ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

DETERMINISTIC STATE TRANSFORMATIONS IN THE RESOURCE THEORY OF SUPERPOSITION

M.Sc. THESIS

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FOREWORD

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ABBREVIATIONS

QRT : Quantum Resource Theory

RTC : Resource Theory of Coherence

RTS : Resource Theory of Superposition

LOCC : Local Operation and Classical Communication

CPTP : Completely Positive Trace Preserving

SYMBOLS

$\mathcal{H}$	: Hilbert Space
$\mathbb{1}$	: Identity Operator
G	: Gram Matrix
D	: Doubly Stochastic Matrix
$\mathcal{F}$	: Set of Free States
$\mathcal{FO}$	: Set of Free Operations
$\mathcal{R}$	: Non-free/Resource States
$\otimes$	: Tensor Product
$\mathbb{R}$	: Set of Real Numbers
$\mathbb{C}$	: Set of Complex Numbers

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DETERMINISTIC STATE TRANSFORMATIONS IN THE RESOURCE THEORY OF SUPERPOSITION

SUMMARY

In this thesis, we investigated deterministic transformations of the resource theory of superposition including maximal states. We fulfilled the gap between the resource theory of coherence and the resource theory of superposition in the context of deterministic transformations and maximal states by establishing a continuous relation by means of a Gram matrix. We present our work as four propositions. The first proposition states a sufficient condition for a deterministic transformation in two and three level systems by means of superposition-free operations of the resource theory of superposition. This sufficient condition for the deterministic transformations is a compound proposition of two subconditions and both subconditions are expressed in terms of a special matrix called a doubly stochastic matrix. Doubly stochastic matrices are used in various resource theories such as bipartite entanglement and coherence to express the conditions for deterministic transformations. In our work, a doubly stochastic matrix is used as a building block in the generalization of the resource theory of coherence. We first show that vectors subject to the doubly stochastic matrix differ from the vectors in the coherence theory in general. After that we show that in the orthonormal case one recovers the resource theory of coherence. Moreover, it was shown that one cannot introduce the majorization theory directly in the resource theory of superposition in contrast to the resource theory of coherence because of the fact that the vectors subject to the doubly stochastic matrix are not necessarily real in general. However, it is possible to introduce the majorization theory if one considers only the transformations between states having only real components in a real scalar product case of linearly independent basis states. It was also shown that one requires an additional condition (which we call condition on completeness (CoC)) in addition to the majorization condition which is stated as the second subcondition. However in the orthogonal case the additional condition is satisfied trivially. Therefore; it is only required to satisfy the majorization as desired. The CoC can be interpreted in the context of positive definiteness. In contrast to the resource theory of coherence, in the resource theory of superposition a pair of vectors satisfying the majorization condition is not enough for the existence of a trace preserving operation that transforms the initial state to the target state with unit probability. However, if a pair of vectors satisfy both conditions, then the transformation can be completed to a trace preserving operation that transforms the initial state to the target state with unit probability by introducing additional Kraus operators. The CoC guarantees that such an operation can be accomplished. The second proposition shows that if a state is not an eigenvector corresponding to the minimum eigenvalue of the Gram matrix which represents the scalar product setting, then it cannot be a maximal state. This proposition also provides a way of representing candidate maximal states of any dimension. It is also shown that representation of candidate maximal states reduces to the maximal states of

coherence as a result of the continuous picture provided by the Gram matrix. The third proposition provides the maximal states of two level systems for arbitrary settings. In the resource theory of coherence, the relative phases can be eliminated by unitary incoherent operations, and hence states that differ only by relative phases are equal. However in contrast to this situation, in the resource theory of superposition the relative phase terms cannot be changed freely in general. Therefore; there can be at most one maximal state for each setting. Finally the fourth proposition provides the maximal states for specific intervals of a real and equal scalar product setting.

SÜPERPOZİSYON KAYNAK TEORİSİNDE DETERMİNİSTİK DURUM DÖNÜŞÜMLERİ

ÖZET

Kuantum mekaniği çok küçük ölçekte gerçekleşen ve klasik fizik çerçevesinde açıklanamayan olayları açıklamak adına ortaya çıkmış bir kuram, bir çatı yapıdır (framework). Bu çok küçük ölçekte gerçekleşen olaylar ve olaya konu olan parçacıklar, Hilbert uzayı olarak adlandırılan kompleks bir vektör uzayında yaşayan vektörler ve bunlara etki eden operatörler ile temsil edilmektedir. Bu uzayda bulunan ve fiziksel durumu temsil eden vektörlere kuantum durumu (state) denilmektedir. Herhangi bir ölçüm gerçekleşmediği takdirde, bir durum başka bir duruma üniter dönüşümler ile evrilmektedir. Her ne kadar bu durumların evrimi deterministik olsa da, bu durumlar üzerinde yapılacak herhangi bir ölçüm determistik olmayan bir süreç ile sonuçlanabilmektedir. Bunun temel sebebi ise, kuantum mekaniğinde incelenen sistemi temsil eden bir durumun, ölçüm ile gözlemlenebilecek durumların bir süperpozisyonu (superposition) halinde olabilmesidir. Süperpozisyon ilkesi, kuantum mekaniğini klasik fizikten ayıran özelliklerden biri olup, klasik fizik kapsamında gerçekleştirilemeyecek bir takım operasyonların gerçekleştirilebilmesine olanak sağlayan olgulardan biridir. Kuantum mekaniksel sistemler, bu bağlamda klasik fizikte karşılığı olmayan ilintiler (non-classical correlations) içermektedir. Dolanıklık (entanglement), uyuşmazlık (discord) gibi ilinti örnekleri, kuantum ışınlama, yoğun kodlama vb. gibi klasik fizik ile gerçekleştirilemeyecek bilgi işlem operasyonları gerçekleştirmek için kullanılmaktadır. Örneğin, kuantum ışınlama operasyonunu gerçekleştirmek için maksimal dolanıklığa sahip bir kuantum durumu gerekmektedir. Bu operasyon maksimal dolaşık olmayan bir durum ile gerçekleştirildiği takdirde başarısızlık ile sonuçlanabilmektedir. Bu ilintilerin kuantum mekaniksel bir operasyonu başarı ile gerçekleştirmek veya başarısızlık şansını azaltmak adına kullanılması, onları bir kaynak statüsüne getirmektedir.

Kuantum kaynak teorisi (quantum resource theory), kuantum bilgi işlemede kullanılabilecek kaynakları tespit etme, sınıflandırma ve hiyerarşik bir yapıya sokma amacı ile ortaya çıkmış bir alt çatı yapıdır. Dolanıklık kaynak teorisi, eşevrelilik (coherence) kaynak teorisi vs. gibi konusu olan kaynak türüne bağlı olarak kendi içinde dallanmaktadır. Bu çalışmada süperpozisyon kaynak teorisi olarak bilinen, eşevrelilik kaynak teorisinin bir genelleştirmesi olan kaynak teorisinin deterministik durum dönüşümleri ve maksimal durumları incelenmektedir.

Bir kaynak teorisi, kullanılması serbest olan durumların ve serbest operasyonların tanımlanması ile oluşturulur. Daha sonra bu operasyonlar aracılığı ile ne gibi dönüşümlerin gerçekleştirilebileği incelenir. Süperpozisyon kaynak teorisinde serbest durumlar, $\{|c_i\rangle\}_{i=1}^{d}$ lineer bağımsız bir baz vektör kümesi olmak üzere $\rho = \sum_i \rho_i |c_i\rangle\langle c_i|$ formunda yazılabilecek tüm durumlar kümesi olarak verilmektedir. Serbest operasyonlar ise oluşturulan bu serbest durumlar kümesini kendisine veya

alt kümesine götüren operasyonlar olarak tanımlanmaktadır. Bu küme dışında kalan, örneğin $|\psi\rangle = \psi_1 |c_1\rangle + \psi_2 |c_2\rangle + \ldots$, durumlar serbest olmayan, kaynak olarak alınan durumdur. Süperpozisyon kaynak teorisi çerçevesinde cevaplanmamış sorulardan biri iki kaynak durum arasında determistik bir dönüşüm gerçekleştirilebilip gerçekleştirilemeyeceğidir. Bir diğer soru ise bütün durumlara deterministik olarak dönüştürülebilecek bir maksimal durumun var olup olmadığı problemidir. Bu çalışmada bu sorular kısmi olarak cevaplanmış olup, alanda yapılacak ilerleyen çalışmalar için bir başlangıç noktası oluşturulmuştur. Bu tezde öncelikle iki ve üç boyutlu kuantum sistemlerinde, saf (pure) kaynak durumları arasında serbest operasyonlar aracılığı ile determistik dönüşümler incelenmiştir. Daha sonra ise bütün saf durumlara dönüşebilen maksimal durumlar incelenmiştir. Bu iki alt başlık tezin özgün içeriğini oluşturmaktadır.

Özgün çalışmanın içeriği dört önerme cümlesi ile sunulmuştur. Birinci önerme cümlesinde iki ve üç boyutlu saf durumların deterministik dönüşümü için yeter şart verilmiştir. Bu yeter şart iki alt şarttan oluşan bileşik bir önermedir. Bu şartlar öncelikle en genel iç çarpım değerlerini (inner/scalar product) kapsayacak şekilde verilmiştir (Baz vektörlerinin arasındaki iç çarpım değerleri bir Gram matrisi ile temsil edilebilir). En genel iç çarpım değerleri için bu şartlar iki yönden stokastik matris (doubly stochastic matrix) kullanılarak ifade edilmiştir. İki yönden stokastik matris doğrudan majorizasyon teorisi ile ilgili bir matris olup, eşevrelilik kaynak teorisinde de deterministik dönüşümler için gerek ve yeter şartı ifade etmekte kullanılmaktadır. Süperpozisyon kaynak teorisi ile eşevrelilik arasındaki fark ise bu matrisin etki ettiği vektörlerin farkından kaynaklanmaktadır. Süperpozisyon kaynak teorisinde bu vektörlerin bileşenleri reel sayılara dönüştürülemediği için majorizasyon teorisi doğrudan ifade edilememektedir. Ancak, deterministik dönüşümlerin sadece bileşenleri reel olan vektörler arasında gerçekleştirildiği takdirde bu şartların majorizasyon ile ifade edilebileceği gösterilmiştir. Bileşke şartların ikincisi ise, eşevrelilik kaynak teorisinde doğrudan karşılığı bulunmayan, ortogonal olmayan baz vektörlerinin iç çarpımlarından gelen katkılar dolayısıyla ortaya çıkan bir şarttır. Bu şart ortogonal durumda çözümü bayağı (trivial) olarak sağlanan denklemler haline gelmektedir. Bu bağlamda birinci önerme cümlesi eşevrelilik kaynak teorisinin deterministik dönüşüm şartını özel bir durum olarak içermektedir. Bu durum da süperpozisyon kaynak teorisinin, eşevrelilik kaynak teorisini genelleyici niteliğine katkı sağlamaktadır.

İkinci, üçüncü ve dördüncü önerme cümleleri maksimal durumlar bağlamında sunulmuştur. İkinci önerme cümlesinde maksimal durumunun sağlaması gereken bir gerek şart verilmiştir. Bu gerek şart bir durumun verilen süperpozisyon kaynak teorisini temsil eden Gram matrisinin en küçük özdeğerine karşılık gelen özvektörü değilse, maksimal durum olamayacağını ifade etmektedir. Ayrıca bu gerek şart, aday maksimal durumların nasıl bir formda olması gerektiğini dikte etmektedir. Üçüncü önerme cümlesi, iki boyutlu bir sisteminin her iç çarpım durumuna karşılık gelen bir maksimal durum bulunduğunu ifade etmektedir. Örneğin, λ iç çarpım değerlerini temsil eden Gram matrisinin en küçük özdeğeri ve $\langle c_1|c_2\rangle = e^{i\omega}\mu$ olmak üzere, $|\psi_1\rangle = \sqrt{1/(2\lambda)}(|c_1\rangle - e^{-i\omega}|c_2\rangle$ durumu $0 \leq \mu < 1$ iç çarpım aralığı için maksimal durumdur. $|\psi_2\rangle = \sqrt{1/(2\lambda)}(|c_1\rangle + e^{-i\omega}|c_2\rangle$ durumu ise $-1 < \mu \leq 0$ iç çarpım aralığı için maksimal durumdur. Ortonormal durumda Gram matrisi birim matrisine eşittir ve bütün özdeğerleri bir değerine sahip olmaktadır. Bu durumda görülmektedir

ki verilen maksimal durumlar, eşevrelilik kaynak teorisinin maksimal durumlarına dönüşmektedir. Bu çalışmanın eşevrelilik ile süperpozisyon kaynak teorileri ile kurduğu sürekli bağıntı burada da ortaya çıkmaktadır. Bu süreklilik sayesinde süperpozisyon kaynak teorisinin eşevrelilik kaynak teorisini genelleme bağlamında sahip olduğu nitelik sayısı tekrar artmaktadır. Dördüncü ve son önerme cümlesinde, yüksek boyutlu sistemlerin maksimal durumları ifade edilmiştir. Yüksek boyutlardaki bu çalışma bazı matematiksel zorluklar nedeni ile sadece reel ve eşit iç çarpım değerleri için ifade edilmiştir. İki boyutlu sistemlerin maksimal durumlarının aksine, buradaki maksimal durumlar sadece reel ve eşit olan iç çarpımların belirli bir aralığı için verilmiştir. Örneğin, d boyutlu bir sistemde iç çarpımlar eşit $(\langle c_i|c_j\rangle = \mu, i \neq j)$ ve $1/(1-d) < \mu \leq 0$ aralığında olmak üzere $|\psi\rangle = \sqrt{1/d\lambda}\sum_{i=1}^{d}|c_i\rangle$ durumu verilen iç çarpım durumlar için bir maksimal durumdur. İki boyutlu sistemlerde olduğu gibi bu durum da ortogonal durumda eşevrelilik kaynak teorisinin maksimal durumunu vermektedir. Verilen son üç önerme, iki boyutlu sistemlerin hangi iç çarpım değerleri için maksimal durumlara sahip olup olmadığı sorusunu tam olarak cevaplamakta, yüksek boyutlarda ise kısmı olarak cevaplamaktadır. Eşevrelilik ve süperpozisyon kaynak teorilerinin bir Gram matrisi ile birbirine bağlanması, gelecekte yapılacak çalışmalar için bir referans oluşturmaktadır.

1. INTRODUCTION

In quantum mechanics, there are certain tasks that have no classical counterpart such as quantum teleportation [1] and dense coding [2]. An entangled state shared between two parties is required to implement these tasks. In this sense entanglement can be treated "as a resource as real as energy" (Horodecki et al, 2009, p. 865). Further developments in quantum information theory showed that phenomena that do not have classical counterpart can be treated as resources in a broader perspective. The consideration of certain quantum states and quantum operations as resources, has led to the formation of quantum resource theories.

