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# MAC WILLIAMS IDENTITY OF COMPLETE M-SPOTTY ROSENBLOOM TSFASMAN WEIGHT ENUMERATORS

**TARIK ARABACI**

**MSc. THESIS**
**DEPARTMENT OF MATHEMATICS**
**PROGRAM OF MATHEMATICS**

**ADVISOR**
**ASSOC. PROF. EMRE KOLOTOĞLU**

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A thesis submitted by Tarık ARABACI in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE** is approved by the committee on 30.05.2017 in Department of Mathematics, Program of Mathematics.

**Thesis Advisor**

Assoc. Prof. Dr. Emre KOLOTOĞLU

Yıldız Technical University

**Approved By the Examining Committee**

Assoc. Prof. Dr. Emre KOLOTOĞLU

Yıldız Technical University _______________________

Assoc. Prof. Dr. Fatih DEMİRKALE

Yıldız Technical University _______________________

Assoc. Prof. Dr. Cafer ÇALIŞKAN

Kadir Has University _______________________

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Praise be to Almighty God.

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Tarık ARABACI

# TABLE OF CONTENTS

Page

# LIST OF SYMBOLS

| | |
|---|---|
| $F_2$ | Field of two elements |
| $\mathbb{N}$ | The set of natural numbers |
| $\mathbb{Z}$ | The set of integers |
| $\mathbb{Z}[x]$ | Polynomial ring with integer coefficients |
| $\lvert C \rvert$ | Number of elements in C |

# LIST OF ABBREVIATIONS

| | |
|---|---|
| m- | m-spotty |
| Cm | complete |
| RT | Rosenbloom-Tsfasman |
| WE | Weight enumerator |
| YTU | Yıldız Technical University |

## LIST OF TABLES

Page

# ABSTRACT

## MAC WILLIAMS IDENTITY OF COMPLETE M-SPOTTY ROSENBLOOM STFASMAN WEIGHT ENUMERATORS

Tarık ARABACI

Department of Mathematics

MSc. Thesis

Advisor: Assoc. Prof. Emre KOLOTOĞLU

Information transfer is usually done on bit sequences via electronic channels. Errors occur due to various reasons during transmission of information. These errors may result in misunderstandings or noise in the communication. They are detected and corrected by using error correction codes which are attached to the communication system. Mathematics plays an important role for designing codes and comparing them by considering their efficiency.

Error detection and correction capability of a code depends on the minimum distance of that code. Minimum distance is equal to the minimum weight in a linear code. For this and a number of other criteria for comparing codes, weight distributions of the codes are remarkable. Polynomials called weight enumerators are used for expressing the weight distribution of a code briefly.

Dual of a code is determined by the inner product defined in the space where codewords live. Weights and weight enumerators are determined by the inner product defined in that space. Some weight enumerators and inner products are so closely matched that there is an identity between the weight enumerator of any code that can be generated in that space and the weight counter of the dual of that code. It is much easier and quicker to calculate the weight enumerator of the small dual code than to calculate the weight enumerator of a large code. In such a case, smaller one is chosen between a code and its dual to calculate its weight enumerator. Then weight enumerator of larger code is produced by using that identity.

An identity between the weight enumerators of the codes and their duals was first established by Jessie Mac Williams using the classical vectorial inner product of Euclidean space and Hamming metric. Similar identities have been established for different metrics and inner products in subsequent years. These identities are called Mac Williams identities in memory of Jessie Mac Williams.

There are several weight enumerators used in coding theory for different uses. While the Hamming metric and derived weight enumerators focus on the number of the errors/nonzero symbols in the vectors, the Rosenbloom-Tsfasman metric and derived weight enumerators deal with the position of that errors/nonzero symbols. Whereas both of them take codewords as a single unit and define the distance between them, m-spotty type of weight enumerators divide the codewords into pieces. These m-spotty type weight enumerators also have different types according to block contributions of the pieces. Weight enumerators that maintain the information about the weight contribution of pieces are called "complete" and can only be expressed by multivariable polynomials. In this study, Mac Williams identity will be established for complete m-spotty Rosenbloom-Tsfasman (CmRT) weight enumerators.

**Key words:** Weight enumerators of linear codes, Mac Williams identity, Rosenbloom-Tsfasman metric, m-spotty weight enumerators, complete weight enumerators

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# ÖZET

## TAM ÇOKBENEKLİ ROSENBLOOM-TSFASMAN AĞIRLIK SAYAÇLARI İÇİN MAC WILLIAMS ÖZDEŞLİĞİ

Tarık ARABACI

Matematik Anabilim Dalı
Yüksek Lisans Tezi

Tez Danışmanı: Doç. Dr. Emre KOLOTOĞLU

Elektronik yöntemlerle yapılan bilgi aktarımı genelde bit dizileri üzerinden yapılmaktadır. Bilgi aktarımı sırasında çeşitli sebeplerle bu diziler üzerinde bozulmalar olabilmektedir. Bu bozulmalar bazen yanlış anlaşılmaya bazen ise iletişimde gürültüye sebep olmaktadır. Bit dizilerinde oluşan hatalar iletişim sistemine eklenen hata düzelten kodlardan yararlanılarak düzeltilmektedir. Bu kodların en verimli olacak şekilde dizayn edilmesinde ve kodların yarar-maliyet açısından ihtiyaca göre kıyaslanmasını sağlayabilecek kriterler belirlenmesinde matematik devreye girmektedir.

Bir blok kodun sistemde meydana gelebilecek hataların kodlama ve dekodlama sayesinde ne kadarını farkedip, ne kadarını düzeltebileceği tamamen o kodun içindeki kodsözlerin minimum uzaklığına bağlıdır. Minimum uzaklık da bir lineer kodda en küçük ağırlıklı kodsözün ağırlığına eşittir. Bu ve bir takım başka karşılaştırmalar için bir kod içindeki kodsözlerin ağırlık dağılımları dikkate değerdir. Bir kodun ağırlık dağılımını kısa ve öz bir şekilde göstermek için ağırlık sayacı denen polinomlar kullanılır.

Bir kodun ağırlık sayacı oluşturulurken koddaki tüm kodsözlerin ağırlıkları hesaplanıp,

tasnif edilir. Çok sayıda kodsöz içeren kodlar için bu zahmetli bir iştir. Ancak bir koddaki kodsöz sayısı, o kodun dual kodundaki kodsözlerin sayısıyla ters orantılıdır. Yani çok büyük kodların dualleri çok küçüktür. Ağırlık sayacını hesaplamanın çok zahmetli olacağı bir kodun dualinin ağırlık sayacı çok daha kolay ve hızlı bir şekilde hesaplanabilir.

Bir kodun dualinin ne olduğu o kodun içinde yaşadığı uzayda tanımlanmış iç çarpıma göre belirlenir. Ağırlıklar ve ağırlık sayaçları da yine o uzayda tanımlanan metriğe göre değişir. Bazı ağırlık sayaçları ve iç çarpımlar birbirleriyle o kadar uyumludur ki o uzayda üretilebilen herhangi bir kodun ağırlık sayacı ile o kodun dualinin ağırlık sayacı arasında bir özdeşlik vardır. Böyle bir durumda kod ve dual kod ikilisinden küçük olanın ağırlık sayacı oluşturulduktan sonra özdeşlikten yararlanılarak büyük olan kodun ağırlık sayacı hesaplanabilir.

Kodlar ve duallerinin ağırlık sayaçları arasındaki bir özdeşlik ilk olarak Hamming metrikte klasik Öklid uzayının vektörel iç çarpımı kullanılarak Jessie Mac Williams tarafından bulunmuştur. Sonraki yıllarda farklı metrik ve iç çarpımlar için de benzer özdeşlikler tanımlanmıştır. Bu özdeşlikler Jessie Mac Williams'ın anısına Mac Williams özdeşlikleri olarak adlandırılır.

Kodlama teorisi içerisinde kullanılan çeşitli ağırlık sayaçları vardır. Hamming metriği ve ondan geliştirilmiş metrikler hata vektöründeki hataların adedi üzerine odaklanmışken, Rosenbloom-Tsfasman metriği ve ondan geliştirilmiş metrikler hataların pozisyonu ile ilgilenir. Vektörleri tek parça olarak ele alıp aralarında uzaklık tanımlayan bu metriklerden, vektörleri parçalara ayırarak hataları analiz eden ağırlık sayaçları da geliştirilmiştir. Bu ağırlık sayaçları önlerine "çokbenekli" önadı alırlar. Çokbenekli tipteki ağırlık sayaçlarının da parçacıkların sayaca katkısını önemseyen ve önemsemeyen farklı tipleri vardır. Parçacıkların katkısını önemsemeyen ağırlık sayaçları genelde tek değişkenli polinomlardır. Parçacıkların katkısıyla ilgili bilgileri de koruyan sayaçlar ise ancak çok değişkenli polinomlarla ifade edilebilir. Parçacıkların katkısıyla ilgili detayı da göz önünde bulunduran bu sayaçlar "tam" önadı alırlar.

Her metrik ve ağırlık sayacı Mac Williams özdeşliği kurmaya uygun değildir. Hamming ve Rosenbloom-Tsfasman metrikleri üzerinde uygun iç çarpımlar altında MacWilliams özdeşlikleri bilinmektedir. Çokbenekli Hamming metriği ile yapılan sayaçlarda aynı ağırlık sayacına sahip kodların oluşturdukları denklik sınıflarının dağılımı yüzünden

hiçbir iç çarpım altında tüm kodlar için geçerli olabilecek bir Mac Williams özdeşliği kurulamaz. Ancak tam çokbenekli Hamming sayaçları için böyle bir özdeşlik kurulmuştur. Bundan yararlanarak Rosenbloom-Tsfasman metriğinin çokbenekli geliştirmesi için de özdeşlik kurulmuştur. Bu çalışmada ise tam çokbenekli Rosenbloom-Tsfasman ağırlık sayaçları için Mac Williams özdeşliği kurulmaktadır.

**Anahtar Kelimeler:** Lineer kodların ağırlık sayaçları, Mac Williams özdeşliği, Rosenbloom-Tsfasman metriği, çokbenekli ağırlık sayaçları, tam ağırlık sayaçları

# CHAPTER 1

# INTRODUCTION

## 1.1 Literature Review

An identity between the weight enumerators of the codes and their duals was first established by Jessie Mac Williams [1] using the classical vectoral inner product of Euclidean space and Hamming metric. Similar identities have been established for different metrics and inner products in subsequent years. These identities are called Mac Williams identities in memory of Jessie Mac Williams.

There are several weight enumerators used in coding theory for different uses. While the Hamming metric and derived weight enumerators focus on the number of the errors/nonzero symbols in the vectors, the Rosenbloom-Tsfasman metric and derived weight enumerators deal with the position of that errors/nonzero symbols. Whereas both of them take codewords as a single unit and define the distance between them, m-spotty type of weight enumerators divide the codewords into pieces. These m-spotty types weight enumerators also have different types according to preserving information about the pieces. Weight enumerators that maintain the information about the pieces are called "complete" and can only be expressed by multivariable polynomials.

Relation between weight distribution of linear codes and their dual codes was first discovered and Mac Williams identity for Hamming weight enumerator was established in [1]. After further researches in Coding Theory such as m-spotty byte errors, other metrics and weight enumerators are developed. Mac Williams identity for m-spotty Hamming weight enumerators could not be established directly but adapted to complete m-spotty Hamming weight enumerator and Mac Williams identity for complete m-spotty Hamming weight enumerator was established in [2]. In Rosenbloom Tsfasman type metrics Mac Williams identity of m-spotty RT weight enumerator was established similarly in [3].

## 1.2 Objective of the Thesis

In order to create a weight enumerator of a code, weights of all codewords in that code need to be calculated and classified. This is a laborious activity for codes containing a large number of codewords. However, the number of codewords in a code is inversely proportional to the number of codewords in its dual code. So, the very large codes have very small dual codes. It is much easier and quicker to calculate the weight enumerator of the small dual code than to calculate the weight enumerator of a large code.

Weight enumerators that preserve the weight information about pieces are called "complete" and can only be expressed by multivariable polynomials. Complete m-spotty Rosenbloom-Tsfasman (CmRT) weight enumerators give more information about weight types of codewords than m-spotty Rosenbloom-Tsfasman weight enumerators. In this study, Mac Williams identity will be adapted for complete m-spotty Rosenbloom-Tsfasman weight enumerators.

## 1.3 Hypothesis

Aim of this study is to establish a Mac Williams identity between binary linear codes for complete m-spotty Rosenbloom-Tsfasman weight. We claim that CmRT weight enumerator of a code $C$ is related to CmRT weight enumerator of its dual code. To find the CmRT weight enumerator of the dual code, we suggest calculating CmRT weight enumerator of $C$ with special combination of variables. CmRT weight enumerator of $C$ with special combination of variables will be $|C|$ multiple of CmRT weight enumerator of the dual code.

# CHAPTER 2

# DEFINITIONS

Let $F_2$:={0,1} be the field of two elements with usual operations + and × modulo 2.

$F_2^N$ = $F_2$×$F_2$×…×$F_2$ (Cartesian product of N times $F_2$) is a vector space over $F_2$. In general, any element of $F_2^N$ is represented by an N-tuple: $(x_1, x_2,\ldots, x_N)$ where $x_i \in F_2$, but in Coding Theory we use this notation without comma or parenthesis. That is, instead of $(x_1, x_2,\ldots, x_N)$ we write $x_1x_2\ldots x_N$.

Any subset $C \subseteq F_2^N$, is called a (binary) code. Any element u ∈ $C$ is called a codeword. When a code $C \subseteq F_2^N$ is also a vector space, we say that $C$ is a linear code. In that case number of codewords in $C$ must be a power of 2=$|F_2|$ i.e. $|C|$=$2^k$ for some integer k. That integer k is called the dimension of the code $C$.

## 2.1 Distance and Weight Types

The concepts of metrics and norms in linear algebra correspond to the concepts of distances and weights in Coding Theory.

