On Extended Selmer Groups for Coleman Deformations

by

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A Dissertation Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Mathematics

KOÇ
UNIVERSITY

August 14, 2017

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ABSTRACT

In 2017, Bharathwaj Palvannan proved a factorization formula involving Selmer groups, predicted by the main conjectures of Iwasawa theory for several Galois representations, corresponding to a factorization formula involving p-adic L-functions proved by Samit Dasgupta in 2014.

Bharathwaj Palvannan started with the Galois representation associated to a Hida family that satisfies a highly restrictive ordinariness hypothesis, namely Panchiskin condition.

The purpose of this thesis is to discuss a potential non-ordinary generalization of Bharathwaj Palvannan's result following Jonathan Pottharst who gave a vast generalization of Panchiskin condition to the non-ordinary setting. Our ultimate goal is to form the extended Selmer groups emerging from the non-ordinary Galois representation attached to a Coleman family.

ÖZETÇE

Bharathwaj Palvannan 2017 yılında Selmer grupları içeren bir ayrıştırma formülü ispatladı. Bu formül, bazı Galois temsilleri için Iwasawa kuramının ana sanıları gereğince, 2014 yılında Samit Dasgupta'nın ispatladığı p-sel L-fonksiyonları içeren bir ayrıştırma formülünün karşılığıdır.

Bharathwaj Palvannan bir Hida ailesinden gelen ve oldukça kısıtlayıcı bir sıradanlık özelliği, Panchiskin koşulu, sağlayan bir Galois temsili ile başladı.

Bu tezin amacı, Panchiskin koşulunu geniş çapta genelleştiren Jonathan Pottharst'ı takip ederek, Bharathwaj Palvannan'ın sonucunun olası bir genelleştirmesini tartışmaktır. Nihai amacımız ise bir Coleman ailesinden gelen ve sıradan olmayan Galois temsilinden ortaya çıkan genişletilmiş Selmer gruplarını oluşturmaktır.

ACKNOWLEDGMENTS

First, I would like to thank my advisor Kâzım Büyükboduk for introducing me to this wonderful area of mathematics.

I am extremely grateful to Ayhan Günaydın, my senior theses supervisor, for his willingness to help whenever needed. The projects that I completed under his guidance has shaped my approach towards mathematics.

I express my deep gratitude to Şafak Özden. As an undergraduate student, I received valuable assistance from him. His investment in my learning has cost him countless hours.

For many helpful and inspiring conversations throughout the years, I thank Emre Alkan, Feza Arslan, Alp Bassa, Barış Coşkunüzer, Ayhan Dil, Uğur Efem, Haydar Göral, Berke Karagöz, Haydar Can Kaya, Özgür Kişisel, Ekin Özman, Özer Öztürk, Cihan Pehlivan, Doğacan Sertbaş, Haluk Şengün, Seyfi Türkelli and Sinan Ünver.

I wish to thank Bharathwaj Palvannan who kindly answered my questions about his work. I also would like to thank Victor Rotger for inviting me STNB2017 during which I had the chance to meet with B. Palvannan.

This thesis could have not be completed without the hospitality offered to me at the Nesin Mathematics Village since 2011. My thanks go to all staff and volunteers who have made the village a unique place.

Finally, I owe an enormous debt of gratitude towards my family and friends. The times I have shared with them make my life worthwhile. *Teşekkürler!*

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

1.1 Galois Representations

Galois representations play a major role in modern arithmetic algebraic geometry: a Galois representation which arises from algebraic geometry carries a great deal of arithmetic and geometric information. For example, the action of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ on the Tate module of an elliptic curve E defined over $\mathbb{Q}$ captures a lot of information about E, such as the nature of the reduction types of E at different primes.

The primary objects of interest are representations of the global Galois group $G_{\mathbb{Q}}$. However, because this group is extremely complicated, we generally work with representations of its "local pieces" $G_{\mathbb{Q}_\ell}$ with ℓ prime. Thus, it is crucial to understand representations of local Galois groups $G_{\mathbb{Q}_\ell}$.

Let p denote a prime number and let V denote a finite dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous linear action of $G_{\mathbb{Q}_p}$ where ℓ is some prime number. As we explain below, whether p equals ℓ or not makes an impression on the underlying representation theory.

In the case $\ell \neq p$, the topologies on $G_{\mathbb{Q}_p}$ and V are mostly incompatible. The representation V is therefore comparatively simple to study. In this case, V is called an *ℓ-adic representation of* $G_{\mathbb{Q}_p}$.

When $\ell = p$, in which case V is called a *p-adic representation of* $G_{\mathbb{Q}_p}$, the situation is more tricky. The topologies on $G_{\mathbb{Q}_p}$ and V are compatible and hence there are many more representations. In order to identify and study the interesting ones, it is necessary to use p-adic Hodge theory.

p-adic Hodge theory associates to a p-adic representation V a (φ, Γ)-module which is a linear algebraic object originally associates to V by Fontaine [Fon90] and subsequently refined by many others. The upshot is the following. In general, it is very hard to describe p-adic representations in an explicit way; but (φ, Γ)-modules are very explicit objects. Moreover, the (φ, Γ)-module corresponding to an irreducible p-adic representation can become reducible. Also, thanks to Berger-Colmez [BC08] and Kedlaya-Liu [KL10], the theory of (φ, Γ)-modules is compatible with analytic families of p-adic representations.

1.2 p-adic Families of Modular Forms

The theory of p-adic families of modular forms gives an important source of Galois representations. Let N be an integer prime to p. Roughly speaking, a *p-adic family of modular forms of tame level N* is a formal power series

$$\sum_{n=1}^{\infty} a_n q^n \in R[[q]]$$

where R is the ring of p-adic analytic functions over a p-adic disc D defined over $\mathbb{C}_p$, with the property that for all sufficiently large integers $k \in D$, the specialization at k, i.e., the formal sum

$$\sum_{n=1}^{\infty} a_n(k) q^n \in \mathbb{C}_p[[q]]$$

is the q-expansion of a classical modular form of weight k and level Np. Here, as usual, q denotes $e^{2\pi i z}$.

In [Hid86a, Hid86b], Hida constructed certain p-adic families cuspidal eigenforms. By a *p-adic family of cuspidal eigenforms*, we mean a p-adic family of modular forms whose specializations at k, for all large enough integers k, are cuspidal eigenforms. These families are known as *Hida families.* Other types of families exist, as we see below.

Hida's results had a certain limitation. Let f be a cuspidal eigenform of level Np. In particular it is an eigenvector for the U_p-operator. Then f appears in a Hida family (as a specialization at a suitable integer) if and only if f is *ordinary at* p, that is its U_p-eigenvalue is a p-adic unit. This restriction was mostly removed by Coleman in [Col96, Col97]: if f has finite *slope*, that is p-adic valuation of its U_p-eigenvalue is finite, then there exists a p-adic family of cuspidal eigenforms in which f appears. These families are referred as *Coleman families*.

Notice that, for a U_p-eigenform, having finite slope is a generalization of being ordinary. Therefore we may say that Coleman families are non-ordinary prototypes of Hida families.

1.3 Non-Ordinary Iwasawa Theory

In his pioneering work [Gre89], Greenberg developed a theory of Selmer groups for Galois representations which are assumed "ordinary" at p. This theory allowed him to formulate the main conjecture of Iwasawa theory in a general context. His ordinariness hypothesis drastically simplifies the underlying theory, on the other hand it is too restrictive for applications.

In his seminal work [Pot13], Pottharst formed a theory of Selmer groups for Galois representations which are only assumed "ordinary" on the level of their associated (φ, Γ)-modules, by generalizing the Greenberg's ordinariness hypothesis and using the machinery of Selmer complexes introduced by Nekovář in [Nek06]. His result provided new foundations for the algebraic side of non-ordinary Iwasawa theory. The upshot is that this new ordinariness assumption is not that restrictive: any refined family in the sense of [BC09], such as a Coleman family, is automatically satisfies this hypothesis away from a Zariski-closed subset.

Structure of the thesis. For convenience we provide the plan of thesis.

In Chapter 2, after starting with a Hida family, we obtain a certain Galois decomposition deforming the Galois representation attached to the Hida family. Then we state Dasgupta's factorization theorem on p-adic L-functions which is the analytic phenomenon of that decomposition. Finally, we state and mostly prove Palvannan's factorization theorem involving Selmer groups attached to the Galois representations appeared in the same decomposition.

In Chapter 3, we disscuss a potential non-ordinary generalization of Palvannan's result. For this, we start with a Coleman family and mimic Palvannan's setup. The big difference is the following. The property that Hida families satisfy in order to define Selmer groups does not hold for Coleman families in general. Therefore, following both Pottharst and Nekovář, we bring more general devices into play, namely (φ, Γ)-modules and Selmer complexes. At the end, we form the extended Selmer groups corresponding to the Galois representations appeared in the decomposition we obtained.

In Appendix A, after introducing some of the basic objects in p-adic Hodge theory, we explain the Panchishkin condition. We use this condition and some of its properties in Chapter 2 repeatedly.

In Appendix B, after giving a sufficient background on non-archimedean functional analysis, we give the basic notions of rigid analytic geometry. In particular, we study affinoid algebras. These objects are used in Chapter 3 as coefficients rings of Galois representations.

Notations and conventions. Throughout the thesis, we use the following notations and conventions.

As usual, $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the field of rational, real and complex numbers, respectively. For any prime ℓ, we let $\mathbb{Z}_\ell$ denote the ring of ℓ-adic integers and $\mathbb{Q}_\ell$ denote the field of ℓ-adic numbers.

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and set $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For a finite set S of primes in $\mathbb{Q}$, we denote the maximal algebraic extension of $\mathbb{Q}$ unramified outside S by $\mathbb{Q}_S$ and set $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$.

For every prime number ℓ, fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and set $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. Also choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ which gives rise to a map $G_{\mathbb{Q}_\ell} \to G_{\mathbb{Q}}$. This map is injective and its image is the decomposition group D_ℓ of a place above

ℓ. Let I_ℓ be the inertia group inside $G_{\mathbb{Q}_\ell}$ and set $\Gamma_\ell = G_{\mathbb{Q}_\ell}/I_\ell$.

For each $\ell \in S$, consider the natural morphism $i_\ell : G_{\mathbb{Q}_\ell} \to G_{\mathbb{Q},S}$. Let V be a representation of $G_{\mathbb{Q},S}$, say $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_d(V)$. The *restriction* of V to $G_{\mathbb{Q}_\ell}$ is given by the composition $\rho \circ i_\ell : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_d(V)$.

For any prime ℓ, let $\mathbb{C}_\ell$ be the ℓ-adic completion of $\overline{\mathbb{Q}}_\ell$. The action of $G_{\mathbb{Q}_\ell}$ on $\overline{\mathbb{Q}}_\ell$ extends by continuity to $\mathbb{C}_\ell$. We denote by $\mathrm{ord}_\ell : \mathbb{C}_\ell \to \mathbb{R} \cup \{\infty\}$ the ℓ-adic valuation on $\mathbb{C}_\ell$ normalized so that $\mathrm{ord}_\ell(\ell) = 1$ and set $|x|_\ell = (1/\ell)^{\mathrm{ord}_\ell(x)}$.

All rings we are interested in are commutative with unity.

Fix a prime number $p \geq 5$.

Chapter 2

ORDINARY RESULTS

The main goal of this chapter is to recall Palvannan's factorization on Selmer groups. We follow mainly his paper [Pal17].

Notations and conventions. Throughout the chapter, we use the following additional notations and conventions:

Let $\mathbb{Q}_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and set $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. The group Γ is a pro-p-group isomorphic to $\mathbb{Z}_p$, fix a topological generator γ of Γ.

Let $\mathcal{O}$ denote the ring of integers in a finite extension of $\mathbb{Q}_p$. We have the character $\kappa : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \hookrightarrow \mathcal{O}[[\Gamma]]^\times$.

Let R be a profinite ring and let M be a module over R. The Pontryagin duals of R and M are denoted by $\widehat{R}$ and $M^\vee$, respectively. A discrete R-module M is called *cotorsion*, *cofinitely generated* if the R-module $M^\vee$ is torsion, finitely generated respectively.

2.1 Hida Families

In this section, we recall the summary of Hida theory used in this thesis following Section 2 of [EPW06] and Section 2 of [Och03]. See also [Laf] for a more elementary approach.

Let $F = \sum_{n=1}^{\infty} a_n q^n \in R[[q]]$ be a Hida family. The ring R is an integrally closed local domain and a finite integral extension of $\mathbb{Z}_p[[x]]$. The ring R is the normalization of an irreducible component of Hida's (ordinary, primitive) Hecke algebra. We may suppose that the ring $\mathcal{O}$ is the integral closure of $\mathbb{Z}_p$ in R.

In [Hid86a], Hida attached a Galois representations to the family F. This Galois representation interpolates (in a suitable sense) all the Galois representations attached to the classical modular forms occurring in the Hida family, see Remark 2.2.3 in [EPW06].

In general, the Galois representation attached to F is only known to take values in the fraction field of R, see Theorem 2.2.1 in [EPW06] or Theorem 2.8 in [Och03] for a more precise statement. But, in order to the study the Iwasawa theory for cyclotomic deformations, it convenient to assume that the representation is integral. The following hypothesis gives us what we need:

IRR The residual representation associated to F (see Definition 2.9 in [Och03]) is absolutely irreducible.