A quantum resource theory (QRT) is a framework built by defining a set of states and a set of operations. In general, the set of states is called free states and it is denoted by $\mathcal{F}$. The set of operations is called free operations and is denoted by $\mathcal{FO}$. The set of elements $(\mathcal{F},\mathcal{FO})$ represents a QRT [3]. Elements of a QRT can be defined based on practical reasons (available operations in a laboratory) or in a purely axiomatic manner. For instance, in the resource theory entanglement [4–6], the set of free states $\mathcal{F}$ are formed by separable states with respect to the separate parties, i.e. $\rho_A \otimes \rho_B$ with A and B being (spatially) separate systems and the free operations are defined as local operations and classical communication (LOCC). Then the elements $(\mathcal{F},\mathrm{LOCC})$ form a resource theory. To express the power of the resource theory of entanglement we give a practical example. One requires one of the Bell states to perform a perfect quantum teleportation for a qubit system [1]. For any other type of entangled state the process is probabilistic [7,8]. Consider that Alice and Bob share the following entangled state

$$|\phi\rangle = \sqrt{\frac{2}{3}}\,|1\rangle_A\,|1\rangle_B + \sqrt{\frac{1}{3}}\,|2\rangle_A\,|2\rangle_B\,, \tag{1.1}$$

which is an entangled state but not a Bell state. Alice wants to send a qubit state with unit probability to Bob using the quantum teleportation protocol. To this end, she would like to convert the state $|\phi\rangle$ to one of Bell States. What she needs to do is to determine whether it is possible to implement such a transformation by using

$(\mathcal{F}, \mathrm{LOCC})$ with unit probability. It turns out that such a transformation is not possible with unit probability. However it is possible to transform a Bell state to any other state (in the corresponding Hilbert Space) by means of LOCC [9]. Here the resource theory of entanglement provides a framework to determine whether it is possible to achieve certain tasks or to determine required resources to implement such tasks. For instance one may try to obtain a Bell state by using multiple copies of the state above, such an operation is called entanglement (resource) distillation [10, 11]. As mentioned above the state above cannot be transformed into a Bell state with unit probability, however a Bell state can be transformed to the state above. In this context a Bell state has a higher hierarchical order. With this motivation, one can establish a hierarchical order with respect to resourcefulness of states by introducing entanglement measures [5, 11, 12] with the relative entropy being the well known example.

The above example displays only a minor feature of a resource theory. There are wide range of resource theories of different topics such as coherence [13–17] where coherent superposition of (orthogonal) states are considered as resources and the complementary set is considered as free. Operations that maps free states to free states are defined as free operations. A generalization of the resource theory of coherence which is called resource theory of superposition is introduced in Ref. [18] where the orthogonality condition is relaxed so that superposition of linearly independent states are allowed. A key concept in quantum mechanics and thermodynamics is redefined in the context of resource theories. The cyclic process of an heat engine (in microscopic scale) is scrutinized and the four Laws of Thermodynamics are derived in Refs. [19, 20] in this respect. In Ref. [21], it is shown that work and heat can be treated in equal footing by introducing a resource theory of work and heat. Resource theories are also used to investigate underlying mathematical structure of quantum mechanics and phenomena. For instance the resource theory of imaginarity allows one to treat imaginarity in an operational standpoint [22, 23] and provides a point of view in understanding the role of complex numbers. For further types of resource theories see Refs. [24–41].

The outline of thesis is as follows. In the following section of this chapter we provide some preliminary work such as majorization theory and Gram matrix. In chapter 2, we first introduce the resource theory of superposition and then we introduce our work.

We stress that the section "Deterministic State Transformations" of chapter 2 is based on our published work [42]. An additional original work is given in the "Maximal States" section of the same chapter which is aimed to be published.

1.1 Preliminaries

In this section, we introduce various mathematical properties of vectors and matrices that are used throughout the thesis. We first comment on Kraus operators very briefly and state transformations and then continue with the majorization theory which has a wide range of usage in physics but especially in resource theories such as the deterministic transformation of bipartite states [9] and the resource theory of coherence [43]. Then we introduce the Gram matrix which we use to establish a continuous relation between the resource theory of coherence and the superposition. Finally, we introduce a theorem to find the upper and lower bounds of a quadratic expression under a constraint expressed in terms of a Gram matrix and another theorem to determine whether a given matrix is positive semidefinite.

1.1.1 Quantum operations

In quantum information theory, the dynamics of a state is represented by quantum operations. A quantum operation is a completely positive trace preserving or non-trace preserving map depending on whether an additional information about the system of interest is gained as a result of a selective measurement. A selective measurement is a measurement that one obtains/reads the result of the experiment. On the other hand if one does not read the result of the experiment, then it is called a non-selective measurement. (For instance, assume that a detector that beeps when a projective measurement is completed and it produces 0 or 1 depending on the result. After the beep, the experimenter knows that the resultant state is $\sigma = P_0\rho P_0 + P_1\rho P_1$ where P_i's are projection operators. If the experimenter does not read the detector's result then such a process corresponds to a non-selective measurement.) If no selective measurement takes place then the operation is a trace preserving map, otherwise the operation is a non-trace preserving map. In this context, a trace preserving operation corresponds to a deterministic process similar to the unitary evolution, however a trace preserving operation is not necessarily reversible. A quantum operation consists of set

of operators called Kraus operators and it is generally written in the following form

$$\Phi(\rho^A) = \sum_n K_n \rho^A K_n^\dagger \tag{1.2}$$

where ρ^A is a density operator of the input Hilbert space $\mathcal{H}^A$ and K_n is a Kraus operator. A quantum operation can involve arbitrary number of Kraus operators. A quantum operation must map a valid density matrix to a valid density matrix, therefore; one requires that $\mathrm{tr}(\Phi(\rho^A)) = \mathrm{tr}(\sum_n K_n \rho^A K_n^\dagger) = 1$ given that there is no selective measurement that took place. The condition on the Kraus operators to be a trace preserving operation is then given as

$$\sum_n K_n^\dagger K_n = \mathbb{1}_B \tag{1.3}$$

where $\mathbb{1}_B$ is the identity operator of the output Hilbert space $\mathcal{H}^B$. If a selective measurement takes place, that is one obtains additional information about the system of interest, then the quantum operation is non-trace preserving. The condition on the Kraus operators becomes

$$\sum_n K_n^\dagger K_n \leq \mathbb{1}_B. \tag{1.4}$$

Here the inequality taken to be in the context of positive definiteness that is $\mathbb{1}_B - \sum_n K_n^\dagger K_n$ is a positive definite or a positive semidefinite operator. A non-trace preserving map can be completed to a trace preserving map by introducing additional Kraus operators.

In contrast to projective measurement or unitary evolution, the Kraus operators are not necessarily orthogonal projection operators or unitary operators . The Kraus operators can be any set of operators as long as they satisfy Eq. (1.3). In analogy with the Hermitian picture, a measurement in this formalism causes a state to collapse into a state $\rho_n^B = K_n \rho^A K_n^\dagger / \mathrm{tr}(K_n \rho^A K_n^\dagger)$ (where tr is the trace operator) among N number of possible outcomes corresponding to the total number of Kraus operators. The probability of occurrence of an outcome resulting in $\rho_n^B = K_n \rho^A K_n^\dagger$ is given as $p_n = \mathrm{tr}(K_n \rho^A K_n^\dagger)$. In this sense, a quantum operation takes a state as an argument and maps it into a convex combination of states that is $\Phi(\rho^A) = \sum_n p_n \rho_n^B$ where p_n's form a probability distribution.

The quantum operation formalism is a natural result of the postulates of the quantum mechanics, however its power comes from the Stinespring's dilation theorem [44].

Stinespring's dilation theorem states that for a map in the form of Eq. (1.2) with Kraus operators satisfying Eq. (1.3), there exists a compound (extended) system equipped with orthogonal projective operators and unitary operators so that the resultant operation gives the desired quantum operation [45, 46]. In other words, the map defined in Eq. (1.2) can also be represented as

$$\Phi(\rho) = \frac{\mathrm{tr}_B\left(P_i U\left(\rho^A \otimes \rho^B\right) U^\dagger P_i\right)}{\mathrm{tr}(P_i U\left(\rho^A \otimes \rho^B\right) U^\dagger P_i)}, \tag{1.5}$$

where $\mathrm{tr}_B(A \otimes B) = \mathbb{1}_A \otimes \mathrm{tr}(B)$ is the partial trace operator, where U is a unitary operator acting on the extended system $\mathcal{H}^{A\otimes B}$ and where P_i is an element of set of orthogonal projective operators acting $\mathcal{H}^{A\otimes B}$. This allows one to use a broader set of operators compared to orthogonal projective measurement operators and unitary evolution.

1.1.2 State transformations

State transformations of resource theories can be divided into two groups; probabilistic transformations and deterministic transformations. A completely positive trace preserving (CPTP) map represented by a set of Kraus operators $\{K_n\}$ has an output in the following form

$$\Phi(\rho) = \sum_n K_n \rho K_n^\dagger = \sum_n p_n \sigma_n \tag{1.6}$$

where ρ and σ_n are density operators and p_n forms a probability distribution. Then, a transformation is said to be a deterministic state transformation if an initial state can be transformed into a target state with unit probability, that is,

$$\Phi(\rho) = \sigma \tag{1.7}$$

otherwise, it is said to be a probabilistic transformation. If the deterministic transformation is considered for pure states only, that is $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, then individual Kraus operators must satisfy $K_n|\psi\rangle = e^{i\theta_n}\sqrt{p_n}|\phi\rangle$ with $0 \le p_n \le 1$ and $\sum_n p_n = 1$.

1.1.3 Majorization

Let $\vec{x} = (x_1, \cdots, x_d)^\mathsf{T}$ and $\vec{y} = (y_1, \cdots, y_d)^\mathsf{T}$ be two vectors in $\mathbb{R}^d$ and let $\vec{x}^\downarrow$ and $\vec{y}^\downarrow$ be vectors obtained by rearranging the components of $\vec{x}$ and $\vec{y}$ in decreasing order respectively. For instance, for $\vec{x}^\downarrow = (x_1^\downarrow, \cdots, x_d^\downarrow)^\mathsf{T}$, then $x_1^\downarrow \ge x_2^\downarrow \cdots \ge x_d^\downarrow$. Then it is said

that $\vec{x}$ is majorized by $\vec{y}$ and denoted by $\vec{x} \prec \vec{y}$, if the following relation holds

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \tag{1.8}$$

for each $1 \leq k \leq d-1$, and equality holding for $k = d$ [47,48]. As an example, let $\vec{x} = (1/7, 3/7, 3/7)^{\mathsf{T}}$ and $\vec{y} = (2/7, 4/7, 1/7)^{\mathsf{T}}$. Then it follows that $\vec{x}^{\downarrow} = (3/7, 3/7, 1/7)^{\mathsf{T}}$ and $\vec{y}^{\downarrow} = (4/7, 2/7, 1/7)^{\mathsf{T}}$. It is clear that Eq. (1.8) holds. As shown in the example, we stress that $\vec{x}$ and $\vec{y}$ does not have to be in decreasing order.

The majorization condition can also be expressed in terms of a special matrix called a doubly stochastic matrix. Let D be a $d \times d$ matrix with the entries d_{ij}. Then D is called a doubly stochastic matrix if

$$0 \leq d_{ij} \leq 1 \quad \text{for all} \quad i, j, \tag{1.9}$$

$$\sum_{i=1}^{d} d_{ij} = 1 \quad \text{for all} \quad j, \tag{1.10}$$

$$\sum_{j=1}^{d} d_{ij} = 1 \quad \text{for all} \quad i, \tag{1.11}$$

holds simultaneously. In words, a doubly stochastic matrix is a matrix whose entries are non-negative and whose columns and rows sum to one. Then it follows that, $\vec{x} \prec \vec{y}$ if and only if there exist a doubly stochastic matrix D such that [47,48]

$$\vec{x} = D\vec{y} \tag{1.12}$$

Further, a doubly stochastic matrix can be decomposed into the following form

$$D = \sum_{n=1}^{d!} p_n P_n \tag{1.13}$$

where P_n are permutation matrices and $0 \leq p_n \leq 1$ with $\sum_n p_n = 1$ [48]. A doubly stochastic matrix in the above form is also called Birkhoff-von Neumann form.

1.1.4 Gram matrix

Gram matrix is a positive semidefinite matrix whose entries are inner products (we use the terms inner product and scalar product interchangeably) of a given set of vectors [49,50] which is defined as follows: Let $\{|v_i\rangle\}_{i=1}^{d}$ be a set of vectors in an inner product space V with an inner product rule $\langle .|. \rangle$, then the corresponding $d \times d$ Gram matrix of the given set of vectors is defined as;

$$G_{ij} = \langle v_i | v_j \rangle . \tag{1.14}$$

The Gram matrix is a useful tool to determine whether a given set of vectors is linearly independent or to compute the total number of linearly independent vectors [18, 51–53]. The number of linearly independent vectors in a given set is equal to the rank of the Gram matrix. A set of vectors is linearly independent if and only if the determinant of the Gram matrix is positive. If one constructs the Gram matrix by using a set of vectors $\{|c_i\rangle\}_{i=1}^{d}$ that span an inner product space, which is given by;

$$G = \begin{pmatrix} 1 & c_{12} & \cdots & c_{1d} \\ c_{12}^* & 1 & \cdots & c_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1d}^* & c_{2d}^* & \cdots & 1 \end{pmatrix}, \tag{1.15}$$

where $c_{ij} = \langle c_i | c_j \rangle$ which is the overlap between normalized basis states such that $\langle c_i | c_i \rangle = 1$. Then, the given set of vectors is a valid basis if and only if $\det(G) > 0$. In the case of $d = 2$; $\det(G) = 1 - |c_{12}|^2 > 0$. For $d \geq 3$ the inner products are constrained with a certain inequality. Consider the case $d = 3$, we have,

$$\det(G) = 1 - |c_{12}|^2 - |c_{13}|^2 - |c_{23}|^2 + 2\mathrm{Re}[c_{12}c_{13}^* c_{23}] > 0, \tag{1.16}$$

where the Re[.] is the real part of a complex number. Moreover, for two given vectors $|\psi\rangle = \sum_i \psi_i |c_i\rangle$, $|\varphi\rangle = \sum_i \varphi_i |c_i\rangle$ the inner product can be expressed as $\langle \psi | \varphi \rangle = \sum_{ij} G_{ij} \psi_i^* \varphi_j$. In this context, the Gram matrix is also a metric tensor that represents the geometric structure of the inner product space. Since the Gram matrix is a positive semidefinite matrix, it can be decomposed into $G = QAQ^\dagger$ where Q is a $d \times d$ matrix whose columns are eigenvectors of the Gram matrix and A is a diagonal $d \times d$ matrix whose diagonal entries are eigenvalues of the Gram matrix. Therefore; defining $\vec{x} := Q^\dagger \vec{\psi}$, the inner product can be written as

$$\langle \psi | \psi \rangle = \vec{\psi}^\dagger G \vec{\psi} = \vec{\psi}^\dagger QAQ^\dagger \vec{\psi} = \vec{x}^\dagger A \vec{x}. \tag{1.17}$$

In this picture, for a normalized vector it is obvious that the inner product,

$$\langle \psi | \psi \rangle = \frac{|x_1|^2}{\left(\lambda_1^{-1/2}\right)^2} + \ldots + \frac{|x_n|^2}{\left(\lambda_n^{-1/2}\right)^2} = 1 \tag{1.18}$$

is an element of (hyper)-ellipsoid. Furthermore, the eigenvectors of the Gram matrix are the principal axes of the (hyper)-ellipsoid and eigenvalues are inverse of the half

length of the corresponding principal axes. These two properties of the Gram matrix will be very useful to determine maximal states (see section 2.3). It will be shown that any candidate maximal state has to be eigenvector of the Gram matrix.

An orthonormal basis $\{|i\rangle\}_{i=1}^{d}$ can be transformed into a linearly independent basis $\{|c_i\rangle\}_{i=1}^{d}$ via transformation matrix V with $V|i\rangle = |c_i\rangle$, hence the basis vectors can be chosen with a reference to an orthonormal basis. In this case the Gram matrix equals to $V^{\dagger}V$. This transition from an orthonormal basis to a linearly independent basis rather than fixing the basis vectors without a reference to an orthonormal basis also allows us to write free operations (superposition-free operations) of the resource theory of superposition in terms of free operations (incoherent operations) of the resource theory of coherence [18]. However, we stress that one does not need to introduce an orthonormal basis to build the resource theory of superposition. We show that the majorization conditions in the resource theory of coherence and superposition are related by a Gram matrix. For this purpose we define the following vector for a future use.

$$\tilde{x} := \mathrm{diag}(x_1^*,\ldots,x_n^*)G(x_1\ldots,x_n)^{\mathsf{T}} = (\tilde{x}_1,\ldots,\tilde{x}_n)^{\mathsf{T}} \tag{1.19}$$

where diag() is a diagonal matrix and where G is a Gram matrix. Realize that in the orthogonal limit i.e., $G \to \mathbb{1}$, the $\tilde{x}$ becomes the vector subjected to the doubly stochastic matrix in the resource theory of coherence [43].