**Definition 2.1** For each pair of codewords Hamming distance indicates the number of positions where they differ. Let $u_i$, $v_i \in F_2$ for 1 ≤ i ≤ N and $u = u_1u_2\ldots u_N$, $v = v_1v_2\ldots v_N \in C \subseteq F_2^N$.

$d_H: C\times C \to \mathbb{N}$ and $wt_H: C \to \mathbb{N}$ are defined as $d_H(u,v) \coloneqq |\{i|\ u_i \neq v_i\}|$ and $wt_H(u) \coloneqq d_H(u,0) = |\{i|\ u_i \neq 0\}|$ .

**Example 2.1** Let's take two codewords of $F_2^6$: 111101 and 100001. They have same value at their $1^{st}$, the $5^{th}$ and the last positions.

So $d_H$ (111101, 100001) = |{2, 3, 4}| = 3.

Also $wt_H$ (111101) = 5 and $wt_H$ (100001) = 2.

Another distance operation on binary codes is RT metric which is introduced by Rosenbloom and Tsfasman [4]. It gives the rightmost position where given two vectors have different values.

**Definition 2.2** Let $u_i, v_i \in F_2$ for 1 ≤ i ≤ N and $u = u_1u_2\ldots u_N, v = v_1v_2\ldots v_N \in C \subseteq F_2^N$.

Then RT distance $d_{RT}$ : $C\times C \to \mathbb{N}$ and RT weight $wt_{RT}$ : $C \to \mathbb{N}$ are defined as

$$d_{RT}(u,v) \coloneqq \begin{cases} 0, \ if\ u = v \\ max\{i|\ u_i \neq v_i\},\ if\ u\ \neq v \end{cases} \tag{2.1}$$

$$wt_{RT}(u) \coloneqq d_{RT}(u,0) = \begin{cases} 0, \ if\ u = 0 \\ max\{i|\ u_i \neq 0\},\ if\ u\ \neq 0 \end{cases} \tag{2.2}$$

**Example 2.2** Let's take two codewords of $F_2^6$ as in Example 2.1: 111101 and 100001.

They have same the values at their the 1st, the 5th and the last positions.

So $d_{RT}$ (111101, 100001) = max {2, 3, 4} = 4. Also $wt_{RT}$ (111101) = 6 and $wt_{RT}$ (100001) = 6.

Let N=nb where n and b are integers and let u be a codeword of length N. Then we can separate these codewords into n parts. We call those parts as blocks.

So, each block has length b. From now on $u_i$ denotes the ith block of a given codeword u. So $u = u_1u_2\ldots u_n$ and $u_i$ = $u_{i_1}\ldots u_{i_b}$ and $u_{i_j} \in F_2$ for 1 ≤ i ≤ n, 1 ≤j ≤ b.

**Example 2.3** Let's take two codewords of $F_2^6$ as in Example 2.1:

u=111101 and v=100001.

Choose n=2 and b=3. That is, we separate our codewords into two blocks. Each block has length 3.

$u_1$ = 111 and $u_2$ = 101.

$v_1$ = 100 and $v_2$ = 001.

Observe that the blocks of codewords of $F_2^N$ are the codewords of $F_2^b$. So, Hamming and RT metrics can also be defined on blocks. By using these facts for an integer t ∈ [1,b] m-spotty Hamming distance and similarly m-spotty RT distance for u, v ∈ $F_2^N$ is defined:

**Definition 2.3** Let N=nb, 1 ≤ t ≤ b for n, b, t ∈ ℕ; $u_i, v_i \in F_2^b$ for 1 ≤ i ≤ n and $u = u_1u_2\ldots u_n,\ v = v_1v_2\ldots v_n \in C \subseteq F_2^N$. Then m-spotty Hamming distance is defined as

$d_{mH}: C\times C \to \mathbb{N}$, $d_{mH}(u,v) \coloneqq \sum_{i=1}^{n} \left\lceil \frac{d_H(u_i,v_i)}{t} \right\rceil$ where $u_i$ and $v_i$ are the $i^{th}$ blocks of u and v, respectively.

**Definition 2.4** Let N=nb, 1 ≤ t ≤ b for n, b, t ∈ ℕ; $u_i$, $v_i \in F_2^b$ for 1 ≤ i ≤ n and $u = u_1u_2\ldots u_n$, $v = v_1v_2\ldots v_n \in C \subseteq F_2^N$. Then m-spotty RT distance is defined as $d_{mRT}: C\times C \to \mathbb{N}$ , $d_{mRT}(u,v) \coloneqq \sum_{i=1}^{n} \left\lceil \frac{d_{RT}(u_i,v_i)}{t} \right\rceil$ where $u_i$ and $v_i$ are the $i^{th}$ blocks of u and v, respectively.

M-spotty Hamming and m-spotty RT distances will determine m-spotty Hamming and m-spotty RT weights:

$$wt_{mH}(u) \coloneqq d_H(u,0) = \sum_{i=1}^{n} \left\lceil \frac{d_H(u_i,0)}{t} \right\rceil \ . \tag{2.3}$$

$$wt_{mRT}(u) \coloneqq d_{RT}(u,0) = \sum_{i=1}^{n} \left\lceil \frac{d_{RT}(u_i,0)}{t} \right\rceil . \tag{2.4}$$

**Example 2.4** Let N=12, n=3, b=4, t=2 and u=010110101000, v=110111110111. So $wt_H(u)$ = 5 and $wt_H(v)$= 10.

When dealing with m-spotty weights we see u as 3 blocks of each length 4. For convenience, we can write the blocks of u separately: 0101 1010 1000, i.e. $u_1$ = 0101, $u_2$ = 1010, $u_3$ = 1000.

Similarly, v=1101 1111 0111 i.e. $v_1$ = 1101, $v_2$ = 1111, $v_3$ = 0111.

Let's find m-spotty Hamming and RT distances between u and v.

$$d_{mH}(u,v)=\sum_{i=1}^{3} \left\lceil \frac{d_H(u_i,v_i)}{t} \right\rceil$$

$$=\left\lceil \frac{d_H(u_1,v_1)}{t} \right\rceil + \left\lceil \frac{d_H(u_2,v_2)}{t} \right\rceil + \left\lceil \frac{d_H(u_3,v_3)}{t} \right\rceil$$

$$=\left\lceil \frac{d_H(0101,1101)}{2} \right\rceil + \left\lceil \frac{d_H(1010,1111)}{2} \right\rceil + \left\lceil \frac{d_H(1000,0111)}{2} \right\rceil$$

$$=\left\lceil \frac{1}{2} \right\rceil + \left\lceil \frac{2}{2} \right\rceil + \left\lceil \frac{4}{2} \right\rceil=4.$$

Since m-spotty Hamming distance will determine m-spotty Hamming weight, we can also find m-spotty Hamming weight.

$$wt_{mH}(u)=d_{mH}(u,0)=\sum_{i=1}^{3} \left\lceil \frac{d_H(u_i,0)}{t} \right\rceil=\left\lceil \frac{wt_H(u_1)}{t} \right\rceil + \left\lceil \frac{wt_H(u_2)}{t} \right\rceil + \left\lceil \frac{wt_H(u_3)}{t} \right\rceil$$

$=\left\lceil\frac{wt_H(0101)}{2}\right\rceil+\left\lceil\frac{wt_H(1010)}{2}\right\rceil+\left\lceil\frac{wt_H(1000)}{2}\right\rceil=\left\lceil\frac{2}{2}\right\rceil+\left\lceil\frac{2}{2}\right\rceil+\left\lceil\frac{1}{2}\right\rceil=3.$

Similarly, $wt_{MH}(v)=d_{MH}(v,0)=\sum_{i=1}^{3}\left\lceil\frac{d_H(v_i,0)}{t}\right\rceil=\left\lceil\frac{wt_H(v_1)}{t}\right\rceil+\left\lceil\frac{wt_H(v_2)}{t}\right\rceil+\left\lceil\frac{wt_H(v_3)}{t}\right\rceil$

$=\left\lceil\frac{wt_H(1101)}{2}\right\rceil+\left\lceil\frac{wt_H(1111)}{2}\right\rceil+\left\lceil\frac{wt_H(0111)}{2}\right\rceil=\left\lceil\frac{3}{2}\right\rceil+\left\lceil\frac{4}{2}\right\rceil+\left\lceil\frac{3}{2}\right\rceil=6.$

Next, we will calculate m-spotty RT distance between u and v:

$d_{mRT}(u,v)=\sum_{i=1}^{3}\left\lceil\frac{d_{RT}(u_i,v_i)}{t}\right\rceil=\left\lceil\frac{d_{RT}(u_1,v_1)}{t}\right\rceil+\left\lceil\frac{d_{RT}(u_2,v_2)}{t}\right\rceil+\left\lceil\frac{d_{RT}(u_3,v_3)}{t}\right\rceil$

$=\left\lceil\frac{d_{RT}(0101)}{2}\right\rceil+\left\lceil\frac{d_{RT}(1010)}{2}\right\rceil+\left\lceil\frac{d_{RT}(1000)}{2}\right\rceil=\left\lceil\frac{1}{2}\right\rceil+\left\lceil\frac{4}{2}\right\rceil+\left\lceil\frac{4}{2}\right\rceil=5.$

Since m-spotty RT distance will determine m-spotty RT weight, we can also find m-spotty RT weight.

$wt_{mRT}(u)=d_{mRT}(u,0)=\sum_{i=1}^{3}\left\lceil\frac{d_{RT}(u_i,0)}{t}\right\rceil=\left\lceil\frac{wt_{RT}(u_1)}{t}\right\rceil+\left\lceil\frac{wt_{RT}(u_2)}{t}\right\rceil+\left\lceil\frac{wt_{RT}(u_3)}{t}\right\rceil$

$=\left\lceil\frac{wt_{RT}(0101,0000)}{2}\right\rceil+\left\lceil\frac{wt_{RT}(1010,0000)}{2}\right\rceil+\left\lceil\frac{wt_{RT}(1000,0000)}{2}\right\rceil=\left\lceil\frac{4}{2}\right\rceil+\left\lceil\frac{3}{2}\right\rceil+\left\lceil\frac{1}{2}\right\rceil=5.$

And similarly, $wt_{mRT}(v)=d_{mRT}(v,0)=\sum_{i=1}^{3}\left\lceil\frac{d_{RT}(v_i,0)}{t}\right\rceil=\left\lceil\frac{wt_{RT}(v_1)}{t}\right\rceil+\left\lceil\frac{wt_{RT}(v_2)}{t}\right\rceil+$
$\left\lceil\frac{wt_{RT}(v_3)}{t}\right\rceil=\left\lceil\frac{wt_{RT}(1101,0000)}{2}\right\rceil+\left\lceil\frac{wt_{RT}(1111,0000)}{2}\right\rceil+\left\lceil\frac{wt_{RT}(0111,0000)}{2}\right\rceil=\left\lceil\frac{4}{2}\right\rceil+\left\lceil\frac{4}{2}\right\rceil+\left\lceil\frac{4}{2}\right\rceil=6.$

We can choose t any integer between 1 and b.

If we have chosen t=1, $wt_{mRT}(u)$ would be $\left\lceil\frac{4}{1}\right\rceil+\left\lceil\frac{3}{1}\right\rceil+\left\lceil\frac{1}{1}\right\rceil=8.$

If we have chosen t=3, $wt_{mRT}(u)$ would be $\left\lceil\frac{4}{3}\right\rceil+\left\lceil\frac{3}{3}\right\rceil+\left\lceil\frac{1}{3}\right\rceil=4.$

If we have chosen t=4, $wt_{mRT}(u)$ would be $\left\lceil\frac{4}{4}\right\rceil+\left\lceil\frac{3}{4}\right\rceil+\left\lceil\frac{1}{4}\right\rceil=3.$

As we see in the above example, m-spotty distance or weights depend on the choice of t.

## 2.2 Weight Enumerator Types

For decoding it is important to know the number of codewords of each weight a code has. We can classify codes by using their weight distributions. We use polynomials to describe weight distributions of codes. We will call these polynomials as weight enumerators.

Generally, a weight enumerator of a code $C$ is a polynomial with nonnegative integer coefficients that is related to weights of codewords of $C$. It gives information about weight distribution of a code $C$. Of course, the definition of this polynomial depends on the choice of weight.

There are different sorts of weight enumerators. We will give the definition of some of them which measure the weights defined above in a linear code.

**Definition 2.5** Hamming Weight Enumerator, $W_H(\mathrm{C})(\mathrm{z})$, counts the number of codewords of each weight with respect to Hamming distance. It is a one variable polynomial such that coefficient of $z^i$ shows the number of codewords that has Hamming weight i.

$$W_H(\mathrm{C})(\mathrm{z}) := \sum_{u \in C} z^{wt_H(u)}. \quad (2.5)$$

**Example 2.5** Let $C = \langle 010111100001; 101010101010 \rangle$,

i.e. $C$ = {000000000000; 010111100001; 101010101010; 111101001011}. Then $wt_H(000000000000)$=0, $wt_H(010111100001)$=6, $wt_H(101010101010)$=6, $wt_H(111101001011)$=8. So $W_H(\mathrm{C})(\mathrm{z}) = z^0 + z^6 + z^6 + z^8 = 1 + 2z^6 + z^8$

Now assume that we do not know anything about $C$ but only its weight enumerator:

$$W_H(\mathrm{C})(\mathrm{z}) = z^0 + 2z^6 + z^8. \quad (2.6)$$

We can deduce from this polynomial those:

There is no codeword of weights 1, 2, 3, 4, 5 or 7 in $C$.

The least weighted nonzero codeword has weight 6 and there are two such codewords. (Since $C$ is linear, this number (6) corresponds to the minimum distance of $C$. It gives error detection and correction capacities of the code.) Remaining codeword's weight is 8.

**Definition 2.6** RT Weight Enumerator, $W_{RT}(\mathrm{C})(\mathrm{z})$, counts the number of codewords of each weight with respect to RT distance. It is a one variable polynomial such that coefficients of $z^i$ shows the number of codewords that has RT weight i.

$$W_{RT}(\mathrm{C})(\mathrm{z}) := \sum_{u \in C} z^{wt_{RT}(u)}. \quad (2.7)$$

**Example 2.6** Let $C$ be a linear code as in Example 2.5.

$wt_{RT}(000000000000)$=0, $wt_{RT}(010111100001)$=12, $wt_{RT}(101010101010)$=11, $wt_{RT}(111101001011)$=12.

