OPTIMAL INVENTORY ALLOCATION AND PRICING POLICIES UNDER A TWO-TIER CUSTOMER MARKET AND RANDOM DEMAND

by

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ABSTRACT

OPTIMAL INVENTORY ALLOCATION AND PRICING POLICIES UNDER A TWO-TIER CUSTOMER MARKET AND RANDOM DEMAND

Along with the shortening of product life cycles, the segmentation in customer markets become more distinct. In the face of segmentation, firms determine varied pricing policies to appeal to each customer markets. Also, with a strong matching ability of supply and demand, firms enhance their profits. Hence, from the point of firms, handling the decisions of pricing and inventory jointly gains prominence. In this thesis, we analyze a model that considers joint ordering, pricing, and inventory allocation decisions of a national distributor for a technology intensive product under two-tier customer market, which consisting of non-overlapping consecutive time intervals. The demand is price dependent and random where the randomness is provided by an additive error term. For the secondary period, the demand is also considered reference price dependent where the price of the primary period is taken as reference price. At the beginning of the primary period, the distributor makes the decision of pricing and order quantity. The distributor, at the beginning of the secondary period, determines the price and any additional items to order, and how many stocks to allocate to the retailers. We employ two-stage dynamic programming algorithm for the problem and conclude that a base-stock list-price policy is optimal providing that the demand functions and transformed expected revenue functions are concave in their variables. Near-explicit expressions are presented for the optimal decisions. We further conduct a computational study to examine the effect of the several system parameters on the optimal decisions.

ÖZET

İKİ KATMANLI MÜŞTERİ PAZARI VE RASTGELE TALEP ALTINDA OPTİMAL ENVANTER TAHSİSİ VE FİYATLANDIRMA POLİTİKALARI

Ürün yaşam döngülerindeki kısalmalarla birlikte, müşteri pazarlarındaki segmentasyon daha belirgin hale gelmektedir. Segmentasyonlar karşısında firmalar, her bir müşteri pazarına hitap etmek için çeşitli fiyatlandırma politikaları belirlemektedir. Ayrıca, güçlü bir arz ve talep eşleştirme becerisiyle de firmalar karlarını artırılar. Dolayısıyla, firmalar açısından, fiyatlandırma ve envanter kararlarının birlikte ele alınması ön plana çıkmaktadır. Bu tezde, birbiriyle çakışmayan ardışık zaman aralıklarından oluşan iki katmanlı müşteri pazarı altında, teknoloji yoğun bir ürün için ulusal bir distribütörün ortak sipariş, fiyatlandırma ve stok tahsisi kararlarını dikkate alan bir modeli analiz ediyoruz. Rastgeleliğin toplamsal bir hata terimi tarafından sağlandığı durumlarda talep, fiyata bağlıdır ve rastgeledir. İkincil dönem için talep, birincil dönemin fiyatının referans fiyat olarak alındığı durumda, referans fiyata da bağlıdır. Birincil dönemin başında distribütör, fiyatlandırma ve sipariş miktarı kararını verir. Distribütör, ikincil dönemin başında, fiyatı ve sipariş edilecek ek kalemleri, ve perakendecilere kaç adet stok tahsis edeceğini belirler. Bu problem için iki aşamalı dinamik programlama algoritması kullandık ve temel-stok liste-fiyat politikasının, talep fonksiyonlarının ve dönüştürülmüş beklenen gelir fonksiyonlarının kendi değişkenlerinde içbükey olması koşuluyla optimal olduğu sonucuna vardık. En uygun kararlar için neredeyse açık ifadeler sunuldu. Sonrasında, çeşitli sistem parametrelerinin optimum kararlar üzerindeki etkisini incelemek için bir hesaplama çalışması yürüttük.

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A	Average market potential for the primary season
A_0	Average market potential for the secondary season
A_i	Average market potential for the secondary season for retailer i
b_1	Backordering cost for the primary season
b_2	Backordering cost for the secondary season
B	Marginal impact of price over the demand for the primary season
B_0	Marginal impact of price over the demand for the secondary season
B_i	Marginal impact of price over the demand for retailer i
c	Positive constant
c_1	Unit ordering cost for the primary season
c_2	Unit ordering cost for the secondary season
C_0	Reference price effect over the demand for the secondary season
C_i	Reference price effect over the demand for retailer i
d_1	Demand variable for the primary season
d_{11}	A feasible demand value for the primary season
d_{12}	A feasible demand value for the primary season
d_1^l	Lower bound on $\hat{d}_1(p_1)$
d_1^u	Upper bound on $\hat{d}_1(p_1)$
d_2	Demand variable for the secondary season
d_{21}	A feasible demand value for the secondary season
d_{22}	A feasible demand value for the secondary season
d_{23}	A feasible demand value for the secondary season
d_2^l	Lower bound on $\hat{d}_2(p_2, p_1)$
d_2^u	Upper bound on $\hat{d}_2(p_2, p_1)$
$\hat{d}_1(p_1)$	Mean demand function for the primary season

$\hat{d}_2$	Short denotation of $\hat{d}_2(p_2, p_1)$
$\hat{d}_2(p_2, p_1)$	Mean demand function for the secondary season
$\hat{d}_{2,i}$	Short denotation of $\hat{d}_{2,i}(p_2, p_1)$
$\hat{d}_{2,i}(p_2, p_1)$	Mean demand function for the secondary season at retailer i
$\mathbb{D}_1$	Set of d_1 for the transformed problem
$D_1(p_1)$	Demand through the primary season
$\mathbb{D}_2$	Set of d_2 and p_1 for the transformed problem
$D_2(p_2, p_1)$	Demand through the secondary season
$D_{2,i}$	Short denotation of $D_{2,i}(p_2, p_1)$
$D_{2,i}(p_2, p_1)$	Demand through the secondary season at retailer i
$\hat{G}(.)$	Expected inventory holding and backorder cost function
$\hat{G}_1(y_1, d_1)$	Transformed expected inventory holding and backorder cost function for the primary season
$\hat{G}_1(y_1, p_1)$	Expected inventory holding and backorder cost function for the primary season
$\hat{G}_2(s_{2,i}, p_2, p_1)$	Expected inventory holding and backorder cost function for retailer i for the secondary season
$\hat{G}_2(y_2, d_2)$	Transformed expected inventory holding and backorder cost function for the secondary season
$\hat{G}_2(y_2, p_2, p_1)$	Expected inventory holding and backorder cost function for the secondary season
h_1	Inventory holding cost for the primary season
h_2	Inventory holding cost for the secondary season
I_1	Inventory level at the beginning of the primary season
I_2	Inventory level at the beginning of the secondary season
i	Index for retailers
j	Index for retailers (in case of need)
J	Optimal expected profit of the entire problem
J_1	Expected profit of the primary season of the optimal solution
J_2	Expected profit of the secondary season of the optimal solution
$J(p_1, y_1)$	Optimal expected profit of the entire problem given p_1 and y_1

$J_1(I_1)$	Optimal expected profit of the entire problem given I_1
$J_1(p_1, y_1)$	Optimal expected profit of the primary season given p_1 and y_1
$J_2(I_2, p_1)$	Optimal expected profit of the secondary season given I_2 and p_1
$J_2(k)$	Optimal profit of the secondary season for k^{th} simulation
$J_2(p_1, y_1)$	Optimal expected profit of the secondary season given p_1 and y_1
k	Index for the simulation in the algorithm
K	Number of simulation for the realizing demand
L	Lagrange function
n	Index for seasons
N	Number of retailers
p_1	Unit price for the primary season *or* reference price for the secondary season
p_{11}	A feasible price value for the primary season
p_{12}	A feasible price value for the primary season
$\bar{p}_1$	A feasible price value for the primary season
p_1^*	Optimal value of p_1
p_1^l	Lower bound on p_1
p_1^{lower}	Lower bound on p_1 for coding purpose
p_1^u	Upper bound on p_1
p_1^{upper}	Upper bound on p_1 for coding purpose
p_2	Unit price for the secondary season
p_{21}	A feasible price value for the secondary season
p_{22}	A feasible price value for the secondary season
$\bar{p}_2$	A feasible price value for the secondary season
p_2^*	Optimal value of p_2
p_2^l	Lower bound on p_2
p_2^{lower}	Lower bound on p_2 for coding purpose
p_2^u	Upper bound on p_2
p_2^{upper}	Upper bound on p_2 for coding purpose

$\hat{p}_1(d_1)$	Inverse demand function for the primary season
$\hat{p}_2(d_2, p_1)$	Inverse demand function for the secondary season
$\mathbb{P}_1$	Price set for the primary season
$\mathbb{P}_2$	Price set for the secondary season
$R(x)$	Unit loss function of x
$s_{2,i}$	Stock allocation amount to retailer i
$s^*_{2,i}$	Optimal value of stock allocation amount to retailer i
T_1	Beginning of the primary season
T_2	Ending of the primary season *or* beginning of the secondary season
T_3	Ending of the secondary season
$V(y_2, d_2, p_1)$	Transformed optimal expected profit acquired after stock allocation
$V(y_2, p_2, p_1)$	Optimal expected profit acquired after stock allocation
y_1	Order-up-to level for the primary season
y_1^*	Optimal value of y_1
y_1^{lower}	Lower bound on y_1 for coding purpose
y_1^{upper}	Upper bound on y_1 for coding purpose
y_2	Order-up-to level for the secondary season
y_2^*	Optimal value of y_2

Δ_1	A positive value
Δ_2	A positive value
Δ_3	A positive value
ε	Standard normal error term
ε_1	Error term for the demand function for the primary season
ε_2	Error term for the demand function for the secondary season
$\varepsilon_{2,i}$	Error term for the demand function of retailer i
λ	Lagrange multiplier
μ	Expected value
$\Pi(d_1)$	Transformed expected revenue function for the primary season

$\Pi(d_2, p_1)$	Transformed expected revenue function for the secondary season
$\Pi(p_1)$	Expected revenue function for the primary season
$\Pi(p_2, p_1)$	Expected revenue function for the secondary season
σ	Standard deviation
σ_1	Standard deviation of ε_1
σ_2	Standard deviation of ε_2
$\sigma_{2,i}$	Standard deviation of $\varepsilon_{2,i}$
$\phi(x \mid \mu, \sigma)$	PDF of random variable x with the expected value of μ and the standard deviation of σ
$\phi(z)$	PDF of standard normal random variable z
$\Phi(x)$	CDF of standard normal random variable x

LIST OF ACRONYMS/ABBREVIATIONS

BSLP	Base-Stock List-Price
CDF	Cumulative Distribution Function
CIRPE	Cost of Ignoring Reference Price Effect
KKT	Karush-Kuhn-Tucker
NP-I	No-pooling with Identical Retailers
NP-NI	No-pooling with Non-identical Retailers
PDF	Probability Density Function
POOL	Pooling of the Retailers Demand

1. INTRODUCTION

Many products follow the stages of product life cycle, which are defined as introduction, growth, maturity, and decline. On the customer side, demand is located in the life cycle as two stages with non-overlapping consecutive time segments: primary and secondary customer markets. The market with the customer demand realizing right after the introduction of the product is referred to as primary market, which lasts until the end of the maturity stage. Primary market customers are typically considered more loyal to the brand and less sensitive to the price of the product. The secondary market refers to the customer demand realizing in the decline period of the product life cycle, where the companies usually change the marketing strategies via either introducing new version of the product or marking down the prices. As opposed to the primary market customers, the secondary market customers are generally more price sensitive. For technology intensive products, the distinction between these two market customers is more apparent. The product life cycle stages and their effects on customer demand for such products are discussed in detail [1–3].

The secondary market customers account for the former price as a reference to evaluate the current price of the product. This incident affecting the customer perception on purchasing the product is referred to as *reference price effect* on demand. It also signifies that the pricing for the primary market will also influence the secondary market demand.

Supply and demand match brings a significant increase in profit. The firm may not determine the demand of the product; nevertheless, designating the price helps the firm to regulate the demand, accordingly the supply. Therefore, joint pricing and inventory planning decisions have received substantial attention in the literature [4]. The benefits originated from joint decision of inventory replenishment and pricing are discussed in [5–7].

In this thesis, our goal is to build a model for finding optimal ordering, pricing, and inventory allocation decisions of a national distributor for a technology intensive product under the price dependent stochastic demand and two-tier customer market, and to analyze the effect of reference price on the optimal decisions. As the primary and the secondary market demands occur in different time intervals, from the point of view of the company these time periods can be also referred to as the primary and the secondary selling seasons, respectively. Through the primary selling season, the product is marketed through the sales channels of the distributor and the inventory is centralized. The decision maker can track the stock level information and hence can intervene in supply disruptions by transshipping the product from a location to another requested location. Therefore, the primary sales season is modeled as a single stocking point problem. At the beginning of the secondary sales season, complete supply availability in replenishment is considered. Through the secondary selling season, the product is marketed through the discount stores (or retailers). The inventory is decentralized, and the stock level information is more difficult to track after stock allocation to the retailers contrary to the primary season. Hence it is limited to intervene in transshipment of the product between demand locations. Therefore, the secondary sales season is modeled as a multiple stock-keeping problem.

According to the system to be considered, at the beginning of the primary sales season, the firm decides on the pricing and the initial order quantity. Then, the primary market demand realizes through the season. At the beginning of the secondary sales season, the firm decides on the pricing, any additional items to order and how many stocks to allocate to each discount store. Then, the secondary market realizes at discount stores through the season.

