APRIL 2017

M.Sc. in Mathematics

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UNIVERSITY OF GAZİANTEP
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CHARACTERIZATION OF THE EIGENVALUES OF NON-NEGATIVE MATRICES

M.Sc. THESIS
IN
MATHEMATICS

BY
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APRIL 2017

Characterization Of The Eigenvalues Of Non-Negative Matrices

M.Sc. Thesis
in
Department of Mathematics
University of Gaziantep

Supervisor
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April 2017

REPUBLIC OF TURKEY
UNIVERSITY OF GAZIANTEP
GRADUATE SCHOOL OF NATURAL & APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

Name of the thesis: Characterization of the Eigenvalues of Non-Negative Matrices

Name of the student: Dana Mawlood MUHAMMED

Exam date: 6 April 2017

Approval of the Graduate School of Natural and Applied Sciences

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This is to certify that we have read this thesis and that in our consensus opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Dana Mawlood MUHAMMED

ABSTRACT

CHARACTERIZATION OF THE EIGENVALUES OF NON-NEGATIVE MATRICES

MUHAMMED, DANA MAWLOOD
M.sc. in Mathematics
Supervisors: Assoc. Prof. Dr. NECATİ OLGUN
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April 2017
33 pages

Inverse eigenvalue problems constitute an important subclass of inverse problems that arise in the context of mathematical modelling and parameter identification.

The inverse eigenvalue problem for non-negative matrices has a very simple formulation: given a list $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers, find necessary and sufficient conditions for the existence of an n-square non-negative matrix A with spectrum Λ. This problem is a very difficult one and it remains unsolved for any positive integer n.

In this work, we will reconstruct the non-negative matrices induced by; Lowey and London for $n=3$, Reams for $n=4, 5$, Laffey and Meehan for $n=5$; by using Newton's identities defined in linear algebra by Dan Kalman. Also, we use Newton's identities to construct the non-negative matrices for $n=6,7,8\ldots$

Keywords: Nonnegative Matrix, Inverse eigenvalue problem.

ÖZET

NEGATİF OLMAYAN MATRİSLERİN ÖZDEĞERLERİNİN KAREKTERİZASYONU

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Yüksek Lisans Tezi, Matematik
Danışman: Doç. Dr. NECATİ OLGUN

Yardımcı Tez Danışmanı:
Yrd. Doç. Dr. MUDHAFAR FATTAH HAMA
Nisan 2017
33 sayfa

Tersinir özdeğer problemleri, matematik modelleme ve parametre tanımlama bağlamında ortaya çıkan tersinir problemlerin önemli bir alt sınıfını oluşturmaktadır. Negatif olmayan matrisler için tersinir özdeğer problemi çok basit bir formülasyona sahiptir: kompleks sayıların bir $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ listesi verildiğinde, Λ spektrumu ile birlikte negatif olmayan bir n-kare A matrisinin varlığı için gerekli ve yeterli koşullarını bulmaktır. Bu problem çözümü zordur ve herhangi bir n pozitif tamsayısı için çözümlenmemiştir.

Bu çalışmada, Dan Kalman'ın Lineer Cebir kitabında tanımlandığı üzere Newton özdeşliklerini kullanarak, n = 3 için Lowey ve London tarafından n = 4, 5 için Reams tarafından, n = 5 için Laffey ve Meehan tarafından indirgenmiş negatif olmayan matrisleri yeniden oluşturacağız. Ayrıca n = 6,7,8 için negatif olmayan matrisleri oluşturmak için de yine Newton'un özdeşliklerini kullanacağız.

Anahtar Kelimeler: Negatif olmayan matris, Tersinir özdeğer problemi.

DEDICATION

I dedicate this thesis to my parents, family, teachers, friends, everyone who support me, Who taught me to think, understand and express, without their support will not be able to complete this thesis.

Dana Mawlood Muhammed

ACKNOWLEDGEMENTS

I'm thankful for ALLAH to give me this kind of ability to have such a degree.

I'm grateful to my supervisor (Assoc. Prof.Dr. Necati OLGUN) and co-seupervoser (Assist. Prof. Dr. Mudhafar Fattah HAMA) who for his recommendation in selecting the research proposals and his tedious work in fulfilling the research.

I would like to thank my parents, brothers, and sisters who supported me to continue my study life.

I thank privately my wife whom I call her MY Life. She is my dearest and nearest person in my life. I'm appreciative of her support of encouraging me to study more.

I'm grateful to all my dear friends.

Finally;

The following are greatly acknowledged for their help:

- Assoc. Prof. Dr. Memet Sahin
- Assoc. Prof. Dr. Kuddusi Kayaduman
- Assist. Prof. Dr. Ozge Oztekin
- Assoc. Prof. Dr. Abdullah Kablan
- Prof. Dr. Adil Kadir Jabbar

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CHAPTER 1

INTRODUCTION

Nonnegative matrices have long been a source of interesting and challenging mathematical problems. They are square real matrices with all their entries being nonnegative and prevalent in many areas of study. Statistics, Economics, Chemistry, computer sciences, and machine learning, are examples of disciplines that use nonnegative matrices.

An inverse eigenvalue problem indicates to the reconstruction of a matrix from prearranged spectral data. The spectral data might be composed of the whole or only some information of eigenvalues or eigenvectors. The target of an inverse eigenvalue problem is to construct a matrix that keeps a certain specific structure and gives spectral property. The problems associated with the inverse eigenvalue are two fundamental questions: the theoretic issue on solvability and the practical issue on computability. A significant effort in solvability has been to find out an essential or an adequate condition under which an inverse eigenvalue problem has a solution. The main concern in computability, on the other hand, has been to develop a procedure by which, the given spectral data are feasible and a matrix can be constructed numerically. Both questions are difficult and challenging. Studies on inverse eigenvalue problems have been intensive and ranging from engineering application to algebraic theorization.

A significant and familiar phenomenon in the all above applications is that the physical parameters of a definite system are to be reconstructed from information of its dynamical behaviour and above all, its ordinary frequencies and/or normal modes. If the physical parameters can be (and often are) described mathematically in the form of a matrix, then we have an inverse eigenvalue problem. In order to make the consequential model physically realizable, it should be illustrated that extra stipulations sometimes must be imposed upon the matrix.

Inverse eigenvalue problems constitute a central subclass of inverse problems which arise in the context of mathematical modelling and parameter identification.

The inverse eigenvalue problem for nonnegative matrices has a very simple formulation: given a list $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers, find necessary and sufficient conditions for the existence of an n-square nonnegative matrix A with spectrum Λ. This problem is a very difficult one and it remains unsolved for any positive integer n.