The hypothesis **IRR** allows us to find an integral model for that Galois representation as wished, see Proposition 2.2.7 in [EPW06] or Proposition 2.10 in [Och03]. More precisely, let S be a finite set of primes in $\mathbb{Q}$ containing p, ∞, the primes dividing the tame level of F and a prime $l \neq p$. In this case, we have the following Galois representation associated to F:

$$\rho_F : G_{\mathbb{Q},S} \to \mathrm{GL}_2(R).$$

Let L_F be the underlying free R-module of rank 2. Suppose F also satisfies the following hypothesis:

p-**DIS** The restriction of the residual representation to $G_{\mathbb{Q}_p}$, which is reducible, has non-scalar semi-simplification.

Due to the hypothesis p-**DIS**, we have the following short exact sequence of free R-modules that is $G_{\mathbb{Q}_p}$-equivariant:

$$0 \to \mathrm{Fil}^+ L_F \to L_F \to \frac{L_F}{\mathrm{Fil}^+ L_F} \to 0 \qquad (\mathrm{Fil}\text{-}\rho_F)$$

which is extremely crucial for the rest of the chapter. Here $\mathrm{Fil}^{+}L_F$ and $\frac{L_F}{\mathrm{Fil}^{+}L_F}$ are free R-modules of rank 1. Without assuming the hypothesis p-**DIS**, we only have a filtration, similar to the one given above, over the fraction field of R, see Proposition 2.2.9 in [EPW06].

Remark 2.1. We would like to make some remarks about the hypotheses **IRR**, p-**DIS** and the prime $l \neq p$ in S.

1. Thanks to the hypotheses **IRR** and p-**DIS**, the representation ρ_F satisfies the Panchishkin condition. Besides this, none of Palvannan's methods require these hypotheses.
2. Including the auxiliary prime l in S is not usual, the reason Palvannan includes it is to use some results in [Gre10] and [Gre16].

We let $T = R\widehat{\otimes}R$. The ring T is a complete integrally closed local domain and a finite integral extension of $\mathbb{Z}_p[[x_1, x_2]]$ where x_1 and x_2 are identified with the "weight variables". We denote the natural map $\pi_{F,F} : T \to R$ obtained by sending an elementary tensor $a \otimes b$ to ab. This map is a surjective $\mathcal{O}$-algebra homomorphism. We have $\pi_{F,F}(x_1) = \pi_{F,F}(x_2) = x$.

Consider the two natural maps $i_1 : R \to T$ and $i_2 : R \to T$ and set $L_{F,F} = \mathrm{Hom}_T(L_F \otimes_{i_1} T, L_F \otimes_{i_2} T)$ which is a free T-module of rank 4. The action of $G_{\mathbb{Q},S}$ on $L_{F,F}$ gives us a 4-dimensional Galois representation, say $\rho_{F,F} : G_{\mathbb{Q},S} \to \mathrm{GL}_4(T)$. Composing $\rho_{F,F}$ with $\pi_{F,F}$ gives us the adjoint representation of ρ_F and we have the following decomposition of Galois representations:

$$\mathrm{Ad}(\rho_F) \cong \mathrm{Ad}^0(\rho_F) \oplus \mathbf{1} \tag{2.1}$$

where

- $\mathrm{Ad}(\rho_F)$ denotes $M_2(R)$ with the action of $G_{\mathbb{Q},S}$ given by conjugation via ρ_F,

- $\mathrm{Ad}^0(\rho_F)$ denotes the trace-zero matrices in $M_2(R)$ with the action of $G_{\mathbb{Q},S}$ given by conjugation via ρ_F,

- $\mathbf{1}$ denotes the scalar matrices in $M_2(R)$ with the trivial action of $G_{\mathbb{Q},S}$.

We need a "deformed" version of the decomposition given in (2.1). The following section explains this issue from a general perspective.

2.2 Cyclotomic Deformations

Let R be an integrally closed local domain that is a finite integral extension of $\mathbb{Z}_p[[u_1,\ldots,u_n]]$. Let S be a finite set of primes in $\mathbb{Q}$ containing p, ∞ and a prime $\ell \neq p$. Assume we have a Galois representation

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_d(R)$$

whose underlying free R-module of rank d is denoted by L_ρ. In this section we define a Galois representation

$$\rho \otimes \kappa^{-1} : G_{\mathbb{Q},S} \to \mathrm{GL}_d(R[[\Gamma]]).$$

Here $R[[\Gamma]]$ is, as usual, the Iwasawa algebra over R and $\kappa : G_{\mathbb{Q}} \to \mathcal{O}[[\Gamma]]^\times$. There is an isomorphism $R[[\Gamma]] \cong R[[s]]$ obtained by sending the topological generator γ to $s+1$.

Let $R[[\Gamma]](\kappa^{-1})$ denote the free $R[[\Gamma]]$-module of rank 1 with the action of $G_{\mathbb{Q},S}$ given by the character κ^{-1}. The free $R[[\Gamma]]$-modules

$$L_{\rho\otimes\kappa^{-1}} = L_\rho \otimes_R R[[\Gamma]](\kappa^{-1}), \quad \mathrm{Fil}^+ L_{\rho\otimes\kappa^{-1}} = \mathrm{Fil}^+ L_\rho \otimes_R R[[\Gamma]](\kappa^{-1}),$$

give the following isomorphism:

$$\frac{L_{\rho\otimes\kappa^{-1}}}{\mathrm{Fil}^+ L_{\rho\otimes\kappa^{-1}}} \cong \frac{L_\rho}{\mathrm{Fil}^+ L_\rho} \otimes_R R[[\Gamma]](\kappa^{-1}).$$

The deformation $\rho \otimes \kappa^{-1}$ is then defined by the action of $G_{\mathbb{Q},S}$ on $L_{\rho\otimes\kappa^{-1}}$. More precisely, we have a Galois representation

$$\rho \otimes \kappa^{-1} : G_{\mathbb{Q},S} \to \mathrm{GL}_d(R[[\Gamma]])$$

whose underlying free $R[[\Gamma]]$-module of rank d is $L_{\rho\otimes\kappa^{-1}}$.

This new representation is related to the cyclotomic deformation of ρ which, as defined in [Gre94], is given by the action $G_{\mathbb{Q},S}$ on $L_\rho \otimes_R R[[\Gamma]](\kappa)$. These kind of deformations frequently arise in Iwasawa theory.

2.3 A Deformed Galois Decomposition

We pick up where we left off. In this section, we obtain a Galois decomposition which is extremely important both on algebraic side involving Selmer groups and on analytic side involving p-adic L-functions.

Let $\chi : G_{\mathbb{Q},S} \to \mathcal{O}^\times$ be a Dirichet character. Applying the construction given in Section 2.2, we may consider the following representations:

- Let $\rho_{4,3} : G_{\mathbb{Q},S} \to \mathrm{GL}_4(T[[\Gamma]])$ be the 4-dimensional Galois representation given by $\rho_{F,F}(\chi) \otimes \kappa^{-1}$.

- Let $\rho_{3,2} : G_{\mathbb{Q},S} \to \mathrm{GL}_3(R[[\Gamma]])$ be the 3-dimensional Galois representation given by $\mathrm{Ad}^0(\rho_F)(\chi) \otimes \kappa^{-1}$.

- Let $\rho_{1,2} : G_{\mathbb{Q},S} \to \mathrm{GL}_1(R[[\Gamma]])$ be the 1-dimensional Galois representation given by $\mathbf{1}(\chi) \otimes \kappa^{-1}$.

Remark 2.2. Notice that two different numbers appear in the subscripts for these Galois representations. Clearly, the first one represents the dimension of the representation. See Section 2.4 for the meaning of the second number.

One can extend the map $\pi_{F,F} : T \to R$ to the completed group rings. More precisely, the map $\pi_{F,F}$ induces a surjective $\mathcal{O}[[\Gamma]]$-algebra homomorphism

$$\pi : T[[\Gamma]] \to R[[\Gamma]].$$

The decomposition (2.1) gives rise to the desired decomposition of Galois representations

$$\pi \circ \rho_{4,3} \cong \rho_{3,2} \oplus \rho_{1,2}. \tag{2.2}$$

2.4 Dasgupta's Factorization

In this section, we state Dasgupta's theorem concerning p-adic L-functions which is the analytic side of the decomposition given in (2.2).

It turns out, see [Pal17], that the Galois representations $\rho_{4,3}$, $\rho_{3,2}$ and $\rho_{1,2}$ satisfy the Panchishkin condition. With this in mind, consider the following p-adic L-functions:

- Let $\theta_{4,3}$ be the (primitive) 3-variable p-adic L-function associated to $\rho_{4,3}$. This function is an element in $\mathrm{Frac}(T[[\Gamma]])$. See [Das16] and [Hid88] for the properties that $\theta_{4,3}$ satisfies.

- Let $\theta_{3,2}$ be the (primitive) 2-variable p-adic L-function associated to $\rho_{3,2}$. This function is an element in $\mathrm{Frac}(R[[\Gamma]])$. See [Das16] and [Hid90] for the properties that $\theta_{3,2}$ satisfies.

- We have a 1-variable p-adic L-function $\theta_{1,1}$, thanks to [KL64], associated to $\rho_{1,2}$. This function is an element in $\mathrm{Frac}(\mathcal{O}[[\Gamma]])$. Let $\theta_{1,2}$ be the image of $\theta_{1,1}$ under the natural inclusion $\mathcal{O}[[\Gamma]] \hookrightarrow R[[\Gamma]]$. This function is an element in $\mathrm{Frac}(R[[\Gamma]])$.

Remark 2.3. Following Remark 2.2, the second number indicates then the number of variables in the p-adic L-function.

We have the following factorization theorem due to Dasgupta:

Theorem 2.4. *After extending the map* π *to the fraction field of* $T[[\Gamma]]$*, we have the following equality in the fraction field of* $R[[\Gamma]]$*:*

$$\pi(\theta_{4,3}) = \theta_{3,2} \cdot \theta_{1,2}.$$

Proof. This is the main result of [Das16]. □

Along the spirit of Iwasawa theory, Palvannan proved the expected result on the algebraic side. In the following section, we define Selmer groups which are the primary objects in Palvannan's factorization.

2.5 Construction of Selmer Groups

Selmer groups are basically defined as the kernel of a natural map from a global Galois cohomology group to a (product of) local Galois cohomology group(s). As in Section 2.2, we approach the topic from a general perspective.

Let R be an integrally closed local domain that is a finite integral extension of $\mathbb{Z}_p[[u_1, \ldots, u_n]]$. Let S be a finite set of primes in $\mathbb{Q}$ containing p, ∞ and a prime $\ell \neq p$. Put $S_0 = S \setminus \{p\}$. Assume we have a Galois representation

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_d(R)$$

whose underlying free R-module of rank d is denoted by L_ρ. Suppose the following hypothesis is satisfied:

FIL There exists a short exact sequence of free R-modules that is $G_{\mathbb{Q}_p}$-equivariant:

$$0 \to \mathrm{Fil}^+ L_\rho \to L_\rho \to \frac{L_\rho}{\mathrm{Fil}^+ L_\rho} \to 0. \tag{Fil-ρ}$$

We call the short exact sequence Fil-ρ as the *filtration* associated to ρ. The filtration Fil-ρ is an extra datum for the Galois representation ρ. A typical example to keep in mind can be given as follows.

Let E be an elliptic curve defined over $\mathbb{Q}$ that has good ordinary reduction at p. Let $\overline{E}$ denote the reduction of E modulo p. We have a surjective reduction homomorphism

$$T_p(E) \xrightarrow{\text{red}} T_p(\overline{E})$$

of $G_{\mathbb{Q}_p}$-modules. Here, $T_p(E)$ and $T_p(\overline{E})$ are the p-adic Tate modules associated to the elliptic curves E and $\overline{E}$, respectively. Since $\overline{E}$ is ordinary, $T_p(\overline{E})$ is a $\mathbb{Z}_p$-module of rank 1. Therefore we have a short exact sequence

$$0 \to \ker(\text{red}) \to T_p(E) \xrightarrow{\text{red}} T_p(\overline{E}) \to 0$$

of free $\mathbb{Z}_p$-modules that is $G_{\mathbb{Q}_p}$-equivariant.

We associate a primitive Selmer group $\mathrm{Sel}_\rho(\mathbb{Q})$ and a non-primitive Selmer group $\mathrm{Sel}_\rho^{S_0}(\mathbb{Q})$ to the Galois representation ρ using Fil-ρ as follows.