So $W_{RT}(\mathrm{C})(\mathrm{z}) = z^0 + z^{11} + z^{12} + z^{12} = 1 + z^{11} + 2z^{12}$

**Definition 2.7** M-spotty Hamming weight enumerator, $W_{mH}(\mathrm{C})(\mathrm{z})$, counts the number of codewords of each weight with respect to m-Spotty Hamming distance. It is a one variable polynomial such that coefficient of $z^i$ shows the number of codewords that has m-Spotty Hamming weight i.

$$W_{mH}(\mathrm{C})(\mathrm{z}) = \sum_{u\in C} z^{wt_{mH}(u)}. \qquad (2.8)$$

**Example 2.7** Let $C$ be a linear code as in Example 2.5. Choose n=3, b=4 and t=2 as in Example 2.4. Then,

$wt_{mH}(0000\ 0000\ 0000)=\left\lceil\frac{0}{2}\right\rceil + \left\lceil\frac{0}{2}\right\rceil + \left\lceil\frac{0}{2}\right\rceil=0$, $wt_{mH}(0101\ 1110\ 0001)= \left\lceil\frac{2}{2}\right\rceil + \left\lceil\frac{3}{2}\right\rceil + \left\lceil\frac{1}{2}\right\rceil=4$, $wt_{mH}(1010\ 1010\ 1010)=\left\lceil\frac{2}{2}\right\rceil + \left\lceil\frac{2}{2}\right\rceil + \left\lceil\frac{2}{2}\right\rceil=3$, $wt_{mH}(1111\ 0100\ 1011)=\left\lceil\frac{4}{2}\right\rceil + \left\lceil\frac{1}{2}\right\rceil + \left\lceil\frac{3}{2}\right\rceil=5$.

So $W_{mH}(\mathrm{C})(\mathrm{z}) = z^0 + z^3 + z^4 + z^5 = 1 + z^3 + z^4 + z^5$.

**Definition 2.8** M-spotty RT weight enumerator, $W_{mRT}(\mathrm{C})(\mathrm{z})$, counts the number of codewords of each weight with respect to m-Spotty RT distance. It is a one variable polynomial such that coefficient of $z^i$ shows the number of codewords that has m-Spotty RT weight i.

$$W_{mRT}(\mathrm{C})(\mathrm{z}) = \sum_{u\in C} z^{wt_{mRT}(u)}. \qquad (2.9)$$

**Example 2.8** Let $C$ be the linear code in Example 2.5. Choose n=3, b=4 and t=2 as in Example 2.4. Then,

$wt_{mRT}(0000\ 0000\ 0000)=\left\lceil\frac{0}{2}\right\rceil + \left\lceil\frac{0}{2}\right\rceil + \left\lceil\frac{0}{2}\right\rceil=0$, $wt_{mRT}(0101\ 1110\ 0001)= \left\lceil\frac{4}{2}\right\rceil + \left\lceil\frac{3}{2}\right\rceil + \left\lceil\frac{4}{2}\right\rceil=6$, $wt_{mRT}(1010\ 1010\ 1010)=\left\lceil\frac{3}{2}\right\rceil + \left\lceil\frac{3}{2}\right\rceil + \left\lceil\frac{3}{2}\right\rceil=6$, $wt_{mRT}(1111\ 0100\ 1011)=\left\lceil\frac{4}{2}\right\rceil + \left\lceil\frac{2}{2}\right\rceil + \left\lceil\frac{4}{2}\right\rceil=5$.

So $W_{mRT}(\mathrm{C})(\mathrm{z}) = z^0 + z^6 + z^6 + z^5 = 1 + z^5 + 2z^6$.

Other types of weight enumerators are complete weight enumerators. They are multivariable polynomials. Weight information of each block of codewords are stored in different variables in these polynomials.

**Definition 2.9** Complete m-Spotty Hamming Weight Enumerator, $W_{CmH}(\mathrm{C})(z_0, \dots, z_b)$, counts the number of codewords of each block-weight distribution type. It is a multivariable polynomial such that powers of $z_i$ in each term shows the number of blocks that has m-Spotty Hamming weight i. Coefficients of each term shows the number of codewords in the block-weight distribution type corresponding to that term.

$$W_{CmH}(\mathrm{C})(z_0, \dots, z_b) := \sum_{u \in C} \prod_{i=0}^{b} z_i^{\propto_{u,i}} \quad (2.10)$$

where $\propto_{u,i}$ is the number of blocks in the codeword $u \in C$ that has Hamming weight i.

**Example 2.9** Let $C$be the linear code as in Example 2.5. Choose n=3, b=4 and t=2 as in Example 2.4. Let's start with u=0101 1110 0001 of $C$. Then u has 3 blocks.

$wt_H(0101) = 2, wt_H(1110) = 3, wt_H(0001) = 1.$

Since none of these 3 blocks has Hamming weight 0, $\propto_{u,0} = 0$. There is only one block of Hamming weight 1. So $\propto_{u,1} = 1$. Similarly, $\propto_{u,2} = 1$ and $\propto_{u,3} = 1$. Since there is no block of u of Hamming weight 4, $\propto_{u,4} = 0$. So, contribution of u=0101 1110 0001 to complete m-spotty Hamming weight enumerator of $C$is:

$\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{\propto_{u,0}} z_1^{\propto_{u,1}} z_2^{\propto_{u,2}} z_3^{\propto_{u,3}} z_4^{\propto_{u,4}}$

$= z_0^{0} z_1^{1} z_2^{1} z_3^{1} z_4^{0}$

$= z_1 z_2 z_3$

Contribution of u=0000 0000 0000 to complete m-spotty Hamming weight enumerator of $C$is: $\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{3} z_1^{0} z_2^{0} z_3^{0} z_4^{0} = z_0^{3}$.

Contribution of u=1010 1010 1010 to complete m-spotty Hamming weight enumerator of $C$is: $\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{0} z_1^{0} z_2^{3} z_3^{0} z_4^{0} = z_2^{3}$.

Contribution of u=1111 0100 1011 to complete m-spotty Hamming weight enumerator of $C$is: $\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{0} z_1^{1} z_2^{0} z_3^{1} z_4^{1} = z_1 z_3 z_4$.

Hence $W_{CmH}(\mathrm{C})(z_0, z_1, z_2, z_3, z_4) = \sum_{u \in C} \prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{3} + z_1 z_2 z_3 + z_2^{3} + z_1 z_3 z_4$.

**Definition 2.10** Complete m-Spotty RT Weight Enumerator, $W_{CmH}(\mathrm{C})(z_0, \dots, z_b)$, counts the number of codewords of each block-weight distribution type. It is a multivariable polynomial such that power of $z_i$ in each term shows the number of blocks that has m-Spotty RT weight i and coefficients of each term shows the number of codewords in the block-weight distribution type corresponding to that term.

$$W_{CmRT}(\mathrm{C})(z_0, \ldots, z_b) := \sum_{u \in C} \prod_{i=0}^{b} z_i^{\propto_{u,i}} \qquad (2.11)$$

where $\propto_{u,i}$ is the number of blocks in the codeword $u \in C$ that has RT weight i.

**Example 2.10** Let $C$ be the linear code as in Example 2.5. Choose n=3, b=4 and t=2 as in Example 2.4. Let's start with u=0101 1110 0001 of $C$ again. Then u has 3 blocks.

$wt_{RT}(0101) = 4, wt_{RT}(1110) = 3, wt_{RT}(0001) = 4.$

Since none of these 3 blocks has RT weight 0, $\propto_{u,0} = 0$. There is only one block of RT weight 3. So $\propto_{u,3} = 1$. There are two blocks of RT weight 4. So $\propto_{u,4} = 2$. But there is no block of u of RT weight 1 or 2, so $\propto_{u,1} = 0$ and $\propto_{u,2} = 0$.

So, contribution of u=0101 1110 0001 to complete m-spotty RT weight enumerator of $C$ is: $\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{\propto_{u,0}} z_1^{\propto_{u,1}} z_2^{\propto_{u,2}} z_3^{\propto_{u,3}} z_4^{\propto_{u,4}} = z_0^{0} z_1^{0} z_2^{0} z_3^{1} z_4^{2} = z_3 z_4^{2}$.

Contribution of u=0000 0000 0000 to complete m-spotty RT weight enumerator of $C$ is: $\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{3} z_1^{0} z_2^{0} z_3^{0} z_4^{0} = z_0^{3}$.

Contribution of u=1010 1010 1010 to complete m-spotty RT weight enumerator of $C$ is: $\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{0} z_1^{0} z_2^{0} z_3^{3} z_4^{0} = z_3^{3}$.

Contribution of u=1111 0100 1011 to complete m-spotty RT weight enumerator of $C$ is: $\prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{0} z_1^{0} z_2^{1} z_3^{0} z_4^{2} = z_2 z_4^{2}$. Hence,

$$W_{CmRT}(\mathrm{C})(z_0, z_1, z_2, z_3, z_4) = \sum_{u \in C} \prod_{i=0}^{4} z_i^{\propto_{u,i}} = z_0^{3} + z_3 z_4^{2} + z_3^{3} + z_2 z_4^{2}. \qquad (2.12)$$

Complete m-spotty weight enumerators are homogeneous polynomials of degree n. For homogeneous polynomials sum of the powers of the factors of each term is a constant. This number is called the degree of that polynomial. This degree of given weight enumerator gives us the number of blocks of codewords of its codes. Like the other weight enumerators, sum of the coefficients gives the number of codewords in the code.

**Example 2.11** Assume $W_{CmRT}(\mathrm{C})(z_0, z_1, z_2, z_3) = z_0^{2} + 4 z_2 z_3 + 3 z_1^{2} + z_1 z_2$ is given and we do not know another information about the code $C$. We can deduce some information about $C$ by going through weight enumerator of $C$. Sum of the coefficients of this polynomial is 1+4+3+1=9. So, $C$ has 9 codewords. Degree of the polynomial is 2 for each term. So, n=2 i.e. every codeword of $C$ consists of 2 blocks. The polynomial has 4 variables so that b=4-1=3 i.e. the length of every block is 3. Also, N=n·b=2·3=6 i.e. the length of each codeword is 6.

We can learn lots of things from this polynomial about the binary code $C$. We can say that $C$ is a subset of $F_2^6$. $C$ has 9 codewords. They were divided into two blocks. We can classify these 9 codewords into 4 types. First type of codewords consists of two 0-weighted blocks. There is only one codeword of this type. Actually, this is the zero vector. Every linear code must have the zero vector. This is not an unexpected information. Another type of codewords consists of one 2-weighted and one 3-weighted block. There are 4 codewords of this type. Another type of codewords consists of two 1-weighted blocks. There are 3 codewords of this type. The last type of codewords in $C$ consists of one 1-weighted and one 2-weighted blocks. There is only 1 codeword of this type. We know this information from the polynomial with no need of writing down all the codewords or calculating their weights one by one.

**Remark 2.1:** Multivariable weight enumerators carry more information about codes. Nevertheless, one variable weight enumerators may be preferred due to being simple and easy to calculate. If someone prefers one variable, then complete m-spotty weight enumerators can transform to m-spotty weight enumerators uniquely by replacing each $z_i$ with $z^{\lceil \frac{i}{t} \rceil}$ at cost of sacrificing some detailed information about weight types of the code. However, this process is not reversible. A one variable m-spotty weight enumerator can correspond to various complete m-spotty weight enumerators.

**Example 2.12** Let $C$ be the linear code as in Example 2.5. Choose n=3, b=4 and t=2 as in Example 2.4. We know $W_{CmRT}(\mathrm{C})(z_0, z_1, z_2, z_3, z_4)$=${z_0}^3 + z_3{z_4}^2 + {z_3}^3 + z_2{z_4}^2$ by Example 2.10.

By letting $z_0 = 1$, $z_1 = z_2 = z$, $z_3 = z_4$=$z^2$, we obtain $1 + z^5 + 2z^6$. This polynomial corresponds to $W_{mRT}(\mathrm{C})(\mathrm{z}) = 1 + z^5 + 2z^6$ in Example 2.8. Now consider another linear code $C_2 = \langle 010111100001;\ 101010111010 \rangle$ with complete m-spotty RT weight enumerator ${z_0}^3 + {z_3}^2 z_4 + {z_3}^3 + {z_4}^3$. Although weight distributions of $C$ and $C_2$ are different in complete sense, they turn into same one variable polynomial $1 + z^5 + 2z^6$. Therefore, reverse process is not well defined.

**Remark 2.2:** Sharing weight enumerator of a code instead of all elements of that code is useful and practical especially for large codes, but calculating a weight enumerator of a large code directly is unpractical.

## 2.3 Inner Products

Since linear codes are linear spaces, we can define inner products on them. One of them is defined classically:

**Definition 2.11** For $1 \leq i \leq n$ and for $u = u_1u_2 \ldots u_n$, $v = v_1v_2 \ldots v_n$ codewords of same length n where $u_i, v_i \in F_2$, $\langle u, v\rangle_1 : F_2^n \times F_2^n \to F_2$ is defined as

$$\langle u, v\rangle_1 \coloneqq \sum_{i=1}^{n} u_i \cdot v_i \tag{2.13}$$

where $u_i \cdot v_i$ is multiplication of $u_i$ and $v_i$ in $F_2$.

**Example 2.13** For u=0101 and v=1110:

$\langle u, v\rangle_1 = \sum_{i=1}^{4} u_i \cdot v_i = 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 1$ in $F_2$.

