MONOMIAL GROUPS

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ÖZGE ALMAŞ

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submitted by **ÖZGE ALMAŞ** in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics Department, Middle East Technical University** by,

Prof. Dr. Gülbin Dural Ünver
Dean, Graduate School of **Natural and Applied Sciences** ______________

Prof. Dr. Mustafa Korkmaz
Head of Department, **Mathematics** ______________

Prof. Dr. Mahmut Kuzucuoğlu
Supervisor, **Mathematics Department, METU** ______________

Assoc. Prof. Dr. Ebru Solak
Co-supervisor, **Mathematics Department, METU** ______________

Examining Committee Members:

Prof. Dr. Ali Erdoğan
Mathematics Department, Hacettepe University ______________

Prof. Dr. Mahmut Kuzucuoğlu
Mathematics Department, METU ______________

Prof. Dr. Yıldıray Ozan
Mathematics Department, METU ______________

Assist. Prof. Dr. Mustafa Gökhan Benli
Mathematics Department, METU ______________

Assist. Prof. Dr. Cansu Betin Onur
Mathematics Department, Atılım University ______________

Date: ______________

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Name, Last Name: ÖZGE ALMAŞ

Signature :

ABSTRACT

MONOMIAL GROUPS

Almaş, Özge
M.S., Department of Mathematics
Supervisor : Prof. Dr. Mahmut Kuzucuoğlu
Co-Supervisor : Assoc. Prof. Dr. Ebru Solak

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A group G is called a permutation group if it is a subgroup of a symmetric group on a set Ω. G is called a linear group if it is a subgroup of the general linear group GL(n, F) for a field F.

Monomial groups are generalization of permutation groups and restriction of linear groups. In matrix terminology, monomial groups of degree n over a group H are the n$\times$ n invertible matrices in which each row and each column contains only one element of H all the other entries are zero.

Basic properties of finite degree monomial groups are studied by Ore in [2]. Infinite degree monomial groups over an arbitrary group H is studied by Crouch in [1]. This thesis is a survey of the Crouch paper, in particular we will give a complete classification of the structure of centralizers of arbitrary elements in complete monomial groups $\Sigma(H; B, B^+, B^+)$ and conjugacy of the elements in $\Sigma(H; B, B^+, B^+)$.

Keywords: Monomial groups, Infinite permutation groups, Centralizer of monomial elements, Splitting of monomial groups

ÖZ

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Bir Ω kümesi üzerindeki simetrik grupların altgruplarına permütasyon grupları denir. F bir cisim olmak üzere genel lineer grup GL(n, F)'nin altgruplarına lineer grup denir. Monomial gruplar ise permütasyon grupların genelleştirmesi lineer grupların da kısıtlamasıdır. Bir H grubu üzerinde tanımlı n dereceli monomial gruplar her satırında ve her sütununda H'den sadece bir eleman içeren, diğer tüm girdileri 0 olan tersinir matrislerdir. Sonlu dereceli monomial grupların temel özellikleri Ore [2] tarafından araştırılmıştır . Herhangi bir H grubu üzerinde tanımlı, sonsuz dereceli monomial gruplarla ilgili çalışmalar da Crouch [1] tarafından yapılmıştır . Bu tez, Crouch'un [1] makalesinin bir incelemesidir. Bu tezde özellikle tam monomial grupların elemanlarının merkezleyenlerinin yapısının tam olarak sınıflandırılması ve $\Sigma(H; B, B^{+}, B^{+})$ grubunun içindeki herhangi iki elemanın eşleniğinin bulunması gösterilmiştir .

Anahtar Kelimeler: Monomial gruplar, Sonsuz permütasyon grupları, Monomial elemanların merkezleyenleri, Monomial gruplarda ayrışma

To my parents & to my sister

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CHAPTER 1

INTRODUCTION

Mainly there are three kinds of representations of groups; permutation representation, linear representation, and monomial representation.

Permutation representation is a homomorphism from the group into symmetric group, linear representation is a homomorphism from group into the group of invertible linear transformations of a vector space over a field F.

Monomial representation is a generalization of permutation representation and restriction of linear representation. If V is a finite dimensional vector space, say dimV=n over a field F, then GL(V) is isomorphic to the general linear group GL(n,F), n$\times$ n invertible matrices over a field F.

A monomial matrix is an invertible $n \times n$ matrix where each row and each column contains only one nonzero entry and this entry comes from a fixed group H.

If G is a group with a subgroup H of index n, then G has a monomial representation of degree n over the subgroup H.

Therefore, the study of monomial groups is the study of the structure of groups which has a subgroup of finite index.

In fact, the famous Kaluźnin - Krasner Theorem which states that, if a group G has a non-trivial subgroup H, then G can be embedded into a monomial group over H, but the degree could be infinite. In this respect study of monomial groups is the study of group extensions.

The basic properties of monomial groups of finite degree are studied by Kerber [3], and Ore [2]. In particular, Ore determined conjugacy of the two elements in complete monomial group. Moreover, he finds the structure of a centralizer of an element in complete monomial groups.

The work of Ore is extended to infinite degree monomial groups by R. B. Crouch [1].

Crouch defines monomial groups (symmetries) of arbitrary degree, over an arbitrary group H in the following way.

Let B be an infinite cardinality, and U be a set with cardinality B. We think of U as an ordered set. By B^+ we denote the successor cardinal of B.

Let d be the cardinality of natural numbers, i.e., $d = \aleph_0$. A monomial substitution over H is a transformation of the form

$$c = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon x_{i_\epsilon} & \cdots \end{pmatrix} \tag{1.1}$$

where the map $x_\epsilon \longmapsto x_{i_\epsilon}$ is a permutation of the set U, and $h_\epsilon \in H$. The product $h_\epsilon x_{i_\epsilon}$ is a formal product satisfying $h_\epsilon(h_\beta x_i) = (h_\epsilon h_\beta)x_i$, where $x_i \in U$.

The set of all monomial substitutions $\Sigma(H; B, B^+, B^+)$ forms a group with composition of substitutions.

In the first part of this thesis, we find the structure of the centralizer of an arbitrary element in $\Sigma(H; B, B^+, B^+)$. Namely we prove the following theorem:

Let y be conjugate to y_1 written in the normal form $y_1 = \prod_i \delta_i$, $\delta_i = \prod_\epsilon \delta^i_\epsilon$, where for a fixed i the δ^i_ϵ are the normalized cycles of the same length n, and the same determinant class a if $n < d$. Let ϵ run over a set of cardinal μ where $0 \leqq \mu \leqq B$. Then the centralizer $C_{\Sigma(H;B,B^+,B^+)}(y)$ is isomorphic to the strong direct product of symmetries

$$C_{\Sigma(H;B,B^+,B^+)}(y) \cong \prod_i (\Sigma(C_{(H}(a) < \delta >, \mu_i, \mu_i^+, \mu_i^+)) \times \Sigma_\kappa(H \times \mathbb{Z}; \kappa, \kappa^+, \kappa^+).$$

The group $C_H(a) < \delta >$ consists of all elements y_1 of the form $y_1 = \{k_i\}(c_1^i)^j$ where k belongs to the centralizer of a in H. The second direct product arises if δ is a product of κ infinite cycles where $\kappa \leq B.$

Let G be a group and N be a normal subgroup of G. If there exists a subgroup $H \leq G$ such that G=NH and $N \cap H = 1$, then we say that G splits over N and H is called complement of N in G. Clearly if H is a complement of N, then all conjugates H^g, $g \in G$ are also complement of N in G. It is a natural question whether all complements of N are conjugate in G, i.e., if G=NT and $N \cap T = 1$ does there exist $x \in G$, such that $T = H^g$. If all complements of N are conjugate, then we say that G splits regularly. Observe that if G has two complements H_1 and H_2 then $G = NH_1$ and $G = NH_2$, where $H_1 \cap N = 1$ and $H_2 \cap N = 1$. It follows that $G/N = H_1N/N = H_1/H_1 \cap N \cong H_1$, on the other hand $G/N = H_2N/N = H_2/H_2 \cap N \cong H_2$. Hence any two complements of N are isomorphic. So we are interested in when they are conjugate. Indeed in the following example we have a group G with a normal subgroup N such that it has two non-conjugate complements. So the above problem makes sense.

Example:
Let $G = S_6$. We know that A_6 is a normal subgroup of S_6. Let $T_1 =< (1, 2) >$. T_1 is a subgroup of G where $G = A_6T_1$ and $A_6 \cap T_1 = 1$. So T_1 is a complement of A_6. Assume $T_2 =< (1, 2), (3, 4), (5, 6) >$. T_2 is also a subgroup of G. Moreover, $G = A_6T_2$ and $A_6 \cap T_2 = 1$. But T_1 and T_2 are not conjugate. So, S_6 does not split regularly.

As the following simple observation shows, if a group G has normal subgroup N, then it may not split. Indeed $G = Q_8$ quaternion group of order 8. $Q_8 = \{1, i, j, k, -1, -i, -j, -k\}$. All subgroups of Q_8 are normal subgroup. Indeed the subgroups $< i >$, $< j >$, $< k >$ are cyclic subgroups of order 4 and $\{1, -1\}$ is the center of Q_8. But the subgroup $< i > \lhd Q_8$ does not split, because there exists no non-trivial subgroup H such that $< i > \cap H = 1$. Because all non-trivial subgroups of Q_8 contain the center $Z(Q_8) = \{1, -1\}$.

If we come back to monomial groups over H, $\Sigma(H; B, B^+, C)$ splits over the base group $V(B, B^+)$ and S(B,C) is a complement of $V(B, B^+)$ in $\Sigma(H; B, B^+, C)$.

In chapter 4, we will discuss the splitting of $\Sigma(H; B, B^+, C)$ and regularity of this splitting. We prove the following:

A necessary and sufficient condition for $\Sigma(H; B, B^+, C)$ where $d^+ \leq C \leq B^+$ to split regularly over the basis group is that H contains no subgroup isomorphic to S(B,C).

An immediate corollary of this result is that $\Sigma(H; B, B^+, C)$ to split regularly over the basis group if and only if H contains no element of order 2.

In the last section we discuss the splitting of alternating groups over the basis group. This discussion is separated into two cases, namely splitting of monomial alternating groups of finite degree n, i.e., $\Sigma_{n,A}(H)$ and splitting of monomial alternating groups of infinite degree B where B is an infinite cardinal number. The main difference between the splitting if complete monomial group and monomial alternating group is the following:

In the former one symmetric group S_n is generated by permutations of the form $(1\ i)$ where i=2,..,n and we study the images of these elements and in the latter one. Alternating groups are generated by permutations of type $(i\ j\ k)$ where i, j, k are pairwise distinct elements of $\{1, 2, ..., n\}$. Therefore we study the images of elements of this type.

The examples are given by $\Sigma_{3,A}(H)$ as a special case and in this case the splitting of $\Sigma_{3,A}(H)$ over the basis group is always regular. The second example of the splitting is $\Sigma_{4,A}(H)$ over the basis group is also given. When n=4, A_4 has a proper normal subgroup is isomorphic to elementary abelian 2-group of order 4. The image of this subgroup into H is studied and for $\Sigma_{4,A}(H)$, we show that there are two types of complement; one comes from the conjugates of S_n and the other complements arises from the homomorphic images of A_4 into a cyclic subgroup of order 3 of H. In particular, if H has no subgroup of order 3, then all complements of $\Sigma_{4,A}(H)$ over the basis group are conjugate i.e., $\Sigma_{4,A}(H)$ splits regularly over the basis group $V_4(H)$. For the general case we show that, $\Sigma_{n,A}(H)$ splits over the basis group and $\Sigma_{n,A}(H)$ splits regularly over the basis group if and only if H does not contain any subgroup isomorphic to A_{n-1}, for $n \geq 6$.

The splitting of infinite alternating groups is studied with a similar technique and we prove that $\Sigma_A(H; B, B^+, d)$ split regularly over the basis group if and only if H does not have any subgroup isomorphic to A(B,d).

CHAPTER 2

THE SYMMETRIES

In this section we define symmetries not only on the finite sets as in the case of Ore [2], but also on the sets arbitrarily large. The group H will be arbitrary.

Let d be the cardinal of the set of integers, i.e., d=$\aleph_0$, B be any infinite cardinal; B^+, the successor of B, U is a set with the cardinal B, and let C be a cardinal such that

$$d \leqq C \leqq B^+.$$

Definition 2.0.1. *The set of all permutations s of the set U onto itself is a group. It is denoted by $S(B, B^+)$, and is called the infinite symmetric group on the set U.*

Let $s \in S(B, B^+)$. For s, we define support of s. Namely

$$supp(s) = \{x_i \in U | s(x_i) \neq x_i\}$$

By $|supp(s)|$ we define the cardinality of the set supp(s).

Now, we define the subgroup S(B, C) of the group $S(B, B^+)$:

$$S(B, C) = \{\sigma \in S(B, B^+) : |supp(\sigma)| < C\}.$$

LEMMA 2.0.2. *The set S(B, C) is a subgroup of $S(B, B^+)$.*

Proof. Let $\sigma_1, \sigma_2 \in S(B, C)$. Then, $|supp(\sigma_1)| < C$ and $|supp(\sigma_2)| < C$. If

$|supp(\sigma_1)| < C$, since $supp(\sigma_1) = supp(\sigma_1^{-1})$, $|supp(\sigma_1^{-1})| = |supp(\sigma_1)| < C$.

So, $\sigma_1^{-1} \in S(B,C)$.

Claim: $supp(\sigma_1\sigma_2) \subseteq supp(\sigma_1) \cup supp(\sigma_2)$.

Assume a$\in$supp($\sigma_1\sigma_2$). Then a$\cdot\sigma_1\sigma_2\neq$ a. Let a$\notin$supp(σ_2). If a$\notin$ supp(σ_2), then we show that a$\in$ $supp(\sigma_1)$.

Since a$\notin$supp(σ_2), a$\cdot\sigma_2$=a, and a$\cdot\sigma_1\sigma_2 \neq$ a. It follows that a$\cdot\sigma_1\neq$ a. So, a$\in$supp(σ_1).

Therefore, supp($\sigma_1\sigma_2$)$\subseteq$ supp(σ_1)$\cup$ supp(σ_2). Since C is infinite,

$$|supp(\sigma_1) \cup supp(\sigma_2)| \leq \{|supp(\sigma_1)| + |supp(\sigma_2)|\} = \\ max\{|supp(\sigma_1)|, |supp(\sigma_2)|\} \leq C$$

This implies $|supp(\sigma_1\sigma_2)| \leq C$. So, $\sigma_1\sigma_2 \in S(B,C)$.

Thus, S(B, C) is a subgroup of $S(B, B^+)$. □

Definition 2.0.3. *If the number of x moved by s is finite, then the group S(B,d) is called finitary symmetric group where $supp(s) = \{x_i \in U | s(x_i) \neq x_i\}$. We denote this set as*

$$FSym(U) = \{s \in S(B, B^+)) : \ |supp(\sigma)| < \infty\}$$

Definition 2.0.4. *Here we put the constraint that the number of moving elements of U by $s \in$ A(B, d), is less than the cardinality of natural numbers. Since we mention alternating groups we should have evenly many transpositions. So, to define evenly many transpositions, the largest cardinality of supp(s) should be finite, and so this s comes from S(B,d). The group A(B, d) has elements s's where those s's comes from S(B,d) and each of which have evenly many transpositions. The group A(B,d) is called the infinite alternating group on the set U.*

In $S(B, B^+)$ every element s determines a set of cycles of the form

$$c = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & \cdots & x_n & x_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

or

$$c = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & x_0 & x_1 & x_2 & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \end{pmatrix}.$$

By well-ordering principle, every set can be well- ordered. Therefore, every permutation or cycle as in the above notation is meaningful.

LEMMA 2.0.5. *Every permutation $s \in S(B, B^+)$ can be written as a disjoint product of commutative cycles of finite length or infinite cycles.*

Proof. Let U be the set as above and $\sigma \in S(B, B^+)$. So, U be a set of cardinality B. Define a relation on U. Two elements x_i, $x_j \in U$ are related $x_i \sim x_j$ if and only if there exists $n \in \mathbb{Z}$ such that $\sigma^n(x_i) = x_j$.

Claim: " $\sim$ " is an equivalence relation on U.

(i) $x_i \sim x_i$ as $\sigma^0 = id$ and $id(x_i) = \sigma^0(x_i) = x_i$.

(ii) $x_i \sim x_j$ implies that there exists $n \in \mathbb{Z}$ such that $\sigma^n(x_i) = x_j$. Then $(\sigma^{-1})^n(x_j) = \sigma^{-n}(x_j) = x_i. So, x_j \sim x_i$ as $-n \in \mathbb{Z}$.

(iii) Assume that $x_i \sim x_j$ and $x_j \sim x_k$, $\{x_i, x_j, x_k\} \subseteq U$. Then there exists n and m in $\mathbb{Z}$ such that $\sigma^n(x_i) = x_j$, and $\sigma^m(x_j) = x_k$. It follows that

$\sigma^{m+n}(x_i) = \sigma^m(\sigma^n(x_i)) = \sigma^m(x_j) = x_k$, $n + m \in \mathbb{Z}$. Hence, $x_i \sim x_k$.

Consequently, $\sim$ is an equivalence relation.