We define

$$\mathcal{D}_\rho = L_\rho \otimes_R \widehat{R}$$

where the Pontryagin dual $\widehat{R}$ endows with a trivial action of $G_{\mathbb{Q},S}$. That Galois group acts on $\mathcal{D}_\rho$ through its action on the first factor. Thus, $\mathcal{D}_\rho$ is a discrete abelian group which is isomorphic to $\widehat{R}^d$ as an R-module and which has a continuous R-linear action of $G_{\mathbb{Q},S}$. Similarly, we define the following discrete module:

$$\mathrm{Fil}^+\mathcal{D}_\rho = \mathrm{Fil}^+L_\rho \otimes_R \widehat{R}.$$

Notice that we have the following isomorphism:

$$\frac{\mathcal{D}_\rho}{\mathrm{Fil}^+\mathcal{D}_\rho} \cong \frac{L_\rho}{\mathrm{Fil}^+L_\rho} \otimes_R \widehat{R}.$$

Definition 2.5. Using the same notations above, the *primitive Selmer group* associated to ρ is defined to be

$$\mathrm{Sel}_\rho(\mathbb{Q}) = \ker\left(H^1(G_{\mathbb{Q},S}, \mathcal{D}_\rho) \xrightarrow{\phi_\rho} H^1\Big(I_p, \frac{\mathcal{D}_\rho}{\mathrm{Fil}^+\mathcal{D}_\rho}\Big)^{\Gamma_p} \times \prod_{\ell \in S_0} H^1(I_\ell, \mathcal{D}_\rho)^{\Gamma_\ell} \right).$$

Now we define the non-primitive Selmer groups. These are defined just as the primitive Selmer groups except that we omit the local conditions away from p. More precisely,

Definition 2.6. With the same notations above, the *non-primitive Selmer group* associated to ρ is defined as

$$\mathrm{Sel}_\rho^{S_0}(\mathbb{Q}) = \ker\left(H^1(G_{\mathbb{Q},S}, \mathcal{D}_\rho) \xrightarrow{\phi_\rho^{S_0}} H^1\Big(I_p, \frac{\mathcal{D}_\rho}{\mathrm{Fil}^+\mathcal{D}_\rho}\Big)^{\Gamma_p} \right).$$

Remark 2.7. Let us make some remarks regarding these Selmer groups.

1. Notice that the maps for the cohomology groups occurring in the definitions depend on the chosen embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ for $\ell \in S$. However, the Selmer groups are independent of the choices made.

2. The Galois cohomology group $H^1(G_{\mathbb{Q},S}, \mathcal{D}_\rho)$ can be considered as a discrete R-module too. It is a cofinitely generated R-module, see Proposition 3.2 in [Gre06]. It is clear that both Selmer groups are R-submodules of $H^1(G_{\mathbb{Q},S}, \mathcal{D}_\rho)$ and so are also discrete cofinitely generated R-modules.

3. By definition, we have the inclusion $\mathrm{Sel}_\rho(\mathbb{Q}) \subset \mathrm{Sel}_\rho^{S_0}(\mathbb{Q})$. Moreover, it is possible to study the differences between these two Selmer groups. See [Gre10] for more details. For the cases Palvannan is interested in, these differences can be evaluated explicitly in terms of certain local factors away from p, see Proposition 5.1 in [Pal17]. He prefers working with non-primitive Selmer groups since it is easier to establish that they satisfy better algebraic properties.

Recall the deformation $\rho \otimes \kappa^{-1}$ that we constructed in Section 2.2. We have the following filtration associated to it:

$$0 \to \mathrm{Fil}^+ L_{\rho\otimes\kappa^{-1}} \to L_{\rho\otimes\kappa^{-1}} \to \frac{L_{\rho\otimes\kappa^{-1}}}{\mathrm{Fil}^+ L_{\rho\otimes\kappa^{-1}}} \to 0. \qquad (\text{Fil-}\rho\otimes\kappa^{-1})$$

Thanks to the isomorphism $R[[\Gamma]] \cong R[[s]]$, one may view $R[[\Gamma]]$ as a finite integral extension of $\mathbb{Z}_p[[u_1, \dots, u_n, s]]$. Moreover, $R[[\Gamma]]$ is an integrally closed domain. Therefore, one can form the Selmer group $\mathrm{Sel}_{\rho\otimes\kappa^{-1}}(\mathbb{Q})$ corresponding to Fil-$\rho \otimes \kappa^{-1}$. When $\rho \otimes \kappa^{-1}$ satisfies Panchishkin condition additionally, one can associate (conjecturally) a p-adic L-function $\theta_{\rho\otimes\kappa^{-1}}$ to $\rho \otimes \kappa^{-1}$. In this setup, Greenberg formulated a remarkable conjecture, known as the "Main Conjecture" of Iwasawa theory, that relates the Selmer group $\mathrm{Sel}_{\rho\otimes\kappa^{-1}}(\mathbb{Q})$ on the algebraic side to the p-adic L-function $\theta_{\rho\otimes\kappa^{-1}}$ on the analytic side. See [Gre94] for further details.

2.6 Selmer Groups for Hida Deformation

In this section, we associate Selmer groups to the Galois representations appeared in the decomposition given in (2.2) following the previous section. We continue from the end of Section 2.4.

Consider the following filtrations:

- Consider the 4-dimensional Galois representation $\pi \circ \rho_{4,3} : G_{\mathbb{Q},S} \to \mathrm{GL}_4(R([[\Gamma]])$. Let $L_{\pi\circ\rho_{4,3}}$ be the underlying free $R[[\Gamma]]$-module of rank 4. More explicitly,

  $$L_{\pi\circ\rho_{4,3}} = \mathrm{Hom}_R(L_F, L_F)(\chi) \otimes_R R[[\Gamma]](\kappa^{-1}).$$

  Consider the following free $R[[\Gamma]]$-module of rank 2:

  $$\mathrm{Fil}^+ L_{\pi\circ\rho_{4,3}} = \mathrm{Hom}_R(L_F, \mathrm{Fil}^+ L_F)(\chi) \otimes_R R[[\Gamma]](\kappa^{-1}).$$

  Then we have the following filtration associated to $\pi \circ \rho_{4,3}$:

  $$0 \to \mathrm{Fil}^+ L_{\pi\circ\rho_{4,3}} \to L_{\pi\circ\rho_{4,3}} \to \frac{L_{\pi\circ\rho_{4,3}}}{\mathrm{Fil}^+ L_{\pi\circ\rho_{4,3}}} \to 0. \qquad (\text{Fil-}\pi\circ\rho_{4,3})$$

- Consider the 3-dimensional Galois representation $\rho_{3,2} : G_{\mathbb{Q},S} \to \mathrm{GL}_3(R([[\Gamma]])$. Let $L_{\rho_{3,2}}$ be the underlying free $R[[\Gamma]]$-module of rank 3. Also let $\mathrm{Ad}^0(L_F)$ be the underlying free R-module of rank 3 corresponding to the 3-dimensional Galois representation $\mathrm{Ad}^0(\rho_F) : G_{\mathbb{Q},S} \to \mathrm{GL}_3(R)$. Consider the following free R-modules of $\mathrm{Ad}^0(L_F)$ that are $G_{\mathbb{Q}_p}$-equivariant:

  $$0 \subsetneq \underbrace{\mathrm{Fil}^{\mathrm{even}}\mathrm{Ad}^0(L_F)}_{\text{rank } 1} \subsetneq \underbrace{\mathrm{Fil}^{\mathrm{odd}}\mathrm{Ad}^0(L_F)}_{\text{rank } 2} \subsetneq \mathrm{Ad}^0(L_F)$$

  where

  $\mathrm{Fil}^{\mathrm{even}}\mathrm{Ad}^0(L_F) = \mathrm{Hom}_R\left(\dfrac{L_F}{\mathrm{Fil}^+ L_F}, \mathrm{Fil}^+ L_F\right)$,

  $\mathrm{Fil}^{\mathrm{odd}}\mathrm{Ad}^0(L_F) = \ker\left(\mathrm{Hom}_R(L_F, L_F) \xrightarrow{\mathrm{Res}} \mathrm{Hom}_R\left(\mathrm{Fil}^+ L_F, \dfrac{L_F}{\mathrm{Fil}^+ L_F}\right)\right) \cap \mathrm{Ad}^0(L_F)$.

  Put

  $$\mathrm{Fil}^+ L_{\rho_{3,2}} = \begin{cases} \mathrm{Fil}^{\mathrm{even}}\mathrm{Ad}^0(L_F)(\chi) \otimes_R R[[\Gamma]](\kappa^{-1}), & \text{if } \chi \text{ is even} \\ \mathrm{Fil}^{\mathrm{odd}}\mathrm{Ad}^0(L_F)(\chi) \otimes_R R[[\Gamma]](\kappa^{-1}), & \text{if } \chi \text{ is odd.} \end{cases}$$

  Then we have the following filtration (which depends on the parity of χ) associated to $\rho_{3,2}$:

  $$0 \to \mathrm{Fil}^+ L_{\rho_{3,2}} \to L_{\rho_{3,2}} \to \frac{L_{\rho_{3,2}}}{\mathrm{Fil}^+ L_{\rho_{3,2}}} \to 0. \qquad (\text{Fil-}\rho_{3,2})$$

- Consider 1-dimensional Galois representation $\rho_{1,2} : G_{\mathbb{Q},S} \to \mathrm{GL}_1(R([[\Gamma]])$. Let $L_{\rho_{1,2}}$ be the corresponding free $R[[\Gamma]]$-module of rank 1. More explicitly, $L_{\rho_{1,2}} = R[[\Gamma]](\chi\kappa^{-1})$. The free $R[[\Gamma]]$-module

  $$\mathrm{Fil}^+ L_{\rho_{1,2}} = \begin{cases} L_{\rho_{1,2}}, & \text{if } \chi \text{ is even} \\ 0, & \text{if } \chi \text{ is odd} \end{cases}$$

  gives the following filtration (which depends on the parity of χ) associated to ρ_1:

  $$0 \to \mathrm{Fil}^+ L_{\rho_{1,2}} \to L_{\rho_{1,2}} \to \frac{L_{\rho_{1,2}}}{\mathrm{Fil}^+ L_{\rho_{1,2}}} \to 0. \qquad (\text{Fil-}\rho_{1,2})$$

Similar to (2.2), we also have the following isomorphism of $R[[\Gamma]]$-modules that is $G_{\mathbb{Q},S}$-equivariant:

$$L_{\pi\circ\rho_{4,3}} \cong L_{\rho_{3,2}} \oplus L_{\rho_{1,2}}. \tag{2.3}$$

Following Section 2.5 and using Fil-$\pi\circ\rho_{4,3}$, Fil-$\rho_{3,2}$ and Fil-$\rho_{1,2}$, we associate Selmer groups $\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})$, $\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})$ and $\mathrm{Sel}^{S_0}_{\rho_{1,2}}(\mathbb{Q})$ to $\pi\circ\rho_{4,3}$, $\rho_{3,2}$ and $\rho_{1,2}$, respectively.

We need one more definition to state Palvannan's factorization. In the following section, we explain the notion of divisor.

2.7 Divisors Attached to Modules

In Iwasawa theory "characteristic ideals" play a major role. When dealing with these objects, we usually work with regular local rings. But, in the setup of Hida theory, the rings we are interested in are not always known to be regular. In this section, following [Bou89], we introduce an object that is a substitute for the notion of characteristic ideal. Again, we pursue a general context.

Let R be an integrally closed Noetherian domain. We denote the set of prime ideals of R of height 1 by P. The *divisor group* $\mathrm{Div}(R)$ of R is defined as the free abelian group on P.

We are interested in the structure of finitely generated torsion modules over R. One of the methods of understanding such modules consists of localizing the modules with respect to all $\mathfrak{p} \in P$. Because $R_{\mathfrak{p}}$ is a discrete valuation ring (see Chapter VII, Section 1.6, Theorem 4 of [Bou89]) and the structure of finitely generated torsion $R_{\mathfrak{p}}$-modules is well known, this method gives information about the structure of finitely generated torsion modules over R. Below, we follow this strategy.

Let M be a finitely generated torsion R-module. Then for all $\mathfrak{p} \in P$, $M_{\mathfrak{p}}$ is a finitely generated torsion $R_{\mathfrak{p}}$-module and thus a module of finite length (see Chapter

IV, Section 2.5, Corollary 2 to Proposition 7 of [Bou89]); we denote this length by $l_{\mathfrak{p}}(M)$. On the other hand, clearly $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ such that $\mathrm{Ann}_R(M) \not\subset \mathfrak{p}$ hence for all but finitely many $\mathfrak{p}$ (see Chapter VII, Section 1.6, Theorem 4 of [Bou89]). These two observations justify the following definition:

Definition 2.8. The *divisor* of a finitely generated torsion R-module M is defined to be

$$\mathrm{Div}(M) = \sum_{\mathfrak{p} \in P} l_{\mathfrak{p}}(M) \cdot \mathfrak{p}.$$

Notice that $\mathrm{Div}(M)$ is an element of $\mathrm{Div}(R)$.

The following proposition explains why this is a substitute for the notion of characteristic ideal.

Proposition 2.9. *For $? \in \{1, 2, 3, \emptyset\}$, let $M_?$ be a finitely generated torsion R-module.*

1. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence, then*

$$Div(M_2) = Div(M_1) + Div(M_3).$$

2. *Let $f : M_1 \to M_2$ be a homomorphism with finite kernel and cokernel, then $Div(M_1) = Div(M_2)$.*

3. *In order that $Div(M) = 0$, it is necessary and sufficient that M be pseudo-zero.*

Proof. See Chapter VII, Section 4.5, Proposition 10 of [Bou89]. □

2.8 Palvannan's Factorization

In this section, we state Palvannan's theorem concerning Selmer groups which is the algebraic side of the decomposition given in (2.2). We also give a sketch of the proof. We continue from the end of Section 2.6.

Recall the Selmer groups $\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})$, $\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})$ and $\mathrm{Sel}^{S_0}_{\rho_{1,2}}(\mathbb{Q})$. By Remark 2.7, these Selmer groups are cofinitely generated $R[[\Gamma]]$-modules. The Pontryagin dual of $\mathrm{Sel}^{S_0}_{\rho_{1,2}}(\mathbb{Q})$ can be described in terms of classical Iwasawa modules which are already known to be torsion due to [Iwa73]. Such a description is detailed in [Gre89] and [Gre94]. Thus, $\mathrm{Sel}^{S_0}_{\rho_{1,2}}(\mathbb{Q})$ is a cotorsion $R[[\Gamma]]$-module. Suppose the following hypothesis is satisfied:

TOR $\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})$ and $\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})$ are cotorsion $R[[\Gamma]]$-modules.

