$S(i)$ is called the **simple representation** at vertex i.

A simple object in a category is a nonzero object and has no proper subobjects. The name of the representations $S(i)$ comes from this fact since these representations have this property. For more details, see Schiffler (2014).

Proposition 3.3.5. *Let M be a representation of Q. Then M is simple representation if and only if $M \cong S(i)$ for some $i \in Q$.*

Proof. It is clear that if $M \cong S(i)$ for some $i \in Q_0$, then M is simple. Conversely, let $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ be a simple representation. Fix a vertex $i \in Q_0$ such that $M_i \neq 0$ and $M_j = 0$, whenever there is an arrow $\alpha : i \to j$ in Q. Note that such a vertex exists since Q is acyclic. Now, let $f_i : S(i)_i \cong k \to M_i$ be an injective linear map. Extending it trivially to a morphism $f : S(i) \to M$ by letting $f_j = 0$ if $i \neq j$. f is a morphism since the following diagram commutes for all arrows $\beta : l \to i$ and $\alpha : i \to j$ in Q:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & S(i)_i \arrow[r] \arrow[d, "f_i"'] & 0 \arrow[d] \\
M_l \arrow[r, "M_\beta"] & M_i \arrow[r, "M_\alpha"] & 0
\end{tikzcd}$$

This shows that $S(i)$ is a subrepresentation of M since f is injective. Therefore $S(i) \cong M$. □

This proposition proves that $\{S(i) \mid i \in Q_0\}$ is a complete set of simple representations in $\operatorname{rep} Q$, up to isomorphism.

Remark 3.3.6. This proposition is satisfied only for an acyclic quiver Q. Because a quiver with oriented cycles has infinitely many pairwise nonisomorphic simple representations of finite dimension, distinct from the representations $S(i)$ with $i \in Q_0$.

For example, let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

Then

$$\begin{tikzcd} S(1) & k \arrow[r, shift left, "0"] & 0 \arrow[l, shift left, "0"] \ , \end{tikzcd}$$

$$\begin{tikzcd} S(2) & 0 \arrow[r, shift left, "0"] & k \arrow[l, shift left, "0"] \ , \end{tikzcd}$$

and also

$$\begin{tikzcd} S_\lambda & k \arrow[r, shift left, "1"] & k \arrow[l, shift left, "\lambda"] \ , \end{tikzcd}$$

with $\lambda \in k \setminus \{0\}$, are all simple. Observe that $S_\lambda \ncong S_\mu$ whenever $\lambda \neq \mu$. For more details, see Assem et al. (2006).

3.3.2 Projective and Injective Representations

Projective and injective representations are one of the central concepts in representation theory. These representations help to attain the goal of representation theory. Therefore, this part is devoted to giving information about these important representations.

Definition 3.3.7. Let i be a vertex of Q and let $P(i) = (P(i)_j, P(i)_\alpha)_{j \in Q_0, \alpha \in Q_1}$ be the representation such that $P(i)_j$ is the k-vector space with basis the set of all paths from i to j in Q and, for an arrow $\alpha : j \to l$ in Q, $P(i)_\alpha : P(i)_j \to P(i)_l$ is the k-linear map defined on the basis by composing the paths from i to j with the arrow $\alpha : j \to l$. That is, if $\omega = (i \mid \alpha_1, \alpha_2, \cdots, \alpha_s \mid j)$ is a basis vector of $P(i)_j$, then the composition of ω with the arrow $\alpha : j \to l$ is $\omega\alpha = (i \mid \alpha_1, \alpha_2, \cdots \alpha_s, \alpha \mid l)$. We write the elements of $P(i)_j$ as

$\sum_\omega \lambda_\omega \omega$ where $\lambda_\omega \in k$. Then $P(i)_\alpha$ is defined by $P(i)_\alpha(\sum_\omega \lambda_\omega \omega) = \sum_\omega \lambda_\omega \omega\alpha$. $P(i)$ is called the **projective representation** at vertex i.

The following remark is necessary to know.

Remark 3.3.8. Let $P(i) = (P(i)_j, P(i)_\alpha)_{j\in Q_0, \alpha\in Q_1}$ be the projective representation at vertex i and let ω be a path starting at i given by

$$\omega = (i \mid \alpha_1, \alpha_2, \cdots, \alpha_s \mid j).$$

Then we can define the map

$$P(i)_\omega : P(i)_i \to P(i)_j$$

as the composition of the maps in the representation $P(i)$ along the path ω, that is, $P(i)_\omega = P(i)_{\alpha_s} \cdots P(i)_{\alpha_2} P(i)_{\alpha_1}$. If e_i denotes the constant path at vertex i, then by the definition of $P(i)$, we obtain that

$$P(i)_\omega(e_i) = \omega.$$

Definition 3.3.9. Let i be a vertex of Q and let $I(i) = (I(i)_j, I(i)_\alpha)_{j\in Q_0, \alpha\in Q_1}$ be the representation such that $I(i)_j$ is the k-vector space with basis the set of all paths from j to i in Q and, for an arrow $\alpha : j \to l$ in Q, $I(i)_\alpha : I(i)_j \to I(i)_l$ is the k-linear map defined on the basis by deleting the arrow $\alpha : j \to l$ from those paths j to i that start with α and sending to zero the paths that do not start with α. That is, if $\omega = (j \mid \alpha_1, \alpha_2, \cdots, \alpha_s \mid i)$ is a basis vector of $I(i)_j$, then there exists a surjective map f between $I(i)_j$ and $I(i)_l$ such that $f(\omega) = (l \mid \alpha_2, \cdots \cdot \alpha_s \mid i)$ if $\alpha_1 = \alpha$, and $f(\omega) = 0$ otherwise. We write the elements of $I(i)_j$ as $\sum_\omega \lambda_\omega \omega$ where $\lambda_\omega \in k$. Then $I(i)_\alpha$ is defined by $I(i)_\alpha(\sum_\omega \lambda_\omega \omega) = \sum_\omega \lambda_\omega f(\omega)$. $I(i)$ is called the **injective representation** at vertex i.

Remark 3.3.10. A vertex in Q is called *sink* if there is no arrow α in Q such that $s(\alpha) = i$.

Then, it is observed that

$$S(i) = P(i) \Leftrightarrow i \text{ is a sink in } Q.$$

Also, a vertex in Q is called *source* if there is no arrow α in Q such that $t(\alpha) = i$. Then, it is observed that

$$S(i) = I(i) \Leftrightarrow i \text{ is a source in } Q.$$

Definition 3.3.11. Let $P(i) = (P(i)_j, P(i)_\alpha)_{j\in Q_0,\alpha\in Q_1}$ be the projective representation at vertex i. The **radical** of $P(i)$ is the representation $\operatorname{rad} P(i) = (R_j, R_\alpha)_{j\in Q_0,\alpha\in Q_1}$ defined as follows: $R_i = 0$, $R_j = P(i)_j$ for $i \neq j$, and $R_\alpha = 0$ if $s(\alpha) = i$, and $R_\alpha = P_\alpha$ if $s(\alpha) \neq i$.

Example 3.3.12. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 & 3 \arrow[l, "\beta"'] \end{tikzcd}$$

The indecomposable projective representations of Q are given by

$$P(1) \cong (k \longrightarrow k \longleftarrow 0)$$

$$P(2) \cong (0 \longrightarrow k \longleftarrow 0)$$

$$P(3) \cong (0 \longrightarrow k \longleftarrow k)$$

and also note that $P(2) \cong S(2) \cong \operatorname{rad} P(1) \cong \operatorname{rad} P(3)$ and $\operatorname{rad} P(2) = 0$. The indecomposable injective representations of Q are given by

$$I(1) \cong (k \longrightarrow 0 \longleftarrow 0)$$

$$I(2) \cong (k \longrightarrow k \longleftarrow k)$$

$$I(3) \cong (0 \longrightarrow 0 \longleftarrow k)$$

and also note that $I(1) \cong S(1)$ and $I(3) \cong S(3)$.

We can also denote the above representations as follows:

$$P(1) = \begin{matrix}1\\2\end{matrix} \qquad P(2) = \begin{matrix}2\end{matrix} \qquad P(3) = \begin{matrix}3\\2\end{matrix}$$

and

$$I(1) = 1 \qquad I(2) = \begin{matrix}1 & & 3\\ & 2 & \end{matrix} \qquad I(3) = 3.$$

We write these last notations by considering the shape of the quiver and the dimension vectors of the corresponding representations.

In category theory, a projective object is an object P such that the functor $\mathrm{Hom}(P,-)$ maps surjective morphisms to surjective morphisms. Dually, an injective object is an object I such that the functor $\mathrm{Hom}(-,I)$ maps injective morphisms to surjective morphisms. The following two propositions 3.3.13 and 3.3.14 show that the projective and injective representations satisfy these conditions, respectively. The names of these representations come from this fact.

Proposition 3.3.13. *(Schiffler, 2014, Proposition 2.3) Let $g : M \to N$ be a surjective morphism between representations of Q, and let $P(i)$ be the projective representation at vertex i. Then the map*

$$g_* : Hom(P(i),M) \to Hom(P(i),N)$$

is surjective.

In other words, if $f : P(i) \to N$ is any morphism, then there exists a morphism $h : P(i) \to M$ such that the following diagram

$$\begin{tikzcd}
& P(i) \arrow[dl, "h"'] \arrow[d, "f"] & \\
M \arrow[r, "g"'] & N \arrow[r] & 0
\end{tikzcd}$$

commutes, that is, $f = g \circ h = g_*(h)$.

Dually,

Proposition 3.3.14. *(Schiffler, 2014, Proposition 2.5) Let $g : L \to M$ be a injective morphism between representations of Q, and let $I(i)$ be the injective representation at vertex i. Then the map*

$$g^* : Hom(M, I(i)) \to Hom(L, I(i))$$

is surjective.

In other words, if $f : L \to I(i)$ is any morphism, then there exists a morphism $h : M \to I(i)$ such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "g"] \arrow[d, "f"'] & M \arrow[dl, "h"] \\
 & I(i) &
\end{tikzcd}$$

commutes, that is, $f = h \circ g = g^*(h)$.

Corollary 3.3.15. *If P is projective, then any exact sequence of the form*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{g} P \longrightarrow 0$$

splits.

Proof. It follows directly by choosing $f = 1_P$ as the identity morphism in Proposition 3.3.13. For more details, see Schiffler (2014),(Corollary 2.4). □

Dually,

Corollary 3.3.16. *If I is injective, then any exact sequence of the form*

$$0 \longrightarrow I \xrightarrow{g} M \longrightarrow N \longrightarrow 0$$

splits.

Proof. It follows directly by choosing $f = 1_I$ as the identity morphism in Proposition 3.3.14. For more details, see Schiffler (2014),(Corollary 2.6). $\square$

Now, we give a well-known proposition from category theory. This proposition holds for any additive category. But here our study deals with $\operatorname{rep} Q$, so we introduce for this category.

Proposition 3.3.17. *(Schiffler, 2014, Proposition 2.7)*

(i) Let P, P_1 and P_2 be representations of Q such that $P = P_1 \oplus P_2$. Then P is projective if and only if P_1 and P_2 are projective.

(ii) Let I, I_1 and I_2 be representations of Q such that $I = I_1 \oplus I_2$. Then I is injective if and only if I_1 and I_2 are injective.

From Proposition 3.3.17, we obtain that if we know the indecomposable projective and injective representations, then we know all projective and injective representations, respectively. Moreover, in this chapter, we prove that all indecomposable projective and injective representations for any acyclic quiver are known.

Proposition 3.3.18. *The representations $S(i), P(i)$ and $I(i)$ are indecomposable.*

Proof. Since $S(i)$ is simple, the result comes from the definition of $S(i)$ directly. Now, we prove it for $P(i)$. Let $P(i) = (P(i)_j, P(i)_\alpha)_{i \in Q_0, \alpha \in Q_1}$. We have $P(i)_i = k$, since Q is acyclic. Suppose that $P(i) = M \oplus N$ for some $M, N \in \operatorname{rep} Q$. We may suppose without loss of generality $P(i)_i = M_i$ and $N_i = 0$. Now let $j \in Q_0$ be such that $N_j \neq 0$. Then $P(i)_j$ has a basis consisting of the paths from i to j in Q. Let $\omega = (i \mid \alpha_1, \cdots, \alpha_s \mid j)$ be such a path. Then $P(i)_\omega = P(i)_{\alpha_s} \cdots P(i)_{\alpha_1}$ denote the composition of the linear maps of the representation $P(i)$ along the path ω. Then, since $P(i) = M \oplus N$, the map

$$P(i)_\omega = M_i \oplus 0 \to M_j \oplus N_j$$

sends e_i to $P(i)_\omega(e_i) \in M_j$, where e_i is the unique basis element of M_i. By Remark 3.3.8, we have $P(i)_\omega(e_i) = \omega$. Hence, every basis element ω of $P(i)_j$ lies in M_j. This is a contradiction.

The proof for $I(i)$ is similar. $\square$

The following important theorem states that the vector space at vertex i of any representation can be described as a space of morphisms using the projective representation $P(i)$.

Theorem 3.3.19. *Let $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ be a representation of Q. Then, there is an isomorphism of vector space*

$$\mathrm{Hom}(P(i), M) \cong M_i,$$

for any vertex i in Q.

Proof. Let e_i be the constant path at i. Then $\{e_i\}$ is a basis of the vector space $P(i)_i$. The map between $\mathrm{Hom}(P(i), M)$ and $\cong M_i$ is defined by

$$\begin{aligned} \phi : \mathrm{Hom}(P(i), M) &\to \cong M_i \\ f = (f_j)_{j\in Q_0} &\mapsto f_i(e_i). \end{aligned}$$

For more details and continuation of the proof, see Schiffler (2014),(Theorem 2.11). $\square$

The next remark is an immediate consequence of the last theorem.

Remark 3.3.20. The vector space $\mathrm{Hom}(P(i), P(j))$ has a basis consisting of all paths from j to i in Q since $\mathrm{Hom}(P(i), P(j)) \cong P(j)_i$. In particular,

$$\mathrm{End}(P(i)) = \mathrm{Hom}(P(i), P(i)) \cong P(i)_i \cong k.$$

We end this section with the following corollary.

Corollary 3.3.21. *(Schiffler, 2014, Corollary 2.14) The representation $P(j)$ is a simple representation if and only if* $\mathrm{Hom}(P(i), P(j)) = 0$ *for all $i \neq j$.*

Proof. By Remark 3.3.10, the representation $P(j)$ is simple if and only if j is a sink, that is, there are no paths from j to any other vertex i. The statement follows from Remark 3.3.20. ◻

3.3.3 Projective and Injective Resolutions

The aim of this part is to introduce another way of describing arbitrary representations by means of projective representations, that is, the projective resolutions. And dually, injective resolutions.

Definition 3.3.22. Let M be a representation of Q. Let P_i be a projective representation. An exact sequence

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is called a **projective resolution** of M. Let I_i be a injective representation. An exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots$$

is called an **injective resolution** of M.