There are other inner products:

**Definition 2.12** For $1 \leq i \leq n$ and for $u = u_1u_2 \ldots u_n$, $v = v_1v_2 \ldots v_n$ codewords of the same length n where $u_i, v_i \in F_2$, $\langle\ ,\ \rangle_2 : F_2^n \times F_2^n \to F_2$ is defined as

$$\langle u, v\rangle_2 \coloneqq \sum_{i=1}^{n} u_i \cdot v_{n+1-i} \tag{2.14}$$

where $u_i \cdot v_j$ is multiplication of $u_i$ and $v_j$ in $F_2$.

**Example 2.14** For u=0101 and v=1110:

$\langle u, v\rangle_2 = \sum_{i=1}^{4} u_i \cdot v_{n+1-i} = 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 = 0$ in $F_2$.

**Definition 2.13** Let N=nb, for n, b $\in \mathbb{N}$; $u_i, v_i \in F_2^b$ for $1 \leq i \leq n$ and $u = u_1u_2 \ldots u_n$, $v = v_1v_2 \ldots v_n \in F_2^N$, $\langle\ ,\ \rangle_3 : F_2^N \times F_2^N \to F_2$ is defined as

$$\langle u, v\rangle_3 \coloneqq \sum_{i=1}^{n} \langle u, v\rangle_1 \ . \tag{2.15}$$

**Definition 2.14** Let N=nb, for n, b $\in \mathbb{N}$; $u_i, v_i \in F_2^b$ for $1 \leq i \leq n$ and $u = u_1u_2 \ldots u_n$, $v = v_1v_2 \ldots v_n \in F_2^N$, $\langle u, v\rangle_4 : F_2^N \times F_2^N \to F_2$ is defined as

$$\langle u, v\rangle_4 \coloneqq \sum_{i=1}^{n} \langle u, v\rangle_2 \ . \tag{2.16}$$

**Example 2.15** For u=0101 1101 1100 and v=1110 0100 0110 in $F_2^{12}$, n=3, b=4;

$\langle \mathrm{u}, \mathrm{v}\rangle_3 = \sum_{i=1}^{3} u_i \cdot v_i$ $=\langle 0101, 1110\rangle_1 + \langle 1101, 0100\rangle_1 + \langle 1100, 0110\rangle_1$=1+1+1=1 in $F_2$.

$\langle \mathrm{u}, \mathrm{v}\rangle_4 = \sum_{i=1}^{3} u_i \cdot v_i$ $=\langle 0101, 1110\rangle_2 + \langle 1101, 0100\rangle_2 + \langle 1100, 0110\rangle_2$=0+0+1=1 in $F_2$.

**Definition 2.15** Let ⟨ , ⟩ be an inner product and u, v ∈ $F_2^n$. If ⟨ $u$ , v ⟩=0, we say u is orthogonal to v under inner product ⟨ , ⟩.

Let $C \subseteq F_2^N$ be a code. The orthogonal complement of $C$ (or the dual of C) is the set of all vectors in $F_2^N$ that are orthogonal to every codeword in $C$.

$C^{\perp}$:={v ∈ $F_2^N$ | $\forall u \in C$ ⟨ $u$ , v ⟩=0 }. (2.17)

**Example 2.16** Let $C$ be the linear code in Example 2.5. Let $u_1 = 000000000000$, $u_2 = 010111100001$, $u_3 = 101010101010$, $u_4 = 111101001011$ and C={$u_1, u_2, u_3, u_4$} =<$u_2, u_3$>. Choose n=3, b=4 and t=2 as in Example 2.4. Use $\langle u, v\rangle_4$ of Definition 2.14 for inner product. Obviously $\langle u_1, v\rangle_4$=0 ∀v∈$F_2^{12}$.

If both $\langle u_2, v\rangle_4 = 0$ and $\langle u_3, v\rangle_4 = 0$ , then $\langle u_4, v\rangle_4 = 0$ ∀ v ∈ $F_2^{12}$. So, the dual of $C$ is the set of all vectors in $F_2^{12}$ that are orthogonal to both $u_2$ and $u_3$.

Consider $v_1$=1111 1111 1100: $\langle u_2, v_1\rangle_4$=⟨0101 1110 0001, 1111 1111 1100⟩$_4$=0 i.e. $v_1$ is orthogonal to $u_2$. But $\langle u_3, v_1\rangle_4$=⟨1010 1010 1010, 1111 1111 1100⟩$_4$=1. So $v_1 \notin C^{\perp}$.

Consider $v_2$=1011 1010 1001: $\langle u_2, v_2\rangle_4$=⟨0101 1110 0001, 1011 1010 1001⟩$_4$=0 i.e. $v_2$ is orthogonal to $u_2$. Also $\langle u_3, v_2\rangle_4$=⟨1010 1010 1010, 1011 1010 1001⟩$_4$=0. So $v_2$ is orthogonal to $u_3$, too. Hence $v_2$ is orthogonal to every codeword of $C$ i.e. $v_2 \in C^{\perp}$.

All space, $F_2^{12}$ has $2^{12}$=4096 vectors. Among them $2^{10}$=1024 vectors belong to $C^{\perp}$.

Actually $C^{\perp}$=<1000 0000 0010, 0100 0000 0001, 0010 0000 0010, 0001 0000 0001, 0000 1000 0011, 0000 0100 0001, 0000 0010 0011, 0000 0001 0000, 0000 0000 1010, 0000 0000 0100>.

Recall Remark 2.2. There is a practical method of calculating weight enumerators of large codes. If a code $C$ has large number of codewords, then $C^{\perp}$ has small number of codewords. So calculating weight enumerators of $C^{\perp}$ is easier. If we have an identity between weight enumerators of a code and its dual, then by using this identity we can obtain weight enumerators of a large code $C$ from weight enumerators of the small code $C^{\perp}$ . If such an identity exists, then it is called Mac Williams identity named after Jessie Mac Williams who has first discovered an identity between weight enumerators of a code and its dual. In her identity Hamming weight enumerator in Definition 2.5 and inner product $\langle \mathrm{u}, \mathrm{v}\rangle_1$ in Definition 2.11 are used [5]. Of course, existence and formulation of this identity depends on both weight enumerator type and inner product type.

We can consider MacWilliams identity as a function on a set of weight enumerators of codes of $F_2^N$. For N$\in \mathbb{N}$, define:

$A_{\mathbb{N},N,k}$:={p$(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k] | there\ exists$ [N, k]code $C$ such that$W(C) = p(x_1, \dots, x_k)$} where a [N,k] code means a linear code of dimension k, with each codeword of that code has length N.

Then Mac Williams identity becomes a function from $A_{\mathbb{N},N,k}$ to $A_{\mathbb{N},N,N-k}$.

MWI: $A_{\mathbb{N},N,k} \longrightarrow A_{\mathbb{N},N,N-k}$

MWI($W$(C)) $= W(C^{\perp})$

This function is well-defined if and only if $W(C_1) = W(C_2)$ implies $W\left({C_1}^{\perp}\right) = W\left({C_2}^{\perp}\right)$.

Define a relation on [N, k] codes such that $C_1 \sim C_2 \Leftrightarrow W(C_1) = W(C_2)$ . Then $\sim$ is an equivalence relation on [N, k] codes and the equivalence classes partition [n, k] codes. On the other hand, the duals of [N, k] codes are [N, N-k] codes and the set of [N, N-k] codes has a partition like the set of [N, k] codes. Let denote equivalence class of $C$ by [C]. Therefore, MWI is well defined if and only if MWI($W([C])) = W([C^{\perp}])$.

If W(C)=$p$ and W($C^{\perp}$)=$p'$, then MWI($p$)= $p'$. So we need the dual of each linear code in [C] has weight enumerator $p'$. This means the dual of each linear code in [C] must be in $[C^{\perp}]$. Since every linear code has a unique dual code, this requires |[C]|≤|$[C^{\perp}]$|. Similarly |$[C^{\perp}]$|≤ |$[(C^{\perp})^{\perp}]$|=|[C]| . So |[C]|=|$[C^{\perp}]$|.

The set $A_{\mathbb{N},N,k}$, equivalence classes ([C]) and so their sizes are determined by metrics. However, duality bijection is determined by an inner product. Therefore, inner product should be chosen so wisely that well definedness of MWI is satisfied. However, sometimes sizes of equivalence classes in [N, k] codes and [N, N-k] codes are incompatible. In that case defining a “wise” inner product is impossible.

**Example 2.17** For RT weight enumerator in Definition 2.6, $A_{\mathbb{N},3,2} = \{1 + z + 2z^2, 1 + z + 2z^3,\ 1 + z^2 + 2z^3\}$. We can represent each equivalence class by unique polynomial p ∈ $A_{\mathbb{N},N,k}$. If W(C)=p we can notate [C] by [p]. We have 7 [3,2] codes.

$|[1 + z + 2z^2]|$=1; $|[1 + z + 2z^3]|$=2; $|[\ 1 + z^2 + 2z^3]|$=4. On the other hand, $A_{\mathbb{N},3,1} = \{1 + z,\ 1 + z^2,\ 1 + z^3\}$ and $|[1 + z]|$=1; $|[\ 1 + z^2]|$=2; $|[\ 1 + z^3\ ]|$=4.

There is only one code in $[1 + z + 2z^2]$ and luckily, we have only one equivalence class in [3, 1] codes: $[1 + z]$. We can check that $[1 + z + 2z^2]$= { < 100 , 010 > } and $[1 + z]$= { < 100 > }. So, we can define inner product such wise that $< 100, 010 >^{\perp} = < 100 >$. But also $[1 + z + 2z^3]$= { <100, 001>, <100, 011> } and $[1 + z^2]$={ <110>, <010>}. So, dual of <100, 001> must be either <110> or <101> and the other one must be dual of <100, 011>. Similarly, duals of codes in $[\ 1 + z^2 + 2z^3]$ must be in $[\ 1 + z^3\ ]$. Although inner product $\langle u, v\rangle_1$ in Definition 2.11 does not satisfy these correspondences, $\langle u, v\rangle_2$ in Definition 2.12 satisfies.

**Example 2.18** For m-spotty RT weight enumerator in Definition 2.8, if we choose n=1, b=3, t=2; then $A_{\mathbb{N},3,2} = \{1 + z + 2z^2, 1 + 3z\}$. Its equivalence class’ sizes are $|[1 + z + 2z^2]|$ =6 and $|[1 + 3z]|$=1.

However, $A_{\mathbb{N},3,1} = \{1 + z^2, 1 + z\}$ with equivalence class’ sizes$|[1 + z^2]| = 4$ and $|[1 + z]|$=3. The sizes of classes are incompatible. Thus, that classes cannot be matched. So there is no way of finding an inner product that makes establishing a Mac Williams identity possible.

# CHAPTER 3

## LEMMA'S AND THEOREMS

An identity between the weight enumerators of a code and its dual was first established by Jessie Mac Williams using inner product defined in Definition 2.11 and Hamming weight enumerator defined in Definition 2.5.

**Theorem 3.1** [1] Let $C \leq F_2^N$ be a linear code, $C^{\perp}$ be the dual code of $C$ with respect to $\langle \, , \, \rangle_1$. Let $A_i$, $B_i$ denote the number of codewords of weight i in $C$ and $C^{\perp}$ respectively. Then these quantities are related by the equation

$$\sum_{i=0}^{n} A_i (1+z)^{n-i} (1-z)^i = |C| \sum_{i=0}^{n} B_i z^i \, . \qquad (3.1)$$

Similar identities have been established for different metrics and inner products in subsequent years. These identities are called Mac Williams identities in memory of Jessie Mac Williams.

**Theorem 3.2** [6] Let $C \leq F_2^N$ be a linear code, N=nb, $1 \leq t \leq b$, $C^{\perp}$ be the dual code of $C$ with respect to $\langle \, , \, \rangle_3$ of Definition 2.13, and $W_{mH}(\mathrm{C})(\mathrm{z})$ be the weight enumerator in Definition 2.7. Then,

$$W_{mH}(\mathrm{C})(\mathrm{z}) = \sum_{\substack{(\alpha_0,\alpha_1,\ldots,\alpha_b) \\ (\alpha_0,\alpha_1,\ldots,\alpha_b \geq 0) \\ \alpha_0+\alpha_1+\cdots+\alpha_b = n}} A^{\perp}_{(\alpha_0,\alpha_1,\ldots,\alpha_b)} \prod_{j=0}^{b} \left(z^{\left\lceil \frac{j}{t} \right\rceil}\right)^{\alpha_j}$$

$$= \frac{1}{|C|} \sum_{\substack{(\alpha_0,\alpha_1,\ldots,\alpha_b) \\ (\alpha_0,\alpha_1,\ldots,\alpha_b \geq 0) \\ \alpha_0+\alpha_1+\cdots+\alpha_b = n}} A_{(\alpha_0,\alpha_1,\ldots,\alpha_b)} \prod_{j=0}^{b} \left(g_j^t(z)\right)^{\alpha_j} \qquad (3.2)$$

where $(\alpha_0, \alpha_1, \ldots, \alpha_b)$ is the weight distribution vector of codewords which has $\alpha_i$ blocks of weight i; $A_{(\alpha_0,\alpha_1,\ldots,\alpha_b)}$ and $A^{\perp}_{(\alpha_0,\alpha_1,\ldots,\alpha_b)}$ are the number of codewords which has $(\alpha_0, \alpha_1, \ldots, \alpha_b)$ weight distribution in $C$ and $C^{\perp}$, respectively; and

$$g_j^t(z) = \sum_{p=0}^{b} \left\{ \sum_{s=0}^{p} (-1)^{p-s} \binom{j}{p-s} \binom{b-j}{s} \right\} z^{\left\lceil \frac{p}{t} \right\rceil} \, . \qquad (3.3)$$

**Theorem 3.3** [7] Let $C \leq F_2^N$ be a linear code, N=nb, $1 \leq t \leq b$, $C^{\perp}$ be the dual code of $C$ with respect to $\langle\ ,\ \rangle_3$ of Definition 2.13, and $W_{mH}(C)(z)$ be the weight enumerator in Definition 2.9. Then

$$W_{CmH}(C^{\perp})(X_0, X_1, \dots, X_b) = \frac{1}{|C|} W_{CmH}(C)((X_0, X_1, \dots, X_b)^t M_b) \quad (3.4)$$

where $M_b = \begin{bmatrix} \gamma_{0,0} & \cdots & \gamma_{0,b} \\ \vdots & \ddots & \vdots \\ \gamma_{b,0} & \cdots & \gamma_{b,b} \end{bmatrix}$ is a (b+1) x (b+1) matrix where $\gamma_{j,l}$ is the coefficient of $x^l$ in the polynomial $(1+x)^{b-j}(1-x)^j$.