Then, following Section 2.7, we may consider the divisors of $\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})^\vee$, $\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})^\vee$ and $\mathrm{Sel}^{S_0}_{\rho_{1,2}}(\mathbb{Q})^\vee$. We have the following factorization theorem due to Palvannan:

Theorem 2.10. *We have the following equality in the divisor group of* $R[[\Gamma]]$*:*

$$Div(Sel^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})^\vee) = Div(Sel^{S_0}_{\rho_{3,2}}(\mathbb{Q})^\vee) + Div(Sel^{S_0}_{\rho_{1,2}}(\mathbb{Q})^\vee).$$

We also have the following decomposition of local factors away from p*:*

$$H^1(I_\ell, \mathcal{D}_{\pi\circ\rho_{4,3}})^{\Gamma_\ell} \cong H^1(I_\ell, \mathcal{D}_{\rho_{3,2}})^{\Gamma_\ell} \oplus H^1(I_\ell, \mathcal{D}_{\rho_{1,2}})^{\Gamma_\ell}, \quad \textit{for all } \ell \in S_0.$$

Proof. This is Theorem 2 in [Pal17]. We outline the the proof here, details can be found in Chapter 4.2 in [Pal17].

The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(G_{\mathbb{Q},S}, \mathcal{D}_{\rho_{1,2}}) \arrow[r] \arrow[d, "\phi^{S_0}_{\rho_{1,2}}"] & H^1(G_{\mathbb{Q},S}, \mathcal{D}_{\pi\circ\rho_{4,3}}) \arrow[r] \arrow[d, "\phi^{S_0}_{\pi\circ\rho_{4,3}}"] & H^1(G_{\mathbb{Q},S}, \mathcal{D}_{\rho_{3,2}}) \arrow[r] \arrow[d, "\phi^{S_0}_{\rho_{3,2}}"] & 0 \\
0 \arrow[r] & H^1\left(I_p, \frac{\mathcal{D}_{\rho_{1,2}}}{\mathrm{Fil}^+\mathcal{D}_{\rho_{1,2}}}\right)^{\Gamma_p} \arrow[r] & H^1\left(I_p, \frac{\mathcal{D}_{\pi\circ\rho_{4,3}}}{\mathrm{Fil}^+\mathcal{D}_{\pi\circ\rho_{4,3}}}\right)^{\Gamma_p} \arrow[r] & H^1\left(I_p, \frac{\mathcal{D}_{\rho_{3,2}}}{\mathrm{Fil}^+\mathcal{D}_{\rho_{3,2}}}\right)^{\Gamma_p} &
\end{tikzcd} \tag{2.4}$$

satisfies the following properties:

- The top row in (2.4) is exact,

- The bottom row in (2.4) is coexact in dimension 1 (Let R be an integrally closed local domain which is a finite integral extension of $\mathbb{Z}_p[[u_1, \ldots, u_n]]$. A sequence $M_1 \to \cdots \to M_n$ of discrete R-modules, whose Pontryagin duals are finitely generated R-modules, is *coexact in dimension* 1 if for every height 1 prime ideal $\mathfrak{p}$ of R, the sequence

  $$(M_n^\vee)_\mathfrak{p} \to \cdots \to (M_1^\vee)_\mathfrak{p}$$

  of $R_\mathfrak{p}$-modules is exact.).

Take the Pontryagin dual of all the modules in (2.4) and consider the localizations with respect to every height 1 prime ideal $\mathfrak{p}$ in $R[[\Gamma]]$. Combining the properties of the diagram (2.4) with the fact that the map $\phi^{S_0}_{\rho_{1,2}}$ is surjective (see Proposition 1.7 in [Pal17]), the snake lemma gives rise to the following exact sequence for any such prime ideal:

$$0 \to \left(\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})^\vee\right)_\mathfrak{p} \to \left(\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})^\vee\right)_\mathfrak{p} \to \left(\mathrm{Sel}^{S_0}_{\rho_{1,2}}(\mathbb{Q})^\vee\right)_\mathfrak{p} \to 0$$

which gives us the following equality:

$$l_\mathfrak{p}\big(\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})^\vee\big) = l_\mathfrak{p}\big(\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})^\vee\big) + l_\mathfrak{p}\big(\mathrm{Sel}^{S_0}_{\rho_{1,2}}(\mathbb{Q})^\vee\big)$$

for every height 1 prime ideal $\mathfrak{p}$ in $R[[\Gamma]]$. The desired equality of divisors in the divisor group of $R[[\Gamma]]$ follows from this equality.

The desired decomposition of local factors away from p is a direct consequence of the decomposition in (2.3). □

Remark 2.11. We would like to make some remarks about Palvannan's paper [Pal17].

1. In the hypothesis **TOR**, we have assumed that both $\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})^\vee$ and $\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})^\vee$ are $R[[\Gamma]]$-torsion. This is unnecessary: In order that $\mathrm{Sel}^{S_0}_{\pi\circ\rho_{4,3}}(\mathbb{Q})^\vee$ is $R[[\Gamma]]$-torsion, it necessary and sufficient that $\mathrm{Sel}^{S_0}_{\rho_{3,2}}(\mathbb{Q})^\vee$ is $R[[\Gamma]]$-torsion. See Theorem 2 in [Pal17].

2. Palvannan also proves that Theorem 2.4 and Theorem 2.10 are completely consistent with the main conjectures for $\rho_{4,3}$, $\rho_{3,2}$ and $\rho_{1,2}$, see Theorem 3 in [Pal17]. The main conjecture is known for $\rho_{1,2}$ thanks to [MW84]. Under certain hypotheses, he indicates how one can use work of Urban [Urb01] to deduce main conjecture for $\rho_{4,3}$ and $\rho_{3,2}$, see Theorem 4 in [Pal17]. As a result, Dasgupta's and Palvannan's results provide evidence for the main conjecture for the Galois representations $\rho_{4,3}$ and $\rho_{3,2}$.

Chapter 3

NON-ORDINARY DISCUSSIONS

In this section, we discuss a possible non-ordinary generalization of Palvannan's factorization on Selmer groups. We start with a Coleman family, instead of a Hida family.

Notations and conventions. Throughout the chapter, we use the following additional notations and conventions.

Fix a system of primitive p^n-th roots of unity $\varepsilon = (\zeta_{p^n})_{n \geqslant 0}$ such that

$$\zeta_p \neq 1, \quad \zeta_{p^{n+1}}^p = \zeta_{p^n} \text{ for all n} \geqslant 0.$$

Let $\mathbb{Q}_p(\zeta_{p^\infty}) = \bigcup_{n=0}^{\infty} \mathbb{Q}(\zeta_{p^n})$ and set $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$. The p-adic cyclotomic character $\chi_p : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is defined by

$$g(\zeta_{p^n}) = \zeta_{p^n}^{\chi_p(g)} \text{ for all } g \in G_{\mathbb{Q}_p},\ n \geqslant 0.$$

χ_p is surjective with kernel $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))$ and thus it induces an isomorphism $\Gamma \to \mathbb{Z}_p^\times$ which we again denote by χ_p.

The group Γ decomposes canonically into the direct sum $\Gamma = \Delta \times \Gamma_0$ where $\Delta = \mathrm{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ and $\Gamma_0 = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p(\zeta_p))$. The group Γ_0 is a pro-p-group isomorphic to $\mathbb{Z}_p$. We fix a topological generator γ of Γ_0.

Fix a coefficient field E which is a finite extension of $\mathbb{Q}_p$, denote its ring of integers by $\mathcal{O}_E$. Let $\Lambda = \mathcal{O}_E[[\Gamma_0]]$ denote the Iwasawa algebra over $\mathcal{O}_E$. Then we have the isomorphism $\Lambda \cong \mathcal{O}_E[[X]]$ such that $\gamma \mapsto 1 + X$.

The algebra $\mathcal{H}$ is defined by

$$\mathcal{H} = \left\{ f(X) = \sum_{i=0}^{\infty} a_i X^i \in E[[X]] : f \text{ is convergent for } |X|_p < 1 \right\}.$$

We consider Λ as a subalgebra of $\mathcal{H}$ via the isomorphism $\Lambda \cong \mathcal{O}_E[[X]]$. Note that $\mathcal{H}$ is the large Iwasawa algebra introduced in [PR94].

If G is a topological group and M is a continuous G-module, we write $C^\bullet(G, M)$ for the complex of continuous cochains of G with coefficients in M.

3.1 Coleman Families

In this section, we recall the summary of Coleman theory used in this thesis following Section 2 of [NO17].

Let $\mathbf{f} = \sum_{n=1}^{\infty} a_n q^n \in A[[q]]$ be a Coleman family. Here the coefficient ring A is an affinod algebra.

Using a similar procedure explained in Secion 2.1, it is possible to produce an integral Galois representation associated to $\mathbf{f}$. More precisely, if S is a finite set of primes in $\mathbb{Q}$ containing p, ∞ and the primes dividing the tame level of $\mathbf{f}$, then we have the following Galois representation associated to $\mathbf{f}$:

$$\rho_{\mathbf{f}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(A).$$

This Galois representation interpolates (in a suitable sense) all the Galois representations attached to the classical modular forms occurring in the Coleman family. See Theorem 2.12 in [NO17] for more details. Say $V_{\mathbf{f}}$ for the free A-module of rank 2 on which $G_{\mathbb{Q},S}$ acts to let us obtain $\rho_{\mathbf{f}}$.

Put $B = A\widehat{\otimes}A$ which is again an affinoid E-algebra by Corollary B.4. Then we have a natural surjection $\pi_{\mathbf{f},\mathbf{f}} : B \to A$ obtained by sending an elementary tensor $a \otimes b$ to ab.

Consider the two natural maps $i_1 : A \to B$ and $i_2 : A \to B$ and set $L_{\mathbf{f},\mathbf{f}} = \mathrm{Hom}_B(V_{\mathbf{f}} \otimes_{i_1} B, V_{\mathbf{f}} \otimes_{i_2} B)$ which is a free B-module of rank 4. The action of $G_{\mathbb{Q},S}$ on $L_{\mathbf{f},\mathbf{f}}$ gives us a 4-dimensional Galois representation, say $\rho_{\mathbf{f},\mathbf{f}} : G_{\mathbb{Q},S} \to \mathrm{GL}_4(B)$.

Composing $\rho_{\mathbf{f},\mathbf{f}}$ with $\pi_{\mathbf{f},\mathbf{f}}$ gives us the adjoint representation of $\rho_{\mathbf{f}}$ and we have the following decomposition of Galois representations:

$$\mathrm{Ad}(\rho_{\mathbf{f}}) \cong \mathrm{Ad}^0(\rho_{\mathbf{f}}) \oplus \mathbf{1}. \tag{3.1}$$

As before, we need a “deformed” version of the decomposition given in (3.1). But, in general, the restriction of $V_{\mathbf{f}}$ to $G_{\mathbb{Q}_p}$ is irreducible and hence an exact sequence similar to Fil-ρ_F does not exist. As a result, the method explained in Section 2.2 does not work here. The following three sections explain, from a general perspective, how to obtain such a decomposition.

3.2 (φ, Γ)-*modules*

In this section, we collect fundamental facts about (φ, Γ)-modules over the Robba ring. In particular, we explain the importance of these modules with regard to p-adic representations.

Definition 3.1. The *Robba ring* $\mathcal{R}_E$ *of power series with coefficients in* E is defined to be

$$\mathcal{R}_E = \bigcup_{0\leq r<1} \mathcal{R}_E^{(r)}$$

where the latter ring $\mathcal{R}_E^{(r)}$ is defined by

$$\mathcal{R}_E^{(r)} = \left\{ f(X) = \sum_{i=-\infty}^{\infty} a_i X^i : a_i \in E,\ f \text{ is convergent for } p^{-1/r} \leq |X|_p < 1 \right\}$$

for any $0 \leq r < 1$.

Remark 3.2. Notice that $\mathcal{R}_E$ is not Noetherian, but the situation is not too unpleasant. By a theorem of Lazard, it has the Bézout property: each finitely generated ideal over $\mathcal{R}_E$ is principal. See [Laz62] for details.

We equip $\mathcal{R}_E$ with an E-linear continuous action of Γ defined by

$$\tau \cdot f(X) = f((1+X)^{\chi_p(\tau)} - 1), \qquad \tau \in \Gamma$$

and a linear operator φ, called the *Frobenius*, given by

$$\varphi(f(X)) = f((1+X)^p - 1).$$

For example, the series $t = \log(1+X) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} X^i$ is a typical element of the Robba ring $\mathcal{R}_E$. Moreover, an easy calculation shows that, we have $\varphi(t) = pt$ and $\tau \cdot t = \chi_p(\tau)t$ for all $\tau \in \Gamma$.

Notice that the actions of Γ and φ commute with each other. This fact is very crucial for the following definition.

Definition 3.3. A (φ,Γ)-*module over* $\mathcal{R}_E$ is a free $\mathcal{R}_E$-module $\mathbb{D}$ of finite rank d equipped with commuting semilinear actions of Γ and φ and such that $\mathcal{R}_E\varphi(\mathbb{D}) = \mathbb{D}$. In other words, if $e_1, \ldots, e_d$ is an $\mathcal{R}_E$-basis of $\mathbb{D}$, then $\varphi(e_1), \ldots, \varphi(e_d)$ is also an $\mathcal{R}_E$-basis of $\mathbb{D}$.