In 1937, Kolomogrov [11] posed the question: When is a given complex number z an eigenvalue of some nonnegative matrix? Kolomogrov question's have been extended by Sulaimanova in 1949 [23] to what is now called nonnegative Inverse eigenvalue problems (NIEP). If such nonnegative matrix A which has the spectrum Λ exists, then we say that A realizes Λ.

Kolmogorov's question is easily answered since there is a positive 3 × 3 circulant matrix that has the given complex number z as an eigenvalue [17] (see also [4,5,9]). On the other hand, the NIEP has challenged mathematicians for over half a century.

In recent year, many papers about eigenvalues of nonnegative or positive matrices have appeared. The geometric method of Sulaimanova [23] was essentially applied by H. Perefect [18], Ciarlet [3], and B.Kellogg [10] to obtain sufficient conditions for n real numbers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be eigenvalues of a nonnegative or positive n square matrix.

Important progresses have been done in the case of a real prescribed spectrum by Suleimanova [23], Perfect [19], Kellogg [10], Salzmann [22], Fiedler [7], Borobia [2] and others, while in the complex prescribed spectrum case the problem was solved for $n = 3$ by Loewy and London [14] and for matrices with trace 0 of order $n = 4$ and $n = 5$ by Reams [21] and Laffey and Meehan [13], respectively.

It is interesting to notice that the sufficient condition given by Laffey and Meehan [13, Theorem 3.1] also holds for matrices with positive trace in the following sense:

If $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\sum \lambda_i > 0$ is given, then we may consider $\overline{\Lambda} = (\mu_1, \mu_2, \dots, \mu_n)$, where $\mu_i = \lambda_i - \alpha$ with $\alpha = \frac{1}{n}\sum \lambda_i$. Thus, if $\overline{\Lambda}$ satisfies the sufficient condition in [13], there are a nonnegative matrix B with spectrum $\overline{\Lambda}$ and a non-negative matrix $A = B + \alpha I$ with spectrum Λ. Sufficient conditions have also been obtained for a normal nonnegative matrix of order n by Xu [25] and Radwan [20] in the complex case. A relevant contribution to the study problem is the fundamental paper by Guo Wuwen [24]. There, Guo Wuwen states the following result:

Theorem 1 (Guo Wuwen). If $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a list of complex numbers with $\Lambda = \overline{\Lambda}$, then there exist a real number λ_0,

$$\max_{2 \le j \le n} \lambda_j \le \lambda_0 \le 2n \max_{2 \le j \le n} \lambda_j, \qquad (1)$$

such that Λ is realized by a non-negative matrix A of order n if and only if $\lambda_1 \ge \lambda_{0.}$

Although a solution may exist ($\lambda_1 \ge \lambda_0$), we cannot compute it from Theorem 1. The problem of finding the real number λ_0 is a difficult one and unsolved in [24].

Many necessary conditions for an n list of numbers to be realizable are known, and there are also several known sufficient conditions. Nevertheless, for $n > 4$, the realizable n lists have not yet been fully characterized.

Reams [21] has demonstrated a matrix factorization of a companion matrix leads to a solution of the NIEP for 4×4 and 5×5 matrices of trace zero And Laffey and Meehan for $n=5$.

In this work, second chapter is about a group of definitions and theorems needed to understand NIEP. In chapter three, we will reconstruct the nonnegative matrices induced by Lowey and London [14], Laffey and Meehan [13], and Reams [21] by using Newton's identities defined in linear algebra by Dan Kalman [8]. In the forth chapter we will construct the nonnegative matrices prescribed by a list of complex eigenvales using Newton's identities and then we conclude that the number of sufficient conditions depend on the number of λ_i's. Chapter five is about conclusion and references, That is, if we increase the number of eigenvalues, the complexity of operation will be increased and the sufficient condition will be exploded.

CHAPTER 2

NON-NEGATIVE INVERSE EIGENVALUES PROBLEMS

This chapter contains all definitions and theorems needed to understand negative inverse eigenvalue problems and they will be used in the next chapters.

Definition 2.1:(Trace of Matrices)[16**]**

The trace of the square matrix A of order n, is denoted by *tr(A)*, is defined by the sum of its diagonal entries, that is, $\operatorname{tr}(\mathrm{A}) = \sum_{i=1}^{n} a_{ii}$.

Definition 2.2:(Permutation Matrix)[16**]**

A square matrix P of order n is called a permutation matrix if P can be obtained from the identity matrix of order n.

Example 2.3:

The matrix $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{4\times 4}$ is a permutation matrix of order 4.

Definition 2.4:(Reducible Matrix)[16**]**

A square matrix A of order n is said to be reducible if there exists a permutation matrix P such that

$$PAP^{T} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where B and D are square matrices and 0 is a zero matrix. The square matrix A is irreducible if it is not reducible

.

Definition 2.5:(Similar Matrix)[16]

Let A and B be two square matrices of order n. We say that A is similar to B if there is an invertible square matrix S such that $S^{-1}AS = B$.

Example 2.6:

The matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ is similar to the matrix $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ because there is a square matrix $S = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ such that $S^{-1}AS = D$.

Definition 2.7:(Eigenvalues and Eigenvectors)[16]

A scalar λ is called an eigenvalue of the square matrix A is there is a nontrivial solution x of $Ax = \lambda x$. Such an x is called an eigenvector corresponding to the eigenvalue λ.

Remark 2.8:

To find the eigenvalues of A we need a condition on λ that is equivalent to the equation $(A-\lambda I)x = 0$ having a nontrivial solution.

Definition 2.9: (Characteristic Polynomial of Matrices)[16]

Let A be a square matrix of order n. Then $det(A - \lambda I)$ is a polynomial in the variable λ of degree n. We call this polynomial the characteristic polynomial of A.

Theorem 2.10: [16**]**

A scalar λ is an eigenvalue of a square matrix A if and only if λ satisfies the characteristic equation $det(A - \lambda I) = 0$.

Example 2.11:

For $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ the characteristic polynomial is $det(A- \lambda I) =0$. That is,

$$0=\det(A-\lambda I)=\det\left(\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & 1\\ 2 & 2 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)$$

$$=\det\begin{pmatrix}1-\lambda & 2 & 2\\ 2 & 1-\lambda & 1\\ 2 & 2 & 1-\lambda\end{pmatrix}=-(\lambda-5)(\lambda+1)^2$$

Thus, eigenvalues are 5, -1, -1.