As indicated in Example 2.18, there is no way to construct a suitable inner product which makes establishing a MacWilliams identity under m-spotty weight enumerator defined in Definition 2.8 possible. However, Mac Williams identities are established for complete m-spotty both Hamming and RT weight enumerators. Absence of Mac Williams identities for Definition 2.8 are filled with a transformation of weight enumerators of Definition 2.10. These multivariable weight enumerators transform to one variable weight enumerators by letting $z_j = z^{\lceil \frac{j}{t} \rceil}$.

To the best of my knowledge a Mac Williams identity has not been established for weight enumerators defined in Definition 2.6.

We establish a Mac Williams identity for complete m-spotty RT weight enumerator defined in Definition 2.10 and under inner product $\langle u, v \rangle_4$ in Definition 2.14.

### 3.1 Statement of Main Theorem

**Theorem 3.4** Let $C \leq F_2^N$ be a linear code, b, n, j $\in \mathbb{N}$, N=nb, and $C^{\perp}$ be the dual code of $C$ with respect to $\langle\ ,\ \rangle_4$ . Then Mac Williams identity between complete m-spotty RT weight enumerators of $C$ and $C^{\perp}$ is established as

$$W_{CmRT}(C^{\perp})(z_0, \dots, z_b) = MWI_{CmRT}\left(W_{CmRT}(C)(z_0, \dots, z_b)\right) = \frac{1}{|C|} W_{CmRT}(C)(\tau_0, \dots, \tau_b)$$

where $\propto_{u,i}$ is as defined in Definition 2.10, and $\tau_0 := z_0 + \sum_{i=1}^{b} 2^{i-1} z_i$, $\tau_j := z_0 + \sum_{i=1}^{b-j} 2^{i-1} z_i - 2^{b-j} z_{b-j+1}$ for $1 \leq j \leq b$. So

$$W_{CmRT}(C^{\perp})(z_0, \dots, z_b) = \frac{1}{|C|} \sum_{u \in C} \prod_{i=0}^{b} \tau_i^{\propto_{u,i}}. \quad (3.5)$$

**Example 3. 1** Let b=4. Then,

$\tau_0 = z_0 + z_1 + 2z_2 + 4z_3 + 8z_4$, $\tau_1 = z_0 + z_1 + 2z_2 + 4z_3 - 8z_4$, $\tau_2 = z_0 + z_1 + 2z_2 - 4z_3$, $\tau_3 = z_0 + z_1 - 2z_2$, $\tau_4 = z_0 - z_1$.

So $W_{CmRT}(C^{\perp})(z_0, z_1, z_2, z_3, z_4) = \frac{1}{|C|} W_{CmRT}(C)(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4) = \frac{1}{|C|} W_{CmRT}(C)(z_0 + z_1 + 2z_2 + 4z_3 + 8z_4,\ z_0 + z_1 + 2z_2 + 4z_3 - 8z_4, z_0 + z_1 + 2z_2 - 4z_3,\ z_0 + z_1 - 2z_2,\ z_0 - z_1)$.

### 3.2 Lemmas

**Definition 3.1** Let G be a group, and f be any function from $F_2^N$ to G. Hadamard function of f is defined as

$$\tilde{f}\,(\mathrm{u}) := \sum_{v \in F_2^N} (-1)^{\langle \mathrm{u,v} \rangle_4} f(v) \tag{3.6}$$

We will give an example of this function before stating an important result about this function.

**Example 3.1** Let N=6, b=3, u=000 010, $G_1 = \mathbb{Z}$, $G_2 = \mathbb{Z}[s,t]$, $x = x_1x_2x_3x_4x_5x_6 \in F_2^N$, $x_i \in F_2$ , $\overline{x_i} = \begin{cases} 0 \in \mathbb{Z},\ x_i = 0 \in F_2 \\ 1 \in \mathbb{Z},\ x_i = 1 \in F_2 \end{cases}$ for 1≤ i ≤ N.

Let $f_1: F_2^N \longrightarrow \mathbb{Z}$ , $f_1(\mathrm{x}) = \overline{x_1} + \overline{x_3}$ ; $f_2: F_2^N \longrightarrow \mathbb{Z}$ , $f_2(\mathrm{x}) = 5\overline{x_1} + \overline{x_4 x_5}$; $f_3: F_2^N \longrightarrow \mathbb{Z}[s,t]$ , $f_3(\mathrm{x}) = s^{f_1(x)} \cdot t^{f_2(x)}$ .

Then $\widetilde{f_1}$ (u)= 0, $\widetilde{f_2}$ (u)=-16 and $\widetilde{f_3}$ (u)= 4 -4t +4s - 4st + 4s$t^5$- 4s$t^6$ + 4$s^2t^5$- 4$s^2t^6$

(See Table 3.1).

**Lemma 3.1** [3] Let G be a group, f be any function from $F_2^N$ to G, $C \leq F_2^N$ be a linear code. Then $\sum_{v \in C^{\perp}} f(v) = \frac{1}{|C|} \sum_{u \in C} \tilde{f}\,(\mathrm{u})$. (3.7)

We will prove this lemma after verifying it by an example.

**Example 3.2** Let $C$= {000 000, 000 010, 001 111, 001 101}, $f = f_3$.

Then $C^{\perp}$ = {000000, 000101, 001000, 001101, 010000, 010101, 011000, 011101, 100001, 100100, 101001, 101100, 110001, 110100, 111001, 111100}.
$\sum_{u \in C} \tilde{f}\,(\mathrm{u}) = \tilde{f}\,(000\ 000) + \tilde{f}\,(000010) + \tilde{f}\,(001\ 111) + \tilde{f}\,(001\ 101)$
=(12 + 4t + 12s + 4st + 12s$t^5$ + 4s$t^6$ + 12$s^2t^5$+ 4$s^2t^6$)+(4 -4t +4s - 4st + 4s$t^5$- 4s$t^6$ + 4$s^2t^5$- 4$s^2t^6$) +(0)+(0)

Table 3.1 Calculation of $\widetilde{f_3}$ (000 010)

| | **Additive contribution to $\widetilde{f_1}$ (000 010)** | **Additive contribution to $\widetilde{f_2}$ (000 010)** | **Additive contribution to $\widetilde{f_3}$ (000 010)** |
|---|---|---|---|
| **Vectors (v)** | **i.e.** $(-1)^{\langle u,v \rangle_4} f_1(v)$ | **i.e.** $(-1)^{\langle u,v \rangle_4} f_2(v)$ | **i.e.** $(-1)^{\langle u,v \rangle_4} f_3(v)$ |
| 000 000 | 0 | 0 | 1 |
| 000 001 | 0 | 0 | 1 |
| 000 010 | 0 | 0 | -1 |
| 000 011 | 0 | 0 | -1 |
| 000 100 | 0 | 0 | 1 |
| 000 101 | 0 | 0 | 1 |
| 000 110 | 0 | -1 | -t |
| 000 111 | 0 | -1 | -t |
| 001 000 | 1 | 0 | s |
| 001 001 | 1 | 0 | s |
| 001 010 | -1 | 0 | -s |
| 001 011 | -1 | 0 | -s |
| 001 100 | 1 | 0 | s |
| 001 101 | 1 | 0 | s |
| 001 110 | -1 | -1 | -s t |
| 001 111 | -1 | -1 | -s t |
| 010 000 | 0 | 0 | 1 |
| 010 001 | 0 | 0 | 1 |
| 010 010 | 0 | 0 | -1 |
| 010 011 | 0 | 0 | -1 |

Table 3.1 (cont'd)

| | | | |
|---|---|---|---|
| 010 100 | 0 | 0 | 1 |
| 010 101 | 0 | 0 | 1 |
| 010 110 | 0 | -1 | -t |
| 010 111 | 0 | -1 | -t |
| 011 000 | 1 | 0 | s |
| 011 001 | 1 | 0 | s |
| 011 010 | -1 | 0 | -s |
| 011 011 | -1 | 0 | -s |
| 011 100 | 1 | 0 | s |
| 011 101 | 1 | 0 | s |
| 011 110 | -1 | -1 | -s t |
| 011 111 | -1 | -1 | -s t |
| 100 000 | 1 | 5 | $st^5$ |
| 100 001 | 1 | 5 | $st^5$ |
| 100 010 | -1 | -5 | - $st^5$ |
| 100 011 | -1 | -5 | - $st^5$ |
| 100 100 | 1 | 5 | $st^5$ |
| 100 101 | 1 | 5 | $st^5$ |
| 100 110 | -1 | -6 | -s $t^6$ |
| 100 111 | -1 | -6 | -s $t^6$ |
| 101 000 | 2 | 5 | $s^2t^5$ |
| 101 001 | 2 | 5 | $s^2t^5$ |
| 101 010 | -2 | -5 | -$s^2t^5$ |

Table 3.1 (cont'd)

| | | | |
|---|---|---|---|
| 101 011 | -2 | -5 | -$s^2t^5$ |
| 101 100 | 2 | 5 | $s^2t^5$ |
| 101 101 | 2 | 5 | $s^2t^5$ |
| 101 110 | -2 | -6 | -$s^2 t^6$ |
| 101 111 | -2 | -6 | -$s^2t^6$ |
| 110 000 | 1 | 5 | s$t^5$ |
| 110 001 | 1 | 5 | s$t^5$ |
| 110 010 | -1 | -5 | - s$t^5$ |
| 110 011 | -1 | -5 | - s$t^5$ |
| 110 100 | 1 | 5 | s$t^5$ |
| 110 101 | 1 | 5 | s$t^5$ |
| 110 110 | -1 | -6 | -s $t^6$ |
| 110 111 | -1 | -6 | -s $t^6$ |
| 111 000 | 2 | 5 | $s^2 t^6$ |
| 111 001 | 2 | 5 | $s^2t^5$ |
| 111 010 | -2 | -5 | -$s^2t^5$ |
| 111 011 | -2 | -5 | -$s^2t^5$ |
| 111 100 | 2 | 5 | $s^2t^5$ |
| 111 101 | 2 | 5 | $s^2t^5$ |
| 111 110 | -2 | -6 | -$s^2 t^6$ |
| 111 111 | -2 | -6 | -$s^2t^6$ |
| **Sum** | **0** | **-16** | **4 - 4t + 4s - 4st + 4 s$t^5$- 4s $t^6$+ 4$s^2t^5$- 4$s^2t^6$** |

$=16+16s+16st^5+16s^2t^5$. Calculations can be checked from Table 3.1, Table 3.2, Table 3.3 and Table 3.4. So,

$$\frac{1}{|C|}\sum_{u\in C}\tilde{f}\,(u)=\frac{\sum_{u\in C}\tilde{f}\,(u)}{4}=4 + 4s + 4st^5 + 4s^2t^5. \quad (3.8)$$

$$\sum_{v\in C^{\perp}} f(v) = 4 + 4s + 4st^5 + 4s^2t^5. \quad (3.9)$$

These calculations can be checked from Table 3.5.

Table 3.2 Calculation of $\widetilde{f_3}$ (000 000)

| Vectors (v) | **Additive contribution to $\widetilde{f_1}$ (000 000)** i.e. $(-1)^{\langle u,v \rangle_4} f_1(v)$ | **Additive contribution to $\widetilde{f_2}$ (000 000)** i.e. $(-1)^{\langle u,v \rangle_4} f_2(v)$ | **Additive contribution to $\widetilde{f_3}$ (000 000)** i.e. $(-1)^{\langle u,v \rangle_4} f_3(v)$ |
|---|---|---|---|
| 000 000 | 0 | 0 | 1 |
| 000 001 | 0 | 0 | 1 |
| 000 010 | 0 | 0 | 1 |
| 000 011 | 0 | 0 | 1 |
| 000 100 | 0 | 0 | 1 |
| 000 101 | 0 | 0 | 1 |
| 000 110 | 0 | 1 | t |
| 000 111 | 0 | 1 | t |
| 001 000 | 1 | 0 | s |
| 001 001 | 1 | 0 | s |
| 001 010 | 1 | 0 | s |
| 001 011 | 1 | 0 | s |
| 001 100 | 1 | 0 | s |
| 001 101 | 1 | 0 | s |
| 001 110 | 1 | 1 | s t |
| 001 111 | 1 | 1 | s t |
| 010 000 | 0 | 0 | 1 |
| 010 001 | 0 | 0 | 1 |
| 010 010 | 0 | 0 | 1 |
| 010 011 | 0 | 0 | 1 |

Table 3.2 (cont’d)

| | | | |
|---|---|---|---|
| 010 100 | 0 | 0 | 1 |
| 010 101 | 0 | 0 | 1 |
| 010 110 | 0 | 1 | t |
| 010 111 | 0 | 1 | t |
| 011 000 | 1 | 0 | s |
| 011 001 | 1 | 0 | s |
| 011 010 | 1 | 0 | s |
| 011 011 | 1 | 0 | s |
| 011 100 | 1 | 0 | s |
| 011 101 | 1 | 0 | s |
| 011 110 | 1 | 1 | s t |
| 011 111 | 1 | 1 | s t |
| 100 000 | 1 | 5 | $s\,t^5$ |
| 100 001 | 1 | 5 | $s\,t^5$ |
| 100 010 | 1 | 5 | $s\,t^5$ |
| 100 011 | 1 | 5 | $s\,t^5$ |
| 100 100 | 1 | 5 | $s\,t^5$ |
| 100 101 | 1 | 5 | $s\,t^5$ |
| 100 110 | 1 | 6 | $s\,t^6$ |
| 100 111 | 1 | 6 | $s\,t^6$ |
| 101 000 | 2 | 5 | $s^2t^5$ |
| 101 001 | 2 | 5 | $s^2t^5$ |
| 101 010 | 2 | 5 | $s^2t^5$ |