Theorem 3.3.23. *Let M be a representation of Q.*

(i) There exists a projective resolution of the form

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

(ii) There exists an injective resolution of the form

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0.$$

Proof. The proof of the part (i) of this theorem is given in Schiffler (2014),(Theorem 2.15). But here, it is necessary to know that there is a construction in the proof, and this construction gives the following:

Let $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where d_i denotes the dimension of M_i, and

$$P_1 = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P(t(\alpha)), \qquad P_0 = \bigoplus_{i \in Q_0} d_i P(i),$$

where $d_i P(i)$ stands for the direct sum of d_i copies of $P(i)$.

Part (*ii*) is similar. □

These projective (injective) resolution given in the last theorem is called *standart projective (injective) resolution* of M, respectively.

Example 3.3.24. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 & 3 \arrow[l, "\beta"'] \end{tikzcd}$$

given in 3.3.12, and M be the injective representation at vertex 2, that is,

$$M = I(2) = \begin{smallmatrix} 1 & & 3 \\ & 2 & \end{smallmatrix}.$$

Consider the construction given in the proof of Theorem 3.3.23. Then $d_1 = 1$, $d_2 = 1$ and $d_3 = 1$. Thus,

$$P_0 = d_1 P(1) \oplus d_2 P(2) \oplus d_3 P(3) = P(1) \oplus P(2) \oplus P(3).$$

Then, we obtain

$$P_0 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}.$$

Also, $d_{s(\alpha)} = d_1 = 1$, $d_{s(\beta)} = d_3 = 1$, and $P(t(\alpha)) = P(2) = P(t(\beta))$. Thus,

$$P_1 = d_1 P(2) \oplus d_3 P(2) = P(2) \oplus P(2).$$

Then we obtain

$$P_1 = 2 \oplus 2.$$

Thus, the standard projective resolution of M is

$$\begin{tikzcd}
0 \arrow[r] & 2\oplus 2 \arrow[r] & \substack{1\\2}\oplus 2\oplus \substack{3\\2} \arrow[r] & \substack{1\ \ 3\\2} \arrow[r] & 0.
\end{tikzcd}$$

Definition 3.3.25. Let M be a representation of Q. A projective representation P together with a surjective morphism $g : P \to M$ is called a **projective cover** of M if whenever $g' : P' \to M$ is surjective morphism with P' is projective representation, then there exists a surjective morphism $h : P' \twoheadrightarrow P$ such that $gh = g'$.

Definition 3.3.26. Let M be a representation of Q. An injective representation I together with an injective morphism $f : M \to I$ is called an **injective envelope** of M if whenever $f' : M \to I'$ is injective morphism with I' is injective representation, then there exists an injective morphism $h : I \hookrightarrow I'$ such that $hf = f'$.

Remark 3.3.27. Projective covers are unique up to isomorphism. For more information, see Schiffler (2014),(Proposition 2.18).

Definition 3.3.28. Let $A = \oplus_{i\in Q_0} P(i)$. A representation $F \in \operatorname{rep} Q$ is called **free** if $F \cong A \oplus \cdots \oplus A$.

Proposition 3.3.29. *(Schiffler, 2014, Proposition 2.20) A representation $M \in \operatorname{rep} Q$ is projective if and only if there exists a free representation $F \in \operatorname{rep} Q$ such that M is isomorphic to a direct summand of F.*

Corollary 3.3.30. *Let P be a projective representation in $\operatorname{rep} Q$. Then $P \cong P(i_1) \oplus \cdots \oplus P(i_t)$ with $i_1, \cdots i_t$ not necessarily distinct.*

Proof. This is a direct consequence of Proposition 3.3.29. □

Consequently, the only indecomposable projective and injective representations are $P(i)$ and $I(i)$, respectively, by Proposition 3.3.18 and Corollary 3.3.30. The next aim of this section is to state that subrepresentations of projective representations are projective in $\operatorname{rep} Q$. The following theorem are introduced for this aim.

Theorem 3.3.31. *Subrepresentations of projective representations in $\operatorname{rep} Q$ are projective.*

Proof. See Schiffler (2014),(Theorem 2.24). □

Remark 3.3.32. Categories which have the property given in Theorem 3.3.31 are called **hereditary**.

As a consequence of Theorem 3.3.31, we obtain the following result:

Corollary 3.3.33. *Let M be an indecomposable, P be a projective representation and $f : M \to P$ be a nonzero morphism in* $\operatorname{rep} Q$*. Then M is projective and f is injective.*

Proof. Since $\operatorname{Im} f$ is a subrepresentation of P, it is projective by Theorem 3.3.31. Therefore, the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r] & M \arrow[r] & \operatorname{Im} f \arrow[r] & 0 \end{tikzcd}$$

splits by Corollary3.3.15, and also Proposition 3.2.18 implies that $\operatorname{Im} f$ is isomorphic to a direct summand of M. But M is indecomposable, so $M \cong \operatorname{Im} f$ is projective and $\operatorname{Ker} f = 0$, that is, f is injective. □

This corollary plays an important role when the Auslander-Reiten quiver of Q is constructed since it suggests that we must start with the projective representations and that the projective representations are partially ordered by inclusion.

CHAPTER FOUR
PATH ALGEBRAS

This chapter introduces path algebras, bound quiver algebras and also give a proof of the equivalence of the notions of modules over the bound quiver algebra and representations of the bound quiver. More clearly, first section is devoted to defining path algebras which are an important class of algebras. In second section, bound quiver algebras are established. In third section, one of the main results in this thesis are explained. Briefly, this result states that the category $\operatorname{rep} Q$ of finite-dimensional representations of the quiver Q is equivalent to the category $\operatorname{mod} kQ$ of finitely generated modules over the path algebra kQ.

4.1 Path Algebras

In Section 3.3, the definition of paths are given. It is clear that the composition of paths is a partially defined operation on the set of all paths in a quiver. It is used to define an algebra as follows.

Definition 4.1.1. Let Q be a quiver. The **path algebra** kQ of Q is the k-algebra that has basis the set of all paths in the quiver Q and such that the multiplication of two basis $\omega = (i \mid \alpha_1,\alpha_2,...,\alpha_s \mid j)$ and $\omega' = (l \mid \beta_1,\beta_2,...,\beta_t \mid k)$ of kQ defined by

$$\omega\omega' = (i \mid \alpha_1,\alpha_2,...,\alpha_s \mid j)(l \mid \beta_1,\beta_2,...,\beta_t \mid k) = \delta_{jl}(i \mid \alpha_1,\alpha_2,...,\alpha_s,\beta_1,\beta_2,...,\beta_t \mid k)$$

where δ_{jl} is the Kronecker delta. That is, the multiplicatication of two basis vectors ω and ω' of kQ is equal to zero if $t(\alpha_s) \neq s(\beta_1)$ and is equal to composed path $\omega.\omega'$ if $t(\alpha_s) = s(\beta_1)$.

Thus the product of arbitrary elements $\sum_{\omega} \lambda_{\omega}\omega$, $\sum_{\omega'} \lambda_{\omega'}\omega'$ of kQ is defined by $\sum_{\omega\omega'} \lambda_{\omega}\lambda_{\omega'}\omega\omega'$, where $\lambda_{\omega}, \lambda_{\omega'} \in k$.

Also, there is a direct sum decomposition

$$kQ = kQ_0 \oplus kQ_1 \oplus kQ_2 \oplus \cdots \oplus kQ_l \oplus \cdots$$

of the k-vector space kQ for each $l \geq 0$ where kQ_l is the subspace of kQ which is generated by all paths of length l.

Lemma 4.1.2. *Let kQ be the path algebra of a given quiver Q. Then the unity element is given by*

$$1 = \sum_{i \in Q_0} e_i$$

where e_i is the constant path.

Proof. Let $a \in kQ$. Then we write $a = \sum_{\omega} \lambda_\omega \omega$ for some $\lambda_\omega \in k$. Then $a \sum_{i \in Q_0} e_i = \sum_{\omega} \lambda_\omega \omega \sum_{i \in Q_0} e_i = \sum_{i \in Q_0} \sum_{\omega} \lambda_\omega \omega$ and $\omega e_i = 0$ if the path ω does not end in the vertex i, and $\omega e_i = \omega$ if the path ω ends in i. Hence $a \sum_{i \in Q_0} = \sum_{i \in Q_0} \sum_{\omega : t(\omega) = i} \lambda_\omega \omega = \sum_{\omega} \lambda_\omega \omega = a$. Similarly, we can show that $\sum_{i \in Q_0} e_i a = a$, and the lemma is proved. □

Example 4.1.3. Let Q be the quiver

$$\begin{tikzcd} i \arrow[loop right, "\alpha"] \end{tikzcd}$$

Then the basis of the path algebra kQ is $\{e_1, \alpha, \alpha^2, \cdots, \alpha^l, \cdots\}$ and the multiplication of basis vectors is given by $e_1 \alpha^l = \alpha^l e_1 = \alpha^l$ for all $l \geq 0$, and $\alpha^l \alpha^k = \alpha^{l+k}$ for all $l, k \geq 0$ where $\alpha^0 = e_1$. It follows that kQ is isomorphic to the algebra of polynomials $k[x]$ such that the isomorphism is obtained by the k-linear map such that

$$e_1 \mapsto 1 \quad \text{and} \quad \alpha \mapsto x.$$

Example 4.1.4. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3. \end{tikzcd}$$

The basis of the path algebra kQ is $\{e_1, e_2, e_3, \alpha, \beta, \alpha\beta\}$. Thus, the path algebra kQ is isomorphic to the $3x3$ upper triangular matrix algebra where the isomorphism is induced by the k-linear map such that

$$e_1 \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\alpha \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \beta \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \alpha\beta \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We end this section with the following important results which are often used in this thesis.

Lemma 4.1.5. *(Assem et al., 2006, Lemma II.1.4) Let Q be a quiver and kQ be its path algebra. Then*

(i) kQ is an associative algebra,

(ii) kQ has an identity element if and only if Q_0 is finite, and

(iii) kQ is finite dimensional if and only if Q is finite and acyclic.

Corollary 4.1.6. *(Assem et al., 2006, Corollary II.1.5) Let Q be a finite quiver. The element $1 = \sum_{i \in Q_0} e_i$ is the identity element of kQ and the set of constant paths $\{e_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents for kQ.*

Note that, if Q is a finite connected quiver, then the path algebra kQ of Q is a connected k-algebra by Assem et al. (2006),(Lemma II.1.7).

4.2 Admissible Ideals and Quotients of the Path Algebras

In previous chapter, we assumed that Q is an acyclic quiver. But here, Q does not need to be acyclic. So, from now on, we let Q be any quiver. Throughout this section, the definitions and properties of admissible ideals and the quotients of the path algebras are given. These definitions and properties help to define the bound quiver algebras of a given quiver. A bound quiver algebra of the quiver Q is the quotient of a path algebra

kQ by an ideal I which is required to satisfy a certain admissibility condition. We prove in Section 4.3 that the category modkQ/I is equivalent to the category repQ/I of all those representations of the quiver Q which satisfy the relations induced by the ideal I. Thus, the bound quiver algebras play an important role in representation theory.

4.2.1 Bound Quiver Algebras

Let Q be a finite quiver, and kQ be its path algebra. If Q has oriented cycles, then kQ is infinite-dimensional. We want to have finite-dimensional algebras here. Thus we take the quotient of the path algebra kQ by certain ideals, and obtain finite-dimensional algebras. These certain ideals have some special properties which lead to the concept of admissible ideals. Before defining the admissible ideals, it is necessary to introduce a special ideal as follows.

Definition 4.2.1. Let Q be finite quiver. The two-sided ideal of the path algebra kQ generated (as an ideal) by all arrows in Q is called the **arrow ideal** of kQ and is denoted by R_Q.

Note that there is a direct sum decomposition as a k-vector space

$$R_Q = kQ_1 \oplus kQ_2 \oplus \cdots \oplus kQ_l \oplus \cdots$$

where kQ_l is the subspace of kQ generated by the set Q_l of all paths of length l. The l-th power of the arrow ideal can be written as

$$R_Q^l = \bigoplus_{m \geq l} kQ_m$$

and so R_Q^l is the ideal of kQ generated (as a k-vector space) by the set of all paths of length greater or equal to l.

Now, we can give the definition of admissible ideals.

Definition 4.2.2. Let Q be a finite quiver and R_Q be the arrow ideal of the path algebra kQ. A two-sided ideal I of kQ is called an **admissible ideal** if there exists an integer

$m \geq 2$ such that

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

Example 4.2.3. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop right, "\beta"] \end{tikzcd}$$

Then the ideal $I = \langle \alpha\beta^2, \beta^3 \rangle$ is admissible.

To observe that let us take $m = 3$. Then any path of length greater or equal to 3 contains β^3 or $\alpha\beta^2$ as a subpath. Thus $R_Q^3 \subset I$. Also, it is clear that $I \subset R_Q^2$ since the generators of I are of length 3.

We are now able to introduce the main definition of this section.

Definition 4.2.4. Let Q be a finite quiver and I be an admissible ideal of the path algebra kQ. The pair (Q,I) is called **bound quiver** and the quotient algebra kQ/I is called **bound quiver algebra**.

Note that for a k-algebra A with a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents, the algebra A is called **basic** if $e_iA \ncong e_jA$ for all $i \neq j$. Also, if A is not basic, define $e_A = e_{s_1} + \cdots + e_{s_t}$ to be the sum of a maximal set of primitive orthogonal idempotents such that $e_{s_i}A \cong e_{s_j}A$ if and only if $i = j$. By Assem et al. (2006, I.6), the algebra e_AAe_A is basic and that the module categories of A and e_AAe_A are equivalent. Thus, from the point of view of representation theory, it suffices to consider only basic algebras. Moreover, we have the following theorem.

Theorem 4.2.5. *Let A be a finite-dimensional basic and connected k-algebra. Then there exists an admissible ideal I of kQ such that $A \cong kQ/I$.*

Proof. See Assem et al. (2006, II.3.7). □

Now, we define an admissible ideal in terms of its generators. These are called *relations*. Before giving the definition of relation, it is necessary to know that two paths ω, ω' in Q are called **parallel** if $s(\omega) = s(\omega')$ and $t(\omega) = t(\omega')$.

Definition 4.2.6. Let Q be a quiver. A **relation** ρ in Q is a linear combination $\rho = \sum_{\omega} \lambda_{\omega}\omega$ of parallel paths each of which has length at least two.

4.2.2 Projective Representations of Bound Quiver Algebras

In Section 3.3, we defined the indecomposable projective representations $P(i)$ and the indecomposable injective representations $I(i)$ for a quiver without relations. Now, in this section, we define these indecomposable representations for bound quivers.

Let (Q, I) be a bound quiver and $A = kQ/I$ be its bound quiver algebra.