The equivalence class containing an element $x_i \in U$ is of the form $\{..., \sigma^{-2}(x_i), \sigma^{-1}(x_i), x_i, \sigma(x_i), \sigma^2(x_i), ...\}$. If this set is finite, then there exists $n \in \mathbb{N}$ such that $\sigma^n(x_i) = x_i$. Then, we have a finite cycle of the form

$$(x_i, \sigma(x_i), ..., \sigma^{n-1}(x_i))$$

If the equivalence class containing x_i is an infinite set, then we have an infinite cycle of the form

$$(..., \sigma^{-2}(x_i), \sigma^{-1}(x_i), x_i, \sigma(x_i), \sigma^2(x_i), ..., \sigma^{n-1}(x_i), ...)$$

This type of cycles are called infinite cycles.

Since " $\sim$ " is an equivalence relation, the equivalence classes are disjoint, and union of equivalence classes is the set U. Hence, every element of U is contained in a cycle, and one may observe that disjoint cycles commute. Hence, every permutation σ can be written in a unique way as a product of disjoint cycles of length finite or infinite up to order. □

Definition 2.0.6. *A cycle with n distinct x's is called an n- cycle; n =1, 2, ...,k .*

Definition 2.0.7. *A monomial substitution over H is a transformation of the form*

$$y = \begin{pmatrix} \cdots & x_l & \cdots \\ \cdots & h_l x_{i_l} & \cdots \end{pmatrix} \tag{2.1}$$

where the mapping $x_l \longrightarrow x_{i_l}$ is a one to one mapping of U onto itself and h_l belongs to H. The h_l will be called factors of y.

If y is given by equation (2.1) and y_1 is given by

$$y_1 = \begin{pmatrix} \cdots & x_l & \cdots \\ \cdots & k_l x_{j_l} & \cdots \end{pmatrix}, \tag{2.2}$$

then the product yy_1 is defined by

$$yy_1 = \begin{pmatrix} \cdots & x_l & \cdots \\ \cdots & h_l k_{i_l} x_{j_{i_l}} & \cdots \end{pmatrix}. \tag{2.3}$$

The inverse of y is

$$y^{-1} = \begin{pmatrix} \cdots & x_{i_l} & \cdots \\ \cdots & h_l^{-1} x_l & \cdots \end{pmatrix}. \tag{2.4}$$

The identity element will be

$$E = \begin{pmatrix} \cdots & x_l & \cdots \\ \cdots & ex_l & \cdots \end{pmatrix}. \tag{2.5}$$

Definition 2.0.8. *By above multiplication, the set of monomial substitution is a group, that will be denoted by* $\Sigma(H; B, B^+, B^+)$ *, and called the monomial group of* H *of degree* B *or, more simply, the symmetry of* H.

If H consists only of the identity element, then $\Sigma(H; B, B^+, B^+)$ is the symmetric group $S(B, B^+)$.

Definition 2.0.9. *A permutation in* $\Sigma(H; B, B^+, B^+)$ *is a substitution of the form*

$$s = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{i_\epsilon} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \epsilon & \cdots \\ \cdots & i_\epsilon & \cdots \end{pmatrix}. \tag{2.6}$$

LEMMA 2.0.10. *The set of permutations forms a subgroup of* $\Sigma(H; B, B^+, B^+)$ *and it is denoted by* $S(B, B^+)$. *We call this subgroup as permutation subgroup of* $\Sigma(H; B, B^+, B^+)$.

Proof. : Let α , $\beta \in S(B, B^+)$, where

$$\alpha = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{i_\epsilon} & \cdots \end{pmatrix} \tag{2.7}$$

$$\beta = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{j_\epsilon} & \cdots \end{pmatrix} \textit{ and } \beta^{-1} = \begin{pmatrix} \cdots & x_{j_\epsilon} & \cdots \\ \cdots & ex_\epsilon & \cdots \end{pmatrix}, \tag{2.8}$$

$$\alpha\beta = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & ex_{j_{i_\epsilon}} & \cdots \end{pmatrix} \tag{2.9}$$

This product is in $S(B, B^{+})$. Therefore, $S(B, B^{+})$ is a subgroup of $\Sigma(H; B, B^{+}, B^{+})$. $\square$

Definition 2.0.11. *A multiplication in $\Sigma(H; B, B^{+}, B^{+})$ is a substitution of the form*

$$v = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon x_\epsilon & \cdots \end{pmatrix} = \{..., h_\epsilon, ...\}. \tag{2.10}$$

LEMMA 2.0.12. *The set of multiplications forms a normal subgroup of $\Sigma(H; B, B^{+}, B^{+})$, denoted by $V(B, B^{+})$, and it is called the basis group of $\Sigma(H; B, B^{+}, B^{+})$.*

Proof. Let κ, $\alpha \in V(B, B^{+})$, and $\theta \in \Sigma(H; B, B^{+}, B^{+})$ where

$$\kappa = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon x_\epsilon & \cdots \end{pmatrix} \tag{2.11}$$

$$\alpha = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & l_\epsilon x_\epsilon & \cdots \end{pmatrix} \tag{2.12}$$

$$\theta = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & k_\epsilon x_{i_\epsilon} & \cdots \end{pmatrix}. \tag{2.13}$$

(i) Since

$$\kappa^{-1} = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon^{-1} x_\epsilon & \cdots \end{pmatrix}, \tag{2.14}$$

clearly $\kappa^{-1} \in V(B, B^{+})$.

(ii) The composition of κ and α will be

$$\kappa\alpha = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon l_\epsilon x_\epsilon & \cdots \end{pmatrix} \tag{2.15}$$

So, the composition belongs to the basis group $V(B, B^{+})$.

Thus, $V(B, B^{+})$ is a subgroup of $\Sigma(H; B, B^{+}, B^{+})$.

Moreover,

$$\theta^{-1}\kappa\theta = \begin{pmatrix} \cdots & x_{i_\epsilon} & \cdots \\ \cdots & k_\epsilon^{-1}h_\epsilon k_\epsilon x_{i_\epsilon} & \cdots \end{pmatrix} \tag{2.16}$$

So, this product is also in the basis group. Therefore, $V(B, B^+)$ is a normal subgroup in $\Sigma(H; B, B^+, B^+)$. □

LEMMA 2.0.13. *The basis group is isomorphic to the Cartesian product of B groups, each of which is isomorphic to H.*

Proof. Let $v \in V(B, B^+)$, $v = \{h_1, h_2, ...\}$.

Assume

$$\theta : V(B, B^+) \longrightarrow \prod H$$
$$v \longmapsto (h_1, h_2, ...)$$

.

• θ is a homomorphism:

Let $v_1, v_2 \in V(B, B^+)$, where

$v_1 = \{h_1, h_2, ...\}$ and

$v_2 = \{k_1, k_2, ...\}, h_i, k_i \in H.$

$\theta(v_1v_2) = (h_1k_1, h_2k_2, ...) = \theta(v_1)\theta(v_2).$

So, θ is a homomorphism.

• θ is one to one:

$$Ker\theta = \{v \in V(B, B^+) | \theta(v) = id_{\prod(H)}\}$$
$$= \{v \in V(B, B^+) | (h_1, h_2, ...) = (e_H, e_H, ...)\}$$

Then $h_i = e_H$ for i=1, 2, . . .

So, θ is one to one.

• θ is onto:

For any element $(h_1, h_2, ...) \in \prod H$, there exists a $v \in V(B, B^+)$ such that

$v = \{h_1, h_2, ...\}$.

So θ is onto.

Consequently, θ is an isomorphism, and the basis group is isomorphic to $H \times H \times ... \times H \times ...$ where the number of H is B many.

□

Definition 2.0.14. *A scalar in $\Sigma(H; B, B^+, B^+)$ is a multiplication with each factor is the same. Scalars are of the form {. . . , h, h, . . . } and are denoted by v={h}.*

LEMMA 2.0.15. *Scalars are the only elements that commute with permutations.*

Proof. Let s be an arbitrary element of $S(B, B^+)$. It is of the form

$$s = \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ x_{j_1} & x_{j_2} & \cdots & x_{j_k} & \cdots \end{pmatrix}. \tag{2.17}$$

Let $y \in \Sigma(H; B, B^+, B^+)$ be arbitrary.

$$y = \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_1 x_{i_1} & h_2 x_{i_2} & \cdots & h_t x_{i_t} & \cdots \end{pmatrix}. \tag{2.18}$$

If ys=sy, then

$$ys = \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_1 x_{i_1} & h_2 x_{i_2} & \cdots & h_t x_{i_t} & \cdots \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ x_{j_1} & x_{j_2} & \cdots & x_{j_k} & \cdots \end{pmatrix} \tag{2.19}$$

$$= \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_1 x_{j_{i_1}} & h_2 x_{j_{i_2}} & \cdots & h_t x_{j_{i_t}} & \cdots \end{pmatrix}, \tag{2.20}$$

and

$$sy = \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ x_{j_1} & x_{j_2} & \cdots & x_{j_k} & \cdots \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_t & \cdots \\ h_1 x_{i_1} & h_2 x_{i_2} & \cdots & h_t x_{i_t} & \cdots \end{pmatrix} \tag{2.21}$$

$$= \begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ h_{j_1} x_{i_{j_1}} & h_{j_2} x_{i_{j_2}} & \cdots & h_{j_k} x_{i_{j_k}} & \cdots \end{pmatrix} \tag{2.22}$$

ys=sy implies that

$h_{j_1} = h_1$

$h_{j_2} = h_2$

.

.

.

$h_{j_t} = h_t$

and

$x_{i_{j_1}} = x_{j_{i_1}}$

$x_{i_{j_2}} = x_{j_{i_2}}$

.

.

.

$x_{i_{j_k}} = x_{j_{i_k}}$

means that

$$i_{j_k} = j_{i_k} \Longrightarrow i_j = j_i \Longrightarrow j = i.$$

This shows that scalars commute with all permutations.

Now, assume ys=sy for all $s \in S(B, B^+)$. We should show y is a scalar. Let h_a, h_b be arbitrary factors of y which occur position a, and position b, respectively.

Consider $y(x_a, x_b)$ and $(x_a, x_b)y$. Since y commutes with all permutations, $y(x_a, x_b) = (x_a, x_b)y$. If we calculate $y(x_a, x_b)$ and $(x_a, x_b)y$, we get

$$y(x_a, x_b) = \begin{pmatrix} \cdots & x_a & \cdots & x_b & \cdots \\ \cdots & h_a x_b & \cdots & h_b x_a & \cdots \end{pmatrix} \tag{2.23}$$

$$(x_a, x_b)y = \begin{pmatrix} \cdots & x_a & \cdots & x_b & \cdots \\ \cdots & h_b x_b & \cdots & h_a x_a & \cdots \end{pmatrix}. \tag{2.24}$$

As a result, we see that $h_a = h_b$ for arbitrary a and b. Thus, in y all factors are the same.

Now, consider $y(x_{i_\epsilon}, x_a)$ where $i_\epsilon \neq a$. By equation (2.18) we see that with y, x_ϵ goes to x_{i_ϵ}. Assume also $i_\epsilon \neq \epsilon$. If we calculate $y(x_{i_\epsilon}, x_a)$ and $(x_{i_\epsilon}, x_a)y$ we should get the same result, since y commutes with all permutation.

$$y(x_{i_\epsilon}, x_a) = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon x_a & \cdots \end{pmatrix} \tag{2.25}$$

$$(x_{i_\epsilon}, x_a)y = \begin{pmatrix} \cdots & x_\epsilon & \cdots \\ \cdots & h_\epsilon x_a & \cdots \end{pmatrix} \tag{2.26}$$

Here since $x_a = x_{i_\epsilon}$ we get a contradiction.

Therefore, y is a scalar.

Consequently, we get that scalars are the only elements commute with permutations.

□

LEMMA 2.0.16. *The center* $Z(\Sigma(H; B, B^+, B^+))$ *of* $\Sigma(H; B, B^+, B^+)$ *is the set of all scalars v={k} where k belongs to the center of H, and* $Z(\Sigma(H; B, B^+, B^+))$ *is isomorphic to the center of H.*

Proof. By Lemma 2.0.15, scalars are the only elements that commute with permutations, and permutations are contained in $\Sigma(H; B, B^+, B^+)$. The elements of

$Z(\Sigma(H; B, B^+, B^+))$ are contained in scalars.

Moreover, $m\{h_1, h_2, ...\} = \{h_1, h_2, ...\}m$where $m = \{m, m, ...\}$ implies that $mh_i = h_im$ for all $h_i \in H$.

Hence, $m \in Z(H)$.

$\varphi : Z(\Sigma(H; B, B^+, B^+)) \longrightarrow Z(H)$

$v = \{k\} \longmapsto k$

Let $v_1 = \{k_1, ...\}$, and $v_2 = \{k_2, ...\}$.

• φ is a homomorphism:

$\varphi(v_1v_2) = k_1k_2 = \varphi(v_1)\varphi(v_2)$.

So φ is a homomorphism.

• φ is a one to one:

$Ker\varphi = \{v \in Z(\Sigma(H; B, B^+, B^+))|\varphi(v) = 1_H\} = \{1, 1, ...\}$.

So φ is one to one.

• φ is a onto:

For any $k \in Z(H)$, there exists $v \in Z(\Sigma(H; B, B^+, B^+))$ such that

$v = \{k, k, ...\}$.

Thus, φ is onto.

Hence φ is an isomorphism.

□

Definition 2.0.17. *A group G splits over a normal subgroup N if there exists a subgroup M of G such that G=$\langle M, N\rangle$=MN , N ∩ M=E.*

The group M may be replaced by any of its conjugates and the relations will still hold. Indeed, for any element $g \in G$, we have $G^g = (NT)^g = N^gT^g$ since N is normal we obtain $N^g = N$. Hence, T^g which is conjugate of T is also a complement of N in G.

Therefore, all conjugates of T will be a complement of N in G. But for every subgroup T such that G=$\langle N, T\rangle$, N$\cap$ T=E it follows that T is conjugate to M, then we say that G splits regularly over N.

LEMMA 2.0.18. *Any substitution y of $\Sigma(H; B, B^+, B^+)$ can be written as a multiplication multiplied by a permutation uniquely.*

Proof. Let

$$y = \begin{pmatrix} \cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1}\cdots & \\ \cdots & h_{\alpha-1}x_{i_{\alpha-1}} & h_\alpha x_{i_\alpha} & h_{\alpha+1}x_{i_{\alpha+1}} & \cdots \end{pmatrix}, \tag{2.27}$$

and y $\in \Sigma(H; B, B^+, B^+)$. Then y=vs where v=$\{\ldots, h_{\alpha-1}, h_\alpha, h_{\alpha+1}, \ldots\}$ and

$$s = \begin{pmatrix} \cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1}\cdots & \\ \cdots & x_{i_{\alpha-1}} & x_{i_\alpha} & x_{i_{\alpha+1}} & \cdots \end{pmatrix}, \tag{2.28}$$

$$y = \{..., h_{\alpha-1}, h_\alpha, h_{\alpha+1}, ...\} \begin{pmatrix} \cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1}\cdots & \\ \cdots & x_{i_{\alpha-1}} & x_{i_\alpha} & x_{i_{\alpha+1}} & \cdots \end{pmatrix} \tag{2.29}$$

So y can be written as a product of a permutation and a multiplication.

Assume that there exist

$$s' = \begin{pmatrix} \cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1}\cdots & \\ \cdots & x_{j_{\alpha-1}} & x_{j_\alpha} & x_{j_{\alpha+1}} & \cdots \end{pmatrix} \tag{2.30}$$

a permutation and $v' = \{..., k_{\alpha-1}, k_\alpha, k_{\alpha+1}, ...\}$ a multiplication such that $y = v's'$. Then

$$v's' = \begin{pmatrix} \cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1}\cdots & \\ \cdots & k_{\alpha-1}x_{j_{\alpha-1}} & k_\alpha x_{j_\alpha} & k_{\alpha+1}x_{j_{\alpha+1}} & \cdots \end{pmatrix} \tag{2.31}$$

$$= \begin{pmatrix} \cdots & x_{\alpha-1} & x_\alpha & x_{\alpha+1}\cdots & \\ \cdots & h_{\alpha-1}x_{i_{\alpha-1}} & h_\alpha x_{i_\alpha} & h_{\alpha+1}x_{i_{\alpha+1}} & \cdots \end{pmatrix} \tag{2.32}$$

If we look at $V(B, B^+) \cap S(B, B^+)$;

v=s implies

$$\begin{pmatrix} \cdots & x_{\alpha-1} & x_{\alpha} & x_{\alpha+1} \cdots & \\ \cdots & k_{\alpha-1}x_{\alpha-1} & k_{\alpha}x_{\alpha} & k_{\alpha+1}x_{\alpha+1} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & x_{\alpha-1} & x_{\alpha} & x_{\alpha+1} \cdots & \\ \cdots & x_{j_{\alpha-1}} & x_{j_{\alpha}} & x_{j_{\alpha+1}} & \cdots \end{pmatrix} \tag{2.33}$$

Then, $x_{i_1} = x_1$

.

.

.

$x_{i_t} = x_t$.

So, $i_t = t$ for any t=1, 2, . . . and $h_t = e$.