Table 3.2 (cont'd)

| | | | |
|---|---|---|---|
| 101 011 | 2 | 5 | $s^2t^5$ |
| 101 100 | 2 | 5 | $s^2t^5$ |
| 101 101 | 2 | 5 | $s^2t^5$ |
| 101 110 | 2 | 6 | $s^2t^6$ |
| 101 111 | 2 | 6 | $s^2t^6$ |
| 110 000 | 1 | 5 | $s\,t^5$ |
| 110 001 | 1 | 5 | $s\,t^5$ |
| 110 010 | 1 | 5 | $s\,t^5$ |
| 110 011 | 1 | 5 | $s\,t^5$ |
| 110 100 | 1 | 5 | $s\,t^5$ |
| 110 101 | 1 | 5 | $s\,t^5$ |
| 110 110 | 1 | 6 | $s\,t^6$ |
| 110 111 | 1 | 6 | $s\,t^6$ |
| 111 000 | 2 | 5 | $s^2t^5$ |
| 111 001 | 2 | 5 | $s^2t^5$ |
| 111 010 | 2 | 5 | $s^2t^5$ |
| 111 011 | 2 | 5 | $s^2t^5$ |
| 111 100 | 2 | 5 | $s^2t^5$ |
| 111 101 | 2 | 5 | $s^2t^5$ |
| 111 110 | 2 | 6 | $s^2t^6$ |
| 111 111 | 2 | 6 | $s^2t^6$ |
| **Sum** | **64** | **176** | $\mathbf{12 + 4t + 12s + 4st + 12st^5 + 4st^6 + 12s^2t^5 + 4s^2t^6}$ |

Table 3.3 Calculation of $\widetilde{f_3}$ (001 101)

| | **Additive contribution to** $\widetilde{f_1}$ **(001 101)** | **Additive contribution to** $\widetilde{f_2}$ **(001 101)** | **Additive contribution to** $\widetilde{f_3}$ **(001 101)** |
|---|---|---|---|
| **Vectors (v)** | **i.e.** $(-1)^{\langle u,v\rangle_4} f_1(v)$ | **i.e.** $(-1)^{\langle u,v\rangle_4} f_2(v)$ | **i.e.** $(-1)^{\langle u,v\rangle_4} f_3(v)$ |
| 000 000 | 0 | 0 | 1 |
| 000 001 | 0 | 0 | -1 |
| 000 010 | 0 | 0 | 1 |
| 000 011 | 0 | 0 | -1 |
| 000 100 | 0 | 0 | -1 |
| 000 101 | 0 | 0 | 1 |
| 000 110 | 0 | -1 | -t |
| 000 111 | 0 | 1 | t |
| 001 000 | 1 | 0 | s |
| 001 001 | -1 | 0 | -s |
| 001 010 | 1 | 0 | s |
| 001 011 | -1 | 0 | -s |
| 001 100 | -1 | 0 | -s |
| 001 101 | 1 | 0 | s |
| 001 110 | -1 | -1 | -s t |
| 001 111 | 1 | 1 | s t |
| 010 000 | 0 | 0 | 1 |
| 010 001 | 0 | 0 | -1 |
| 010 010 | 0 | 0 | 1 |
| 010 011 | 0 | 0 | -1 |

Table 3.3 (cont'd)

| | | | |
|---|---|---|---|
| 010 100 | 0 | 0 | -1 |
| 010 101 | 0 | 0 | 1 |
| 010 110 | 0 | -1 | -t |
| 010 111 | 0 | 1 | t |
| 011 000 | 1 | 0 | s |
| 011 001 | -1 | 0 | -s |
| 011 010 | 1 | 0 | s |
| 011 011 | -1 | 0 | -s |
| 011 100 | -1 | 0 | -s |
| 011 101 | 1 | 0 | s |
| 011 110 | -1 | -1 | -s t |
| 011 111 | 1 | 1 | s t |
| 100 000 | -1 | -5 | - $st^5$ |
| 100 001 | 1 | 5 | $st^5$ |
| 100 010 | -1 | -5 | - $st^5$ |
| 100 011 | 1 | 5 | $st^5$ |
| 100 100 | 1 | 5 | $st^5$ |
| 100 101 | -1 | -5 | - $st^5$ |
| 100 110 | 1 | 6 | $st^6$ |
| 100 111 | -1 | -6 | - $st^6$ |
| 101 000 | -2 | -5 | -$s^2t^5$ |
| 101 001 | 2 | 5 | $s^2t^5$ |
| 101 010 | -2 | -5 | -$s^2t^5$ |

Table 3.3 (cont'd)

| | | | |
|---|---|---|---|
| 101 011 | 2 | 5 | $s^2t^5$ |
| 101 100 | 2 | 5 | $s^2t^5$ |
| 101 101 | -2 | -5 | -$s^2t^5$ |
| 101 110 | 2 | 6 | $s^2t^6$ |
| 101 111 | -2 | -6 | -$s^2t^6$ |
| 110 000 | -1 | -5 | - s$t^5$ |
| 110 001 | 1 | 5 | s$t^5$ |
| 110 010 | -1 | -5 | - s$t^5$ |
| 110 011 | 1 | 5 | s$t^5$ |
| 110 100 | 1 | 5 | s$t^5$ |
| 110 101 | -1 | -5 | - s$t^5$ |
| 110 110 | 1 | 6 | s$t^6$ |
| 110 111 | -1 | -6 | - s$t^6$ |
| 111 000 | -2 | -5 | -$s^2t^5$ |
| 111 001 | 2 | 5 | $s^2t^5$ |
| 111 010 | -2 | -5 | -$s^2t^5$ |
| 111 011 | 2 | 5 | $s^2t^5$ |
| 111 100 | 2 | 5 | $s^2t^5$ |
| 111 101 | -2 | -5 | -$s^2t^5$ |
| 111 110 | 2 | 6 | $s^2t^6$ |
| 111 111 | -2 | -6 | -$s^2t^6$ |
| **Sum** | **0** | **0** | **0** |

Table 3.4 Calculation of $\widetilde{f_3}$ (001 111)

| | **Additive contribution to $\widetilde{f_1}$ (001 111)** | **Additive contribution to $\widetilde{f_2}$ (001 111)** | **Additive contribution to $\widetilde{f_3}$ (001 111)** |
|---|---|---|---|
| **Vectors (v)** | **i.e.** $(-1)^{\langle u,v\rangle_4} f_1(v)$ | **i.e.** $(-1)^{\langle u,v\rangle_4} f_2(v)$ | **i.e.** $(-1)^{\langle u,v\rangle_4} f_3(v)$ |
| 000 000 | 0 | 0 | 1 |
| 000 001 | 0 | 0 | -1 |
| 000 010 | 0 | 0 | -1 |
| 000 011 | 0 | 0 | 1 |
| 000 100 | 0 | 0 | -1 |
| 000 101 | 0 | 0 | 1 |
| 000 110 | 0 | 1 | t |
| 000 111 | 0 | -1 | -t |
| 001 000 | 1 | 0 | s |
| 001 001 | -1 | 0 | -s |
| 001 010 | -1 | 0 | -s |
| 001 011 | 1 | 0 | s |
| 001 100 | -1 | 0 | -s |
| 001 101 | 1 | 0 | s |
| 001 110 | 1 | 1 | s t |
| 001 111 | -1 | -1 | -s t |
| 010 000 | 0 | 0 | 1 |
| 010 001 | 0 | 0 | -1 |
| 010 010 | 0 | 0 | -1 |
| 010 011 | 0 | 0 | 1 |

Table 3.4 (cont'd)

| | | | |
|---|---|---|---|
| 010 100 | 0 | 0 | -1 |
| 010 101 | 0 | 0 | 1 |
| 010 110 | 0 | 1 | t |
| 010 111 | 0 | -1 | -t |
| 011 000 | 1 | 0 | s |
| 011 001 | -1 | 0 | -s |
| 011 010 | -1 | 0 | -s |
| 011 011 | 1 | 0 | s |
| 011 100 | -1 | 0 | -s |
| 011 101 | 1 | 0 | s |
| 011 110 | 1 | 1 | s t |
| 011 111 | -1 | -1 | -s t |
| 100 000 | -1 | -5 | - $st^5$ |
| 100 001 | 1 | 5 | $st^5$ |
| 100 010 | 1 | 5 | $st^5$ |
| 100 011 | -1 | -5 | -$st^5$ |
| 100 100 | 1 | 5 | $st^5$ |
| 100 101 | -1 | -5 | - $st^5$ |
| 100 110 | -1 | -6 | -$st^6$ |
| 100 111 | 1 | 6 | $st^6$ |
| 101 000 | -2 | -5 | -$s^2t^5$ |
| 101 001 | 2 | 5 | $s^2t^5$ |
| 101 010 | 2 | 5 | $s^2t^5$ |

Table 3.4 (cont'd)

| | | | |
|---|---|---|---|
| 101 011 | -2 | -5 | -$s^2t^5$ |
| 101 100 | 2 | 5 | $s^2t^5$ |
| 101 101 | -2 | -5 | -$s^2t^5$ |
| 101 110 | -2 | -6 | -$s^2t^6$ |
| 101 111 | 2 | 6 | $s^2t^6$ |
| 110 000 | -1 | -5 | - s$t^5$ |
| 110 001 | 1 | 5 | s$t^5$ |
| 110 010 | 1 | 5 | $st^5$ |
| 110 011 | -1 | -5 | -s$t^5$ |
| 110 100 | 1 | 5 | s$t^5$ |
| 110 101 | -1 | -5 | -s$t^5$ |
| 110 110 | -1 | -6 | -s$t^6$ |
| 110 111 | 1 | 6 | s$t^6$ |
| 111 000 | -2 | -5 | -$s^2t^5$ |
| 111 001 | 2 | 5 | $s^2t^5$ |
| 111 010 | 2 | 5 | $s^2t^5$ |
| 111 011 | -2 | -5 | -$s^2t^5$ |
| 111 100 | 2 | 5 | $s^2t^5$ |
| 111 101 | -2 | -5 | -$s^2t^5$ |
| 111 110 | -2 | -6 | -$s^2t^6$ |
| 111 111 | 2 | 6 | $s^2t^6$ |
| **Sum** | **0** | **0** | **0** |

Table 3.5 Calculation of $\sum_{v \in C^{\perp}} f(v)$

| **Vectors** | $\boldsymbol{f_1(v)}$ | $\boldsymbol{f_2(v)}$ | $\boldsymbol{f_3(v)}$ |
|---|---|---|---|
| 000 000 | 0 | 0 | 1 |
| 000 101 | 0 | 0 | 1 |
| 001 000 | 1 | 0 | s |
| 001 101 | 1 | 0 | s |
| 010 000 | 0 | 0 | 1 |
| 010 101 | 0 | 0 | 1 |
| 011 000 | 1 | 0 | s |
| 011 101 | 1 | 0 | s |
| 100 001 | 1 | 5 | s $t^5$ |
| 100 100 | 1 | 5 | s $t^5$ |
| 101 001 | 2 | 5 | $s^2$ $t^5$ |
| 101 100 | 2 | 5 | $s^2t^5$ |
| 110 001 | 1 | 5 | s $t^5$ |
| 110 100 | 1 | 5 | s $t^5$ |
| 111 001 | 2 | 5 | $s^2$ $t^5$ |
| 111 100 | 2 | 5 | $s^2t^5$ |
| **Sum** | **16** | **40** | **4 + 4s + 4s $t^5$+ 4$s^2$ $t^5$** |

**Proof of Lemma 3.1** We want to show $\sum_{u\in C}\tilde{f}(\mathrm{u}) = |\boldsymbol{C}|\sum_{v\in C^{\perp}} f(v)$. (3.10)

$\sum_{u\in C}\tilde{f}(\mathrm{u}) = \sum_{u\in C}\sum_{v\in F_2^N}(-1)^{\langle \mathrm{u,v}\rangle_4} f(v)$ (by definition of $\tilde{f}$ (u).) (3.11)

$= \sum_{v\in F_2^N} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4}$ (change order of sum) (3.12)

$=\sum_{v\in C^{\perp}\cup(F_2^N\backslash C^{\perp})} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4}$ (since $C^{\perp}\cup(F_2^N\backslash C^{\perp}) = F_2^N$) (3.13)

$=\sum_{v\in C^{\perp}} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4} + \sum_{v\in(F_2^N\backslash C^{\perp})} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4}$ (3.14)

$=\sum_{v\in C^{\perp}} f(v)\sum_{u\in C}(-1)^{0} + \sum_{v\in(F_2^N\backslash C^{\perp})} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4}$ (if $u\in C$ *and* $v\in C^{\perp}$, then $\langle \mathrm{u,v}\rangle_4$=0 ) (3.15)

$=\sum_{v\in C^{\perp}} f(v)\sum_{u\in C} 1 + \sum_{v\in(F_2^N\backslash C^{\perp})} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4}$ (3.16)

$=\sum_{v\in C^{\perp}} f(v)|\boldsymbol{C}| + \sum_{v\in(F_2^N\backslash C^{\perp})} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4}$ (since $\sum_{u\in C} 1 = |\boldsymbol{C}|$) (3.17)

So, it is enough to show that $\sum_{v\in(F_2^N\backslash C^{\perp})} f(v)\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4} = 0$.

({-1,1}, · ) is a multiplicative group with 2 elements. Fix $v\in(F_2^N\backslash C^{\perp})$ and define: φ: $C$⟶({-1,1}, · ) , φ(u)= $(-1)^{\langle \mathrm{u,v}\rangle_4}$. Then φ (0) =1. Also there exists $u'\in C$ such that $\langle u',\mathrm{v}\rangle_4$=1, otherwise v would be in $C^{\perp}$. So φ($u'$)= -1 and φ is onto.