Now the eigenvector of the eigenvalue can be found as:

$$\begin{pmatrix}0\\0\\0\end{pmatrix}=\left(\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & 1\\ 2 & 2 & 1\end{pmatrix}-(-1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2 & 2 & 2\\ 2 & 2 & 1\\ 2 & 2 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}2x_1+2x_2+2x_3\\ 2x_1+2x_2+2x_3\\ 2x_1+2x_2+2x_3\end{pmatrix}$$

This implies that $x_1+x_2+x_3=0$ and we have infinite solutions

Consequently, we get that, if $x_2=1,\ \ x_3=-1$, then $x_1=-x_2-x_3=0$. Thus we have a non zero eigenvector

$$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

corresponding to the eigenvalue -1. But, since we have another eigenvalue of value -1, then we must take different value of x_i's as: $x_2=-1,\ \ x_3=1$, then

$$x_1=-x_2-x_3=0.$$

In this case, the second eigenvector corresponding the second eigenvalue -1 is

$$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\-1\\1\end{pmatrix}.$$

By the same way we can find the eigenvector corresponding to the eigenvalue 5 which is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Definition 2.12: (Spectrum of a Matrix)[16]

The set of all eigenvalues of a square matrix *A*, is denoted by $\sigma(A)$, is called the spectrum of *A*.

Definition 2.13: (Spectral Radius)[16]

The spectral radius of a square matrix *A* is the nonnegative real number $\rho(A)$ which is defined by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

In example (2.11) the spectrum of *A* is $\sigma(A) = \{-1, 5\}$ and the spectral radius $\rho(A) = 5$.

Remark 2.14:

If we have a square matrix *A* of order *n*, then finding the eigenvalues and the corresponding eigenvectors is called Eigenvalue problems.

Definition 2.15: (Nonnegative Matrix)[17]

A matrix *A* is said to be non negative if all entries are non negative. That is, $A = (a_{i,j})_{n\times m} \geq 0$ if and only if $a_{i,j} \geq 0$ for all $i = 1,\dots,n$ and $j = 1,\dots,m$.

In our thesis, we supposed that $n=m$.

Remark 2.16:

If we have a list of eigenvalues $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, then finding a square non negative matrix *A* of order *n* which has these eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, is called nonnegative inverse eigenvalue problems (NIEP).

Definition 2.17:(Companion Matrix)[16]

For each monic polynomial $p(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$, the companion matrix of $p(x)$ is defined to be

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}$$

The polynomial $p(x)$ is both the characteristic and minimum polynomial for A.

Newton's Identities:

Newton's identities relate sums of powers of roots of a polynomial with the coefficients of the polynomial. Newton's identities also have a natural expression in the context of matrix algebra, where the trace of k^{th} power of matrix is the sum of the k^{th} powers of the eigenvalues. The matrix interpretation of Newton's identities is familiar in the linear algebra literature, providing a means of computing the characteristic polynomial of a matrix in terms of the traces of the powers of the matrix. However, using the matrix setting to derive Newton's identities doesn't seem to be well known.[8]

Definition 2.18:(Elementary Symmetric Polynomial)[15]

The elementary symmetric polynomials in n variables $\lambda_1, \ldots, \lambda_n$, written $P_k(\lambda_1, \ldots, \lambda_n)$ for $k = 0, 1, \ldots, n$, are defined by

$P_0(\lambda_1, \ldots, \lambda_n) = 1$,

$P_1(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le i \le n} \lambda_i$

$P_2(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le i < k \le n} \lambda_i \lambda_k$

$P_3(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le i < k < j \le n} \lambda_i \lambda_k \lambda_j$

$P_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n$.

In general, for $k \geq 0$ we define

$$P_k(\lambda_1,\ldots,\lambda_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$$

so that $P_k(\lambda_1,\ldots,\lambda_n) = 0$ if $k > n$.

Definition 2.19: (Newton's Identities)[8]

Let $p(x) = P_0x^n + P_1x^{n-1} + \ldots + P_{n-1}x + P_n$ have roots λ_j , $j= 1, \ldots ,n$ where

$P_i= P_i(\lambda_1,\ldots,\lambda_n)$ defined as above. Define

$$S_k = S_k(\lambda_1,\cdots\lambda_n) \equiv \sum_{j=1}^{n} \lambda_j^k .$$

Then Newton's identities can be stated as

$$kP_k(\lambda_1,\cdots\lambda_n) = \sum_{i=1}^{k}(-1)^{i-1}P_{k-i}(\lambda_1,\cdots\lambda_n)S_i(\lambda_1,\cdots\lambda_n)$$

valid for all $n \geq 1$ and $k \geq 1$.

Also, one has

$$0 = \sum_{i=k-n}^{k}(-1)^{i-1}P_{k-i}(\lambda_1,\cdots\lambda_n)S_i(\lambda_1,\cdots\lambda_n)$$

for all $k > n \geq 1$.

Identifying the Coefficients of the Characteristic Polynomial in Terms of Traces

Now let A be a square matrix of order n with characteristic polynomial equal to $p(x)$. For example, A might be

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -P_n & -P_{n-1} & -P_{n-2} & \cdots & -P_1 \end{pmatrix}$$

the companion matrix of p. Then the roots of p are the eigenvalue of A, and more generally, the k^{th} powers of the roots of p are the eigenvalue of A^k.

The polynomial with roots λ_i may be expanded as

$$\prod_{i=1}^{n}(x-\lambda_i)=\sum_{k=0}^{n}(-1)^{n+k}P_{n-k}x^k$$

where the coefficients P_i's are the symmetric polynomials defined above. Given the power sums S_i of the roots. Then from Newton's identities the coefficients of the polynomial with may be expressed recursively in terms of the power sums as

$P_0 = 1,$

$P_1 = S_1,$

$$2P_2 = P_1S_1 - S_2 \rightarrow P_2 = \frac{1}{2}(P_1S_1 - S_2) = \frac{1}{2}(S_1^2 - S_2)$$

$$\begin{aligned}3P_3 = P_2S_1 - P_1S_2 + S_3 \rightarrow P_3 &= \frac{1}{3}(P_2S_1 - P_1S_2 + S_3)\\ &= \frac{1}{6}S_1^3 - \frac{1}{2}S_1S_2 + \frac{1}{3}S_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3),\end{aligned}$$

$$4P_4 = P_3S_1 - P_2S_2 + P_1S_3 - S_4 \rightarrow P_4 = \frac{1}{24}(S_1^4 - 6S_1^2S_2 + 3S_2^2 + 8S_1S_3 - 6S_4),$$

Thus the general form of P_i's can be expressed as:

$$P_n = (-1)^n \sum_{\substack{m_1+2m_2+\cdots+nm_n=n\\ m_1\ge 0, m_2\ge 0\cdots, m_n\ge 0}} \prod_{i=1}^{n}\frac{(-S_i)^{m_i}}{m_i!\,i^{m_i}}. \tag{1.1}$$

The following Theorem is crucial to the study of nonnegative matrices.