Definition 4.2.7. Let i be a vertex of Q and let $P(i) = (P(i)_j, P(i)_{\alpha})_{j\in Q_0, \alpha\in Q_1}$ be the representation such that $P(i)_j$ is the k-vector space with basis the set of all residue classes $\omega + I$ of paths ω from i to j in Q; and, for an arrow $\alpha : j \to l$ in Q, $P(i)_{\alpha} : P(i)_j \to P(i)_l$ is the k-linear map defined on the basis by composing the paths from i to j with the arrow $\alpha : j \to l$, that is, $P(i)_{\alpha}(\omega + I) = \omega\alpha + I$. $P(i)$ is called the **projective representation** of (Q, I) at vertex i.

Definition 4.2.8. Let i be a vertex of Q and let $I(i) = (I(i)_j, I(i)_{\alpha})_{j\in Q_0, \alpha\in Q_1}$ be the representation such that $I(i)_j$ is the k-vector space with basis the set of all residue classes $\omega + I$ of paths ω from j to i in Q; and, for an arrow $\alpha : j \to l$ in Q, $I(i)_{\alpha} : I(i)_j \to I(i)_l$ is the k-linear map defined on the basis by deleting the arrow $\alpha : j \to l$ from those paths j to i which start with α and sending to zero the paths that do not start with α, that is, if $\omega = \alpha\omega'$, $I(i)_{\alpha}(\omega + I) = \omega' + I$, otherwise, $I(i)_{\alpha}(\omega + I) = 0$. $I(i)$ is called the **injective representation** at vertex i.

Definition 4.2.9. Let A be a finite-dimensional k-algebra, and let $M \in \operatorname{mod} A$.

- (i) The intersection of all maximal submodules of M is called the **radical** of the module M and is denoted by $\operatorname{rad} M$.
- (ii) The quotient $M/\operatorname{rad} M$ is called the **top** of M and is denoted by $\operatorname{top} M$.
- (iii) The submodule which is generated by all simple submodules of M is called the **socle** of M, and is denoted by $\operatorname{soc} M$.

The following lemma gives the radical of the projective representation $P(i)$.

Lemma 4.2.10. *(Schiffler, 2014, Lemma 5.9) Let* $A = kQ/I$ *be a bound quiver algebra and* $P(i) = (P(i)_j, P(i)_\alpha)_{j\in Q_0,\alpha\in Q_1}$ *be the indecomposable projective representation at vertex i. Then*

$$\operatorname{rad} P(i) = (P(i)'_j, P(i)'_\alpha)_{j\in Q_0,\alpha\in Q_1},$$

where $P(i)'_j = P(i)_j$ *if* $i \neq j$*, and* $P(i)'_i$ *is the vector space spanned by* $\{c + I \mid c$ *is a nonconstant path from i to* $j\}$*; and* $P(i)'_\alpha = P(i)_\alpha|_{P(i)'_{s(\alpha)}}$.

In particular, $\operatorname{top} P(i) \cong S(i)$.

We end this section with the following example.

Example 4.2.11. Let Q be the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & & \\ 1 \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & 4 \arrow[r, "\rho"] & 5 \\ & 3 \arrow[ur, "\delta"'] & & \end{tikzcd}$$

and $I = \langle \alpha\beta - \gamma\delta, \beta\rho, \delta\rho\rangle$.

Then

$$P(1) = \begin{array}{c} 1 \\ 2\ 3 \\ 4 \end{array} \qquad P(2) = \begin{array}{c} 2 \\ 4 \end{array} \qquad P(3) = \begin{array}{c} 3 \\ 4 \end{array} \qquad P(4) = \begin{array}{c} 4 \\ 5 \end{array} \qquad P(5) = 5$$

and

$$\operatorname{rad} P(1) = \begin{array}{c} 2\ 3 \\ 4 \end{array} \qquad \operatorname{rad} P(2) = 4 \qquad \operatorname{rad} P(3) = 4 \qquad \operatorname{rad} P(4) = 5 \qquad \operatorname{rad} P(5) = 0.$$

4.3 Equivalence of Categories

In this thesis, we are interested in two categories: the category of finite-dimensional quiver representations $\operatorname{rep} Q$ of a given quiver Q and the category of finitely generated kQ-module $\operatorname{mod} kQ$. In this section, we give a proof of an important theorem which is one of the most central theorem of representation theory. This theorem states that these two categories $\operatorname{rep} Q$ and $\operatorname{mod} kQ$ are equivalent. Actually, we give this theorem for more general result, that is, for any admissible ideal I, the category $\operatorname{rep}(Q,I)$ of finite-dimensional bound quiver representations is equivalent to the category of $\operatorname{mod} kQ/I$ of finitely generated kQ/I-modules. By this theorem, we can switch between two descriptions and use which one is convenient for our needs. For example, if we compute Auslander-Reiten quivers, it is more convenient to use the graphical notation coming from the representations of quivers, and if we use algebraic arguments, using modules is generally easier.

Now, the main theorem of this section is given as follows.

Theorem 4.3.1. *(Schiffler, 2014, Theorem 5.4) Let $A = kQ/I$, where Q is a finite connected quiver and I is an admissible ideal of kQ. Then the category of finitely generated right A-modules* $\operatorname{mod} A$ *and the category of finite-dimensional bound quiver representations* $\operatorname{rep}(Q,I)$ *are equivalent, that is,*

$$\operatorname{mod} A \cong \operatorname{rep}(Q,I).$$

Proof. We need to construct two functors

$$F : \operatorname{mod} A \to \operatorname{rep}(Q,I) \ \text{ and } \ G : \operatorname{rep}(Q,I) \to \operatorname{mod} A$$

such that $F \circ G \cong 1_{\operatorname{rep}(Q,I)}$ and $G \circ F \cong 1_{\operatorname{mod} A}$.

(i) Construction of $F : \operatorname{mod} A \to \operatorname{rep}(Q,I)$. We have to define the functor F on A-modules and on morphisms.

Let M be an A-module. Then we define the representation $F(M) = (M_i, M_\alpha)_{i\in Q_0, \alpha\in Q_1}$ of (Q,I) as follows: if $i \in Q_0$, let e_i be the corresponding primitive idempotent in A, then set $M_i = Me_i$ where $M_i = Me_i$ is the vector space consisting of all me_i, with $m \in M$;

if $\alpha : i \to j \in Q_1$, the map $M_\alpha : M_i \to M_j$ is defined by

$$M_\alpha(me_i) = m(e_i\alpha) = \begin{cases} m\alpha & \text{if } s(\alpha) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we must show that the representation $F(M)$ satisfies the relations in I. Let $\rho = \sum \lambda_\omega \omega$ be a relation in I, where $\omega = (i|\alpha_1, \alpha_2, \cdots, \alpha_t|j)$ is a path. Let the map M_ω be the composition $M_{\alpha_t} \circ \cdots \circ M_{\alpha_1}$. Then we must show that $\sum \lambda_\omega M_\omega = 0$. So, for me_i in M_i, we have $M_\omega(me_i) = me_i\alpha_1\alpha_2 \cdots \alpha_t = m\omega$, and therefore $\sum \lambda_\omega M_\omega(me_i) = \sum \lambda\omega m\omega = m\sum \lambda\omega\omega = m\rho$. But $\rho \in I$, so $\rho = 0$ in A, and therefore $m\rho = 0$. This shows that $F(M)$ is an object in $\operatorname{rep}(Q, I)$ and defines our functor on the objects.

Let $f : M \to M'$ be a morphism in $\operatorname{mod}A$. We want to define a morphism $F(f) : F(M) \to F(M')$ of $\operatorname{rep}(Q, I)$. For $i \in Q_0$ and $m = me_i \in Me_i = M_i$, we have $f(me_i) = f(me_i^2) = f(me_i)e_i \in M'e_i = M'_i$. So the restriction f_i of f to M_i is a linear map $f_i : M_i \to M'_i$. Observe that the map $f_i : M_i \to M'_i$ sending me_i to $f(m)e_i$. Then we put $F(f) = (f_i)_{i \in Q_0}$.

Finally, we have to show that $M'_\alpha f_i = f_j M_\alpha$, for any arrow $\alpha : i \to j$. Let $me_i \in M_i$, then $f_j M_\alpha(me_i) = f_j(m(e_i\alpha)) = f(m)(e_i\alpha) = f_i(m)(e_i\alpha) = M'_\alpha f_i(me_i)$ which shows that the diagram

$$\begin{tikzcd}
Me_i \arrow[r, "M_\alpha"] \arrow[d, "f_i"'] & Me_j \arrow[d, "f_j"] \\
M'e_i \arrow[r, "M'_\alpha"] & M'e_j
\end{tikzcd}$$

commutes and $F(f)$ is a morphism of representations.

Clearly, $F(1_M) = 1_{F(M)}$ for every A-module M and also, $F(fg) = F(f)F(g)$ for any two morphisms $f : M \to M'$ and $g : M' \to M''$ in $\operatorname{mod}A$. Thus F is a functor.

(ii) Construction of $G : \operatorname{rep}(Q, I) \to \operatorname{mod}A$. Let $M = (M_i, M_\alpha)$ be an object of $\operatorname{rep}(Q, I)$. We set $G(M) = \bigoplus_{i \in Q_0} M_i$ and we define an A-module structure on the k-vector space $G(M)$ as follows:

Let $m = \oplus_{i\in Q_0} m_i \in G(M)$ and let a be an element in A, say $a = \sum \lambda_\omega \omega + I$, where the sum is over all paths ω in the quiver Q. Define

$$m.a = \sum \lambda_\omega M_\omega(m),$$

where $M_\omega(m) = (0,\cdots,0,M_\omega(m_{s(\omega)}),0,\cdots,0)$ with the unique nonzero entry at position $t(\omega)$. Now, we show that this A-module structure is well-defined. So, let $\sum \mu_\omega \omega$ be an another representative of the coset a, then $a = \sum \lambda_\omega \omega + I = \sum \mu_\omega \omega + I$. Thus $\sum \lambda_\omega \omega - \sum \mu_\omega \omega \in I$. Then $\sum \lambda_\omega M_\omega(m) - \sum \mu_\omega M_\omega(m) \in I = \sum(\lambda_\omega - \mu_\omega) M_\omega(m)$. But since $M = (M_i, M_\alpha)$ is a representation of the bound quiver (Q,I), the map $\sum(\lambda_\omega - \mu_\omega)M_\omega(m)$ is the zero map. Thus the above A-module structure is well-defined.

Now, we show that $G(M)$ is an A-module. Since it is clearly a k-vector space, it only remains to show that the four axioms of the module structure hold. Let $m, m' \in G(M)$, $\lambda \in k$ and $a, a' \in A$. It is sufficient to show that the axioms for the special case where $a = \omega + I$, $a' = \omega' + I$ are represented by paths ω, ω'. Then

(1) $(m+m')a = M_\omega(m+m') = M_\omega(m) + M_\omega(m') = ma + m'a.$

(2) $m(a+a') = (M_\omega + M'_\omega)(m) = M_\omega(m) + M_{\omega'}(m) = ma + ma'.$

(3) $m(aa') = M_{\omega\omega'}(m) = M_{\omega'} \circ M_\omega(m) = (ma)a'.$

(4) $m1 = m\sum_{i\in Q_0} e_i = \sum_{i\in Q_0} Me_i(m) = \sum_{i\in Q_0} m_i = m.$

Thus $G(M)$ is an A-module. This defines our functor G on the objects.

Now, we have to show that G is defined on morphisms. Then, let $f = (f_i)_{i\in Q_0}$ be a morphism from $M = (M_i, M_\alpha)$ to $M' = (M'_i, M'_\alpha)$ in $\operatorname{rep}(Q,I)$. We want to construct a homomorphism $G(f) : G(M) \to G(M')$ of A-modules. Since $G(M) = \bigoplus_{i\in Q_0} M_i$ and $G(M') = \bigoplus_{i\in Q_0} M'_i$ as k-vector spaces, there exists a k-linear map $G(f) = \oplus_{i\in Q_0} f_i : G(M) \to G(M')$. We claim that $G(f)$ is an A-module homomorphism, that is, for any $m \in G(M)$ and any $a \in A$, we have $G(f)(ma) = G(f)(m)a$. Then, let $m = \oplus_{i\in Q_0} m_i \in G(M)$ and $a = \omega + I \in A$. The map $G(f)$ is linear, since each f_i is linear. Observe that $G(f)(ma) = \oplus_{i\in Q_0} f_i(M_\omega(m_i))$ is equal to $f_{t(\omega)}(M_\omega(m_{s(\omega)}))$ at position $t(\omega)$; and zero elsewhere. Also, observe that $G(f)(m)a = (\oplus_{i\in Q_0} f_i(m_i))a$ is equal to $M'_\omega(f_{s(\omega)}(m_{s(\omega)}))$

at position $t(\omega)$; and zero elsewhere. Since f is a morphism of quiver representations, it follows that $M'_\omega \circ M_{s(\omega)} = M_{t(\omega)} \circ M_\omega$ for every path ω. So, $G(f)(ma) = G(f)(m)a$. This proves our claim. Thus our functor G on the morphisms is defined.

It is clear that $G(1_M) = 1_{G(M)}$ and $G(fg) = G(f)G(g)$ for all morphisms $f, g \in \operatorname{rep}(Q, I)$. Thus G is a functor.

Lastly, it is necessary to show that $F \circ G \cong 1_{\operatorname{rep}(Q,I)}$ and $G \circ F \cong 1_{\operatorname{mod} A}$.

Let $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \operatorname{rep} Q$ and denote its image under G by $G(M)$. Then the representation $F(G(M))$ at vertex i is $G(M)e_i = (\oplus M_i)e_i = M_i$, and the linear map $M_i \to M_j$ on an arrow $\alpha : i \to j$ in $F(G(M))$ maps m_i to $m_i\alpha = M_\alpha(m_i)$. Thus $F(G(M)) = (M_i, M_\alpha)$, and $F \circ G \cong 1_{\operatorname{rep}(Q,I)}$.

Also, let $M \in \operatorname{mod} A$ and denote $F(M)$ by (M_i, M_α). Then the underlying vector space of the module $G \circ F(M)$ is $\oplus M_i = \oplus M e_i \cong M$, and the A-module structure on it is given by $m. \sum \lambda_\omega \omega = \sum \lambda_\omega M_\omega(m) = \sum \lambda_\omega m\omega$. Thus $G \circ F \cong 1_{\operatorname{mod} A}$, and it proves that F and G are equivalences of categories. □

As an immediate result of this theorem, we obtain the following corollary.

Corollary 4.3.2. *Let Q be a finite, connected and acyclic quiver. There exists an equivalences of categories* $\operatorname{mod} kQ \cong \operatorname{rep} Q$.