This implies

$$V(B, B^+) \cap S(B, B^+) = E.$$

Then $vs = v's'$ implies $(v')^{-1}v = s's^{-1} \in V(B, B^+) \cap S(B, B^+) = E$. Hence, v=v' and s=s'.

So, this multiplication is unique. □

Thus,

$$\Sigma(H; B, B^+, B^+) = < S(B, B^+), V(B, B^+) > .$$

Let B, C, D be infinite cardinal such that

$$d \leqq C \leqq B^+,$$

$$d \leqq D \leqq B^+.$$

Let $\Sigma(H; B, C, D)$ be the set of all y=vs where v$\in V(B, B^+)$, s$\in S(B, B^+)$ and v has less than C non identity factors, s moves less than D of the x's. Then we get the following Lemma.

LEMMA 2.0.19. *$\Sigma(H;B,C,D)$ is a subgroup of $\Sigma(H;B,B^{+},B^{+})$.*

Proof. Let y_1, $y_2 \in \Sigma(H;B,C,D)$ where y_1= $v_1 s_1$ and y_2=$v_2 s_2$. We know that v_1, v_2 have less than C non identity factors; and

|supp(s_1)|, |supp(s_2)|<D

i) y_1^{-1}∈$\Sigma(H;B,C,D)$:

y_1^{-1}=$s_1^{-1}v_1^{-1}$= $s_1^{-1}v_1^{-1}s_1 s_1^{-1}$= $(v_1^{-1})^{s_1}s_1^{-1}$

Since s_1 moves only the components of v_1 according to the action of s_1 on the set U, we have in the elements $(v_1^{-1})^{s_1}$ the elements of v_1^{-1} permuted with respect to the action of s_1. Hence the cardinality of moved elements will not increase. On the other hand, $supp(s_1) = supp(s_1^{-1})$.

Thus, y_1^{-1}∈$\Sigma(H;B,C,D)$

ii) y_1y_2∈$\Sigma(H;B,C,D)$:

y_1y_2=$v_1s_1v_2s_2$= $v_1s_1v_2s_1^{-1}s_1s_2$= $v_1v_2^{s_1^{-1}}s_1s_2$= $v_1v_1^{1}s_1s_2$ since basis group is normal subgroup.

v_1 is of the form v_1=[$..., h_{-1}, h_0, h_1, ...$] and v_1^1 is of the form v_2=[$..., k_{-1}, k_0, k_1, ...$] then $v_1v_1^1$=[$..., h_{-1}k_{-1}, h_0k_0, h_1k_1, ...$]. Since v_1 and v_1^1 has less than C non identity factors $v_1v_1^1$ also has less than C non identity factors.

By Lemma 2.0.2, |supp(s_1s_2)|<D.

Hence, $y_1y_2 \in \Sigma(H;B,C,D)$, and $\Sigma(H;B,C,D)$ is a subgroup of $\Sigma(H;B,B^{+},B^{+})$. □

The set $\Sigma_A(H;B,C,d)$ of all y=vs where v less than C non identity factors and s belongs to A(B, d) forms a subgroup of $\Sigma(H;B,B^{+},B^{+})$.

Let o(U)=n where n is a finite cardinal. Then the symmetry over H of U will be denoted by $\Sigma(H;n,n+1,n+1)$= $\Sigma_n(H)$. Then $\Sigma_{n,A}(H)$ where elements of this group can be written as y=vs, and this s belongs to A_n, is a subgroup. Here basis group is denoted by V(n, n+1)=V_n.

CHAPTER 3

CYCLES, TRANSFORMATIONS AND CENTRALIZERS

Let y be an arbitrary element of $\Sigma(H; B, B^{+}, B^{+})$. It has been shown that y has a unique decomposition y=vs where v belongs to $V(B, B^{+})$ and s belongs to $S(B, B^{+})$. Throughout this section we will mention more about cycles, transformations, and we will give the centralizers of finite and infinite monomial groups, which are written by Ore [2] and Crouch [1].

By Lemma 2.0.5, we know that any permutation s in $S(B, B^{+})$ can be written as a disjoint product of commutative cycles. This decomposition induces a decomposition of v such that to each cycle c_ϵ of s there corresponds a multiplication v_ϵ with all factors e in those positions corresponding to x that s does not move and factors the same as in v for the x that s moves. Thus $v_\epsilon c_\epsilon$ has one of the two forms

$$v_\epsilon c_\epsilon = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 x_2 & h_2 x_3 & \cdots & h_n x_1 \end{pmatrix} \text{ when n<d} \tag{3.1}$$

or

$$v_\epsilon c_\epsilon = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & h_{-1} x_0 & h_0 x_1 & h_1 x_2 & \cdots \end{pmatrix} \text{ when n=d.} \tag{3.2}$$

If c is a cycle of length n and of the form

$$c = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 x_2 & h_2 x_3 & \cdots & h_n x_1 \end{pmatrix}, \tag{3.3}$$

observe that

$$c^2 = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 h_2 x_3 & h_2 h_3 x_4 & \cdots & h_n h_1 x_2 \end{pmatrix}. \tag{3.4}$$

Then,

$$c^n = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ h_1 h_2 ... h_n x_1 & h_2 h_3 ... h_n h_1 x_2 & \cdots & h_n h_1 ... h_{n-1} x_n \end{pmatrix} \tag{3.5}$$

$$= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \delta_1 x_1 & \delta_2 x_2 & \cdots & \delta_n x_n \end{pmatrix} \tag{3.6}$$

The factor of n^{th} power of c is $\{\delta_1, \delta_2, \ldots, \delta_n\}$ where $\delta_1 = h_1 \ldots h_n$,

$\delta_2 = h_2 \ldots h_n\ h_1$, $\delta_n = h_n\ h_1 \ldots h_{n-1}$.

Definition 3.0.20. *These δ_i's are called the determinants of c.*

Note that, δ_i's are conjugate. Indeed,

$h_n^{-1}\ \delta_n\ h_n = \delta_1$.

$h_2\ \delta_3\ h_2^{-1} = \delta_2$.

$h_3\ \delta_4\ h_3^{-1} = \delta_3$.

.

.

.

$h_n\ \delta_{n-1}\ h_n^{-1} = \delta_1$

Since δ_i's are conjugate, there exists a unique determinant class for each cycle. Above, we have defined determinant class of a finite cycle.

Theorem 3.0.21. *Two finite cycles are conjugate if and only if they have the same length and determinant class.*

Proof. Let,

$$\kappa = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ k_1 x_{j_1} & k_2 x_{j_2} & \cdots & k_m x_{j_m} \end{pmatrix} \tag{3.7}$$

$$\kappa^{-1} = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_m} \\ k_1^{-1} x_1 & k_2^{-1} x_2 & \cdots & k_m^{-1} x_m \end{pmatrix} \tag{3.8}$$

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ c_1 x_2 & c_2 x_3 & \cdots & c_n x_1 \end{pmatrix} \tag{3.9}$$

When we consider conjugation of κ with γ there are three cases:

<u>Case 1:</u> If m=n,

$$\kappa^{-1}\gamma\kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_n} \\ k_1^{-1} c_1 k_2 x_{j_2} & k_2^{-1} c_2 k_3 x_{j_3} & \cdots & k_n^{-1} c_n k_n x_{j_1} \end{pmatrix} \tag{3.10}$$

<u>Case 2:</u> If m<n,

$$\kappa^{-1}\gamma\kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_m} & \cdots & x_{j_n} \\ k_1^{-1} c_1 k_2 x_{j_2} & k_2^{-1} c_2 k_3 x_{j_3} & \cdots & k_m^{-1} c_m x_{j_{m+1}} & \cdots & c_n k_1 x_{j_1} \end{pmatrix} \tag{3.11}$$

<u>Case 3:</u> If m>n,

$$\kappa^{-1}\gamma\kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_n} & x_{j_{n+1}} & \cdots & x_{j_m} \\ k_1^{-1} c_1 k_2 x_{j_2} & k_2^{-1} c_2 k_3 x_{j_3} & \cdots & k_n^{-1} c_n k_1 x_{j_1} & x_{j_{n+1}} & \cdots & x_{j_m} \end{pmatrix} \tag{3.12}$$

Above, it can be seen that $\kappa^{-1}\gamma\kappa$ has the same form in three cases; they have the same length and the same determinant class. Namely, for case (1) in equation (3.10) determinant class of $\kappa\gamma^{-1}\kappa$ is the product of $(k_1^{-1}c_1k_2)(k_2^{-1}c_2k_3)...(k_n^{-1}c_nk_n) = (k_1^{-1}c_1c_2...c_nk_1) = k_1^{-1}\delta_1k_1$. It is a conjugate of determinant class of γ by the element k_1 in H. For case (2), in equation (3.11) determinant class of $\kappa\gamma^{-1}\kappa$ is the product of $(k_1^{-1}c_1k_2)(k_2^{-1}c_2k_3)...(k_m^{-1}c_mc_{m+1}...c_nk_1) = (k_1^{-1}c_1c_2...c_nk_1) = k_1^{-1}\delta_1k_1$. It is again conjugate of determinant class of γ by the element k_1 in H. For case (3), in equation (3.12) determinant class of $\kappa\gamma^{-1}\kappa$ is the product of $(k_1^{-1}c_1k_2)(k_2^{-1}c_2k_3)...(k_n^{-1}c_nk_1) = (k_1^{-1}c_1c_2...c_nk_1) = k_1^{-1}\delta_1k_1$. It is a conjugate of determinant class of γ by the element k_1 in H also.

□

Ore [2] has investigated the result of transforming a finite cycle of an element of monomial group to its normal form. We will state that in the following theorem.

Theorem 3.0.22. *Any cycle of length n may be transformed to the normal form*

$$\gamma = \begin{pmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_n} \\ x_{i_2} & x_{i_3} & \cdots & x_{i_1} \end{pmatrix} = \{x_{i_2}, \cdots, x_{i_n}, ax_{i_1}\} \tag{3.13}$$

where a is any element in the determinant class of γ. Any monomial substitution ρ is similar to a product of cycles without common variables $\rho = \gamma_1...\gamma_r$ where each cycle is in normal form.

Proof. Let

$$\kappa = \begin{pmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_n} \\ c_1x_{i_2} & c_2x_{i_3} & \cdots & c_nx_{i_1} \end{pmatrix} \tag{3.14}$$

have the same determinant class of the cycle γ.

If we can find a β such that $\beta^{-1}\kappa\beta = \gamma$ then we will get the result.

$\Delta_\kappa = c_1c_2...c_n$, and $\Delta_\gamma = a$ are determinants of κ and γ. By our assumption, Δ_κ and Δ_γ are in the same determinant class, so there exists p_1 in H such that $\Delta_\kappa^{p_1} = \Delta_\gamma$. By Theorem 1 in the paper of Ore [2] there exist p_1 such that

$p_1^{-1}c_1p_2 = 1$, $p_2^{-1}c_2p_3 = 1$,..., $p_{n-1}^{-1}c_{n-1}p_n = 1$, $p_n^{-1}c_np_1 = a$.

Choose

$$\beta = \begin{pmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_n} \\ p_1 x_{i_1} & p_2 x_{i_2} & \cdots & p_n x_{i_n} \end{pmatrix}. \tag{3.15}$$

Then, $\beta^{-1}\kappa\beta = \gamma$. Hence, each cycle may be transformed into normal form.
Since the transformation of γ into normal form may be performed by means of a substitution involving only the same variables, all cycles in ρ may be transformed into normal form simultaneously. □

Example: Let n=4, i.e., the set U has 4 elements, and $H = S_3$. Let

$\sigma \in \Sigma$(H;4, 5, 5)=Σ_4 , where

$$\sigma = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ (12)x_2 & (123)x_3 & (1)x_4 & (23)x_1 \end{pmatrix} \tag{3.16}$$

Then,

$$\sigma^2 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ (12)(123)x_3 & (123)(1)x_4 & (1)(23)x_1 & (23)(12)x_2 \end{pmatrix} \tag{3.17}$$

$$= \begin{pmatrix} x_1 & x_3 \\ (13)x_3 & (23)x_1 \end{pmatrix} \begin{pmatrix} x_2 & x_4 \\ (123)x_4 & (123)x_2 \end{pmatrix} \tag{3.18}$$

$$\sigma^3 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ (13)x_4 & (13)x_1 & (123)x_2 & (132)x_3 \end{pmatrix} \tag{3.19}$$

$$\sigma^4 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ (123)x_1 & (132)x_2 & (132)x_3 & (132)x_4 \end{pmatrix} \tag{3.20}$$

In this case $\delta_i' s$ will be:

$\delta_1 = (123)$

$\delta_2 = (132)$

$\delta_3 = (132)$

$\delta_4 = (132)$

We know that conjugacy classes of S_3 are [(1)], [(1 2)], [(1 2 3)] where

[(1)]={(1)}

[(1 2)]={(1 2), (1 3), (2 3)}

[(1 2 3)]={(1 2 3), (1 3 2)}.

Since $\delta_i's$ are in the same conjugacy class, they are conjugate.

Now, we should consider infinite cycles. If a cycle is infinite, then below we will show that any infinite cycle in $\Sigma(H; B, B^+, B^+)$ is conjugate to an infinite permutation in $S(B, B^+)$. Indeed, let κ be an arbitrary substitution and γ be an infinite cycle in $\Sigma(H; B, B^+, B^+)$.

$$\kappa = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & k_{-1}x_{j_{-1}} & k_0 x_{j_0} & k_1 x_{j_1} & \cdots \end{pmatrix} \tag{3.21}$$

$$\gamma = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & h_{-1}x_0 & h_0 x_1 & h_1 x_2 & \cdots \end{pmatrix}. \tag{3.22}$$

Then,

$$\kappa^{-1}\gamma\kappa = \begin{pmatrix} \cdots & x_{j_{-1}} & x_{j_0} & x_{j_1} \cdots & \\ \cdots & k_1^{-1}h_1 k_0 x_{j_0} & k_0^{-1}h_0 k_1 x_{j_1} & k_1^{-1}h_1 k_2 x_{j_2} & \cdots \end{pmatrix}. \tag{3.23}$$

It shows that $\kappa^{-1}\gamma\kappa$ has an infinite cycle form.

THEOREM 3.0.23. *Two cycles of length d are conjugate if and only if they leave the same number of x fixed.*

Proof. Let γ and θ be conjugate infinite cycles such that γ has n fixed points. Then

$$\gamma = \begin{pmatrix} \cdots & x_{i_{-1}} & x_{i_0} & x_{i_1} & \cdots \\ \cdots & h_{i_{-1}} x_{i_0} & h_{i_0} x_{i_1} & h_{i_1} x_{i_2} & \cdots \end{pmatrix} (a_1)(a_2)...(a_n) \tag{3.24}$$

then, for some monomial substitution κ,

$$\kappa = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & k_{-1} x_{j_{-1}} & k_0 x_{j_0} & k_1 x_{j_1} & \cdots \end{pmatrix} \tag{3.25}$$

$$\theta = \kappa^{-1} \gamma \kappa \tag{3.26}$$

$$= \begin{pmatrix} \cdots & x_{j_{-1}} & x_{j_0} & j_{i_1} & \cdots \\ \cdots & k_{i_{-1}}^{-1} h_{-1} x_{i_0} & h_{i_0} x_{i_1} & h_{i_1} x_{i_2} & \cdots \end{pmatrix} (a_1)^{\kappa} (a_2)^{\kappa} ...(a_n)^{\kappa}. \tag{3.27}$$

So, θ has n fixed points.

Conversely, let

$$c = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & h_{-1} x_0 & h_0 x_1 & h_1 x_2 & \cdots \end{pmatrix}, \tag{3.28}$$

and

$$c^{'} = \begin{pmatrix} \cdots & x_{i_{-1}} & x_{i_0} & x_{i_1} & \cdots \\ \cdots & r_{-1} x_{i_0} & r_0 x_{i_1} & r_1 x_{i_2} & \cdots \end{pmatrix} \tag{3.29}$$

c and $c^{'}$ leave the same number of x fixed. We should consider if there exists a $y \in \Sigma(H; B, B^+, B^+)$ such that $y^{-1}cy$= $c^{'}$ where

$$y = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & k_{-1} x_{j_{-1}} & k_0 x_{j_0} & k_1 x_{j_1} & \cdots \end{pmatrix} \tag{3.30}$$

$$y^{-1}cy = \begin{pmatrix} \cdots & x_{j_{-1}} & x_j & x_{j_1} & \cdots \\ \cdots & k_{-1}^{-1} h_{-1} k_0 x_j & k_0^{-1} h_0 k_1 x_{j_1} & k_1^{-1} h_1 k_2 x_{j_2} & \cdots \end{pmatrix} \tag{3.31}$$

then,

. . . .

$k_{-i}^{-1}h_{-i}k_{-i+1}= r_{-i}$

. . . .

$k_{-1}^{-1}h_{-1}k_0= r_{-1}$

$k_0^{-1}h_0k_1= r_0$

$k_1^{-1}h_1k_2= r_1$

. . . .

$k_i^{-1}h_ik_{i+1}= r_i$

. . . .

Let k_0= t, where t is arbitrary.