The (φ,Γ)-modules over $\mathcal{R}_E$ form a category in the obvious way: if $\mathbb{D}_1$ and $\mathbb{D}_2$ are two such modules, we define a homomorphism

$$\mathbb{D}_1 \to \mathbb{D}_2$$

as a $\mathcal{R}_E$-linear map commuting with the actions of φ and Γ. We denote by $\mathbf{M}_{\mathcal{R}_E}^{\varphi,\Gamma}$ this category; it is obviously additive, but it is not abelian since the quotient of two (φ,Γ)-modules is not necessarily free (consider, for example, the inclusion map $t\mathcal{R}_E \hookrightarrow \mathcal{R}_E$).

The category $\mathbf{M}_{\mathcal{R}_E}^{\varphi,\Gamma}$ contains an important subcategory, namely the subcategory of “étale” modules. In order to define these modules, we need Kedlaya’s theory of “slopes”.

Let $\mathbb{D}$ be a φ-module over $\mathcal{R}_E$, i.e., a free $\mathcal{R}_E$-module of finite rank d with a semilinear action of φ such that $\mathcal{R}_E\varphi(\mathbb{D}) = \mathbb{D}$. In [Ked04], Kedlaya associates to $\mathbb{D}$ a sequence of rational numbers $s_1 \leq \cdots \leq s_d$, called the *slopes* of $\mathbb{D}$. The φ-module $\mathbb{D}$ is said *étale* if $s_1 = \cdots = s_d = 0$. The following definition extends this notion to an arbitrary (φ, Γ)-module.

Definition 3.4. A (φ, Γ)-module over $\mathcal{R}_E$ is called *étale* if its underlying φ-module (forgetting the action of Γ) is.

The subcategory of étale modules is denoted by $\mathbf{M}_{\mathcal{R}_E}^{\varphi,\Gamma,\text{ét}}$.

Definition 3.5. A *p-adic representation of $G_{\mathbb{Q}_p}$ with coefficients in E* is a finite-dimensional E-vector space equipped with a continuous E-linear action of $G_{\mathbb{Q}_p}$. Let $\mathbf{Rep}_E(G_{\mathbb{Q}_p})$ denote the category of such representations.

The following theorem (which combines work of Fontaine, Cherbonnier-Colmez and Kedlaya) is the reason that (φ, Γ)-modules are important in the study of p-adic Galois representations.

Theorem 3.6. *There is a fully faithful functor*

$$\mathbb{D}^{\dagger}_{rig,E} : \mathbf{Rep}_E(G_{\mathbb{Q}_p}) \to \mathbf{M}_{\mathcal{R}_E}^{\varphi,\Gamma}$$

whose essential image is $\mathbf{M}_{\mathcal{R}_E}^{\varphi,\Gamma,\acute{e}t}$.

Proof. See Theorem 3.4.3 of [Fon90], Corollary III.5.2 of [CC98] and Theorem 6.3.3 of [Ked05]. □

3.3 Families of (φ, Γ)-modules

In Section 3.2, we have considered the p-adic representations over vector spaces. But the families of p-adic representations emerging from number theory are usually over rigid analytic spaces. Therefore, in this section, we review the theory of (φ, Γ)-modules in families and its relationship to families of p-adic representations over affinoid algebras. We keep the notations in the previous section.

The rings $\mathcal{R}_E^{(r)}$ are equipped with a canonical Fréchet topology, see [Ber02]. Let A be an affinoid E-algebra and set

$$\mathcal{R}_A^{(r)} = \mathcal{R}_E^{(r)} \widehat{\otimes}_E A, \qquad \mathcal{R}_A = \bigcup_{0 \leq r < 1} \mathcal{R}_A^{(r)}.$$

The actions of φ and Γ on $\mathcal{R}_E$ extend to $\mathcal{R}_A$ by linearity.

We want to define (φ, Γ)-modules over $\mathcal{R}_A$. Notice that $\mathcal{R}_A^{(r)}$ is stable under the action of Γ; on the other hand that $\varphi(\mathcal{R}_A^{(r)}) \not\subset \mathcal{R}_A^{(r)}$, but instead $\varphi(\mathcal{R}_A^{(r)}) \subset \mathcal{R}_A^{(r^{1/p})}$. With this in mind, in order to obtain a reasonable theory, we define first (φ, Γ)-modules over $\mathcal{R}_A^{(r)}$. (φ, Γ)-modules over $\mathcal{R}_A$ are defined by extension of coefficients from $\mathcal{R}_A^{(r)}$ to $\mathcal{R}_A$.

Definition 3.7.

1. A (φ, Γ)-*module over* $\mathcal{R}_A^{(r)}$ is a finitely generated projective $\mathcal{R}_A^{(r)}$-module $\mathbb{D}^{(r)}$ with the following structures:

    (a) A φ-semilinear map

    $$\varphi : \mathbb{D}^{(r)} \to \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A^{(r^{1/p})}$$

    such that the induced linear map

    $$\varphi^* : \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}, \varphi} \mathcal{R}_A^{(r^{1/p})} \to \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A^{(r^{1/p})}$$

    is an isomorphism of $\mathcal{R}_A^{(r^{1/p})}$-modules,

    (b) A semilinear continuous action of Γ on $\mathbb{D}^{(r)}$ commuting with φ.

2. We say that $\mathbb{D}$ is a (φ, Γ)-*module over* $\mathcal{R}_A$ if

    $$\mathbb{D} = \mathbb{D}^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A$$

    for some (φ, Γ)-module $\mathbb{D}^{(r)}$ over $\mathcal{R}_A^{(r)}$.

Let $\mathbf{M}_{\mathcal{R}_A}^{\varphi,\Gamma}$ denote the category of (φ, Γ)-modules over $\mathcal{R}_A$.

The following proposition shows that if $A = E$, then this definition is compatible with Definition 3.4.

Proposition 3.8. *Let* $\mathbb{D}$ *be a* (φ, Γ)*-module over* $\mathcal{R}_E$*. There exists* $0 \leq r(\mathbb{D}) < 1$ *such that for any* r *with* $r(\mathbb{D}) \leq r < 1$*, there exists a unique free* $\mathcal{R}_E^{(r)}$*-submodule* $\mathbb{D}^{(r)}$ *of* $\mathbb{D}$ *satisfying the following properties:*

1. $\mathbb{D}^{(s)} = \mathbb{D}^{(r)} \otimes_{\mathcal{R}_E^{(r)}} \mathcal{R}_E^{(s)}, \quad r \leq s < 1.$

2. $\mathbb{D} = \mathbb{D}^{(r)} \otimes_{\mathcal{R}_E^{(r)}} \mathcal{R}_E.$

3. *The Frobenius* φ *induces isomorphisms*

$$\varphi^* : \mathbb{D}^{(r)} \otimes_{\mathcal{R}_E^{(r)}, \varphi} \mathcal{R}_E^{(r^{1/p})} \to \mathbb{D}^{(r)} \otimes_{\mathcal{R}_E^{(r)}} \mathcal{R}_E^{(r^{1/p})}, \quad r(\mathbb{D}) \leq r < 1.$$

Proof. This is Théoréme 1.3.3 of [Ber08]. □

Definition 3.9. A *p-adic representation of* $G_{\mathbb{Q}_p}$ *with coefficients in* A is a finitely generated projective A-module of finite rank equipped with an A-linear continuous action of $G_{\mathbb{Q}_p}$. Let $\mathbf{Rep}_A(G_{\mathbb{Q}_p})$ denote the category of such representations.

Remark 3.10. As A is a Noetherian ring (see Corollary B.4), a finitely generated A-module is projective if and only if it is flat.

The construction of the functor given in Theorem 3.6 can be directly generalized to families. More precisely,

Theorem 3.11. *There exists a fully faithul functor*

$$\mathbb{D}^{\dagger}_{rig,A} : \mathbf{Rep}_A(G_{\mathbb{Q}_p}) \to \mathbf{M}^{\varphi,\Gamma}_{\mathcal{R}_A}$$

which commutes with base change. More precisely, let $X = Max(A)$*. For each* $x \in X$*, set* $E_x = A/x$ *which is a finite extension of* E*. Then the diagram*

$$\begin{tikzcd}
\mathbf{Rep}_A(G_{\mathbb{Q}_p}) \arrow[r, "\mathbb{D}^{\dagger}_{rig,A}"] \arrow[d, "\otimes_A E_x"] & \mathbf{M}^{\varphi,\Gamma}_{\mathcal{R}_A} \arrow[d, "\otimes_A E_x"] \\
\mathbf{Rep}_{E_x}(G_{\mathbb{Q}_p}) \arrow[r, "\mathbb{D}^{\dagger}_{rig,E_x}"] & \mathbf{M}^{\varphi,\Gamma}_{\mathcal{R}_{E_x}}
\end{tikzcd}$$

commutes, i.e., for a p-adic representation V of $G_{\mathbb{Q}_p}$ with coefficients in A, we have the following isomorphism:

$$\mathbb{D}^{\dagger}_{rig,A}(V) \otimes_A E_x \cong \mathbb{D}^{\dagger}_{rig,E_x}(V \otimes_A E_x).$$

Proof. This holds thanks to the main results of [BC08] and [KL10]. □

Remark 3.12. Note that, in contrast with the classical theory, the essential image of $\mathbb{D}^{\dagger}_{\mathrm{rig},A}$ does not coincide with the subcategory of étale modules generally. For further discussion, see [BC08] and [KPX14].

For each continuous character $\delta : \mathbb{Q}_p^{\times} \to A^{\times}$, we denote by $\mathcal{R}_A(\delta) = \mathcal{R}_A \cdot e_\delta$ the (φ, Γ)-module of rank 1 having e_δ as a basis such that

$$\begin{aligned} \varphi(e_\delta) &= \delta(p)e_\delta, \\ \tau(e_\delta) &= \delta(\chi_p(\tau))e_\delta, \quad \tau \in \Gamma. \end{aligned}$$

The following proposition shows that, if $A = E$ then every (φ, Γ)-module of rank 1 is of “character type”.

Proposition 3.13. *Every (φ, Γ)-module over $\mathcal{R}_E$ of rank 1 is isomorphic to $\mathcal{R}_E(\delta)$ for a unique $\delta : \mathbb{Q}_p^{\times} \to E^{\times}$.*

Proof. This is Proposition 3.1 of [Col08]. □

3.4 Cyclotomic Deformations

Let A be an affinoid E-algebra and set $\mathcal{H}_A = \mathcal{H}\widehat{\otimes}_E A$. Assume that we have a Galois representation $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_d(A)$ with underlying free A-module V. In this section, we shall define a Galois representation

$$\overline{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_d(\mathcal{H}_A)$$

following [Pot12] and [KPX14].

The $\mathcal{O}_E$-linear involution $\iota : \mathcal{O}_E[[\Gamma_0]] \to \mathcal{O}_E[[\Gamma_0]]$ obtained by sending γ to γ^{-1} extends to $\mathcal{H}_A$. Let $\mathcal{H}_A^\iota$ denote $\mathcal{H}_A$ equipped with the $\mathcal{H}_A$-module structure given by

$$\alpha \cdot \lambda = \iota(\alpha)\lambda, \quad \alpha, \lambda \in \mathcal{H}_A.$$

We write $V \otimes_A \mathcal{H}_A^\iota$ for $V \widehat{\otimes}_A \mathcal{H}_A$ equipped with the diagonal action of $G_{\mathbb{Q},S}$ and the $\mathcal{H}_A$-module structure given by

$$\alpha \cdot (v \otimes \lambda) = v \otimes \iota(\alpha)\lambda, \qquad \alpha \in \mathcal{H}_A, \quad v \otimes \lambda \in V \widehat{\otimes}_A \mathcal{H}_A.$$

We set

$$\overline{V} = V \otimes_A \mathcal{H}_A^\iota.$$

It is called the *cyclotomic deformation* of V. The action of $G_{\mathbb{Q},S}$ on $\overline{V}$ gives rise to the desired Galois representation $\overline{\rho}$. More precisely, we have a Galois representation

$$\overline{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_d(\mathcal{H}_A)$$

whose underlying free $\mathcal{H}_A^\iota$-module of rank d is $\overline{V}$.

Similarly, for any (φ, Γ)-module $\mathbb{D}$ over $\mathcal{R}_A$, we define its *cyclotomic deformation* to be

$$\overline{\mathbb{D}} = \mathbb{D} \otimes_{\mathcal{R}_A} \mathbb{D}^\dagger_{\mathrm{rig},A}(\mathcal{H}_A^\iota).$$

In particular, for a p-adic representation V with coefficients in A, we have the following equality:

$$\mathbb{D}^\dagger_{\mathrm{rig},A}(\overline{V}) = \overline{\mathbb{D}^\dagger_{\mathrm{rig},A}(V)}. \tag{3.2}$$

3.5 A Deformed Galois Decomposition

Let us continue from where we left off. In this section, we obtain a decomposition of Galois representations, similar to the one given in (2.2).