Proof. Since Q is finite and acyclic, the algebra kQ is finite-dimensional. It follows by choosing $I = 0$ in Theorem 4.3.1. □

Proposition 4.3.3. *Path algebras of acyclic quivers are hereditary.*

Proof. By Theorem 3.3.31, the category $\operatorname{rep} Q$ is hereditary. This follows from Theorem 4.3.1. □

Now, let (Q, I) be a bound quiver and $A = kQ/I$ be its bound quiver algebra. The vector spaces e_iA and Ae_j have as bases the set of paths starting in i and the set of paths ending in j, respectively. Moreover, e_iAe_j has as a basis the set of paths starting in i and ending in j. By Assem et al. (2006), we have

$$A = \bigoplus_{i=1}^{n} e_iA.$$

Clearly, each e_iA is a right A-module.

Corollary 4.3.4. *The A-module e_iA is indecomposable.*

Proof. Since the only idempotents in $\mathrm{End}(e_iA)$ are 0 and 1, it follows by Lemma 2.1.10 and Corollary 2.1.11. □

We now introduce the following important proposition which is necessary in this thesis and end this chapter.

Proposition 4.3.5. *Let (Q,I) be a bound quiver and $A = kQ/I$ be its bound quiver algebra. Under the equivalence of categories of Theorem 4.3.1, $P(i)$ corresponds to the indecomposable projective module e_iA, and $I(i)$ corresponds to the indecomposable injective module DAe_i.*

Proof. See Schiffler (2014),(Proposition 5.7). □

CHAPTER FIVE
AUSLANDER-REITEN THEORY

In this chapter, our essential objective is to introduce the Auslander-Reiten quiver. The Auslander-Reiten quiver is a very good approximation of the module category. If the number of isoclasses of the indecomposable representations of a given quiver Q is finite, then the Auslander-Reiten quiver gives the complete picture of the module category. To understand the theoretical background of the Auslander-Reiten quiver, we need some definitions and properties. Therefore, step by step we give the necessary information and reach the Auslander-Reiten quiver. Thus we begin with introducing a series of notions and results named after Auslander and Reiten such that all of them create the Auslander-Reiten theory. More clearly, in the first section, we present irreducible morphisms and almost split sequences. Second section explains the Auslander-Reiten translations. Well-known Auslander-Reiten formulas are given in the third section; and in the last section, we reach the Auslander-Reiten quiver which is the main goal in this chapter.

5.1 Irreducible Morphisms and Almost Split Sequences

This section is devoted to introducing the notions of irreducible morphisms and almost split sequences. Recall that the aim of representation theory is to classify the indecomposable modules and the morphisms between them. Irreducible morphisms play an important role to achieve this aim.

We first need to present the concept of almost split morphisms to define the almost split sequences.

Definition 5.1.1. Let $L, M, N \in \operatorname{mod} A$. A morphism $f : L \to M$ in $\operatorname{mod} A$ is called **left minimal** if each morphism $h : M \to M$ such that $hf = f$ is an automorphism of M.

Similarly, a morphism $g : M \to N$ in $\operatorname{mod} A$ is called **right minimal** if each morphism $h : M \to M$ such that $gh = h$ is an automorphism of M.

Definition 5.1.2. A morphism $f : L \to M$ in $\operatorname{mod} A$ is called **left almost split** if

(i) f is not section, and

(ii) for each morphism $u : L \to U$ in mod$A$ which is not a section, there exists a morphism $u' : M \to U$ such that $u'f = u$.

Similarly, a morphism $g : M \to N$ in mod$A$ is called **right almost split** if

(i) g is not retraction, and

(ii) for each morphism $v : V \to N$ in mod$A$ which is not a retraction, there exists a morphism $v' : V \to M$ such that $gv' = v$.

Definition 5.1.3. A morphism $f : L \to M$ is called **left minimal almost split morphism** if it is both left minimal and left almost split morphism.

Similarly, a morphism $g : M \to N$ is called **right minimal almost split morphism** if it is both right minimal and right almost split morphism.

Now, we can define the almost split sequences.

Definition 5.1.4. A short exact sequence in mod$A$

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

is called an **almost split sequence** if f is a left minimal almost split morphism and g is a right minimal almost split morphism.

Note that an almost split sequence is not split by Definition 5.1.2.

The other goal of this section is to define irreducible morphisms.

Definition 5.1.5. A morphism $f : X \to Y$ in mod$A$ is called **irreducible** if

(i) f is neither a section nor a retraction, and

(ii) If $f = f_1 f_2$ for some $f_1 : X \to Z$ and $f_2 : Z \to Y$, then either f_1 is a retraction or f_2 is a section.

Observe that an irreducible morphism in $\operatorname{mod} A$ is either injective or surjective (but not both). Moreover, it admits no nontrivial factorization. For more detailed information, see Assem et al. (2006).

Proposition 5.1.6. *(Assem et al., 2006, Proposition IV.3.5) (a) Let P be an indecomposable projective module in modA. An A-module homomorphism $g : M \to P$ is right minimal almost split if and only if g is a monomorphism with image equal to* $\operatorname{rad} P$.

(b) Let I be an indecomposable injective module in $\operatorname{mod} A$*. An A-module homomorphism $f : I \to M$ is left minimal almost split if and only if f is an epimorphism with kernel equal to* $\operatorname{soc} I$.

Corollary 5.1.7. *(Assem et al., 2006, Corollary IV.3.9) (a) Let S be a simple projective noninjective module in* $\operatorname{mod} A$*. If $f : S \to M$ is irreducible, then M is projective.*

(b) Let S be a simple injective nonprojective module in $\operatorname{mod} A$*. If $g : M \to S$ is irreducible, then M is injective.*

Remark 5.1.8. By Corollary 5.1.7, we construct examples of almost split sequences. That is, let S be simple, projective, noninjective and $f : S \to P$ be left minimal almost split. By Corollary 5.1.7, P is projective and by Proposition 5.1.6, for each indecomposable summand P' of P, the corresponding component $f' : S \to P'$ of f is a monomorphism with image a summand of $\operatorname{rad} P'$. It follows that, if P is the direct sum of all such indecomposable projectives P', then the sequence

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "f"] & P \arrow[r] & \operatorname{Coker} f \arrow[r] & 0 \end{tikzcd}$$

is almost split.

Example 5.1.9. Let A be a k-algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4. \end{tikzcd}$$

Observe that $S(2)$ is a simple projective noninjective summand of $\operatorname{rad} P(3)$ and is equal to $\operatorname{rad} P(1)$. Then we have an almost split sequence

$$0 \longrightarrow S(2) \longrightarrow P(1) \oplus P(3) \longrightarrow (P(1) \oplus P(3))/S(2) \longrightarrow 0.$$

The next proposition is used in last section. Thus it is necessary to know.

Proposition 5.1.10. *(Assem et al., 2006, Proposition IV.3.11) Let P be a nonsimple indecomposable projective-injective module,* $S = \operatorname{soc} P$, *and* $R = \operatorname{rad} R$. *Then the sequence*

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} q \\ i \end{bmatrix}} R/S \oplus P \xrightarrow{\begin{bmatrix} -j & p \end{bmatrix}} P/S \longrightarrow 0$$

is almost split, where i, j *are the inclusions and* p, q *are the projections.*

Example 5.1.11. Let A be the k-algebra given by the quiver

$$\begin{tikzcd}
 & 2 \arrow[rd, "\beta"] & \\
1 \arrow[ru, "\alpha"] \arrow[r, "\gamma"] \arrow[d, "\lambda"] & 3 \arrow[r, "\delta"] & 6 \\
4 \arrow[rr, "\mu"] & & 5 \arrow[u, "\nu"']
\end{tikzcd}$$

bound by the relations $\alpha\beta = \gamma\delta$ and $\gamma\delta = \lambda\mu\nu$. Observe that the A-module $P(1) = I(6)$ is projective-injective and the almost split sequence described in Proposition 5.1.10 with $P = P(1)$ is of the form

$$0 \longrightarrow \operatorname{rad} P(1) \longrightarrow S(2) \oplus S(3) \oplus \frac{P(4)}{S(6)} \oplus P(1) \longrightarrow \frac{P(1)}{S(6)} \longrightarrow 0.$$

The following lemma states that there is an equivalent definition of almost split sequences in terms of irreducible morphisms.

Lemma 5.1.12. *A short exact sequence in* $\operatorname{mod} A$

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is an almost split sequence if and only if L and N are indecomposable modules and f and g are irreducible morphisms. Moreover, in this situation, we have $L \cong \tau N$.[1]

Proof. See Auslander et al. (1997). □

By Lemma 2.1.5, for a ring R, $a \in \operatorname{rad} R$ if and only if $1 - ab$ is invertible for all $b \in R$. Now, we define the radical of a category by using this description of the radical of a ring.

Definition 5.1.13. (a) Let C be an additive k-category and X, Y be any pair of objects in C. The (Jacobson) **radical** of C is the ideal rad_C in C defined by

$$\operatorname{rad}_C(X,Y) = \{f \in \operatorname{Hom}_C(X,Y) \mid (1_X - gf) \text{ is invertible for any } g \in \operatorname{Hom}_C(Y,X)\}.$$

(b) Given $m \geq 1$, we define the mth power $\operatorname{rad}_C^m \subseteq \operatorname{rad}_C$ of rad_C by taking for rad_C^m the subspace of $\operatorname{rad}_C(X,Y)$ consisting of all finite sums of morphisms of the form

$$\begin{tikzcd} X = X_0 \arrow[r, "f_1"] & X_1 \arrow[r, "f_2"] & X_2 \arrow[r] & \cdots \arrow[r] & X_{m-1} \arrow[r, "f_m"] & X_m = Y, \end{tikzcd}$$

where $f_j \in \operatorname{rad}_C(X_{j-1}, X_j)$.

Now, let us consider the category $\operatorname{mod} A$.

Remark 5.1.14. For simplicity, we write rad and $\operatorname{rad}(X,Y)$ instead of $\operatorname{rad}_{\operatorname{mod} A}$ and $\operatorname{rad}_{\operatorname{mod} A}(X,Y)$, respectively.

Our current aim is to characterize irreducible morphisms in terms of the radical of $\operatorname{mod} A$. Thus, note that if X and Y are modules in $\operatorname{mod} A$, we define $\operatorname{rad}^2(X,Y)$ by Definition 5.1.13 as follows: $\operatorname{rad}^2(X,Y)$ consists of all A-module morphisms of the form gf, where $f \in \operatorname{rad}(X,Z)$ and $g \in \operatorname{rad}(Z,Y)$ for some (not necessarily indecomposable) A-module Z. Clearly, $\operatorname{rad}^2(X,Y) \subseteq \operatorname{rad}(X,Y)$.

Now, we can characterize irreducible morphisms by the next lemma.

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[1] τ is called the *Auslander-Reiten translation* which is introduced in Section 5.2.

Lemma 5.1.15. *Let X, Y be indecomposable modules in* $\operatorname{mod} A$. *A morphism $f : X \to Y$ is irreducible if and only if $f \in \operatorname{rad}(X,Y) \setminus \operatorname{rad}^2(X,Y)$.*

Proof. See Assem et al. (2006),(Lemma IV.1.6). □

Note that this lemma shows that the number of irreducible morphisms between indecomposable modules X and Y is measured by the quotient space $\operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$.

We indicated that the main goal of this chapter is to introduce the Auslander-Reiten quiver. Before introducing the Auslander-Reiten quiver in details, we define it by using the main tools given in this section.

Definition 5.1.16. Let A be a finite-dimensional k-algebra, and let X and Y be indecomposable A-modules. The **Auslander-Reiten quiver** of A is the quiver whose vertices consist of isoclasses of indecomposable A-modules, and the number of arrows from X to Y is equal to the dimension of $\operatorname{rad}(X,Y)/\operatorname{rad}^2(X,Y)$.

5.2 Auslander-Reiten Translations

This section aims to define the Auslander-Reiten translation which is fundamental for Auslander-Reiten theory and Auslander-Reiten quivers. Moreover, in this section, we introduce the Coxeter transformation which plays an important role in Auslander-Reiten theory.

5.2.1 Duality, Transposition and Nakayama Functor

Let A be a finite-dimensional k-algebra and $\operatorname{mod} A$ be the category of finitely generated A-modules. Firstly, consider the A-dual functor

$$(-)^t = \operatorname{Hom}_A(-, A) : \operatorname{mod} A \to \operatorname{mod} A^{op}.$$

Note that if P is a projective right A-module, then $P^t = \operatorname{Hom}_A(P, A)$ is a projective left A-module; indeed, if $P \cong eA$ for some primitive idempotent $e \in A$, then

$P^t = \mathrm{Hom}_A(eA, A) \cong Ae$, and our statement thus follows from the additivity of $(-)^t$. By Assem et al. (2006, IV.2), the functor $(-)^t$ induces a duality, also denoted by $(-)^t$, between the category $\mathrm{proj}\, A$ of projective right A-modules and the category $\mathrm{proj}\, A^{op}$ of projective left A-modules. We use this new duality to define a duality on an appropriate quotient of $\mathrm{mod}\, A$, and this duality is called the transposition.

Now, we start by approximating each module M by projective modules. So, let

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & M \arrow[r] & 0 \end{tikzcd}$$

be an exact sequence of A-modules, where P_0, P_1 are projective. This exact sequence is called a *projective presentation* of M.[2]

Remark 5.2.1. In above projective presentation, $p_0 : P_0 \to M$ and $p_1 : P_1 \to \mathrm{Ker}\, p_0$ are required to be projective covers. This projective presentation is also called **minimal projective presentation** of M.

Example 5.2.2. Let A be the k-algebra given by the 2-Kronecker quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2, \end{tikzcd}$$

and M be the representation

$$\begin{tikzcd} k \arrow[r, shift left, "1"] \arrow[r, shift right, "0"'] & k, \end{tikzcd}$$

where 1 denotes the identity morphism and 0 denotes the zero morphism. Now, observe that an endomorphism f of M is given by a pair (c_1, c_2) of scalars such that $c_1.1 = 1.c_2$ and $c_1.0 = 0.c_2$. These two conditions give $f = c.1_M$, where $c = c_1 = c_2 \in k$. Thus $\mathrm{End}\, M \cong k$, and so the only idempotent of $\mathrm{End}\, M$ is 0 and 1. Then, by Lemma 2.1.10, $\mathrm{End}\, M$ is local, and by Corollary 2.1.11, M is indecomposable. A minimal projective

---

[2] Dually,

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i_0"] & I_0 \arrow[r, "i_1"] & I_1 \end{tikzcd}$$

be an exact sequence of A-modules, where I_0, I_1 are injective. This exact sequence is called a *injective presentation* of M.

presentation of M is given by

$$\begin{tikzcd} P(2) \arrow[r, "p_1"] & P(1) \arrow[r, "p_2"] & M \arrow[r] & 0, \end{tikzcd}$$

where $P(1) = \big(k \underset{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightrightarrows}} k^2 \big)$ and $P(2) = S(2) = (0 \rightrightarrows k)$ are the indecomposable projective A-modules, p_1 is an isomorphism of $P(2)$ onto the direct summand of $\operatorname{rad} P(1)$ equal to $0 \rightrightarrows \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]k$, and p_2 is its cokernel homomorphism. Thus, in particular, M is not projective.