$k_{-1}^{-1} = r_{-1}t^{-1}h_{-1}^{-1}$

$k_1 = h_0^{-1}tr_0$

$k_2 = h_1^{-1}h_0^{-1}tr_0r_2$

Since we can solve this equations, any two cycles of infinite length are conjugate. In particular, if $c^{'}$ is a permutation still we can solve $y^{-1}cy = c^{'}$, and $c^{'}$ is a permutation. Hence, every infinite cycle in $\Sigma(H; B, B^+, B^+)$ can be made conjugate to an infinite permutation.

□

Now, in the light of the Theorem 3.0.23 and Theorem 3.0.21, we can state the following Theorem.

Theorem 3.0.24. *Two monomial substitutions y and y_1 are conjugate if and only if in their cyclic decomposition the finite cycles can be made to correspond in a one to* *one manner such that corresponding cycles have the same lengths and determinant class and cardinality of the set of infinite cycles is the same for both y and y_1.*

Proof. This Theorem is consequence of Theorem 3.0.21 and Theorem 3.0.23.

□

3.1 Centralizers of Elements in Monomial Group

Monomial groups appear naturally as centralizer of an element in symmetric groups. The structure of centralizers of elements in finite symmetric groups is well known. If α is an n-cycle in finite symmetric group S_n on n-letters, then

$C_{S_n}(\alpha) =< \alpha >$. Indeed, $< \alpha > \leq C_{S_n}(\alpha)$. Moreover, if $\beta \in C_{S_n}(\alpha)$, then $\alpha^\beta = \alpha$. Since under conjugation cycle type of a permutation is preserved α^β must be an n-cycle and conjugation sends

$$(a_1, a_2, ..., a_n)^\beta = (a_1^\beta, a_2^\beta, ..., a_n^\beta) = (a_1, a_2, ..., a_n)$$

implies that if $a_1^\beta = a_j$ for some j, then $a_2^\beta = a_{j+1}$, $a_3^\beta = a_{j+2}$, $\cdots$, $a_{n-(j-1)}^\beta = a_n$, $a_{n-j}^\beta = a_1$, $\cdots$, $a_n^\beta = a_{j+n-1} = a_{j-1}$. It shows that

$$y = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-(j-1)} & a_{n-j} & \cdots & a_n \\ a_j & a_{j+1} & a_{j+2} & \cdots & a_n & a_1 & \cdots & a_{j-1}. \end{pmatrix} \qquad (3.32)$$

So, $\beta = \alpha^{j-1}$. Hence, $\beta \in< \alpha >$, i.e., $C_{S_n}(\alpha) =< \alpha >$.

Now, if α is in S_n and α is a product of cycles of the same length k, then α is of the form
$\alpha = (a_1, a_2, ..., a_k)(a_{k+1}, a_{k+2}, ..., a_{2k})...(a_{(m-1)k+1}, a_{(m-k)+2}, ..., a_{mk})$.

So, mk=n. In this case $C_{S_n}(\alpha) \cong (C_k \times C_k \times ... \times C_k) \rtimes S_m$ where C_k is a cyclic group of order k, and S_m is the finite symmetric group on m-letters. The elements of S_m permute the cycles in this case.

Let $g \in C_{S_n}(\alpha)$. Then, $\alpha^g = \alpha$. Since conjugation of a permutation by another permutation preserves the cycle type. α^g will be again a permutation of the same type as α. Hence α^g is a product of m cycles of length k. Moreover, $\alpha^g = \alpha$ implies that the cycles are the same except the order of cycles in α, because distinct disjoint cycles commute. Hence we can multiply g with permutation g_1 where $g_1 \in S_m$ and $g_1^{-1}g$ fixes each cycle of g. Since $g_1^{-1}g$ is again an element of the centralizer and $g_1^{-1}g$ fixes each cycle. By above paragraph we know the centralizer of a k-cycle in S_k, namely $C_{S_k}(a_1, ..., a_k) =< (a_1, ..., a_k) >$. We multiply by elements $c_{i,k}$ of cyclic group C_k for each cycle of α. Hence $c_{i,k}^{-1}c_{2,k}^{-1}...c_{m,k}^{-1}g_1^{-1}g = id$. Hence $g \in C_{S_n}(\alpha) \cong (C_k \times C_k \times ... \times C_k) \rtimes S_m \cong \Sigma_m(C_k)$ which is a monomial group of degree m over the cyclic group C_k of order k.

If α is a product of cycles of different length then α is of the form $\alpha = (a_{1,1}, ..., a_{1,k_1})...(a_{m,1}, ..., a_{m,k_m})$. We need to find $C_{S_n}(\alpha)$. Assume that we have l different lengths of cycles in the cycle decomposition of α, and let Y_m be the union of the orbits of the same length m, where m=1,2,...,l. The $Y_m's$ are a partition of the set with n elements into disjoint sets. Let $x \in C_{S_n}(\alpha)$. It is clear that $Y_m's$ are α invariant and also x invariant by previous paragraph. Conversely, if we have a permutation x_m of Y_m such that x_m commutes with restriction α_m of an element α to Y_m, then x_m is in $C_{S_n}(\alpha)$. Therefore $C_{S_n}(\alpha) = C_{S_{|Y_m|}}(\alpha_1) \times ... \times C_{S_{|Y_l|}}(\alpha_l)$ where $S_{|Y_m|}$ is the symmetric group of degree $|Y_m|$. Therefore it is enough to calculate $C_{S_{|Y_k|}}(\alpha)$ where α has a fixed cycle length k.

Let $\alpha = (a_{1,1}, ..., a_{1,k})...(a_{m,1}, ..., a_{m,mk})$. We want to show that $C_{S_n}(\alpha) \cong (C_k \wr S_m)$.

Define r' by $a_{lj}r' = a_{lj,r}$. Then,

$$f : S_m \to C_{S_{mk}}(\alpha)$$

$$r \mapsto r'$$

is a homomorphism.

Let $\theta_l : a_{l,1} \to a_{l,2} \to ... \to a_{l,k} \to a_{l,1}$. Certainly θ_l is in $C_{S_{mk}}(\alpha)$ and $W_i =< f(r), \theta_l : l = 1, 2, ..., m; r \in S_m >\cong (C_k \wr S_m)$.

Conversely, if $g \in C_{S_{mk}}(\alpha)$, then g permutes the cycles of α, so there exists $r \in S_m$

such that gr' fixes every cycle of α. Since the centralizer in S_k of a cycle of length k is a cyclic group of order k.

$(gr')\prod_{l=1}^{m}\theta_l^{k_l} = 1$ where $1 \leq k_l \leq k-1$.

Hence $g \in W_m$, so $C_{S_{mk}}(\alpha) = W_m \cong C_k \wr S_m \cong \Sigma_m(C_k)$.

In the case α is a product of cycles of different length since each cycle type will be preserved under conjugation, $C_{S_n}(\alpha) = \Sigma_{m_1}(C_{k_1}) \times ... \times \Sigma_{m_l}(C_{k_l})$ where we have l different length each length k_i have m_i cycles of length k_i.

Theorem 3.1.1. *Let y be conjugate to y_1 written in the normal form $y_1 = \prod_i \delta_i$, $\delta_i = \prod_\epsilon \delta_\epsilon^i$, where for a fixed i the δ_ϵ^i are the normalized cycles of the same length n, and the same determinant class a if $n < d$. Let ϵ run over a set of cardinal μ where $0 \leqq \mu \leqq B$. Then the centralizer $C_{\Sigma(H;B,B^+,B^+)}(y)$ is isomorphic to the strong direct product of symmetries*

$$C_{\Sigma(H;B,B^+,B^+)}(y) \cong \prod_i (\Sigma(C_{(H}(a) < \delta >, \mu_i, \mu_i^+, \mu_i^+)) \times \Sigma_\kappa(H \times \mathbb{Z}; \kappa, \kappa^+, \kappa^+).$$

The group $C_H(a) < \delta >$ consists of all elements y_1 of the form $y_1 = \{k_i\}(c_1^i)^j$ where k belongs to the centralizer of a in H. The second direct product arises if δ is a product of κ infinite cycles where $\kappa \leq B$

Proof. Let y be an arbitrary element of $\Sigma(H; B, B^+, B^+)$. Then by taking conjugate of y by elements of $\Sigma(H; B, B^+, B^+)$, we may assume that y is a product of cycles in its normal form, say y_1.

Since $C_{\Sigma(H;B,B^+,B^+)}(y) \cong C_{\Sigma(H;B,B^+,B^+)}(y_1)$, it is enough to find the structure of centralizer of the elements y_1, which is a product of symmetries in its normal form.

The element y_1 may contain finite cycles and infinite cycles. We prove the theorem case by case. In the case of finite cycles we follow the proof of Ore [2] and in the case of infinite cycles we follow Crouch [1].

Since conjugation of an element $y_1 \in \Sigma(H; B, B^+, B^+)$ by an element $g \in \Sigma(H; B, B^+, B^+)$ preserves cycle length and the determinant class, the centralizer of an element y_1 will be direct product of centralizers of elements for each cycle length and determinant class. For this reason we will find the structure of centralizers of elements for each cycle length and determinant class.

Step 1: (a) Assume that y_1 is just a cycle of length n in $\Sigma_n(H)$, for an arbitrary group H, and y_1 has determinant class a. In this case y_1 is of the normal form

$$y_1 = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_3 & x_4 & \cdots & ax_1. \end{pmatrix}. \tag{3.33}$$

Then for

$$\kappa = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ k_1 x_{j_1} & k_2 x_{j_2} & \cdots & k_m x_{j_m} \end{pmatrix} \tag{3.34}$$

and by the calculation

$$\kappa^{-1} y_1 \kappa = \begin{pmatrix} x_{j_1} & x_{j_2} & \cdots & x_{j_n} & x_{j_{n+1}} & \cdots & x_{j_m} \\ k_1^{-1} k_2 x_{j_2} & k_2^{-1} k_3 x_{j_3} & \cdots & k_n^{-1} a k_1 x_{j_1} & x_{j_{n+1}} & \cdots & x_{j_m} \end{pmatrix} \tag{3.35}$$

$$= \{k_n^{-1} a k_1 x_{j_1}, k_1^{-1} k_2 x_{j_2}, k_2^{-1} k_3 x_{j_3}, \cdots, k_{n-1}^{-1} k_n x_{j_n}\} \tag{3.36}$$

Since κ is in the centralizer of y_1 implies that $\kappa^{-1} y_1 \kappa = y_1$ we may solve the unknowns $k_1, k_2, ..., k_n$ in H and so by Theorem (8) in [2]

$$\kappa = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-j+1} & x_{n-j+2} & \cdots & x_n \\ r x_j & r x_{j+1} & \cdots & r x_n & r a x_1 & \cdots & r a x_{j-1} \end{pmatrix} = \{r\} y_1^j \tag{3.37}$$

where $r \in C_H(a)$. Clearly, the powers of $y_1, y_1^m \in C_{\Sigma(H;B,B^+,B^+)}(y_1)$. Hence, we obtain $\kappa = \{r\} y_1^j = y_1^j \{r\}$.

Hence, $C_{\Sigma_n(H)}(y_1) \cong C_H(a) < y_1 >=< y_1 > C_H(a)$. $C_{\Sigma_n(H)}(y_1)$ is an extension of $C_H(a)$ by the group $< y_1 >$ of degree n.

(b) If y_1 is a product of k cycles of length n and each cycle in the normal form has the same determinant class a.

In this case y_1 is of the form

$$y_1 = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_3 & \cdots & ax_1 \end{pmatrix} \begin{pmatrix} x_{n+1} & x_{n+2} & \cdots & x_{2n} \\ x_{n+2} & x_{n+3} & \cdots & ax_{n+1} \end{pmatrix} ... \tag{3.38}$$

$$\begin{pmatrix} x_{(k-1)n+1} & x_{(k-1)n+2} & \cdots & x_{kn} \\ x_{(k-1)n+2} & x_{(k-1)n+3} & \cdots & x_{(k-1)n+1} \end{pmatrix} \tag{3.39}$$

any permutation of cycles of type $(x_1, x_{n+1})(x_2, x_{n+2})...(x_n, x_2)$ and

$(x_1, x_{n+1}, x_{2n+1}, ..., x_{jn+1})(x_2, x_{n+2}, x_{2n+2}, ..., x_{jn+2})...(x_n, x_{2n}, x_{3n}, ..., x_{jn+n})$ commute with the given symmetry y_1 and these type of permutations generate a subgroup isomorphic to S_k, permutations permuting the cycles.

Moreover, for each fixed cycle centralizer be as in then case (a) and these centralizers commute with each other. Hence the centralizer of y_1 will be isomorphic to

$$C_H(a) < \delta_1 > \times C_H(a) < \delta_2 > \times ... \times C_H(a) < \delta_k > \rtimes S_k$$

$$\cong C_H(a) < \delta_1 > \wr S_k \cong \Sigma_k(C_H(a) < \delta_1 >).$$

Since in our symmetry B might be infinite in the case y_1 is a product of infinitely many cycles of length n and determinant class a and the cardinality of the cycles is say μ then $C_{\Sigma(H;B,B^+,B^+)}(y_1) \cong \Sigma(C_H(a) < \delta >; \mu, \mu^+, \mu^+)$.

Step 2: Now, we find the centralizer of a cycle of infinite length.
First observe by Lemma 3.0.23 that any infinite cycle of the form

$$\begin{pmatrix} \cdots & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & \cdots \\ \cdots & h_{-1}x_{-2} & h_{-1}x_{-1} & h_0x_0 & h_1x_1 & h_2x_2 & \cdots \end{pmatrix} \tag{3.40}$$

by taking its conjugate to a permutation cycle

$$\begin{pmatrix} \cdots & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & \cdots \\ \cdots & x_{-1} & x_0 & x_1 & x_2 & x_3 & \cdots \end{pmatrix}. \tag{3.41}$$

Therefore, in the infinite cycle case we may assume that in the normal form y_1 is a product of say μ_2 infinite permutations as above.

First we find the centralizer of an infinite permutation $\Sigma(H;\aleph_0,\aleph_0^+,\aleph_0^+)$ without fixed point and all elements are moved.

We follow Crouch [1] for find the structure of centralizer of a cycle product of μ_2 infinite cycles as above will be the trivial consequence of the same argument.

Now,

$$\kappa^{-1}c\kappa = \begin{pmatrix} \cdots & x_{i_{-1}} & x_{i_0} & x_{i_1} & \cdots \\ \cdots & k_{-1}^{-1}k_0x_{i_{-0}} & k_0^{-1}k_1x_{i_1} & k_1^{-1}k_2x_{i_2} & \cdots \end{pmatrix}, \tag{3.42}$$

$$\text{where} \quad c = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & x_{-0} & x_1 & x_2 & \cdots \end{pmatrix}. \tag{3.43}$$

So, we can solve ..., k_{-2}, k_{-1}, k_0, k_1, ...

Then we observe that $\kappa^{-1}c\kappa = c$ implies that

$$\kappa = \begin{pmatrix} \cdots & x_{-1} & x_0 & x_1 & \cdots \\ \cdots & kx_{j-1} & x_j & kx_{j+1} & \cdots \end{pmatrix}. \tag{3.44}$$

Hence $\kappa = \{k\}c^j = c^j\kappa$ where κ is not a true scalar on the set U, but k is a scalar only on the variables which appear in the cycle. {k} will be identity on the elements of U which c does not move. It follows that $C_{\Sigma(H;d,d^+,d^+)}(c) \cong H \times \mathbb{Z}$, where $\mathbb{Z}$ is an infinite cyclic group which comes from the isomorphism $< c > \cong \mathbb{Z}$. It is independent of c as all isomorphic to $\mathbb{Z}$.

Now, if y is a product of $\mu_2 d - cycles$ then

$C_{\Sigma(H;B,B^+,B^+)}(y) \cong \Sigma(H \times Z;\mu_2,\mu_2^+,\mu_2^+)$. Now, from case 1 and case 2 theorem follows.

$\square$

CHAPTER 4

THE SPLITTING OF THE SYMMETRY

Definition 4.0.2. *Let G be a group, and N be a normal subgroup of G. If there exists a subgroup H≤G such that G=NH and N∩H=1, then we say that G splits over N and H is called as complement of N in G.*

If G splits over N, then for any g∈G, g=nh for some n∈N, h∈H. Moreover, this writing is unique.

Indeed assume that if g=nh=n_1h_1 where n_1, n∈N, h_1, h∈H, then nh=n_1h_1 implies n_1^{-1}n=h_1h^{-1}∈ N∩H=1. So, n=n_1 and h=h_1. Hence, the writing g=nh is unique, i. e. , n∈N, h∈H is unique in the writing g=nh.

Observe that for any g∈G,

G^g=$(NH)^g$=N^gH^g=NH^g as $N \trianglelefteq G$, $N^g = N$.

Therefore, all conjugates of H are also complement of N in G i.e., G=NH^g and $N \cap H^g = 1$.

Definition 4.0.3. *A group G splits regularly over N, if every complement of N in G is conjugate of H.*

If G splits regularly, G=NT, and N∩T=1, then there exists g∈ G such that T=H^g.

Let H and T be two complements of N in G such that H is not conjugate to T. Then by above, all conjugates of T in G are also complement of N in G.

Therefore, we may decompose the set of all complements of N in G.

Let C={T≤G| TN=G, T∩N=1} be the set of all complements of N in G. We may define an equivalence relation on C.

T_1∼T_2 iff T_1=T_2^g for some g∈ G.