Theorem 2.20: (Guo Wuwen)[24]

If $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a list of complex numbers with $\Lambda = \overline{\Lambda}$, then there exist a real number λ_0,

$$\max_{2\le j\le n}\lambda_j \le \lambda_0 \le 2n\max_{2\le j\le n}\lambda_j, \tag{1.2}$$

such that Λ is realized by a nonnegative matrix A of order n if and only if $\lambda_1 \ge \lambda_0$.

Theorem 2.21: (Perron–Frobenius Theorem) [6]

Let *A* be a square matrix of order *n* and suppose *A* is non-negative. Define the Perron root $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. If *A* is irreducible, then

(i) $\rho(A) > 0$.

(ii) $\rho(A)$ is an eigenvalue of *A*.

(iii) there is an entry-wise positive vector *x* such that $Ax = \rho(A)x$.

(iv) $\rho(A)$ is an algebraically simple(unique eigenvalue) eigenvalue of *A*.

Let $\sigma(A) = (\lambda_1, \lambda_2, \cdots \lambda_n)$ be a list of *n* complex numbers. We have four necessary conditions for the NIEP.

1. The Perron root $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ belongs to $\sigma(A)$.
2. $\sigma(A) = \overline{\sigma(A)}$.
3. $S_k \geq 0, \quad k = 1,2,\cdots$
4. (Johnson Laffay & Loewy condition; JLL condition) is

$$S_k^m \leq n^{m-1} S_{km} \qquad for\ all \quad k, m = 1,2,\cdots$$

Remark 2.22:

For *n=4*, these necessary conditions are not sufficient, for example:

The list of eigenvalue $\sigma(A) = (\sqrt{2}, \sqrt{2}, i, -i)$ satisfies condition 1-4 given above. If $\sigma(A)$ is realized by a nonnegative matrix *A*, then *A* must be reducible by the Perron-Frobenius theorem. Hence $\{\sqrt{2}, i, -i\}$ must be realizable. But $\{\sqrt{2}, i, -i\}$ does not satisfies the JLL condition because

$$S_1^2 = (\sqrt{2} + i - i)^2 = 2 > 3^{2-1} S_{1*2} = 3*((\sqrt{2})^2 + (i)^2 + (-i)^2 = 3*(2-1-1) = 0.$$

A refinement of the JLL condition in the case of trace zero ($S_1=0$) lists of odd length comes from Laffay and Meehan.

Theorem 2.23:[12]

Let *A* be a nonnegative square matrix of order *n* with $S_1=tr(A)=0$. If *n* is odd, then $S_2^2 \leq (n-1)S_4$.

Theorem 2.24:[14]

Let $\sigma(A)=(\lambda_1,\lambda_2,\lambda_3)$ be a list of three complex numbers, and assume that σ satisfies the following conditions:

1. $\rho(A)=\max\{|\lambda|:\lambda\in\sigma(A)\}\in\sigma,$
2. $\sigma(A)=\overline{\sigma(A)},$
3. $S_1=\lambda_1+\lambda_2+\lambda_3\geq 0,$
4. $S_1^2\leq 3S_2.$

Then σ is realized by a non-negative matrix *A*.

Theorem 2.25:[21]

Let $\sigma(A)=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be a list of four complex numbers. If $S_1=0,\ S_2\geq 0,$ $S_3\geq 0,$ and $S_2^2\leq 4S_4,$ then there exists a nonnegative square matrix of order 4 that realizes σ.

Theorem 2.26:[12]

Let $\sigma(A)=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ be a list of five complex numbers. Assume $S_1=0.$ Then $\sigma(A)=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$ is the spectrum of a nonnegative square matrix of order 5 if and only if the following conditions hold:

1. $S_k\geq 0 \quad k=2,\ 3,\ 4,\ 5$
2. $S_2^2\leq 4S_4$ and
3. $12S_5-5S_2S_3+5S_3\sqrt{4S_4-S_2^2}\geq 0.$

Theorem 2.27:[1]

Suppose that $\lambda_1,\lambda_2,\cdots,\lambda_n\in C$ are nonzero and that $\lambda_1=\max_{2\leq j\leq n}|\lambda_j|$. Then $\{\lambda_1,\lambda_2,\cdots,\lambda_n\}$is the non zero spectrum of a nonnegative matrix of order $m\geq n$, if and only if

1. $\lambda_1>|\lambda_i|, \quad for \quad i=2,3,\cdots,n$
2. $S_k\geq 0, \quad for \quad k=1,2,\cdots, \quad and$
3. All coefficient of the characteristic polynomial $f(x)=\prod_{i=1}^{n}(x-\lambda_i)$ are real.

CHAPTER 3

RECONSTRUCT THE NONNEGATIVE MATRICE FOR $n \le 5$ USING NEWTON'S IDENTITIES

3.1 Introduction

This chapter contains the reconstruction of all nonnegative matrices; of order less than or equal to 5, induced by Lowey and London [14], Laffey and Meehan [13], and Reams [21] by using Newton's identities explained in the previous chapter.

In case *n*=*1* the nonnegative matrix of order one is itself such that the the eigenvalue must be nonnegative which is the **trivial and clear**.

3.2 Reconstruction of Nonnegative Matrix of Order *n*=*2*.

Let $\sigma = (\lambda_1, \lambda_2)$ be two real numbers. Then the characteristic polynomial of two real numbers is $0 = f(x) = (x-\lambda_1)(x-\lambda_2) = x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2$

The realization of nonnegative matrix with respect to λ_1, λ_2 is

$$A = \frac{1}{2}\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix}.,$$

In addition if λ_1, λ_2 are any two complex number satisfying $\lambda_1 + \lambda_2 \ge 0$ and $\lambda_1 \ge |\lambda_2|$, then we have the same formula to realize the nonnegative matrix [6].

Remark 3.2.1:

We remark that:

1. $\lambda_1 + \lambda_2 > 0$ Means the two eigenvalues must be conjugates each to the other because we can not compare two complex numbers.

2. $\lambda_1 + \lambda_2 = 0$ Means the two eigenvalues are not conjugate each to the other but $\lambda_1 = -\lambda_2$ and this case can not deduce a nonnegative matrix *A*. For example, let $\lambda_1 = i,\ \lambda_2 = -i$ satisfies $\lambda_1 + \lambda_2 = 0$ but the matrix *A* is not nonnegative:

$$A=\frac{1}{2}\begin{pmatrix}\lambda_1+\lambda_2 & \lambda_1-\lambda_2\\ \lambda_1-\lambda_2 & \lambda_1+\lambda_2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0 & 2i\\ 2i & 0\end{pmatrix}.$$

3. From the components $\lambda_1+\lambda_2$ and $\lambda_1-\lambda_2$ we must have the second condition which is $\lambda_1 \geq |\lambda_2|$ in case of real eigenvalues.