We apply (left exact, contravariant) functor $(-)^t$ to the minimal projective presentation of M and we obtain an exact sequence of left A-modules

$$\begin{tikzcd} 0 \arrow[r] & M^t \arrow[r, "p_o^t"] & P_0^t \arrow[r, "p_1^t"] & P_1^t \arrow[r] & \operatorname{Coker} p_1^t \arrow[r] & 0. \end{tikzcd}$$

Now, we define $\operatorname{Tr} M$ by $\operatorname{Tr} M = \operatorname{Coker} p_1^t$ and call it the **transpose** of M. The main properties of the transpose Tr is given in the following proposition.

Proposition 5.2.3. *(Assem et al., 2006, Proposition IV.2.1) Let M be an indecomposable module in* $\operatorname{mod} A$.

(a) The left A-module $\operatorname{Tr} M$ *has no nonzero projective direct summands.*

(b) If M is not projective, then the sequence

$$\begin{tikzcd} P_0^t \arrow[r, "p_1^t"] & P_1^t \arrow[r] & \operatorname{Tr} M \arrow[r] & 0 \end{tikzcd}$$

induced from the minimal projective presentation

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & M \arrow[r] & 0 \end{tikzcd}$$

of M is a minimal projective presentation of the left A-module $\operatorname{Tr} M$.

(c) M is projective if and only if $\operatorname{Tr} M = 0$. *If M is not projective, then* $\operatorname{Tr} M$ *is indecomposable and* $\operatorname{Tr}(\operatorname{Tr} M) \cong M$.

(d) If M and N are indecomposable nonprojective, then $M \cong N$ if and only if $\mathrm{Tr}\, M \cong \mathrm{Tr}\, N$.

As a result of this proposition, the transpose Tr maps modules of $\mathrm{mod}\, A$ to modules of $\mathrm{mod}\, A^{op}$. However, Tr does not define a duality $\mathrm{mod}\, A \to \mathrm{mod}\, A^{op}$ since it annihilates the projectives. In order to make this correspondence a duality, the projectives are annihilated from $\mathrm{mod}\, A$ and $\mathrm{mod}\, A^{op}$. So, now let $X, Y \in \mathrm{mod}\, A$ and let $\mathcal{P}(X, Y)$ be the subset of $\mathrm{Hom}_A(X, Y)$ consisting of all morphisms that factor through a projective A-modules. This defines an ideal $\mathcal{P}$ in the category $\mathrm{mod}\, A$. For details and constructions, see Assem et al. (2006, IV.2). This yields the following definition.

Definition 5.2.4. The quotient category

$$\underline{\mathrm{mod}}A = \mathrm{mod}\, A/\mathcal{P}$$

is called the *projectively stable category*. Its objects are the objects of $\mathrm{mod}\, A$, but the k-vector space $\underline{\mathrm{Hom}}_A(X, Y)$ of morphism from X to Y in $\underline{\mathrm{mod}}A$ is defined to be the quotient vector space

$$\underline{\mathrm{Hom}}_A(X, Y) = \mathrm{Hom}_A(X, Y)/\mathcal{P}(X, Y).$$

Dually, we may construct an ideal $\mathcal{I}$ in $\mathrm{mod}\, A$, by considering for each pair X, Y of A-modules, the k-subspace $\mathcal{I}(X, Y)$ of $\mathrm{Hom}_A(X, Y)$ consisting of all morphisms that factors through an injective A-module.

Definition 5.2.5. The quotient category

$$\overline{\mathrm{mod}}A = \mathrm{mod}\, A/\mathcal{I}$$

is called the *injectively stable category*. Its objects are the objects of $\mathrm{mod}\, A$, but the k-vector space $\overline{\mathrm{Hom}}_A(X, Y)$ of morphism from X to Y in $\overline{\mathrm{mod}}A$ is defined to be the quotient vector space

$$\overline{\mathrm{Hom}}_A(X,Y) = \mathrm{Hom}_A(X,Y)/\mathcal{I}(X,Y).$$

As a result of these, we conclude that there exists a functor $\mathrm{mod}A \to \underline{\mathrm{mod}}A$ that is the identity on objects and associates to a homomorphism $f : X \to Y$ in $\mathrm{mod}A$ its residual class modulo $\mathcal{P}(X,Y)$ in $\underline{\mathrm{mod}}A$.

Dually, we conclude the residue class functor $\mathrm{mod}A \to \overline{\mathrm{mod}}A$.

Thus we reach our aim as follows:

Proposition 5.2.6. *(Assem et al., 2006, Proposition IV.2.2) The correspondence* $M \mapsto \mathrm{Tr}\,M$ *induces a k-linear duality functor*

$$\mathrm{Tr} : \underline{\mathrm{mod}}A \to \underline{\mathrm{mod}}A^{op}.$$

The duality Tr defined in Proposition 5.2.6 is called the *transposition*. It transforms the right A-modules to left A-modules and conversely. Thus, if we want to define an endofunctor of $\mathrm{mod}A$, we compose it with another duality between right and left A-modules, that is, the standard duality $D = \mathrm{Hom}_k(-,k)$. Actually, the composition of D with Tr defines the Auslander-Reiten translation which is given in Section 5.2.2.

We need to define another important functor which plays an important role in the definition of Auslander-Reiten translation and so Auslander-Reiten quiver.

Definition 5.2.7. The functor $\nu = D(-)^t : \mathrm{mod}A \to \mathrm{mod}A$ is called the **Nakayama functor**.

Now, we end this section by the following proposition.

Proposition 5.2.8. *(Assem et al., 2006, Proposition III.2.10) The restriction of the Nakayama functor* $\nu : \mathrm{mod}A \to \mathrm{mod}A$ *to* $\mathrm{proj}A$ *is an equivalence of categories* $\mathrm{proj}A \to \mathrm{inj}A$ *whose quasi-inverse is given by* $\nu^{-1} = \mathrm{Hom}_A(D({}_AA),-) : \mathrm{inj}A \to \mathrm{proj}A$.

For more information, see (Assem et al., 2006; Auslander et al., 1997).

5.2.2 Auslander-Reiten Translations

In Section 5.2.1, we have defined two duality functor Tr and D. The composition of D and Tr gives the definition of Auslander-Reiten translation as follows.

Definition 5.2.9. The composition $D\mathrm{Tr}$ of D and Tr is called the **Auslander-Reiten translation** and it is denoted by $\tau = D\mathrm{Tr}$. Also, $\tau^{-1} = \mathrm{Tr}D$ denotes the **inverse Auslander-Reiten translation**.

The following proposition gives the construction method for the Auslander-Reiten translate of a module.

Proposition 5.2.10. *(Assem et al., 2006, Proposition IV.2.4) (a) Let*

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & M \arrow[r] & 0 \end{tikzcd}$$

be a minimal projective presentation of an A-module M. Then there exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \tau M \arrow[r] & \nu P_1 \arrow[r, "\nu p_1"] & \nu P_0 \arrow[r, "\nu p_0"] & \nu M \arrow[r] & 0. \end{tikzcd}$$

(b) Let

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i_0"] & I_0 \arrow[r, "i_1"] & I_1 \end{tikzcd}$$

be a minimal injective presentation of an A-module N. Then there exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \tau^{-1}N \arrow[r, "\nu^{-1}i_0"] & \nu^{-1}I_0 \arrow[r, "\nu^{-1}i_1"] & \nu^{-1}I_1 \arrow[r] & \nu^{-1}N \arrow[r] & 0. \end{tikzcd}$$

Remark 5.2.11. If M is projective, then $\tau M = 0$ since in the minimal projective presentation of M, $M = P_0$ and $P_1 = 0$. Dually, if M is injective, then $\tau^{-1}M = 0$ since in the minimal injective presentation of M, $M = I_0$ and $I_1 = 0$.

Example 5.2.12. Consider the minimal projective presentation

$$\begin{tikzcd} P(2) \arrow[r, "p_1"] & P(1) \arrow[r, "p_2"] & M \arrow[r] & 0 \end{tikzcd}$$

of M given in Example 5.2.2. Now, apply the Nakayama functor ν to this exact sequence. By 5.2.10, we obtain a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \tau M \arrow[r] & I(2) \arrow[r, "\nu p_1"] & I(1) \arrow[r] & 0, \end{tikzcd}$$

where $I(1) = S(1) = (\, k \rightrightarrows 0 \,)$ and $I(2) = (\, k^2 \underset{[0\ 1]}{\overset{[1\ 0]}{\rightrightarrows}} k \,)$ are the indecomposable injective A-modules. An obvious computation shows that the homomorphism νp_1 induces an isomorphism of the quotient module of $I(2)$ defined by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}\ k \rightrightarrows 0$ onto $I(1)$. Then $\tau M = \operatorname{Ker} \nu p_1$ is given by $k \underset{0}{\overset{1}{\rightrightarrows}} k$, that is, $\tau M \cong M$.

The following lemma gives useful characterization of modules of projective or injective dimension at most one in terms of their Auslander-Reiten translates.

Lemma 5.2.13. *(Assem et al., 2006, Lemma IV.2.7) Let M be a module in* $\operatorname{mod} A$.

(a) $\operatorname{pd} M \leq 1$ *if and only if* $\operatorname{Hom}(DA, \tau M) = 0$.

(b) $\operatorname{id} M \leq 1$ *if and only if* $\operatorname{Hom}(\tau^{-1} M, A) = 0$.

For more information, see Assem et al. (2006). However, briefly, this lemma gives formulas for the dimension vector of the Auslander-Reiten translation in terms of an important transformation which is called Coxeter transformation of any algebra A of finite global dimension. This transformation is defined in the next section.

Now, we introduce a proposition which gives some important properties of Auslander-Reiten translation as follows.

Proposition 5.2.14. *(Assem et al., 2006, Proposition IV.2.10) Let M and N be indecomposable modules in* $\operatorname{mod} A$.

(a) The module τM is zero if and only if M is projective.

(a') The module $\tau^{-1} N$ is zero if and only if N is injective.

(b) If M is a nonprojective module, then τM is indecomposable noninjective and $\tau^{-1}\tau M \cong M$.

(b') If N is a noninjective module, then $\tau^{-1}N$ is indecomposable nonprojective and $\tau\tau^{-1}N \cong N$.

(c) If M and N are nonprojective, then $M \cong N$ if and only if there is an isomorphism $\tau M \cong \tau N$.

(c') If M and N are noninjective, then $M \cong N$ if and only if there is an isomorphism $\tau^{-1}M \cong \tau^{-1}N$.

Proof. Since τ and τ^{-1} are compositions of the transposition Tr and the duality D, it follows from Proposition 5.2.3, the properties of duality D and the definitions. □

Corollary 5.2.15. *(Assem et al., 2006, Corollary IV.2.11) The Auslander-Reiten translations τ and τ^{-1} induce mutually inverse equivalences*

$$\begin{tikzcd} \underline{\operatorname{mod}}A \arrow[r, shift left, "\tau"] & \overline{\operatorname{mod}}A \arrow[l, shift left, "\tau^{-1}"] \end{tikzcd} .$$

Proof. It follows from Proposition 5.2.6 and Proposition 5.2.14. □

At this point, we consider the almost split sequences and give the next theorem which states the existence of almost split sequences. For more details, see (Assem et al., 2006; Auslander et al., 1997). Moreover, the theory of almost split sequences has been developed in (Auslander & Reiten, 1975, 1977a,b, 1978).

Theorem 5.2.16. *(Assem et al., 2006, Theorem IV.3.1) (a) For any indecomposable nonprojective A-module M, there exists an almost split sequence*

$$\begin{tikzcd} 0 \arrow[r] & \tau M \arrow[r] & E \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

in $\operatorname{mod}A$.

(b) For any indecomposable noninjective A-module N, there exists an almost split sequence

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & F \arrow[r] & \tau^{-1}N \arrow[r] & 0 \end{tikzcd}$$

in $\operatorname{mod}A$.

5.2.3 Coxeter Transformation

In this section, we introduce Coxeter transformation which plays an important role in Auslander-Reiten theory. But first we define main steps for this transformation.

Definition 5.2.17. Let A be a finite-dimensional k-algebra of finite global dimension, and let $\{e_1,\cdots,e_n\}$ be complete set of primitive orthogonal idempotents such that $1_A = e_1+e_2+\cdots+e_n$. The **Cartan matrix** of A is defined to be the $n\times n$ matrix:

$$C_A = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & & c_{nn} \end{pmatrix},$$

where $c_{ji} = \dim e_iAe_j$, for $i,j = 1,\cdots,n$.

By Proposition 4.3.5, $P(i)$ corresponds to the indecomposable projective module e_iA. Also, by Theorem 3.3.19, we write

$$\begin{aligned} e_iAe_j &\cong \mathrm{Hom}(e_jA, e_iA) \\ &\cong \mathrm{Hom}(P(j), P(i)). \end{aligned} \tag{5.1}$$

Thus, the ith column of C_A is the dimension vector of the indecomposable projective $P(i)$ at vertex i and so the ith row of C_A is the dimension vector of the indecomposable projective $I(i)$ at vertex i. Moreover, the entry c_{ji} corresponds to the number of paths from i to j.

Example 5.2.18. Let Q be the 2-Kronecker quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2. \end{tikzcd}$$

The corresponding k-algebra of Q is the Kronecker algebra

$$A = \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}.$$

Note that

$$P(1) = \begin{matrix} 1 \\ 2\ 2 \end{matrix} \qquad P(2) = 2.$$

Also, by Definition 5.2.17, the Cartan matrix of A has the form

$$C_A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Observe that C_A is invertible and

$$C_A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Moreover, we have the following proposition:

Proposition 5.2.19. *(Schiffler, 2014, Proposition 7.12) Let* $A = kQ/I$ *be a bound quiver algebra of finite global dimension. Then* $\det C_A = \{1, -1\}$*. In particular,* C_A *is invertible in the matrix ring* $\mathbb{M}_n(\mathbb{Z})$ *over the integers.*

We use the Cartan matrix C_A to define a nonsymmetric $\mathbb{Z}$-bilinear form on the group $\mathbb{Z}^n$ as follows:

Definition 5.2.20. Let A be a finite-dimensional algebra of finite global dimension, and let C_A be the Cartan matrix of A with respect to a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents of A. Then the $\mathbb{Z}$-bilinear (nonsymmetric) form $\langle -, - \rangle_A : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by $\langle x, y \rangle_A = x^t (C_A^{-1})^t y$ for $x, y \in \mathbb{Z}^n$ is called the **Euler characteristic** of A.

Also, the quadratic form $q_A : \mathbb{Z}^n \to \mathbb{Z}$ defined by $q_A(x) = \langle x, x \rangle_A$ for $x \in \mathbb{Z}^n$ is called the **Euler quadratic form** of an algebra A.

Example 5.2.21. Let A be the 2-Kronecker algebra given in Example 5.2.18. Then $n = 2$,

$$C_A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and

$$(C_A^{-1})^t = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

and the Euler characteristic of A is given by $\langle x, y\rangle_A = x_1y_1 + x_2y_2 - 2x_1y_2$.