∼ is an equivalence relation:

i) T_1∼T_1, since id∈G and T_1^{id}=T_1.

ii) T_1∼T_2 implies that there exists g∈G such that T_1=T_2^g if and only if $T_1^{g^{-1}}$=T_2 implies T_2∼T_1

iii) T_1∼T_2 and T_2∼T_3 implies that there exists g_1, g_2 ∈ G such that T_1=$T_2^{g_1}$, and T_2=$T_3^{g_2}$, and so $T_3^{g_2g_1}$= $(T_3^{g_2})^{g_1}$=$T_2^{g_1}$=T_1. g_2g_1 ∈ G, so ∼ is an equivalence relation.

The equivalence class containing T_1,

[T_1]= {T| T^g=T_1 for some g∈G}

If C has only one equivalence class, then G splits regularly, otherwise G does not split regularly.

If we go back to $\Sigma(H; B, B^+, B^+)$ we should notice that $\Sigma(H; B, B^+, B^+)$ splits over $V(B, B^+)$ with complement $S(B, B^+)$.

Now, we will investigate the splitting of $\Sigma(H; B, B^+, C)$, i.e., the splitting of the subgroup of $\Sigma(H; B, B^+, B^+)$ such that order of support of permutations is less than C where d≤C≤B^+.

Clearly, $\Sigma(H; B, B^+, C)$=$V(B, B^+)S(B, C)$, and $V(B, B^+)$∩$S(B, C) = 1$. Hence, $\Sigma(H; B, B^+, C)$ is a splitting of $V(B, B^+)$ with the complement S(B, C).

We are interested in the following question:

Find the necessary and sufficient conditions that C has only one equivalence class, i.e., whether all complements of $V(B, B^+)$ in $\Sigma(H; B, B^+, C)$ are conjugate or not.

If H and T are two complements of N in G, then H is isomorphic to T.

Let G=HN=NH and G=NT. Then, G／N= HN／N= TN／N.

HN／N≅H／H∩N≅H since H∩N=1.

HN／N=TN／N≅T／T∩N≅T since T∩N=1.

Hence, we have shown that any two complements of N in G are isomorphic.

We want to find out when H is conjugate to T.

Back to $\Sigma(H; B, B^{+}, C)$. Assume that T is a complement of $V(B, B^{+})$, Then by the above paragraph T≅S(B, C). Moreover,

$\Sigma(H; B, B^{+}, C)$=T$V(B, B^{+})$=S(B, C)$V(B, B^{+})$.

Denote θ by the natural isomorphism θ: S(B, C) $\rightarrow$ T such that θ(s)=vs=t where v∈$V(B, B^{+})$, s∈ S(B, C) and v, s are unique satisfying θ(s)=vs=t.

By using the above natural isomorphism, the elements s=(1, α) is a transposition in $\Sigma(H; B, B^{+}, C)$, and

$$t_{\alpha} = \theta(s) = \{h_{1,\alpha}, h_{2,\alpha}, ..., h_{n,\alpha}, ...\}(1, \alpha)$$

Since we can find the elements up to conjugacy of T, say $T^{'}$=vTv^{-1} where v∈ $V(B, B^{+})$.

If κ={k_1, k_2, . . . , k_n, . . . } where $\kappa \in V(B, B^{+})$, then T' has elements whose first factors are identity. Indeed

$$t'_{\alpha} = \{k_1^{-1}, k_2^{-1}, ..., k_n^{-1}, ...\}\{h_{1,\alpha}, h_{2,\alpha}, ..., h_{n,\alpha}, ...\}(1, \alpha)\{k_1, k_2, ..., k_n, ...\}$$

$$= \{k_1^{-1}h_{1,\alpha}k_{\alpha}, k_2^{-1}h_{2,\alpha}k_2, k_3^{-1}h_{3,\alpha}k_3, ..., k_{\alpha}^{-1}h_{\alpha,\alpha}k_1, ...\}(1, \alpha)$$

We can choose the first factor $k_1^{-1}h_{1,\alpha}k_{\alpha} = 1$ since $k_1^{-1}h_{1,\alpha}k_{\alpha} = 1$ implies $k_1 = h_{1,\alpha}k_{\alpha}$. For each $\alpha \in B$, $\alpha \neq 1$, we can do this conjugation and choose $k_{\alpha} = h_{1,\alpha}^{-1}k_1$. Then simultaneously we can solve this equation and obtain the first component of each t'_{α} is identity for all $\alpha \neq 1$. $k_2^{-1}h_{2,\alpha}k_2$ is a conjugate of $h_{2,\alpha}$ and other $h_{j,\alpha}$ are conjugate except the α^{th} component.

Then we have $t'_\alpha = \{e, h_{2,\alpha}, ..., h_{\alpha,\alpha}, ..., h_{\epsilon,\alpha}, ...\}(1, \alpha)$.

As $(t'_\alpha)^2 = E$, we have

$(t'_\alpha)^2 = \{h_{\alpha,\alpha}, h^2_{2,\alpha}, ..., h^2_{\alpha-1,\alpha}, k^{-1}_\alpha h_{1,\alpha} k_\alpha, ...\} = E$

where E is the identity of $\Sigma(H; B, B^+, B^+)$.

So, $h_{\alpha,\alpha} = e$ and, $h^2_{j,\alpha} = e$. Then,

$$t_\alpha = \theta(s) = \{1_H, h_{2,\alpha}, ..., 1_H, ...\}(1, \alpha).$$

So, we can write

(i) $t'_\alpha = \{e, h_{2,\alpha}, ..., , h_{\epsilon,\alpha}, ...\}(1, \alpha)$
(ii) $h_{\alpha,\alpha} = e$
(iii) $h^2_{\epsilon,\alpha} = e$ for $\epsilon \neq 1, \epsilon \neq \alpha$.

Let S_1(B, C) be the set of all elements of S(B, C) where

S_1(B, C)={g ∈S(B, C)| x_1·g=x_1 }, i.e., the stabilizer of the point x_1 in U.

Since stabilizer of a point is a subgroup, S_1(B, C) is a subgroup of S(B, C).

Observe that, if U={1, 2, . . . , n}, then

$$S_1(B, C) \cong S_{n-1} \cong S(n-1, n) \leq S(n, n+1).$$

Moreover if U is an infinite set, then $S_1(B, C) \cong S(B, C)$ where d≤C≤B^+.

LEMMA 4.0.4. *If s∈ S(B, C) and s moves x_1, then s can be written uniquely as $s = (1, \alpha)s_1$ where s_1 leaves x_1 fixed.*

Proof. Let s∈ S(B, C). Then we may write s as a product of disjoint cycles. In the writing of s as a disjoint product of cycles we are interested in only the cycle containing 1 as the other cycles already fixes 1 and the product of elements which fixes 1 is again an element fixing 1. For this reason we consider the cycle which contains (moves) 1. If this cycle is finite then we have the following:

Observe first that, if s=(1, 2, . . . , n),

s=(1, n)(2, 3, . . . , n), (2, 3, . . . , n)$\in S_1$(B, C).

We can not write S=(1, k)(α_1, α_2, . . . , α_l) where

(α_1, α_2, . . . , α_l)$\in S_1$(B, C).

Think of the permutation (1, 2, 3, 4)=(1, 3)β where $\beta\in S_4$. There exists no such $\beta\in S_4$.

This observation can be generalized for all S(B, C).

If s=(1, n)α_1=(1, n)α_2, then α_1=α_2. So, the writing is unique, i.e.,

$\alpha_1\in S_1$(B, C) is unique.

If s is an infinite cycle, and s moves x_1, then

$$s = \begin{pmatrix} x_1 & x_2 & ... & x_\beta & \cdots \\ x_\alpha & x_\epsilon & ... & x_1 & \cdots \end{pmatrix} = (1, \beta) \begin{pmatrix} x_2 & x_3 & ... & x_\beta & \cdots \\ x_\epsilon & x_\lambda & ... & x_\alpha & \cdots \end{pmatrix}.$$

On the other hand,

$$s = \begin{pmatrix} x_2 & x_3 & ... & x_\alpha & \cdots & x_\beta & \cdots \\ x_\epsilon & x_\lambda & ... & x_\delta & \cdots & x_\alpha & \cdots \end{pmatrix} (1, \alpha)$$

where $\begin{pmatrix} x_2 & x_3 & ... & x_\alpha & \cdots & x_\beta & \cdots \\ x_\epsilon & x_\lambda & ... & x_\delta & \cdots & x_\alpha & \cdots \end{pmatrix} \in S_1(B, C).$

Assume that there exists $x_\alpha\in$ U such that $\alpha\neq 1$, and $x_\alpha s_1$=x_α, i.e., s_1 fixes x_α for some $\alpha\neq 1$. So, $x_1 s_1$=x_1, and $x_\alpha s_1$=x_α. Then, consider s=(1, α)s_1=s_1(1, α) $\Rightarrow$ s_1=(1, α)s = s(1, α)

$$\theta(1, \alpha) = \{e, h_{2,\alpha}, ..., h_{\epsilon,\alpha}, ...e, ...\}(1, \alpha),$$

where e occurs as a factor in the first and α^{th} positions.

$$\theta(s_1) = \theta((1, \alpha)s) = \theta(1, \alpha)\theta(s) = \theta(s(1, \alpha)) = \theta(s)\theta(1, \alpha)$$

If s belongs to S(B,C) and moves x_1, then by Lemma 4.0.4, s can be written uniquely as $s = (1, \alpha)s_1$ where $s_1 \in S_1(B, C)$.

The image of $(1, \alpha)$ under θ has been described above as

$$\theta(1, \alpha) = \{e, h_{2,\alpha}, ..., h_{\epsilon,\alpha}, ...e, ...\}(1, \alpha).$$

To find the image of any element of S(B,C) it is sufficient to discuss those elements in $S_1(B, C)$.

Let $s_1 \in S_1(B, C)$ such that $x_\alpha s_1 = x_\alpha$ for some $x_\alpha, \alpha \neq 1$ i.e., s_1 fixes x_α where $\alpha \neq 1$ i.e., $s_1 \in S_1(B, C) \cap S_\alpha(B, C)$.
Let $s = (1, \alpha)s_1$. Then $s_1 = (1, \alpha)s = s(1, \alpha)$ where s sends x_1 into x_α, and x_α into x_1 .

Let, θ(s)=$\{k_1, k_2, \ldots, k_\epsilon, \ldots\}$s.

So,

$$\theta(s_1) = \{e, h_{2,\alpha}, ..., h_{\epsilon,\alpha}, ...e, ...\}(1, \alpha)\{k_1, k_2, ..., k_\epsilon, ...\}s \tag{4.1}$$

$$= \begin{pmatrix} x_1 & x_2 & ... & x_\alpha & \cdots \\ k_\alpha x_1 & h_{2,\alpha}k_2 x_\delta & ... & k_1 x_\alpha & \cdots \end{pmatrix}. \tag{4.2}$$

$$\theta(s_1) = \theta(s)\theta(1, \alpha) = \begin{pmatrix} x_1 & ... & x_\alpha & \cdots \\ k_1 x_1 & ... & k_\alpha x_\alpha & \cdots \end{pmatrix} \Rightarrow k_1 = k_\alpha. \tag{4.3}$$

This shows that if s_1 belongs to S_1(B, C), then the factors of v where θ(s_1)=vs_1 in the positions corresponding to those x which s_1 leaves fixed are equal to the first factor of v.

□

LEMMA 4.0.5. *Let s belongs to S(B, C) , and have the following properties; s moves x_1, i. e, $x_1 s \neq x_1$, and $x_\alpha s = x_\alpha$ where $\alpha \neq 1$, and $x_\beta s = x_1$. Then s has the following form*

$$s = \begin{pmatrix} x_1 & ... & x_\beta & ... & x_\alpha & \cdots \\ x_\delta & ... & x_1 & ... & x_\alpha & \cdots \end{pmatrix} \tag{4.4}$$

where $\delta \neq 1$. Let θ(s)=vs where v$\in V(B, B^{+})$. Then the factors which occur in the first and β^{th} positions of v are equal.

Proof. Let θ(s)=vs={c_1, c_2, c_3, . . . , c_β, . . . , c_ϵ, . . . }s, we need to show c_1=c_β.

We may write s in the following form

$$s = (1, \beta) \begin{pmatrix} x_1 & ... & x_\beta & ... & x_\alpha & \cdots \\ x_1 & ... & x_\delta & ... & x_\alpha & \cdots \end{pmatrix} = (1, \beta)s_1$$

where $s_1 \in S_1$(B, C). Also, we may write from right, as

$$s = \begin{pmatrix} x_1 & ... & x_\beta & ... & x_\alpha & \cdots \\ x_1 & ... & x_\delta & ... & x_\alpha & \cdots \end{pmatrix} (1, \delta) = s_1(1, \delta).$$

Observe that when we write s as (1, β)s_1 and s_1(1, δ), the s_1's are the same. We want to find θ(s)=vs. So we need to understand factors of v.

s=(1, β)s_1=s_1(1, δ), so θ(s)=θ(1, β)$\theta(s_1)$. As we discussed on page 40,

$\theta(s_1)$=$\{h_\alpha, ..., h_\beta, ..., h_\alpha, ..., h_\delta, ...\}s_1$

θ(1, β)=$\{e, ..., e, ..., h_{\alpha,\beta}, ..., h_{\delta,\beta}, ...\}(1, \beta)$

θ(1, δ)=$\{e, ..., h_{\beta,\delta}, ..., h_{\alpha,\delta}, ..., e, ...\}(1, \delta)$

θ(s)=θ(1, β)$\theta(s_1)$= $\begin{pmatrix} x_1 & ... & x_\beta & ... \\ h_\beta x_\delta & ... & h_\alpha x_1 & ... \end{pmatrix}$

θ(s)=$\theta(s_1)\theta$(1, δ)= $\begin{pmatrix} x_1 & ... & x_\beta & ... \\ h_\alpha x_\delta & ... & h_\beta x_1 & ... \end{pmatrix}$

So, we have $h_\beta = h_\alpha$.

We can do the above computation for any α which is fixed by s. So, we may conclude that all the corresponding factors which are fixed by s, the factors of v are equal to h_β where x_βs=x_1.

<u>Claim:</u> Let $s_1 \in S_1(B, C)$ $x_\alpha s_1 = x_\alpha$, $\alpha \neq 1$ and $\theta(s_1) = vs_1$. Then v is a scalar.

It is sufficient to show that the factors occupying positions corresponding to x which s_1 moves are the same as the first factor of v.

We have shown that if s does not move x_α, then $h_\alpha = h_1, \forall\alpha$. If we can show that $h_1 = h_\beta$, where s_1 moves x_β, v will be a constant, so that, the factors coming from the x's such that s_1 fixes x the factors are equal to h_1 and the factors of x which s_1 moves x also equal to h_1 implies v is a constant.

Let

$$s_1 = \begin{pmatrix} x_1 & ... & x_\beta & ... & x_\alpha & \cdots \\ x_1 & ... & x_\delta & ... & x_\alpha & \cdots \end{pmatrix} = \begin{pmatrix} x_1 & ... & x_\beta & ... & x_\alpha & \cdots \\ x_\delta & ... & x_1 & ... & x_\alpha & \cdots \end{pmatrix} (1, \delta)$$

where $\delta \neq \beta$ and $\delta \neq 1$.

By the Lemma 4.0.4,

$$\theta(s_1) = \{h_1, ..., h_\beta, ..., h_\alpha, ..., h_\delta, ...\}s_1$$

where $h_\alpha = h_1$.

Furthermore, $\theta(1, \delta)=\{e, ..., h_{\beta,\delta}, ..., h_{\alpha,\delta}, ..., e, ...\}(1, \delta)$.

Using the decomposition of s_1 and the fact that θ is an isomorphism.

$$\theta(s_1) = \theta(s)\theta(1, \delta) = \begin{pmatrix} x_1 & ... & x_\beta & ... \\ h_\alpha x_1 & ... & h_\alpha x_\delta & ... \end{pmatrix}$$

Recall that, above if $s_1 \in S_1(B, C)$, and $x_\alpha s_1 = x_\alpha$, then the first factor and α^{th} factor are the same.

So far

$$\theta(s_1) = \{h_1, ..., h_\beta, ..., h_1, ..., h_\delta, ...\}s_1$$

By the above calculation, we show $h_1 = h_\alpha = h_\beta$. Hence, under the above condition, v is a constant.

It remains to discuss the case where there exists no x_α such that $x_\alpha s_1 = x_\alpha$, i.e., s_1 moves all x_β for all $\beta \neq 1$. We need to show under the condition

$\theta(s_1) = vs_1$ and v is a constant. Assume

$$s_1 = \begin{pmatrix} x_1 & x_2 & ... & x_\beta & ... \\ x_1 & x_\beta & ... & x_\alpha & ... \end{pmatrix} = (2,\beta) \begin{pmatrix} x_1 & x_2 & ... & x_\beta & \cdots \\ x_1 & x_\alpha & ... & x_\beta & \cdots \end{pmatrix} = \bar{s_1} s_1^{'}$$

where $\bar{s_1}, s_1^{'} \in S_1(B,C)$.

By the above calculation, $\theta(\bar{s_1}) = v_{\bar{s_1}} \bar{s_1}$ and $\theta(s_1^{'}) = v_{s_1^{'}} s_1^{'}$ where $v_{s_1^{'}}, v_{\bar{s_1}}$ are constant.