1.1.5 Rayleigh quotient

Rayleigh quotient: Let G be a Gram matrix and $\vec{v}$ be an eigenvector corresponding to the minimum eigenvalue $\lambda_{\min}$ and $\vec{u}$ be an eigenvector corresponding to the maximum eigenvalue $\lambda_{\max}$. Then upper and lower bounds of a quadratic expression in the form $\vec{x}^{\dagger}\cdot\vec{x}$ under elliptic constraint represented by $\vec{x}^{\dagger}G\vec{x} = 1$ are given as

$$\max\{\sum_i |x_i|^2\} = \sum_i |v_i|^2 = \frac{1}{\lambda_{\min}}, \tag{1.20}$$

$$\min\{\sum_i |x_i|^2\} = \sum_i |u_i|^2 = \frac{1}{\lambda_{\max}}, \tag{1.21}$$

respectively. Therefore one has

$$\frac{1}{\lambda_{\max}} \leq \sum_i |x_i|^2 \leq \frac{1}{\lambda_{\min}}. \tag{1.22}$$

For a proof see Theorem 4.2.2 in chapter 4 of [49]. (In the book's notation, replace $A = G$ and decompose $G = V^{\dagger}V$, then $x' = Vx$ and $x = V^{-1}x'$. Obtain G^{-1} for the denominator term and modify the eigenvalues accordingly. This theorem also goes by the name Min-Max Theorem.) For a future use, we give the following inequality (multiply the above equation by $\lambda_{\min}$),

$$1 - \frac{\lambda_{\min}}{\lambda_{\max}} \geq 1 - \lambda_{\min} \sum_i |x_i|^2 \geq 0 \tag{1.23}$$

since $\lambda_{\max} \geq \lambda_{\min}$ it is always equal or grater than zero.

Diagonally Dominant Matrix: Let A be a $d \times d$ matrix whose entries $a_{ij} \in \mathbb{C}$, if $|a_{ii}| \geq \sum_{i \neq j} |a_{ij}|$ for all $1 \leq i \leq d$ then A is called a diagonally dominant matrix. Moreover, if A is Hermitian with real non-negative diagonal entries , then A is positive semidefinite. [49, 54] For instance, for a matrix given as

$$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}, \tag{1.24}$$

from the above theorem it can be deduced that it is a positive semidefinite matrix.

2. RESOURCE THEORY OF SUPERPOSITION

In this chapter, we first briefly summarize a part of the resource theory of superposition introduced by [18], mostly the definition of free states and operations and transformations which our work is based on. Then we continue to the deterministic state transformations of the resource theory of superposition.

2.1 Introduction

Resource theory of superposition (RTS) is a generalization of the resource theory of coherence (RTC). In contrast to the RTC, the orthogonality condition of basis states is relaxed by letting basis states to be linearly independent which leads to a possible overlap between basis states. Consequences of this relaxation can be investigated in two parts which occur to be a generalization of two aspects of the RTC. The first one is the classification of resourcefulness of a state. The measures used to calculate and to classify resourcefulness of a state are also generalized that is a measure that quantifies resourcefulness of a given state is also a valid measure in the coherence theory. For instance, the l_1 norm [18]

$$l_1(\rho) = \sum_{i \neq j} \left|\rho_{ij}\right| \tag{2.1}$$

is a both valid measure of RTS and RTC. Even though it may seem that generalization of this step is because of the generic character of the measure functions, one may still arrive at the conclusion that introducing only the RTS and then defining the measures automatically involves the RTC itself, hence any tool introduced in the RTS can be used in the RTC. The second one is manipulation of resources that is deterministic or probabilistic conversion of states to each other. In Ref. [18], probabilistic transformations are investigated and it is shown that any non-trace preserving operation can be completed to a trace preserving operation by introducing additional free operations which again coherence occurs as a special case. In this work, we investigate the deterministic transformations of single copy pure states and extent the generalization of the RTS, that is we provide sufficient conditions for a

deterministic transformation between states under the free operations. Moreover, based on this work, we also investigate the maximal states and show that there exist settings for d level systems in general which admit a maximal state.

As in the case of coherence, the free states in the RTS are defined with respect to a fixed basis. Let $\{|c_i\rangle\}_{i=1}^{d}$ be a normalized linearly independent basis of a d dimensional Hilbert space/complex Euclidean space denoted by $\mathbb{C}^d$, then the states in the following form

$$\rho = \sum_{i=1}^{d} \rho_i |c_i\rangle \langle c_i|, \tag{2.2}$$

where $\rho_i \geq 0$ and $\sum_i \rho_i = 1$ are called superposition-free states. The set that contains all superposition-free states $\mathcal{F} = \{\rho : \rho = \sum_{i=1}^{d} \rho_i |c_i\rangle \langle c_i|\}$ forms the superposition-free states. Any state $\sigma \notin \mathcal{F}$ is called a superposition state that is a resourceful state hence, the complementary set of $\mathcal{F}$ denoted by $\mathcal{R}$ forms the set of resourceful states. Since any non-trivial superposition of free states such as $|\psi\rangle = \sum_{i=1}^{d} \psi_i |c_i\rangle$ with $d \geq 2$ is a superposition state, a notion similar to the classical rank [55, 56] called superposition rank is introduced to differentiate the number of superposition of free states that spans a pure resource state. A superposition rank of a state denoted by $r_s(|.\rangle)$ defined as the minimum number of free states to span given state [18] (i.e., $r_s\left(\sum_{i=1}^{d} \psi_i |c_i\rangle\right) = d$, given that $\psi_i \neq 0$ for all i).

The free operations of the resource theory of superposition called superposition-free operation are defined as a subset of CPTP maps that satisfy a statistical constraint. A Kraus operator that satisfies the constraint $K_n \rho K_n^\dagger \in \mathcal{F}$ for all $\rho \in \mathcal{F}$ is called superposition-free Kraus operator and a CPTP quantum operation $\Phi(\rho)$ written in the following form,

$$\Phi(\rho) = \sum_{n} K_n \rho K_n^\dagger \tag{2.3}$$

by means of superposition-free Kraus operators is defined to be a superposition-free operation. The set of free operations denoted by $\mathcal{FO}$ is formed by the superposition-free operators. Moreover, the constraint on the Kraus operators also dictates the form of the operators. A Kraus operator is superposition-free if and only if it is in the following form

$$K_n = \sum_{i=1}^{d} \alpha_{i,n} \left|c_{f_n(i)}\right\rangle \left\langle c_i^{\perp}\right| \tag{2.4}$$

where the $\alpha_{i,n} \in \mathbb{C}$ and the $f_n(i)$ are arbitrary index functions and $(\{|c_i^{\perp}\rangle\})_{i=1}^{d}$ is a (non-normalized) basis satisfying $\langle c_i^{\perp}|c_j\rangle = \delta_{ij}$. Two superposition-free Kraus operators sharing the same index function are said to be the same type and a superposition-free operation can involve arbitrary number of Kraus operators of the same type. One can also write the superposition-free Kraus operators in terms of incoherent operators of the RTC. As stated in section 1.1.4, one can transform an orthonormal basis to a linearly independent basis via an invertible matrix V and since the incoherent Kraus operators have the form $\tilde{K}_n = \sum_i \beta_i |f_n(i)\rangle\langle i|$, thus; the superposition-free Kraus operators can be written as,

$$K_n = V\tilde{K}_n V^{-1}. \tag{2.5}$$

Further the condition to be a trace preserving operation reads, $\sum_n (V^{\dagger})^{-1}\tilde{K}_n^{\dagger}V^{\dagger}V\tilde{K}_n V^{-1} = \mathbb{1}_d$ where $\mathbb{1}_d$ the identity operator of $\mathbb{C}^d$. Notice that by multiplying this condition with $V^{\dagger}$ and V from left and right, respectively, the equation becomes [42]

$$\sum_n \tilde{K}_n^{\dagger} G \tilde{K}_n = G \tag{2.6}$$

where $V^{\dagger}V = G$ is the Gram matrix that is constructed with respect to the set of basis vectors $|c_i\rangle$. Notice that due to the off-diagonal entries of the Gram matrix, total number of equations to be satisfied is increased compared to the RTC.

In our framework we use two sets of superposition-free operators. The first set is used to transform an initial state to a target state with unit probability. However, this set alone does not produce a trace preserving operation in general due to the non-orthogonality. We add another set of rank(1) superposition-free Kraus operators that outputs zero to have enough degree of freedom to make the operation trace preserving while transforming states with unit probability. Recall that a superposition-free Kraus operator is in the form (Eq. (2.4))

$$K_n = \sum_{i=1}^{d} \alpha_{i,n} \left|c_{f_n(i)}\right\rangle \left\langle c_i^{\perp}\right|.$$

Here we choose the first set of Kraus operators as all permutations of an index set $f_n = P_n(1,2,\ldots,d)$, hence the set involves $d!$ elements for dimension d (where P_n denotes the nth permutation). For the second set of Kraus operators, we choose a set of operators containing d operators whose index functions are repeating elements

i.e. $f_n = (n-d!, n-d!, \ldots, n-d!)$ for $d! < n \leq d! + d$ so that they are rank(1) superposition-free Kraus operators. With the defined two sets of operators, we define the following map

$$\Phi(\rho) = \sum_{n=1}^{d!+d} K_n \rho K_n^\dagger \tag{2.7}$$

with $K_n|\psi\rangle = \sqrt{p_n}|\phi\rangle$ for $1 \leq n \leq d!$ and $K_n|\psi\rangle = 0$ for $d! < n \leq d! + 1$ (Here phase terms are embedded into the coefficients of Kraus operators). The superposition-free Kraus operators in the first set of the defined map has the following property, the probability terms p_n form a doubly stochastic matrix, that is

$$D = \sum_{n=1}^{d!} p_n P_n \tag{2.8}$$

where P_n is a permutation matrix. Notice that the doubly stochastic matrix is in the Birkhoff-von Neumann form given by the Eq. (1.13). Here we denoted both the permutation operator of the index functions and the permutation matrix in purpose, because the nth permutation matrix corresponds to the nth permutation of the index set. For instance, a doubly stochastic matrix of dimension two has the form

$$D = p_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}. \tag{2.9}$$

Further details of this picture is given in the following section.

2.2 Deterministic State Transformations

In this section, we investigate the deterministic transformations of two and three dimensional single copy pure states under superposition-free operations. Here we first propose a sufficient condition to transform an initial state to a target state deterministically and comment on the general structure by giving various applications of the proposition. Then we proceed to derivation of the proposition in both two and three dimensional systems.

Proposition 1: *For $d = 2$ and $d = 3$, let $|\psi\rangle = \sum_i e^{i\theta_i}\psi_i|c_i\rangle \in \mathbb{C}^d$ and $|\phi\rangle = \sum_i e^{i\lambda_i}\phi_i|c_i\rangle \in \mathbb{C}^d$ be pure states where $\theta_i, \lambda_i \in [0, \pi)$ and $\psi_i, \phi_i \in \mathbb{R}$. Further, let D be a $d \times d$ doubly stochastic matrix. Then, $|\psi\rangle$ deterministically transforms into $|\phi\rangle$ (denoted by $|\psi\rangle \to |\phi\rangle$) by means of the map Φ if the following conditions are satisfied*

simultaneously:

$$\text{C1:} \quad \tilde{\psi} = D\tilde{\phi} \tag{2.10}$$

$$\text{C2:} \quad \left(\psi_1^2, \ldots, \psi_n^2\right)^{\mathsf{T}} \geq D\left(\phi_1^2, \ldots, \phi_n^2\right)^{\mathsf{T}} \tag{2.11}$$

where the inequality is taken to be element-wise and where $\tilde{\psi}, \tilde{\phi}$ defined in Eq. (1.19). Realize that, in C1, the entries of the vector subject to the doubly stochastic matrix are not necessarily real, hence one cannot introduce the majorization theory. We stress that it is practically two separate equation systems since there are both real and imaginary parts. However, in the real setting where the components of the states are real, then one can introduce the majorization theory, since a real vector $\vec{x}$ is majorized by a real vector $\vec{y}$ if and only if $\vec{x} = D\vec{y}$ [47] for some doubly stochastic matrix D. Hence, in such a setting condition C1 turns into the majorization condition, thereby; C1 reads $\tilde{\psi} \prec \tilde{\phi}$ and Eq. (1.8) can be used to determine whether it is satisfied that is,

$$\sum_{i=1}^{k} \tilde{\psi}_i^{\downarrow} \leq \sum_{i=1}^{k} \tilde{\phi}_i^{\downarrow} \tag{2.12}$$

for each $1 \leq k \leq d-1$, and equality holding for $k = d$. We stress two points for the conditions C1 and C2. Realize that for a normalized state, the statement in the majorization condition "*equality holding for* $k = d$" is redundant, hence only $d-1$ comparisons are required. A similar picture also appears in C2. For a given pair of normalized states, if C1 and $d-1$ elements of C2 are satisfied then the remaining element of C2 is satisfied trivially due to the normalization and the majorization conditions.

In Ref. [18], it is shown that superposition rank of a state cannot be increased by means of superposition-free operations, hence; a probabilistic or a deterministic transformation only possible if the initial state's superposition rank is greater than or equal to the target state's superposition rank, that is, $r_s(|\psi\rangle) \geq r_s(|\phi\rangle)$. Even though the Proposition 1 is a sufficient condition, it can be conjectured that the underlying reason is due to the majorization condition. Since for two vectors $\vec{x}$ and $\vec{y}$, if $\vec{x}$ has more non-zero elements than $\vec{y}$, then $\vec{x}$ cannot be majorized by $\vec{y}$ which can be seen from the Eq. (2.12).

The condition C2 guarantees that sum of Kraus operators that perform the actual transformation (the operators in the first set) can be completed to a trace preserving

map by additional rank(1) superposition-free operators that output zero. To see this consider the following; if one wants that $K_n|\psi\rangle = 0$ for all $d!+1 \leq n \leq d!+d$, then the expression (recall the map given by Eq. (2.7))

$$\mathbb{1}_d - \sum_{n=1}^{d!} K_n^\dagger K_n \geq 0 \tag{2.13}$$

must be positive semidefinite with $|\psi\rangle$ being an eigenvector corresponding to a zero valued eigenvalue that is

$$\left(\mathbb{1}_d - \sum_{n=1}^{d!} K_n^\dagger K_n\right)|\psi\rangle = 0 \tag{2.14}$$

since $K_n|\psi\rangle = 0$ for all $d!+1 \leq n \leq d!+d$. Therefore, one can decompose it into positive semidefinite operators, that is,

$$\sum_{n=d!+1}^{d!+d} K_n^\dagger K_n = \mathbb{1}_d - \sum_{n=1}^{d!} K_n^\dagger K_n. \tag{2.15}$$

If two given states satisfy only C1, then Eq. (2.13) is not positive semidefinite, hence one can not make the operation trace preserving by adding another set of superposition-free operators. Due to this picture we call the condition C2 *condition on completeness* (CoC). The CoC has further constraint on the set of settings that admit a maximal state (see section 2.3).