We show that a base-stock list-price (BSLP) policy is optimal for concave demand models and concave transformed expected revenue functions as in Guler *et al.* [8]. Then, we perform a computational study to observe the effect of various parameters on the optimal policy. In this study, we use linear demand functions with additive error term and absolute difference reference effect. Our major observations are as follows:

(i) As the reference price effect rises, price and order-up-to level for the primary season decreases, and price and order-up-to level for the secondary season increases.

(ii) As the number of retailers increases, variability in demand increases and consequently, order-up-to level for the secondary season considerably increases. This also eventuate in the decrease in the optimal expected profit.

(iii) When demand is pooled, demand variability goes down and accordingly order-up-to level decreases. Since there is centralization, an increase in the profit occurred. Benefits of risk-pooling increases as the number of retailers or coefficient of variation increases.

(iv) An increase in coefficient of variation leads to an increase in demand variability. Accordingly, order-up-to levels raise, and the optimal expected profit declines. The % penalty of cost of ignoring reference price effect increases as well.

(v) An increase or a decrease in the average market potential compared to the primary sales season decreases the % penalty of cost of ignoring reference price effect.

In this thesis, we study ordering, pricing, and stock allocation decisions under the presence of price and reference price dependent stochastic demand. Within the area of two-tier customer market, this is the first study accommodating all factors previously stated. This is our major motivation and contribution. In this context, we provide near-explicit expressions for the optimal decisions. Also, we introduce the cases where the retailer demands are pooled in order to observe the effect of pooling on the optimal decisions. Another contribution of this thesis is to investigate the realized loss in the profit where reference price effect is ignored by the decision maker.

The remainder of the thesis is organized as follows. In Chapter 2, we provide a literature review of two-tier customer market, inventory and pricing models, reference price, and stock allocation. In Chapter 3, we present the dynamic programming formulation of the problem, the transformation made for providing the sufficient conditions for optimality, and the optimal policy. Computational environment is introduced, and then the numerical results and discussion are presented in Chapter 4. Finally, we summarize the study and our findings, and point out possible extensions in Chapter 5.

2. LITERATURE REVIEW

There are four streams of research related to this study, which are two-tier customer markets, inventory and pricing models, reference price effect, and allocation problem with stochastic demand. We review the related literature in each stream individually.

2.1. Literature on Two-tier Customer Market

Kouvelis and Gutierrez [9] is one of the first papers where two markets with non-overlapping selling seasons with stochastic demand is introduced. They develop centralized and decentralized control policies under the exchange rate uncertainty, considering a global newsvendor. In [9], they propose a nonlinear pricing scheme for production coordination by means of an intermediary firm as an alternative for the decentralized control policy.

Petruzzi and Dada [10] study on an inventory and pricing problem in the presence of two non-overlapping retail market. In their setup, stochastic additive demand is used, and foreign exchange rate uncertainty is included similar to [9]. They obtain a centralized decision policy for the secondary market through the demand information observed from the primary market.

Lee and Whang [11] focus primarily on a secondary market where resellers can trade excess inventories among each other in the secondary market. They develop a two-period model with a single manufacturer and multiple resellers and derive the optimal decisions for the resellers and the equilibrium price for the secondary market. Lee and Whang [11] also investigate the effects of the secondary market on the manufacturer and supply chain performance and mention the options of the manufacturer to increase sales in the existence of the secondary market.

Maiti and Giri [12] presents a two-period supply chain model comprised of one manufacturer and one retailer where Stackelberg strategies are followed. They investigate the manufacturer's pricing strategies and the retailer's decision strategies where the manufacturer is Stackelberg leader and the retailer is the follower. Maiti and Giri [12] also analyze the value of adherence to the contract and the effect of the contracts on the players both analytically and with a computational study.

Kyparisis and Koulamas [13] model a price setting newsvendor with nonlinear salvage revenue and shortage cost under linear additive demand in the presence of non-overlapping selling seasons. Kyparisis and Koulamas [13] investigate the equivalence of their model to the results of Petruzzi and Dada [10], utilizing their notation.

Nagare and Dutta [14] study the single period ordering and markdown pricing policies where the period is divided into two non-overlapping segments as the prime period and markdown period. Also, market segmentation is considered in terms of price sensitivity of the customers. In [14], the demand is modeled as deterministic and a function of inventory level, price, and time. Nagare and Dutta [14] show the benefits of market segmentation and the superiority of a markdown policy to a single pricing policy.

Raza *et al.* [15] consider the primary and the secondary market concept in the process mean selection context. In [15], besides the joint pricing and production decisions, quality decision is made in the presence of demand leakage and price-dependent stochastic demand.

Rajabi *et al.* [16] investigate the inventory planning and control problem under random demand in a two consecutive selling seasons. In [16], purchasing is realized in a preseason period and time dependent. They put emphasis on the selling price of end-of-period products for the sales of the secondary season.

Gullu and Tabanli [17] examine the joint ordering, pricing, and inventory allocation problem in the existence of non-overlapping consecutive selling seasons. They use demand models having additive and multiplicative type price dependency where the randomness is introduced with additive error terms. Giving near-explicit expressions for the optimal decisions, they provide that a base-stock list-price is optimal for the problem.

2.2. Literature on Inventory Control and Pricing

In this area, Whitin [18] is one of the first papers addressing to the connection between pricing and inventory control. Elmaghraby and Keskinocak [19], Yano and Gilbert [20], Chan *et al.* [21], and Chen and Simchi-Levi [22] present extensive surveys on benefits of coordinating pricing and inventory decisions. Petruzzi and Dada [23] examine the newsvendor problem by simultaneous decisions of stocking quantity and pricing under the price dependent random demand for a single period problem.

Multi-period extensions of Petruzzi and Dada [23] are investigated in Federgruen and Heching [5] and Chen and Simchi-Levi [6]. Federgruen and Heching [5] show the base-stock list-price policy is optimal when the demand function is linear. In [5], the unsatisfied demand is considered backordered and no setup cost is included. On the other hand, Chen and Simchi-Levi [6] provide a version of [5] with fixed ordering cost. The problem in Chen and Simchi-Levi [6] is extended for the infinite horizon case and continuous-review models [24, 25]. Polatoglu and Sahin [26] examine the multi-period model with finite horizon case where the unsatisfied demand eventuates in a lost sale. Kocabiyikoglu and Popescu [27] introduce *the lost-sales rate elasticity* which captures structural properties of the newsvendor with pricing model for a given price and inventory level.

Feng *et al.* [4] provide sufficient conditions for the optimality of the base-stock list-price where the demand follows a generalized additive model.

In a recent study, Chen *et al.* [28] study the joint pricing and inventory control problem with fixed ordering cost and incomplete demand information where the unsatisfied demand is backlogged in the presence of price-dependent random demand.

Combined decisions of inventory control and pricing recently evolve into several directions. One of them is game-theoretic approach. Huang *et al.* [29] consider a dual-channel system under random demand with one manufacturer and one retailer where Stackelberg strategies are followed. While the manufacturer acts as Stackelberg leader, making decisions of price in both channels, the retailer is considered as the follower, which decides on the order quantity and retail price. In Tao *et al.* [30], inventory inaccuracy is considered for the joint pricing and inventory replenishment strategies where this time the retailer acts as Stackelberg leader in the existence of one supplier and one retailer. Mahmoodi [31] also studies coordinated inventory and pricing decisions with Stackelberg framework where the manufacturer determines on the wholesale price and the retailer decides the retail price and the replenishment cycle. Later, Mahmoodi [32] extends the model in [31] to duopoly supply chains where both chains consist of one manufacturer and one retailer.

Rubio-Herrero and Baykal-Gursoy [33] investigate the price-setting newsvendor problem for a single item and single-period, focusing on the risk attitudes and mean-variance consideration, where the demand is introduced as price-dependent, additive, and random. Fan *et al.* [34] study the joint pricing and replenishment problem for fresh products under consumer behaviour. Jadidi *et al.* [35] examine the combined pricing and inventory policy with price discount aspect. In [35], the supplier offers all-unit quantity discount due to obsolescence of the product in time, and demand is random and price-dependent. Marand *et al.* [36] study the joint inventory and pricing model in queueing theory perspective under continuous-review system where customer arrival rate is price-dependent. For dynamic pricing and inventory control problem, Yao [37] brings a different approach where the demand is modeled as Brownian motion alongside price dependency with the framework of infinite horizon continuous-review. In Qin *et al.* [38], Chen *et al.* [39], Chen *et al.* [40], and Chen *et al.* [41], the dynamic pricing and inventory control problem is handled in the censored demand concept.

Chen *et al.* [42] tackle the joint pricing and inventory control problem examining constant-order policy. According to this policy, a constant order quantity is placed in every period and the pricing decision is made in accordance with on-hand inventory.

Feng *et al.* [43] study the coordinated pricing and inventory decisions where the unsatisfied demand eventuates in a lost sale and the price has inventory level dependence. In Chen and Shi [44], integrating decisions of pricing and inventory control is analyzed within the context of strategic customers under infinite horizon. In [44], the customers are offered a set of purchase options where the delivery time is different for each option. Hu *et al.* [45] present joint decisions of pricing and inventory replenishment in a periodic-review framework where a fixed cost and a concave or convex variable cost are taken into account. While Hu *et al.* [45] provide the structure of optimal policy for the single-period problem, they advance a heuristic for the multi-period problem.

Joint inventory replenishment and pricing models for perishable products is highly researched recently. Since these products have different characteristics than technology intensive products, distinctive considerations emerge such as waste and deterioration rate. Therefore, demand is considered time-dependent for perishable products alongside other dependencies of demand. For the recent papers in the area of joint pricing and inventory control for perishable products, we refer the reader to [46–58].

The studies in this stream, involving the reference price effect, are reviewed in Section 2.3.

2.3. Literature on Reference Price Effect in Supply Chains

In recent years, along with noticing the actual price is not the only determinant for modelling the demand, the research stream on reference price effect arises. In the supply chain management context, Popescu and Wu [59], Urban [60] and Gimpl-Heersink [7] are remarkable papers, which analyze the coordinating pricing and inventory decisions under reference price effects. Gimpl-Heersink [7] demonstrates that base-stock list-price policy is optimal under one-period and two-period settings by using a demand function that is linear in price and the reference price.

In Maiti and Giri [12], absolute difference reference effect is used in the demand function of the secondary period similar to our study.

Guler *et al.* [8] examine a multi-period problem with reference price effect and obtain dynamic pricing and inventory policies. Guler *et al.* [8] also shows that a state-dependent order-up-to policy is optimal for concave demand models and concave transformed expected revenue functions. In Guler *et al.* [61], the same problem is considered specific to additive random error term. Chen *et al.* [62] investigate the problem in Guler *et al.* [8] when the customers are loss averse. Chen *et al.* [63] study the dynamic pricing problem with stochastic reference price effect. In a very recent study, Cao and Duan [64] examine joint production and pricing policies under stochastic reference price effect through an infinite horizon in a continuous-time framework.

Den Boer and Keskin [65] investigate the dynamic pricing problem for loss-averse customers, incorporating the reference price effect and demand learning. Chen *et al.* [66] uses the reference price effect in a two-stage pricing model in the existence of strategic consumers. Duan and Liu [67] handle the dynamic pricing problem for perishable items under reference price effect. In [67], demand is affected by sales price, the reference price as well as the quality of the product. In the model of Hsieh and Dye [68], dynamic pricing for perishable items is tackled where the inventory level influences the demand rate in addition to price and the reference price. All types of risk behavior are considered in [68].

In a very recent study, Wang *et al.* [69] analyze joint pricing and ordering policies under the existence of reference price effect and loss-averse customers. In [69], demand is dependent on price and reference price via purchasing utility of customer. Song *et al.* [70] study on the same problem by modelling the stochastic customer demand in a different way, a choice model, which consists of utility function, loss aversion and reference price. Similar to [69, 70], Zhang *et al.* [71] investigate the loss-aversion and reference price effect on the joint decisions of pricing and production policies. In Zhang *et al.* [71], demand has also time dependency and the reference price exhibits an asymmetric characteristic.

2.4. Literature on Allocation Problem with Stochastic Demand

Allocation of a central inventory to demand locations under random demand is first investigated by the seminal paper Eppen and Schrage [72]. The model of Eppen and Schrage [72] were extended by other researchers (see, for example, Bollapragada *et al.* [73] and Diks and De Kok [74]) in several directions. Federgruen and Zipkin [75] provide the dynamic version of the allocation problem under periodic-review system where the unsatisfied demand is backordered and linear cost of holding and backorder is incurred. Corbett and Rajarm [76] investigate the effect of pooling under nonnormal and dependent demand structures. In this stream, Federgruen [77] and De Kok and Graves [78] present notable surveys on the allocation problem with random demand.

Pramudyo and Luong [79] study vendor managed inventory problem in the presence of one vendor and multiple retailers where the difficulties that the vendor face due to lack of information sharing is no longer a problem.

In Kloos *et al.* [80], a single-period allocation planning problem is handled in a sales hierarchy context. They provide centralized and decentralized approaches to the problem under the supply scarcity and uncertain demand with the target of service-level targets. Similar to Kloos *et al.* [80], Fleischmann *et al.* [81] tackle the problem of allocation of scarce supply under the presence of stochastic demand and supply lead times where the customer hierarchy is considered.