Since C_A is invertible, we use its inverse to define another matrix.

Definition 5.2.22. The **Coxeter matrix** Φ_A of A is the $n \times n$ integer matrix:

$$\Phi_A = -C_A^t C_A^{-1}.$$

Also, the linear map $\Phi_A : \mathbb{Z}^n \to \mathbb{Z}^n$ defined by $\Phi_A(x) = \Phi_A.x$ for all $x \in \mathbb{Z}^n$ is called the **Coxeter transformation**.

Example 5.2.23. The Coxeter matrix of the 2-Kronecker algebra given in Example 5.2.18 is

$$\Phi_A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}.$$

The following lemma gives a basic property of the Coxeter matrix:

Lemma 5.2.24. *(Assem et al., 2006, Lemma III.3.16 (a))*

$$\Phi_A(\underline{\dim} P(i)) = -\underline{\dim} I(i) \quad \text{for each} \quad i \in \{1, \cdots, n\}.$$

Now, we give an application of the Coxeter transformation Φ in Auslander-Reiten theory. We use the Coxeter transformation to compute the dimension vector of the Auslander-Reiten translate of a given module.

Proposition 5.2.25. *(Schiffler, 2014, Proposition 7.15) Let A be a finite-dimensional k-algebra of finite global dimension.*

(1) Let M be an indecomposable nonprojective A-module and let

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & M \arrow[r] & 0 \end{tikzcd}$$

be a minimal projective presentation. Then

$$\underline{\dim}\tau M = \Phi_A\underline{\dim}M - \Phi_A\underline{\dim}\ker p_1 + \underline{\dim}\nu M.$$

(2) Let M be an indecomposable noninjective A-module and let

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i_0"] & I_0 \arrow[r, "i_1"] & I_1 \end{tikzcd}$$

be a minimal injective presentation. Then

$$\underline{\dim}\tau^{-1}M = \Phi_A^{-1}\underline{\dim}M - \Phi_A^{-1}\underline{\dim}\,\mathrm{Coker}\, i_1 + \underline{\dim}\nu^{-1}M.$$

For more information, see Schiffler (2014).

Corollary 5.2.26. *(Assem et al., 2006, Corollary IV.2.9)*

(a) If M is an indecomposable module in $\mathrm{mod}A$ *such that* $\mathrm{pd}\,M \leq 1$ *and* $\mathrm{Hom}(M,A) = 0$*, then* $\underline{\dim}\tau M = \Phi_A\underline{\dim}M$.

(b) If N is an indecomposable module in $\mathrm{mod}A$ *such that* $\mathrm{id}\,M \leq 1$ *and* $\mathrm{Hom}(DA,N) = 0$*, then* $\underline{\dim}\tau^{-1}N = \Phi_A^{-1}\underline{\dim}N$.

We use this fact in order to construct Auslander-Reiten quivers in Section 5.4.

5.3 Auslander-Reiten Formulas

This section is devoted to introducing the Auslander-Reiten formulas. These formulas describe a relation between short exact sequences and morphisms in the module category, and also provides a powerful computational tool, since it provides a way to calculate Ext^1 in terms of morphisms.

Remark 5.3.1. Let M be an A-module and let φ^M be the functorial homomorphism such that

$$\varphi^M : (-)\otimes_A M^t \to \mathrm{Hom}_A(M,-)$$

and defined on a right A-module N by

$$\begin{aligned} \varphi_N^M : N \otimes_A M^t &\to \operatorname{Hom}_A(M,N) \\ n \otimes f &\mapsto (m \mapsto nf(m)), \end{aligned}$$

where $m \in M$, $n \in N$ and $f \in M^t$. By Assem et al. (2006), if M is projective, then φ^M is a functorial isomorphism and that if N is projective, then φ_N^M is an isomorphism. Moreover, the cokernel of φ_N^M coincides with $\underline{\operatorname{Hom}}_A(M,N)$.

The main theorem of this section is given as follows.

Theorem 5.3.2. *(***Auslander-Reiten formulas.***) Let M,N be A-modules. Then there exist isomorphisms*

$$\operatorname{Ext}^1(M,N) \cong D\underline{\operatorname{Hom}}(\tau^{-1}N,M) \cong D\overline{\operatorname{Hom}}(N,\tau M)$$

that are functorial in both variables, that is, the following functors are isomorphic:

$$\operatorname{Ext}^1(-,N) \cong D\underline{\operatorname{Hom}}(\tau^{-1}N,-) \cong D\overline{\operatorname{Hom}}(N,\tau -)$$

$$\operatorname{Ext}^1(M,-) \cong D\underline{\operatorname{Hom}}(\tau^{-1}-,M) \cong D\overline{\operatorname{Hom}}(-,\tau M).$$

Proof. We only prove the first isomorphism since the second one is similar. Clearly, it is sufficient to prove the claimed isomorphism for modules N having no injective direct summand. By Proposition 5.2.14, it can be supposed that $N = \tau L$, where $L = \tau^{-1}N$. Now, let

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] & P_0 \arrow[r, "p_0"] & L \arrow[r] & 0 \end{tikzcd}$$

be a minimal projective presentation of L. We apply the functor $\nu = D(-)^t$ to this presentation and by Proposition 5.2.10, we obtain the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \tau L \arrow[r] & DP_1^t \arrow[r, "Dp_1^t"] & DP_0^t \arrow[r, "Dp_0^t"] & DL^t \arrow[r] & 0, \end{tikzcd}$$

where both DP_1^t and DP_0^t are injective. Now, we apply the functor $\operatorname{Hom}_A(M,-)$ to this exact sequence and obtain the complex

$$0 \longrightarrow \mathrm{Hom}_A(M,\tau L) \longrightarrow \mathrm{Hom}_A(M,DP_1^t) \xrightarrow{p_1'} \mathrm{Hom}_A(M,DP_0^t) \xrightarrow{p_0'} \mathrm{Hom}_A(M,DL^t),$$

where $p_1' = \mathrm{Hom}_A(M,Dp_1^t)$ and $p_0' = \mathrm{Hom}_A(M,Dp_0^t)$. So

$$\mathrm{Ext}_A^1(M,N) = \mathrm{Ext}_A^1(M,\tau L) = \mathrm{Ker}\, p_0' / \mathrm{Im}\, p_1'.$$

Also, we apply the right exact functor $D\mathrm{Hom}_A(-,M)$ to the minimal projective presentation of L and obtain an exact sequence

$$D\mathrm{Hom}_A(P_1,M) \xrightarrow{p_1''} D\mathrm{Hom}_A(P_0,M) \xrightarrow{p_0''} D\mathrm{Hom}_A(L,M) \longrightarrow 0,$$

where $p_1'' = D\mathrm{Hom}_A(p_1,M)$ and $p_0'' = D\mathrm{Hom}_A(p_0,M)$. From Remark 5.3.1, there exists a functorial morphism $\varphi^M : (-)\otimes_A M^t \to \mathrm{Hom}_A(M,-)$ which is associated to an A-module M. Now, consider the composition of the dual homomorphism $D\varphi^M : D\mathrm{Hom}_A(M,-) \to D((-)\otimes_A M^t)$ with the adjunction isomorphism $\eta^M : D((-)\otimes_A M^t) \to \mathrm{Hom}_A(-,DM^t)$ gives a functorial morphism

$$\omega^M = \eta^M D\varphi^M : D\mathrm{Hom}_A(M,-) \to \mathrm{Hom}_A(-,DM^t),$$

which is an isomorphism whenever M is projective. Thus we obtain a commutative diagram with exact lower row

$$\begin{tikzcd}
\mathrm{Hom}_A(M,DP_1^t) \arrow[r, "p_1'"] & \mathrm{Hom}_A(M,DP_0^t) \arrow[r, "p_0'"] & \mathrm{Hom}_A(M,DL^t) & \\
D\mathrm{Hom}_A(P_1,M) \arrow[r, "p_1''"] \arrow[u, "\varphi_M^{P_1}", "\cong"'] & D\mathrm{Hom}_A(P_0,M) \arrow[r, "p_0''"] \arrow[u, "\varphi_M^{P_0}", "\cong"'] & D\mathrm{Hom}_A(L,M) \arrow[r] \arrow[u, "\varphi_M^{L}"] & 0.
\end{tikzcd}$$

We obtain a induced homomorphism $\psi : \mathrm{Ker}\, p_0' \to \mathrm{Ker}\, \omega_M^L$ by the homomorphism $p_0''(\omega_M^{P_0})^{-1}$ of A-modules. Since p_0'' is an epimorphism and $\omega_M^{P_0}$ an isomorphism, ψ must be an epimorphism. Also, since $\mathrm{Ker}\, p_0'' = \mathrm{Im}\, p_1''$, and the maps $\omega_M^{P_0}$, $\omega_M^{P_1}$ are isomorphisms, we have that $\operatorname{Ker}\psi = \operatorname{Im} p_1'$. Thus, we obtain

$$\begin{aligned}\operatorname{Ker} p_0' / \operatorname{Im} p_1' &\cong \operatorname{Ker} p_0' / \operatorname{Ker}\psi \\ &\cong \operatorname{Ker}\omega_M^L \\ &= \operatorname{Ker} D\varphi_M^L \cong D\operatorname{Coker}\varphi_M^L.\end{aligned}$$

Hence there exists an isomorphism $\operatorname{Ext}_A^1(M,N) \cong D\operatorname{Coker}\varphi_M^L$ and by Remark 5.3.1,

$$\operatorname{Coker}\varphi_M^L \cong \underline{\operatorname{Hom}}_A(L,M) = \underline{\operatorname{Hom}}_A(\tau^{-1}N,M).$$

Thus the proof is completed. □

5.4 Auslander-Reiten Quivers

We have stated that Auslander-Reiten quivers provide information about the representation theory of quivers, and also algebras. So far we have developed the theory to able to compute Auslander-Reiten quivers. Let A be a finite-dimensional k-algebra. Here, we want to get information about the category $\operatorname{mod} A$ in the form of a quiver.[3] Our aim is to classify all representations of a given algebra in $\operatorname{mod} A$ and all morphisms between them up to isomorphism. By using the *Krull-Remak-Schmidt theorem*, it can be stated that it is sufficient to classify all indecomposable modules and irreducible morphisms between them. It leads to construct the Auslander-Reiten quiver whose vertices consist of indecomposable modules and arrows represent the irreducible morphisms between them. Also, we have given in Lemma 5.1.15 that a homomorphism $f : M \to N$ in $\operatorname{mod} A$ is irreducible if and only if $f \in \operatorname{rad}(M,N) \setminus \operatorname{rad}^2(M,N)$. Thus the quotient

$$\operatorname{Irr}(M,N) = \operatorname{rad}(M,N) / \operatorname{rad}^2(M,N)$$

measures the number of irreducible morphisms from M to N. It is called the *space of irreducible morphisms*.

---

[3] We can use the category $\operatorname{rep}(Q,I)$ instead of $\operatorname{mod} A$ by Theorem 4.3.1 to get information about $\operatorname{rep}(Q,I)$.

Now, we define the Auslander-Reiten quiver.

Definition 5.4.1. Let A be a finite-dimensional connected k-algebra. The quiver Γ_A of $\operatorname{mod} A$ is defined as follows:

(a) The vertices of Γ_A are the isomorphism classes $[M]$ of indecomposable modules M in $\operatorname{mod} A$.

(b) Let $[M], [N]$ be the vertices in Γ_A corresponding to the indecomposable modules $M, N \in \operatorname{mod} A$. The arrows $[M] \to [N]$ are in bijective correspondence with the vectors of a basis of the k-vector space $\operatorname{Irr}(M, N)$.

The quiver Γ_A of the module category $\operatorname{mod} A$ is called the **Auslander-Reiten quiver** of A.

Corollary 5.4.2. *(Assem et al., 2006, Corollary IV.4.4 (a)) Let*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] & \bigoplus_{i=1}^{t} M_i^{n_i} \arrow[r, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence in $\operatorname{mod} A$ *with* L, N *indecomposable and the* M_i *indecomposable and pairwise nonisomorphic. Write* $f = \begin{bmatrix} f_1 \\ \vdots \\ f_t \end{bmatrix}$ *and* $g = [g_1 \cdots g_t]$, *where* $f_i = \begin{bmatrix} f_{i1} \\ \vdots \\ f_{in_i} \end{bmatrix} : L \to M_i^{n_i}$ *and* $g_i = [g_{i1} \cdots g_{in_i}] : M_i^{n_i} \to N$. *If the given sequence is almost split, then for each i,*

$$\dim_k \operatorname{Irr}(L, M_i) = \dim_k \operatorname{Irr}(M_i, N).$$

We now give the following observations:

(a) From Definition 5.4.1, the vertices of Γ_A are the isomorphism classes of indecomposable A-modules, and there exists an arrow $[L] \to [M]$ if and only is $\operatorname{Irr}(L, M) \neq 0$, that is, if and only if there exists an irreducible morphism $L \to M$.

(b) By Proposition 5.1.6, Theorem 5.2.16 and Corollary 5.4.2, the set $[M]^-$ of the direct predecessors of $[M]$ coincides with the set of those vertices $[L]$ such that

  (i) If M is projective, then L is an indecomposable direct summand of $\operatorname{rad} M$.

  (ii) If M is not projective, then L is an indecomposable direct summand of the middle term of the almost split sequence ending with M.

Similarly,

the set $[M]^+$ of the direct successors of $[M]$ coincides with the set of those vertices $[N]$ such that

(i) If M is injective, then N is an indecomposable direct summand of $M/\operatorname{soc} M$.

(ii) If M is not injective, then L is an indecomposable direct summand of the middle term of the almost split sequence starting with M.

In particular, for every M, the sets $[M]^+$ and $[M]^-$ are finite. Thus each vertex of Γ_A has only finitely many neighbours. This type of quivers is called *locally finite*.

Clearly, Γ_A is finite, that is, has finitely many vertices if and only if A is representation-finite, that is, the number of the isomorphism classes of indecomposable A-modules is finite.

(c) Every irreducible morphism $f : M \to N$ is either a proper monomorphism or a proper epimorphism. Hence, if $M = N$, then f is an isomorphism since M is finite dimensional as a k-vector space. So we obtain that the source and the target of an irreducible homomorphism must be distinct and thus an Auslander-Reiten quiver has no loops.

(d) Let Γ_P (or Γ_I) denote the set of those vertices in Γ_A that correspond to a projective (or an injective, respectively) indecomposable module. For each $[N] \in \Gamma_A \setminus \Gamma_P$, the Auslander-Reiten translate τN of N exists, and by Proposition 5.2.14, we obtain $[\tau N] \in \Gamma_A \setminus \Gamma_I$. This defines a bijection

$$\tau : \Gamma_A \setminus \Gamma_P \to \Gamma_A \setminus \Gamma_I.$$

So for each indecomposable nonprojective module N, we have $\tau[N] = [\tau N]$.