This brings us to the following problem. How does one find a doubly stochastic matrix for a pair of states to test whether they satisfy conditions C1 and C2 simultaneously? Two real vectors can be tested by using the Eq. (2.12) to determine whether the majorization condition holds, however to test whether C1 and C2 holds simultaneously, one must find a doubly stochastic. Fortunately, a doubly stochastic matrix can be constructed for a subset of initial and target state pairs by using order relation of the states components as discussed in [57]. In section 2.2.1 and section 2.2.2 we utilize this framework to build doubly stochastic matrices for given states.

2.2.1 Two level (qubit) systems

In this section we give various applications of the Proposition 1 for two dimensional (qubit) systems. First consider the general form of an initial and a target states which

can be written as

$$|\psi\rangle = e^{i\theta_1}\psi_1 |c_1\rangle + e^{i\theta_2}\psi_2 |c_2\rangle , \tag{2.16}$$

$$|\phi\rangle = e^{i\lambda_1}\phi_1 |c_1\rangle + e^{i\lambda_2}\phi_2 |c_2\rangle \tag{2.17}$$

where $\theta_i, \lambda_i \in [0, \pi)$ and $\psi_i, \phi_i \in \mathbb{R}$. Then the condition C1 (Eq. (2.10)) reads

$$\begin{pmatrix} \psi_1^2 + \psi_1\psi_2 c_{12} e^{i(\theta_2-\theta_1)} \\ \psi_2^2 + \psi_1\psi_2 c_{12}^* e^{-i(\theta_2-\theta_1)} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix} \begin{pmatrix} \phi_1^2 + \phi_1\phi_2 c_{12} e^{i(\lambda_2-\lambda_1)} \\ \phi_2^2 + \phi_1\phi_2 c_{12}^* e^{-i(\lambda_2-\lambda_1)} \end{pmatrix}. \tag{2.18}$$

Without loss of generality we set $\theta_1 = \lambda_1 = 0$. Further, we parameterize the scalar product as $c_{12} = \mu e^{i\omega}$ where $\mu \in (-1, 1)$ and $\omega \in [0, \pi)$ to investigate the imaginary part of the equation system. The imaginary part is found to be (two identical equations)

$$\psi_1\psi_2\mu \sin(\omega + \theta_2) = \phi_1\phi_2\mu(p_1 - p_2)\sin(\omega + \lambda_2), \tag{2.19}$$

$$\psi_1\psi_2\mu \sin(\omega + \theta_2) = \phi_1\phi_2\mu(p_1 - p_2)\sin(\omega + \lambda_2). \tag{2.20}$$

Even though the Proposition 1 holds for the most general setting (i.e, both the scalar product and the components of the initial and the target state are not constrained), the set of possible deterministic transformations may be an empty set for a particular setting. Consider the following case where $\omega = 0$, hence the scalar product is real and $\theta_2 \neq 0$, $\lambda_2 = 0$ that is, a transformation from a complex state to a real state. In that case $\mu \sin(\theta_2) = 0$ which holds only if $\mu = 0$, i.e., in the orthonormal limit, hence there is no possible deterministic transformation except the orthonormal case which corresponds to the deterministic transformations of the RTC. This situation also shows that in contrast to the RTC, the operations acting on the relative phase terms are not free in general as stated in Ref. [18]. Even if one has an initial state and a target state with equal components up to a phase factor, one cannot accomplish a deterministic transformation. For instance, a transformation given by

$$\psi_1 |c_1\rangle + e^{i\theta_2}\psi_2 |c_2\rangle \nrightarrow \psi_1 |c_1\rangle + \psi_2 |c_2\rangle \tag{2.21}$$

is not possible for a real scalar product setting.

However realize that, if one sets $p_1 = p_2 = 1/2$ the resultant expression of the right hand side of the Eq. (2.18) is always $(1/2, 1/2)^\intercal$ since $\tilde{\phi}_1 + \tilde{\phi}_2 = 1$. This suggest that by taking the probabilities of transformations implemented by the first set of Kraus

operators $1/d!$ in general, one may obtain maximal states. In section 2.3 we show that this is in fact possible.

In the completely real case where both the scalar product and the components of the states are real, the vectors subject to the doubly stochastic matrix become real and hence one can introduce the majorization theory. For a given pair of an initial and a target state

$$|\psi\rangle = \psi_1 |c_1\rangle + \psi_2 |c_2\rangle \tag{2.22}$$

$$|\phi\rangle = \phi_1 |c_1\rangle + \phi_2 |c_2\rangle \tag{2.23}$$

respectively and where $\psi_i, \phi_i \in \mathbb{R}$. Then the majorization condition reads (see Eq. (2.12)),

$$\tilde{\psi}_1^{\downarrow} \leq \tilde{\phi}_1^{\downarrow} \tag{2.24}$$

since $\tilde{\psi}_1^{\downarrow} + \tilde{\psi}_2^{\downarrow} = \tilde{\phi}_1^{\downarrow} + \tilde{\phi}_2^{\downarrow} = 1$ for normalized states. As an application of the Proposition 1, consider the following set of state pairs given for the setting where the inner product of the basis states taken to be $\langle c_1|c_2\rangle = c_{12} = 1/2$.

$$S_1 : \left\{ \frac{3|c_1\rangle - |c_2\rangle}{\sqrt{7}}, \frac{4|c_1\rangle - |c_2\rangle}{\sqrt{13}} \right\}, \tag{2.25}$$

$$S_2 : \left\{ \frac{3|c_1\rangle + |c_2\rangle}{\sqrt{13}}, \frac{4|c_1\rangle - |c_2\rangle}{\sqrt{13}} \right\}, \tag{2.26}$$

$$S_3 : \left\{ \frac{4|c_1\rangle + |c_2\rangle}{\sqrt{21}}, \frac{3|c_1\rangle + |c_2\rangle}{\sqrt{13}} \right\}, \tag{2.27}$$

$$S_4 : \left\{ \frac{3|c_1\rangle + |c_2\rangle}{\sqrt{13}}, \frac{4|c_1\rangle + |c_2\rangle}{\sqrt{21}} \right\}, \tag{2.28}$$

$$S_5 : \left\{ \frac{4|c_1\rangle - |c_2\rangle}{\sqrt{13}}, \frac{3|c_1\rangle + |c_2\rangle}{\sqrt{13}} \right\}. \tag{2.29}$$

Here members of a pair on the left is the initial state and the right one is the target state. The conditions satisfied by the given pairs of states are summarized in Table 2.1. Among the given sets of pairs, only the S_1 (Eq. (2.25)) can be transformed deterministically. Realize that

$$\tilde{\psi} = \left(\frac{15}{14}, \frac{-1}{14} \right)^{\mathsf{T}}, \quad \tilde{\phi} = \left(\frac{14}{13}, \frac{-1}{13} \right)^{\mathsf{T}}, \tag{2.30}$$

here one can check that the majorization condition holds by using Eq. (2.24). A doubly stochastic matrix connecting the two vectors above can be found using the framework introduced in Ref. [57] by replacing components of the initial and target states with

the components of $\tilde{\psi}$ and $\tilde{\phi}$ (the power of this framework will be demonstrated in the following section). The probabilities are are found to be,

$$p_1 = \frac{\tilde{\phi}_1 - \tilde{\psi}_2}{\tilde{\phi}_1 - \tilde{\phi}_2}, \quad p_2 = \frac{\tilde{\psi}_2 - \tilde{\phi}_2}{\tilde{\phi}_1 - \tilde{\phi}_2}. \tag{2.31}$$

(Here we omit $\downarrow$ since the vectors themselves are sorted.) Then the doubly stochastic matrix can be constructed as

$$D = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix} = \begin{pmatrix} \frac{209}{210} & \frac{1}{210} \\ \frac{1}{210} & \frac{209}{210} \end{pmatrix} \tag{2.32}$$

therefore; $\tilde{\psi} = D\tilde{\phi}$. Moreover, the CoC can be tested as,

$$\left(\frac{9}{7}, \frac{1}{7}\right)^{\intercal} - \begin{pmatrix} \frac{209}{210} & \frac{1}{210} \\ \frac{1}{210} & \frac{209}{210} \end{pmatrix} \left(\frac{16}{13}, \frac{1}{13}\right)^{\intercal} = \begin{pmatrix} \frac{11}{182} \\ \frac{11}{182} \end{pmatrix} \tag{2.33}$$

thus, both the majorization and the CoC conditions are satisfied. The superposition-free Kraus operators can be constructed as,

$$K_1 = \sqrt{\frac{209}{210}} \left[\sqrt{\frac{7}{13}} \left(\frac{4}{3} |c_1\rangle \left\langle c_1^{\perp} \right| + |c_2\rangle \left\langle c_2^{\perp} \right| \right) \right], \tag{2.34}$$

$$K_2 = -\sqrt{\frac{1}{210}} \left[\sqrt{\frac{7}{13}} \left(\frac{1}{3} |c_2\rangle \left\langle c_1^{\perp} \right| + 4 |c_1\rangle \left\langle c_2^{\perp} \right| \right) \right], \tag{2.35}$$

$$K_3 = -\sqrt{\frac{11}{26}} \left(\frac{1}{3} |c_1\rangle \left\langle c_1^{\perp} \right| + |c_1\rangle \left\langle c_2^{\perp} \right| \right), \tag{2.36}$$

where $K_1 |\psi\rangle = \sqrt{\frac{209}{210}} |\phi\rangle$, $K_2 |\psi\rangle = \frac{1}{\sqrt{210}} |\phi\rangle$, and $K_3 |\psi\rangle = 0$. (Here we take $K_4 = 0$.)

A measure of a resource theory classifies how resourceful a given state is, therefore; it is a requirement that if a deterministic transformation is possible, then the initial states must be no less resourceful than the target state that is, $\mathcal{M}(|\psi\rangle\langle\psi|) \geq \mathcal{M}(|\phi\rangle\langle\phi|)$ where $\mathcal{M}$ denotes a measure function. Here, we use l_1 norm (Eq. (2.1)) as a measure for comparison of the resourcefulness of a state. By calculating the l_1 norm of the initial and the target state of the set S_1,

$$l_1(|\psi\rangle\langle\psi|) = \frac{3}{7}, \quad l_1(|\phi\rangle\langle\phi|) = \frac{4}{13} \tag{2.37}$$

one can see that $l_1(|\psi\rangle\langle\psi| > l_1(|\phi\rangle\langle\phi|)$ as expected.

One can check that states in the set S_2 satisfy the majorization condition but violates the CoC. Hence the first set of Kraus operators (Eq. (2.7)) cannot be completed to a

Table 2.1 : List of example set of state pairs with the conditions they satisfy or violate. Here '✓' denotes that the given pair satisfies the corresponding condition and '×' means the pair violates the condition.

Set	Majorization	CoC	l_1 Norm
S_1	✓	✓	✓
S_2	✓	×	×
S_3	×	✓	×
S_4	✓	×	✓
S_5	×	✓	✓

trace preserving map by introducing rank(1) superposition-free Kraus operators. For the set S_3, one can find a doubly stochastic matrix that satisfies the CoC, however in this case the majorization condition is violated hence one cannot construct the first set of Kraus operators. Finally for the sets S_4 and S_5 both pairs' initial states have greater l_1 norm than the target states (hence more resourceful) however, the S_4 violates the CoC and the S_5 violates the majorization condition.

The given set of examples shows a difference between the RTC and RTS. In the RTC, two states are said to be incomparable if neither $|\psi\rangle \to |\phi\rangle$ nor $|\phi\rangle \to |\psi\rangle$ (by means of incoherent operations) [43] which is a notion inherited from the deterministic transformations of bipartite entanglement [9]. In RTC, the majorization condition is the necessary and sufficient condition for deterministic transformations and hence if neither of them majorizes the other then they are incomparable. However, in the RTS, two states may still be incomparable by means of the map Eq. (2.7) even if they satisfy the majorization condition as in the case of S_2 or S_4. Both states satisfy the majorization condition, but they violate the CoC condition hence neither $|\psi\rangle \to |\phi\rangle$ nor $|\phi\rangle \to |\psi\rangle$. We stress a point that the Proposition 1 is a sufficient condition, hence there can be more general maps that can transform pairs of states to each other that cannot be transformed by the map defined in Eq. (2.7). Therefore; it is an open question whether conditions given by the Proposition 1 are necessary and sufficient conditions.

2.2.2 Three level systems

Three dimensional systems or any higher dimensional systems vastly differ from the qubit system due to the number of Kraus operators of the first set. In qubit system there are $2! = 2$ Kraus operator in the first set, hence there are two probability terms to be found. However in dimension three, there are six operators hence there are

six probability terms to be found. As the number of unknown probabilities exceeds the dimension of the system, one cannot find the probabilities trivially, therefore; one cannot construct the corresponding doubly stochastic matrix. The majorization condition only provides the existence of a doubly stochastic matrix. Assume that $\tilde{\psi} \prec \tilde{\phi}$, then there exists a doubly stochastic matrix such that $\tilde{\psi} = D\tilde{\phi}$, however entries of D are unknown. In the case of a qubit, there are two equations (rows) and two probabilities hence it is trivial to find the probability terms, however for $d > 2$ it is non-trivial. This problem can be solved for a subset of initial and target states. If one takes all components of states real and the scalar product terms real and equal, the framework introduced in Ref. [57] can be invoked. However, the subset of the initial and target states must be further split into two cases due to the majorization condition. Assume that a pair of states with real components satisfies the majorization condition. Then, component comparison of the majorization condition reads

$$\tilde{\psi}_1^{\downarrow} \leq \tilde{\phi}_1^{\downarrow}, \tag{2.38}$$

$$\tilde{\psi}_1^{\downarrow} + \tilde{\psi}_2^{\downarrow} \leq \tilde{\phi}_1^{\downarrow} + \tilde{\phi}_2^{\downarrow}. \tag{2.39}$$

Notice that there are two possibilities for a pair of states to satisfy Eq. (2.39). The first one is $\tilde{\psi}_2 \geq \tilde{\phi}_2$ and the second one $\tilde{\psi}_2 \leq \tilde{\phi}_2$. In Ref. [57] it is shown that for each case given above, d operators among $d!$ type of Kraus operators (operators in the first set), can be chosen in such a way that one can find d probability terms for d equations (rows), hence one can construct a doubly stochastic matrix by solving the equation system. For the first case, the Kraus operators are chosen to be such that index functions are given as

$$\{f_1(k)\} = \{1,2,3\}, \quad \{f_2(k)\} = \{3,2,1\}, \quad \{f_3(k)\} = \{2,1,3\}, \tag{2.40}$$

for $1 \leq k \leq 3$, and for the second set, index functions are given as

$$\{f_1(k)\} = \{1,2,3\}, \quad \{f_2(k)\} = \{3,2,1\}, \quad \{f_3(k)\} = \{1,3,2\}, \tag{2.41}$$

for $1 \leq k \leq 3$.