In Chavarro *et al.* [82], integer ratio policy is investigated in a one warehouse and multiple resellers setting where stochastic demand is introduced. Since deterministic demand is unrealistic, when demand stochasticity is ignored, underestimation of the total cost comes into question. Chavarro *et al.* [82] show that as the demand variability and backorder cost increase, this underestimation increases as well. They also propose a method to offer cost savings.

Bayram *et al.* [83] deal with the inventory, capacity, and demand allocation decisions where multiple warehouses exist. In [83], linear and concave cost structures are used, and stochastic demand and stochastic production lead time are considered. They demonstrate that the customer allocations are not necessarily single-sourced by proposing a formulation for demand allocation problem.

Nambiar *et al.* [84] analyze the inventory allocation problem in a two-echelon structure under a multi-period setting for seasonal goods where the unsatisfied demand ends up with lost sale and transshipment between retailers is not allowed. In [84], minimization of lost sales and holding cost is aimed. They develop a monitorable algorithm for the problem using Lagrangian relaxation in the presence of demand learning and demonstrate that the algorithm is asymptotically optimal. Then, with a two-period setting, they investigate the effect of demand learning on the allocation decisions.

Edirisinghe and Atkins [85] analyze the effect of risk-pooling in a two-echelon structure under a two-period allocation setting. In [17], stock allocation to the retailers is considered for the secondary season. By focusing on the secondary season in their numerical study, they mention the pooling effect on the optimal decisions and the effect of various parameters on the benefits of risk-pooling. For details of the risk-pooling concept, the reader is referred to Mak and Shen [86].

Ample studies have been perfromed within these four streams. Although some studies we have pointed out involve more than one streams of our research interest, none of them investigates these four streams together. Among the studies, Gullu and Tabanli [17] analyze the joint ordering, pricing and inventory control decisions within the area of two-tier customer market where the reference price effect is not incorporated into the system. We build such a model that the considerations of Gullu and Tabanli [17] is included alongside the reference price effect.

3. THE MATHEMATICAL MODEL

In this chapter, a dynamic programming approach for joint pricing and inventory allocation problem in the presence of a two-tier customer market is presented. Figure 3.1 and 3.2 together depict the two-tier customer market system and its components to be considered by means of inventory level-time graphs. While 3.1 represents the primary selling season realizing through the distributor, Figure 3.2 represents the secondary selling season realizing through the retailers. We provide the details for the system in Section 3.1. Then stochastic demand and its structure are introduced in Section 3.2. Moreover, the firm that makes decisions of the joint inventory, pricing and allocation is referred as the distributor in Chapter 3.

The model is an extended version of Gullu and Tabanli [17] where our model accounts for the reference price effect on mean demand for the secondary season. We formulate the model as a two-stage dynamic programming similar to the formulation of [17]. To show the candidate policy (BSLP) is optimal, the concavity of the expected profit functions must be fulfilled. Accordingly, the expected inventory holding and backorder costs must be convex, but it is not satisfied unless the mean demand is linear in price and reference price. Nevertheless, in this case the concavity of the expected revenue function is not satisfied. To cope with this hardship, Chen *et al.* [22] propose a transformation technique. When the concavity of the expected profit functions is satisfied, the convexity of the expected inventory holding and backorder costs is not satisfied. With a transformation, also applied in [6, 8, 24, 25], the mean demand turns into a variable and accordingly the price becomes a function of demand and reference price. Under the assumptions of concave demand model and concave transformed expected revenue function, the concavity of transformed expected profit functions is satisfied, and hence we conclude that a BSLP policy is optimal.

3.1. Two Selling Seasons for Two-tier Customer Market

As mentioned in Chapter 1, there are two consecutive non-overlapping time periods referred as the primary and the secondary selling seasons. The distributor orders $y_1 - I_1$ items for the primary selling season in order to increase the inventory level to y_1, assuming that the lead-times are negligible, where I_1 denotes the inventory level of the item at the beginning of the primary season ($y_1 \geq I_1$). The unit price and the unit ordering cost for the product for the primary season are denoted as p_1 and c_1, respectively ($p_1 > c_1$). We assume that unfulfilled demand in the primary season is backordered and the unit backordering and holding costs are given by b_1 and h_1, respectively ($b_1 > c_1$).

Let I_2 be the inventory level at the beginning of the secondary selling season before the placement of orders. Then, I_2 is expressed as in the following equation:

$$I_2 = \big(y_1 - D_1(p_1)\big)^+ \tag{3.1}$$

where $(a)^+$ denotes $max(a, 0)$ and $D_1(p_1)$ is the demand through the primary season.

The distributor orders additional items for the secondary season to increase its inventory level to y_2 where $y_2 \geq I_2$ and lead-times are negligible as in the primary season. Also, we assume that the replenishment decision of the distributor is completely fulfilled, signifying that supply uncertainty is ignored. The distributor designates the price of the product for the secondary season, which is denoted as p_2, and the stock allocation to the retailers for given y_2 and p_2, denoting as $s_{2,i}$ for $i = 1, ..., N$, where N is the number of retailers. Let p_2, c_2, b_2, and h_2 be the unit price, ordering cost, backorder cost, and inventory holding cost through the secondary season where $p_2 \geq c_2$ and $b_2 \geq c_2$, respectively. In our model, it is assumed that p_2, c_2, b_2, and h_2 are equal for all retailers and transportation costs between locations are ignored.

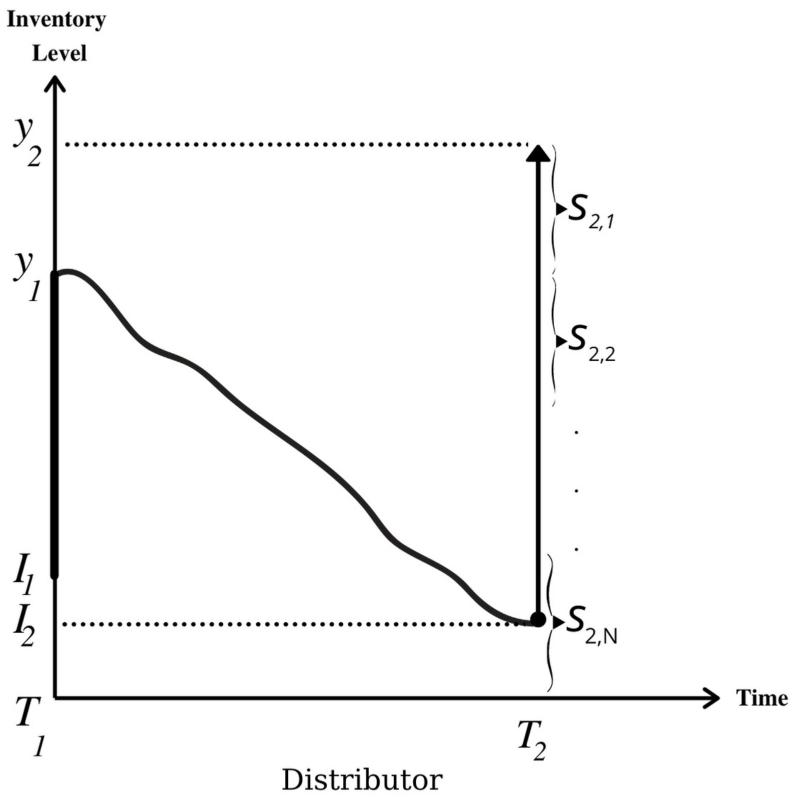

Figure 3.1. Representation of the primary selling season.

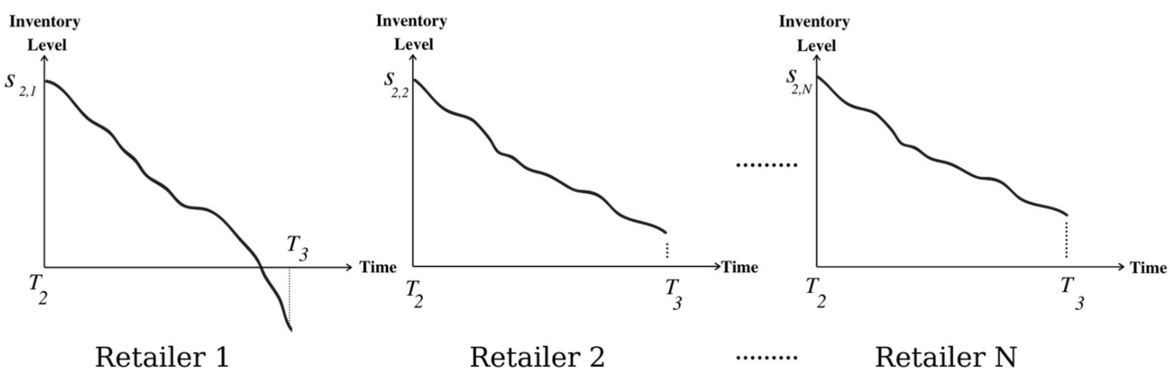

Figure 3.2. Representation of the secondary selling season.

The system to be considered is summarized as inventory level - time graphs in Figure 3.1 and 3.2. In Figure 3.1, T_1 and T_2 denote the beginning and ending of the primary season, respectively. In Figure 3.2, T_2 and T_3 denote the beginning and ending of the secondary season, respectively. Since two seasons are consecutive, the beginning of the secondary season also means the ending of the primary season.

3.2. Structure of Stochastic Demand Functions

Price-dependent demand models with additive or multiplicative random error term are generally used for modelling the demand in the inventory and pricing literature. In our setting, the demand through the primary season, denoted as $D_1(p_1)$, is expressed as the summation of two parts: $\hat{d}_1(p_1)$ and ε_1. $\hat{d}_1(p_1)$ refers to the mean demand, assumed as a decreasing function of p_1, also given by Assumption 3.1. The latter, ε_1, is the additive error term which is normally distributed with mean of 0 and and standard deviation of σ_1. Then the corresponding demand model can be shown as in the following equation:

$$D_1(p_1) = \hat{d}_1(p_1) + \varepsilon_1 \tag{3.2}$$

The demand of secondary season is influenced by the last observed price (p_1), which is called reference price, as well as the current price (p_2). Then the demand function for retailer i is denoted as $D_{2,i}(p_2, p_1)$ for $i = 1, ..., N$ where N is the number of retailers. $D_{2,i}(p_2, p_1)$ mainly consists of two parts in the form of summation, which are $\hat{d}_{2,i}(p_2, p_1)$ and $\varepsilon_{2,i}$ where $\hat{d}_{2,i}(p_2, p_1)$ is the mean demand for retailer i and $\varepsilon_{2,i}$ is the additive error term which is normally distributed with mean of 0 and standard deviation of $\sigma_{2,i}$ for retailer i. Then the corresponding demand function under the reference price effect for retailer i can be expressed as follows:

$$D_{2,i}(p_2, p_1) = \hat{d}_{2,i}(p_2, p_1) + \varepsilon_{2,i} \qquad\qquad i = 1, ..., N \tag{3.3}$$

We can also reach the mean demand through the secondary season by adding the mean demand of retailers up. Let $\hat{d}_2(p_2, p_1)$ denote the mean demand for the secondary season. Then, the mean demand for the secondary season can be determined by the following equation:

$$\hat{d}_2(p_2, p_1) = \sum_{i=1}^{N} \hat{d}_{2,i}(p_2, p_1) \tag{3.4}$$

Following assumption is considered for the mean demand functions:

Assumption 3.1. *$\hat{d}_1(p_1)$ is strictly decreasing in p_1 and $\hat{d}_2(p_2, p_1)$ and $\hat{d}_{2,i}(p_2, p_1)$ are strictly decreasing in p_2 and increasing in p_1. All mean demands are nonnegative, bounded, and continuous.*

The current price and the reference price are assumed to be confined by a lower and an upper bound as a corollary of Assumption 3.1. In the pricing literature, Popescu and Wu [59] and Guler *et al.* [8] also use a bounded price set for the current and the reference prices. In our case, while demand function of the secondary season includes both prices, the demand function of the primary season has only one variable, which is the current price. Therefore, this distinction leads us to evaluate each season separately in terms of demand function and its properties.

For the primary season, establishing a lower (upper) bound to the price corresponds to establishing an upper (lower) bound to $\hat{d}_1(p_1)$ because $\hat{d}_1(p_1)$ is strictly decreasing in p_1 given by Assumption 3.1. Let the lower bound and the upper bound on p_1 denote p_1^l and p_1^u, respectively. Since the mean demand is always nonnegative, p_1^u can be set in the manner where the mean demand $\hat{d}_1(p_1) = 0$. On the other hand, p_1^l can be set as unit ordering cost c_1. On the demand side, d_1^l is already set as 0 and d_1^u can be automatically set in parallel with p_1^l where d_1^l and d_1^u denote the lower and the upper bound on $\hat{d}_1(p_1)$, respectively. Let $\mathbb{P}_1$ denote the set of p_1 defined by the bounds. Then $\mathbb{P}_1 = \{p_1 : p_1 \in [p_1^l, p_1^u], d_1^l \leq \hat{d}_1(p_1) \leq d_1^u\}$ and $\mathbb{P}_1$ evidently forms a convex set.