φ is group homomorphism because φ$(u_1+u_2)$ $=(-1)^{\langle u_1+u_2,\mathrm{v}\rangle_4}$ $=(-1)^{\langle u_1,\mathrm{v}\rangle_4}(-1)^{\langle u_2,\mathrm{v}\rangle_4}=$ φ$(u_1)$ · φ$(u_2)$. Then $C/_{Ker\varphi}\cong(\{-1,1\},\cdot\ )$ by 1st Isomorphism Theorem. So $\frac{|\boldsymbol{C}|}{|Ker\varphi|}=2$. So $|Ker\varphi| = \frac{|\boldsymbol{C}|}{2} = |C\backslash Ker\varphi|$. Then

$\sum_{u\in C}(-1)^{\langle \mathrm{u,v}\rangle_4} = \sum_{u\in Ker\varphi}(-1)^{\langle \mathrm{u,v}\rangle_4} + \sum_{u\notin Ker\varphi}(-1)^{\langle \mathrm{u,v}\rangle_4}$ (3.18)

$= \sum_{u\in Ker\varphi} 1 + \sum_{u\notin Ker\varphi}(-1) = |Ker\varphi| - |C\backslash Ker\varphi| = \frac{|\boldsymbol{C}|}{2} - \frac{|\boldsymbol{C}|}{2} = 0.$ (3.19)

Hence $\sum_{v\in(F_2^N\backslash C^{\perp})} f(v)\cdot 0 = 0.$ (3.20)

This proves Lemma 3.1.

Benefit of Lemma 3.1: Thanks to this lemma we can prove Theorem 3.1. If we choose $f(v)=\prod_{j=0}^{b} {z_j}^{\propto_{v,j}}$, then it will be enough to show $\tilde{f}$ (u)=$\prod_{j=0}^{b} {\tau_j}^{\propto_{u,j}}$ to prove Theorem 3.1.

**Lemma 3.2** [3] Let u, v ∈ $F_2^b$, u=$u_1 \dots u_b$, v=$v_1 \dots v_b$, $u_i, v_i \in F_2$. Assume u is fixed. Let $wt_{RT}(u)$=p. Let k ≠0. Then

$$\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} (-1)^{\langle u,v\rangle_2} = \begin{cases} 2^{k-1} & if\ p+k<b+1 \\ -2^{k-1} & if\ p+k=b+1 \\ 0 & if\ p+k>b+1 \end{cases}. \tag{3.21}$$

**Proof of Lemma 3.2**

$wt_{RT}(v)$=k means that the position of the rightmost nonzero digit of v is k, i.e. $v_k$=1, $v_i$=0 for i > k and $v_i$ can be anything for i < k. If p=0, $\langle u,v\rangle_2$=0 for any vector v. Then the first case of the lemma is valid and

$$\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} (-1)^{\langle u,v\rangle_2} = \sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} 1 = \left|\{v\in F_2^b \,\middle|\, wt_{RT}(v)=k\}\right| = 2^{k-1}. \tag{3.22}$$

Now we can assume p≠0. $wt_{RT}(u)$=p means position of rightmost nonzero digits of u is p; i.e. $u_p$=1, $u_i$=0 for i > p and $u_i$ can be anything for i < p.

We fix a vector u, and take arbitrary vector v of weight k in $F_2^b$. Then

$$(-1)^{\langle u,v\rangle_2} = (-1)^{\sum_{i=1}^{b} u_i\cdot v_{b+1-i}} = \prod_{i=1}^{b}(-1)^{u_i\cdot v_{b+1-i}} \tag{3.23}$$

$$= \left(\prod_{i=1}^{b-k}(-1)^{u_i\cdot v_{b+1-i}}\right)\left(\prod_{i=b-k+1}^{b-k+1}(-1)^{u_i\cdot v_{b+1-i}}\right)\left(\prod_{i=b-k+2}^{b}(-1)^{u_i\cdot v_{b+1-i}}\right) \tag{3.24}$$

$= \left(\prod_{i=1}^{b-k}(-1)^{0}\right)\left(\prod_{i=b-k+1}^{b-k+1}(-1)^{u_i\cdot v_{b+1-i}}\right)\left(\prod_{i=b-k+2}^{b}(-1)^{u_i\cdot v_{b+1-i}}\right)$ since i ≤ b-k implies k ≤ b-i < b-i+1, and so $v_{b+1-i}$=0. Hence,

$$(-1)^{\langle u,v\rangle_2} = 1\cdot(-1)^{u_{b-k+1}\cdot v_k}\left(\prod_{i=b-k+2}^{b}(-1)^{u_i\cdot v_{b+1-i}}\right) \tag{3.25}$$

$$= (-1)^{u_{b-k+1}\cdot 1}\left(\prod_{i=b-k+2}^{b}(-1)^{u_i\cdot v_{b+1-i}}\right) \text{ since } v_k = 1. \tag{3.26}$$

Let's sum this function for each v of weight k over $F_2^b$.

$$\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} = \sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} (-1)^{u_{b-k+1}}\left(\prod_{i=b-k+2}^{b}(-1)^{u_i\cdot v_{b+1-i}}\right) \tag{3.27}$$

$$= (-1)^{u_{b-k+1}}\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} \left(\prod_{i=b-k+2}^{b}(-1)^{u_i\cdot v_{b+1-i}}\right) \tag{3.28}$$

$$= (-1)^{u_{b-k+1}}\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} \left(\prod_{j=1}^{k-1}(-1)^{v_j\cdot u_{b+1-j}}\right) \tag{3.29}$$

$$= (-1)^{u_{b-k+1}}\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=k}} \left((-1)^{v_1\cdot u_b}(-1)^{v_2\cdot u_{b-1}}\dots(-1)^{v_{k-1}\cdot u_{b-k+2}}\right) \tag{3.30}$$

$$= (-1)^{u_{b-k+1}}\sum_{\substack{v_1\in F_2\\ \vdots\\ v_{k-1}\in F_2}} \left[(-1)^{v_1\cdot u_b}(-1)^{v_2\cdot u_{b-1}}\dots(-1)^{v_{k-1}\cdot u_{b-k+2}}\right] \tag{3.31}$$

$= (-1)^{u_{b-k+1}} \left(\sum_{v_1 \in F_2} (-1)^{v_1 \cdot u_b}\right) \left(\sum_{v_2 \in F_2} (-1)^{v_2 \cdot u_{b-1}}\right) \ldots$

$\ldots \left(\sum_{v_{k-1} \in F_2} (-1)^{v_{k-1} \cdot u_{b-k+2}}\right)$ (3.32)

$= (-1)^{u_{b-k+1}} \left(\prod_{j=1}^{k-1} \sum_{v_j \in F_2} (-1)^{v_j \cdot u_{b+1-j}}\right)$ (3.33)

$= (-1)^{u_{b-k+1}} \left(\prod_{j=1}^{k-1} ((-1)^{0 \cdot u_{b+1-j}} + (-1)^{1 \cdot u_{b+1-j}})\right)$ (3.34)

$= (-1)^{u_{b-k+1}} \left(\prod_{j=1}^{k-1} (1 + (-1)^{u_{b+1-j}})\right)$ (3.35)

Case 1: $p + k < b + 1$

b-j+1 > b-k+1 > p for 1 ≤ j ≤ k-1 < k. So $u_{b+1-j} = 0$ and $u_{b-k+1} = 0$. Then $\sum_{\substack{v \in F_2^b \\ wt_{RT}(v)=k}} (-1)^{\langle u,v \rangle_2} = (-1)^0 \left(\prod_{j=1}^{k-1} (1 + (-1)^0)\right) = 1 \cdot \left(\prod_{j=1}^{k-1} 2\right) = 2^{k-1}$. (3.36)

Case 2: $p + k = b + 1$

b -j+1 > b-k+1 = p for 1 ≤ j ≤ k-1 < k. So $u_{b+1-j} = 0$ and $u_{b-k+1} = 1$. Then $\sum_{\substack{v \in F_2^b \\ wt_{RT}(v)=k}} (-1)^{\langle u,v \rangle_2} = (-1)^1 \left(\prod_{j=1}^{k-1} (1 + (-1)^0)\right) = (-1) \cdot \left(\prod_{j=1}^{k-1} 2\right)$

$= -2^{k-1}$. (3.37)

Case 3: $p + k > b + 1$

b-k+1 < b-k+2 ≤ p ≤ b. Then $u_{b-k+1}$ can be anything. Since p $\in [b - k + 2, b]$, there exists $j'$ such that $u_p = u_{b+1-j'}$ for 1 ≤ $j'$ ≤ k-1 < k. Then $j'^{th}$ factor $\left(1 + (-1)^{u_{b+1-j'}}\right) = (1 + (-1)^1) = 0$. So $\prod_{j=1}^{k-1} (1 + (-1)^{u_{b+1-j}}) = 0$.

Hence, $\sum_{\substack{v \in F_2^b \\ wt_{RT}(v)=k}} (-1)^{\langle u,v \rangle_2} = (-1)^{u_{b-k+1}} \cdot 0 = 0$. (3.38)

**Definition 3.2** Let u $\in F_2^b$ and $wt_{RT}(u)$=p. We will call the sum $\sum_{\substack{v \in F_2^b \\ wt_{RT}(v)=k}} (-1)^{\langle u,v \rangle_2}$ which depends on k and p as snack of u for k and denote it by S(k , p).

**Remark 3.1** For any vector u of $F_2^b$, snack of u for 0 is always 1.

S(0 , p)= $\sum_{\substack{v \in F_2^b \\ wt_{RT}(v)=0}} (-1)^{\langle u,v \rangle_2} = \sum_{v \in \{0\}} (-1)^{\langle u,v \rangle_2} = (-1)^{\langle u,0 \rangle_2} = 1$. (3.39)

**Remark 3.2** For k ≥ 1, $S(k,p) = \begin{cases} 2^{k-1} & if\ k < b - p + 1 \\ -2^{k-1} & if\ k = b - p + 1 \\ 0 & if\ k > b - p + 1 \end{cases}$ ,by Lemma 3.2. (3.40)

**Remark 3.3** Sum of all snacks of u is equal to $\sum_{v\in F_2^b}(-1)^{\langle u,v\rangle_2}$.

$$\sum_{v\in F_2^b}(-1)^{\langle u,v\rangle_2} = \sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=0}}(-1)^{\langle u,v\rangle_2} + \sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=1}}(-1)^{\langle u,v\rangle_2} + \cdots + \sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=b}}(-1)^{\langle u,v\rangle_2}$$

$= S(0, p) + S(1, p) + \ldots + S(k, p) = \sum_{k=0}^{b} S(k,p)$. (3.41)

**Definition 3.3** Let $u \in F_2^b$, $wt_{RT}(u)$=p, and $z_k$'s be variables for $1\leq k \leq b$ . The linear combination $\sum_{v\in F_2^b}(-1)^{\langle u,v\rangle_2} z_{wt_{RT}(v)}$ of $z_k's$ is called the dual contributor of block u for complete m-spotty RT weight enumerator and denoted by $\tau_p$.

**Remark 3.4** $\tau_p = \sum_{v\in F_2^b}(-1)^{\langle u,v\rangle_2} z_{wt_{RT}(v)}$ (3.42)

$$=\left(\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=0}}(-1)^{\langle u,v\rangle_2} z_{wt_{RT}(v)}\right)+\cdots+\left(\sum_{\substack{v\in F_2^b\\ wt_{RT}(v)=b}}(-1)^{\langle u,v\rangle_2} z_{wt_{RT}(v)}\right) \quad (3.43)$$

$= S(0, p)\, z_0 + S(1, p)\, z_1 + \ldots + S(b, p)\, z_b$ (3.44)

$=\sum_{k=0}^{b} S(k,p) = z_0 + \sum_{k=1}^{b} S(k,p)\, z_k$ (3.45)

## 3.3 Proof of the Main Theorem

We want to show that $W_{CmRT}(C^{\perp})(z_0,\ldots,z_b) = \frac{1}{|C|} W_{CmRT}(C)(\tau_0,\ldots,\tau_b)$, i.e. we want to show that $\sum_{v\in C^{\perp}} \prod_{i=0}^{b} {z_i}^{\propto_{v,i}} = \frac{1}{|C|}\sum_{u\in C}\prod_{j=0}^{b} {\tau_j}^{\propto_{u,j}}$ (3.46)

Let $f(v) = \prod_{i=0}^{b} {z_i}^{\propto_{v,i}}$ be contributor of a codeword $v$ of $C^{\perp}$.

Consider $v = v_1 \ldots v_n$, where $v_i \in {F_2}^b$ and define $p_i \coloneqq wt_{RT}(v_i)$. Remember $\propto_{v,j}$ is the number of blocks $v_j's$ of $v$ whose RT weight is $j$ . Then the number of factors $z_j$ in the product $\prod_{i=1}^{n} z_{p_i}$ is exactly $\propto_{v,j}$. So,

$\prod_{i=1}^{n} z_{p_i} = \prod_{j=0}^{b} {z_j}^{\propto_{v,j}}$ . (3.47)

Take a codeword u of $C$. Then,

$\tilde{f}\,(u) = \sum_{v\in F_2^N}(-1)^{\langle u,v\rangle_4} f(v)$ (3.48)

$= \sum_{v\in F_2^N}(-1)^{\langle u,v\rangle_4} \prod_{i=0}^{b} {z_i}^{\propto_{v,i}}$ (3.49)

$= \sum_{v\in F_2^N} \prod_{i=1}^{n}(-1)^{\langle u_i,v_i\rangle_2} \prod_{i=0}^{b} {z_i}^{\propto_{v,i}}$ (3.50)

$= \sum_{v\in F_2^N} \prod_{i=1}^{n}(-1)^{\langle u_i, v_i\rangle_2} \prod_{i=1}^{n} z_{wt_{RT}(v_i)}$ (3.51)

$=\sum_{v\in F_2^N} \prod_{i=1}^{n}(-1)^{\langle u_i, v_i\rangle_2} z_{wt_{RT}(v_i)}$ (3.52)

$=\sum_{v_1, v_2,, v_n\in F_2^b} \prod_{i=1}^{n}(-1)^{\langle u_i, v_i\rangle_2} z_{wt_{RT}(v_i)}$ (3.53)

$=\prod_{i=1}^{n} \sum_{v_i\in F_2^b}(-1)^{\langle u_i, v_i\rangle_2} z_{wt_{RT}(v_i)}$ (3.54)

Since $\sum_{v_i\in F_2^b}(-1)^{\langle u_i, v_i\rangle_2} z_{wt_{RT}(v_i)} = \tau_{wt_{RT}(u_i)}$ , the dual contributor of any block $u_i$ of weight p is $wt_{RT}(u_i)$.