Using the framework, the probability terms are found to be

$$p_1 = 1 - p_2 - p_3, \quad p_2 = \frac{\tilde{\psi}_3^{\downarrow} - \tilde{\phi}_3^{\downarrow}}{\tilde{\phi}_1^{\downarrow} - \tilde{\phi}_3^{\downarrow}}, \quad p_3 = \frac{\tilde{\psi}_2^{\downarrow} - \tilde{\phi}_2^{\downarrow}}{\tilde{\phi}_1^{\downarrow} - \tilde{\phi}_2^{\downarrow}} \tag{2.42}$$

for the first case ($\tilde{\psi}_2 \geq \tilde{\phi}_2$). The corresponding doubly stochastic matrix is given as

$$D = \begin{pmatrix} p_1 & p_3 & p_2 \\ p_3 & p_1+p_2 & 0 \\ p_2 & 0 & p_1+p_3 \end{pmatrix}. \tag{2.43}$$

For the second case ($\tilde{\psi}_2 \leq \tilde{\phi}_2$), the probability terms are found as

$$p_1 = 1-p_2-p_3, \quad p_2 = \frac{\tilde{\phi}_1^{\downarrow} - \tilde{\psi}_1^{\downarrow}}{\tilde{\phi}_1^{\downarrow} - \tilde{\phi}_3^{\downarrow}}, \quad p_3 = \frac{\tilde{\phi}_2^{\downarrow} - \tilde{\psi}_2^{\downarrow}}{\tilde{\phi}_2^{\downarrow} - \tilde{\phi}_3^{\downarrow}} \tag{2.44}$$

and the corresponding doubly stochastic matrix is

$$D = \begin{pmatrix} p_1+p_3 & p_2 & 0 \\ p_2 & p_1 & p_3 \\ 0 & p_3 & p_1+p_2. \end{pmatrix}. \tag{2.45}$$

As an application of our findings consider the following examples. To represent the first case we have an initial state

$$|\psi\rangle = \sqrt{\frac{2}{17}}\left(3|c_1\rangle + 2|c_2\rangle + |c_1\rangle\right), \tag{2.46}$$

and a target state

$$|\phi\rangle = \sqrt{\frac{1}{14}}\left(4|c_1\rangle + 2|c_2\rangle + |c_1\rangle\right) \tag{2.47}$$

for a setting where $\langle c_1|c_2\rangle = \langle c_1|c_3\rangle = \langle c_2|c_3\rangle = \mu = -1/4$. It is easy to check that $\tilde{\psi} \prec \tilde{\phi}$ with $\tilde{\psi}_2 \geq \tilde{\phi}_2$ which corresponds to the first case. By using the Eq. (2.42), probabilities are found to be

$$p_1 = \frac{2947}{3519}, \quad p_2 = \frac{1}{153}, \quad p_3 = \frac{61}{391}. \tag{2.48}$$

After constructing the doubly stochastic matrix via Eq. (2.43), it is trivial to show that the CoC is also satisfied. Using the form of the superposition-free Kraus operators (Eq. (2.4)) with the index set define above, one can construct the following Kraus operators to implement the deterministic transformation.

$$K_1 = \sqrt{p_1}\sqrt{\frac{17}{28}}\left(\frac{4}{3}|c_1\rangle\left\langle c_1^{\perp}\right| + |c_2\rangle\left\langle c_2^{\perp}\right| + |c_3\rangle\left\langle c_3^{\perp}\right|\right), \tag{2.49}$$

$$K_2 = \sqrt{p_2}\sqrt{\frac{17}{28}}\left(\frac{1}{3}|c_3\rangle\left\langle c_1^{\perp}\right| + |c_2\rangle\left\langle c_2^{\perp}\right| + 4|c_1\rangle\left\langle c_3^{\perp}\right|\right), \tag{2.50}$$

$$K_3 = \sqrt{p_3}\sqrt{\frac{17}{28}}\left(\frac{2}{3}|c_2\rangle\left\langle c_1^{\perp}\right| + 2|c_1\rangle\left\langle c_2^{\perp}\right| + |c_3\rangle\left\langle c_3^{\perp}\right|\right), \tag{2.51}$$

since the CoC is satisfied one does not need to explicitly built the second set operators.

Proof of Proposition 1

Two Level: In the qubit case, an initial $|\psi\rangle$ and a target $|\phi\rangle$ pure states can be written in the most general form as

$$|\psi\rangle = \psi_1 |c_1\rangle + e^{i\theta_2}\psi_2 |c_2\rangle , \tag{2.52}$$

$$|\phi\rangle = \phi_1 |c_1\rangle + e^{i\lambda_2}\phi_2 |c_2\rangle , \tag{2.53}$$

where $\psi_i, \phi_i \in \mathbb{R}, \forall i$ and $\theta_2, \lambda_2 \in [0, \pi)$ are relative phases of the states. We introduce the set of Kraus operators using the same procedure in Ref [18] that is, we first define the incoherent operators, then transform them to superposition-free operators using the relation given in the Eq. (2.5), thus; $\{K_n\}_{n=1}^4 = \{V\tilde{K}_n V^{-1}\}_{n=1}^4$. Using the index functions $f_1 = \{1,2\}$, $f_2 = \{2,1\}$, $f_3 = \{1,1\}$ and $f_4 = \{2,2\}$ we obtain the following Kraus operators

$$\begin{aligned}
\tilde{K}_1 &= \alpha_1 |1\rangle\langle 1| + \beta_1 |2\rangle\langle 2| \rightarrow K_1 = \alpha_1 |c_1\rangle\langle c_1| + \beta_1 |c_1\rangle\langle c_2| , \\
\tilde{K}_2 &= \alpha_2 |2\rangle\langle 1| + \beta_2 |1\rangle\langle 2| \rightarrow K_2 = \alpha_2 |c_2\rangle\langle c_1| + \beta_2 |c_1\rangle\langle c_2| , \\
\tilde{K}_3 &= \alpha_3 |1\rangle\langle 1| + \beta_3 |1\rangle\langle 2| \rightarrow K_3 = \alpha_3 |c_1\rangle\langle c_1| + \beta_3 |c_1\rangle\langle c_2| , \\
\tilde{K}_4 &= \alpha_4 |2\rangle\langle 1| + \beta_4 |2\rangle\langle 2| \rightarrow K_4 = \alpha_4 |c_2\rangle\langle c_1| + \beta_4 |c_2\rangle\langle c_2| .
\end{aligned} \tag{2.54}$$

To have a trace preserving operation one needs to satisfy the Eq. (2.6), with the coefficients given above. Then Eq. (2.6) reads

$$\begin{aligned}
&\sum_n \tilde{K}_n^\dagger G \tilde{K}_n = \\
&\begin{pmatrix} |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 & \alpha_1^*\beta_1 c_{12} + \alpha_2^*\beta_2 c_{12}^* + \alpha_3^*\beta_3 + \alpha_4^*\beta_4 \\ \alpha_1\beta_1^* c_{12}^* + \alpha_2\beta_2^* c_{12} + \alpha_3\beta_3^* + \alpha_4\beta_4^* & |\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2 + |\beta_4|^2 \end{pmatrix},
\end{aligned} \tag{2.55}$$

where G is a Gram matrix representing the inner product setting. One can see that $G_{13} = G_{23}^*$. Unlike in the RTC, here we have also off-diagonal terms coupled with the inner product due to the non-orthogonality. To carry out the transformation $|\psi\rangle\langle\psi| \rightarrow |\phi\rangle\langle\phi|$ one needs,

$$\Phi(|\psi\rangle\langle\psi|) = \sum_{n=1}^{4} K_n |\psi\rangle\langle\psi| K_n^\dagger = |\phi\rangle\langle\phi| , \tag{2.56}$$

therefore; the output of the operators in the first set must be $K_n |\psi\rangle = \sqrt{p_n}|\phi\rangle$ with $p_1 + p_2 = 1$, (We omitted the overall phases of the target state using the fact that it

can be embedded to Kraus operators) and the remaining rank(1) superposition-free Kraus operators (the second set) must be $K_n|\psi\rangle = 0$. To obtain the output state, the coefficients of the operators in the first set have to be in the following form,

$$\begin{aligned} \alpha_1 &= \sqrt{p_1}\frac{\phi_1}{\psi_1}, \qquad & \beta_1 &= \sqrt{p_1}e^{i(\theta_2-\theta_1)}\frac{\phi_2}{\psi_2}, \\ \alpha_2 &= \sqrt{p_2}e^{i\theta_2}\frac{\phi_2}{\psi_1}, \qquad & \beta_2 &= \sqrt{p_2}e^{-i\theta_1}\frac{\phi_1}{\psi_2}, \end{aligned} \tag{2.57}$$

and for the rank(1) superposition-free operators, the constraints on the coefficients are given as

$$\alpha_3 = -e^{i\theta_1}\beta_3\frac{\psi_2}{\psi_1}, \quad \alpha_4 = -e^{i\theta_1}\beta_4\frac{\psi_2}{\psi_1}. \tag{2.58}$$

After algebraic manipulations, one can obtain the following equations using Eq. (2.55). From G_{11} and from G_{22}, one obtains

$$p_1\phi_1^2 + p_2\phi_2^2 + |\beta_3|^2\psi_2^2 + |\beta_4|^2\psi_2^2 = \psi_1^2 \tag{2.59}$$

and

$$p_2\phi_1^2 + p_1\phi_2^2 + |\beta_3|^2\psi_2^2 + |\beta_4|^2\psi_2^2 = \psi_2^2 \tag{2.60}$$

respectively. From the off-diagonal element $G_{12} = G_{21}^*$ (multiplied by $e^{i\theta_1}$) one obtains the following equation

$$p_1\phi_1\phi_2c_{12}e^{i\theta_2} + p_2\phi_1\phi_2c_{12}^*e^{-i\theta_2} - |\beta_3|^2\psi_2^2 - |\beta_4|^2\psi_2^2 = \psi_1\psi_2c_{12}e^{i\theta_1}. \tag{2.61}$$

By combining Eq. (2.59), Eq. (2.61) and Eq. (2.60), complex conjugate of Eq. (2.61) one can obtain the following equation system

$$\begin{aligned} p_1\left(\phi_1^2 + \phi_1\phi_2c_{12}e^{i\theta_2}\right) + p_2\left(\phi_2^2 + \phi_1\phi_2c_{12}^*e^{-i\theta_2}\right) &= \psi_1^2 + \psi_1\psi_2c_{12}e^{i\theta_1}, \\ p_2\left(\phi_1^2 + \phi_1\phi_2c_{12}e^{i\theta_2}\right) + p_1\left(\phi_2^2 + \phi_1\phi_2c_{12}^*e^{-i\theta_2}\right) &= \psi_2^2 + \psi_1\psi_2c_{12}^*e^{-i\theta_1}. \end{aligned} \tag{2.62}$$

One can re-write the Eq. (2.62) in the matrix form as

$$\begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}\begin{pmatrix} \phi_1^2 + \phi_1\phi_2c_{12}e^{i\theta_2} \\ \phi_2^2 + \phi_1\phi_2c_{12}^*e^{-i\theta_2} \end{pmatrix} = \begin{pmatrix} \psi_1^2 + \psi_1\psi_2c_{12}e^{i\theta_1} \\ \psi_2^2 + \psi_1\psi_2c_{12}^*e^{-i\theta_1} \end{pmatrix}. \tag{2.63}$$

Realize that, for a deterministic transformation one has $p_1, p_2 \geq 0$ with $p_1 + p_2 = 1$, therefore; one can identify the matrix whose entries are the probabilities of the transformation as a doubly stochastic matrix. Moreover, the above equation can be decomposed into

$$D\begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2e^{-i\theta_2} \end{pmatrix}G\begin{pmatrix} \phi_1 \\ \phi_2e^{i\theta_2} \end{pmatrix} = \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2e^{-i\theta_1} \end{pmatrix}G\begin{pmatrix} \psi_1 \\ \psi_2e^{i\theta_1} \end{pmatrix}. \tag{2.64}$$

where D is a doubly stochastic matrix and where G is the Gram matrix defined in Eq. (1.15). Therefore, the condition C1 of the proposition reads

$$\tilde{\psi} = D\tilde{\phi}. \tag{2.65}$$

To obtain C2, one can manipulate Eq. (2.59) and Eq. (2.60), then one obtains

$$\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} - D \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix} = \begin{pmatrix} |\beta_3|^2 \psi_2^2 + |\beta_4|^2 \psi_2^2 \\ |\beta_3|^2 \psi_2^2 + |\beta_4|^2 \psi_2^2 \end{pmatrix}, \tag{2.66}$$

therefore one is also required to satisfy the following due to the non-negative terms

$$\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} \geq D \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix} \tag{2.67}$$

which is the condition C2 of the proposition.

Three Level: We start with defining an initial and a target pure states in the general form,

$$|\psi\rangle = \psi_1 |c_1\rangle + e^{i\theta_2} \psi_2 |c_2\rangle + e^{i\theta_3} \psi_3 |c_3\rangle, \tag{2.68}$$

$$|\phi\rangle = \phi_1 |c_1\rangle + e^{i\lambda_2} \phi_2 |c_2\rangle + e^{i\lambda_3} \phi_3 |c_3\rangle, \tag{2.69}$$

where $\theta_i, \lambda_i \in [0, \pi)$ and $\psi_i, \phi_i \in \mathbb{R} \, \forall \, i$. Further we take the superposition-free Kraus operators in the following form,

$$K_n = V \left(\alpha_n |f_n(1)\rangle\langle 1| + \beta_n |f_n(2)\rangle\langle 2| + \gamma_n |f_n(3)\rangle\langle 3| \right) V^{-1} = V \tilde{K}_n V^{-1} \tag{2.70}$$

where $\tilde{K}_n$ is an incoherent operation of coherence and the index functions, $f_n(k)$'s are in the following order

$$f_1 = \{1,2,3\} \quad f_2 = \{2,1,3\} \quad f_3 = \{3,2,1\}$$
$$f_4 = \{1,3,2\} \quad f_5 = \{2,3,1\} \quad f_6 = \{3,1,2\}$$

To have a trace preserving superposition-free operation one requires $\sum K_n^\dagger K_n = \mathbb{1}_d$ or equivalently $\sum \tilde{K}_n^\dagger G \tilde{K}_n = G$, hence one must satisfy the following equations simultaneously

$$G_{11} = \sum_{i=1}^{9} |\alpha_i|, \quad G_{22} = \sum_{i=1}^{9} |\beta_i|, \quad G_{33} = \sum_{i=1}^{9} |\gamma_i|, \tag{2.71}$$

$$\begin{aligned} G_{12} = G_{21}^* = {} & \alpha_1^* \beta_1 c_{12} + \alpha_2^* \beta_2 c_{12}^* + \alpha_3^* \beta_3 c_{23}^* \\ & + \alpha_4^* \beta_4 c_{13} + \alpha_5^* \beta_5 c_{23}^* + \alpha_6^* \beta_6 c_{13}^* + \sum_{i=7}^{9} \alpha_i^* \beta_i, \end{aligned} \tag{2.72}$$

$$G_{13} = G_{31}^* = \alpha_1^*\gamma_1 c_{13} + \alpha_2^*\gamma_2 c_{23} + \alpha_3^*\gamma_3 c_{13}^* + \alpha_4^*\gamma_4 c_{12} + \alpha_5^*\gamma_5 c_{12}^* + \alpha_6^*\gamma_6 c_{23}^* + \sum_{i=7}^{9} \alpha_i^*\gamma_i, \tag{2.73}$$

$$G_{23} = G_{32}^* = \beta_1^*\gamma_1 c_{23} + \beta_2^*\gamma_2 c_{13} + \beta_3^*\gamma_3 c_{12}^* + \beta_4^*\gamma_4 c_{23}^* + \beta_5^*\gamma_5 c_{13}^* + \beta_6^*\gamma_6 c_{12} + \sum_{i=7}^{9} \beta_i^*\gamma_i. \tag{2.74}$$