Now we can construct a nonnegative matrix having $\lambda_1,\ \lambda_2$. As follows:

From definition (2.17) for $0=f(x)=x^2-(\lambda_1+\lambda_2)x+\lambda_1\lambda_2$ we have

$$A=\begin{pmatrix}0 & 1\\ -P_2 & P_1\end{pmatrix}$$

where $P_2=\lambda_1\lambda_2$ and $P_1=\lambda_1+\lambda_2$.By using the Newton's identity explained in Chapter 2 we have

$P_1=S_1,$

$2P_2=P_1S_1-S_2 \rightarrow P_2=\frac{1}{2}(P_1S_1-S_2)=\frac{1}{2}(S_1^2-S_2).$

So, if suppose that $P_1=0=S_1$, we get

$$A=\begin{pmatrix}0 & 1\\ \frac{S_2}{2} & 0\end{pmatrix}$$

To verify the matrix *A* is nonnegative, we have supposed that $P_1=0=S_1$, this gives that $\lambda_1=-\lambda_2$, $S_2=\lambda_1^2+\lambda_2^2=2\lambda_1^2$ without loss of generality. Consequently, we deduced that *A* is nonnegative matrix and the eigenvalues of *A* are λ_1,λ_2.

So we get the following theorem:

Theorem 3.2.2:

Let $\sigma=(\lambda_1,\lambda_2)$ be two real numbers such that $\lambda_1+\lambda_2=0$, then there exists a nonnegative matrix $A=\begin{pmatrix}0 & 1\\ \frac{S_2}{2} & 0\end{pmatrix}$ with spectrum σ. where $S_2=\lambda_1^2+\lambda_2^2$.

Proof:

Clear from above.

Example 3.2.3:

If $\sigma=(\lambda_1,\lambda_2)=(-1,1)$, then $A=\begin{pmatrix} 0 & 1 \\ \frac{S_2}{2} & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3.3 Reconstruction of Nonnegative Matrix of Order *n=3.*

For *n=3*, Lowewy and London [14] solve the nonnegative inverse eigenvalue for lists of three complex numbers as stated in chapter one theorem (2.23). The matrix which is defined by the lists of complex numbers in theorem ([14]) is

$$A=\frac{1}{3}\begin{pmatrix} \rho+2\cos\theta & \rho-2\cos(\theta+\frac{\pi}{3}) & \rho-2\cos(\frac{\pi}{3}-\theta) \\ \rho-2\cos(\frac{\pi}{3}-\theta) & \rho+2\cos\theta & \rho-2\cos(\theta+\frac{\pi}{3}) \\ \rho-2\cos(\theta+\frac{\pi}{3}) & \rho-2\cos(\frac{\pi}{3}-\theta) & \rho+2\cos\theta \end{pmatrix}$$

where $\sigma=(\rho,e^{i\theta},e^{-i\theta})$ be three complex numbers. But if we have $\sigma=(\frac{1}{2},\frac{-1}{4}-i\frac{1}{4},\frac{-1}{4}+i\frac{1}{4})$, then theorem (2.23) can not be applied because $\rho\geq 1$. Here we will take another condition to realize a nonnegative matrix, but all theorems are not applied for any three complex numbers. In the following we use Newton's identities for case real and complex number separately.

Let $\sigma=(\lambda_1,\lambda_2,\lambda_3)$ be three real numbers. Then the characteristic polynomial of these numbers is

$$0=f(x)=(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)=x^3-(\lambda_1+\lambda_2+\lambda_3)x^2+(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)x-\lambda_1\lambda_2\lambda_3$$

The realization of nonnegative matrix with respect to λ_1, λ_2 and λ_3 is

$$A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ P_3 & -P_2 & P_1 \end{pmatrix}.$$

By using the Newton's identity explained in Chapter 2, we have

$P_1=S_1,$

$P_2=\frac{1}{2}(S_1^2-S_2),$

$P_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3).$

Suppose that $P_1 = 0 = S_1$, we get

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}.$$

To show that the matrix *A* is nonnegative, we have supposed that $P_1 = 0 = S_1$, this gives that $\lambda_1 = -\lambda_2 - \lambda_3$ without loss of generality, $S_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ is always nonnegative, and we must supposed that $S_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = -3\lambda_2^2\lambda_3 - 3\lambda_2\lambda_3^2 \geq 0$. Thus, we induce a nonnegative matrix having the eigenvalues $\sigma = (\lambda_1, \lambda_2, \lambda_3)$.

Let $\sigma = (\lambda_1, \lambda_2, \overline{\lambda_2})$ be three complex numbers. Then the companion matrix with respect to the roots $\sigma = (\lambda_1, \lambda_2, \overline{\lambda_2})$ is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ P_3 & -P_2 & P_1 \end{pmatrix}.$$

By using the Newton's identity explained in Chapter 2, we have

$P_1 = S_1,$

$P_2 = \frac{1}{2}(S_1^2 - S_2),$

$P_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3).$

Suppose that $P_1 = 0 = S_1$, we get

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}$$

To show that the matrix *A* is nonnegative, we must show that S_2 and S_3 are nonnegative we must suppose that $\lambda_2 = -a + ia$ such that $a \geq 0$ that is, if we have

$$\sigma = (\lambda_1, \lambda_2, \overline{\lambda_2}) = (2a,\ -a+ia,\ -a-ia)$$

In this case, we have

$S_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 4a^2 + 2ia^2 - 2ia^2 = 4a^2 \geq 0,$

and

$S_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 8a^3 - 2ia^3 + 2a^3 + 2ia^3 + 2a^3 = 12a^3 \geq 0$.

Thus

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4a^3 & 2a^2 & 0 \end{pmatrix}.$$

So we get the following theorem:

Theorem 3.3.1:

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3)$ be three real numbers such that $S_1 = \lambda_1 + \lambda_2 + \lambda_3 = 0$ and $S_3 \geq 0$, then there exists a nonnegative matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}$ with spectrum σ where S_2, S_3 are defined in the previous chapter.

Proof:

Clear from above.

Theorem 3.3.2:

Let $\sigma = (\ \lambda_1,\ \lambda_2,\ \lambda_3\ )$ be a list of three complex numbers of the form $\sigma = (\ 2a,\ -a+ia,\ -a-ia\ )$. Then there exists a nonnegative matrix A with spectrum $\sigma = (\lambda_1, \lambda_2, \lambda_3)$ as follows:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}.$$

Proof:

Clear from above.