In the presence of reference price, bounds are established differently from the primary season. Let d_2^l, d_2^u, p_2^l, and p_2^u denote the lower and the upper bound on $\hat{d}_2(p_2, p_1)$, and the lower and the upper bound on p_2, respectively. Let $\mathbb{P}_2$ denote the set of p_1 and p_2 defined by the bounds. Bounds on p_1 are already established before. Since $\hat{d}_2(p_2, p_1)$ is strictly decreasing in p_2 and increasing in p_1 given by Assumption 3.1, p_2^u can be set in a way where $\hat{d}_2(p_2, p_1) = 0$, implying that $d_2^l = 0$, and $p_1 = p_1^l$. On the other hand, p_2^l can be set as unit ordering cost c_2. Accordingly, upper bound for $\hat{d}_2(p_2, p_1)$ is automatically set in parallel with p_2^l where $p_2 = p_2^l$ and $p_1 = p_1^u$. Then $\mathbb{P}_2 = \{(p_1, p_2) : p_1 \in [p_1^l, p_1^u], p_2 \in [p_2^l, p_2^u], d_2^l \leq \hat{d}_2(p_2, p_1) \leq d_2^u\}$ and $\mathbb{P}_2$ evidently forms a convex set with a rectangular shape.

Mean demand for the secondary season consist of additively tied two parts. The former only depend on the current price p_2, and the latter depends on both current and reference prices (p_2 and p_1, respectively), which also called reference effect. In the literature, the reference effect on demand is generally modeled in two ways: *absolute difference* and *relative difference* models. To exemplify, if the second part of the mean demand is expressed as $c(p_1 - p_2)$ where c is a positive constant, then this is an example of the absolute difference model. On the other hand, if the second part of the mean demand is expressed as $c(\frac{p_1 - p_2}{p_1})$, then this is an example of the relative difference model. In our setting, we apply absolute difference reference effect on demand.

3.3. Mathematical Formulation

We formulate the optimal order quantity, pricing, and allocation decisions for the distributor as a dynamic programming, consisted of two stages. $J_1(I_1)$ and $J_2(I_2, p_1)$ denote the expected profit of the entire problem and the secondary season, respectively where I_1 and I_2 are the beginning inventory level of two selling seasons. $V(y_2, p_2, p_1)$ is the optimal profit acquired after the stock allocation to the retailer i in the secondary season.

$\hat{G}(.)$ functions express a summation of expected inventory holding and backorder costs. While $\hat{G}_1(y_1, p_1)$ and $\hat{G}_2(y_2, p_2, p_1)$ are the summation of expected inventory holding and backorder costs through the primary season and the secondary season, respectively, $\hat{G}_2(s_{2,i}, p_2, p_1)$ denotes the summation of expected inventory holding and backorder costs for retailer i through the secondary season. Explicit forms of these functions are given as follows:

$$\hat{G}_1(y_1, p_1) = h_1 E[(y_1 - \hat{d}_1(p_1) - \varepsilon_1)^+] + b_1 E[(\hat{d}_1(p_1) + \varepsilon_1 - y_1)^+] \tag{3.5}$$

$$\hat{G}_2(y_2, p_2, p_1) = h_2 E[(y_2 - \hat{d}_2(p_2, p_1) - \varepsilon_2)^+] + b_2 E[(\hat{d}_2(p_2, p_1) + \varepsilon_2 - y_2)^+] \tag{3.6}$$

$$\hat{G}_2(s_{2,i}, p_2, p_1) = h_2 E[(s_{2,i} - \hat{d}_{2,i}(p_2, p_1) - \varepsilon_{2,i})^+] + b_2 E[(\hat{d}_{2,i}(p_2, p_1) + \varepsilon_{2,i} - s_{2,i})^+] \tag{3.7}$$

where ε_2 is the additive error term for the total demand through the secondary season. Then, the mathematical formulation is:

$$\begin{aligned} J_1(I_1) = \max_{p_1 \geq c_1, y_1 \geq I_1} \big\{ & p_1 \hat{d}_1(p_1) - c_1(y_1 - I_1) - \hat{G}_1(y_1, p_1) \\ & + E[J_2\big((y_1 - \hat{d}_1(p_1) - \varepsilon_1)^+, p_1\big)] \big\} \end{aligned} \tag{3.8}$$

$$J_2(I_2, p_1) = \max_{p_2 \geq c_2, y_2 \geq I_2} \big\{ V(y_2, p_2, p_1) - c_2(y_2 - I_2) \big\} \tag{3.9}$$

$$\text{where} \quad V(y_2, p_2, p_1) = \max \big\{ p_2 \hat{d}_2(p_2, p_1) - \sum_{i=1}^{N} \hat{G}_2(s_{2,i}, p_2, p_1) \big\} \tag{3.10}$$

$$\text{subject to} \sum_{i=1}^{N} s_{2,i} = y_2, \qquad s_{2,i} \geq 0 \qquad i = 1, ..., N \tag{3.11}$$

Constraint in Eq. 3.11 ensures the distribution of all items to the retailers. Before optimality conditions are stated, we go over certain details of stock allocation to the retailers in subsequent section.

3.3.1. Reformulation of the Model

Presently consider the maximization problem expressed in Eq. 3.10 and Eq. 3.11. When p_1, y_2, and p_2 are given, $s_{2,i}$ is the only variable remaining in the subproblem. Therefore, the first term of Eq. 3.10 becomes a constant according to $s_{2,i}$. Thus, we can convert this subproblem in Eq. 3.10–3.11 to a minimization problem as in Eq. 3.12–3.13.

$$V(y_2, p_2, p_1) = \min \sum_{i=1}^{N} \hat{G}_2(s_{2,i}, p_2, p_1) \tag{3.12}$$

$$\text{subject to} \quad \sum_{i=1}^{N} s_{2,i} = y_2, \qquad s_{2,i} \geq 0 \qquad i = 1, ..., N \tag{3.13}$$

As a preliminary to solution of optimal stock allocation, on the assumption of given values of p_1, y_2, and p_2, first define notations and certain expressions to be used further in the following items:

(i) $\phi(x \mid \mu, \sigma)$ denotes the probability distribution function (PDF) of normal random variable x with the expected value of μ and the standard deviation of σ. $\phi(z)$ denotes PDF of standard normal random variable.

(ii) $\Phi(x)$ denotes the cumulative distribution function (CDF) of standard normal random variable x.

(iii) λ denotes the Lagrange multiplier of the constraint in Eq. 3.13 and and L denotes the Lagrange function.

(iv) $s^*_{2,i}$ denotes the optimal value of $s_{2,i}$.

(v) For the sake of simplicity, denote $\hat{d}_2(p_2, p_1)$, $\hat{d}_{2,i}(p_2, p_1)$, and $D_{2,i}(p_2, p_1)$ as $\hat{d}_2$, $\hat{d}_{2,i}$, and $D_{2,i}$, respectively.

By removing the expression of expected values, we first rewrite the expression of $\sum_{i=1}^{N} \hat{G}_2(s_{2,i}, p_2, p_1)$ in Eq. 3.12 as follows:

$$\sum_{i=1}^{N} \hat{G}_2(s_{2,i}, p_2, p_1) = h_2 \sum_{i=1}^{N} \Big(\int_0^{s_{2,i}} (s_{2,i} - D_{2,i}) \phi(D_{2,i} \mid \hat{d}_{2,i}, \sigma_{2,i}) dD_{2,i} \Big)$$

$$+ b_2 \sum_{i=1}^{N} \Big(\int_{s_{2,i}}^{\infty} (D_{2,i} - s_{2,i}) \phi(D_{2,i} \mid \hat{d}_{2,i}, \sigma_{2,i}) dD_{2,i} \Big) \qquad (3.14)$$

Then, we build Karush-Kuhn-Tucker (KKT) conditions, which is commonly used in convex optimization problems, for the minimization problem expressed in Eq. 3.12 – 3.13. Corresponding Lagrange function can be written as in Eq. 3.15. We take the derivative of the Lagrange function in Eq. 3.15 with respect to $s_{2,i}$ values for all $i = 1, ..., N$ and equalize it to 0 as in Eq. 3.16. By rearranging Eq. 3.16, we find Eq. 3.17 and then Eq. 3.18. Finally we attain $s^*_{2,i}$ values, which is expressed in Eq. 3.19; however, $s^*_{2,i}$ values contain the variable λ.

$$L = h_2 \sum_{i=1}^{N} \Big(\int_0^{s_{2,i}} (s_{2,i} - D_{2,i}) \phi(D_{2,i} \mid \hat{d}_{2,i}, \sigma_{2,i}) dD_{2,i} \Big) + b_2 \sum_{i=1}^{N} \Big(\int_{s_{2,i}}^{\infty} (D_{2,i} - s_{2,i})$$

$$\phi(D_{2,i} \mid \hat{d}_{2,i}, \sigma_{2,i}) dD_{2,i} \Big) + \lambda \Big(\sum_{i=1}^{N} s_{2,i} - y_2\Big) \qquad (3.15)$$

$$\frac{dL}{ds_{2,i}} = h_2 \int_0^{s_{2,i}} \phi(D_{2,i} \mid \hat{d}_{2,i}, \sigma_{2,i}) dD_{2,i} - b_2 \int_{s_{2,i}}^{\infty} \phi(D_{2,i} \mid \hat{d}_{2,i}, \sigma_{2,i}) dD_{2,i} + \lambda = 0$$

$$i = 1, ..., N \qquad (3.16)$$

Note that $\sum_{i=1}^{N} s_{2,i} = y_2$ for $i = 1, ..., N$ and $\hat{d}_2 = \sum_{i=1}^{N} \hat{d}_{2,i}$. We sum $s^*_{2,i}$ for all retailers up and equalize it to y_2. Afterwards, we obtain the equations in order of Eq. 3.20 and Eq. 3.21. To elude λ, by replacing the expression containing λ in Eq. 3.19

with the right side of 3.21, we obtain the optimal stock allocation to the retailer i ($s^*_{2,i}$) shown in Eq. 3.22.

$$h_2\Phi\Big(\frac{s_{2,i}-\hat{d}_{2,i}}{\sigma_{2,i}}\Big) - b_2\Bigg(1-\Phi\Big(\frac{s_{2,i}-\hat{d}_{2,i}}{\sigma_{2,i}}\Big)\Bigg) + \lambda = 0 \qquad i = 1, ..., N \qquad (3.17)$$

$$(h_2+b_2)\Phi\Big(\frac{s_{2,i}-\hat{d}_{2,i}}{\sigma_{2,i}}\Big) - b_2 + \lambda = 0 \qquad i = 1, ..., N \qquad (3.18)$$

$$s^*_{2,i} = \Phi^{-1}\Big(\frac{b_2-\lambda}{b_2+h_2}\Big)\sigma_{2,i} + \hat{d}_{2,i} \qquad i = 1, ..., N \qquad (3.19)$$

$$\sum_{i=1}^{N}\sigma_{2,i}\Phi^{-1}\Big(\frac{b_2-\lambda}{b_2+h_2}\Big) + \hat{d}_2 = y_2 \qquad (3.20)$$

$$\Phi^{-1}\Big(\frac{b_2-\lambda}{b_2+h_2}\Big) = \frac{y_2-\hat{d}_2}{\sum_{i=1}^{N}\sigma_{2,i}} \qquad (3.21)$$

$$s^*_{2,i} = \hat{d}_{2,i} + \frac{\sigma_{2,i}}{\sum_{j=1}^{N}\sigma_{2,j}}(y_2-\hat{d}_2) \qquad i = 1, ..., N \qquad (3.22)$$

For further referrals to the optimal stock allocation to the retailers, we rewrite Eq. 3.22 in such a manner that the mean demand functions are stated as a function of related variable of price, expressed in Eq. 3.23.

$$s^*_{2,i} = \hat{d}_{2,i}(p_2,p_1) + \sigma_{2,i}\frac{y_2-\hat{d}_2(p_2,p_1)}{\sum_{j=1}^{N}\sigma_{2,j}} \qquad i = 1, ..., N \qquad (3.23)$$

When $s^*_{2,i}$ values are summed up, it is clearly seen that the constraint in Eq. 3.11 is satisfied. Therefore, in the presence of Eq. 3.23, Eq. 3.11 can be discarded, and Eq. 3.10 can be rewritten. As a result, the following equation is obtained:

$$V(y_2, p_2, p_1) = p_2\hat{d}_2(p_2, p_1) + h_2\big(\hat{d}_2(p_2, p_1) - y_2\big) - (b_2 + h_2)\sum_{i=1}^{N}\sigma_{2,i}R\Big(\frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N}\sigma_{2,i}}\Big) \tag{3.24}$$

where $R(x) = \int_x^\infty (z - x)\phi(z)dz$, which also referred as to *unit loss function* in the literature. The derivation of Eq. 3.24 is shown in Appendix A. Since $V(y_2, p_2, p_1)$ no longer depends on $s_{2,i}$, the formulation in Eq. 3.8 – 3.11 can be also redefined as in the following form without using $s_{2,i}$:

$$J_1(I_1) = \max_{p_1 \geq c_1, y_1 \geq I_1}\big\{p_1\hat{d}_1(p_1) - c_1(y_1 - I_1) - \hat{G}_1(y_1, p_1) + E[J_2\big((y_1 - \hat{d}_1(p_1) - \varepsilon_1)^+, p_1\big)]\big\} \tag{3.25}$$

$$J_2(I_2, p_1) = \max_{p_2 \geq c_2, y_2 \geq I_2}\big\{V(y_2, p_2, p_1) - c_2(y_2 - I_2)\big\} \tag{3.26}$$

$$\text{where} \quad V(y_2, p_2, p_1) = p_2\hat{d}_2(p_2, p_1) - \hat{G}_2(y_2, p_2, p_1) \tag{3.27}$$

Thus Eq. 3.10 and Eq. 3.11 are combined as Eq. 3.27. Note that Eq. 3.8 – 3.9 are the same equations as Eq. 3.25 – 3.26, respectively. They are repeated once more for the formulation to stay together.