$\theta(s_1) = \theta(\bar{s_1} s_1^{'}) = \theta(\bar{s_1})\theta(s_1^{'}) = v_{\bar{s_1}} \bar{s_1} v_{s_1^{'}} s_1^{'}$

Since $v_{s_1^{'}}, v_{\bar{s_1}}$ are constant and constants commute with all permutations we can write

$\theta(s) = v_{\bar{s_1}} v_{s_1^{'}} \bar{s_1} s_1^{'} = v_{\bar{s_1}} v_{s_1^{'}} s_1$, as $\bar{s_1} s_1^{'} = s_1$.

And $v_{\bar{s_1}} v_{s_1^{'}} s_1 = v_{s_1} s_1$

$v_{s_1} = v_{\bar{s_1}} v_{s_1^{'}}$ is a constant since product of two constant is constant. So we have shown that if s_1 does not fix any $x_\alpha, \alpha \neq 1$, then $\theta(s_1) = vs_1$ where v is a scalar. Hence, under all conditions for any $s_1 \in S_1(B,C)$ $\theta(s_1) = vs_1$ where v is a constant. □

Define a map $\phi : S_1(B,C) \longrightarrow H$ such that $\phi(s_1) = h_{s_1},$ and

$\theta(s_1) = vs_1$ where $v = \{h_{s_1}\}$

$\theta : S(B,C) \longrightarrow T$ such that $\theta(s) = vs$ where vs∈T.

A computation shows that if

$$\theta(1, \alpha) = \{e, ..., h_{\beta,\alpha}, ..., h_{\alpha,\alpha}, ...\}(1,\alpha)$$

$$\theta(1, \beta) = \{e, ..., h_{\beta,\beta}, ..., h_{\alpha,\beta}, ...\}(1,\beta)$$

where $h_{\beta,\beta} = e$ and $h_{\alpha,\alpha} = e$, then

$$\theta((1,\alpha)(1,\beta)(1,\alpha) = \theta(\alpha,\beta)$$

$$= \{e, ..., h_{\beta,\alpha,...,e,...,}\}(1,\alpha)\{e, ..., e, ..., h_{\alpha,\beta}, ...\}(1,\beta)\{e, ..., h_{\beta,\alpha}, ..., e, ...\}(1,\alpha)$$

$$= \begin{pmatrix} x_1 & ... & x_\alpha & ... & x_\beta & ... \\ h_{\alpha,\beta}x_1 & ... & h_{\beta,\alpha}x_\beta & ... & h_{\beta,\alpha}x_\alpha & ... \end{pmatrix}$$

where $\alpha \neq 1, \beta \neq 1, \alpha \neq \beta$. But as $\alpha \neq 1, \beta \neq 1$ we have

$$\theta(\alpha, \beta) = \{g_{\alpha,\beta}\}(\alpha, \beta).$$

So, $h_{\alpha,\beta} = h_{\beta,\alpha} = g_{\alpha,\beta} \in H.$

Theorem 4.0.6. *The symmetry $\Sigma(H; B, B^{+}, C)$ splits over the basis group, $\Sigma(H; B, B^{+}, C) = V(B, B^{+}) \cup T, \Sigma(H; B, B^{+}, B^{+}) \cap T = E$. Any such group T is the conjugate of some group $T^{'}$ obtained by the following construction. Let G be a subgroup of H that is the homomorphic image of $S_1(B, C)$ where $d \leq C \leq B^{+}$. Let $\phi(s) = g_s$ indicate the homomorphism. In particular, $\phi(\alpha, \beta) = g_{\alpha,\beta}$. Then the elements of $T^{'}$ are obtained from the elements of S(B, C) by the isomorphism defined as follows: Let *: S(B, C)$\longrightarrow T^{'}$ be a map , and $*(s) = \{g_s\}$ for s belonging to $S_1(B, C)$, *(1, α)=$\{e, g_{2,\alpha}, ..., g_{\epsilon,\alpha}, ..., e, ...\}(1, \alpha)$ where e occurs in the first and α^{th} positions.*

In previous pages we have shown that if T is a complement of $V(B, B^{+})$, then the correspondence gives a homomorphism from S(B,C) into H where the above conditions are satisfied. Therefore, we need to prove the converse of the theorem. Namely if there is a correspondence as in the theorem, then it must be an isomorphism.

Proof. We have defined a map $\alpha : S(B, C) \longrightarrow T^{'}$. Now, we want to show that * is an isomorphism . We know that if an element s$\in$S(B, C), then we may write $s = (1, \alpha)s_1$ where $s_1 \in S_1(B, C)$. Indeed if s is already fixing x_1 then s=s_1, and α=1. So, we are done. We may assume that s moves x_1. Then as s is a permutation there exists α such that $x_\alpha s = x_1$. Then s=$(1, \alpha)s_1$, and where $j \neq 1$

$$s = \begin{pmatrix} x_1 & ... & x_\alpha & ... \\ x_j & ... & x_1 & ... \end{pmatrix} = (1, \alpha) \begin{pmatrix} x_1 & ... & x_\alpha & ... \\ x_1 & ... & x_j & ... \end{pmatrix}$$

$$*(s) = *((1, \alpha)s_1) = *(1, \alpha) * (s_1) = \{e, g_{2,\alpha}, ..., g_{\epsilon,\alpha}, ..., e, ...\}(1, \alpha) * (s_1)$$

$$= \{e, g_{2,\alpha}, ..., g_{\epsilon,\alpha}, ..., e, ...\}(1, \alpha)\{h_{s_1}\}s_1$$

Since $\{h_{s_1}\}$ is constant, it commutes with $(1, \alpha)$.

So, $*(s) = \{h_{s_1}, g_{2,\alpha}h_{s_1}, ..., g_{\epsilon,\alpha}h_{s_1}, ..., h_{s_1}, ...\}s$ □

Let $\bar{s} = (1, \beta)\bar{s_1}$ be another element of S(B, C) where $x_\beta \bar{s} = x_1$ and $\bar{s_1} \in S_1(B, C)$. We want to show that * is a homomorphism, i. e,

$$*(s\bar{s}) = *(s) * (\bar{s}) = *(1, \alpha) * (s_1) * (1, \beta)(\bar{s_1}).$$

By Ore [2], it is enough to show that $*(s(1, \beta)) = *(s) * (1, \beta)$. This is equivalent to show that

(i) $*(1, \alpha) * (1, \beta) = *((1, \alpha)(1, \beta)$, and

(ii) $*(s_1(1, \beta)) = *(s_1) * (1, \beta)$ for any $s_1 \in S_1(B, C)$.

$*(s(1, \beta)) = *((1, \alpha)s_1(1, \beta) = *(s) * (1, \beta)$

(ii) $*(s_1(1, \beta)) = *(s_1) * (1, \beta)$

$*((1, \alpha)(1, \beta) = \{g_{\alpha,\beta}, ..., e, ..., g_{\beta,\alpha}, ...\}(1, \alpha, \beta)$

Indeed

$$*(1, \alpha) * (1, \beta) = \{e, ..., g_{\epsilon,\alpha}, ..., e, ...g_{\beta,\alpha}, ...\}(1, \alpha)\{e, ..., g_{\epsilon,\beta}, ..., g_{\alpha,\beta}, ..., e, ...\}(1, \beta)$$
$$= \begin{pmatrix} x_1 & ... & x_\alpha & ... & x_\beta & ... \\ g_{\alpha,\beta}x_\alpha & ... & x_\beta & ... & g_{\beta,\alpha}x_1 & ... \end{pmatrix}$$

For $\alpha = \beta$ the case is trivially true. So we assume that $\alpha \neq \beta$. Then we have $(1, \alpha)(1, \beta) = (1, \alpha, \beta) = (1, \beta)(1, \alpha)$ So,

$*((1, \alpha)(1, \beta) = *(1, \alpha, \beta) = (1, \beta) * (\alpha, \beta) = *(1, \beta)\{g_{\alpha,\beta}\}(\alpha, \beta)$

$= \{e, ..., g_{\alpha,\beta}, ..., e, ...\}\{g_{\alpha,\beta}\}(1, \beta)(\alpha, \beta) = \{g_{\alpha,\beta}, ..., g_{\epsilon,\beta}, ..., g^2_{\alpha,\beta}, ..., g_{\alpha,\beta}, ...\}(1, \beta)(\alpha, \beta)$

$= \{g_{\alpha,\beta}, ..., g_{\epsilon,\beta}, ..., g_{\alpha,\beta}, ..., e, ..., g_{\alpha,\beta}\}(1, \alpha)(1, \beta).$

Now we compute the corresponding factors and obtain $g_{\alpha,\beta}$=e and $g_{\epsilon,\alpha}g_{\epsilon,\beta} = g_{\epsilon,\beta}g_{\alpha,\beta}$ since $(\epsilon, \alpha)(\epsilon, \beta) = (\epsilon, \beta)(\alpha, \beta)$, and where $\phi : S_1(B, C) \longrightarrow H$ such that $\phi(\alpha, \beta) = g_{\alpha,\beta}$, ϕ is a homomorphism.

Now, we should show that

$*(s_1(1,\beta)) = *s_1 * (1,\beta)$ for all $s_1 \in S_1(B,C)$.

There are two cases in this verification i.e., we will analyze it when s_1 moves x_β and when s_1 does not move x_β.

Case 1: If s_1 does not move x_β,

we know $s = (1,\beta)s_1 = s_1(1,\beta)$.

$*s = *(1,\beta) * (s_1)$ since * is a homomorphism.

Also, $*(s(1,\beta)) = *s * (1,\beta) = *(1,\beta) * s_1 * (1,\beta) = *s_1 * (1,\beta) * (1,\beta) = *(s_1(1,\beta)) * (1,\beta) = *s * (1,\beta)$.

Case 2: If s_1 moves x_β, then we can not say anything about $*(s(1,\beta))$ with direct computation. But,

$$s = s_1(1,\beta) = \begin{pmatrix} x_1 & \cdots & x_\beta & \cdots & x_\delta & \cdots \\ x_1 & \cdots & x_\alpha & \cdots & x_\beta & \cdots \end{pmatrix} (1,\beta) = (1,\delta)s_1. \tag{4.5}$$

Here s_1 does not move x_β so we can do computation.

$*s_1 = \{g_{s_1}\}s_1$,

$*(1,\beta) = \{e, ..., e, ..., g_{\delta,\beta}, ..., g_{\epsilon,\beta}, ...\}(1,\beta)$,

$*(1,\delta) = \{e, ..., g_{\beta,\delta}, ..., e, ..., g_{\epsilon,\delta}, ...\}(1,\delta)$ implies that

$*s_1 * (1,\beta) = \{g_{s_1}\}s_1\{e, ..., e, ..., g_{\delta,\beta}, ..., g_{\epsilon,\beta}, ...\}(1,\beta)$

$$= \begin{pmatrix} x_1 & \cdots & x_\beta & \cdots & x_\delta & \cdots & x_\epsilon & \cdots \\ g_{s_1}x_\beta & \cdots & g_{s_1}g_{\alpha,\beta}x_\alpha & \cdots & g_{s_1}x_1 & \cdots & g_{s_1}g_{\epsilon,\beta}x_{i_\epsilon} & \cdots \end{pmatrix} (1,\beta) = (1,\delta)s_1 \tag{4.6}$$

$$= *(s_1(1,\beta))$$

since * is a homomorphism.

Also, $*(s_1(1,\beta)) = *((1,\delta)s_1) = *(1,\delta) * (s_1)$

$$= \begin{pmatrix} x_1 & \cdots & x_\beta & \cdots & x_\delta & \cdots & x_\epsilon & \cdots \\ g_{s_1}x_\beta & \cdots & g_{s_1}g_{\beta,\delta}x_\alpha & \cdots & g_{s_1}x_1 & \cdots & g_{s_1}g_{\epsilon,\delta}x_{i_\epsilon} & \cdots \end{pmatrix} \tag{4.7}$$

Since ϕ is a homomorphism factors of above two computations are the same.

As a consequence, we get that the given correspondence in the theorem preserves the multiplication.

Images of the elements of S(B,C) form a group T. This T is isomorphic S(B,C). We can say clearly $V(B,B^+) \cap T = E$. Moreover, $V(B,B^+) \cup T = \Sigma(H;B,B^+,C)$ since if $y \in \Sigma(H;B,B^+,C)$, it can be written $y = vv_1^{-1}v_1s = v_2t$ where $*s = v_1s = t$.

Theorem 4.0.7. *A necessary and sufficient condition for $\Sigma(H;B,B^+,C)$ where $d^+ \leq C \leq B^+$ to split regularly over the basis group is that H contains no subgroup isomorphic to S(B, C).*

Remark 1: S(B, C)≅ $S_1(B,C)$ when B is infinite, and this is the case as $d^+ \leq C \leq B^+$.

Remark 2: Let y∈ $\Sigma(H;B,B^+,C)$, y=vs, and $v_1 \in V(B,B^+)$, if we take the conjugate of y by v_1, we have $v_1^{-1}yv_1 = v_1^{-1}vsv_1 = v_1^{-1}vsv_1s^{-1}s = v_1^{-1}vv_2s = v_3s$, where $v_2 = sv_1s^{-1}$ and $v_3 = v_1^{-1}vv_2 \in V(B,B^+)$.

So, s is fixed.

Remark 3: $s^{-1}ys \in \Sigma(H;B,B^+,C)$, where s∈ $S(B,C)$

Proof. Assume that $\Sigma(H;B,B^+,C)$ splits regularly over the basis group. Let T' be another complement oft he basis group . Then by assumption there exists $y \in \Sigma(H;B,B^+,C)$ such that

$$(T')^y = y^{-1}(T')y = S(B,C)$$

Every element $t \in T^{'}$ can be written in the form t=vs for some $v \in V(B, B^{+})$, and $s \in S(B, C)$. By remark 2 and 3, $t^{y} = (vs)^{y} \in S(B, C)$, we may take the element y$\in V(B, B^{+})$ because taking conjugate of an element by a permutation only permutes the factors. Therefore, if we want to obtain by taking conjugate we must take conjugation by an element of $V(B, B^{+})$. Therefore we may assume that y$\in V(B, B^{+})$. Say y=$\{k_1, k_2, ..., k_\alpha, ...\}$.

In order to understand the elements of $T^{'}$ we may consider the elements $t = \{g_{s_1}\}s_1$ where $s_1 \in S_1(B, C)$.

Consider the element $t = \{g_{s_1}\}s_1$ of T' where g_{s_1} is a constant element of $V(B, B^{+})$ and $s_1 \in S_1(B, C)$ i. e. s_1 fixes the symbol x_1. S_1 is the stabilizer of a point x_1 in S(B, C).

$$yty^{-1} = \{k_1, k_2, ...\}\{g_{s_1}\}s_1\{k_1^{-1}, k_2^{-1}, ...\} = \begin{pmatrix} x_1 & x_2 & ... \\ k_1 g_{s_1} k_1^{-1} x_1 & ... & ... \end{pmatrix} \in S_1(B, C)$$

$k_1 g_{s_1} k_1^{-1} = e$ implies $g_{s_1} = e \quad \forall s_1 \in S_1(B, C)$, and consider

$\theta : T^{'} \longrightarrow H$ such that $\theta(\{g_{s_1}\}s_1) = g_{s_1} = e$

So, t=$\{e\}s_1$ then every element of $T^{'}$ which is of the form $\{g_{s_1} s_1\} = \{e\}s_1$. Hence elements of T coming from $S_1(B, C)$ i. e. constant term is actually coming from H is identity.

Since θ sends all g_{s_1} into identity the above homomorphism sends all elements of the form $\{g_{s_1}\}$ to identity. Hence H does not contain a subgroup isomorphic to $S_1(B, C)$. Since by remark 1 , S(B, C)$\cong S_1(B, C)$, H contains no subgroup isomorphic to S(B, C).

Conversely, assume H contains no subgroup isomorphic to S(B, C) and that $\Sigma(H; B, B^{+}, C)$ does not split regularly. Then H contains no subgroup G which is the homomorphic image of $S_1(B, C)$. Scott has shown that this implies that G contains a subgroup isomorphic to S(B, C), contradicting the hypothesis. Therefore, $\Sigma(H; B, B^{+}, C)$ splits regularly.

□

Theorem 4.0.8. *A necessary and sufficient condition for* $\Sigma(H; B, B^{+}, d)$ *to split* *regularly over its basis group is that H contain no element of order 2.*

Proof. If $\Sigma(H; B, B^+, d)$ splits regularly, then by Theorem 4.0.6, it contains an isomorphic copy of $S_1(B, d) \simeq S(B, d)$. Since by assumption H contains no element of order 2, then the map $\gamma : T' \longrightarrow H$ is the trivial projection i. e.

$$t_\alpha = \{h_{1,\alpha}, h_{2,\alpha}, ..., h_{\epsilon,\alpha}, ...\}(1, \alpha),$$

and $\gamma : \pi a_\pi \longmapsto a_\pi$ where $a_\pi = 1$. Then T'=S(B, d).

By Baer's Theorem the only normal subgroup of $S(B, B^+)$ are the subgroup S(B, C) where $d \leq C \leq B^+$, S(B, d) and Alt(B, d). $S(B, d)/A(B, d) \simeq \mathbb{Z}_2$ and $|\mathbb{Z}_2| = 2$.