So $\prod_{i=1}^{n} \sum_{v_i\in F_2^b}(-1)^{\langle u_i, v_i\rangle_2} z_{wt_{RT}(v_i)} = \prod_{i=1}^{n} \tau_{wt_{RT}(u_i)}$ (3.55)

$=\prod_{\tau_{wt_{RT}(u_i)}=0}^{b} \tau_{wt_{RT}(u_i)}{}^{\propto_{u, wt_{RT}(u_i)}} = \prod_{p=0}^{b} \tau_p{}^{\propto_{u,p}}.$ (3.56)

Hence $\tilde{f}$ (u)=$\prod_{p=0}^{b} \tau_p{}^{\propto_{u,p}}$. (3.57)

Applying Lemma 3.1, we get

$\sum_{v\in C^\perp} f(v) = \frac{1}{|C|}\sum_{u\in C} \tilde{f}$ (u), $\sum_{v\in C^\perp} \prod_{i=0}^{b} z_i{}^{\propto_{v,i}} = \frac{1}{|C|}\sum_{u\in C} \prod_{p=0}^{b} \tau_p{}^{\propto_{u,p}}$. (3.58)

So $W_{CmRT}(C^\perp)(z_0, \dots, z_b) = \frac{1}{|C|} W_{CmRT}(C)(\tau_0, \dots, \tau_b)$. (3.59)

It only remains to show that the dual contributor of p-weighted blocks, $\tau_p$, is $z_0 + \sum_{i=1}^{b-p} 2^{i-1} z_i - 2^{b-p} z_{b-p+1}$ for p>0 and $z_0 + \sum_{i=1}^{b} 2^{i-1} z_i$ for p=0.

$\tau_p = z_0 + \sum_{k=1}^{b} S(k, p)\, z_k$ by Remark 3.4. (3.60)

Then $\tau_0 = z_0 + \sum_{k<b+1} \mathrm{S}(\mathrm{k}, 0) z_k$. By using Remark 3.2, $\tau_0 = z_0 + \sum_{k<b+1} 2^{k-1} z_k$

i.e. $\tau_p = z_0 + \sum_{k=1}^{b} 2^{k-1} z_k$ . (3.61)

For p>0 $\tau_p = z_0 + \sum_{k<b-p+1} \mathrm{S}(\mathrm{k}, \mathrm{p}) z_k + \sum_{k=b-p+1} \mathrm{S}(\mathrm{k}, \mathrm{p}) z_k + \sum_{k>b-p+1} \mathrm{S}(\mathrm{k}, \mathrm{p}) z_k$

By using Remark 3.2, $\tau_p = z_0 + \sum_{k<b-p+1} 2^{k-1} z_k + \sum_{k=b-p+1}(-2^{k-1}) z_k + \sum_{k>b-p+1} 0 \cdot z_k$, i.e. $\tau_p = z_0 + \sum_{k=1}^{b-p} 2^{k-1} z_k - 2^{k-1} z_{b-p+1}$ . (3.62)

Therefore, we establish the identity.

# CHAPTER 4

## AN APPLICATION OF THE MAC WILLIAMS IDENTITY FOR COMPLETE m-SPOTTY ROSENBLOOM-TSFASMAN WEIGHT ENUMERATORS

Let $C$ be the linear code as in Example 2.5 i.e. $C$= <010111100001, 101010101010> and $C^{\perp}$=<1000 0000 0010, 0100 0000 0001, 0010 0000 0010, 0001 0000 0001, 0000 1000 0011, 0000 0100 0001, 0000 0010 0011, 0000 0001 0000, 0000 0000 1010, 0000 0000 0100>.

C consists of only 4 codewords and we can quickly find $W_{CmRT}(C)(\tau_0, \ldots, \tau_b)$.

Contribution of 0000 0000 0000 is $\tau_0\tau_0\tau_0 = {\tau_0}^3$; contribution of 0101 1110 0001 is $\tau_4\tau_3\tau_4 = \tau_3{\tau_4}^2$; contribution of 1010 1010 1010 is $\tau_3\tau_3\tau_3 = {\tau_3}^3$; and lastly contribution of 1111 0100 1011 is $\tau_4\tau_2\tau_4 = \tau_2{\tau_4}^2$. So $W_{CmRT}(C)(\tau_0, \ldots, \tau_b)$= ${\tau_0}^3 + {\tau_3}^3 + \tau_2{\tau_4}^2 + \tau_3{\tau_4}^2$.

Since b=4, we can use from Example 3.1:

$$\tau_0 = z_0 + z_1 + 2z_2 + 4z_3 + 8z_4 \tag{4.1}$$

$$\tau_1 = z_0 + z_1 + 2z_2 + 4z_3 - 8z_4 \tag{4.2}$$

$$\tau_2 = z_0 + z_1 + 2z_2 - 4z_3 \tag{4.3}$$

$$\tau_3 = z_0 + z_1 - 2z_2 \tag{4.4}$$

$$\tau_4 = z_0 - z_1 \tag{4.5}$$

We only need to calculate ${\tau_0}^3$, ${\tau_3}^3$, $\tau_2{\tau_4}^2$, $\tau_3{\tau_4}^2$.

$\tau_0{}^3 = (z_0 + z_1 + 2z_2 + 4z_3 + 8z_4)^3 = z_0{}^3 + 3\, z_1\, z_0{}^2 + 6\, z_2\, z_0{}^2 + 12\, z_3 z_0{}^2 +$
$24\, z_4\, z_0{}^2 + 3\, z_1{}^2\, z_0 + 12\, z_2{}^2\, z_0 + 48\, z_3{}^2\, z_0\; 192\, z_4{}^2\, z_0 + 12\, z_1\, z_2 z_0 +$
$24\, z_1\, z_3\, z_0\; 48\, z_2\, z_3\, z_0 + 48\, z_1\, z_4\, z_0 + 96\, z_2\, z_4\, z_0 + 192\, z_3\, z_4\, z_0 + z_1{}^3 +$
$8\, z_2{}^3 + 64\, z_3{}^3 + 512\, z_4{}^3 + 12\, z_1\, z_2{}^2 + 48\, z_1\, z_3{}^2 + 96\, z_2\, z_3{}^2 + 192\, z_1 z_4{}^2 +$
$384\, z_2\, z_4{}^2 + 768\, z_3\, z_4{}^2 + 6\, z_1{}^2\, z_2 + 12\, z_1{}^2\, z_3 + 48\, z_2{}^2\, z_3 + 48\, z_1\, z_2\, z_3 +$
$24\, z_1{}^2\, z_4 + 96\, z_2{}^2 z_4 + 384\, z_3{}^2 z_4 + 96\, x_1 z_2 z_4 + 192\, z_1 z_3 z_4 + 384\, z_2 z_3 z_4;$
(4.6)

$\tau_3{}^3 = (z_0 + z_1 - 2z_2)^3 = z_0{}^3 + 3\, z_1\, z_0{}^2 - 6\, z_2\, z_0{}^2 + 3\, z_1{}^2\, z_0 + 12\, z_2{}^2\, z_0 -$
$12\, z_1\, z_2\, z_0 + z_1{}^3 - 8\, z_2{}^3 + 12\, z_1\, z_2{}^2 - 6\, z_1{}^2\, z_2\,;$ (4.7)

$\tau_4{}^2 = (z_0 - z_1)^2 = z_0{}^2 - 2\, z_0 z_1 + z_1{}^2\,;$ (4.8)

$\tau_2\tau_4{}^2 = (z_0 + z_1 + 2z_2 - 4z_3)(z_0{}^2 - 2\, z_0 z_1 + z_1{}^2) = z_0{}^3 - z_1\, z_0{}^2 + 2\, z_2\, z_0{}^2 -$
$4\, z_3\, z_0{}^2 - z_1{}^2\, z_0 - 4\, z_1\, z_2\, z_0 + 8\, z_1\, z_3\, z_0 + z_1{}^3 + 2\, z_1{}^2\, z_2 - 4\, z_1{}^2\, z_3\,;$ (4.9)

$\tau_3\tau_4{}^2 = (z_0 + z_1 - 2z_2)(z_0{}^2 - 2\, z_0 z_1 + z_1{}^2) = z_0{}^3 - z_1\, z_0{}^2 - 2\, z_2\, z_0{}^2 -$
$z_1{}^2\, z_0 + 4\, z_1\, z_2\, z_0 + z_1{}^3 - 2\, z_1{}^2\, z_2\,.$ (4.10)

So $W_{CmRT}(C)(\tau_0, \dots, \tau_b) = \tau_0{}^3 + \tau_3{}^3 + \tau_2\tau_4{}^2 + \tau_3\tau_4{}^2$ (4.11)

$= 4\, z_0{}^3 + 4\, z_1 z_0{}^2 + 8\, z_3\, z_0{}^2 + 24\, z_4\, z_0{}^2 + 4\, z_1{}^2 z_0 + 24\, z_2{}^2 z_0 + 48\, z_3{}^2 z_0 +$
$192\, z_4{}^2\, z_0 + 32\, z_1\, z_3\, z_0 + 48\, z_2\, z_3\, z_0 + 48\, z_1\, z_4\, z_0 + 96\, z_2\, z_4\, z_0 + 192\, z_3\, z_4\, z_0 +$
$4\, z_1{}^3 + 64\, z_3{}^3 + 512\, z_4{}^3 + 24\, z_1\, z_2{}^2 + 48\, z_1\, z_3{}^2 + 96\, z_2\, z_3{}^2 + 192\, z_1\, z_4{}^2 +$
$384\, z_2\, z_4{}^2 + 768\, z_3\, z_4{}^2 + 8\, z_1{}^2\, z_3 + 48\, z_2{}^2\, z_3 + 48\, z_1\, z_2\, z_3 + 24\, z_1{}^2\, z_4 +$
$96\, z_2{}^2\, z_4 + 384\, z_3{}^2\, z_4 + 96\, z_1\, z_2\, z_4 + 192\, z_1\, z_3\, z_4 + 384\, z_2\, z_3\, z_4\,.$ (4.12)

Now apply Theorem 3.1 to C:

$W_{CmRT}(C^{\perp})(z_0, \dots, z_b) = \frac{1}{4}\Big(\; W_{CmRT}(C)(\tau_0, \dots, \tau_b)\Big)$ (4.13)

$= z_0{}^3 + z_1\, z_0{}^2 + 2\, z_3\, z_0{}^2 + 6\, z_4\, z_0{}^2 + z_1{}^2\, z_0 + 6\, z_2{}^2\, z_0 + 12\, z_3{}^2\, z_0 +$
$48\, z_4{}^2\, z_0 + 8\, z_1\, z_3\, z_0 + 12\, z_2\, z_3\, z_0 + 12\, z_1\, z_4\, z_0 + 24\, z_2\, z_4\, z_0 + 48\, z_3\, z_4\, z_0 +$
$z_1{}^3 + 16\, z_3{}^3 + 128\, z_4{}^3 + 6\, z_1\, z_2{}^2 + 12\, z_1\, z_3{}^2 + 24\, z_2\, z_3{}^2 + 48\, z_1\, z_4{}^2 +$
$96\, z_2\, z_4{}^2 + 192\, z_3\, z_4{}^2 + 2\, z_1{}^2\, z_3 + 12\, z_2{}^2\, z_3 + 12\, z_1\, z_2\, z_3 + 6\, z_1{}^2\, z_4 +$
$24\, z_2{}^2\, z_4 + 96\, z_3{}^2\, z_4 + 24\, z_1\, z_2\, z_4 + 48\, z_1\, z_3\, z_4 + 96\, z_2\, z_3\, z_4\,.$ (4.14)

## CHAPTER 5

## RESULTS AND DISCUSSION

Mac Williams identities provide us an advantage of quicker calculations of weight enumerators of higher dimensional linear codes. We established a Mac Williams identity for complete m-spotty RT weight enumerators where the duality is defined under inner product of Definition 2.14.

Complete m-spotty RT weight enumerators are multi-dimensional weight enumerators. When one variable weight enumerators are needed, we can turn them into one variable polynomials by letting $z_j = z^{\lceil \frac{j}{t} \rceil}$. But for RT based metrics it is important to bear in mind that the obtained weight enumerator is not actually (noncomplete) m-spotty weight enumerator. Although letting $z_j$ as above produces m-spotty Hamming weight enumerator from complete m-spotty Hamming weight enumerator, this is not the case for m-spotty RT weight enumerator and complete m-spotty Hamming weight enumerator. Actually, we explained the impossibility of establishing Mac Williams identity between them. So, identity in Theorem 3.4 does not give rise a general relation between m-spotty RT weight distribution of a code and its dual. We can use that identity only for complete m-spotty RT weights.

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# CURRICULUM VITAE

## PERSONAL INFORMATION

**Name Surname** : Tarık ARABACI

**Date of birth and place** : 18.09.1987 Bakırköy/İSTANBUL

**Foreign Languages** : English

**E-mail** : tarikarabaci@hotmail.com

## EDUCATION

| Degree | Department | University | Date of Graduation |
|---|---|---|---|
| Master | Mathematics | Yıldız Technical University | 2017 |
| Undergraduate | Mathematics | Boğaziçi University | 2012 |
| High School | Science | Üsküdar H. Avni Sözen A.L. | 2005 |

## WORK EXPERIENCE

| Year | Corporation/Institute | Enrollment |
|---|---|---|
| 2015 | İstanbul Bilgi University | Research Assistant |

**PUBLISHMENTS**

**Conference Papers**

1. “Mac Williams identity for complete m-spotty Rosenbloom-Tsfasman weight enumerators”,(2017), XIX. Antalya Algebra Days, 17-21 May, İzmir.