3.4. Optimality Conditions

The procedure to be applied to find the optimal solution of the problem can be summarized in the following steps:

(i) For any given values of p_1 and y_2, first solve Eq. 3.10 and Eq. 3.11, and obtain the optimal values of p_2, and subsequently $s_{2,i}$ where $i = 1, ..., N$.

(ii) Then, for any given values of I_2 and p_1; solve Eq. 3.9, and obtain the optimal value of y_2.

(iii) Solve Eq. 3.8, and obtain the optimal value of p_1 and y_1.

In order to follow through on these steps, the first three items must be satisfied in the following conditions:

(i) The joint concavity of $V(y_2, p_2, p_1)$ function in Eq. 3.10 over p_1, p_2, and $s_{2,i}$.

(ii) The concavity of $J_2(I_2, p_1)$ function in Eq. 3.9 over y_2.

(iii) The joint concavity of $J_1(I_1)$ function in Eq. 3.8 over p_1 and y_1.

(iv) The joint concavity of $V(y_2, p_2, p_1)$ function in Eq. 3.27 over p_1, p_2, and y_2.

The fact that these conditions (i), (ii), and (iii) are fulfilled shows that a base-stock list-price (BSLP) policy is optimal. As stated in Section 3.3, we can convert Eq. 3.10 and Eq. 3.11 to Eq. 3.27 so as for $s_{2,i}$ to be taken out of the equation. Namely, satisfying the condition (i) corresponds to satisfy the condition (iv). Hence the determined BSLP policy is optimal when the conditions (ii), (iii), and (iv) are satisfied as well. Therefore, we proceed with showing the optimality conditions for Eq. 3.25 – 3.27.

In the remaining parts, concavity (or convexity) of a function implies the joint concavity (or convexity) of the corresponding function over each of its variables separately, unless otherwise specified. The sufficient conditions for optimality are to show the concavity of expected profit functions ($J_1(I_1)$ and $J_2(I_2, p_1)$) and $V(y_2, p_2, p_1)$ function in Eq. 3.27. Ordering costs are linear for both seasons; therefore, their concavity is apparent. Hence in the inventory literature there is more focus on the concavity of the expected revenue function, and the convexity of expected inventory holding and backorder cost function. Expected revenue functions for the primary and the secondary seasons, denoted as $\Pi(p_1)$ and $\Pi(p_2, p_1)$ are expressed in Eq. 3.28 and Eq. 3.29, respectively.

$$\Pi(p_1) = p_1\hat{d}_1(p_1) \tag{3.28}$$

$$\Pi(p_2, p_1) = p_2\hat{d}_2(p_2, p_1) \tag{3.29}$$

Also, expected inventory holding and backorder cost functions for the primary and the secondary season are $\hat{G}_1(y_1, p_1)$ and $\hat{G}_2(y_2, p_2, p_1)$, which their explicit forms are expressed in Eq. 3.5 and Eq. 3.6, respectively.

$\hat{G}_1(y_1, p_1)$ and $\hat{G}_2(y_2, p_2, p_1)$ are convex only where the mean demand function is linear in p_1, and p_1 and p_2, respectively [6, 8]. In this case, however; the concavity of the expected revenue function of the related season is not fulfilled. To handle this situation, Chen *et al.* [22] develop a transformation technique, and demonstrate the concavity of the expected profit function after transformation is employed.

On the other side, in case of a mean demand model with concave expected revenue function, $\hat{G}_1(y_1, p_1)$ and $\hat{G}_2(y_2, p_2, p_1)$ eventuate in nonconvexity. To elude such a challenge, transformation in the subsequent expressions is employed. We take the mean demand as the variable instead of the price by defining an inverse demand function. For the present, say, the primary season is considered. Then the price is expressed as a function of demand, denoted as $\hat{p}_1(d_1)$. Let the transformed $\hat{G}_1(y_1, p_1)$ be denoted as $\hat{G}_1(y_1, d_1)$, and $\hat{G}_1(y_1, d_1)$ turns into a linear function of y_1 and the mean demand d_1. Considering the mean demand is strictly decreasing in the price, given in Assumption 3.1, a one-to-one mapping between the mean demand and the price exists. Chen and Simchi-Levi [6] and Guler *et al.* [8] also apply such a transformation.

Differently from the mean demand of the primary season, the mean demand of the secondary season includes a reference price (p_1) along with the current price (p_2). This transformation is also accomplishable for the secondary season. The mean demand for the secondary season, $\hat{d}_2(p_2, p_1)$, is assumed to be strictly decreasing in p_2 as indicated in

Assumption 3.1. Therefore, for a given reference price p_1, one-to-one mapping between the mean demand and the current price p_2 exists. Then the current price is expressed as a function of demand and reference price, denoted as $\hat{p}_2(d_2, p_1)$ where $\hat{p}_2(d_2, p_1)$ is inverse demand function for a given p_1. With this transformation, $\hat{G}_2(y_2, p_2, p_1)$ turns into a linear function of y_2 and the mean demand d_2, denoted as $\hat{G}_2(y_2, d_2)$. Since $\hat{G}_1(y_1, d_1)$ and $\hat{G}_2(y_2, d_2)$ become a linear function of their corresponding variables, their convexity is satisfied. For details of the demonstration, please see Guler *et al.* [8]. Accordingly, expected revenue functions for the primary and the secondary seasons can be rewritten as $\Pi(d_1)$ and $\Pi(d_2, p_1)$, expressed in Eq. 3.30 and Eq. 3.31, respectively.

$$\Pi(d_1) = \hat{p}_1(d_1)d_1 \tag{3.30}$$

$$\Pi(d_2, p_1) = \hat{p}_2(d_2, p_1)d_2 \tag{3.31}$$

However, the concavity of the expected revenue function for both seasons is not satisfied yet. Before analyzing the expected revenue functions, we first recognize the structure of inverse demand function, and its connection with the original demand function.

Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be the domain of $\hat{p}_1(d_1)$ and $\hat{p}_2(d_2, p_1)$ functions after transformation, respectively. Then define $\mathbb{D}_1 = \{d_1 : d_1 \in [d_1^l, d_1^u], p_1^l \leq \hat{p}_1(d_1) \leq p_1^u\}$ and $\mathbb{D}_2 = \{(d_2, p_1) : d_2 \in [d_2^l, d_2^u], p_1 \in [p_1^l, p_1^u], p_2^l \leq \hat{p}_2(d_2, p_1) \leq p_2^u\}$. In the original problem $\mathbb{D}_1$ and $\mathbb{D}_2$ correspond to the range of $\hat{d}_1(p_1)$ and $\hat{d}_2(p_2, p_1)$, respectively. Since one-to-one mapping between the mean demand and the price exists as stated before, there exists one-to-one mapping between sets $\mathbb{D}_1$ and $\mathbb{P}_1$ either. Accordingly, there exists one-to-one mapping between sets $\mathbb{D}_2$ and $\mathbb{P}_2$ for every p_1 within its boundaries.

For $\Pi(d_1)$ and $\Pi(d_2, p_1)$ to be concave, $\hat{p}_1(d_1)$ and $\hat{p}_2(d_2, p_1)$ must be concave on $\mathbb{D}_1$ and $\mathbb{D}_2$, respectively. Furthermore, $\mathbb{D}_1$ and $\mathbb{D}_2$ must form a convex set. To show these statements, first we assume that $\hat{d}_1(p_1)$ is concave on $\mathbb{P}_1$ and $\hat{d}_2(p_2, p_1)$ is concave on $\mathbb{P}_2$, which is also expressed in Assumption 3.2.

Assumption 3.2. *$\hat{d}_1(p_1)$ is concave on $\mathbb{P}_1$ and $\hat{d}_2(p_2, p_1)$ is concave on $\mathbb{P}_2$.*

The propositions subsequently presented enable these statements to be fulfilled given Assumption 3.1 — 3.2.

Proposition 3.1. *(i) $\hat{p}_1(d_1)$ is strictly decreasing in d_1, and (ii) $\hat{p}_2(d_2, p_1)$ is strictly decreasing in d_2 and increasing in p_1.*

Proposition 3.2. *(i) $\mathbb{D}_1$ is a convex set, and (ii) $\hat{p}_1(d_1)$ is concave on $\mathbb{D}_1$.*

Proposition 3.3. *(i) $\mathbb{D}_2$ is a convex set, and (ii) $\hat{p}_2(d_2, p_1)$ is concave on $\mathbb{D}_2$.*

For the proof of Proposition 3.1, please see Appendix B. Guler *et al.* [8] give a similar proposition corresponding to Proposition 3.2 and Proposition 3.3. For the details of the proposition and its proof, we refer the reader to Proposition 2 of Guler *et al.* [8].

Statements indicated in Proposition 3.2 and Proposition 3.3 are not sufficient to show the concavity of the corresponding expected revenue function. Hence, we perform the analysis on an interval that the expected revenue function for both seasons is concave in their variables, shown by Assumption 3.3, to provide a sufficient condition for optimality.

Assumption 3.3. *$\Pi(d_1)$ is concave in d_1, and $\Pi(d_2, p_1)$ is concave in d_2 and p_1.*

Under the given assumptions, the sufficient conditions for optimality are fulfilled. The next step is to demonstrate the transformed dynamic problem formulation, given in the subsequent section.

3.5. Mathematical Formulation of Transformed Model

Along with the transformation, the demand for the primary and the secondary seasons, which are formerly expressed by a function, become decision variables as d_1 and d_2 instead of $\hat{d}_1(p_1)$ and $\hat{d}_2(p_2, p_1)$, respectively. Accordingly, p_1 is converted to $\hat{p}_1(d_1)$ for the primary season while it, behaving as a reference price, remains as a variable for the secondary season. p_2 is transformed into a function of d_2 and p_1, denoting as $\hat{p}_2(d_2, p_1)$. $\hat{G}_1(y_1, p_1)$ and $\hat{G}_2(y_2, p_2, p_1)$ turn into $\hat{G}_1(y_1, d_1)$ and $\hat{G}_2(y_2, d_2)$ functions, respectively. Explicit forms of these functions are shown in the Eq. 3.32 and Eq. 3.33.

$$\hat{G}_1(y_1, d_1) = h_1 E[(y_1 - d_1 - \varepsilon_1)^+] + b_1 E[(d_1 + \varepsilon_1 - y_1)^+] \tag{3.32}$$

$$\hat{G}_2(y_2, d_2) = h_2 E[(y_2 - d_2 - \varepsilon_2)^+] + b_2 E[(d_2 + \varepsilon_2 - y_2)^+] \tag{3.33}$$

The expected inventory holding and backorder cost function for the secondary season is expressed by only y_2 and d_2 after this transformation as in Eq. 3.33. By becoming d_2 a variable, we start the profit function for the secondary season $V(y_2, p_2, p_1)$ to denote as $V(y_2, d_2, p_1)$. After these transformations are made, the dynamic formulation of the model stated in Eq. 3.25 – 3.27 can be equivalently written as follows:

$$\begin{aligned} J_1(I_1) = \max\nolimits_{d_1 \in \mathbb{D}_1, y_1 \geq I_1} \big\{ &\hat{p}_1(d_1) d_1 - c_1(y_1 - I_1) - \hat{G}_1(y_1, d_1) \\ &+ E[J_2\big((y_1 - d_1 - \varepsilon_1)^+, p_1\big)] \big\} \end{aligned} \tag{3.34}$$

$$J_2(I_2, p_1) = \max\nolimits_{d_2 \in \mathbb{D}_2, y_2 \geq I_2} \big\{ V(y_2, d_2, p_1) - c_2(y_2 - I_2) \big\} \tag{3.35}$$

$$\text{where} \quad V(y_2, d_2, p_1) = \hat{p}_2(d_2, p_1) d_2 - \hat{G}_2(y_2, d_2) \tag{3.36}$$

3.6. Optimal Policy

In the previous sections, we demonstrate that a BSLP policy, also known as order-up-to policy, is optimal under certain given assumptions. Since the conditions for optimality are satisfied, we can apply the procedure mentioned in Section 3.4.

Note that $J_1(I_1)$ and $J_2(I_2, p_1)$, stated in Eq. 3.8 and Eq. 3.9, are the expected profit of the distributor for the entire problem and the secondary market problem, respectively, and also $V(y_2, p_2, p_1)$ function in Eq. 3.10 is rewritten in Eq. 3.24. The solution of the secondary market problem in Eq. 3.9 can be obtained by using partial derivatives of Eq. 3.24. Let p_2^*, y_2^*, and $s_{2,i}^*$ be the optimal values of p_2, y_2, and $s_{2,i}$, respectively. Then, p_2^* is found by taking partial derivative of Eq. 3.24 with respect to p_2. Partial derivative of Eq. 3.24 with respect to y_2 gives y_2^* as follows:

$$y_2^* = \hat{d}_2(p_2, p_1) + \sum_{i=1}^{N} \sigma_{2,i} \Phi^{-1}\left(\frac{b_2 - c_2}{b_2 + h_2}\right) \tag{3.37}$$

$s_{2,i}^*$ is also found by Eq. 3.23 where p_1, p_2, and y_2 are known.