Example 3.3.3:

If $\sigma=(\lambda_1,\lambda_2,\lambda_3)=(2,\ -1-i,\ -1+i)$, then $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{pmatrix}$

3.4 Reconstruction of Nonnegative Matrix of Order *n=4.*

Robert Reams in [21] have demonstrate a matrix factorization of a companion matrix, which leads to a solution of the nonnegative inverse eigenvalue problem for a square matrix of order *n=4.*

Let $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be a list of four complex numbers. If $S_1=0,\ \ S_2\geq 0, S_3\geq 0$ and $4S_4\geq S_2^2$ then there exists a nonnegative square matrix of order 4 that realizes σ as follows:

From Newton's identity in Chapter 2, we have

$$P_1=S_1,$$

$$P_2=\frac{1}{2}(S_1^2-S_2)$$

$$P_3=\frac{1}{6}(S_1^3-3S_1S_2+2S_3)$$

$$P_4=\frac{1}{24}(S_1^4-6S_1^2S_2+3S_2^2+8S_1S_3-6S_4)$$

We can write these equations in matrix form with a companion matrix as

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -P_4 & P_3 & -P_2 & P_1 \end{pmatrix}\begin{pmatrix} -4 & 0 & 0 & 0 \\ -S_1 & -3 & 0 & 0 \\ -S_2 & -S_1 & -2 & 0 \\ -S_3 & -S_2 & -S_1 & -1 \end{pmatrix}=\begin{pmatrix} -S_1 & -3 & 0 & 0 \\ -S_2 & -S_1 & 0 & 0 \\ -S_3 & -S_2 & -S_1 & 0 \\ -S_4 & -S_3 & -S_2 & -S_1 \end{pmatrix}$$

This system of matrices are translated to a $UAU^{-1}=B$ where

$$A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-P_4&P_3&-P_2&P_1\end{pmatrix},\ B=\begin{pmatrix}0&1&0&0\\\frac{S_2}{4}&0&1&0\\\frac{S_3}{12}&0&0&1\\\frac{4S_4-S_2^2}{16}&\frac{S_3}{12}&\frac{S_2}{4}&0\end{pmatrix}, \text{ and } U=\begin{pmatrix}1&0&0&0\\0&1&0&0\\\frac{S_3}{4}&0&1&0\\\frac{3S_3}{4}&\frac{5S_3}{12}&0&1\end{pmatrix}.$$

For the detail of the proof see [21, Theorem 3].

Let $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ be a list of four complex numbers. If $S_1=0,\ S_2\geq 0, S_3\geq 0$ and $2S_4\geq S_2^2$ then there exists a nonnegative square matrix of order 4 that realizes σ as follows:

From Newton’s identity in Chapter 2, we have

$$P_1=S_1,$$

$$P_2=\frac{1}{2}(S_1^2-S_2),$$

$$P_3=\frac{1}{6}(S_1^3-3S_1S_2+2S_3)$$

$$P_4=\frac{1}{24}(S_1^4-6S_1^2S_2+3S_2^2+8S_1S_3-6S_4)$$

The characteristic polynomial of $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is $f(x)=\prod_{i=1}^{4}(x-\lambda_i)$ and from theorem (2.25) the coefficient of *f(x)* are real. So the companion matrix *A* of *f(x)* is (If $S_1=0$)

$$A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-P_4&P_3&-P_2&P_1\end{pmatrix}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\\frac{2S_4-S_2^2}{8}&\frac{S_3}{3}&\frac{S_2}{2}&0\end{pmatrix}$$

But $S_2\geq 0, S_3\geq 0,$ and $2S_4\geq S_2^2$ are given. So *A* is non-negative matrix of spectrum $\lambda_1,\lambda_2,\lambda_3,\lambda_4$.

Theorem 3.4.1:

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be a list of four complex numbers. If $S_1 = 0,\ S_2 \geq 0, S_3 \geq 0$ and $2S_4 \geq S_2^2$ then there exists a non-negative square matrix of order 4 that realizes σ as:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2S_4 - S_2^2}{8} & \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}$$

Proof:

Clear from above.

The difference between our technique and the technique of Reams is about the complexity of the construction of the nonnegative matrix and we have not use the idea of similarity of matrices.

Example 3.4.2:

If $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-1 + \frac{i}{2}, -1 - \frac{i}{2}, 3, -1)$, then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2S_4 - S_2^2}{8} & \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3.75 & 8.5 & 5.75 & 0 \end{pmatrix}$$

3.5 Reconstruction of nonnegative matrix of order *n*=5.

Robert Reams in [21] , Thomas J.Laffey and Eleanor Meehan in [13] have given sufficient conditions for the existence of a nonnegative square matrix of order 5 of trace zero. He has used a matrix factorization of a companion matrix, which leads to a solution of the nonnegative inverse eigenvalue problem.

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be a list of five complex numbers. If $S_1 = 0,\ S_2 \geq 0,$ $S_3 \geq 0,\ 4S_4 \geq S_2^2$ and $2S_5 \geq S_2 S_3$ then there exists a non-negative square matrix of order 5 with spectrum σ as follows:

From Newton’s identity in Chapter 2, we have

$$P_1 = S_1,$$

$$P_2 = \frac{1}{2}(S_1^2 - S_2)$$

$$P_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3)$$

$$P_4 = \frac{1}{24}(S_1^4 - 6S_1^2S_2 + 3S_2^2 + 8S_1S_3 - 6S_4)$$

$$P_5 = \frac{1}{5}(P_4S_1 - P_3S_2 + P_2S_3 - P_1S_4 + S_5)$$

He has factorized the companion matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ P_5 & -P_4 & P_3 & -P_2 & 0 \end{pmatrix}$$

as the simpliar matrix to *B* where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{5} & 0 & 1 & 0 & 0 \\ \frac{s_3}{5} & \frac{s_2}{20} & 0 & 1 & 0 \\ 0 & \frac{s_3}{20} & 0 & 0 & 1 \\ \frac{2s_5 - s_2s_3}{10} & \frac{4s_4 - s_2^2}{16} & \frac{s_3}{12} & \frac{s_2}{4} & 0 \end{pmatrix}.$$

For the detail of the proof see [21, Theorem 4]. He have written this similarity is that if the entries of a companion matrix, for a given σ are nonnegative, then the matrix *B* is non-negative but the conversely is not true. But our result the conversely is also true.