Let $\chi : G_{\mathbb{Q},S} \to \mathcal{O}_E$ be a Dirichlet character. Applying the construction given in Section 3.4, we may consider the following Galois representations:

- Let $\rho_4 : G_{\mathbb{Q},S} \to \mathrm{GL}_4(\mathcal{H}_B)$ be the 4-dimensional Galois representation given by $\overline{\rho_{\mathbf{f},\mathbf{f}}(\chi)}$.
- Let $\rho_3 : G_{\mathbb{Q},S} \to \mathrm{GL}_3(\mathcal{H}_A)$ be the 3-dimensional Galois representation given by $\overline{\mathrm{Ad}^0(\rho_{\mathbf{f}})(\chi)}$.
- Let $\rho_1 : G_{\mathbb{Q},S} \to \mathrm{GL}_1(\mathcal{H}_A)$ be the 1-dimensional Galois representation given by $\overline{\mathbf{1}(\chi)}$.

The map $\pi_{\mathbf{f},\mathbf{f}}$ induces a surjective homomorphism

$$\pi : \mathcal{H}_B \to \mathcal{H}_A.$$

The isomorphism (3.1) gives us the following decomposition of Galois representations:

$$\pi \circ \rho_4 \cong \rho_3 \oplus \rho_1. \tag{3.3}$$

We want to associate Selmer groups to the Galois representations appeared in decomposition (3.3). Because of the irreducibility issue explained at the end of Section 3.1, the method explained in Section 2.5 does not work here. The following two sections explain a more general method.

3.6 Construction of Selmer Complexes

In this section, we review the construction of Selmer complexes. This notion was introduced by Nekovář in [Nek06].

Let A be an affinoid E-algebra and suppose that we have a Galois representation

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_d(A)$$

where S is a finite set of primes in $\mathbb{Q}$ containing p and ∞. Say V for the free A-module of rank d on which $G_{\mathbb{Q},S}$ acts to let us obtain ρ.

A *local condition* at $\ell \in S$ is a morphism of complexes

$$i_\ell : U_\ell^\bullet(V) \to C^\bullet(G_{\mathbb{Q}_\ell}, V).$$

For each collection $U^\bullet(V) = (U_\ell^\bullet(V), i_\ell)_{\ell \in S}$ of local conditions, consider the following diagram

$$\begin{tikzcd}
C^\bullet(G_{\mathbb{Q},S}, V) \arrow[r, "\mathrm{res}_S"] & \bigoplus\limits_{\ell \in S} C^\bullet(G_{\mathbb{Q}_\ell}, V) \\
 & \bigoplus\limits_{\ell \in S} U_\ell^\bullet(V) \arrow[u, "i_S"']
\end{tikzcd}$$

where $i_S = (i_\ell)_{\ell \in S}$ and res_S denotes the restriction maps.

Definition 3.14. The *Selmer complex* associated to these data is defined as

$$S^\bullet(V, U^\bullet(V)) = \mathrm{cone}\left[C^\bullet(G_{\mathbb{Q},S}, V) \oplus \left(\bigoplus_{\ell \in S} U_\ell^\bullet(V)\right) \xrightarrow{\mathrm{res}_S - i_S} \bigoplus_{\ell \in S} C^\bullet(G_{\mathbb{Q}_\ell}, V)\right][-1].$$

More precisely, each element of $S^i(V, U^\bullet(V))$ may be written as a triple

$$\big(x, (x_\ell)_{\ell \in S}, (\lambda_\ell)_{\ell \in S}\big)$$

where

$$x \in C^i(G_{\mathbb{Q},S}, V), \quad (x_\ell)_{\ell \in S} \in \bigoplus_{\ell \in S} U_\ell^i(V), \quad (\lambda_\ell)_{\ell \in S} \in \bigoplus_{\ell \in S} C^{i-1}(G_{\mathbb{Q}_\ell}, V)$$

and the differential $d^i : S^i(V, U^\bullet(V)) \to S^{i+1}(V, U^\bullet(V))$ is given by

$$d^i\big(x, (x_\ell)_{\ell \in S}, (\lambda_\ell)_{\ell \in S}\big) = \big(d(x), d((x_\ell)_{\ell \in S}), -\mathrm{res}_S(x) + i_S((x_\ell)_{\ell \in S}) - d((\lambda_\ell)_{\ell \in S})\big).$$

We denote by $\mathbf{R}\Gamma(V, U^\bullet(V))$ the class of $S^\bullet(V, U^\bullet(V))$ in the derived category of A-modules and define

$$H^i(V, U^\bullet(V)) = \mathbf{R}^i\Gamma(V, U^\bullet(V)).$$

It turns out, see for example Chapter 3 of [Ben14], that these cohomology groups are finitely generated A-modules.

Remark 3.15. Let us briefly mention the relationship of such a construction to the objects defined in Chapter 2. When V satisfies the Panchishkin condition, the natural local condition at p in this context is the derived version of Palvannan's local condition at p and therefore the first cohomology group given above is closely related to the Selmer group defined by Palvannan. On the other hand, the construction of Selmer complexes has many advantages over the traditional constructions. See [Nek06] for further details.

3.7 An Example of Local Conditions

We keep the assumptions and notations in the previous section. In this section, we define a collection of local conditions for non-ordinary Galois representations following [Ben14], [BB15] and [Pot13].

Recall the Galois representation $\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_d(A)$. We denote by $\mathbb{D}^{\dagger}_{\mathrm{rig},A}(V)$ the (φ,Γ)-module over $\mathcal{R}_A$ associated to the restriction of V to $G_{\mathbb{Q}_p}$. Suppose the following hypothesis is satisfied:

SUB There exists a (φ,Γ)-submodule $\mathbb{D}$ of $\mathbb{D}^{\dagger}_{\mathrm{rig},A}(V)$ such that $\mathbb{D}$ is a $\mathcal{R}_A$-direct summand of $\mathbb{D}^{\dagger}_{\mathrm{rig},A}(V)$.

Local conditions at $\ell \neq p$: For each $\ell \in S \setminus \{p\}$, we set

$$U^{\bullet}_{\ell}(V) = \left[V^{I_\ell} \xrightarrow{\mathrm{Fr}_\ell - 1} V^{I_\ell} \right]$$

where the terms are places in degree 0 and 1. Here Fr_ℓ denotes the geometric Frobenius. One can easily check that,

$$H^0(U^{\bullet}_{\ell}(V)) = H^0(G_{\mathbb{Q}_\ell}, V), \quad H^1(U^{\bullet}_{\ell}(V)) = \ker\bigl(H^1(G_{\mathbb{Q}_\ell}, V) \to H^1(I_\ell, V)\bigr).$$

Define

$$i_\ell : U^{\bullet}_{\ell}(V) \to C^{\bullet}(G_{\mathbb{Q}_\ell}, V)$$

by

$$\begin{aligned} i_\ell(x) &= x && \text{in degree } 0, \\ (i_\ell(x))(\mathrm{Fr}_\ell) &= x && \text{in degree } 1. \end{aligned}$$

These maps are the local conditions away from p.

Local condition at p: This is more tricky to define.

For any (φ, Γ)-module $\mathbb{D}$ over $\mathcal{R}_A$, define

$$C^\bullet_\gamma(\mathbb{D}) : \mathbb{D}^\Delta \xrightarrow{\gamma-1} \mathbb{D}^\Delta$$

where the terms are places in degree 0 and 1. Consider the Fontaine-Herr complex

$$C^\bullet_{\varphi,\gamma}(\mathbb{D}) = \mathrm{Tot}\Big(C^\bullet_\gamma(\mathbb{D}) \xrightarrow{\varphi-1} C^\bullet_\gamma(\mathbb{D})\Big).$$

More precisely,

$$C^\bullet_{\varphi,\gamma}(\mathbb{D}) : \mathbb{D}^\Delta \xrightarrow{d_0} \mathbb{D}^\Delta \oplus \mathbb{D}^\Delta \xrightarrow{d_1} \mathbb{D}^\Delta$$

where $d_0(x) = ((\varphi-1)x, (\gamma-1)x)$ and $d_1(y,z) = (\gamma-1)y - (\varphi-1)z$.

In [Ber02], Berger constructed the ring of p-adic periods $\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}}$ which is the completion of a certain ring with respect to Fréchet topology. Put

$$\widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},A} = \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} A, \quad \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},A} = \bigcup_{r>0} \widetilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},A}.$$

It turns out (see, for example, Section 2.4 of [Ben14]) that we have the following exact sequence:

$$0 \to A \to \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},A} \xrightarrow{\varphi-1} \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},A} \to 0. \tag{3.4}$$

Set $V^\dagger_{\mathrm{rig}} = V \otimes_A \widetilde{\mathbb{B}}^{\dagger}_{\mathrm{rig},A}$ and consider the complex $C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}})$. Then (3.4) induces the following exact sequence:

$$0 \to C^\bullet(G_{\mathbb{Q}_p}, V) \to C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \xrightarrow{\varphi-1} C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \to 0.$$

Define

$$K^\bullet_p(V) = \mathrm{Tot}\left(C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \xrightarrow{\varphi-1} C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}})\right).$$

We have a morphism of complexes

$$\alpha : C^\bullet_\gamma(\mathbb{D}^\dagger_{\mathrm{rig},A}(V)) \to C^\bullet(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}})$$

defined by

$$\begin{aligned} \alpha(x) &= x, & x &\in C^0_\gamma(\mathbb{D}_{\mathrm{rig},A}(V)), \\ (\alpha(x))(g) &= \frac{g-1}{\gamma-1}(x), & x &\in C^1_\gamma(\mathbb{D}_{\mathrm{rig},A}(V)), \quad g \in G_{\mathbb{Q}_p}, \end{aligned}$$

which induces the following morphism (which we denote again by α):

$$\alpha : C^\bullet_{\varphi,\gamma}(\mathbb{D}^\dagger_{\mathrm{rig},A}(V)) \to K^\bullet_p(V).$$

On the other hand, The canonical map $V \to V^\dagger_{\mathrm{rig}}$ induces a morphism

$$\xi : C^\bullet(G_{\mathbb{Q}_p}, V) \to K^\bullet_p(V)$$

given by

$$x \mapsto (0, x) \in C^{n-1}(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}) \oplus C^n(G_{\mathbb{Q}_p}, V^\dagger_{\mathrm{rig}}), \quad x \in C^n(G_{\mathbb{Q}_p}, V).$$

Proposition 3.16. *The maps α and ξ are quasi-isomorphisms and we have the following diagram:*

$$\begin{tikzcd} C^\bullet(G_{\mathbb{Q}_p}, V) \arrow[r, "\xi", "\cong"'] & K^\bullet_p(V) \\ & C^\bullet_{\varphi,\gamma}(\mathbb{D}^\dagger_{rig,A}(V)). \arrow[u, "\cong", "\alpha"'] \end{tikzcd}$$

Proof. See Proposition 9 in [Ben15]. □

Finally, we are ready to define the local condition at p. Set $U^\bullet_p(V, \mathbb{D}) = C^\bullet_{\varphi,\gamma}(\mathbb{D})$. Composing the quasi-isomorphism $\alpha : C^\bullet_{\varphi,\gamma}(\mathbb{D}^\dagger_{\mathrm{rig},A}(V)) \to K^\bullet_p(V)$ with the canonical morphism $U^\bullet_p(V, \mathbb{D}) \to C^\bullet_{\varphi,\gamma}(\mathbb{D}^\dagger_{\mathrm{rig},A}(V))$, we obtain a map

$$i_p : U^\bullet_p(V, \mathbb{D}) \to K^\bullet_p(V)$$

which is the desired local condition.

To uniformize notation, we write $K_\ell^\bullet(V)$ for $C^\bullet(G_{\mathbb{Q}_\ell}, V)$ and $U_\ell(V, \mathbb{D})$ for $U_\ell(V)$ if $\ell \in S \setminus \{p\}$. Set

$$K_S^\bullet(V) = \bigoplus_{\ell \in S} K_\ell(V), \qquad U_S^\bullet(V, \mathbb{D}) = \bigoplus_{\ell \in S} U_\ell(V, \mathbb{D}).$$

Then we have the diagram

$$\begin{tikzcd} C^\bullet(G_{\mathbb{Q},S}, V) \arrow[r, "\mathrm{res}_S"] & K_S^\bullet(V) \\ & U_S^\bullet(V, \mathbb{D}) \arrow[u, "i_S"] \end{tikzcd}$$

where $i_S = (i_\ell)_{\ell \in S}$ and res_S denotes the restriction maps.

Consider the corresponding Selmer complex. Since this is the only collection of local conditions that we use in this thesis, we denote that Selmer complex by $S^\bullet(V, \mathbb{D})$. Similarly, we denote by $\mathbf{R}\Gamma(V, \mathbb{D})$ the class of $S^\bullet(V, \mathbb{D})$ in the derived category and put

$$H^i(V, \mathbb{D}) = \mathbf{R}^i\Gamma(V, \mathbb{D}).$$

We call $H^1(V, \mathbb{D})$ the *extended Selmer group.*

3.8 Extended Selmer Groups for Coleman Deformation

We continue from the end of Section 3.5. In this section, we associate extended Selmer groups to the Galois representations appeared in the decomposition given in (3.3) by copying ideas of Palvannan.

We have the Galois representation $\rho_{\mathbf{f}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(A)$ with underlying free A-module $V_{\mathbf{f}}$ of rank 2. Denote by $\mathbb{D}^\dagger_{\mathrm{rig},A}(V_{\mathbf{f}})$ the (φ, Γ)-module over $\mathcal{R}_A$ associated to the restriction of $V_{\mathbf{f}}$ to $G_{\mathbb{Q}_p}$.