To carry out the transformation $K_n|\psi\rangle = \sqrt{p_n}|\phi\rangle$, coefficients of the Kraus operators must be

$$\begin{aligned}
&\alpha_1 = \sqrt{p_1}\phi_1/\psi_1, && \beta_1 = \sqrt{p_1}e^{i(\lambda_2-\theta_2)}\phi_2/\psi_2, && \gamma_1 = \sqrt{p_1}e^{i(\lambda_3-\theta_3)}\phi_3/\psi_3,\\
&\alpha_2 = \sqrt{p_2}e^{i\lambda_2}\phi_2/\psi_1, && \beta_2 = \sqrt{p_2}e^{-i\theta_2}\phi_1/\psi_2, && \gamma_2 = \sqrt{p_2}e^{i(\lambda_3-\theta_3)}\phi_3/\psi_3,\\
&\alpha_3 = \sqrt{p_3}e^{i\lambda_3}\phi_3/\psi_1, && \beta_3 = \sqrt{p_3}e^{i(\lambda_2-\theta_2)}\phi_2/\psi_2, && \gamma_3 = \sqrt{p_3}e^{-i\theta_3}\phi_1/\psi_3,\\
&\alpha_4 = \sqrt{p_4}\phi_1/\psi_1, && \beta_4 = \sqrt{p_4}e^{i(\lambda_3-\theta_2)}\phi_3/\psi_2, && \gamma_4 = \sqrt{p_4}e^{i(\lambda_2-\theta_3)}\phi_2/\psi_3\\
&\alpha_5 = \sqrt{p_5}e^{i\lambda_2}\phi_2/\psi_1, && \beta_5 = \sqrt{p_5}e^{i(\lambda_3-\theta_2)}\phi_3/\psi_2, && \gamma_5 = \sqrt{p_5}e^{-i\theta_3}\phi_1/\psi_3\\
&\alpha_6 = \sqrt{p_6}e^{i\lambda_3}\phi_3/\psi_1, && \beta_6 = \sqrt{p_5}e^{-i\theta_2}\phi_1/\psi_2, && \gamma_6 = \sqrt{p_6}e^{-i(\lambda_2-\theta_3)}\phi_2/\psi_3
\end{aligned}$$

From $F_n|\psi\rangle = 0$ one obtains,

$$\alpha_i = -\left(\beta_i e^{i\theta_2}\psi_2 + \gamma_i e^{i\theta_3}\psi_3\right)/\psi_1, \text{ for } i = 7,8,9. \tag{2.75}$$

By combining the coefficients with the equations above, one obtains the following equation from Eq. (2.71) (G_{11} multiplied by ψ_1^2),

$$p_1\phi_1^2 + p_2\phi_2^2 + p_3\phi_3^2 + p_4\phi_1^2 + p_5\phi_2^2 + p_6\phi_3^2 + \sum_{i=7}^{9}\left|\beta_i e^{i\theta_2}\psi_2 + \gamma_i e^{i\theta_3}\psi_3\right|^2 = \psi_1^2. \tag{2.76}$$

Similarly from G_{22} (multiplied by ψ_2^2) the equation reads,

$$p_1\phi_2^2 + p_2\phi_1^2 + p_3\phi_2^2 + p_4\phi_3^2 + p_5\phi_3^2 + p_6\phi_1^2 + \sum_{i=5}^{7}|\beta_i|^2\psi_2^2 = \psi_2^2, \tag{2.77}$$

and from the last diagonal element G_{33}(multiplied by ψ_3^2) the equation found to be

$$p_1\phi_3^2 + p_2\phi_3^2 + p_3\phi_1^2 + p_4\phi_2^2 + p_5\phi_1^2 + p_6\phi_2^2 + \sum_{i=5}^{7}|\gamma_i|^2\psi_3^2 = \psi_3^2. \tag{2.78}$$

Realize that the three equations above involve convex combinations of the components of target states with additional real non-negative terms. Using $G_{12} = G_{21}^*$ (multiplied

by $e^{i\theta_2}$) one obtains

$$\begin{aligned} & p_1 e^{i\lambda_2}\phi_1\phi_2 c_{12} + p_2 e^{-i\lambda_2}\phi_1\phi_2 c_{12}^* + p_3 e^{-i(\lambda_3-\lambda_2)}\phi_2\phi_3 c_{23}^* \\ & + p_4 e^{i\lambda_3}\phi_1\phi_3 c_{13} + p_5 e^{i(\lambda_3-\lambda_2)}\phi_2\phi_3 c_{23} + p_6 e^{-i\lambda_3}\phi_1\phi_3 c_{13}^* \\ & - \sum_{i=7}^{9}\left(|\beta_i|^2\psi_2^2 + \gamma_i^*\beta_i e^{-i(\theta_3-\theta_2)}\psi_2\psi_3\right) = \psi_1\psi_2 c_{12} e^{i\theta_2}, \end{aligned} \tag{2.79}$$

similarly, from $G_{13} = G_{31}^*$ (multiplied by $e^{i\theta_3}$) one obtains

$$\begin{aligned} & p_1 e^{i\lambda_3}\phi_1\phi_3 c_{13} + p_2 e^{i(\lambda_3-\lambda_2)}\phi_2\phi_3 c_{23} + p_3 e^{-i\lambda_3}\phi_1\phi_3 c_{13}^* \\ & + p_4 e^{i\lambda_2}\phi_1\phi_2 c_{12} + p_5 e^{-i\lambda_2}\phi_1\phi_2 c_{12}^* + p_6 e^{-i(\lambda_3-\lambda_2)}\phi_2\phi_3 c_{23}^* \\ & - \sum_{i=7}^{9}\left(\beta_i^*\gamma_i e^{i(\theta_3-\theta_2)}\psi_2\psi_3 + |\gamma_i|^2\psi_3^2\right) = \psi_1\psi_3 c_{13} e^{i\theta_3}, \end{aligned} \tag{2.80}$$

From $G_{23} = G_{32}^*$ (multiplied by $e^{i(\theta_3-\theta_2)}$) the equation reads

$$\begin{aligned} & p_1 e^{i(\lambda_3-\lambda_2)}\phi_2\phi_3 c_{23} + p_2 e^{i\lambda_3}\phi_1\phi_3 c_{13} + p_3 e^{-i\lambda_2}\phi_1\phi_2 c_{12}^* \\ & + p_4 e^{-i(\lambda_3-\lambda_2)}\phi_2\phi_3 c_{23}^* + p_5 e^{-i\lambda_3}\phi_1\phi_3 c_{13}^* + p_6 e^{i\lambda_2}\phi_1\phi_2 c_{12}^* \\ & + \sum_{i=7}^{9}\beta_i^*\gamma_i e^{i(\theta_3-\theta_2)}\psi_2\psi_3 = \psi_2\psi_3 c_{23} e^{i(\theta_3-\theta_2)}. \end{aligned} \tag{2.81}$$

By combining Eq. (2.76), Eq. (2.79), Eq. (2.80) one obtains

$$p_1\tilde{\phi}_1 + p_2\tilde{\phi}_2 + p_3\tilde{\phi}_3 + p_4\tilde{\phi}_1 + p_5\tilde{\phi}_2 + p_6\tilde{\phi}_3 = \tilde{\psi}_1, \tag{2.82}$$

similarly from Eq. (2.77), complex conjugate of Eq. (2.79), Eq. (2.81) one obtains

$$p_1\tilde{\phi}_2 + p_2\tilde{\phi}_1 + p_3\tilde{\phi}_2 + p_4\tilde{\phi}_3 + p_5\tilde{\phi}_3 + p_6\tilde{\phi}_1 = \tilde{\psi}_2, \tag{2.83}$$

and from Eq. (2.78), complex conjugate of Eq. (2.80), complex conjugate of Eq. (2.81) the equation reads

$$p_1\tilde{\phi}_3 + p_2\tilde{\phi}_3 + p_3\tilde{\phi}_1 + p_4\tilde{\phi}_2 + p_5\tilde{\phi}_1 + p_6\tilde{\phi}_2 = \tilde{\psi}_3. \tag{2.84}$$

The above equations can be written in the matrix form as follows,

$$\begin{pmatrix} p_1+p_4 & p_2+p_5 & p_3+p_6 \\ p_2+p_6 & p_1+p_3 & p_4+p_5 \\ p_3+p_5 & p_4+p_6 & p_1+p_2 \end{pmatrix} \begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} \tag{2.85}$$

Here the matrix on the left hand side is a doubly stochastic matrix as long as $\sum p_i = 1$ with $0 \leq p_i \leq 1$ [42] and it can be decomposed into Birkhoff-von Neumann form that

is $D = \sum_i p_i P_i$ where P_i's are permutation matrices. Therefore; the condition C1 in the proposition reads

$$\tilde{\psi} = D\tilde{\phi} \tag{2.86}$$

for a doubly stochastic matrix D. Realize that we eliminate the non-negative terms in the diagonal entries (Eq. (2.76), Eq. (2.77) and Eq. (2.78)), therefore the equation above is valid only if

$$\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \\ \psi_3^2 \end{pmatrix} \geq D \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \\ \phi_3^2 \end{pmatrix} \tag{2.87}$$

which is the condition C2 of the proposition.

2.3 Maximal States

In this section we investigate maximal states of the RTS. We divide the problem into two parts. We first investigate two dimensional system and then arbitrary $d > 2$ dimensional systems. For two dimensional systems we solve the problem for the most general setting, however in $d > 2$ we solve the problem for an interval in the real and equal scalar product settings. We first provide a necessary condition that must be satisfied by a maximal state candidate subject to the map Φ (Eq. (2.7)). This condition provides a way of representing the candidate maximal state so that one can test whether it is a valid maximal state for a given setting. Then we show that every setting in dimension two admits a maximal state, however in dimension three while some scalar product values do not admit a maximal state, there are settings where a maximal state exists.

Before stating our findings, we change our notation to express the superposition-free operators and states to fully utilize the matrix picture so that matrix properties of the operators become vivider. From now on, the scalar product expression will be represented in terms of Gram matrix for simplicity that is, $\langle \phi | \psi \rangle = \vec{\phi}^\dagger G \vec{\psi}$ with vectors having the components of the states i.e., $\vec{\psi} = (\psi_1, \ldots, \psi_n)^\intercal$ for a state in the form $|\psi\rangle = \psi_1 |c_1\rangle + \cdots + \psi_n |c_n\rangle$. Then the normalization condition reads

$$\vec{\psi}^\dagger G \vec{\psi} = 1. \tag{2.88}$$

Moreover, consider an arbitrary superposition-free Kraus operator acting on a state product, that is

$$\begin{aligned} K_n |\psi\rangle &= \sum_{i=1}^{d} \alpha_{i,n} \left|c_{f_n(i)}\right\rangle \left\langle c_i^{\perp}\right| (\psi_1 |c_1\rangle + \psi_2 |c_2\rangle + \cdots) \\ &= \alpha_{1,n}\psi_1 \left|c_{f_n(1)}\right\rangle + \alpha_{2,n}\psi_2 \left|c_{f_n(2)}\right\rangle + \cdots \end{aligned} \tag{2.89}$$

since $\left\langle c_i^{\perp}\middle|c_j\right\rangle = \delta_{ij}$, expressions for operators acting on states can be expressed as an ordinary matrix multiplication that is,

$$K_n |\psi\rangle \to \tilde{K}_n \vec{\psi} = \begin{pmatrix} \tilde{\alpha}_{f_n(i),i} & \cdots \\ \cdots & \tilde{\alpha}_{f_n(i'),i'} \end{pmatrix} \vec{\psi} \tag{2.90}$$

where $\tilde{\alpha}_{f_n(i),i}$ is entry of the matrix $\tilde{K}_n$ for corresponding indexes. We note that $\tilde{K}_n$ is a matrix in the form of an incoherent Kraus operator of the RTC [18]. Then, to be a trace preserving equation one can state that

$$\sum_{n=1}^{d!+d} \tilde{K}_n^{\dagger} G \tilde{K}_n = G \tag{2.91}$$

since $\sum_n \langle\psi| K_n^{\dagger} K_n |\psi\rangle = \sum_n \vec{\psi}^{\dagger} \tilde{K}_n^{\dagger} G \tilde{K}_n \vec{\psi} = \vec{\psi}^{\dagger} G \vec{\psi} = 1$. The properties of the operators will be found in this picture, however we will express the final form of the maximal states in the Dirac's bra-ket notation.

As mentioned previously, one cannot introduce majorization theory by using Eq. (2.10) since the vectors subject to the doubly stochastic matrix are not necessarily real in contrast to the RTS. In the RTC, one can eliminate phase terms of the coefficients and make them real by using incoherent operations [43]. However, it turns out that if one sets all entries of the doubly stochastic matrix to $1/d!$, one can obtain maximal states since such a matrix maps all vectors (given that vector satisfies $\sum x_i = 1$) to $(1/d, \ldots, 1/d)^{\mathsf{T}}$. The additional condition, namely CoC, also adds additional restriction which we propose as a necessary condition. Consider the following to capture the details. For a qubit system with $\vec{\psi} = (\psi_1, e^{i\theta_2}\psi_2)^{\mathsf{T}}$ being an initial state and $\vec{\phi} = (\phi_1, e^{i\lambda_2}\phi_2)^{\mathsf{T}}$ being a target state, conditions for a deterministic transformation, namely C1 and C2, can be written as (Eq. (2.10), (2.11))

$$\begin{pmatrix} \psi_1^2 + \psi_1\psi_2 c_{12} e^{i\theta_2} \\ \psi_2^2 + \psi_1\psi_2 c_{12}^* e^{-i\theta_2} \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix} \begin{pmatrix} \phi_1^2 + \phi_1\phi_2 c_{12} e^{i\lambda_2} \\ \phi_2^2 + \phi_1\phi_2 c_{12}^* e^{-i\lambda_2} \end{pmatrix}, \tag{2.92}$$

and

$$\begin{pmatrix} \psi_1^2 \\ \psi_2^2 \end{pmatrix} \geq \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix} \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix} \tag{2.93}$$

respectively. Realize that if one sums the terms of Eq. (2.93), then the expression becomes,

$$\psi_1^2 + \psi_2^2 \geq \phi_1^2 + \phi_2^2. \tag{2.94}$$

(Since $p_1 + p_2 = 1$ for a deterministic transformation). It is clear that, a condition that does not satisfy this relation cannot satisfy each component of the inequality simultaneously. For a maximal state this relation must always hold, therefore; using the Rayleigh quotient theorem (Eq. (1.20)) one concludes that a maximal state has to be an eigenvector of the Gram matrix corresponding to the minimum eigenvalue. Even though this is a necessary condition, i.e., a state that satisfies this conditions does not have to be a maximal state, it enables us to shrink the set of maximal state candidates by combining it with the following picture. Setting the probabilities to $1/2!$, Eq. (2.93) reads,

$$\begin{pmatrix} \psi_1^2 + \psi_1\psi_2 c_{12} e^{i\theta_2} \\ \psi_2^2 + \psi_1\psi_2 c_{12}^* e^{-i\theta_2} \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \tag{2.95}$$

so the resultant expression does not depend on the components of the target vector. Now assume that $G\vec{\psi} = \lambda_{\min}\vec{\psi}$ and $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2)^\intercal = (1/2, 1/2)^\intercal$. It follows that

$$\mathrm{diag}(\psi_1, e^{-i\theta_2}\psi_2) G\vec{\psi} = \tilde{\psi} = \lambda_{\min}(|\psi_1|^2, |\psi_2|^2)^\intercal \tag{2.96}$$

therefore; $\tilde{\psi}_i = \lambda_{\min}|\psi_i|^2$ which dictates $|\psi_1| = |\psi_2|$. Therefore, a maximal state is necessarily in the following form

$$|\psi\rangle = N(e^{i\omega_1}|c_1\rangle + e^{i\omega_2}|c_2\rangle) \tag{2.97}$$

where N is the normalization constant which we comment in the next paragraph. Motivated by the above example, we modify the probability terms of the map defined in Eq. (2.7). That is, all probability terms are set to $1/(d!)$, therefore; one has $\tilde{K}_n\vec{\psi} = \sqrt{1/(d!)}\vec{\phi}$ for all $1 \leq n \leq d!$ (the operators in the first set). Having described the picture, we state our proposition for d level systems as follows.