Also, there is another theorem to generate the nonnegative matrix of order 5 with respect to $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ as follows:

Theorem 3.5.1: [13]

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be a list of five complex numbers. Then $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is the spectrum of the non-negative square matrix *A* of order 5 if and only if the following conditions hold:

1. $S_1 = 0,$ and $S_k \geq 0,$ $\quad for \quad k = 2,3,4,5,$

2. $S_2^2 \leq 2S_4,$ and

3. $12S_5 - 5S_2S_3 + 5S_3\sqrt{4S_4 - S_2^2} \geq 0.$

Where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{s_5}{5} & \frac{2s_4 - s_2^2}{8} & \frac{s_3}{3} & 0 & 0 \end{pmatrix}.$$

Now, let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ *be a list of five complex numbers. If* $S_1 = 0,$ $S_2 \geq 0,$ $S_3 \geq 0,$ $2S_4 \geq S_2^2,$ and $6S_5 \geq 5S_2S_3,$ then there exists a nonnegative square matrix of order 5 with spectrum σ as follows:

From Newton’s identity in Chapter 2, we have

$P_1 = S_1,$

$$P_2 = \frac{1}{2}(S_1^2 - S_2),$$

$$P_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3)$$

$$P_4 = \frac{1}{24}(S_1^4 - 6S_1^2 S_2 + 3S_2^2 + 8S_1 S_3 - 6S_4)$$

$$P_5 = \frac{1}{5}(P_4 S_1 - P_3 S_2 + P_2 S_3 - P_1 S_4 + S_5)$$

The characteristic polynomial of $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is $f(x) = \prod_{i=1}^{5}(x - \lambda_i)$ and from theorem (2.25) the coefficient of *f(x)* are real, so the companion matrix *A* of *f(x)* is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ P_5 & -P_4 & P_3 & -P_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{6S_5 - 5S_2 S_3}{30} & \frac{2S_4 - S_2^2}{8} & \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}$$

But since $S_2 \geq 0,\ S_3 \geq 0,\ 2S_4 \geq S_2^2$ and $6S_5 \geq 5S_2 S_3$ are given. So *A* is nonnegative matrix of spectrum $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$.

Theorem 3.5.2:

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be a list of five complex numbers. If $S_1 = 0,\ S_2 \geq 0,$ $S_3 \geq 0,\ 2S_4 \geq S_2^2$ and $6S_5 \geq 5S_2 S_3$ then there exists a nonnegative square matrix of order 5 with spectrum σ as

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{6S_5 - 5S_2 S_3}{30} & \frac{2S_4 - S_2^2}{8} & \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}$$

Proof:

Clear from above.

Example 3.5.3:

If $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (\frac{i}{2}, -\frac{i}{2}, 1, -0.5, -0.5)$, then

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{6S_5 - 5S_2S_3}{30} & \frac{2S_4 - S_2^2}{8} & \frac{S_3}{3} & \frac{S_3}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.0625 & 0.1875 & 0.25 & 0.5 & 0 \end{pmatrix}$$

Remark 3.5.4:

If we have $\sigma = (6,1,1,-4,-4)$, then $S_1 = 6+1+1-4-4 = 0$, $S_2 = 70$, $S_3 = 90$, $S_4 = 1810$ and $S_5 = 5730$. All conditions of theorem Reams [21, Theorem 4] are satisfied but the companion matrix with respect to this list is not nonnegative because $P_5 = \frac{1}{5}(P_4S_1 - P_3S_2 + P_2S_3 - P_1S_4 + S_5) = \frac{6S_5 - 5S_2S_3}{30} = \frac{-5248}{30}$ which is not nonnegative entry.

This list is not satisfy our conditions if we have the case of satisfactions, then the companion matrix with respect to the list is nonnegative certainly.

CHAPTER 4

CONSTRUCTION OF NONEGATIVE SQUARE MATRIX OF ORDER *n=6,7,8*

In this chapter we will try to construct a nonnegative square matrix of order n=6,7,8 using Newton's identities equations. Finally, we conclude that we can obtain nonnegative matrices by these identities for any order *n* but when we increase the number of complex number (real number) *n*, then we must put more conditions and we get more restriction on its eigenvalues which is not supportable in computation programming because of its complexity.

Theorem 4.1:

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ be a list of six complex numbers. If $S_1 = 0$, $S_2 \geq 0$, $S_3 \geq 0$, $S_2^2 \leq 2S_4$, $5S_2S_3 \leq 6S_5$, and $3S_2^3 + 24S_6 \geq 18S_2S_4 + 8S_3^2$, then there exists a nonnegative square matrix of order 6 with spectrum σ.

Proof:

From Newton's identity in Chapter 2, we have

$$P_1 = S_1,$$

$$P_2 = \frac{1}{2}(S_1^2 - S_2),$$

$$P_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3)$$

$$P_4 = \frac{1}{24}(S_1^4 - 6S_1^2S_2 + 3S_2^2 + 8S_1S_3 - 6S_4)$$

$$P_5 = \frac{1}{5}(P_4S_1 - P_3S_2 + P_2S_3 - P_1S_4 + S_5)$$

$$P_6 = \frac{1}{6}(P_5S_1 - P_4S_2 + P_3S_3 - P_2S_4 + P_1S_5 - S_6)$$

The characteristic polynomial of $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)$ is $f(x)=\prod_{i=1}^{6}(x-\lambda_i)$ and from theorem (2.25) the coefficient of *f(x)* are real, this means that if we have a complex root, then we must have its conjugate. So the companion matrix *A* of *f(x)* is

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -p_6 & p_5 & -p_4 & p_3 & -p_2 & p_1 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ f_1 & \frac{6s_5-5s_2s_3}{30} & \frac{2s_4-s_2^2}{8} & \frac{s_3}{3} & \frac{s_2}{2} & 0 \end{pmatrix}$$

where $f_1=\frac{24S_6+3s_2^3-8s_3^2-18s_2s_4}{144}$. But from the hypothesis we can say that *A* is a nonnegative matrix of order 6 with spectrum $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)$.

Example 4.2:

If $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)=(-0.25,-0.25,-0.15,-0.2,-0.15,1)$, then

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ f_1 & \frac{6s_5-5s_2s_3}{30} & \frac{2s_4-s_2^2}{8} & \frac{s_3}{3} & \frac{s_2}{2} & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.00192290625 & 0.007125 & 0.06959375 & 0.318 & 0.605 & 0 \end{pmatrix}$$

Theorem 4.3:

Let $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7)$ be a list of seven complex numbers. If $S_1=0$, $S_2\geq 0$, $S_3\geq 0$, $S_2^2\leq 2S_4$, $5S_2S_3\leq 6S_5$, $3S_2^3+24S_6\geq 18S_2S_4+8S_3^2$,and $120S_7+35S_2^2S_3\geq 70S_3S_4+84S_2S_5$ then there exists a nonnegative square matrix of order 7 with spectrum σ.