For the solution of the entire problem, we use Theorem 2 of [4], and conclude that a BSLP policy is optimal under given assumptions. Let p_1^* and y_1^* be the optimum values of p_1 and y_1 obtained by maximizing the profit for the entire problem given by Eq. 3.8. According to this policy, at the beginning of the primary season, the inventory level is raised to base-stock level y_1^* on the assumption that the inventory level I_1 is below y_1^*, and the selling price is set as p_1^*. For the secondary season, if $I_2 \leq y_2^*$, then $y_2^* - I_2$ items are ordered so as for the inventory level to raise to y_2^* at the beginning of the secondary season, and the selling price is set as p_2^* by taking partial derivative of Eq. 3.24 with respect to p_2 as stated before. Items are allocated to the retailer i ($i = 1, ..., N$) in the amount of $s_{2,i}$, obtained from Eq. 3.23. If $I_2 > y_2^*$, no additional items are ordered, and the selling price p_2^* is set as a function of I_2. Items are allocated to the retailer i ($i = 1, ..., N$) in the amount of $s_{2,i}$, obtained by replacing y_2 with I_2 in Eq. 3.23.

Observing the effects of various parameters on optimal ordering, pricing, and inventory allocation decisions is considerably challenging by analytical configuration only. Therefore, we make a computational analysis to monitor these effects, and the behavior of the optimal decisions, shown in the subsequent chapter.

4. COMPUTATIONAL STUDY

In this chapter, a computational study is performed to analyze the effects of parameters on the optimal decisions. Firstly, linear mean demand functions given in Chapter 3 are introduced in an explicit form where Assumption 3.1 holds. Note that while the mean demand for the primary season is only price-dependent, the mean demand for the secondary season has the price and the reference price dependency. Reference price effect is measured by absolute difference model as specified in Section 3.2. Since one of our aim is to analyze the pooling effect, we include the cases where the retailers are not considered for the secondary season to the setting. We implement dynamic programming approach proposed in Eq. 3.25 – 3.27 by discretizing the decision variables of p_1, p_2, and y_1. Note that given these values, the optimal value of y_2 is found analytically by Eq. 3.37, and then Eq. 3.23 gives the optimal allocation decisions. In this computational environment, 1512 runs are performed.

4.1. Parameter Setting

In our setting, we use a simple mean demand function $\hat{d}_1(p_1)$, obeying Assumption 3.1, which is $\hat{d}_1(p_1) = A - Bp_1$ where A is the average market potential and B is marginal impact of price over the demand for the primary season. Note that $D_1(p_1) = \hat{d}_1(p_1) + \varepsilon_1$, where ε_1 is defined as the additive error term which is normally distributed with mean of 0 and standard deviation of σ_1. For the secondary selling season, the mean demand function ($\hat{d}_{2,i}(p_2, p_1)$) is expressed as follows:

$$\hat{d}_{2,i}(p_2, p_1) = A_i - B_i p_2 + C_i(p_1 - p_2) \qquad i = 1, ..., N \tag{4.1}$$

where $p_1 \geq p_2$, and A_i, B_i, and C_i denotes the average market potential for the secondary season, marginal impact of the current price and the reference price effect over the demand for retailer i, respectively. We use absolute difference model to show the reference effect on the demand through the secondary season as can be seen in Eq. 4.1. Extended version of the linear demand function for the secondary season is also expressed as follows:

$$D_{2,i}(p_2, p_1) = A_i - B_i p_2 + C_i(p_1 - p_2) + \varepsilon_{2,i} \qquad i = 1, ..., N \tag{4.2}$$

Note that $\varepsilon_{2,i}$ is the additive error term which is normally distributed with mean of 0 and standard deviation of $\sigma_{2,i}$ for retailer i. It can be reached the aggregate demand through the secondary season by adding demand of retailers up. For the aggregate mean demand, related calculations are as follows:

$$\sum_{i=1}^{N} \hat{d}_{2,i}(p_2, p_1) = \sum_{i=1}^{N} A_i - \sum_{i=1}^{N} B_i p_2 + \sum_{i=1}^{N} C_i(p_1 - p_2) \tag{4.3}$$

We can rewrite the equation above in the following way for the sake of simplicity:

$$\hat{d}_2(p_2, p_1) = A_0 - B_0 p_2 + C_0(p_1 - p_2) \tag{4.4}$$

where $\hat{d}_2(p_2, p_1) = \sum_{i=1}^{N} \hat{d}_{2,i}(p_2, p_1)$, $A_0 = \sum_{i=1}^{N} A_i$, $B_0 = \sum_{i=1}^{N} B_i$, and $C_0 = \sum_{i=1}^{N} C_i$.

Table 4.1 presents the various system parameters and their levels. For the number of retailers (N), we use three different levels: 2, 5, and 10. We fix A as 100, and use three different B value, which are 1, 2, and 5. We consider three cases for the variation through the primary season (σ_1): $0.8A$, $0.5A$, and $0.25A$, which we can call high, mediate, and low variation, respectively. We set σ_1 as a parameter dependent on A so that the coefficient of variation for the market potential ($\frac{\sigma_1}{A}$) remains constant. A_0, B_0, and C_0 are the parameters which belongs to the secondary season, and we choose these parameters in accordance with the primary season in order not to encounter any inconsistency. Since the market potential may change for the secondary season, we consider three possibilities for A_0 as $0.8A$, A, and $1.2A$. Respectively they stand for the decrease, stability, and increase in the market potential compared to the primary season. Furthermore, the marginal impact for the current price B_0 is fixed as B. Since one of our aim is to observe the reference price effect, we choose two different values for C_0 in a way to depend on B_0.

Also, we set two different values for b_1 and b_2 where $b_2 = b_1$ as shown in Table 4.1. Similarly, for inventory holding cost, we set two different values where $h_2 = h_1$. Ordering cost takes only one value as $c_1 = 0.8$, and $c_2 = c_1$. While determining b_1, b_2, h_1, h_2, c_1, and c_2, we consider the ratio $\frac{b_n - c_n}{b_n + h_n}$, called the critical ratio in the literature, is close to one where $n = 1, 2$. Finally, we choose the beginning inventory for the primary season (I_1) as 0.

As stated before, we consider three possibilities for the number of retailers. To demonstrate the effect of pooling the retailer demands, we also set pooled demand for the secondary season.

4.1.1. Pooling and No-pooling

In this subsection, we construct three different settings to monitor the effect of number of retailers and pooling the demand where the total variance of the demand function is set fixed for these three cases. Let the variance of the overall demand for

Table 4.1. System parameters and their levels.

Parameter	**Level 1**	**Level 2**	**Level 3**
N	2	5	10
A	100		
B	1	2	5
A_0	$0.8A$	A	$1.2A$
B_0	B		
C_0	$0.5B$	B	
σ_1	$0.8A$	$0.5A$	$0.25A$
b_1	10	15	
b_2	b_1		
h_1	0.5	1	
h_2	h_1		
c_1	0.8		
c_2	c_1		
I_1	0		

the secondary season be σ_2^2, which is also set as σ_1^2. We set A_i, B_i, and C_i values in proportional to the variance of retailer i, and as so A_0, B_0, and C_0 values are preserved.

Case 1 - Pooling (POOL): The retailer demands are centralized in this case. Therefore, retailers are not considered, and the variance is equal to σ_2^2. Eq. 4.4 holds for the pooling case. Also, specific to pooling case, Eq. 3.37 turns out to be Eq. 4.5.

$$y_2^* = \hat{d}_2(p_2, p_1) + \sigma_2 \Phi^{-1}\Big(\frac{b_2 - c_2}{b_2 + h_2}\Big) \tag{4.5}$$

Case 2 - No-pooling with identical retailers (NP-I): If there are N retailers, the parameters are set as in the following way: $A_i = \frac{A_0}{N}$, $B_i = \frac{B_0}{N}$, and $C_i = \frac{C_0}{N}$. Since all retailers are regarded as identical, $\sigma_{2,i}^2 = \frac{\sigma_2^2}{N}$ holds for any retailer.

Case 3 - No-pooling with non-identical retailers (NP-NI): For this case, we consider that there is one big retailer and $N-1$ small retailers where N is the number of retailers. When $N = 2$, we set the variances for the retailers as follows: $\sigma_{2,1}^2 = 0.75\sigma_2^2$ and $\sigma_{2,1}^2 = 0.25\sigma_2^2$. When $N = 5$ or $N = 10$, we set the variances for the retailers as follows: $\sigma_{2,1}^2 = 0.5\sigma_2^2$ and $\sigma_{2,i}^2 = \frac{0.5}{N-1}\sigma_2^2$ for $i = 2, ..., N$. The coefficients of A_i, B_i, and C_i are set in proportional to the variance of retailer i.

4.1.2. Base Case Scenario

In this subsection, our aim is to examine a case in more detail where we call it base case. The levels of the base case are presented in Table 4.2. In the base case scenario, the retailers are considered identical (NP-I), which implies that demand function and its variance are equal.

$$\hat{d}_{2,i}(p_2, p_1) = 50 - p_2 + 0.5(p_1 - p_2) \qquad i = 1, 2 \tag{4.6}$$

$$\hat{d}_2(p_2, p_1) = 100 - 2p_2 + (p_1 - p_2) \tag{4.7}$$

$$\hat{d}_1 = 100 - 2p_1 \tag{4.8}$$

As stated in subsection 4.1.1, $\sigma_2^2 = \sigma_1^2 = 50^2$. Since the retailers are identical and the total variance is preserved, then $\sigma_{2,1}^2 = \sigma_{2,2}^2 = 1250$. Similarly, mean demand function for the secondary season for these two retailers are expressed in Eq. 4.6. Here $A_1 = A_2 = 50$, $B_1 = B_2 = 1$, and $C_1 = C_2 = 0.5$. Thus, Eq. 4.4 holds where $A_0 = \sum_{i=1}^{2} A_i$, $B_0 = \sum_{i=1}^{2} B_i$, and $C_0 = \sum_{i=1}^{2} C_i$, and for this case, mean demand for the secondary season can be written as in Eq. 4.7. Mean demand for the primary season is also expressed in Eq. 4.8.

Through the base case, we show the boundary for the price and demand, and the corresponding price sets defined in Section 3.2. For the primary season, since p_1^u is established where $\hat{d}_1(p_1) = 0$, then $p_1^u = 50$. On the other hand, $p_1^l = 0.8$ (c_1) and the corresponding $d_1^u = 98.4$. For the secondary season, since p_2^u is established where $\hat{d}_2(p_2, p_1) = 0$ and $p_1 = p_1^l$, then $p_2^u = 33.6$. On the other hand, $p_2^l = 0.8$ (c_2) and the corresponding $d_2^u = 147.6$ where $p_1 = p_1^u$. Also note that $p_1 \geq p_2$. Thus, the price sets for the base case are defined as $\mathbb{P}_1 = \{p_1 : p_1 \in [0.8, 50], 0 \leq \hat{d}_1(p_1) \leq 98.4\}$ and $\mathbb{P}_2 = \{(p_1, p_2) : p_1 \in [0.8, 50], p_2 \in [0.8, 33.6], p_1 \geq p_2, 0 \leq \hat{d}_2(p_2, p_1) \leq 147.6\}$. Either sets clearly form a convex set.

Table 4.2. Levels of Base Case Scenario.

Parameter	**Base Case**
N	2
A	100
B	2
A_0	100
B_0	2
C_0	1
σ_1	50
b_1	10
b_2	10
h_1	0.5
h_2	0.5
c_1	0.8
c_2	0.8
I_1	0

4.2. Implementation

We set the parameters and demand functions as in Section 4.1. In this section, we demonstrate how we implement the dynamic programming approach to the problem. We obtain 1512 different instances with the combination of these parameters as stated before. While 216 of the instances belong to POOL case, 648 instances are created for each of NP-I and NP-NI cases. We proceed with explaining the implementation through one instance with NP-I or NP-NI.

Decision variables should be discretized to implement dynamic programming approach. Since optimal value of y_2 can be calculated analytically for given values of p_1 and p_2 as in Eq. 3.37, y_2 is not necessarily to be a discrete value. Similarly, optimal value of $s_{2,i}$ for retailer i is calculated analytically for given values of p_1, p_2, and y_2 as in Eq. 3.23. However, p_1, p_2, and y_1 are required to be discrete values, and hence we discretize p_1, p_2, and y_1 at intervals of one decimal. There are still numerous combinations of p_1, p_2, and y_1, and thus preliminary elimination is necessary to narrow down the boundaries. We consider both periods separately and optimize them individually since we anticipate that optimal solution should be close to the solution obtained by this way. For this purpose, first we obtain solutions for the primary period itself by using Eq. 4.9 and Eq. 4.10 for every possible p_1 values, and determine the optimum values for y_1 and p_1. Eq. 4.10 expresses the expected profit of the primary period for given p_1 and y_1. By using p_1 value obtained, for every possible p_2 we optimize the second period, and determine the optimum values for y_2 and p_2 with the formulas in Eq. 3.37 and Eq. 3.26. Now, we have optimal values of p_1, p_2, y_1, and y_2 in case of separate optimization of both periods. Eq. 4.9 and Eq. 4.10 are derived in a similar way to Eq. 3.37 and Eq. 3.26, respectively.