The inverse bijection is denoted by the following:

$$\tau^{-1} : \Gamma_A \setminus \Gamma_I \to \Gamma_A \setminus \Gamma_P,$$

and for each indecomposable noninjective module L, we obtain $\tau^{-1}[L] = [\tau^{-1} L]$.

τ is called *translation* of the quiver Γ_A. Now, let N be an indecomposable nonprojective A-module, and let

$$\begin{tikzcd} 0 \arrow[r] & \tau N \arrow[r] & \bigoplus_{i=1}^{t} M_i^{n_i} \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

be an almost split sequence ending with N, with the M_i indecomposable and pairwise nonisomorphic. By Corollary 5.4.2, we have

$$n_i = \dim_k \mathrm{Irr}(M_i, N) = \dim_k \mathrm{Irr}(\tau N, M_i).$$

As a result of the above observations, we obtain the following *mesh* corresponding to above almost split sequence in Γ_A:

$$\begin{tikzcd}
 & {[M_1]} \arrow[dr, shift left=2, "\beta_{11}"] \arrow[dr, shift right=2, "\beta_{1n_1}"'] \arrow[d, phantom, "\vdots"] & \\
{[\tau N]} \arrow[ur, shift left=2, "\alpha_{11}"] \arrow[ur, shift right=2, "\alpha_{1n_1}"'] \arrow[dr, shift right=2, "\alpha_{t_1}"'] \arrow[dr, shift left=2, "\alpha_{1n_t}"] & \cdots\cdots\cdots & {[N]} \\
 & {[M_t]} \arrow[ur, shift left=2, "\beta_{tn_t}"] \arrow[ur, shift right=2, "\beta_{t_1}"'] \arrow[u, phantom, "\vdots"] &
\end{tikzcd}$$

We see that $[\tau N]^+ = [N]^-$ and for each $[M_i]$ in this set, there exists a bijection between the set $\{\alpha_{i_1}, \cdots, \alpha_{i_{n_i}}\}$ of arrows from $[\tau N]$ to $[M_i]$ and the set $\{\beta_{i_1}, \cdots, \beta_{i_{n_i}}\}$ of arrows from $[M_i]$ to $[N]$. Now, we are able to construct the Auslander-Reiten quiver with an example.

Example 5.4.3. Let A be the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3. \end{tikzcd}$$

We can write the indecomposable simple, projective and injective A-modules, given as representations:

$$\begin{aligned}
P(1) &= \quad (k \longrightarrow k \longrightarrow k) \quad &= I(3)\\
P(2) &= \quad (0 \longrightarrow k \longrightarrow k) & \\
P(3) &= \quad (0 \longrightarrow 0 \longrightarrow k) \quad &= S(3)\\
I(1) &= \quad (k \longrightarrow 0 \longrightarrow 0) \quad &= S(1)\\
I(2) &= \quad (k \longrightarrow k \longrightarrow 0) & \\
S(2) &= \quad (0 \longrightarrow k \longrightarrow 0). &
\end{aligned}$$

Moreover, observe that

$$P(2) = \operatorname{rad} P(1), \quad P(3) = \operatorname{rad} P(2), \quad I(2) = I(3)/S(3) = P(1)/S(3), \quad I(1) = I(2)/S(2).$$

Since $P(3)$ is simple, projective and noninjective, by Corollary 5.1.7, the target of each irreducible morphism starting with $P(3)$ is projective. Since $P(3) = \operatorname{rad} P(2)$, and $P(3)$ is not a summand of $\operatorname{rad} P(1)$, the inclusion $i : P(3) \to P(2)$ is the only irreducible morphism and is actually the only right minimal almost split morphism ending with $P(2)$. Thus we have an almost split sequence

$$0 \longrightarrow P(3) \xrightarrow{i} P(2) \longrightarrow \operatorname{Coker} i \longrightarrow 0.$$

Observe that $\operatorname{Coker} i = P(2)/P(3) = S(2)$. Now consider $P(2)$. We have just seen that there exists an irreducible morphism $P(2) \to S(2)$. On the other hand $\operatorname{rad} P(1) = P(2)$, hence there exists an irreducible (inclusion) morphism $P(2) \to P(1)$. Now $P(1) = I(3)$ is projective-injective, hence by Proposition 5.1.10, we have an almost split sequence of the form

$$0 \longrightarrow P(2) \longrightarrow P(1) \oplus S(2) \longrightarrow I(2) \longrightarrow 0.$$

On the other hand, the homomorphism $I(2) \to I(2)/S(2) = I(1) = S(1)$ is a left minimal almost split, with kernel $S(2)$, so that we have an almost split sequence

$$0 \longrightarrow S(2) \longrightarrow I(2) \longrightarrow S(1) \longrightarrow 0.$$

Now, we put together the information and we obtain Γ_A is the quiver

$$\begin{tikzcd}
{[P(3)]} \arrow[dr] & & {[S(2)]} \arrow[dr] & & {[S(1)]} \\
& {[P(2)]} \arrow[ur] \arrow[dr] & & {[I(2)]} \arrow[ur] & \\
& & {[P(1)]} \arrow[ur] & &
\end{tikzcd}$$

It is customary that when we draw Γ_A, to put the translate τL of a nonprojective point L on the same horizontal line as L. We always follow this convention.

If we replace each indecomposable module in the above Auslander-Reiten quiver by its dimension vector, we obtain

$$\begin{tikzcd}
001 \arrow[dr] & & 010 \arrow[dr] & & 100 \\
& 011 \arrow[ur] \arrow[dr] & & 110 \arrow[ur] & \\
& & 111 \arrow[ur] & &
\end{tikzcd}$$

Remark 5.4.4. By Assem et al. (2006), for each mesh of Γ_A of the form

$$\begin{tikzcd}
& {[M_1]} \arrow[ddr] & \\
& \vdots & \\
{[\tau N]} \arrow[uur] \arrow[ddr] \arrow[rr, dotted, no head] & & {[N]} \\
& \vdots & \\
& {[M_t]} \arrow[uur] &
\end{tikzcd}$$

$\underline{\dim} N + \underline{\dim} \tau N = \sum_{i=1}^{t} \underline{\dim} M_i$; this follows from the fact that the corresponding almost split sequence is exact.

This remark gives a method of construction called *knitting with dimension vectors*.

For more details and information about knitting technique, see Barot (2015). Now, we illustrate the knitting with dimension vectors in the following example.

Example 5.4.5. Let A be the k-algebra given by the quiver

$$\begin{tikzcd}
 & & 3 & \\
1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] & & 5 \arrow[ul, "\delta"'] \\
 & & 4 \arrow[ul, "\gamma"] \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

bound by $\varepsilon\delta = \gamma\beta$, $\alpha\beta = 0$. Observe that $P(3)$ is the only one simple projective module. Note that the dimension vector of $P(3)$ is $00{}^{1}_{0}0$. We know that no arrow of Γ_A ends in $P(3)$ and that the target of each arrow starting at $P(3)$ is projective. In our case, we find two such arrows, namely $[P(3)] \to [P(2)]$ and $[P(3)] \to [P(5)]$ (indeed, $P(3) = \operatorname{rad} P(2) = \operatorname{rad} P(5)$), which are our first two arrows. Moreover, these are the only arrows of targets $P(2)$ and $P(5)$, respectively. Because $P(3)$ is not injective, we have an almost split sequence

$$0 \to P(3) \to P(2) \oplus P(5) \to \tau^{-1}P(3) \to 0.$$

Moreover,

$$\underline{\dim}\tau^{-1}P(3) = \underline{\dim}P(2) + \underline{\dim}P(5) - \underline{\dim}P(3) = 01{}^{1}_{0}0 + 00{}^{1}_{0}1 - 00{}^{1}_{0}0 = 01{}^{1}_{0}1.$$

We see that $\tau^{-1}P(3) = \operatorname{rad} P(4)$, and hence there is a unique arrow of target $P(4)$, that is, $[\tau^{-1}P(3)] \to [P(4)]$. This gives us the beginning of Γ_A (where the isomorphism classes of indecomposable A-modules are replaced by their dimension vectors):

$$\begin{tikzcd}
 & 01{}^{1}_{0}0 \arrow[dr] & & \\
00{}^{1}_{0}0 \arrow[ur] \arrow[dr] & & 01{}^{1}_{0}1 \arrow[r] & 01{}^{1}_{1}1 \\
 & 00{}^{1}_{0}1 \arrow[ur] & &
\end{tikzcd}$$

The calculation of the almost split sequences starting at $P(2)$ and $P(5)$, respectively, gives

$$\begin{tikzcd}
 & 01^{1}_{0}0 \arrow[dr] & & 00^{0}_{0}1 \\
00^{1}_{0}0 \arrow[ur] \arrow[dr] & & 01^{1}_{0}1 \arrow[ur] \arrow[r] \arrow[dr] & 01^{1}_{1}1 \\
 & 00^{1}_{0}1 \arrow[ur] & & 01^{0}_{0}0
\end{tikzcd}$$

Since $S(2) = \operatorname{rad} P(1)$, there exists a unique arrow of target $P(1)$, that is, $[S(2)] \to [P(1)]$. In this way, all the projectives have been obtained. All other indecomposable modules are thus of the form $\tau^{-1}L$, with L indecomposable: to obtain the dimension vector of such a module, we consider the almost split sequence

$$0 \to L \to M_1 \oplus \cdots \oplus M_t \to \tau^{-1}L \to 0.$$

Since we can assume by induction that $\underline{\dim} L$ and $\underline{\dim} M_i$ (for all i with $1 \leq i \leq t$) are known, we obtain $\underline{\dim}\tau^{-1}L = \sum_{i=1}^{t} \underline{\dim} M_i - \underline{\dim} L$. This allows us to construct the rest of Γ_A. The construction stops when we reach the injectives; indeed, the left minimal almost split morphism starting at an indecomposable injective $I(i)$ is the projection onto its socle factor $I(i)/S(i)$, and

$$\dim_k I(i) = 1 + \dim_k I(i)/S(i) > \dim_k I(i)/S(i).$$

Thus the previous method would give a dimension vector with negative coordinates, a contradiction. Continuing the construction gives the Auslander-Reiten quiver Γ_A

$$\begin{tikzcd}
 & 01^{1}_{0}0 \arrow[dr] & & 00^{0}_{0}1 \arrow[dr] & & 01^{0}_{1}0 \arrow[dr] & & 10^{0}_{0}0 \\
00^{1}_{0}0 \arrow[ur] \arrow[dr] & & 01^{1}_{0}1 \arrow[ur] \arrow[r] \arrow[dr] & 01^{1}_{1}1 \arrow[r] & 01^{0}_{1}1 \arrow[ur] \arrow[dr] & & 11^{0}_{1}0 \arrow[ur] \arrow[dr] & \\
 & 00^{1}_{0}1 \arrow[ur] & & 01^{0}_{0}0 \arrow[ur] \arrow[dr] & & 11^{0}_{1}1 \arrow[ur] \arrow[dr] & & 00^{0}_{1}0 \\
 & & & & 11^{0}_{0}0 \arrow[ur] & & 00^{0}_{1}1 \arrow[ur] &
\end{tikzcd}$$

Remark 5.4.6. In this last example, we finished when we have constructed the full

Auslander-Reiten component. Could there be others with more modules? At this point, we refer Brauer-Thrall conjectures in Section 6.1 since Auslander's proof for the first Brauer-Thrall conjecture shows exactly what we need here, that is, this gives that if a finite Auslander-Reiten component contains all projectives and all injectives, it contains all indecomposables.

For more examples and constructions about Auslander-Reiten quiver, see (Assem et al., 2006; Gabriel, 1980).

CHAPTER SIX
INDECOMPOSABLES AND DIMENSIONS

Some deepest results in the representation theory of quivers, and also algebras are introduced in this chapter. More clearly, first section is devoted to presenting two important conjectures called the Brauer-Thrall conjectures. Next section introduces a well-known theorem which is Gabriel's theorem and gives some necessary definitions and results for this theorem. Moreover, another important theorem called Kac's theorem is presented in last section.

6.1 The Brauer-Thrall Conjectures

In this section, we introduce the two early statements, brought into life by Brauer around 1940 as exercise for his students and which remained unproved for a long time. At the origin of many recent developments of representation theory are these statements which are known as the **Brauer-Thrall Conjectures**. Brauer-Thrall conjectures include two conjectures given as follows:

Conjecture 1. A finite dimensional k-algebra is either representation-finite or there exist indecomposable modules with arbitrarily large dimension.

This conjecture was proved by Roiter (1968) for algebraically closed fields and later generalized by Auslander (1974).

Conjecture 2. A finite dimensional algebra over an infinite field k is either representation-finite or there exists an infinite sequence of numbers $d_i \in \mathbb{N}$ such that, for each i, there exists an infinite number of nonisomorphic indecomposable modules with k-dimension d_i.

The first complete proof of the second Brauer-Thrall conjecture was given by Bautista (1985) for algebraically closed fields of characteristic different from 2. For arbitrary fields, the problem is still open. For a detailed discussion about the developments in the proofs of the Brauer-Thrall conjectures, see also Ringel (1980).

In previous section, we referred the Brauer-Thrall Conjectures to explain why the construction in Example 5.4.5 is finished. To give explanation, we need to give the proof of Auslander for the first Brauer-Thrall conjecture. But we first give some necessary notions and results for the proof.

Definition 6.1.1. Let A be a finite-dimensional k-algebra. A sequence of irreducible morphisms in $\operatorname{mod} A$ of the form

$$\begin{tikzcd} M_0 \arrow[r, "f_1"] & M_1 \arrow[r, "f_2"] & \cdots \arrow[r, "f_t"] & M_t, \end{tikzcd}$$

with all the M_i indecomposables is called a **chain of irreducible morphisms** from M_0 to M_t of length t.

Lemma 6.1.2. *(Assem et al., 2006, Lemma IV.5.1) Let $t \in \mathbb{N}$ and let M and N be indecomposable right A-modules with $\operatorname{Hom}_A(M,N) \neq 0$. Assume that there exists no chain of irreducible morphisms from M to N of length $< t$.*

(a) There exists a chain of irreducible morphisms

$$\begin{tikzcd} M = M_0 \arrow[r, "f_1"] & M_1 \arrow[r, "f_2"] & \cdots \arrow[r, "f_t"] & M_t \end{tikzcd}$$

and a homomorphism $g : M_t \to N$ with $g f_t \cdots f_2 f_1 \neq 0$.

(b) There exists a chain of irreducible morphisms

$$\begin{tikzcd} N_t \arrow[r, "g_t"] & N_{t-1} \arrow[r, "g_{t-1}"] & \cdots \arrow[r] & N_1 \arrow[r, "g_1"] & N_0 = N \end{tikzcd}$$

and a homomorphism $f : M \to N_t$ with $g_1 \cdots g_t f \neq 0$.