Recall that in order to define Selmer complex, we need a (φ, Γ)-submodule $\mathbb{D}$ of $\mathbb{D}^\dagger_{\mathrm{rig},A}(V_{\mathbf{f}})$ that is an $\mathcal{R}_A$-direct summand of $\mathbb{D}^\dagger_{\mathrm{rig},A}(V_{\mathbf{f}})$. With this in mind, the following proposition gives us a “triangulation” of the (φ, Γ)-module $\mathbb{D}^\dagger_{\mathrm{rig},A}(V_{\mathbf{f}})$:

Proposition 3.17. *The* (φ,Γ)*-module* $\mathbb{D}^{\dagger}_{rig,A}(V_{\mathbf{f}})$ *sits in an exact sequence*

$$0 \to \mathbb{D}_{\mathbf{f}} \to \mathbb{D}^{\dagger}_{rig,A}(V_{\mathbf{f}}) \to \frac{\mathbb{D}^{\dagger}_{rig,A}(V_{\mathbf{f}})}{\mathbb{D}_{\mathbf{f}}} \to 0 \qquad (\text{Sub-}\rho_{\mathbf{f}})$$

where $\mathbb{D}_{\mathbf{f}}$ *and* $\frac{\mathbb{D}^{\dagger}_{rig,A}(V_{\mathbf{f}})}{\mathbb{D}_{\mathbf{f}}}$ *are* (φ,Γ)*-modules of rank* 1*.*

Proof. This follows from the main theorem of [Liu15]. See also Chapter 2 and Chapter 4 of [BC09]. □

Consider the following submodules:

- Let $V_{\pi\circ\rho_4}$ be free $\mathcal{H}^{\iota}_{A}$-module of rank 4 on which $G_{\mathbb{Q},S}$ acts to obtain $\pi\circ\rho_4$. More explicitly,

  $$V_{\pi\circ\rho_4} = \overline{\mathrm{Hom}_A(V_{\mathbf{f}}, V_{\mathbf{f}})(\chi)}.$$

  Then we have

  $$\mathbb{D}^{\dagger}_{\mathrm{rig},A}(V_{\pi\circ\rho_4}) = \overline{\mathrm{Hom}_{\mathcal{R}_A}(\mathbb{D}^{\dagger}_{\mathrm{rig},A}(V_{\mathbf{f}}), \mathbb{D}^{\dagger}_{\mathrm{rig},A}(V_{\mathbf{f}}))(\chi)}$$

  by the equality given in (3.2). We set

  $$\mathbb{D}_{\pi\circ\rho_4} = \overline{\mathrm{Hom}_{\mathcal{R}_A}(\mathbb{D}^{\dagger}_{\mathrm{rig},A}(V_{\mathbf{f}}), \mathbb{D}_{\mathbf{f}})(\chi)}$$

  which gives the following exact sequence:

  $$0 \to \mathbb{D}_{\pi\circ\rho_4} \to \mathbb{D}^{\dagger}_{\mathrm{rig},A}(V_{\pi\circ\rho_4}) \to \frac{\mathbb{D}^{\dagger}_{\mathrm{rig},A}(V_{\pi\circ\rho_4})}{\mathbb{D}_{\pi\circ\rho_4}} \to 0. \qquad (\text{Sub-}\pi\circ\rho_4)$$

- Let V_{ρ_3} be the free $\mathcal{H}^{\iota}_{A}$-module on which $G_{\mathbb{Q},S}$ acts to let us obtain ρ_3. Define $\mathrm{Ad}^0(V_{\mathbf{f}})$ as the free A-module of rank 3 on which $G_{\mathbb{Q},S}$ acts to let us obtain $\mathrm{Ad}^0(\rho_{\mathbf{f}}) : G_{\mathbb{Q},S} \to \mathrm{GL}_3(A)$. Consider the following free $\mathcal{R}_A$-modules of $\mathbb{D}^{\dagger}_{\mathrm{rig},A}(\mathrm{Ad}^0(V_{\mathbf{f}}))$:

  $$0 \subsetneq \underbrace{\mathbb{D}^{\mathrm{even}}\mathrm{Ad}^0(V_{\mathbf{f}})}_{\text{rank }1} \subsetneq \underbrace{\mathbb{D}^{\mathrm{odd}}\mathrm{Ad}^0(V_{\mathbf{f}})}_{\text{rank }2} \subsetneq \mathbb{D}^{\dagger}_{\mathrm{rig},A}(\mathrm{Ad}^0(V_{\mathbf{f}}))$$

where

$$\mathbb{D}^{\text{even}}\text{Ad}^0(V_{\mathbf{f}}) = \text{Hom}_{\mathcal{R}_A}\left(\frac{\mathbb{D}^{\dagger}_{\text{rig},A}(V_{\mathbf{f}})}{\mathbb{D}_{\mathbf{f}}}, \mathbb{D}_{\mathbf{f}}\right),$$

$$\mathbb{D}^{\text{odd}}\text{Ad}^0(V_{\mathbf{f}}) = \ker\left(\text{Hom}_{\mathcal{R}_A}(\mathbb{D}^{\dagger}_{\text{rig},A}(V_{\mathbf{f}}), \mathbb{D}^{\dagger}_{\text{rig},A}(V_{\mathbf{f}})) \xrightarrow{\text{Res}} \text{Hom}_{\mathcal{R}_A}\left(\mathbb{D}_{\mathbf{f}}, \frac{\mathbb{D}^{\dagger}_{\text{rig},A}(V_{\mathbf{f}})}{\mathbb{D}_{\mathbf{f}}}\right)\right)$$
$$\cap\, \mathbb{D}^{\dagger}_{\text{rig},A}(\text{Ad}^0(V_{\mathbf{f}})).$$

Set

$$\mathbb{D}_{\rho_3} = \begin{cases} \overline{\mathbb{D}^{\text{even}}\text{Ad}^0(V_{\mathbf{f}})(\chi)}, & \text{if } \chi \text{ is even} \\ \overline{\mathbb{D}^{\text{odd}}\text{Ad}^0(V_{\mathbf{f}})(\chi)}, & \text{if } \chi \text{ is odd.} \end{cases}$$

Then we have the following exact sequence (which depends on the parity of χ):

$$0 \to \mathbb{D}_{\rho_3} \to \mathbb{D}^{\dagger}_{\text{rig},A}(V_{\rho_3}) \to \frac{\mathbb{D}^{\dagger}_{\text{rig},A}(V_{\rho_3})}{\mathbb{D}_{\rho_3}} \to 0. \tag{Sub-ρ_3}$$

- Let V_{ρ_1} be free $\mathcal{H}^{\iota}_A$-module of rank 1 on which $G_{\mathbb{Q},S}$ acts to obtain ρ_1. More explicitly, $V_{\rho_1} = \mathcal{H}^{\iota}_A(\chi)$. Set

$$\mathbb{D}_{\rho_1} = \begin{cases} \mathbb{D}^{\dagger}_{\text{rig},A}(V_{\rho_1}), & \text{if } \chi \text{ is even} \\ 0, & \text{if } \chi \text{ is odd.} \end{cases}$$

Then we have the following exact sequence (which depends on the parity of χ):

$$0 \to \mathbb{D}_{\rho_1} \to \mathbb{D}^{\dagger}_{\text{rig},A}(V_{\rho_1}) \to \frac{\mathbb{D}^{\dagger}_{\text{rig},A}(V_{\rho_1})}{\mathbb{D}_{\rho_1}} \to 0. \tag{Sub-ρ_1}$$

Similar to (3.3), we have the following isomorphism of $\mathcal{H}^{\iota}_A$-modules that is $G_{\mathbb{Q},S}$-equivariant:

$$V_{\pi\circ\rho_4} \cong V_{\rho_3} \oplus V_{\rho_1}$$

which gives the following isomorphism of $\mathbb{D}^{\dagger}_{\text{rig},A}(\mathcal{H}^{\iota}_A)$-modules:

$$\mathbb{D}^{\dagger}_{\text{rig},A}(V_{\pi\circ\rho_4}) \cong \mathbb{D}^{\dagger}_{\text{rig},A}(V_{\rho_3}) \oplus \mathbb{D}^{\dagger}_{\text{rig},A}(V_{\rho_1}).$$

Following Sections 3.6 & 3.7 and using the exact sequences Sub-$\pi \circ \rho_4$, Sub-ρ_2 and Sub-ρ_1, we associate Selmer complexes $S^\bullet(V_{\pi\circ\rho_4}, \mathbb{D}_{\pi\circ\rho_4})$, $S^\bullet(V_{\rho_3}, \mathbb{D}_{\rho_3})$ and $S^\bullet(V_{\rho_1}, \mathbb{D}_{\rho_1})$ to $\pi \circ \rho_4$, ρ_3 and ρ_1, respectively. Consider the corresponding objects $\mathbf{R}\Gamma(V_{\pi\circ\rho_4}, \mathbb{D}_{\pi\circ\rho_4})$, $\mathbf{R}\Gamma(V_{\rho_3}, \mathbb{D}_{\rho_3})$ and $\mathbf{R}\Gamma(V_{\rho_1}, \mathbb{D}_{\rho_1})$ in the derived category which give the extended Selmer groups $H^1(V_{\pi\circ\rho_4}, \mathbb{D}_{\pi\circ\rho_4})$, $H^1(V_{\rho_3}, \mathbb{D}_{\rho_3})$ and $H^1(V_{\rho_1}, \mathbb{D}_{\rho_1})$, respectively. Recall, see Section 3.6, that these are finitely generated modules over A.

Remark 3.18. Let us make some remarks regarding a possible continuation.

1. Recall that Palvannan's factorization theorem includes non-primitive Selmer groups. Following his methods, maybe it is necessary to define the notion of *non-primitive* extended Selmer groups in the obvious sense.

2. A device similar to divisor defined in Section 2.7 is presently unknown in this context. We need such a notion in order to assert an equality containing extended Selmer groups.

Appendix 1

PANCHISHKIN HYPOTHESIS

The main goal of this appendix is to define the Panchishkin condition. But first, we need to introduce some of Fontaine's period rings. The book [FO] is an excellent reference for these topics. See also [Ber04] for a quick introduction. Below, we follow both.

A.1 p-adic Hodge Theory

In this section, we define some of the basic objects in p-adic Hodge theory which are necessary to define the Panchishkin condition. Recall the p-adic cyclotomic character $\chi_p : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ defined in Chapter 3.

Let $\mathbb{Z}_p(i) = \mathbb{Z}_p \cdot t^i$ denote the i^{th} *Tate twist* of $\mathbb{Z}_p$, which is isomorphic to $\mathbb{Z}_p$ as a $\mathbb{Z}_p$-module, and its $G_{\mathbb{Q}_p}$-action given by the i^{th}-power of the p-adic cyclotomic character:

$$g \cdot t^i = \chi_p^i(g)t^i, \quad g \in G_{\mathbb{Q}_p}.$$

It is possible to extend this notion to an arbitrary $\mathbb{Z}_p$-module as follows. Let M be a $\mathbb{Z}_p$-module. The i^{th} *Tate twist* of M is defined to be $M(i) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$. Notice that the map

$$M \to M(i), \quad m \mapsto m \otimes t^i$$

is a $\mathbb{Z}_p$-module isomorphism. Furthermore, assume that $G_{\mathbb{Q}_p}$ acts on M. Then it also acts on $M(i)$ through

$$g \cdot (m \otimes t^i) = gm \otimes gt^i = \chi_p^i(g)gm \otimes t^i, \quad m \otimes t^i \in M(i), \quad g \in G_{\mathbb{Q}_p}.$$

Recall that $\mathbb{C}_p$ is equipped with a continuous $G_{\mathbb{Q}_p}$-action.

Definition A.1. The *Hodge-Tate ring* $\mathbb{B}_{\mathrm{HT}}$ is defined as

$$\mathbb{B}_{\mathrm{HT}} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i) \cong \mathbb{C}_p[t, 1/t].$$

Let V be a p-adic representation of $G_{\mathbb{Q}_p}$, define

$$\mathbb{D}_{\mathrm{HT}}(V) = (\mathbb{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}.$$

Proposition A.2. $\mathbb{B}_{HT}^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$.

Proof. See Proposition 5.2 of [FO]. □

Thanks to Proposition A.2, $\mathbb{D}_{\mathrm{HT}}(V)$ is a $\mathbb{Q}_p$-vector space. It is always true (see Proposition 5.2 of [FO]) that $\dim_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V$. V is called *Hodge-Tate* if the equality holds. This is equivalent to saying that there is a decomposition of $G_{\mathbb{Q}_p}$-modules

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)^{k_i}$$

where $k_i \in \mathbb{Z}_{\geq 0}$.

Remark A.3. For a p-adic representation V, the $\mathbb{Q}_p$-vector space $\mathbb{D}_{\mathrm{HT}}(V)$ is actually a graded vector space since

$$\mathbb{D}_{\mathrm{HT}}(V) = \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}^i \mathbb{D}_{\mathrm{HT}}(V)$$

where $\mathrm{gr}^i\mathbb{D}_{\mathrm{HT}}(V) = (\mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$.

There is a slightly more sophisticated period ring $\mathbb{B}_{\mathrm{dR}}^+$, called the *de Rham ring.* See Chapter 5 of [FO] for its construction. Let us recall some of the properties that $\mathbb{B}_{\mathrm{dR}}^+$ satisfies:

- It has a continuous action of $G_{\mathbb{Q}_p}$.

- It is a complete discrete valuation ring whose residue field is $\mathbb{C}_p$.