As we anticipate that these values are close to the optimal values in the original problem, we set lower and upper boundaries for p_1, p_2, and y_1 around the corresponding values we have obtained. Thus, we determine the lower and upper values for p_1, p_2, and y_1 for the original problem for the purpose of using in the algorithm. Let the lower and upper boundaries of p_1, p_2, and y_1 be denoted as p_1^{lower}, p_1^{upper}, p_2^{lower}, p_2^{upper}, y_1^{lower}, and y_1^{upper}.

$$y_1^* = \hat{d}_1(p_1) + \sigma_1 \Phi^{-1}\Big(\frac{b_1 - c_1}{b_1 + h_1}\Big) \tag{4.9}$$

$$J_1(p_1, y_1) = p_1 \hat{d}_1(p_1) - c_1(y_1 - I_1) + h_1\big(\hat{d}_1(p_1) - y_1\big) - (b_1 + h_1)\sigma_1 R\Big(\frac{y_1 - \hat{d}_1(p_1)}{\sigma_1}\Big) \tag{4.10}$$

Figure 4.1 present the pseudocode of the dynamic programming algorithm for the problem after rearranging the boundaries for a given instance. Since one of our major aim is to observe cost of ignoring the reference price effect under different conditions, this algorithm is run twice. In the first one C_0 is taken as zero so that we can obtain the optimal decisions where the system ignores the reference price. Then, we run this algorithm for the second time with C_0 value of the corresponding instance. When we place the variables obtained where C_0 into the original model, we reach to the expected profit when we ignore the reference price effect.

In the pseudocode in Figure 4.1, previously undefined terms exist, which are K, ε, J, $J_2(k)$, $J(p_1, y_1)$, $J_1(p_1, y_1)$, and $J_2(p_1, y_1)$. For the calculations of secondary period, we need the beginning inventory for the secondary season I_2, and accordingly realized demand for the primary period $D_1(p_1)$ should be known. We simulate the secondary season with K-many different $D_1(p_1)$ by generating K-many error terms. Note that ε_1 denotes the error term of $D_1(p_1)$ with mean of $\hat{d}_1(p_1)$ and standard deviation of σ_1 as in Eq. 3.2. Here K refers to the number of the simulation and ε refers to the standard normal error term. J and $J_2(k)$ denote the optimal expected profit of the entire problem and optimal profit of the secondary period for k^{th} simulation, respectively. Also, $J(p_1, y_1)$, $J_1(p_1, y_1)$, and $J_2(p_1, y_1)$ respectively denote the optimal expected profit for the entire problem, the primary period and the secondary period given values of p_1 and y_1.

```
Set parameters;
Initialize all required vectors and matrices;
for p_1 = p_1^{lower} to p_1^{upper} do
  \hat{d}_1(p_1) ⇐ a − b × p_1;
  for y_1 = y_1^{lower} to y_1^{upper} do
    Compute the profit J_1(p_1, y_1) using Eq. 4.10;
    for k = 1 to K do
      ε ⇐ Φ^{−1}(−1/(2K) + k/K) (standard normal error term);
      D_1(p_1) ⇐ \hat{d}_1(p_1) + σ_1 × ε;
      if D_1(p_1) ≤ 0 then
        D_1(p_1) ⇐ 0;
      end if
      I_2 ⇐ y_1 − D_1(p_1);
      for p_2 = p_2^{lower} to p_2^{upper} do
        Set function \hat{d}_2(p_2, p_1) as defined in Eq. 4.4;
        Set y_2 as defined in Eq. 3.37;
        if y_2 < I_2 then
          y_2 ⇐ I_2;
        end if
        Compute the profit of the secondary period using Eq. 3.26 and Eq. 3.24;
      end for
      Find p_2, y_2, and J_2(k) where the secondary period profit is maximum;
    end for
    Set J_2(p_1, y_1) by taking average of J_2(k) vector;
    J(p_1, y_1) ⇐ J_1(p_1, y_1) + J_2(p_1, y_1);
  end for
end for
Set J where J(p_1, y_1) is maximum;
Find p_1, p_2, y_1, and y_2 where J(p_1, y_1) is maximum;
```

Figure 4.1. Pseudocode of dynamic programming algorithm.

In the algorithm presented in Figure 4.1, we first set p_1 and y_1 values, and find the corresponding expected profit for the primary period by using Eq. 4.10. Then, for the secondary period for given p_1 and y_1, we approximate the optimal solution by simulation. We generate K different standard normal error terms in an orderly manner as in Figure 4.1 and $D_1(p_1)$ values are generated according to these error terms. In order to find exact values, K should be infinite; however, it is not possible to code. There is a trade-off between efficiency and precision. As K increases, while precision increases running time increases as well. After testing several values of K, we consider an appropriate K value as 10. Then, for every $D_1(p_1)$, I_2 values are calculated. Later, for every given p_2, order-up-to level y_2 and the profit of secondary period are computed for given I_2. We choose p_2 value with the maximum profit. Thus, we find optimal values of p_2, y_2, and the profit of the secondary period given p_1, y_1, and I_2. Then, by taking the average of profits we have found, we obtain an approximation of the optimal expected profit for the secondary season given p_1 and y_1. By summing up the optimal expected profit for the primary and the secondary season, we reach the optimal expected profit values for every given p_1 and y_1. The one with the maximum value is the optimal solution. Optimal values of p_1, p_2, y_1, and y_2, and J (optimal expected profit) are acquired.

With the values obtained after the algorithm stated in Figure 4.1, we calculate the stock allocation to the retailers using Eq. 3.23 shown in Figure 4.2.

Inputs: p_1, p_2, y_1, y_2
for $i = 1$ to N **do**
  Calculate $s_{2,i}$ using Eq. 3.23;
end for

Figure 4.2. Stock allocation to the retailers.

The algorithm works for an instance belonging to POOL case if two changes are made. The first one is that Eq. 4.5 is used instead of Eq. 3.37 in Figure 4.1. Secondly, the stock allocation part, which is stated in Figure 4.2, is removed from the algorithm.

4.3. Results and Discussion

Our major aim is to demonstrate the cost of ignoring the reference price effect under changing parameters and to observe the effect of pooling of retailer demands on the optimal decisions. Besides, this analysis enables us to observe the effects of other various parameters on the optimal decisions such as backordering cost, holding cost, reference price, average market potential, number of retailers. In Table 4.4, 4.9 and 4.10, CIRPE refers to cost of ignoring reference price effect.

Table 4.3. An example of prescription of p_2 for different realized demand.

D_1	I_2	y_2	p_2
0	162.2	136.3	29.7
11.5	150.7	136.3	29.8
32.2	130	136.3	30
52.4	109.8	136.3	30
73.1	89.1	136.3	30
96.3	65.9	136.3	30
125.2	37	136.3	30
173.9	-11.7	136.3	30

Note that optimal value of p_2 is defined as a function of I_2 where $I_2 > y_2$ and hence optimal p_2 may take different values depending on the realized demand for the primary season. In Table 4.3, prescription for the secondary season after demand realizes for the primary season for one instance. The system solves this instance and finds $y_1 = 162.2$, $y_2 = 136.3$ and $p_2 = 30$ where $I_2 \leq y_2$. If $I_2 > y_2$ as in the first two rows in Table 4.3, optimal value of p_2 is determined according to I_2. For the sake of clarity, optimal value of p_2 is considered where $I_2 \leq y_2$ for the further analysis, unless otherwise specified. Also, J denotes the optimal expected profit for the entire problem. J_1 and J_2 respectively denote the expected profit of the primary and the secondary seasons of the optimal solution, which implies that $J = J_1 + J_2$.

Table 4.4. Optimal decisions under changing N, C_0, and retailer type.

Pooling	N	C_0	p_1	y_1	p_2	y_2	J	**% penalty of CIRPE**
POOL	-	1	30.7	122	22.2	121.91	2361.90	2.28
NP-I	2	1	30.7	122	22.2	145.85	2317.44	2.29
NP-I	5	1	30.7	122	22.2	193.36	2229.21	2.38
NP-I	10	1	30.7	122	22.2	246.90	2129.78	2.49
NP-NI	2	1	30.7	122	22.2	143.07	2322.61	2.29
NP-NI	5	1	30.7	122	22.2	186.73	2241.53	2.37
NP-NI	10	1	30.7	122	22.2	227.60	2165.62	2.45
POOL	-	2	35.8	111.8	21.8	142.21	2600.37	7.35
NP-I	2	2	35.8	111.8	21.8	166.15	2555.90	7.46
NP-I	5	2	35.8	111.8	21.9	213.26	2467.67	7.73
NP-I	10	2	35.8	111.8	21.9	266.80	2368.24	8.05
NP-NI	2	2	35.8	111.8	21.8	163.37	2561.07	7.45
NP-NI	5	2	35.8	111.8	21.8	207.03	2479.99	7.69
NP-NI	10	2	35.8	111.8	21.9	247.50	2404.08	7.93

In Table 4.4, we provide the optimal decisions and % penalty of cost of ignoring reference price effect under changing parameters of the number of retailers (N) and the coefficient of reference effect (C_0) as well as pooling of retailer demands (Pooling). Remaining parameters are taken fixed as follows: $A = 100$, $B = 2$, $A_0 = 100$, $B_0 = 2$, $\sigma_1 = \sigma_2 = 50$ $I_1 = 0$, $b_1 = b_2 = 10$, $c_1 = c_2 = 0.8$, and $h_1 = h_2 = 0.5$.

Table 4.5. Optimal allocation for NP-I and NP-NI with five retailers.

	y_2	$s_{2,1}$	$s_{2,2}$	$s_{2,3}$	$s_{2,4}$	$s_{2,5}$
NP-I	193.36	38.67	38.67	38.67	38.67	38.67
NP-NI	186.73	72.93	28.45	28.45	28.45	28.45

In Table 4.4, among the optimal decisions, allocation stock to the retailers is not presented. In Table 4.5, we provide the optimal allocation to the retailers for two cases from Table 4.4, where $N = 5$. In turn, optimal order-up-to level (y_2), shown already in Table 4.4, and allocation these orders to five retailers. Note that NP-I and NP-NI, and their properties are defined in detail in subsection 4.1.1.

Table 4.6. Demand variability in terms of pooling and N.

Pooling	$\boldsymbol{N}$	**Demand variability**
POOL	-	50
NP-I	2	70.71
NP-I	5	111.8
NP-I	10	158.11
NP-NI	2	68.3
NP-NI	5	106.07
NP-NI	10	141.42

In Table 4.6, we present the effect of pooling and number of retailers on demand variability for the secondary season where $\sigma_2 = 50$.

Table 4.7 presents the effect of pooling on the optimal expected profit and the corresponding order-up-to level under changing parameters of the number of retailers (N) and the coefficient of variation ($\frac{\sigma_1}{A}$) where $\sigma_1 = \sigma_2$ and $A = 100$.

In Table 4.8, we provide the optimal decisions under changing inventory costs where other parameters are fixed. Also, note that $b_1 = b_2$ and $h_1 = h_2$.

In Table 4.9, we provide the optimal decisions and % penalty of cost of ignoring reference price effect under changing parameters of the average market potential for the second period (A_0) and the coefficient of variation ($\frac{\sigma_1}{A}$). Note that we set $\sigma_2 = \sigma_1$. Remaining parameters are taken fixed as follows: $N = 5$, $A = 100$, $B = 2$, $B_0 = 2$, $C_0 = 2$, $I_1 = 0$, $b_1 = b_2 = 10$, $c_1 = c_2 = 0.8$, and $h_1 = h_2 = 0.5$.

Table 4.7. Pooling effect under changing N and coefficient of variation.

Pooling	N	$\frac{\sigma_1}{A}$	**% increase in the profit with pooling**	**% decrease in y_2 with pooling**
NP-I	2	0.8	3.12	19.62
	2	0.5	1.88	16.42
	2	0.25	0.91	11.40
	5	0.8	9.38	42.18
	5	0.5	5.62	36.95
	5	0.25	2.71	27.75
	10	0.8	16.42	56.07
	10	0.5	9.83	50.63
	10	0.25	4.75	40.19
NP-NI	2	0.8	2.76	17.73
	2	0.5	1.66	14.79
	2	0.25	0.80	10.21
	5	0.8	8.50	39.82
	5	0.5	5.10	34.71
	5	0.25	2.46	25.84
	10	0.8	13.88	51.91
	10	0.5	8.31	46.44
	10	0.25	4.01	36.23

Table 4.8. Optimal decisions under changing b_1 and h_1.

Pooling	b_1	h_1	p_1	y_1	p_2	y_2	J
POOL	10	0.5	29.8	120.4	18.7	111.51	1968.70
	10	1	29.8	106.2	18.7	102.68	1902.71
	15	0.5	29.8	130.1	18.7	122.67	1952.86
	15	1	29.8	116.5	18.7	114.37	1877.28
NP-I	10	0.5	29.8	123.8	18.7	182.96	1836.45
	10	1	29.8	107.2	18.7	163.23	1734.94
	15	0.5	29.8	132.8	18.7	207.93	1805.60
	15	1	29.8	117.1	18.7	189.36	1688.36

Table 4.9. Optimal decisions under changing A_0 and $\frac{\sigma_1}{A}$ (or $\frac{\sigma_2}{A}$).