H contains no element of order 2. So the map γ is the identity map. □

4.1 The Splitting of $\Sigma_{n,A}(H)$

We first consider the special cases.

Case 1: Splitting of $\Sigma_{3,A}(H)$

Now we will discuss the splitting for n=3 i.e., H will be an arbitrary group and alternating monomial group of degree 3.

We already know that $\Sigma_{3,A}(H) = V_3 A_3$ and $V_3 \cap A_3 = 1$. So, A_3 is a complement of V_3 in $\Sigma_{3,A}(H)$.

Recall that $V_3 = H \times H \times H$. Let T be an arbitrary complement of V_3 in $\Sigma_{3,A}(H)$. Then $V_3 T = \Sigma_{3,A}(H)$ and $V_3 \cap T = 1$. Since arbitrary element of $\Sigma_{3,A}(H)$ can be written as vs where $v \in V_3$ and $s \in A_3$. The elements of T will be $\{1, g, g^2\}$. Let $\theta : A_3 \longrightarrow T$ be an isomorphism. Let $A_3 = \{1, a, a^2\}$. Since $\theta(1) = 1$, the image of a will determine the isomorphism. Let

$\theta(a) = \{h_1, h_2, h_3\}(1\ 2\ 3)$. Since we will find the complement up to conjugacy we may take the conjugate of T by multiplication

$\kappa = \{k_1, k_2, k_3\} \in V_3$. Then $\kappa T \kappa^{-1}$ contains the element of the form

$$\{k_1, k_2, k_3\}\{h_1, h_2, h_3\}(1\ 2\ 3)\{k_1^{-1}, k_2^{-1}, k_3^{-1}\}$$

$$= \{k_1h_1k_2^{-1}, k_2h_2k_3^{-1}, k_3h_3k_1^{-1}\}(1\ 2\ 3).$$

Since k_1, k_2, k_3 are arbitrary elements of H we can choose $k_1 = k$ as arbitrary. Then we may choose $k_2 = kh_1$, $k_3 = k_2h_2 = kh_1h_2$.

It follows that the third component $k_3h_3k_1^{-1} = kh_1h_2h_3k^{-1}$. Then we have $\theta(a) = \{e, e, kh_1h_2h_3k^{-1}\}(1\ 2\ 3) = \{e, e, b\}(1\ 2\ 3)$. Since a has order 3, we have $\theta(a)$ has order 3. Then $1 = \theta(a)^3 = (\{e, e, b\}(1\ 2\ 3))^3 = \{e, e, e\}$. It follows that b=1. Hence the isomorphism θ will be the identity automorphism and so $T = A_3$. Hence all complements of V_3 will be conjugate to A_3 and so $\Sigma_{3,A}(H)$ splits regularly.

Case 2: Splitting of $\Sigma_{4,A}(H)$

Recall that the alternating group A_4 has order 12 and consists of even permutations of symmetric group on 4 letters. The subgroup $\kappa = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ will be a normal subgroup of A_4 which is isomorphic to elementary Abelian group of order 4 in fact $\kappa \cong Z_2 \times Z_2$. Then $A_4 = \kappa < (1\ 2\ 3) >$. So A_4 is a split extension of κ with a cyclic subgroup of order 3. Since $< (1\ 2\ 3) >$ will be a Sylow 3-subgroup ofA_4 and by Sylow theorem, all Sylow 3-subgroups are conjugate. We have all complements of V in A_4 are conjugate. In our terminology, A_4 splits regularly over the normal group κ.
Since the only nontrivial normal subgroup of A_4 is κ, and any homomorphism θ from A_4 to any other group will be either $\theta(A_4) = 1$ trivial homomorphism or $\theta(A_4) = A_4$ isomorphism or $\theta(A_4) =< d >$ where $< d >$ is a cyclic group of order 3. Therefore, in the above cases $Ker(\theta) = \{1\}$ and $Ker(\theta) = \kappa$.

We will prove the following theorem for this special case.

Theorem 4.1.1. *The group $\Sigma_{4,A}(H)$ splits over the basis group $V_4 \cong H \times H \times H \times H$ with complement A_4. Let T' be another complement of $V_4 \in \Sigma_{4,A}(H)$. Then there exists a homomorphism $\phi : A_4 \longrightarrow H$ satisfying $\phi(s) = g_s$ for all $s \in A_4$. Then the isomorphism θ will be $\theta(s) = \{g_s\}s$ for all $s \in A_4$.*

Proof. Let T be a complement of V_4 in $\Sigma_{4,A}(H)$. Then there exists a homomorphism $\theta : A_4 \longrightarrow T$. Since by previous pages A_4 has an elementary Abelian normal subgroup κ isomorphic to $\mathbb{Z}_{\not\simeq} \times \mathbb{Z}_{\not\simeq}$. We will consider the images of κ into T. Since κ

is generated by $\sigma_1 = (1\,2)(3\,4)$, $\sigma_2 = (1\,3)(2\,4)$ and $\sigma_3 = (1\,4)(2\,3)$ the group T will have either homomorphic image of κ i.e., $\theta(\kappa) \cong \kappa$ or $\theta(\kappa) = \{e\}$.

Let $\theta(\sigma_1) = \{h_{11}, h_{12}, h_{13}, h_{14}\}(1\,2)(3\,4)$

$\theta(\sigma_2) = \{h_{21}, h_{22}, h_{23}, h_{24}\}(1\,3)(2\,4)$

$\theta(\sigma_3) = \{h_{31}, h_{32}, h_{33}, h_{34}\}(1\,4)(2\,3)$

As before, since we want to find the complements up to conjugacy we may take the conjugate of T with a product $\kappa = \{k_1, k_2, k_3, k_4\}$. Then

(i) $\kappa\theta(\sigma_1)\kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{11}, h_{12}, h_{13}, h_{14}\}(1\,2)(3\,4)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_1h_{11}k_2^{-1}, k_2h_{12}k_1^{-1}, k_3h_{13}k_4^{-1}, k_4h_{14}k_3^{-1}\}(1\,2)(3\,4)$

(ii) $\kappa\theta(\sigma_2)\kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{21}, h_{22}, h_{23}, h_{24}\}(1\,3)(2\,4)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_1h_{21}k_3^{-1}, k_2h_{22}k_4^{-1}, k_3h_{23}k_1^{-1}, k_4h_{24}k_2^{-1}\}(1\,3)(2\,4)$

(iii)
$\kappa\theta(\sigma_3)\kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{31}, h_{32}, h_{33}, h_{34}\}(1\,4)(2\,3)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_1h_{31}k_4^{-1}, k_2h_{32}k_3^{-1}, k_3h_{33}k_2^{-1}, k_4h_{34}k_1^{-1}\}(1\,4)(2\,3)$.

Then again as k_1, k_2, k_3, k_4 are arbitrary elements of H, choose $k_1 = k$ fixed, then by (i) and (ii), choose $k_2 = kh_{11}$, then $k = k_2h_{12}$. $k_4 = k_3h_{13}$ implies $k_3 = k_4h_{14}$. $\sigma(i)$ has order 2. So $\theta(\sigma_1)^2 = \{h_{11}h_{12}, h_{12}h_{11}, h_{13}h_{14}, h_{14}, h_{13}\} = \{e, e, e, e\}$. Then $h_{12} = h_{11}^{-1}$ and $h_{14} = h_{13}^{-1}$. Hence $\theta(\sigma_1) = \{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)$.

$\theta(\sigma_2)^2 = \{h_{21}h_{33}, h_{22}h_{24}, h_{23}h_{21}, h_{24}, h_{22}\} = \{e, e, e, e\}$. Then $h_{33} = h_{21}^{-1}$ and $h_{24} = h_{22}^{-1}$. Then $\theta(\sigma_2) = \{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4)$.

$\theta(\sigma_3)^2 = \{h_{31}h_{34}, h_{32}h_{33}, h_{33}h_{32}, h_{34}, h_{31}\} = \{e, e, e, e\}$. Then $h_{31} = h_{34}^{-1}$ and $h_{32} = h_{33}^{-1}$. Hence $\theta(\sigma_3) = \{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3)$.

□

Now we use the property that κ is an Abelian group. Therefore $\theta(\kappa)$ is an Abelian group.

$\theta(\sigma_1)\theta(\sigma_2) = \theta(\sigma_1\sigma_2) = \theta(\sigma_2\sigma_1) = \theta(\sigma_2)\theta(\sigma_1)$ implies that

$\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4) =$
$\{h_{11}h_{22}, h_{11}^{-1}h_{21}, h_{13}h_{22}^{-1}, h_{13}^{-1}h_{21}^{-1}\}(1\,4)(2\,3) =$
$\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4)\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)$
$= \{h_{21}h_{13}, h_{22}h_{13}^{-1}, h_{21}^{-1}h_{11}, h_{22}^{-1}h_{11}^{-1}\}(1\,4)(2\,3).$

We obtain

(A) $h_{11}h_{22} = h_{21}h_{13}$

(B) $h_{11}^{-1}h_{21} = h_{22}h_{13}^{-1}$

(C) $h_{13}h_{22}^{-1} = h_{21}^{-1}h_{11}$

(D) $h_{13}^{-1}h_{21}^{-1} = h_{22}^{-1}h_{11}^{-1}$ implies that $A = D^{-1}$ and $B = C^{-1}$.

Only the following equation remains.

$h_{11}h_{22} = h_{21}h_{13}$.

$\theta(\sigma_2)\theta(\sigma_3) = \theta(\sigma_3)\theta(\sigma_2)$

implies that $\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4)\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3) =$
$\{h_{21}h_{32}^{-1}, h_{22}h_{31}^{-1}, h_{21}^{-1}h_{31}, h_{22}^{-1}h_{32}\}(1\,2)(3\,4) =$
$\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3)\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4) =$
$\{h_{31}h_{22}^{-1}, h_{32}h_{21}^{-1}, h_{32}^{-1}h_{22}, h_{31}^{-1}h_{21}\}(1\,2)(3\,4).$

We obtain

(A) $h_{21}h_{32}^{-1} = h_{31}h_{22}^{-1}$

(B) $h_{22}h_{31}^{-1} = h_{32}h_{21}^{-1}$

(C) $h_{21}^{-1}h_{31} = h_{32}^{-1}h_{22}$

(D) $h_{22}^{-1}h_{32} = h_{31}^{-1}h_{21}$

implies that $A = B^{-1}$ and $C = D^{-1}$.

Only the following equation remain.

$h_{21}h_{32}^{-1} = h_{31}h_{22}^{-1}$

$\theta(\sigma_1)\theta(\sigma_3) = \theta(\sigma_3)\theta(\sigma_1)$

implies that $\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3) = \{h_{11}h_{32}, h_{11}^{-1}h_{31}, h_{13}h_{31}^{-1}, h_{13}^{-1}h_{32}^{-1}\}(1\,3)(2\,4) = \{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3)\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4) = \{h_{31}h_{13}^{-1}, h_{32}h_{13}, h_{32}^{-1}h_{11}^{-1}, h_{31}^{-1}h_{11}\}(1\,3)(2\,4)$.

So, we have the following equations

(A) $h_{11}h_{32} = h_{31}h_{13}^{-1}$

(B) $h_{11}^{-1}h_{31} = h_{32}h_{13}$

(C) $h_{13}h_{31}^{-1} = h_{32}^{-1}h_{11}^{-1}$

(D) $h_{13}^{-1}h_{32}^{-1} = h_{31}^{-1}h_{11}$

implies that $A = C^{-1}$ and $B = C^{-1}$.

We get only $h_{11}h_{32} = h_{31}h_{13}^{-1}$.

Then we use the property

$\theta(\sigma_1\sigma_2) = \theta(\sigma_3), \theta(\sigma_2\sigma_3) = \theta(\sigma_1), \theta(\sigma_1\sigma_3) = \theta(\sigma_2)$. Then

$\theta(\sigma_1\sigma_2) = \{h_{11}h_{22}, h_{11}^{-1}h_{21}, h_{13}h_{22}^{-1}, h_{13}^{-1}h_{21}^{-1}\}(1\,4)(2\,3) = \{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3) = \theta(\sigma_3)$

implies that

$h_{31}^{-1} = h_{22}^{-1}h_{11}^{-1} = h_{13}^{-1}h_{21}^{-1}$,

$h_{32} = h_{11}^{-1}h_{21}$,

$h_{32}^{-1} = h_{13}h_{22}^{-1} = h_{21}^{-1}h_{11}$.

$\theta(\sigma_2\sigma_3) = \{h_{21}h_{32}^{-1}, h_{22}h_{31}^{-1}, h_{21}^{-1}h_{31}, h_{22}^{-1}h_{32}\}(1\,2)(3\,4) = \{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4) = \theta(\sigma_1)$

implies that

$h_{11}^{-1} = h_{32}h_{21}^{-1} = h_{22}h_{31}^{-1}$,

$h_{13} = h_{21}^{-1}h_{31} = h_{32}^{-1}h_{22}$.

$\theta(\sigma_1\sigma_3) = \{h_{11}h_{32}, h_{11}^{-1}h_{31}, h_{13}h_31^{-1}, h_{13}^{-1}h_{32}^{-1}\}(1\,3)(2\,4) = \{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4) = \theta(\sigma_2)$

implies that

$h_{21} = h_{11}h_{32} = h_{31}h_{13}^{-1}$,

$h_{22} = h_{11}^{-1}h_{31} = h_{32}h_{13}$.

Now, we can take conjugate with $\kappa = \{k_1, k_2, k_3, k_4\}$. Then we obtain

$(i)\kappa\theta(\sigma_1)\kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{11}, h_{11}^{-1}, h_{13}, h_{13}^{-1}\}(1\,2)(3\,4)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_2h_{11}^{-1}k_1^{-1}, k_1h_{11}k_2^{-1}, k_4h_{13}^{-1}k_3^{-1}, k_3h_{13}k_4^{-1}\}(1\,2)(3\,4)$

(ii)
$\kappa\theta(\sigma_2)\kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{21}, h_{22}, h_{21}^{-1}, h_{22}^{-1}\}(1\,3)(2\,4)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_3h_{21}^{-1}k_1^{-1}, k_4h_{22}^{-1}k_2^{-1}, k_1h_{21}k_3^{-1}, k_2h_{22}k_4^{-1}\}(1\,3)(2\,4)$

(iii)
$\kappa\theta(\sigma_3)\kappa^{-1} = \{k_1, k_2, k_3, k_4\}\{h_{31}, h_{32}, h_{32}^{-1}, h_{31}^{-1}\}(1\,4)(2\,3)\{k_1^{-1}, k_2^{-1}, k_3^{-1}, k_4^{-1}\} = \{k_4h_{31}^{-1}k_1^{-1}, k_3h_{32}^{-1}k_2^{-1}, k_2h_{32}k_3^{-1}, k_1h_{31}k_4^{-1}\}(1\,4)(2\,3)$.

Since κ is arbitrary, to do first component of $\kappa\theta(\sigma_i)\kappa^{-1}$, i=1,2,3, is identity we can choose the proper k_1, k_2, k_3, k_4.

Say $k_1 = k$ is fixed and

$k_1h_{11}k_2^{-1} = e$, $k_3h_{21}^{-1}k_1^{-1} = e$, and $k_4h_{31}^{-1}k_1^{-1} = e$ gives us the following equations.

$k_2 = kh_{11}$,

$k_3 = kh_{21}$,

$k_4 = kh_{31}$.

Using the equations we found up to now, we get

(i)
$\kappa\theta(\sigma_1)\kappa^{-1} = \{kh_{11}^{-1}k^{-1}, kh_{11}h_{11}^{-1}k^{-1}, kh_{31}h_{13}^{-1}h_{21}^{-1}k^{-1}, k^{-1}h_{21}k_{13}h_{31}^{-1}\}(1\,2)(3\,4) = \{e, e, e, e\}(1\,2)(3\,4)$

(ii) $\kappa\theta(\sigma_2)\kappa^{-1} = \{k^{-1}h_{21}h_{21}^{-1}k^{-1}, kh_{31}h_{22}^{-1}h_{11}^{-1}k^{-1}, kh_{21}h_{21}^{-1}k^{-1}, kh_{11}h_{22}h_{31}^{-1}k^{-1}\}(1\,3)(2\,4) = \{e, e, e, e\}(1\,3)(2\,4)$

(iii) $\kappa\theta(\sigma_3)\kappa^{-1} = \{kh_{31}h_{31}^{-1}k^{-1}, k^{-1}h_{21}h_{32}^{-1}h_{11}^{-1}k, kh_{11}h_{32}h_{21}^{-1}k^{-1}, kh_{31}h_{31}^{-1}k^{-1}\}(1\,4)(2\,3) = \{e, e, e, e\}(1\,4)(2\,3)$.

As a result, we get that $t' = \theta(s) = \{e, e, e, e\}s = s$. It follows that where $\theta(A_4) = T'$, T' is A_4.