Definition A.4. The *de Rham field* $\mathbb{B}_{\mathrm{dR}}$ is defined to be

$$\mathbb{B}_{\mathrm{dR}} = \mathrm{Frac}(\mathbb{B}_{\mathrm{dR}}^{+}).$$

The field $\mathbb{B}_{\mathrm{dR}}$ contains $\mathbb{Q}_p$. We endow this field with the filtration defined by

$$\mathrm{Fil}^{i}\mathbb{B}_{\mathrm{dR}} = \mathfrak{m}^{i}_{\mathbb{B}^{+}_{\mathrm{dR}}}$$

where $\mathfrak{m}^{i}_{\mathbb{B}^{+}_{\mathrm{dR}}}$ is the i^{th}-power of the maximal ideal of $\mathbb{B}^{+}_{\mathrm{dR}}$.

Let V be a p-adic representation of $G_{\mathbb{Q}_p}$, define

$$\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}.$$

Proposition A.5. $\mathbb{B}_{dR}^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$.

Proof. See Proposition 5.24 of [FO]. □

Thanks to Proposition A.5, $\mathbb{D}_{\mathrm{dR}}(V)$ is a $\mathbb{Q}_p$-vector space. It is always true that $\dim_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p}V$. We say that V is *de Rham* if the equality holds.

Remark A.6. The $\mathbb{Q}_p$-vector space $\mathbb{D}_{\mathrm{dR}}(V)$ is actually a filtered vector space with the following filtration:

$$\mathrm{Fil}^{i}\mathbb{D}_{\mathrm{dR}}(V) = (\mathrm{Fil}^{i}\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}.$$

Furthermore, this decreasing filtration is exhausted and separated. That is,

- $\mathrm{Fil}^{i}\mathbb{D}_{\mathrm{dR}}(V) = \mathbb{D}_{\mathrm{dR}}(V)$ for $i \ll 0$,
- $\mathrm{Fil}^{i}\mathbb{D}_{\mathrm{dR}}(V) = 0$ for $i \gg 0$.

Let V be a p-adic representation of $G_{\mathbb{Q}_p}$ and consider the filtration $\mathrm{Fil}\,\mathbb{D}_{\mathrm{dR}}(V) = (\mathrm{Fil}^{i}\mathbb{D}_{\mathrm{dR}}(V))_{i\in\mathbb{Z}}$. Define

$$\mathrm{gr}^{i}\mathbb{D}_{\mathrm{dR}}(V) = \mathrm{Fil}^{i}\mathbb{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{i+1}\mathbb{D}_{\mathrm{dR}}(V), \quad \mathrm{Gr}\mathbb{D}_{\mathrm{dR}}(V) = \bigoplus_{i\in\mathbb{Z}}\mathrm{gr}^{i}\mathbb{D}_{\mathrm{dR}}(V).$$

Since the filtration is exhausted and separated, we have the following equality:

$$\dim_{\mathbb{Q}_p} \mathrm{Gr}\mathbb{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{dR}}(V). \tag{A.1}$$

The following proposition gives the relation between de Rham and Hodge-Tate representations.

Proposition A.7. *Let V be a p-adic representation of $G_{\mathbb{Q}_p}$. Then V is de Rham implies that V is Hodge-Tate.*

Proof. It turns out, see Section 2.3 in Chapter 5 of [FO], that

$$\mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}} = \mathbb{B}_{\mathrm{dR}}^{+}(i)$$

which gives rise to the exact sequence

$$0 \to \mathrm{Fil}^{i+1}\mathbb{B}_{\mathrm{dR}} \to \mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}} \to \mathbb{C}_p(i) \to 0.$$

Here we have used the fact that the residue field of $\mathbb{B}_{\mathrm{dR}}^{+}$ is $\mathbb{C}_p$. Tensoring this exact sequence with V and taking the $G_{\mathbb{Q}_p}$-invariant, we get

$$0 \to \mathrm{Fil}^{i+1}\mathbb{D}_{\mathrm{dR}}(V) \to \mathrm{Fil}^i \mathbb{D}_{\mathrm{dR}}(V) \to (\mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$$

which gives

$$\mathrm{gr}^i \mathbb{D}_{\mathrm{dR}}(V) \hookrightarrow \mathrm{gr}^i \mathbb{D}_{\mathrm{HT}}(V).$$

Hence we have the inclusion

$$\mathrm{Gr}\mathbb{D}_{\mathrm{dR}}(V) \hookrightarrow \mathbb{D}_{\mathrm{HT}}(V).$$

The result follows from the equality (A.1). □

Remark A.8. We have seen that de Rham implies Hodge-Tate, but the converse does not hold in general, see Proposition 5.31 in [FO].

We are ready to define the Panchishkin condition.

A.2 Panchishkin Condition

Definition A.9. Let V be a p-adic representation of $G_{\mathbb{Q}_p}$. V satisfies *Panchishkin condition* if

(i) V is a de Rham representation.

(ii) There exists a short exact sequence of p-adic representations of $G_{\mathbb{Q}_p}$

$$0 \to V^{+} \to V \to V^{-} \to 0$$

such that $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V^{+} \cong \bigoplus_{i \geq 1} \mathbb{C}_p(i)^{k_i}$ and $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V^{-} \cong \bigoplus_{i \leq 0} \mathbb{C}_p(i)^{k_i}$ as $G_{\mathbb{Q}_p}$-modules where $k_i \in \mathbb{Z}_{\geq 0}$.

We call the exact sequence which appeared in Definition A.9, the *Panchishkin filtration* for V.

Remark A.10. If V satisfies the Panchishkin condition, then it satisfies the ordinary condition in the sense of [Gre89].

Appendix 2

RIGID ANALYTIC GEOMETRY

This appendix is devoted to the basic notions of rigid analytic geometry. This necessarily requires us to do a bit of non-archimedean functional analysis. Below, we follow the books [BGR84], [FvdP04] and [Sch02]. For a quick introduction to these topics, see [Sch98].

Let k be a field which is complete with respect to a non-archimedean absolute value denoted by $|\cdot|$. Fix an algebraic closure $\overline{k}$ of k and denote by $\widehat{\overline{k}}$ its completion. Note that the field $\widehat{\overline{k}}$ is also algebraically closed.

B.1 Non-Archimedean Functional Analysis

This section reviews the theory of non-archimedean analysis we need for the following chapters.

A *k-normed space* is k-vector space V with a map $\|\cdot\| : V \to \mathbb{R}$, so-called *norm*, satisfying the following rules:

1. For all $v \in V, \|v\| \geq 0$.
2. If $\|v\| = 0$, then $v = 0$.
3. For $a \in k$ and $v \in V, \|av\| = |a| \cdot \|v\|$.
4. For $v_1, v_2 \in V, \|v_1 + v_2\| \leq \max(\|v_1\|, \|v_2\|)$.

On the other hand, a *semi-norm* is a map $\|\cdot\| : V \to \mathbb{R}$ which has the properties $1, 2$ and 4 given above.

Notice that the norm induces a metric. A k-normed space which is complete with respect to that metric is called a *k-Banach space.*

A locally convex k-vector space (see Section 4 in Chapter 1 of [Sch02]) V is called a *k-Fréchet space* if it is metrizable and complete. As a classical result, every Banach space is a Fréchet space. The converse, however, does not hold in general.

A *k-Banach algebra* is a k-algebra A equipped with a norm $\|\cdot\|$ making it into a k-Banach space with the following properties:

1. $\|1\| \leq 1$
2. For $a_1, a_2 \in A$, $\|a_1 \cdot a_2\| \leq \|a_1\| \cdot \|a_2\|$.

Before move on, let us give a prototypical example. For simplicity, the field k is supposed to algebraically closed. Take $A = k\langle T\rangle$ which is the k-algebra consisting of power series $\sum_{n=0} a_n T^n \in k[[T]]$ such that $|a_n| \to 0$ as $n \to \infty$. The norm on this algebra is defined by

$$\left\| \sum_{n=0} a_n T^n \right\| = \max\{|a_i| : i \geq 0\}.$$

Notice that $\|1\| = 1$ and that $\|f_1 \cdot f_2\| = \|f_1\| \cdot \|f_2\|$ for any $f_1, f_2 \in A$. Then A is a Banach algebra over k. This is the simplest example of an affinoid algebra.

Let V and W be Fréchet spaces over k and consider the tensor product $V \otimes_k W$ on which we define a semi-norm $\|\cdot\|$ by the formula

$$\|x\| = \inf\Big\{ \max_{1\leq i\leq n} \|v_i\| \cdot \|w_i\| : x = \sum_{i=1}^{n} v_i \otimes w_i, v_i \in V, w_i \in W \Big\}.$$

The semi-normed space $V \otimes_k W$, in general, is not complete. Then if we complete it, we again get a Fréchet space over k, which we denote $V \widehat{\otimes}_k W$. This is called the *completed tensor product.*

B.2 Affinoid Algebras

In this section, we define and study a more special type of k-algebras, so-called affinoid algebras.

Definition B.1. For a natural number n, the *Tate algebra* $T_n = k\langle T_1, \dots, T_n\rangle$ *over* k *of dimension* n is the k-algebra consisting of power series

$$\sum_{k_i \geq 0} a_{k_1,\dots,k_n} T_1^{k_1} \dots T_n^{k_n} \in k[[T_1, \dots, T_n]]$$

satisfying $\left|a_{k_1,\dots,k_n}\right| \to 0$ as $k_1 + \cdots + k_n \to \infty$.

Remark B.2. Before move on, let us discuss the naturalness of this definition. Consider the n-dimensional poly-disk

$$\mathbb{B}^n = \{(z_1, \dots, z_n) \in \widehat{\overline{k}}^n : |z_i| \leq 1 \, \text{for} \, i = 1, \dots, n\}.$$

We need to decide the definition of “analytic” functions on this geometric object. The naive answer, follows from classical analysis, is not good enough because of the totally disconnected topology on $\mathbb{B}^n$. On the other hand, for an arbitrary power series $f = f(T_1, \dots, T_n) \in k[[T_1, \dots, T_n]]$, an easy calculation shows that in order that f converges on $\mathbb{B}^n$, it is necessary and sufficient that the coefficients of f tend to 0. With this is mind, we may say that any element of T_n is an *analytic* function on $\mathbb{B}^n$. See [Sch98] for details.

The *Gauss norm* $\|\cdot\|$ on T_n is defined to be

$$\left\| \sum_{k_i \geq 0} a_{k_1,\dots,k_n} T_1^{k_1} \dots T_n^{k_n} \right\| = \max\{\left|a_{k_1,\dots,k_n}\right| : k_1 + \cdots + k_n \geq 0\}.$$

Recall that the case $n = 1$ is discussed before. As in that case, T_n is also a Banach algebra.

Definition B.3. A k-algebra A is called *affinoid* if $A \cong T_n/I$ for some natural number n and some ideal I.

One can show that Weierstrass theory, such as preparation and division theorems, works for the Tate algebra T_n, see Theorem 3.1.1 in [FvdP04] for a more precise statement. Then we have the following corollary which presents some basic properties of T_n and general affinoid algebras that we need.

Corollary B.4. *For* $? \in \{1, 2, \emptyset\}$, *let* $A_?$ *be an affinoid algebra over* k.

1. *A is noetherian.*
2. *Let $\mathfrak{m}$ be a maximal ideal of T_n. Then the residue field $T_n/\mathfrak{m}$ is a finite extension of k.*
3. *The Gauss norm $\|\cdot\|$ on T_n induces a norm on A. With this norm, A is a Banach algebra.*
4. *For $i = 1, 2$, assume A_i is equipped with a norm $\|\cdot\|_i$ for which it is a Banach algebra. Any k-algebra homomorphism $u : A_1 \to A_2$ is continuous with respect to the given norms.*
5. *The completed tensor product $A_1 \widehat{\otimes}_k A_2$ is again an affinoid algebra over k.*

Proof. See Theorem 3.2.1 and Lemma 3.7.1 in [FvdP04]. □

Let A be an affinoid algebra over k. We associate to A the set $\mathrm{Max}(A)$ of its maximal ideals. For each point $x \in \mathrm{Max}(A)$, the field A/x is a finite extension of k by Corollary B.4.

B.3 Rigid Analytic Varieties

In classical algebraic geometry, an algebraic variety is obtained by glueing affine varieties with respect to the Zariski topology. In rigid geometry, a similar procedure occurs. Let us be more precise.

Let A be a k-affinoid algebra. To make up a geometric intuition, we write $X = \mathrm{Max}(A)$. The second property given in Corollary B.4 is an analogue of Hilbert's Nullstellensatz in classical algebraic geometry. It then gives rise to an explicit correspondence

$$\text{Galois orbits in}\,\overline{k}\text{-points of}\,\mathbb{B}^n \to \mathrm{Max}(T_n).$$

Using this description, we obtain an "canonical" topology on X which is not good enough because of the problem we mentioned in Remark B.2. On the other hand, it is possible define a certain Grothendieck topology on X, so-called a *G-topology*, which is considerably coarser than the initial. Moreover, one can define a certain sheaf of k-algebras $\mathcal{O}_X$ on X with respect for the G-topology; it is called the *structure sheaf of X*.

Definition B.5. Using the definitions and notations above, an *affinoid variety over k* is a triple

$$\mathrm{Sp}(A) = (X, G\text{-topology}, \mathcal{O}_X).$$

A *rigid analytic variety* is obtained by glueing affinoid varieties with respect to the G-topology by the usual procedure. See Chapter 9 in [BGR84] or Chapter 4 in [FvdP04] for details.

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