A_0	$\frac{\sigma_1}{A}$	p_1	y_1	p_2	y_2	J	**% penalty of CIRPE**
80	0.8	34.4	164.7	19	279.62	1891.89	7.62
100	0.8	35.7	162.1	21.8	291.02	2284.15	8.45
120	0.8	37.2	159.1	24.7	302.42	2733.53	5.68
80	0.5	34.4	114.6	19	202.06	2075.22	7.01
100	0.5	35.8	111.8	21.9	213.26	2467.67	7.73
120	0.5	37.2	109	24.7	224.86	2917.28	5.26
80	0.25	34.5	72.7	19	137.63	2226.43	6.49
100	0.25	35.9	69.9	21.9	148.83	2619.34	7.39
120	0.25	37.3	67.1	24.7	160.43	3069.36	5.07

We have obtained the results. Our main observations and related discussions as a consequence of the numerical study are as follows:

Observation 1: As the reference price effect increases, the gap between p_1 and p_2 will go up. Then, the system that takes the reference price into account more will experience higher demand and hence the distributor will order more for the secondary period, which implies that y_2 will increase. Regarding the primary season, since an increase in the reference price causes a rise in p_1, the demand, accordingly the order amount, will decrease. Also, we observe that double increase in the reference price effect causes more than double damage in the profit if we ignore the reference price effect for the secondary season.

Observation 2: As the number of retailers increases, the variability in demand increases as well and hence the uncertainty in demand for the secondary season will increase (see Table 4.4). Therefore, the distributor will place more order. Also, demand variability is more when the retailers are identical. On the other hand, since the expected total demand remains the same, the more inventory holding, and backordering cost will be incurred and hence the optimal expected profit will decrease. Furthermore, as the number of retailers increases, the proportional cost of ignoring the reference price increases as well.

Observation 3: When the retailer demands are centralized, the uncertainty will be at the minimum level because pooling reduces the demand variability (see Table 4.6), and accordingly, the order amount y_2 will be at the minimum level and the optimal expected profit will increase. In short, the improvement in terms of both order amount and the profit is made by pooling, which we can call the benefits of risk-pooling. Moreover, instead of N identical retailers, when a big retailer and $N-1$ small identical retailers exist, the system gets closer to the pooling. Therefore, we can order demand variability for the cases as POOL $<$ NP-NI $<$ NP-I.

Observation 4: In Table 4.7, we state two benefits of risk-pooling, which are % increase in the profit and % decrease in the order-up-to level over the systems where the demand is not pooled (NP-I and NP-NI). As the coefficient of variation and the number of retailers increases, these benefits will increase.

Observation 5: Pooling the demand decreases the order-up-to level for the primary period slightly as well (see Table 4.8). However, in certain instances, we are not able to see this effect. An increase in the backordering cost increases order-up-to levels and an increase in the inventory holding cost decreases the corresponding order-up-to level as in any inventory problem.

Observation 6: As the coefficient of variation increases, order-up-to levels increase, and the optimal expected profit decreases due to increasing uncertainty (or demand variability). The % penalty of cost of ignoring reference price effect increases in parallel with the coefficient of variation. Also, we observe that as the coefficient of variation increases, there is a slight decrease in the price for the instances where $A_0 = 100$ and $A_0 = 120$ in Table 4.9.

Observation 7: As the average market potential for the secondary period A_0 increases, the prices of both periods increase; however, an increase in p_2 is more apparent, which implies that the prices approach to each other. For order-up-to level for the secondary period, since the effect of increasing market potential outweighs the effect of increasing p_2, y_2 raises in accordance with A_0. Also, we observe that % penalty of cost of ignoring reference price takes the highest value where the average market potential is stable ($A_0 = 100$). To ensure these findings, we perform additional runs for different A_0 values where other parameters are fixed, provided in Table 4.10. As the market potential for the secondary period increases, the system will pull p_2 up. Since both seasons are tied via the reference price, the system will force p_1 to raise; however, an increase in p_1 will not be as much as p_2. Due to increasing the market potential for the secondary season, total expected profit will increase. However, the expected profit for the primary season will decrease because of the fact that the corresponding price increases while the market potential remains the same ($A = 100$ for all instances). We observe the reference price effect more where there is a stability in the market potential. Two factors may lead this effect to decrease: a decline in the share of optimal profit for the secondary season ($\frac{J_2}{J}$) and a decline in $p_1 - p_2$. In case of declining in the market potential, the former one outweighs for this decline. On the other hand, in case of growth in the market potential, the latter one outweighs for this decline.

Table 4.10. Optimal decisions under changing A_0.

A_0	p_1	y_1	p_2	y_2	J_1	J	% penalty of CIRPE
40	56	160.6	23.1	125.84	2250.12	2899.85	1.04
60	57.7	162.2	30	136.34	2226.31	3416.97	1.50
80	59.4	164.3	37	146.69	2196.21	4072.52	1.92
100	61.1	166.9	43.9	157.19	2159.75	4866.45	2.27
120	62.8	169.6	50.9	167.54	2117.15	5798.75	0.94
150	65.4	168.1	61.3	183.24	2045.04	7457.28	0.09

5. CONCLUSION

In this study, we investigate the joint inventory allocation and pricing problem of a single item under certain assumptions in a two-tier customer market where the demand is stochastic and price and reference price dependent. We formulate the problem by using dynamic programming approach and demonstrate that a BSLP policy is optimal for concave mean demand models and concave transformed expected revenue functions as in Guler *et al.* [8]. Then, we derive optimal allocation amount to retailers after replenishment realizes.

We, later, carry out a computational study with the aim of analyzing the behavior of the optimal decisions under changing parameters. In this numerical study, we consider linear mean demand with an additive error term and absolute difference reference effect. Main focus is to find out the effect of the reference price and the effect of pooling on the optimal decisions. We observe that a reference price increase causes to an increase in the price and replenishment amount for the secondary period while it causes an exact opposite effect for the primary period. Another observation is that an increase in the number of retailers, implying that the system diverge from centralization, reduces the optimal expected profit along with demand variability increase. With the effect of pooling, replenishment quantity decreases, and the expected profit increases.

Also, an increase in the number of retailers or coefficient of variation leads to benefits of risk-pooling increase. Another major observation is that the % penalty of cost of ignoring reference price effect increases in accordance with coefficient of variation. As a more salient observation, we present that the reference price is more influential where the average market potential is closer to its level in the previous period (see, Table 4.10).

This study can be extended in several directions. One can consider lost sales instead of backordering for unsatisfied demands. In this study, we include the reference effect in the demand function as absolute difference. Employing relative difference reference effect is another direction of future work. In our model, we consider that the replenishment order of the supplier (the distributor in our case) is fully satisfied; however, the case where the supplier may not fulfill the entire order due to not having enough stock is more realistic. Therefore, we can integrate the supply uncertainty factor into the model. Finally, we can investigate the effect of the reference price on change in average market potential under different demand settings and/or demand structures.

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APPENDIX A: DERIVATION OF EQUATION 3.24

Proof. Firstly, write Eq. 3.10 once more as follows:

$$V(y_2, p_2, p_1) = p_2\hat{d}_2(p_2, p_1) - \sum_{i=1}^{N} \hat{G}_2(s_{2,i}, p_2, p_1) \tag{A.1}$$

Then, rewrite the second part of Eq. A.1 by substituting $s_{2,i}$ as in Eq. 3.23.

$$\begin{aligned}
\sum_{i=1}^{N} \hat{G}_2(s_{2,i}, p_2, p_1) &= h_2 \sum_{i=1}^{N} \left(E\left[\left(s_{2,i} - \hat{d}_{2,i}(p_2, p_1) - \varepsilon_{2,i} \right)^{+} \right] \right) \\
&\quad + b_2 \sum_{i=1}^{N} \left(E\left[\left(\hat{d}_{2,i}(p_2, p_1) + \varepsilon_{2,i} - s_{2,i} \right)^{+} \right] \right) \\
&= h_2 \sum_{i=1}^{N} \left(E\left[\left(\frac{\sigma_{2,i}}{\sum_{j=1}^{N} \sigma_{2,j}} \left(y_2 - \hat{d}_{2,i}(p_2, p_1) \right) - \varepsilon_{2,i} \right)^{+} \right] \right) \\
&\quad + b_2 \sum_{i=1}^{N} \left(E\left[\left(\varepsilon_{2,i} - \frac{\sigma_{2,i}}{\sum_{j=1}^{N} \sigma_{2,j}} \left(y_2 - \hat{d}_{2,i}(p_2, p_1) \right) \right)^{+} \right] \right) \\
&= h_2 \sum_{i=1}^{N} \sigma_{2,i} E\left[\left(\frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N} \sigma_{2,i}} - \varepsilon \right)^{+} \right] \\
&\quad + b_2 \sum_{i=1}^{N} \sigma_{2,i} E\left[\left(\varepsilon - \frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N} \sigma_{2,i}} \right)^{+} \right] \\
&= h_2 \sum_{i=1}^{N} \sigma_{2,i} \left(R\left(\frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N} \sigma_{2,i}} \right) + \frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N} \sigma_{2,i}} \right) \\
&\quad + b_2 \sum_{i=1}^{N} \sigma_{2,i} R\left(\frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N} \sigma_{2,i}} \right) \\
&= h_2 \left(y_2 - \hat{d}_2(p_2, p_1) \right) + (b_2 + h_2) \sum_{i=1}^{N} \sigma_{2,i} R\left(\frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N} \sigma_{2,i}} \right)
\end{aligned}$$

where $\varepsilon_{2,i} = \sigma_{2,i}\varepsilon$ and ε is a standard normal random variable. Therefore, by replacing the second component of Eq. A.1, we obtain the following expression:

$$V(y_2, p_2, p_1) = p_2\hat{d}_2(p_2, p_1) + h_2\big(\hat{d}_2(p_2, p_1) - y_2\big) - (b_2 + h_2)\sum_{i=1}^{N}\sigma_{2,i}R\Big(\frac{y_2 - \hat{d}_2(p_2, p_1)}{\sum_{i=1}^{N}\sigma_{2,i}}\Big) \tag{A.2}$$

as required. □

APPENDIX B: PROOF OF PROPOSITION 3.1

Proof. These proves are done in a quite similar way of Proof of Proposition 1 in Guler *et al.* [8]. First, prove (i) as follows:

(i) For $\Delta_1 > 0$ and $\bar{p}_1 \in \mathbb{P}_1$, define $d_{11} = \hat{d}_1(\bar{p}_1)$ and $d_{12} = \hat{d}_1(\bar{p}_1 + \Delta_1)$. Since the demand function is strictly decreasing in p_1 given Assumption 3.1, $d_{11} > d_{12}$. Now define $p_{11} = \hat{p}_1(d_{11}) = \bar{p}_1$ and $p_{12} = \hat{p}_1(d_{12}) = \bar{p}_1 + \Delta_1$, which is clearly seen that $p_{12} > p_{11}$. Then, one can conclude that $p_{12} > p_{11}$ if $d_{11} > d_{12}$, which implies that $\hat{p}_1(d_1)$ is strictly decreasing in d_1.

(ii) For $\Delta_2 > 0$ and $\bar{p}_1, \bar{p}_2 \in \mathbb{P}_2$, define $d_{21} = \hat{d}_2(\bar{p}_2, \bar{p}_1)$ and $d_{22} = \hat{d}_2(\bar{p}_2 + \Delta_2, \bar{p}_1)$. Since the demand function is strictly decreasing in p_2 given Assumption 3.1, $d_{21} > d_{22}$. Now define $p_{21} = \hat{p}_2(d_{21}, \bar{p}_1) = \bar{p}_2$ and $p_{22} = \hat{p}_2(d_{22}, \bar{p}_1) = \bar{p}_2 + \Delta_2$, which is seen that $p_{22} > p_{21}$. Then, one can conclude that for a given $\bar{p}_1$, $p_{22} > p_{21}$ if $d_{21} > d_{22}$, which implies that $\hat{p}_2(d_2, p_1)$ is strictly decreasing in d_2.

On the other hand, to show that $\hat{p}_2(d_2, p_1)$ is strictly increasing in p_1, define another point as $d_{23} = \hat{d}_2(\bar{p}_2, \bar{p}_1 + \Delta_2)$. Since the demand function is strictly increasing in p_1 given Assumption 3.1, $d_{23} > d_{21}$. Let $\Delta_3 = d_{23} - d_{21}$ and note that $\bar{p}_2 = \hat{p}_2(d_{21}, \bar{p}_1)$. One can write $\bar{p}_2 = \hat{p}_2(d_{23}, \bar{p}_1 + \Delta_2) = \hat{p}_2(d_{21} + \Delta_3, \bar{p}_1 + \Delta_2)$. Since $\hat{p}_2(d_2, p_1)$ is strictly decreasing in d_2, $\hat{p}_2(d_{21}, \bar{p}_1) = \hat{p}_2(d_{21} + \Delta_3, \bar{p}_1 + \Delta_2) < \hat{p}_2(d_{21}, \bar{p}_1 + \Delta_2)$, which implies that $\hat{p}_2(d_2, p_1)$ is strictly increasing in p_1. □