Case 3: $\Sigma_{n,A}(H)$**,** $n \neq 5$

THEOREM 4.1.2. *The group $\Sigma_{n,A}(H)$ splits over the basis group, $\Sigma_{n,A}(H) = V_n \cup T$, $V_n \cap T = E$. The group T is conjugate to some group T' obtained as follows. Let G be a subgroup of H which is the homomorphic image of A_{n-1}. Let $g_4, ..., g_n$ be generators of G, satisfying the following relations: (i) $g_i^3 = e, i = 4, ..., n$,*

(ii) $(g_i g_j)^2 = e$ where $i \neq j$.

Let $s_i = (1\ i\ 2)$ for i=3,...,n generate the group A_n. Then the elements of A_n with the aid of the isomorphism θ defined by $\theta(s_3) = t_3' = \{e, e, e, g_4, ..., g_n\}(1\ i\ 2)$.

$\theta(s_i) = t_i' = \{e, g_i, g_i^2, g_i^2 g_4, ..., g_i^2 g_{i-1}, ...g_,^2, g_i^2 g_{i+1}, ..., g_i^2 g_n\}(1\ i\ 2)$ *for i=4,...,n.*

Proof. The group $\Sigma_{n,A}(H)$ consists of all symmetries where the permutation part is an element of alternating group A_n. Again the group H is an arbitrary group as in the case of $\Sigma_n(H)$ complete monomial group. $\Sigma_{n,A}(H)$= $(H \times H \times ... \times H) \rtimes A_n \simeq H \wr A_n$. The action of A_n on the direct product as before permutes the factors. Let $V_n = H \times H \times ... \times H$ and A_n is the alternating group on n letters. So,

$$\Sigma_{n,A}(H) = V_n \rtimes A_n \ \ i.e. \ V_n \cap A_n = 1$$

$V_n.A_n = \Sigma_{n,A}(H)$ so $\Sigma_{n,A}(H)$ splits over V_n.

In this section we will consider the splitting problem of $\Sigma_{n,A}(H)$. Since $\Sigma_{n,A}(H) = V_n.A_n$ for any element g$\in$ $\Sigma_{n,A}(H)$, we have $(V_n.A_n)^g = V_n^g.A_n^g = V_n.A_n^g$, and so when A_n is a complement of V_n, then any conjugate of A_n namely A_n^g is also a complement of V_n. But there are cases that there might be other complements T of V_n i. e. $V_n.T = \Sigma_{n,A}(H)$ and $V_n \cap$T=1 but T may not be a conjugate of A_n. (It is clear that $V_n.T = V_n.A_n$ and $V_n.T/V_n = V_n.A_n/V_n \simeq A_n/A_n \cap V_n \simeq A_n$, and $V_n.T/V_n \simeq T/V_n \cap T \simeq T$.) Hence every complement is isomorphic to alternating group A_n. But we are interested in when T and A_n are conjugate If all complements of V_n are conjugate, then we say that $\Sigma_{n,A}(H)$ splits regularly.

Assume that T is a complement of V_n. Then by above, T is isomorphic to

A_n . Moreover, as $\Sigma_{n,A}(H) = V_n.T$ the isomorphism

$$\theta : A_n \longrightarrow T$$

can be written in the form that $\theta(a) = v_a a$ where $v_a \in V_n, a \in A_n$. The natural isomorphism. (Every such isomorphism should be natural isomorphism.)

<u>Claim:</u> For $i \neq j, 1 \neq i, 1 \neq j$ the elements (1 i 2) generate the alternating group A_n where i=3, . . . , n.

By taking conjugate of (1 i 2) with (1 j 2) we have

$(1 \ i \ 2)^{(1 \ j \ 2)} = (j \ i \ 1) = (1 \ j \ i)$.

So we may obtain all 3-cycles of the form (1 i j) where $i \neq j$.

$(1 \ k \ j)^{(1 \ 2 \ i)}$= (k i j) so we may obtain all 3-cycles of the form (i j k). Hence the group $A_n =< (1 \ i \ 2) \mid i = 3, ..., n >$.

Let s_i=(1 i 2). Since A_n is generated by s_i, then T is generated by $\theta(s_i)$. Then $\theta(s_i) = t_i \in T$ and $t_i = s_i v_i$ where $t_i = \{h_{1i}, h_{2i}, ..., h_{ni}\}(1 \ i \ 2)$ where i=3, . . , n. So we have $t_3, t_4, t_5, ..., t_n$ i. e. we have n-2 t_i's.

Since we want to find the complement T up to conjugacy we may take conjugate of all t_i with a fixed product $v = \{k_1, k_2, ..., k_n\}$.

$t'_i = vt_iv^{-1} = \{k_1, k_2, ..., k_n\}\{h_{1i}, h_{2i}, ..., h_{ni}\}(1\ i\ 2)\{k_1^{-1}, k_2^{-1}, ..., k_n^{-1}\}$

$= \{k_1h_{1i}k_i^{-1}, k_2h_{2i}k_1^{-1}, ..., k_ih_{ii}k_2^{-1}, ..., k_jh_{ji}k_j^{-1}, ...\}(1\ i\ 2)$

Since we want to find complement T of V_n up to conjugacy and $k_i's$ are arbitrary, we may substitute $k_i's$. So, let k_1 be arbitrary fixed element in H. Choose $k_i = k_1h_{1i}$ for i=3, . . , n.

Choose $k_2 = k_1h_{23}$. Then T'=vTv^{-1} contains $t'_3 = \{e, e, g_{33}, ..., g_{n3}\}(132)$ $t'_i = \{e, g_{2i}, ..., g_{ni}\}(1\ i\ 2)$ for i=4, . . . , n. So, for i=3 and $k_1 = k$ be an arbitrary $k_2h_{23}k_1^{-1} = e$. We can solve k_2 as $k_2 = k_1h_{23}^{-1}$. Hence from the 1^{st} component $k_1h_{13}k_3^{-1}$=e then $k_3 = k_1h_{13}$.

$t'_3 = \{e, e, g_{33}, ..., g_{n3}\}(1\ 3\ 2)$

Now, for i$\geq$ 4 we have,

$t'_i = \{e, g_{2i}, ..., g_{ni}\}(1\ i\ 2)$ where i=4, . . . , n , and $g_{ni} = k_nh_{ni}k_n^{-1}$

Since s_i=(1 i 2) is a 3-cycle, s_i^3=1 . Then $t_i = \theta(s_i)^3 = 1$.

Consider s_is_j=(1 i 2)(1 j 2)=(1 i)(2 j) where $i \neq j$. Then $(s_is_j)^2 = 1$.

$(t'_i)^3 = \{g_{ii}g_{2i}, g_{2i}g_{ii}, ..., g_{ii}g_{2i}g_{ii}, ..., g_{ji}^3, ...\}$

$(t'_it'_j)^2 = \{g_{ij}g_{ii}g_{2j}, g_{2i}g_{ji}g_{jj}, ..., g_{ii}g_{2j}g_{ij}, ..., g_{ji}g_{jj}g_{2i}, ..., (g_{ki}g_{kj})^2\}$

Recall, we have the isomorphism *: $S_n \longrightarrow T'$ such that $*(s_i) = t'_i$ Here, we have a 3-cycle for all t'_i where i =3, . . . , n for obtaining alternating group A_n. So where $*(s_i) = t'_i$, $|s_i| = |t'_i|$ since * is an isomorphism. Therefore, $(t'_i)^3 = E$. If we look at the order of $(t'_it'_j)$ we should think order of (s_is_j).

(s_is_j)=(1 i 2)(1 j 2)=(1 i)(2 j). We see $|s_is_j|$=2. So, $|(t'_it'_j)|$=2. We get $(t'_it'_j)^2 = E$ where E is the identity of T'.

We have from above calculation $(t'_i)^3 = \{g_{ii}g_{2i}, g_{2i}g_{ii}, ..., g_{ii}g_{2i}g_{ii}, ..., g_{ji}^3, ...\}$ implies that $g_{ii}g_{2i} = e$ and $g_{ij}g_{ii}g_{2j} = e$ where $i \neq j$and i, j$\in \{3, 4, ..., n\}$

Noting that $g_{1i} = e = g_{1j} = e = g_{23} = e$, and writing g_i for g_{2i} we have $g_i = g_{2i} = g_{ii}^{-1}$ and $g_{33} = g_3 = e$ since $g_{33}^{-1} = g_{23} = e$.

Also, $g_{ij} = (g_{ii}g_{2j})^{-1} = g_{2j}^{-1}g_{ii}^{-1} = g_j^{-1}g_i = g_{ji}^{-1}$

$g_{i3}g_{ii}g_{23}$=e implies that $g_{i3} = g_{ii}^{-1} = g_i = g_{3i}^{-1}$ as $g_{ij} = g_{ji}^{-1}$.

If we use these equations then t'_i will have the form

$t'_i = \{e, g_i, g_i^{-1}, g_i^{-1}, g_4, ..., g_i^{-1}g_n\}$.

If $k > 2$ where $k \neq j$, $k \neq i$ in $(t'_i)^3$ and $(t'_it'_j)^2$ the k^{th} factor will satisfy $g_{k_i}^3 = (g_i^{-1}g_k)^3 = e$, $(g_{k_i}g_{k_j})^2 = e$.

For k=3, we found $g_i^3 = (g_ig_j)^2 = e$.

So, the elements g_i where i=3,...,n generate a homomorphic group to A_{n-1}.

The first, second and the i^{th} factors of $(t'_i)^3 are g_i^3$, g_i^3, g_i^3, respectively, and the first, second, i^{th} and the j^{th} factors of $(t'_it'_j)^2$ are $g_j^2g_ig_i^2g_j$, $g_ig_i^2g_jg_j^2$, $g_i^2g_jg_j^2g_i$, $g_i^2g_jg_i^2g_i$, respectively.

Those above factors are e. If $k > 2, the k^{th}$ factors of $(t'_i)^3$ and $(t'_it'_j)^2$ are $(g_i^2g_k)^3$, $(g_i^2g_kg_j^2g_k)^2$ where $k \neq i$ and $k \neq j$, respectively. These factors also e.

Therefore there is n-2 elements in the generating set of the group which is homomorphic image of A_n. Permutation part of T' is in A_n. So T' is isomorphic to A_n.

We found that $T \cong A_n, V \cap T' = E$. Moreover, if y is in $\Sigma_{n,A}(H)$, then $y = vv_1^{-1}v_1s = v_2t$ where $\theta(s) = v_1s$. Hence, $\Sigma_{n,A}(H) = V_n \cup T$.

□

THEOREM 4.1.3. *The group $\Sigma_{n,A}(H)$ splits regularly over the basis group if and only if H contains no non-trivial subgroup which is homomorphic image of A_{n-1}.*

Proof. Assume that H contains no non-trivial subgroup which is isomorphic image of A_{n-1}. Then the complement T' obtained as in Theorem 4.1.2 is simply A_n. Hence splitting is regular over the basis group.

Conversely, if the group $\Sigma_{n,A}(H)$ splits regularly, then by taking the conjugate of T' with an element$v \in V_n$. Where $v = \{k_1, k_2, ...\}$, we obtain $vt'_3v^{-1} = \{k_1k_3^{-1}, k_2k_1^{-1}, k_3k_2^{-1}, ..., k_ng_nk_n^{-1}\}(1\ 3\ 2)$. This element is a permutation means that $k_ig_ik_i^{-1} = e$ for all i=4,...,n. By multiplying from left by k_i^{-1} and from right by k_i we obtain $g_i = e$ for all i=4,...,n. It follows that the image $G = \{e\}$. □

COROLLARY 4.1.4. *The group $\Sigma_{n,A}(H)$ for n=4,5 splits regularly over the basis group if and only if H contains no element of order 3.*

Proof. For n=4, the group H contains an isomorphic copy of $A_{n-1} = A_3$. But there exists no element of order 3 in H implies that $G = \{e\}$. Then the epimorphism between A_{n-1} and the group G in H will be trivial projection and T' will be A_3.

Conversely, if the splitting is regular, then T' will be conjugate of A_4. Then the multiplication part of the complement is trivial. Hence G contains no element of order 3.

For $\Sigma_{5,A}(H)$, $A_{n-1} = A_4$ and the epimorphism $\phi : A_4 \longrightarrow G$, G is a subgroup of H we can say that all 3-cycles will go to identity. But in the elementary Abelian group K in A_4 all elements will be product of two 3 cycles and the homomorphism ϕ will send all elements of A_4 into identity. Hence $G = \{e\}$ and the splitting will be regular. □

Theorem 4.1.3 implies for $n \geq 6$, the following corollary.

COROLLARY 4.1.5. *Let $n \geq 6$. The group $\Sigma_{n,A}(H)$ splits regularly over the basis group if and only if H contains no subgroup isomorphic to A_{n-1}.*

4.2 Splitting of $\Sigma_A(H; B, B^+, d)$

Now we go back to infinite case and discuss the splitting of $\Sigma_A(H; B, B^+, d)$ over the base group $V(B, B^+)$.

In order to be able to talk about infinite alternating groups, each element must have a finite support and so we can talk the permutation is odd or even. For this reason for the splitting $\Sigma_A(H; B, B^+, d)$ we lie inside finitary symmetric group and hence even permutations are in finitary symmetric group.

In the proof of the Theorem 4.1.2, we discuss the isomorphism between S(B,C) and $S_1(B, C)$ when C is an infinite cardinal.

The same proof will work for infinite alternating groups namely if B is infinite cardinal then $A(B, d) \cong A_1(B, d)$ where $A_1(B, d)$ is the alternating group on the set $B \setminus \{1\}$.

By using the similar technique for the infinite case one can prove the following theorem.

Theorem 4.2.1. *The complete alternating group $\Sigma_A(H; B, B^+, d)$ splits over the basis group $V(B, B^+)$.*

Proof. Two conjugate complements T and T' may be obtained by the following method.
Let G be a subgroup of H obtained as a homomorphic image of A(B,d). Let $g_4, ..., g_\epsilon$ be generators of the group G with the following relations

(a) $g_\epsilon^2 = e$, and

(b) $(g_\epsilon g_\delta)^2 = e$ for $\epsilon \neq$.

We choose as generators of A(B,d) the three cycles $s_\alpha = (1\ \alpha\ 2)$ where α=3,4,... Then the elements of the complement T' are obtained by the isomorphism θ where $\theta(s_3) = t'_3 = \{e, e, e, g_4, ..., g_\epsilon, ...\}(1\ 3\ 2)$

$\theta(s_\alpha) = t'_3 = \{e, g_\alpha, g_\alpha^2, g_\alpha g_4, ..., g^{\epsilon 2}, g_\alpha g_\epsilon\}(1\ \alpha\ 2)$

□

Theorem 4.2.2. *The group $\Sigma_A(H; B, B^+, d)$ splits regularly over the basis group $V(B, B^+)$ if and only if the group H does not contain a subgroup isomorphic to the alternating group A(B,d).*

Proof. It is well known that infinite alternating groups are simple. See also [4].

By theorem, there exists a homomorphism $\phi : A(B,d) \longrightarrow T$. Since A(B,d) is simple we have two cases; $\phi(B,d) \cong A(B,d)$ i.e., $Ker\phi = \{1\}$ or $Ker\phi = A(B,d)$.

Case 1: If $Ker\phi = \{1\}$, then $\phi(A(B,d))$ is a subgroup of H isomorphic to A(B,d). But by assumption H does not contain any subgroup isomorphic to A(B,d). Hence this case is impossible.So, the second case happens.

Case 2: In this case $Ker\phi = A(B,d)$, and it follows that $\phi(A(B,d)) = \{1\}$ i.e., $Ker\phi = A(B,d)$. Hence, ϕ is the identity map in this case and so $\Sigma_A(H; B, B^+, d)$ splits regularly.

Conversely, assume that the complete alternating monomial group splits regularly over the basis group $V(B, B^+)$. Then by a conjugate of an element of $\Sigma_A(H; B, B^+, d)$, the complement T may be transformed to A(B,d). But as in the case of finite case, this implies that the subgroup G in H which is the homomorphic image of A(B,d) will be the identity group i.e., $G = \{e\}$. So, H contains no subgroup isomorphic to A(B,d).

□

COROLLARY 4.2.3. *For a given group H, there exists a complete monomial alternating group $\Sigma_A(H; B, B^+, d)$ such that the splitting of the monomial group over the basis group is regular.*

Proof. If we choose the cardinal B such that the order of A(B,d)=B is strictly greater than the order of H, then by the above Theorem 4.2.2), the isomorphism $\phi : A(B,d) \longrightarrow H$ must be an epimorphism i.e., $\phi(A(B,d)) = \{1\}$. Because in the other case as A(B,d) is simple, ϕ must be one to one and hence H must contain an isomorphic copy of A(B,d) which is impossible by the order of H, namely $|H| \lneqq B = |A(B,d)|$. □

REFERENCES

[1] R. Crouch, *Monomial Groups* Trans. Amer. Math. Soc. , volume 80, 187-215, 1955.

[2] O. Ore, *Theory of Monomial Groups* Trans. Amer. Math. Soc. , volume 51, 15-64, 1942.

[3] A. Kerber, *Representations of Permutation Groups, Lecture Notes in Mathematics, No:240, Springer-Verlag, Berlin-Heidelberg-New York, 1971* Trans. Amer. Math. Soc. , volume 80, 187-215, 1955.

[4] R. Baer, *Die Kompositionsreihe der Gruppe alter Eineindeutigen Abbildungen einer Unendlichen Menge auf sich*, Studia Mathematica vol. 5 (1934) pp. 15-17.