Proposition 2: *Let G be a Gram matrix that represents the scalar product settings of the given RTS. A maximal state is necessarily*

1. *an eigenvector corresponding to the minimum eigenvalue of the Gram matrix,*

2. *it must have the following form*

$$|\Psi\rangle = \frac{1}{\sqrt{d\lambda_{\min}}} \sum_{j=1}^{d} e^{i\theta_j} |c_j\rangle, \tag{2.98}$$

where $|c_i\rangle$'s forms a linearly independent, not necessarily orthogonal normalized basis and where $\theta_j \in [0, 2\pi)$. Here, the $\lambda_{\min}$ joins to the normalization constant since $\vec{\Psi}^\dagger G \vec{\Psi} = \lambda_{\min} \vec{\Psi}^\dagger \cdot \vec{\Psi}$, it follows that the normalization constant $N = \left(\vec{\Psi}^\dagger \cdot \vec{\Psi} \lambda_{\min}\right)^{-1/2}$ with $|\Psi_i| = \left|\Psi_j\right|$ for all i, j, therefore $N = 1/\sqrt{d\lambda_{\min}}$. With these information in hand, we seek maximal states by investigating first whether a Gram matrix for a given setting has a eigenvector corresponding to the minimum eigenvalue in the form given above. If such a vector exists, we then investigate whether the first set of Kraus operators can be completed to a trace preserving map by the second set of operators. Due to this fact, we call any state satisfying one of the conditions given in the Proposition 3 a candidate maximal state. Here we first state our finding for two dimensional systems, then continue to d dimensional systems.

Proposition 3: *Let G be a Gram matrix that represents the scalar product settings of a two dimensional RTS, then an eigenvector $|\Psi\rangle$ corresponding to the minimum eigenvalue $\lambda_{\min}$ is a maximal state.*

First, consider the known examples in two dimension given in Refs [18, 42]. For a real scalar product $\langle c_1|c_2\rangle = \mu$ in dimension two, eigenvalues of the Gram matrix are found to be $\lambda_1 = 1 - \mu$ and $\lambda_2 = 1 + \mu$ and the corresponding normalized eigenvectors are

$$|\Psi_1\rangle = \frac{1}{\sqrt{2(1-\mu)}}(|c_1\rangle - |c_2\rangle). \tag{2.99}$$

and

$$|\Psi_2\rangle = \frac{1}{\sqrt{2(1+\mu)}}(|c_1\rangle + |c_2\rangle). \tag{2.100}$$

For $0 \leq \mu < 1$, $\lambda_{\min} = \lambda_1$ hence the maximal state for this interval is $|\Psi_1\rangle$ [18]. Similarly, for $-1 \leq \mu \leq 0$ $\lambda_{\min} = \lambda_2$, hence the maximal state is $|\Psi_2\rangle$ [42]. Finally, consider the general case for dimension two that is, $\langle c_1|c_2\rangle = e^{i\omega}\mu$ where $\mu \in (-1, 1)$ and $\omega \in [0, \pi)$. The maximal state for the general case is found to be

$$|\Psi\rangle = \begin{cases} \frac{1}{\sqrt{2(1-\mu)}} \left(|c_1\rangle - e^{-i\omega}|c_2\rangle\right) & , \quad 0 \leq \mu < 1 \\ \frac{1}{\sqrt{2(1+\mu)}} \left(|c_1\rangle + e^{-i\omega}|c_2\rangle\right) & , \quad -1 < \mu \leq 0 \end{cases} \tag{2.101}$$

Notice that phase of the scalar product of basis states are compensated by the relative phase of the maximal state so that $\tilde{\psi} = (1/2, 1/2)^\intercal$. As one can see that in the

orthogonal limit maximal states given in Eq. (2.99), Eq. (2.100) and the equation above reduces to different form of maximal state of coherence whose general form is given by [14],

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j}^{d} e^{i\tau_j} |j\rangle \tag{2.102}$$

(where $|j\rangle$'s forms an orthonormal basis and $\tau_j \in \mathbb{R}$) in the orthogonal limit i.e, $G \to I$, hence $\lambda_{\min} \to 1$. It is important to note that in the RTC, one has freedom to choose τ_i arbitrarily since the states are equivalent to each other by means of incoherent unitary operations, however, relative phases of the maximal states of resource theory of superposition are fixed because of the fact that operations acting on relative phases are not free operations [18]. As a consequence, the maximal state of the RTS for different settings reduces to different elements of the set of maximal states of coherence as $\mu \to 0$. It is due to the fact that the Gram matrix dictates the relative phases of the eigenvectors (satisfying the Proposition 3). For instance, an eigenvector having relative phase $\theta_2 = 0$ is allowed only in the case of negative values of scalar product, hence only in this setting a maximal states reduces to the canonical form of the maximal state of the coherence that is $\tau_2 = 0$. Since Eq. (2.101) is in the most general case, we conclude the qubit case by stating that there exists a maximal state for all scalar product settings, however in higher dimensions this is not the case. For instance, for $d = 3$ one cannot have a maximal state for positive and equal scalar product values by means of the map Φ (Eq. (2.7)).

Before continuing to higher dimensional systems, we first analyze the Gram matrix for a real and equal product setting to parameterize the eigenvectors and the corresponding eigenvalues. The Gram matrix for such a setting then can be written as

$$G = (1-\mu)\mathbb{1} + \mu\vec{\Psi}\vec{\Psi}^{\intercal}, \tag{2.103}$$

where $\mathbb{1}$ is a $d \times d$ identify matrix and $\vec{\Psi}$ is a column vector whose elements are all one. Then it is clear that $\vec{\Psi}$ is an eigenvector corresponding to the eigenvalue $1 + \mu(d-1)$ with multiplicity one and any vector orthogonal to $\vec{\Psi}$ is an eigenvector corresponding to the eigenvalue $1-\mu$ with multiplicity $d-1$. Moreover, to be a valid linearly independent basis the determinant of the Gram matrix must be positive. Using the eigenvalues of the Gram matrix, one obtains the following interval for the scalar

product

$$-\frac{1}{d-1} < \mu < 1, \tag{2.104}$$

to have $\det(G) > 0$. Then we have $\lambda_{\min} = 1 + \mu(d-1)$ for the interval $-1/(d-1) < \mu \leq 0$. Having identified the eigenvectors and the eigenvalues, we state the following proposition.

Proposition 4: *For a $d > 2$ system with real and equal scalar values settings, the state given by*

$$|\Psi\rangle = \frac{1}{\sqrt{d\,(1+(d-1)\mu)}} \sum_{i}^{d} |c_i\rangle \tag{2.105}$$

is a maximal state for the interval $1/(1-d) < \mu \leq 0$. Realize that the state is normalized version of the vector $\vec{\Psi}$ in the Eq. (2.103), it corresponds to the minimum eigenvalue, therefore; it automatically satisfies the second proposition . It is unclear and thus is an open problem whether the second proposition is also a sufficient condition due to the fact that maximal states we found do not seem to have any additional property other than the conditions given by the second proposition. For instance, in dimension three with a setting given as $\langle c_1|c_2\rangle = \langle c_1|c_3\rangle = \langle c_2|c_3\rangle = \mu = -1/4$

$$|\Psi\rangle = \sqrt{\frac{2}{3}}\,(|c_1\rangle + |c_2\rangle + |c_3\rangle) \tag{2.106}$$

is a maximal state. The state corresponds to the minimum eigenvalue $\lambda_{\min} = 1/2$. However, for the case $\mu = 1/2$, the eigensystem of the Gram matrix is given as

$$|\Psi_1\rangle = \sqrt{\frac{1}{6}}\,(|c_1\rangle + |c_2\rangle + |c_3\rangle)\,, \text{for } \lambda_1 = 2 \tag{2.107}$$

$$|\Psi_2\rangle = |c_1\rangle - |c_3\rangle\,, \text{for } \lambda_2 = \frac{1}{2} \tag{2.108}$$

$$|\Psi_3\rangle = |c_1\rangle - |c_2\rangle\,, \text{for } \lambda_3 = \frac{1}{2} \tag{2.109}$$

Here, the state $|\Psi_1\rangle$ is in the form of Eq. (2.98), however it does not corresponds to the minimum eigenvalue and in Ref. [18] it is proven that there is no maximal state for this setting. Moreover, since $\lambda_{\min} = \lambda_2 = \lambda_3 = 1/2$ for degenerate eigenvalues, any vector in the form

$$a_1\,|\Psi_2\rangle + a_2\,|\Psi_3\rangle \tag{2.110}$$

is an eigenvector corresponding to the minimum eigenvalue. Such a state cannot be written in the form of Eq. (2.98). That is,

$$|\Psi\rangle = N\,[(a_1+a_2)\,|c_1\rangle - a_2\,|c_2\rangle - a_3\,|c_3\rangle] \tag{2.111}$$

for $a_1, a_2 \in \mathbb{C}$ and where $N = \left(|a_1|^2 + |a_2|^2 + a_1 a_2^*/2 + a_1^* a_2/2\right)^{-1/2}$. One can check that there are no a_1 and a_2 that make $|\Psi\rangle$ in the form of Eq. (2.98), hence the state violates the second proposition.

Proof of Propositions

We stress that the proof has three folds which corresponds to the three propositions respectively. First we show that it is necessary for a maximal state to be an eigenvector of the Gram matrix corresponding to the minimum eigenvalue and to be in the required form given by Eq. (2.98) by means of the map defined in Eq. (2.7). Then, we give the proof of both the Proposition 3 and the Proposition 4 by setting the dimension and Gram matrix appropriately when it is it required.

We first define the initial state and the target state for dimension d,

$$\vec{\psi} = \left(e^{i\theta_1}\psi_1, \ldots, e^{i\theta_d}\psi_d\right)^{\mathsf{T}}, \tag{2.112}$$

$$\vec{\phi} = \left(e^{i\lambda_1}\phi_1, \ldots, e^{i\lambda_d}\phi_d\right)^{\mathsf{T}}, \tag{2.113}$$

where $\theta_j, \lambda_j \in [0, 2\pi]$ and $\psi_j, \phi_j \in \mathbb{R}^{\geq 0}$. To transform the initial state to the target state that is $\tilde{K}_n \vec{\psi} = \sqrt{1/d!}\vec{\phi}$, the first set of Kraus operators has to be in the following form, (defining $\varepsilon_{ij} \coloneqq \lambda_i - \theta_j$)

$$(\tilde{K}_n)_{ij} = \sqrt{\frac{1}{d!}} e^{i(\varepsilon_{ij,n})} \frac{\phi_{i,n}}{\psi_{j,n}}. \tag{2.114}$$

First consider the properties of the first set of Kraus operators. They have a matrix rank which equals to the superposition rank of the target state (number of non-zero elements of $\vec{\phi}$). The operators have at most one non-zero entry in each column and row, that is

$$\tilde{K}_n = \begin{pmatrix} \cdots & 0 & 0 & \cdots \\ \cdots & e^{(\lambda_i - \theta_j)} \frac{\phi_i}{\psi_j} & 0 & \cdots \\ \cdots & 0 & e^{(\lambda_i' - \theta_j')} \frac{\phi_i'}{\psi_j'} & \cdots \\ \cdots & 0 & 0 & \cdots \end{pmatrix}. \tag{2.115}$$

It is clear that whether the entry is zero or not depends on the index n. By summing over the index n, one obtains

$$\sum_{n=1}^{d!} (\tilde{K}_n)_{ij} = \frac{d!}{d} \sqrt{\frac{1}{d!}} e^{i(\varepsilon_{ij})} \frac{\phi_i}{\psi_j} \tag{2.116}$$

since the given set of Kraus operators are permutation matrices whose entries are replaced with the corresponding transformation coefficient. Realize that this relation

does not depend on whether the target state has non-zero entries or not. If $\phi_i = 0$, then the corresponding term in the sum is also zero. Also by using the property $(\tilde{K}_n^\dagger)_{ij} = (\tilde{K}_n^*)_{ji}$, here we give the proof by invoking a counting problem. One can construct the matrix entries as follows

$$\left(\sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n\right)_{il} = \sum_{n=1}^{d!} \left(\sum_{j,k}^{d} \frac{1}{d!} e^{i(\varepsilon_{kl,n} - \varepsilon_{ji,n})} G_{jk} \frac{\phi_{j,n}}{\psi_{i,n}} \frac{\phi_{k,n}}{\psi_{l,n}}\right). \tag{2.117}$$

Here we can split the inner sum term on the right hand side to invoke counting problem that is

$$\begin{aligned}\left(\sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n\right)_{il} = \sum_{n=1}^{d!} \Bigg(&\sum_{j=1}^{d} \frac{1}{d!} e^{i(\varepsilon_{n,jl} - \varepsilon_{ji,n})} G_{jj} \frac{\phi_{j,n}}{\psi_{i,n}} \frac{\phi_{j,n}}{\psi_{l,n}} \\ &+ \sum_{\substack{j,k \\ j\neq k}}^{d} \frac{1}{d!} e^{i(\varepsilon_{kl,n} - \varepsilon_{ji,n})} G_{jk} \frac{\phi_{j,n}}{\psi_{i,n}} \frac{\phi_{k,n}}{\psi_{l,n}}\Bigg).\end{aligned} \tag{2.118}$$

Recall that a Kraus operator in the form of an incoherent operator is dictated to have only one non-zero element in each column [14]. Moreover the first set of Kraus operators have also only one non-zero element in each row (Eq. (2.114)), therefore; for $i = l$, the sum over the index set $j \neq k$ is zero for all n, and for $i \neq l$ the sum over the index set $j = k$ is zero for all n. Moreover, for $i = l$ there are $d!/d$ ϕ_j/ψ_i's for each j, therefore; summing over n one obtains

$$\left(\sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n\right)_{ii} = \sum_{j=1}^{d} \frac{1}{d!} \frac{d!}{d} \frac{\phi_j}{\psi_i} \frac{\phi_j}{\psi_i} = \frac{1}{d} \frac{1}{\psi_i^2} \sum_{j=1}^{d} \phi_j^2 \tag{2.119}$$

for each $1 \leq i \leq d$. For the off-diagonal elements, since $i \neq l$ and $j \neq k$, there exist a Kraus operator with the entries ϕ_j/ψ_i and ϕ_k/ψ_l for an arbitrary index n'. Starting from this, among $d!$ number of permutations of n'-th Kraus operator, only $(d-2)!$ does not change the indexes, hence summing over n, one obtains

$$\left(\sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n\right)_{i\neq l} = \frac{(d-2)!}{d!} \frac{e^{i(\theta_i - \theta_l)}}{\psi_i \psi_l} \sum_{\substack{j,k \\ j\neq k}}^{d} e^{i(\lambda_k - \lambda_j)} G_{jk} \phi_j \phi_k. \tag{2.120}$$

By combining the diagonal and off-diagonal elements, one can obtain the matrix entries in the index form

$$\begin{aligned}\left(\sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n\right)_{il} = &\frac{1}{d} \frac{1}{\psi_i^2} \sum_{j=1}^{d} \phi_j^2 \delta_{il} \\ &+ \frac{(d-2)!}{d!} \frac{e^{i(\theta_i - \theta_l)}}{\psi_i \psi_l} \sum_{\substack{j,k \\ j\neq k}}^{d} e^{i(\lambda_k - \lambda_j)} G_{jk} \phi_j \phi_k (1 - \delta_{il}).\end{aligned} \tag{2.121}$$

Now assume that Eq. (2.121) can be completed to a trace preserving map by using the second set of Kraus operators. Recall that a rank(1) superposition-free operator is an operator with all rows zero except the nth row, that is (Here we change the symbol for the second Kraus operators with $\tilde{F}_n$ to be able to start indices from 1.)