Lemma 6.1.3. ***(Harada and Sai).*** *For a natural number b, let*

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] & M_2 \arrow[r, "f_2"] & M_3 \arrow[r] & \cdots \arrow[r] & M_{2^b-1} \arrow[r, "f_{2^b-1}"] & M_{2^b} \end{tikzcd}$$

be a chain of nonzero nonisomorphisms in $\operatorname{mod} A$, with all M_i indecomposables of length $\leq b$. Then $f_{2^b-1} \cdots f_2 f_1 = 0$.

Proof. See Assem et al. (2006),(Lemma IV.5.2). □

We are now able to explain Remark 5.4.6 which was announced in the previous section.

Theorem 6.1.4. *Let A be a finite-dimensional k-algebra. If Γ_A admits a connected component $\mathcal{C}$ whose modules are of bounded length, then $\mathcal{C}$ is finite and $\mathcal{C} = \Gamma_A$. In particular, A is representation-finite.*

Proof. Let b be a bound for the length of the indecomposable modules X with $[X]$ in $\mathcal{C}$. Let M, N be two indecomposable A-modules such that $\mathrm{Hom}_A(M,N) \neq 0$. Let $[M] \in \mathcal{C}_0$, where $\mathcal{C}_0$ is the connected component. Then there exists a chain of irreducible morphisms from M to N of length smaller than $2^b - 1 = t$, and in particular $[N] \in \mathcal{C}_0$. Indeed, if this is not the case, by Lemma 6.1.3, there exists a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t$$

and a homomorphism $g : M_t \to N$ with $g f_t \cdots f_1 \neq 0$. However, from Lemma 6.1.3, we obtain $f_t \cdots f_1 = 0$. This contradiction shows our claim. Similarly, if $[N] \in \mathcal{C}_0$, we have $[M] \in \mathcal{C}_0$.

Now, let $[M] \in \mathcal{C}_0$ be arbitrary. Then there exists an indecomposable projective module P such that $\mathrm{Hom}_A(P,M) \neq 0$; hence we also have $[P] \in \mathcal{C}_0$. By Assem et al. (2006, I.5.17, II.3.4), for any other indecomposable projective P', there exists a sequence of indecomposable projective modules $P = P_0, P_1, \cdots, P_s = P'$ such that $\mathrm{Hom}_A(P_{i-1}, P_i) \neq 0$ or $\mathrm{Hom}_A(P_i, P_{i-1}) \neq 0$ for each $1 \leq i \leq s$ and $P \cong e_i A$ and $P' \cong e_j A$ for some primitive orthogonal idempotents e_i, e_j of A. Also, by Theorem 3.3.19, $\mathrm{Hom}_A(e_i A, e_j A) \cong e_j A e_i$. Hence $[P'] \in \mathcal{C}_0$. Thus we obtain that any indecomposable A-module X corresponds to a point $[X]$ in $\mathcal{C}$ since there exists an indecomposable projective A-module P' such that $\mathrm{Hom}_A(P',X) \neq 0$. This shows that $\mathcal{C} = \Gamma_A$.

On the other hand, for each indecomposable projective A-module P and each indecomposable A-module M such that $\mathrm{Hom}_A(P,M) \neq 0$, there exists a chain of irreducible morphisms from P to M of length smaller than $t = 2^b - 1$. Since there are only finitely many nonisomorphic indecomposable projectives, there are only finitely many nonisomorphic indecomposable modules corresponding to points in C. Hence A is representation-finite. □

As a consequence of Theorem 6.1.4, we obtain the first Brauer-Thrall conjecture.

Corollary 6.1.5. *Any algebra is either representation-finite or admits indecomposable modules of arbitrary length.*

We end this section with the next corollary, which states the importance of the irreducible morphisms and so the importance of Auslander-Reiten quiver, for the description of the module category of a representation-finite algebra.

Corollary 6.1.6. *Let A be representation-finite algebra. Any nonzero nonisomorphism between indecomposable modules in* mod$A$ *is a sum of compositions of irreducible morphisms.*

Proof. See Assem et al. (2006),(Corollary IV.5.6). □

6.2 Gabriel's Theorem

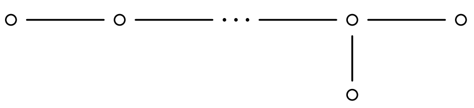
Gabriel's theorem is the first classification result in the representation theory of finite-dimensional algebras. So, this section is devoted to introduce the Gabriel's theorem. Before introducing Gabriel's theorem, it is necessary to give some definitions and results. Thus we first introduce the certain diagrams called the *Dynkin diagrams* which are of particular interest in the Gabriel's theorem.

The Dynkin Diagrams

$\mathbb{A}_m$:

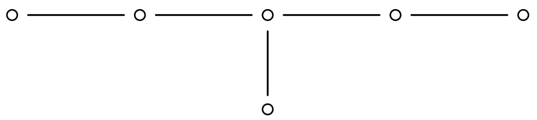
$m \geq 1$

$\mathbb{D}_n$:

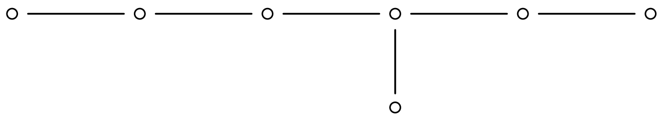


$n \geq 4$

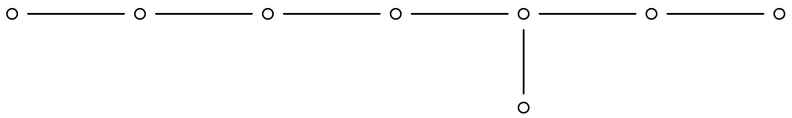
$\mathbb{E}_6$:



$\mathbb{E}_7$:



$\mathbb{E}_8$:



Note that the index in the Dynkin diagrams denotes the number of points in the diagram.

Let Q be an acyclic quiver. Now, our aim is to define a quadratic form q associated to Q and introduce its roots to prepare a substructure to introduce the Gabriel's Theorem.

Definition 6.2.1. A quadratic form $q = q(x_1, \cdots, x_n)$ on $\mathbb{Z}^n$ in n indeterminates $x_1, \cdots, x_n$ is called an **integral quadratic form** if it is of the form

$$q(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j,$$

where $a_{ij} \in \mathbb{Z}$ for all i, j.

An integral quadratic form can be considered as a map:

$$q : \mathbb{Z}^n \to \mathbb{Z},$$
$$x = (x_1, \cdots, x_n) \mapsto q(x).$$

Also, given a quadratic form q, we can define its symmetric bilinear form $(-,-)_q$ by the formula

$$(x, y)_q = q(x+y) - q(x) - q(y).$$

Definition 6.2.2. Let q be a quadratic form. Then

(1) q is called **positive definite** if $q(x) > 0$, for all $x \neq 0$.

(2) q is called **positive semi-definite** if $q(x) \geq 0$, for all x.

(3) q is called **weakly positive** if $q(x) > 0$, for all $x > 0$.

Definition 6.2.3. Let q be an integral quadratic form on $\mathbb{Z}^n$. An element $x \in \mathbb{Z}^n \setminus \{0\}$ is called a **real root** if $q(x) = 1$, and x is called an **imaginary root** if $q(x) = 0$. The set of all roots is denoted by Φ.

Let $\{e_1, \cdots, e_n\}$ denote the standard basis of $\mathbb{Z}^n$. It is clear that every element of $\mathbb{Z}^n$, so in particular every root, is of the form $\sum_i a_i e_i$ with $a_i \in \mathbb{Z}$.

Definition 6.2.4. A root $x = \sum_i a_i e_i$ is called **positive** if $x \neq 0$ and all $a_i \geq 0$, and x is called **negative** if $x \neq 0$ and all $a_i \leq 0$. The set of positive (or negative) roots is denoted by Φ_+ (or Φ_-), respectively.

Definition 6.2.5. Let Q be an acyclic quiver. The **quadratic form** of Q is defined by

$$q_Q : \mathbb{Z}^n \to \mathbb{Z}, \qquad q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}.$$

Note that q_Q does not depend on the actual orientation of the arrows in the quiver Q, but only on the underlying graph.

Note that we want to evaluate q_Q on the dimension vectors of representations of Q. Also, note that the value of the quadratic form q_Q only depends on the dimension vector of a given representation and not on the particular representation itself.

Example 6.2.6. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd}$$

The quadratic form of Q is $q_Q(x) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3$.

The following lemma gives the information about the quadratic form of the path algebra kQ of the quiver Q.

Lemma 6.2.7. *(Assem et al., 2006, Lemma VII.4.1) Let Q be a finite, connected, and acyclic quiver. Then the Euler quadratic form q_A of the path algebra $A = kQ$ and the* *quadratic form q_Q of the quiver Q coincide. Moreover,*

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{i,j \in Q_0} a_{ij} x_i x_j,$$

where $a_{ij} = \dim_k \mathrm{Ext}_A^1(S(i), S(j))$.

Now, we are able to present Gabriel's theorem. Gabriel's theorem, proved by Pierre Gabriel, classifies the quivers of finite representation type in terms of Dynkin diagrams.[1] More precisely, we introduce the Gabriel's theorem as follows.

Theorem 6.2.8. ***(Gabriel's Theorem)*** *Let Q be a finite, connected and acyclic quiver; and $A = kQ$ be the path algebra of Q where k is an algebraically closed field.*

(1) The algebra A is representation-finite if and only if Q is of Dynkin type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$.

(2) If Q is of Dynkin type $\mathbb{A}, \mathbb{D}$ or $\mathbb{E}$, then the dimension vector induces a bijection ψ from isoclasses of indecomposable A-modules to the set of positive roots:

$$\psi : \mathrm{ind}\, kQ \to \Phi_+$$
$$\psi : M \mapsto \underline{\dim} M.$$

(3) The number of the isoclasses of indecomposable A-modules equals $\frac{1}{2}n(n+1)$, $n^2 - n$, 36, 63 and 120, if Q is of Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, with $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$, respectively.

The proof of Gabriel's Theorem can be found in (Bernstein et al., 1973; Gabriel, 1972). For alternative proofs, see (Assem et al., 2006; Auslander et al., 1997; Schiffler, 2014). Moreover, for a generalization of Gabriel's Theorem, see (Dlab & Ringel, 1975).

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[1] A quiver is of finite representation type if it has only finitely many isoclasses of indecomposable representations.

6.3 Kac's Theorem

It was stated that the main problem in the representation theory of quivers is to classify the indecomposable representations up to isomorphism. But, in some cases, it is too difficult since it includes the classification of all finite dimensional indecomposable representations over the complex quivers. However, Kac proved that the set of dimension vectors of indecomposable representations can be classified. Thus this section is devoted to introducing Kac's Theorem.

Definition 6.3.1. Let q be an integral quadratic form on $\mathbb{Z}^n$, and let $(-,-)_q$ be the corresponding symmetric bilinear form on $\mathbb{Z}^n$. For each i with $1 \leq i \leq n$, we define a mapping $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$s_i(x) = x - 2(x, e_i)e_i.$$

This map is called a **reflection** at i.

Remark 6.3.2. Note that

$$\begin{aligned}(s_i(x), s_i(y))_q &= (x - 2(x, e_i)_q e_i, y - 2(y, e_i)_q e_i) \\ &= (x, y)_q - 2(x, e_i)_q(y, e_i)_q - 2(y, e_i)_q(x, e_i)_q + 4(x, e_i)(y, e_i) \\ &= (x, y)_q.\end{aligned}$$

Definition 6.3.3. Let q be a weakly positive integral quadratic form on $\mathbb{Z}^n$. The subgroup W_q of the automorphism group of $\mathbb{Z}^n$ generated by the reflections $s_1, \cdots, s_n$ is called the **Weyl group** of q.

Now, let $x = \sum_{i_1}^n x_i e_i$ be a vector in $\mathbb{Z}^n$. The subset of $\{1, \cdots, n\}$ which is defined by $\operatorname{supp} x = \{i \mid 1 \leq i \leq n, \quad x_i \neq 0\}$ is called **support** of x. Moreover, the set

$$F = \{x \in \mathbb{N}^n \mid \operatorname{supp} x \text{ is connected and } (x, e_i)_q \leq 0 \text{ for } 1 \leq i \leq n\}$$

is called the **fundamental region**, where $(-,-)_q$ is the symmetric bilinear form associated to q.

Definition 6.3.4. A vector $x \in \mathbb{Z}^n$ is called a **Schur root** if it belongs to the W-orbit WS, where $S = \{e_1, \cdots, e_n\} \cup F \cup (-F)$. The elements of the form we_i for some $i = 1, \cdots, n$ and $w \in W$ are called **real**; and the elements of the form $w(\pm f)$ for some $f \in F$ and $w \in W$ are called **imaginary**.

Note that from Remark 6.3.2, we have the Weyl group preserves the bilinear form $(-,-)_q$. Thus $q(x) = 1$ for any real Schur root x. Similary, one can see that $q(x) \leq 0$ holds for any imaginary Schur root. Moreover, if x is a real Schur root, then $\pm x$ are the unique multiples of x which are Schur roots; and if x is imaginary, then every nonzero multiple of x is a Schur root. For more details, see Barot (2015).

Theorem 6.3.5. ***(Kac's Theorem)*** *Let k be any field and Q be a finite, acyclic quiver. If M is an indecomposable representation of Q, then* $\underline{\dim} M$ *is a Schur root. Conversely, we have the following characterization.*

(a) If x is a positive real Schur root, then up to isomorphism there exists a unique indecomposable representation M of Q with $\underline{\dim} M = x$.

(b) If x is a positive imaginary Schur root, then there exist infinitely many pairwise non-isomorphic indecomposable representations M of Q with $\underline{\dim} M = x$.

This theorem was proved by Kac (1980a), and later generalized by Kac (1980b) to quivers with loops.

CHAPTER SEVEN
CONCLUSION

In this thesis, we researched an important theory called the theory of representations of quivers. We investigated and collected the essential results of this theory and mainly focused the goal of representations of quivers. The goal of representation theory of quivers is to classify all representations of a given quiver Q and all morphisms between them up to isomorphism. *Krull-Remak-Schmidt theorem* helps to achieve this goal since it states that every representation can be expressed in a unique way as a direct sum of indecomposable representations. Thus we introduced this important theorem. By using this well-known theorem, we observed that studying with indecomposable representations and morphisms between them is enough for this aim. On the other hand, we focused the equivalence of the category of (finite-dimensional) representations of the quiver Q and the category of (finitely generated) kQ-modules and introduced the theorem of this equivalence. This theorem is very important in representation theory since by using it, we can switch between two descriptions and use which one is convenient for our needs. We later presented Auslander-Reiten theory to prepare a substructure for Auslander-Reiten quiver which provides information about the representation theory of quivers, and also algebras.

On the purpose of classifying all representations by using indecomposable representations and developing the representation theory, some special results were obtained and we introduced them in this thesis. First one is the *Brauer-Thrall conjectures* which include two conjectures and help to construct Auslander-Reiten quiver as well. Another important result is *Gabriel's theorem*, which is the first classification result in the representation theory of finite-dimensional algebras. Lastly, we presented *Kac's theorem*.

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