Proof:

Newton's identity for *n=7* give us:

$$P_1=S_1,$$

$$P_2=\frac{1}{2}(S_1^2-S_2),$$

$$P_3=\frac{1}{6}(S_1^3-3S_1S_2+2S_3)$$

$$P_4=\frac{1}{24}(S_1^4-6S_1^2S_2+3S_2^2+8S_1S_3-6S_4)$$

$$P_5=\frac{1}{5}(P_4S_1-P_3S_2+P_2S_3-P_1S_4+S_5)$$

$$P_6=\frac{1}{6}(P_5S_1-P_4S_2+P_3S_3-P_2S_4+P_1S_5-S_6)$$

$$P_7=\frac{1}{7}(P_6S_1-P_5S_2+P_4S_3-P_3S_4+P_2S_5-P_1S_6+S_7)$$

The characteristic polynomial of $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7)$ is $f(x)=\prod_{i=1}^{7}(x-\lambda_i)$ and from theorem (2.25) the coefficient of *f(x)* are real, this means that if we have a complex root, then we must have its conjugate. Thus all times we have a real number among that list of complex numbers. So the companion matrix *A* of *f(x)*

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ P_7 & -P_6 & P_5 & -P_4 & P_3 & -P_2 & P_1 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & \frac{6S_5-6S_2S_3}{30} & \frac{2S_4-S_2^2}{8} & \frac{S_3}{3} & \frac{\mathrm{S}_2}{2} & 0 \end{pmatrix}$$

where $a_2=\dfrac{24S_6+3s_2^3-8s_3^2-18s_2s_4}{144}$, and $a_1=\dfrac{120S_7+35S_2^2S_3-84S_2S_5-70S_3S_4}{840}$.

Thus from all hypothesis we get a nonnegative square matrix of order *7* with spectrum σ.

Example 4.4:

If $\sigma=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7)=(-0.25,-0.15,-0.05,-0.2,-0.32,-0.03,1)$, then

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & \frac{6S_5-6S_2S_3}{30} & \frac{2S_4-S_2^2}{8} & \frac{S_3}{3} & \frac{\mathrm{S}_2}{2} & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.0000036 & 0.100969211362 & 0.0062126 & 0.6476775 & 0.31336 & 0.6154 & 0 \end{pmatrix}.$$

Theorem 4.5:

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)$ be a list of eight complex numbers. If $S_1 = 0$, $S_2 \geq 0$, $S_3 \geq 0$, $S_2^2 \leq 2S_4$, $5S_2S_3 \leq 6S_5$, $3S_2^3 + 24S_6 \geq 18S_2S_4 + 8S_3^2$,
$120S_7 + 35S_2^2S_3 \geq 70S_3S_4 + 84S_2S_5$,
$180S_4^2 + 15S_2^4 + 480S_2S_6 + 384S_3S_5 \geq 160S_2S_3^2 + 180S_2^2S_4 + 720S_8$ then there exists a nonnegative square matrix of order 8 with spectrum σ.

Proof:

Newton's identity for *n=8* give us:

$$P_2 = \frac{1}{2}(S_1^2 - S_2),$$

$$P_3 = \frac{1}{6}(S_1^3 - 3S_1S_2 + 2S_3)$$

$$P_4 = \frac{1}{24}(S_1^4 - 6S_1^2S_2 + 3S_2^2 + 8S_1S_3 - 6S_4)$$

$$P_5 = \frac{1}{5}(P_4S_1 - P_3S_2 + P_2S_3 - P_1S_4 + S_5)$$

$$P_6 = \frac{1}{6}(P_5S_1 - P_4S_2 + P_3S_3 - P_2S_4 + P_1S_5 - S_6)$$

$$P_7 = \frac{1}{7}(P_6S_1 - P_5S_2 + P_4S_3 - P_3S_4 + P_2S_5 - P_1S_6 + S_7)$$

$$P_8 = \frac{1}{8}(P_7S_1 - P_6S_2 + P_5S_3 - P_4S_4 + P_3S_5 - P_2S_6 + P_1S_7 - S_8)$$

The characteristic polynomial of $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$ is $f(x) = \prod_{i=1}^{7}(x - \lambda_i)$ and from theorem (2.25) the coefficient of *f(x)* are real, this means that if we have a complex root, then we must have its conjugate. Thus all times we have a real number among that list of complex numbers. So the companion matrix *A* of *f(x)* is

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -P_8 & P_7 & -P_6 & P_5 & -P_4 & P_3 & -P_2 & P_1 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 & \frac{S_3}{3} & \frac{S_2}{2} & 0 \end{pmatrix}$$

where

$a_5=\frac{2S_4-S_2^2}{8}$, $a_4=\frac{6S_5-5S_2S_3}{30}$, $a_3=\frac{3S_2^3+24S_6-18S_2S_4+8S_3^2}{144}$, $a_2=\frac{120S_7+35S_2^2S_3-84S_2S_5-70S_3S_4}{840}$, and $a_1=\frac{180S_4^2+15S_2^4+480S_2S_6+384S_3S_5-160S_2S_3^2-180S_2^2S_4-720S_8}{5760}$.

Hence,the hypothesis give us a nonnegative square matrix of order $n=8$ with spectrum σ.

CHAPTER 5

CONCLUSIONS

From the computational work done above we conclude

1. The our construction nonnegative matrix has needed to $\lambda_1 + \lambda_2 = 0$, and we cannot supposed that $\lambda_1 + \lambda_2 > 0$ for any positive λ_1, λ_2 but the construction of $A = \frac{1}{2}\begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix}$ need another condition which is $\lambda_1 \geq |\lambda_2|$. Consequently, put more conditions to have more nonnegative matrix.

2. The construction of nonnegative matrix using Newton's identities for *n=3* we have less conditions than the conditions in the theorem (2.23) but we cannot use our or their theorem to any list of three complex or three real number to realize a nonnegative matrix.

3. For *n=5*, Robert Reams have written the similarity of Newton's identities equations in the matrix form of the companion matrix with respect to $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ in the sense that if the entries of a companion matrix $A=U^{-1}BU$, for a given σ are nonnegative, then the matrix *B* is non negative but the conversely is not true. Our technique; directly induced from Newton's identities equations tell us if the entries of the matrix B are nonnegative, then the companion matrix of trace zero is also nonnegative. Briefly, find a necessary and sufficient condition that σ is the set of eigenvalues of a square nonnegative matrix of order *n* is currently unsolved except in restricted cases.

4. Since the entries of non negative matrix are real number, so it characteristic polynomial has real coefficient and by the fundamental theorem in calculus we get that if we have a root (eigenvalue) of that polynomial, then we will have its conjugate. Consequently, it is clear that for odd number n, the set of eigenvalues must contain at least one real number.

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