$$\left(\tilde{F}_n\right)_{ij} = \alpha_{ij}\delta_{ni} \tag{2.122}$$

and hence the total sum over all operators gives

$$\left(\sum_{n=1}^{d} \tilde{F}_n^\dagger G \tilde{F}_n\right)_{il} = \sum_{n=1}^{d}\left(\sum_{j,k}^{d} \alpha_{ji}^* \delta_{ni} G_{jk} \alpha_{kl} \delta_{nk}\right) = \sum_{n=1}^{d} \alpha_{ni}^* \alpha_{nl}. \tag{2.123}$$

By combining with the first set of operators, one obtains

$$\begin{aligned}\left(\sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n + \sum_{n=1}^{d} \tilde{F}_n^\dagger G \tilde{F}_n\right)_{il} &= \frac{1}{d}\frac{1}{\psi_i^2}\sum_{j=1}^{d}\phi_j^2\delta_{il} \\ &+ \frac{(d-2)!}{d!}\frac{e^{i(\theta_i-\theta_l)}}{\psi_i\psi_l}\sum_{\substack{j,k \\ j\neq k}}^{d} e^{i(\lambda_k-\lambda_j)} G_{jk}\phi_j\phi_k(1-\delta_{il}) \\ &+ \sum_{n=1}^{d}\alpha_{ni}^*\alpha_{nl} = G_{il}.\end{aligned} \tag{2.124}$$

By manipulating the denominator terms, it follows that

$$\begin{aligned}\frac{1}{d}\sum_{j=1}^{d}\phi_j^2\delta_{il} + \frac{(d-2)!}{d!}\sum_{\substack{j,k \\ j\neq k}}^{d} e^{i(\lambda_k-\lambda_j)} G_{jk}\phi_j\phi_k(1-\delta_{il}) \\ + \sum_{n=1}^{d}\alpha_{ni}^*\alpha_{nl}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l = G_{il}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l.\end{aligned} \tag{2.125}$$

By summing over the index l, the equation reads

$$\begin{aligned}\frac{1}{d}\sum_{j=1}^{d}\sum_{l=1}^{d}\phi_j^2\delta_{il} + \frac{(d-2)!}{d!}\sum_{l=1}^{d}\sum_{\substack{j,k \\ j\neq k}}^{d} e^{i(\lambda_k-\lambda_j)} G_{jk}\phi_j\phi_k(1-\delta_{il}) \\ + \sum_{n=1}^{d}\sum_{l=1}^{d}\alpha_{ni}^*\alpha_{nl}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l = \sum_{l=1}^{d} G_{il}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l\end{aligned} \tag{2.126}$$

realize that $\sum_{l=1}^{d}\alpha_{nl}e^{i\theta_l}\psi_l = 0$ since $\tilde{F}_n\vec{\psi} = 0$, hence we obtain

$$\frac{1}{d}\sum_{j=1}^{d}\phi_j^2 + (d-1)\frac{(d-2)!}{d!}\sum_{\substack{j,k \\ j\neq k}}^{d} e^{i(\lambda_k-\lambda_j)} G_{jk}\phi_j\phi_k = \sum_{l=1}^{d} G_{il}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l. \tag{2.127}$$

With further algebraic manipulations, it can be simplified into the following equation

$$\frac{1}{d}\sum_{j=1}^{d}\phi_j^2+\frac{1}{d}\sum_{\substack{j,k\\ j\neq k}}^{d}e^{i(\lambda_k-\lambda_j)}G_{jk}\phi_j\phi_k=\sum_{l=1}^{d}G_{il}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l. \tag{2.128}$$

Finally one can obtain the following equation

$$\frac{1}{d}\left(\sum_{j=1}^{d}\phi_j^2+\sum_{\substack{j,k\\ j\neq k}}^{d}e^{i(\lambda_k-\lambda_j)}G_{jk}\phi_j\phi_k\right)=\frac{1}{d}\vec{\phi}^{\dagger}G\vec{\phi}=\sum_{l=1}^{d}G_{il}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l. \tag{2.129}$$

Then one is required to satisfy the following equation

$$\frac{1}{d}=\sum_{l=1}^{d}G_{il}e^{-i(\theta_i-\theta_l)}\psi_i\psi_l=\tilde{\psi}_i. \tag{2.130}$$

Here, one can introduce the Rayleigh quotient (min-max theorem) with additional manipulations to show that the maximal state is an eigenvector of the Gram matrix corresponding to the minimum eigenvalue. Consider the diagonal elements of the matrix given by the Eq. (2.124)

$$\frac{1}{d}\sum_{j=1}^{d}\phi_j^2+\sum_{j=1}^{d}\left|\alpha_{ji}\right|^2\psi_i^2=\psi_i^2 \tag{2.131}$$

for each $1\leq i\leq d$. Realize that one obtains the following inequality from the sum of the equations over the index i,

$$\sum_{i=1}^{d}\psi_i^2\geq\sum_{i=1}^{d}\phi_i^2 \tag{2.132}$$

due to the non-negative terms. It is obvious that, a condition that does not satisfy this relation cannot satisfy each component of the inequality simultaneously. For a maximal state this relation must always hold, therefore; using the Rayleigh quotient one concludes that a maximal state has to be an eigenvector of the Gram matrix corresponding to the minimum eigenvalue. To obtain the form given by Eq. (2.98), assume that $G\vec{\psi}=\lambda_{\min}\vec{\psi}$ and $\tilde{\psi}=(\tilde{\psi}_1,\tilde{\psi}_2,\ldots,\tilde{\psi}_n)^{\intercal}=(1/d,1/d,\ldots 1/d)^{\intercal}$ (due to Eq. (2.130) and Eq. (2.132)). It follows that

$$\mathrm{diag}(\psi_1^*,\ldots,\psi_n^*)G\vec{\psi}=\tilde{\psi}=\lambda_{\min}(|\psi_1|^2,|\psi_2|^2,\ldots,|\psi_n|^2)^{\intercal}, \tag{2.133}$$

therefore; $\tilde{\psi}_i=\lambda_{\min}|\psi_i|^2$ which dictates $|\psi_i|=\left|\psi_j\right|$ for all i,j which completes the proof of the second proposition.

Now to prove the Proposition 3 and the Proposition 4, we show that the given first set of Kraus operators can be made a trace preserving operation by adding the second set of Kraus operator. To accomplish that one requires that the following matrix must be positive semidefinite

$$G - \sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n \geq 0 \tag{2.134}$$

so that one can decompose into positive semidefinite operators $\sum_{n=1}^{d} \tilde{F}_n^\dagger G \tilde{F}_n$. Since $\tilde{F}_n \vec{\psi} = 0$ for all n one also requires

$$\left(G - \sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n \right) \vec{\psi} = 0. \tag{2.135}$$

First we show that the latter is satisfied, and then show that the given matrix is positive semidefinite. To shorten the expressions, we define $\phi^2 := \sum_{j=1}^{d} \phi_j^2$ and

$$X := \left(G - \sum_{n=1}^{d!} \tilde{K}_n^\dagger G \tilde{K}_n \right). \tag{2.136}$$

Now assume that $\vec{\psi}$ is an eigenvector of the Gram matrix corresponding to the $\lambda_{\min}$, then one has

$$X\vec{\psi} = \sum_{l=1}^{d} G_{il} e^{i\theta} \psi_l - \frac{1}{d} \frac{\phi^2}{\psi_i^2} \sum_{l=1}^{d} - \frac{(d-2)!}{d!} \left(\sum_{l=1}^{d} e^{i(\theta_i - \theta_l)} e^{\theta_i} \psi_l (1 - \delta_{il}) \right) \tag{2.137}$$

$$X\vec{\psi} = \lambda_{\min} e^{\theta_i} \psi_i - \frac{1}{d} \frac{\phi^2}{\psi_i^2} e^{i\theta_i} \psi_i - \frac{(d-2)!}{d!} (1 - \phi^2) \left(\frac{e^{i\theta_i}}{\psi_i} - \frac{e^{i\theta_i}}{\psi_i} \right) \tag{2.138}$$

$$X\vec{\psi} = \lambda_{\min} e^{\theta_i} \psi_i - \lambda_{\min} \phi^2 e^{i\theta_i} \psi_i - \frac{(d-2)!}{d!} (1 - \phi^2) (d-1) \frac{e^{i\theta_i}}{\psi_i} \tag{2.139}$$

$$X\vec{\psi} = \lambda_{\min} e^{\theta_i} \psi_i - \lambda_{\min} \phi^2 e^{i\theta_i} \psi_i - \lambda_{\min} (1 - \phi^2) (d-1) e^{i\theta_i} \psi_i \tag{2.140}$$

$$X\vec{\psi} = \lambda_{\min} e^{\theta_i} \psi_i \left(1 - \phi^2 - (1 - \phi^2) \right) = 0 \tag{2.141}$$

hence $\vec{\psi}$ is also an eigenvector of X with the eigenvalue zero. To show that X is positive definite, first by using Eq. (1.20) and Eq. (2.133) identify that

$$\lambda_{\min} = \frac{1}{d\psi_i^2}, \tag{2.142}$$

and by using Eq. (2.121), X then has the following diagonal entries,

$$X_{ii} = 1 - \lambda_{\min} \phi^2 \tag{2.143}$$

and it has the following off-diagonal entries

$$X_{il} = G_{il} - \frac{\lambda_{\min}}{d-1}\left(1-\phi^2\right)e^{i(\theta_i-\theta_l)} \quad \text{for } i \neq l. \tag{2.144}$$

Notice that Eq. (2.143) is non-negative (see Eq. (1.23)). Here, we first calculate the entries of the matrix X and the minimum eigenvalue of the Gram matrix with the corresponding eigenvector for $d = 2$ in the most general setting and then for $d > 2$ level for the interval given in the Proposition 4.

For $d = 2$ there are two cases for $\lambda_{\min}$ which are $\lambda_{\min} = 1-\mu$ and $\lambda_{\min} = 1+\mu$ depending on the interval of μ. For $\lambda_{\min} = 1-\mu$ case, Eq. (2.143) and Eq. (2.144) reads,

$$X_{11} = X_{22} = 1-(1-\mu)\phi^2 \tag{2.145}$$

and

$$X_{12} = X_{21}^* = \mu-(1-\mu)(1-\phi^2)e^{-i(-\omega+\pi)} \tag{2.146}$$

respectively. It is trivial to show that $X_{11} = X_{22} = |X_{12}| = |X_{21}|$, therefore; X is a diagonally dominant matrix (see section 1.1.5). Since X is also a Hermitian matrix with positive diagonal entries one concludes that it is a positive semidefinite matrix. For $\lambda_{\min} = 1+\mu$ case, it is trivial to show that the resultant matrix is also a positive semidefinite by repeating the steps above identically.

For $d > 2$ level, the Gram matrix then can be written as

$$G = (1-\mu)\mathbb{1} + \mu\vec{\Psi}\vec{\Psi}^{\intercal} \tag{2.147}$$

where $\mathbb{1}$ is an $d \times d$ identify matrix and $\vec{\Psi}$ is a column vector whose elements are all one. Then it is clear that $\vec{\Psi}$ is an eigenvector corresponding to the eigenvalue $1+\mu(d-1)$ with multiplicity one and any vector orthogonal to $\vec{\Psi}$ is an eigenvector corresponding to the eigenvalue $1-\mu$ with multiplicity $d-1$. Then we have that $\lambda_{\min} = 1+\mu(d-1)$ for the interval $-1/(d-1) < \mu \leq 0$. By combining it with Eq. (2.143) and Eq. (2.144), we obtain

$$X_{ii} = 1-(1+\mu(d-1))\phi^2 \tag{2.148}$$

$$X_{il} = \mu - \frac{1+\mu(d-1)}{d-1}\left(1-\phi^2\right) = \frac{-1+(1+\mu(d-1))\phi^2}{d-1} \tag{2.149}$$

Realize that, $|X_{ii}| = \sum_l |X_{il}|$ for all $l \neq i$ and $1 \leq i \leq d$. Hence using the dominant diagonal property of the matrix one can conclude that it is positive semidefinite.

Therefore, X has a spectral decomposition that is

$$X = \sum_{i=1}^{d-1} \lambda_i \vec{u}_i \vec{u}_i^{\dagger}, \tag{2.150}$$

where u_i is an eigenvector of X. Let $\vec{e}_n$ be a vector whose only nth element is non-zero. In that case $\tilde{F}_n = \vec{e}_n \vec{m}^{\dagger}$ is a valid superposition-free rank(1) Kraus operator for an arbitrary $\vec{m}$. Let $\vec{m}$ be a vector in the following form $\vec{m} = \sum_{i=1}^{d-1} a_{ni} \vec{u}_i$ where $a_i \in \mathbb{C}$. Therefore, the rank(1) superposition-free Kraus operators satisfy $\tilde{F}_n \vec{\psi} = 0$ for all n. Moreover, it follows that

$$\sum_{n=1}^{d} \tilde{F}_n^{\dagger} G \tilde{F}_n = \sum_{n=1}^{d} \sum_{i=1}^{d-1} \sum_{i=j}^{d-1} a_{ni} \vec{u}_i \left(\vec{e}_n^{\dagger} G \vec{e}_n \right) a_{nj}^* \vec{u}_j^{\dagger} \tag{2.151}$$

$$\sum_{n=1}^{d} \tilde{F}_n^{\dagger} G \tilde{F}_n = \sum_{n=1}^{d} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} a_{ni} a_{nj}^* \vec{u}_i \vec{u}_j^{\dagger}. \tag{2.152}$$

To have

$$\sum_{n=1}^{d} \tilde{F}_n^{\dagger} G \tilde{F}_n = X \tag{2.153}$$

it is required that $\sum_n a_{ni} a_{nj}^* = 0$ for all $i \neq j$. Since $d > d-1$, take $a_{ij} = 0$ for all $i > d-1$ and take $a_{ij} = 0$ for all remaining $i \neq j$ hence,

$$\sum_{i=1}^{d-1} |a_{ii}|^2 \vec{u}_i \vec{u}_i^{\dagger} = \sum_{i=1}^{d-1} \lambda_i \vec{u}_i \vec{u}_i^{\dagger} \tag{2.154}$$

therefore, $|a_{ii}| = \sqrt{\lambda_i}$ since $\lambda_i > 0$ which completes the proof of the Proposition 3 and 4.

3. CONCLUSION

In this thesis, we investigated the deterministic transformations of the resource theory of superposition along with the maximal states. We fulfilled the gap between the resource theory coherence and the resource theory of superposition in the context of deterministic transformations and maximal states by establishing a continuous relation by means of a Gram matrix. The Gram matrix picture provides an easy way to represent different scalar product settings of a resource theory of superposition. We also showed that in the orthonormal limit the Gram matrix goes to the identity matrix, therefore; one recovers the resource theory of coherence. In this context, our work provides further development in the way of generalizing the resource theory of coherence. It is also shown that the maximal states of the resource theory of superposition can be represented in the Gram matrix picture, i.e. a state cannot be a maximal state if it is not an eigenvector corresponding to the minimum eigenvalue of the Gram matrix that represent the scalar product setting. We found the maximal states of two level systems for arbitrary scalar product setting. Finally we provided maximal states for higher dimensional system that are valid in certain settings.

Our future aim is to prove that the first proposition is also valid for arbitrary dimensional systems. It is also an open question whether this proposition is a necessary and sufficient condition. If one accomplishes to show that the first proposition is true for arbitrary dimension, then it becomes possible to derive the remaining propositions as immediate corollaries. In this case it may become possible to generalize the resource theory of coherence in a broader way. Another open question is the existence of maximal states for settings that are complementary to the settings given in the fourth proposition. We conjecture that if a Gram matrix that represents the given scalar product setting of an RTS admits an eigenvector corresponding to the minimum eigenvalue in the following form $|\psi\rangle = \sqrt{1/d\lambda_{\min}} \sum_{i=1}^{d} e^{i\omega_i} |c_i\rangle$, then it is a maximal state.

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